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HOW TO WIN A WAR IF YOU MUST: OPTIMAL STOPPING BASED ON SUCCESS RUNS[1]

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A coin is tossed repeatedly at a fixed cost per toss. The payoff is the length of the terminal run of heads, less the cost of tossing. Properties of the dynamic programming solution are derived; the exact optimal policy and value of the game are obtained when the game has an infinite horizon, and the rate at which this solution is approached by the sequence of truncated strategies is analyzed numerically.

1. Introduction. In order to negotiate a peace settlement "from a position of strength" a nation desires to maximize, subject to a cost constraint, its terminal number of consecutive field victories. It is assumed that at most n military engagements, each with constant cost and probability of success, may be sequentially staged.

The problem obviously has a simple coin tossing analogue; we shall formalize in this context. Consider a game in which we may make at most n independent tosses of a coin, one at a time. The constant probability p $(0 < p < 1)$ at each toss is known. After each toss we must decide, on the basis of past outcomes only, either to *continue* tossing or to *stop*; in the latter case the game ends. (If all n tosses are made we must stop.) If the game terminates at toss k, $0 \leqq k \leqq n$, then our payoff is

$$\phi(k) = r_k - ck ,$$

where by definition r_k is the length of the terminal success run; that is

$$r = r \qquad \text{if heads resulted on the } (k - r + 1)\text{st thru } k\text{th toss,}$$
$$r_k = 0 \qquad \text{if tails resulted on toss } k, \qquad\qquad (0 < k \leqq n,\ 0 \leqq r \leqq k)$$

and $r_0 = 0$. Here the nonnegative constant cost c of tossing is known, and we assume throughout is at most one, for otherwise $\phi(k) < 0$ and it is trivially optimal to make no tosses. Our objective is to study the stopping policy t for which $E\phi(t)$, the expected value of the payoff, is maximized; we shall call the optimal policy τ_n. As stated, we restrict ourselves to policies t with $0 \leqq t \leqq n$, for which the event $\{t > k\}$ may depend on the first k tosses only, $0 \leqq k < n$.

2. The optimal policy. We may, with very little preparation, write down the functional equations which define the optimal policy. It will be convenient to refer to an n-stage game as one in which we may make at most n tosses, and to say that the game is in state (j, r) if after j tosses we are on succes run of

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length r, $0 \leqq r \leqq j \leqq n$. For fixed $n \geqq 0$, define

(1) $x_n(j, r) =$ (conditional) expected value of $\phi(\cdot)$ using the optimal policy for an n-stage game in state (j, r);

calling $x_n = x_n(0, 0)$ observe that, using the optimal policy, the expected payoff for an n-stage game is by definition

$$E\phi(\tau_n) = x_n\,. \tag{2}$$

Suppose now that consistent with the optimal policy we have made j tosses and are in state (j, r). Either we must stop, in which case our payoff is

$$r - cj\,, \tag{3}$$

or (assuming $j < n$) continue. If we make the $(j + 1)$st toss, then we move to state $(j + 1, r + 1)$ with probability p and to state $(j + 1, 0)$ with probability $q = 1 - p$; thus, if we are in state (j, r) and continue the (conditional) expected value of $\phi(\tau_n)$ will be

$$x_n(j + 1, r + 1)p + x_n(j + 1, 0)q\,. \tag{4}$$

TABLE 1

$p = .25$

Expected payoff $x_n = E\phi(\tau_n)$ *for an n stage game*

$n \backslash c$	.0625	.015625	.0039062	.0009766	.0000002
2	0.3281250	0.4101563	0.4306641	0.4357910	0.4374996
4	0.5126953	0.6408691	0.6729126	0.6809235	0.6835930
6	0.6165161	0.7706451	0.8093719	0.8212395	0.8251942
8	0.6749153	0.8631048	0.9228122	0.9393784	0.9448987
10	0.7077656	0.9468641	1.0243645	1.0452003	1.0521431
15	0.7399788	1.1209526	1.2353230	1.2650328	1.2749338
20	0.7476225	1.2541227	1.3966961	1.4331942	1.4453554
30	0.7498674	1.4339142	1.5201387	1.6602249	1.6754408
50	0.7500000	1.6006737	1.8520765	1.9349222	1.9635849
75	·	1.6647587	2.0680399	2.1938925	2.2366114
100	·	1.6815434	2.2271042	2.3846359	2.4377079
125	·	1.6858530	2.3442631	2.5251284	2.5858231
150	·	1.6870899	2.4305553	2.6286068	2.6952019
175	·	1.6873875	2.4941168	2.7060766	2.7885237
200	·	1.6874609	2.5409307	2.7748709	2.8751669
$n \to \infty$	$x = 0.7500000$	$x = 1.6875000$	$x = 2.6718750$	$x = 3.6679687$	$x = 9.6666670$

Length of run necessary to stop if at most k tosses remain

$k \geqq 1$ 1	$1 < k < 6$ 1	$1 \leqq k < 5$ 1	$1 \leqq k < 5$ 1	$1 \leqq k < 4$ 1
	$k \geqq 6$ 2	$5 \leqq k < 34$ 2	$5 \leqq k < 32$ 2	$4 \leqq k < 30$ 2
		$k \geqq 34$ 3	$32 \leqq k < 163$ 3	$30 \leqq k < 144$ 3
			$k \geqq 163$ 4	$144 \leqq k \leqq 200$ 4
				$k > 200$ *
$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 10$

* computations not performed.

Accordingly, if in an n-stage game we reach state (j, r), $0 \leqq r \leqq j < n$, then we stop or continue according as (3) or (4) is larger (in the case of equality it is immaterial), and our expected payoff is

$$x_n(j, r) = \max(r - cj, x_n(j+1, r+1)p + x_n(j+1, 0)q)\,; \tag{5}$$

after n tosses we must stop and receive

$$x_n(n, r) = r - cn \qquad (0 \leqq r \leqq n)\,. \tag{6}$$

For fixed n, the $x_n(\cdot, \cdot)$ sequence determines the optimal policy: make τ_n tosses, where

$$\tau_n = \text{least integer } 0 \leqq j \leqq n \text{ for which we enter state } (j, r) \text{ with } x_n(j, r) = r - cj\,. \tag{7}$$

As stated, it does not matter whether we stop or continue when (3) and (4) are

TABLE 2

$p = .5$

Expected payoff $x_n = E\phi(\tau_n)$ for an n stage game

n \ c	.25	.125	.0625	.03125	.0004883
2	0.3750000	0.5625000	0.6562500	0.7031250	0.7492676
4	0.4687500	0.7968750	0.9921875	1.0898437	1.1859741
6	0.4921875	0.9531250	1.2109375	1.3398437	1.4667358
8	0.4980469	1.0556641	1.3540039	1.5256348	1.7041969
10	0.4995117	1.1228027	1.4736328	1.6854248	1.9027996
15	0.4999847	1.2059174	1.6964455	1.9856033	2.2759075
20	0.5000000	1.2347412	1.8430948	2.1830168	2.5640478
30	·	1.2481842	2.0030212	2.4540501	3.0025377
50	·	1.2500000	2.1021652	2.7715740	3.5773172
75	·	·	2.1221933	2.9468279	4.0633173
100	·	·	2.1246595	3.0165081	4.4124508
125	·	·	2.1249628	3.0442123	4.6966114
150	·	·	2.1249990	3.0552254	4.9278555
175	·	·	2.1249999	3.0596008	5.1175938
200	·	·	2.1250000	3.0613384	5.2859497
$n \to \infty$	$x = 0.5000000$	$x = 1.2500000$	$x = 2.1250000$	$x = 3.0625000$	$x = 9.0009766$

Length of run necessary to stop if at most k tosses remain

	$k \geqq 1$ 1	$k = 1$ 1	$k = 1$ 1	$k = 1$ 1	$k = 1$ 1
		$k > 1$ 2	$1 < k < 7$ 2	$1 < k < 5$ 2	$1 < k < 5$ 2
			$k \geqq 7$ 3	$5 \leqq k < 18$ 3	$5 \leqq k < 13$ 3
				$k \geqq 18$ 4	$13 \leqq k < 31$ 4
					$31 \leqq k < 73$ 5
					$73 \leqq k < 161$ 6
					$161 \leqq k \leqq 200$ 7
					$k > 200$ *
	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 10$

* computations not performed.

equal; notice that we have (by convention) included this case in the stopping region (7).

3. Some properties of the optimal policy. We shall need a few preliminary results which are summarized as

LEMMAS. *For* $0 \leqq r \leqq j \leqq n$,

(A) $x_n(j, 0) = x_{n-j} - cj$,
(B) $x_n(j, r) \leqq x_{n+1}(j, r)$,
and for $j < n$
(C) $x_n(j, 0) \geqq x_n(j + 1, 0)$.

Assertion (C) follows immediately from (A) and (B). (A) asserts that if in an n-stage game we have made j tosses and are on a success run of length zero (toss j was a tail), then our expected payoff is simply the expected payoff of an $(n - j)$-stage game, less the cj units we have already spent. (B) states that the expected

TABLE 3
$p = .75$
Expected payoff $x_n = E\phi(\tau_n)$ *for an n stage game*

n \ c	.5625	.421875	3164063	.2373047	.0422351
2	0.2343750	0.4687500	0.6796875	0.8378906	1.2280312
4	0.2490234	0.6159668	0.9279785	1.2016754	1.8996611
6	0.2499390	0.6638947	1.0775452	1.4176788	2.4053812
8	0.2500000	0.6797180	1.1599121	1.5701141	2.7920160
10	·	0.6849365	1.2061005	1.6784821	3.1127768
15	·	0.6873627	1.2515259	1.8330688	3.7338362
20	·	0.6875000	1.2623291	1.8994141	4.1919975
30	·	·	1.2654877	1.9400940	4.8485184
50	·	·	1.2656250	1.9490051	5.6470566
75	·	·	·	1.9492187	6.2113447
100	·	·	·	·	6.5403185
125	·	·	·	·	6.7558174
150	·	·	·	·	6.8974333
175	·	·	·	·	6.9904985
200	·	·	·	·	7.0516567
$n = \infty$	$x = 0.2500000$	$x = 0.6875000$	$x = 1.2656250$	$x = 1.9492187$	$x = 7.1689405$

Length of run necessary to stop if at most k tosses remain

	$k \geqq 1$ 1	$k \geqq 1$ 2	$k = 1$ 2	$1 \leqq k < 4$ 3	$k = 1$ 3
			$k > 1$ 3	$k \geqq 4$ 4	$k = 2$ 4
					$3 \leqq k < 6$ 5
					$6 \leqq k < 11$ 6
					$11 \leqq k < 20$ 7
					$20 \leqq k < 38$ 8
					$38 \leqq k < 74$ 9
					$k \geqq 74$ 10
	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 10$

payoff is a non-decreasing function of the number of stages in the game, no matter what state the game is in, and in particular $x_n \leqq x_{n+1}$, $\forall n \geqq 0$. These results are an immediate consequence of the definition (1) of the $x_n(\cdot, \cdot)$ sequence; they may also be proved directly by an induction on the functional equations (5) and (6), which we here omit.

In order to simplify our description of the optimal policy, we shall prove

THEOREM 1. *For* $0 \leqq r \leqq j < n$

$$x_n(j, r) = r - cj \tag{8}$$

if and only if (iff)

$$r - cj \geqq p/q(1 - c) + x_n(j + 1, 0) . \tag{9}$$

The theorem implies that if in an n-stage game we reach state (j, r), $j < n$, then $\tau_n = j$ just in case (9) holds.

To prove the neccessity of (9), suppose (8) holds; then from (5)

$$r - cj \geqq (r + 1 - c(j + 1))p + x_n(j + 1, 0)q \tag{10}$$

and (9) follows upon rewriting (10). Since (9) and (10) are equivalent, we shall prove sufficiency by assuming (10), and arguing by induction. Suppose (10) holds with $j = n - 1$; that is

$$r - c(n - 1) \geqq (r + 1 - cn)p + x_n(n, 0)q . \tag{11}$$

Since by definition (6) $x_n(n, r + 1) = r + 1 - cn$, (11) yields

$$r - c(n - 1) \geqq x_n(n, r + 1)p + x_n(n, 0)q ,$$

and (8) follows from (5), proving sufficiency for $j = n - 1$. Suppose now that for arbitrary j, $0 < j \leqq n - 1$, (10) implies (8), and assume that (10) holds for $j - 1$; that is, for $0 \leqq r \leqq j - 1$,

$$r - c(j - 1) \geqq (r + 1 - cj)p + x_n(j, 0)q . \tag{12}$$

From (12)

$$\begin{aligned} r + 1 - cj &\geqq (r + 2 - c(j + 1))p + x_n(j, 0)q \\ &\geqq (r + 2 - c(j + 1))p + x_n(j + 1, 0)q , \end{aligned}$$

where the last inequality follows from Lemma (C). Thus (10) holds in state $(j, r + 1)$, implying that

$$x_n(j, r + 1) = r + 1 - cj \tag{13}$$

by the induction assumption. Substituting (13) in (12) gives

$$r - c(j - 1) \geqq x_n(j, r + 1)p + x_n(j, 0)q$$

and hence from (5)

$$x_n(j - 1, r) = r - c(j - 1) ,$$

completing the induction.

In passing we mention

COROLLARY 1. *For* $0 \leqq r \leqq j < n$

$$x_n(j, r) = r - cj \to x_n(j+1, r+1) = r + 1 - c(j+1), \tag{14}$$

so that if it is optimal to stop, we should also stop upon making an additional toss resulting in a head; equivalently, if it is optimal to continue beyond the next toss if it is a head, then we should make the next toss.

The proof of (14) is trivial (from (6)) when $j = n - 1$, and for $j < n - 1$ follows upon noting

$$\begin{aligned} x_n(j, r) = r - cj &\to r - cj \geqq (r + 1 - c(j+1))p + x_n(j+1, 0)q \\ &\to r + 1 - c(j+1) \geqq (r + 2 - c(j+2))p + x_n(j+2, 0)q \\ &\to x_n(j+1, r+1) = r + 1 - c(j+1). \end{aligned}$$

A more significant implication of Theorem 1 is that we may now describe the optimal policy for an n-stage game, previously defined in (7), as

$$\begin{aligned} &\tau_n = \text{least integer } 0 \leqq j < n \text{ for which we enter state } (j, r) \\ &\text{with} \\ &\qquad r \geqq \frac{p-c}{q} + x_{n-j-1} \end{aligned} \tag{15}$$

and $\tau_n = n$ if no such state results before toss n. Expression (15) follows immediately from Theorem 1 and Lemma (A), and has several interesting consequences.

COROLLARY 2. *For any n-stage game, $n \geqq 1$, $\tau_n = 0$ iff $p \leqq c$; moreover, if $p > c$ and we reach state $(j, 0)$, $0 \leqq j < n$, then $\tau_n > j$.*

In other words, make no tosses if $p \leqq c$, and always toss on a run of length zero if $p > c$. For proof, suppose $p > c$ and that we enter state $(j, 0)$; since $x_n \geqq 0$ for every n, we have for $p > c$

$$\frac{p-c}{q} + x_{n-j-1} \geqq \frac{p-c}{q} > 0,$$

and $\tau_n > j$ by (15). Conversely, suppose $p \leqq c$, and that $\tau_n > 0$. From (15), for $p \leqq c$

$$\begin{aligned} 0 < \frac{p-c}{q} + x_{n-1} \leqq x_{n-1} \to \tau_{n-1} > 0 \\ &\to 0 < \frac{p-c}{q} + x_{n-2} \leqq x_{n-2} \to \cdots \\ &\to 0 < \frac{p-c}{q} + x_1 \leqq x_1 = \max(0, p-c) = 0, \end{aligned}$$

so that a contradiction results, proving $\tau_n = 0$ when $p \leqq c$.

In succeeding sections, we shall use (15) to develop an asymptotic theory as well as exact numerical results. However, the simple form of (15) invites some intuitive discussion before proceeding. Notice that (15) may be written

$$r - cj \geqq (r+1)p - c(j+1) + x_{n-j-1} \cdot q$$

where x_{n-j-1} is the expected payoff using the optimal policy for an $(n-j-1)$-stage game and

$$(r+1)p - c(j+1)$$

is the expected payoff if having reached state (j, r) we make just one more toss. Accordingly, the optimal policy has the following interesting form:

> stop in state (j, r) iff the present payoff is at least the expected payoff if we make just one more toss plus q times the expected payoff if, beyond that toss, we start a new $((n-j-1)$-stage) game.

We cannot explain why the continuation boundary should be additive in this manner.

4. Approximations. In order to sharpen our insights concerning the optimal policy for an n-stage game, we now let $n \to \infty$ and study the limiting behavior of τ_n and x_n. Our results will follow directly from

LEMMA (D). *Let* $y^+ = \max(0, y)$; *then for* $c > 0$

$$E[\sup_n (r_n - cn)^+] < \infty . \tag{16}$$

For proof, we have for $z > 0$ fixed

$$P(\sup(r_n - cn) > z) \leqq \sum_{n=0}^{\infty} P(r_n \geqq z + cn)$$
$$\leqq \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} p^{z+cn+j} = \frac{1}{q} \cdot \frac{1}{1-p^c} \cdot p^z ,$$

so that

$$E[\sup(r_n - cn)^+] = \int_0^\infty P(\sup(r_n - cn) > z)\, dz \leqq -\frac{1}{q(1-p^c)\log p} < \infty .$$

From (7) and Lemma (B), notice that $0 = \tau_0 \leqq \tau_1 \leqq \cdots$, and define

$$\tau = \sup_n \tau_n = \lim_n \tau_n \leqq \infty .$$

Since for every $n \geqq 0$

$$x_n \leqq E[\sup(r_n - cn)^+]$$

it follows from Lemmas (B) and (D) that as $n \to \infty$

$$x_n \uparrow x < \infty .$$

Thus, as $n \to \infty$ the right side of (15) tends non-decreasingly to

$$\frac{p-c}{q} + x < \infty , \tag{17}$$

implying that

$$P(\tau < \infty) = 1 . \tag{18}$$

Moreover, appealing to Theorem 1 of Chow and Robbins [1], we have from (16) and (18), for $c > 0$

THEOREM 2. *$x = E\phi(\tau) = \sup_{t \in S} E\phi(t)$, where S is the class of all stopping policies t with $0 \leqq t < \infty$, for which the event $\{t > k\}$ depends on the first k tosses only, $k \geqq 0$.*

Theorem 2 asserts the existence of an optimal policy (with a finite expected payoff) for a game in which we may toss indefinitely. We determine the policy and the expected payoff in

THEOREM 3. *τ = least integer $j \geqq 0$ for which $r_j = r$, where by definition $r \geqq 0$ is the least integer satisfying*

$$p^{r+1} \leqq c ; \tag{19}$$

moreover,

$$E\phi(\tau) = r - \frac{c(1-p^r)}{qp^r} . \tag{20}$$

For proof, we observe first that lim τ_n as $n \to \infty$ is

> τ = least integer $j \geqq 0$ for which $r_j = r$, where by definition $r \geqq 0$ is the least integer satisfying

$$r \geqq \frac{p-c}{q} + x . \tag{21}$$

To see that this is so, let $\varepsilon > 0$ and $j \geqq 0$ be fixed; then from (15) and Lemma (B)

$$\begin{aligned} \tau_n \leqq j \Rightarrow r_i &\geqq \frac{p-c}{q} + x_{n-i-1} \qquad \exists i = 0, \cdots, j \\ \Rightarrow r_i &\geqq \frac{p-c}{q} + x_{n-j-1} \qquad \exists i = 0, \cdots, j . \end{aligned}$$

Moreover since $x_n \uparrow x$, $\exists N = N(\varepsilon, j) \ni$ for all $n > N$

$$\begin{aligned} \tau_n \leqq j \Rightarrow r_i &\geqq \frac{p-c}{q} + x - \varepsilon \qquad \exists i = 0, \cdots, j \\ &\Rightarrow \text{(for } \varepsilon \text{ sufficiently small) } r_i \geqq \frac{p-c}{q} + x \qquad \exists i = 0, \cdots, j \\ &\Rightarrow \tau \leqq j \Rightarrow \lim \tau_n \geqq \tau , \end{aligned}$$

and obviously lim $\tau_n \leqq \tau$.

Returning to the proof of the Theorem, we thus have from Feller [2]

$$E\phi(\tau) = E(r_\tau - c\tau) = r - cE(\tau) = r - \frac{c(1-p^r)}{qp^r} . \tag{22}$$

Since from Theorem 2 $x = E\phi(\tau)$, replacing in (21) yields

$$r \geqq \frac{p-c}{q} + r - \frac{c(1-p^r)}{qp^r},$$

which is equivalent to (19); (20) then follows immediately from (22).

Since $\tau_n \uparrow \tau$ and $x_n \uparrow x$, we may approximate the optimal policy and expected payoff for an n-stage game by (19) and (20) respectively, when n is sufficiently large. In the next section we present some numerical results for selected values of p and c, in order to determine what magnitude n must be in order for the approximations to be useful. However, we first conclude this section with a

REMARK. The referee has pointed out that when $c = 0$ (no cost of playing), there exist policies $t \in S$ for which $E\phi(t) = \infty$. To verify his contention, we construct such a policy as follows: let k_j be a sequence of integers satisfying $p^{k_j} = [(j+1)\log(j+1)]^{-1}$, put $m_i = \sum_{j=1}^{i} k_j$, and define the stopping time

$$t = m_j, \text{ where } j \geqq 1 \text{ is the least integer for which } r_{m_j} \geqq k_j;$$

then, $P(t = m_n) = p^{k_n} \prod_{j=1}^{n-1} (1 - p^{k_j}) \sim \log 2/n \log^2 n$ as $n \to \infty$.
Thus $P(t < \infty) = 1$ and since $k_n \sim -(\log P)^{-1} \log n$,

$$E\phi(t) = E(r_t) = \sum_{n=1}^{\infty} E(r_t \mid t = m_n)P(t = m_n) \geqq \sum_{n=1}^{\infty} k_n P(t = m_n) = \infty .$$

5. Numerical results. In the accompanying tables we present values of the expected payoff x_n of an n-stage game for selected values of p, c and n. Notice that the convergence to $x = E\phi(\tau)$ is quite rapid, except in the case p is small and c is very small (that is, the case in which it pays to continue tossing until we get a long run, but the probability of a long run after a short number of tosses is small).

Using (15) we also give the length of the success run we must achieve in order to stop if *at most* k tosses *remain* to us in an n-stage game, $k \leqq n$. The reader should compare these values with the terminal length r (see (19)) of our success run in a game for which there is no upper limit on the number of tosses.

It is instructive to note that only x_{200} was actually computed in compiling each table, since in the course of this computation (using (5) and (6)) it was necessary to obtain $x_{200}(j, 0)$ for $0 \leqq j \leqq 200$, and all tabled values were then easily deduced upon applying Lemma (A) together with (15).

6. Concluding. A related game. Continue to assume that we toss a coin with probability p of heads, at most n times. Now before each toss we may place a bet which is then matched by an equal amount (or "faded"), the total bet being called the *kitty*. We then win or lose the kitty according as we toss a head or tail.

Our betting strategy is as follows: At the first toss we bet one dollar, and for the jth toss, $1 < j \leqq n$ we bet one dollar if toss $j - 1$ was a tail and we bet the amount in the previous kitty or "let it ride" if toss $j - 1$ was a head. If the game is terminated after k tosses $0 \leqq k \leqq n$, this strategy of doubling up on winning bets results in the payoff

$$\phi^*(k) = 2^{r_k} - (k - h_k + 1),$$

where r_k continues to denote the length of the terminal success run, and where h_k is the number of heads resulting from the first k tosses; here $h_0 = 0$.

Call $\tau_n{}^*$ the optimal stopping policy for an n-stage game and let $x_n{}^*$ denote the expected payoff using the optimal policy. By methods analogous to (but somewhat more complicated than) those of Section 3, it may be shown that

$\tau_n{}^* =$ least integer $0 \leqq j < n$ for which we enter state (j, r) with

$$2^r \frac{(q-p)}{q} \geqq x^*_{n-j-1}, \tag{23}$$

and $\tau_n{}^* = n$ if no such state results before toss n. We may thus deduce (as we might also from martingale considerations)

THEOREM 4.

$$\begin{aligned} \tau_n{}^* &= 0 \qquad \text{if } p \leqq \tfrac{1}{2} \\ &= n \qquad \text{if } p > \tfrac{1}{2}. \end{aligned}$$

For $p > \frac{1}{2}$ the result is obvious, since $x_n{}^* \geqq 0$ for all n. For $p \leqq \frac{1}{2}$, suppose $\tau_n{}^* > c$ then from (23) for $q \geqq p$

$$\begin{aligned} 0 \leqq \frac{q-p}{q} < x^*_{n-1} \to \tau^*_{n-1} > 0 \\ \to 0 \leqq \frac{q-p}{q} < x^*_{n-2} \to \tau^*_{n-2} > 0 \\ \to \cdots \to 0 \leqq \frac{q-p}{q} < x_1{}^* = \max(0, p-q) = 0, \end{aligned}$$

and we have a contradiction.

In the case that p is unknown and a Bayes approach is adopted the optimal policy is by no means so trivial; we will study this approach presently.

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