

HOMOGENEOUS MAPPINGS OF REGULARLY VARYING VECTORS

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It is well known that the product of two independent regularly varying random variables with the same tail index is again regularly varying with this index. In this paper, we provide sharp sufficient conditions for the regular variation property of product-type functions of regularly varying random vectors, generalizing and extending the univariate theory in various directions. The main result is then applied to characterize the regular variation property of products of i.i.d. regularly varying quadratic random matrices and of solutions to affine stochastic recurrence equations under nonstandard conditions.

1. Introduction.

1.1. *Closure of regular variation under multiplication—The univariate case.* Consider a nonnegative random variable X and assume that X is regularly varying with index $\alpha > 0$ in the sense that

$$(1.1) \quad \mathbb{P}(X > x) = \frac{L(x)}{x^\alpha}, \quad x > 0,$$

where L denotes some slowly varying function; we refer to Bingham et al. [3] for an encyclopedic treatment of univariate regularly varying functions and to Resnick [20, 21] for the case of regularly varying random vectors.

A natural question appears in this context: given Y is a nonnegative random variable independent of X , under which conditions is the product XY regularly varying with index α ? This is a natural problem indeed: in numerous contexts of applied probability one studies models which involve products of independent random variables. Among those are classical time series models such as the ARCH-GARCH family and the stochastic volatility model; see Andersen et al. [1] for an extensive treatment of these models in financial time series analysis. In both cases, the real-valued time series (X_t) is given via the relation $X_t = \sigma_t Z_t$, where (σ_t) is a strictly stationary sequence of positive random variables which is either predictable with respect to the natural filtration of the i.i.d. sequence (Z_t) (such as for ARCH-GARCH) or (σ_t) and (Z_t) are mutually independent (such as for the stochastic volatility model). In both cases, there is strong interest in the tail behavior of the products $X_t = \sigma_t Z_t$ (notice that, under the aforementioned conditions, σ_t and Z_t are independent). In the ARCH-GARCH the condition $\mathbb{E}[|Z|^\alpha] < \infty$ (Z stands for a generic element of (Z_t)) and the dynamics of the volatility sequence (σ_t) ensure that $\mathbb{P}(\sigma_t > x) \sim cx^{-\alpha}$ for some positive constants c, α ; see Section 3 for more details. In turn, the condition $\mathbb{E}[|Z|^\alpha] < \infty$ and the so-called Breiman lemma imply that

$$(1.2) \quad \mathbb{P}(\pm\sigma_t Z_t > x) \sim \mathbb{E}[(Z^\pm)^\alpha] \mathbb{P}(\sigma_t > x), \quad x \rightarrow \infty.$$

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Breiman’s result [4] is contained in the following useful lemma; for a proof, see Appendix C.3 in [5].

LEMMA 1.1. *Assume X, Y are independent nonnegative random variables, X is regularly varying with index $\alpha > 0$ in the sense of (1.1), and $\mathbb{E}[Y^{\alpha+\delta}] < \infty$ for some $\delta > 0$ or $\mathbb{P}(X > x) \sim cx^{-\alpha}$ for some positive $c > 0$ and $\mathbb{E}[Y^\alpha] < \infty$. Then $\mathbb{P}(XY > x) \sim \mathbb{E}[Y^\alpha]\mathbb{P}(X > x)$ as $x \rightarrow \infty$.*

Thus the regular variation of X is preserved under multiplication with an independent nonnegative random variable Y if the corresponding assumptions on Y hold, ensuring that Y has a lighter tail than X . We already mentioned the case of an ARCH-GARCH process (X_t) when σ_t is regularly varying with index $\alpha > 0$ and X_t inherits this property if $\mathbb{E}[|Z|^\alpha] < \infty$. In the stochastic volatility model, regular variation of X_t may originate from the same property for σ_t or Z_t . In the former case, X_t is regularly varying with index $\alpha > 0$ if σ_t has the same property and $\mathbb{E}[|Z|^{\alpha+\delta}] < \infty$ for some $\delta > 0$, and then (1.2) holds. In the latter case, X_t is regularly varying with index $\alpha > 0$ if Z_t has this property in the sense that it satisfies a *tail balance condition*:

$$(1.3) \quad \mathbb{P}(Z > x) \sim p_+ \frac{L(x)}{x^\alpha} \quad \text{and} \quad \mathbb{P}(Z < -x) \sim p_- \frac{L(x)}{x^\alpha}$$

for constants p_\pm such that $p_+ + p_- = 1$ and a slowly varying function L , and $\mathbb{E}[\sigma_t^{\alpha+\delta}] < \infty$ for some $\delta > 0$, and then

$$\mathbb{P}(\pm X_t > x) \sim \mathbb{E}[\sigma_t^\alpha] \mathbb{P}(\pm Z > x), \quad x \rightarrow \infty.$$

We mention that power-law tail behavior of a stationary sequence (X_t) is essential for the asymptotic behavior of their extremes and partial sums, and related point process convergence and functionals acting on them. For example, if (Z_t) is i.i.d. and regularly varying with index $\alpha > 0$, then the sequence of the maxima $(a_n^{-1}M_n)$, where $M_n = \max_{i=1, \dots, n} Z_i$, and (a_n) satisfies $n\mathbb{P}(Z > a_n) \rightarrow 1$, converges in distribution to a Fréchet distribution $\Phi_\alpha(x) = \exp(-x^{-\alpha})$, $x > 0$; see Embrechts et al. [13], Section 3.3. Moreover, the process of the points $(a_n^{-1}X_i)_{i=1, \dots, n}$ converges in distribution to an inhomogeneous Poisson process on $(0, \infty)$ with intensity function $\alpha x^{-\alpha-1}dx$; see Resnick [20, 21], Embrechts et al. [13], Chapter 5. Similarly, if $\alpha \in (0, 2)$ and Z is regularly varying in the sense of (1.3) then for $S_n = Z_1 + \dots + Z_n$, $(a_n^{-1}(S_n - c_n))$ converges in distribution (with suitable centering constants (c_n)) to an infinite variance α -stable limit; see Feller [14] or Resnick [20]. Moreover, there is a vast literature that extends these results from the i.i.d. to the dependent case.

It is possible to relax the condition $\mathbb{E}[Y^{\alpha+\delta}] < \infty$ in Breiman’s result (Lemma 1.1); see for example, [7, 11]. For completeness of the presentation we mention some related results for independent nonnegative random variables X, Y when one or both are regularly varying with index α . This situation is much more subtle than in the Breiman case but, still, XY is regularly varying:

LEMMA 1.2. *Assume that X, Y are independent nonnegative random variables and X is regularly varying with index $\alpha > 0$. Then the following statements hold:*

- (1) *If either Y is regularly varying with index α or $\mathbb{P}(Y > x) = o(\mathbb{P}(X > x))$ as $x \rightarrow \infty$ then XY is regularly varying with index α .*
- (2) *If $\mathbb{E}[Y^\alpha] = \infty$ then $\lim_{x \rightarrow \infty} \mathbb{P}(XY > x)/\mathbb{P}(X > x) = \infty$.*

(3) If $\mathbb{E}[Y^\alpha] < \infty$ then the following limit relations are equivalent:

$$(1.4) \quad \lim_{x \rightarrow \infty} \frac{\mathbb{P}(XY > x)}{\mathbb{P}(X > x)} = \mathbb{E}[Y^\alpha],$$

$$\lim_{\varepsilon \rightarrow 0} \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(XY > x, X \leq \varepsilon x)}{\mathbb{P}(X > x)} = 0.$$

(4) If Y is also regularly varying with index α , $\mathbb{E}[X^\alpha + Y^\alpha] < \infty$ and

$$c_0 = \lim_{x \rightarrow \infty} \frac{\mathbb{P}(Y > x)}{\mathbb{P}(X > x)} \in [0, +\infty),$$

then the following limit relations are equivalent:

$$(1.5) \quad \lim_{x \rightarrow \infty} \frac{\mathbb{P}(XY > x)}{\mathbb{P}(X > x)} = \mathbb{E}[Y^\alpha] + c_0 \mathbb{E}[X^\alpha],$$

$$\lim_{M \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(XY > x, M < Y \leq x/M)}{\mathbb{P}(X > x)}$$

$$= \lim_{M \rightarrow \infty} \limsup_{x \rightarrow \infty} \int_M^{x/M} \frac{\mathbb{P}(X > x/y)}{\mathbb{P}(X > x)} \mathbb{P}(Y \in dy) = 0.$$

(5) Assume the conditions of (4) and $c_0 > 0$. If $\lim_{x \rightarrow \infty} \mathbb{P}(XY > x)/\mathbb{P}(X > x) = c < \infty$ then $c = \mathbb{E}[Y^\alpha] + c_0 \mathbb{E}[X^\alpha]$.

The proof of this result is given in Appendix A.1. Note that Lemma 1.2(3) includes the Breiman lemma: if $\mathbb{E}[Y^{\alpha+\delta}] < \infty$ for some $\delta > 0$ then (1.4) holds.

REMARK 1.3. Condition (1.5) is a very technical assumption. To verify it one would need to have very precise information about the tail behavior of X . This condition does not follow from the uniform convergence theorem for regularly varying functions; the latter result ensures that for any $M > 0$,

$$\lim_{x \rightarrow \infty} \sup_{y \leq M} \left| \frac{\mathbb{P}(X > x/y)}{\mathbb{P}(X > x)} - y^\alpha \right| = 0.$$

However, for the verification of (1.5) we need information about the deviation of $\mathbb{P}(X > x/y)/\mathbb{P}(X > x)$ from y^α in the range $y \in [M, x/M]$ for any $M > 0$ and large x , that is, for large values of y . Part (3) was proved as Proposition 3.1 by Davis and Resnick [10] in the case when X, Y are i.i.d. In this case, (1.5) is necessary and sufficient for $\mathbb{P}(XY > t)/\mathbb{P}(X > t) \rightarrow 2\mathbb{E}[X^\alpha]$, and the latter constant is the only possible one; see Chover et al. [6], Foss and Korshunov [15].

We mention in passing that regular variation of XY does in general not imply regular variation of X or Y ; see Jacobsen et al. [16].

1.2. Closure of regular variation under multiplication—The multivariate case. Our main goal in this paper is to extend some of the aforementioned results to the multivariate case. We start by introducing regular variation of random vectors. For this reason we equip $\mathbb{R}^{d_{\mathbf{X}}}$ with an arbitrary norm $\|\cdot\|$. A random vector \mathbf{X} has a multivariate regularly varying distribution if $\|\mathbf{X}\|$ has a univariate regularly varying distribution and is asymptotically independent of $\mathbf{X}/\|\mathbf{X}\|$ given $\|\mathbf{X}\| > x$. More precisely, we say that a random vector $\mathbf{X} \in \mathbb{R}^{d_{\mathbf{X}}}$ and its distribution are *regularly varying* if

$$(1.6) \quad \mathbb{P}\left(\frac{\mathbf{X}}{\|\mathbf{X}\|} \in \cdot, \frac{\|\mathbf{X}\|}{x} \in \cdot \mid \|\mathbf{X}\| > x\right) \xrightarrow{w} \mathbb{P}(\Theta_{\mathbf{X}} \in \cdot) \mathbb{P}(Z \in \cdot), \quad x \rightarrow \infty,$$

where Z is Pareto distributed with $\mathbb{P}(Z > y) = y^{-\alpha}$, $y > 1$, and $\Theta_{\mathbf{X}}$ assumes values in the unit sphere $\mathbb{S}^{d_{\mathbf{X}}-1} = \{\mathbf{x} \in \mathbb{R}^{d_{\mathbf{X}}} : \|\mathbf{x}\| = 1\}$. The distribution of $\Theta_{\mathbf{X}}$ is the *spectral distribution* of \mathbf{X} .

We will often refer to an equivalent formulation of multivariate regular variation. Namely, a random vector $\mathbf{X} \in \mathbb{R}^{d_{\mathbf{X}}}$ and its distribution are *regularly varying* if and only if, there exists a nonnull Radon measure $\mu^{\mathbf{X}}$ on $\mathbb{R}_0^{d_{\mathbf{X}}} = \mathbb{R}^{d_{\mathbf{X}}} \setminus \{\mathbf{0}\}$ such that

$$\mu_t^{\mathbf{X}}(\cdot) = \frac{\mathbb{P}(t^{-1}\mathbf{X} \in \cdot)}{\mathbb{P}(\|\mathbf{X}\| > t)} \xrightarrow{v} \mu^{\mathbf{X}}(\cdot), \quad t \rightarrow \infty,$$

where \xrightarrow{v} denotes vague convergence in the space of measures on $\mathbb{R}_0^{d_{\mathbf{X}}}$, that is, for any non-negative continuous compactly supported¹ function f on $\mathbb{R}_0^{d_{\mathbf{X}}}$, for short $f \in C_c^+(\mathbb{R}_0^{d_{\mathbf{X}}})$, we have

$$(1.7) \quad \int f(\mathbf{x})\mu_t^{\mathbf{X}}(d\mathbf{x}) \xrightarrow{w} \int f(\mathbf{x})\mu^{\mathbf{X}}(d\mathbf{x}), \quad t \rightarrow \infty;$$

see Resnick [20], Sections 3.3.5 and 6.1.4. It turns out that the limit measure $\mu^{\mathbf{X}}$ has a *homogeneity property*: there exists $\alpha_{\mathbf{X}} > 0$ such that, for any Borel set $A \subset \mathbb{R}_0^{d_{\mathbf{X}}}$,

$$\mu^{\mathbf{X}}(tA) = t^{-\alpha_{\mathbf{X}}}\mu^{\mathbf{X}}(A), \quad t > 0.$$

We call $\alpha_{\mathbf{X}}$ the *index of regular variation* or *tail index* of \mathbf{X} and write $\mathbf{X} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{X}})$. Of course, we necessarily have

$$(1.8) \quad \mathbb{P}(\|\mathbf{X}\| > x) = \frac{L(x)}{x^{\alpha_{\mathbf{X}}}},$$

for some slowly varying function L . A comparison of (1.6) and (1.7) yields a relation between $\Theta_{\mathbf{X}}$ and $\mu^{\mathbf{X}}$ via the equality, for any $r > 0$ and Borel set $S \subset \mathbb{S}^{d_{\mathbf{X}}-1}$,

$$\mu^{\mathbf{X}}(\{\mathbf{x} : \|\mathbf{x}\| > r, \mathbf{x}/\|\mathbf{x}\| \in S\}) = r^{-\alpha_{\mathbf{X}}}\mathbb{P}(\Theta_{\mathbf{X}} \in S)$$

which further implies

$$(1.9) \quad \int f(\mathbf{x})\mu^{\mathbf{X}}(d\mathbf{x}) = \int_0^\infty \alpha_{\mathbf{X}}r^{-\alpha_{\mathbf{X}}-1}\mathbb{E}[f(r\Theta_{\mathbf{X}})]dr, \quad f \in C_c^+(\mathbb{R}_0^{d_{\mathbf{X}}}).$$

We refer to Resnick [20, 21] as general references to multivariate regular variation and its applications.

Now consider two independent vectors \mathbf{X} and \mathbf{Y} with values in $\mathbb{R}^{d_{\mathbf{X}}}$ and $\mathbb{R}^{d_{\mathbf{Y}}}$, respectively. Our goal is to establish sufficient conditions under which $\mathbf{Z} = \psi(\mathbf{X}, \mathbf{Y})$ is also regularly varying where

$$\psi : \mathbb{R}^{d_{\mathbf{X}}} \times \mathbb{R}^{d_{\mathbf{Y}}} \rightarrow \mathbb{R}^{d_{\mathbf{Z}}}$$

is continuous, $\alpha_{\mathbf{X}}$ -homogeneous with respect to the first argument and $\alpha_{\mathbf{Y}}$ -homogeneous with respect to the second one for positive $\alpha_{\mathbf{X}}, \alpha_{\mathbf{Y}}$, that is, for any $\mathbf{x} \in \mathbb{R}^{d_{\mathbf{X}}}$ and $\mathbf{y} \in \mathbb{R}^{d_{\mathbf{Y}}}$,

$$(1.10) \quad \psi(s\mathbf{x}, t\mathbf{y}) = s^{\alpha_{\mathbf{X}}}t^{\alpha_{\mathbf{Y}}}\psi(\mathbf{x}, \mathbf{y}), \quad s, t \geq 0.$$

Other type of functions where previously treated in [17], see also [9] for the inverse problem.

¹In the context of regular variation, the origin is excluded from consideration. Therefore, a set $K \subset \mathbb{R}_0^{d_{\mathbf{X}}}$ is compact if it is compact in $\mathbb{R}^{d_{\mathbf{X}}}$ but bounded away from zero.

EXAMPLE 1.4 (Product of independent random matrices). If $d_{\mathbf{X}} = n_1 \cdot d_1$ then one can identify $\mathbb{R}_0^{d_{\mathbf{X}}}$ with the set of nonzero $n_1 \times d_1$ matrices $\mathbb{M}_{n_1 \times d_1}$. Similarly, if $d_{\mathbf{Y}} = d_1 \cdot m_1$, $\mathbb{R}_0^{d_{\mathbf{Y}}} = \mathbb{M}_{d_1 \times m_1}$. We define $\psi(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ where $\mathbf{x} \cdot \mathbf{y}$ denotes ordinary matrix multiplication of an $n_1 \times d_1$ matrix \mathbf{x} with a $d_1 \times m_1$ matrix \mathbf{y} . Then $d_{\mathbf{Z}} = n_1 \cdot m_1$, $a_{\mathbf{X}} = a_{\mathbf{Y}} = 1$, and \mathbf{Z} is a product of two independent random matrices \mathbf{X} and \mathbf{Y} .

In the case when one of the matrices \mathbf{X} , \mathbf{Y} is regularly varying, regular variation of \mathbf{Z} was proved in Basrak et al. [2], Proposition 5.1; it is a multivariate analog of the Breiman lemma 1.1: if

$$\mathbf{X} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{X}}) \quad \text{and} \quad \mathbb{E}[\|\mathbf{Y}\|^{\alpha_{\mathbf{X}} + \delta}] < \infty \quad \text{for some } \delta > 0,$$

then

$$\frac{\mathbb{P}(t^{-1}\mathbf{X} \cdot \mathbf{Y} \in \cdot)}{\mathbb{P}(\|\mathbf{X}\| > t)} \xrightarrow{v} \xi(\cdot) := \mathbb{E}[\mu^{\mathbf{X}}(\{\mathbf{x} : \mathbf{x} \cdot \mathbf{Y} \in \cdot\})].$$

In particular, if ξ is nonnull then $\mathbf{Z} = \mathbf{X} \cdot \mathbf{Y} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{Z}})$ where

$$\mu^{\mathbf{Z}}(\cdot) = \frac{\xi(\cdot)}{\xi(\{\mathbf{z} : \|\mathbf{z}\| > 1\})}.$$

EXAMPLE 1.5 (Kronecker products of independent random matrices). Suppose that $d_{\mathbf{X}} = n_1 \cdot d_1$ and $d_{\mathbf{Y}} = d_2 \cdot n_2$, so we can identify $\mathbb{R}_0^{d_{\mathbf{X}}} = \mathbb{M}_{n_1 \times n_2}$, $\mathbb{R}_0^{d_{\mathbf{Y}}} = \mathbb{M}_{d_1 \times d_2}$. Now define $\psi : \mathbb{R}^{d_{\mathbf{X}}} \times \mathbb{R}^{d_{\mathbf{Y}}} \rightarrow \mathbb{R}^{n_1 d_1 n_2 d_2} = \mathbb{M}_{n_1 d_1 \times n_2 d_2}$ via the Kronecker product $\psi(\mathbf{x}, \mathbf{y}) = \mathbf{x} \otimes \mathbf{y}$. As for ordinary matrix multiplication, we have $a_{\mathbf{X}} = a_{\mathbf{Y}} = 1$.

EXAMPLE 1.6 (Random quadratic form). If $d_{\mathbf{Y}} = d_{\mathbf{X}}^2$, identifying $\mathbb{R}_0^{d_{\mathbf{Y}}} = \mathbb{M}_{d_{\mathbf{X}} \times d_{\mathbf{X}}}$, we define $\psi : \mathbb{R}^{d_{\mathbf{X}}} \times \mathbb{R}^{d_{\mathbf{Y}}} \rightarrow \mathbb{R}$ by $\psi(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top \mathbf{y} \mathbf{x}$. In this case, $a_{\mathbf{X}} = 2$ and $a_{\mathbf{Y}} = 1$.

1.3. *Organization of the article.* Our main result (Theorem 2.1) yields sharp sufficient conditions for regular variation of the homogeneous function $\psi(\mathbf{X}, \mathbf{Y})$ acting on independent regularly varying random vectors \mathbf{X} , \mathbf{Y} . The proof is given in Section 4. We apply these results in Section 3. In particular, in Section 3.1 we derive the regular variation properties of products of i.i.d. regularly varying quadratic matrices while, in Section 3.2, we prove regular variation of solutions to affine stochastic recurrence equations under nonstandard conditions.

2. Main result. In what follows, \mathbf{X} and \mathbf{Y} are independent random vectors with values in $\mathbb{R}^{d_{\mathbf{X}}}$ and $\mathbb{R}^{d_{\mathbf{Y}}}$, respectively, and we always assume $\mathbf{X} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{X}})$. We will study the regular variation property of the $a_{\mathbf{X}}-a_{\mathbf{Y}}$ -homogeneous function $\mathbf{Z} = \psi(\mathbf{X}, \mathbf{Y})$ (see (1.10)). In particular, we are interested in the vague limit relation

$$(2.1) \quad \xi_t(\cdot) := \frac{\mathbb{P}(t^{-1}\mathbf{Z} \in \cdot)}{\mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}} \cdot \|\mathbf{Y}\|^{a_{\mathbf{Y}}} > t)} \xrightarrow{v} \xi(\cdot),$$

in $\mathbb{R}_0^{d_{\mathbf{Z}}}$, and we will characterize the Radon measure ξ .

We work under three conditions on \mathbf{X} , \mathbf{Y} which we introduce next.

CONDITION (H).

(H1) $\mathbf{X} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{X}})$

(H2) $c_0 := \lim_{x \rightarrow \infty} \mathbb{P}(\|\mathbf{Y}\|^{a_{\mathbf{Y}}} > x) / \mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}} > x) = 0$

(H3) $\lim_{x \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}} \|\mathbf{Y}\|^{a_{\mathbf{Y}}} > x)}{\mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}} > x)} = \mathbb{E}[\|\mathbf{Y}\|^{\alpha_{\mathbf{X}} a_{\mathbf{Y}} / a_{\mathbf{X}}}] < \infty.$

Condition (H) is satisfied if $\mathbf{X} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{X}})$ and $\mathbb{E}[\|\mathbf{Y}\|^{a_{\mathbf{Y}}\alpha_{\mathbf{X}}/a_{\mathbf{X}}+\delta}] < \infty$ for some $\delta > 0$. Under (H) one may expect that the tail behavior of \mathbf{Z} is mainly influenced by that of \mathbf{X} . But we also want to cover the situation when $\|\mathbf{Z}\|$, $\|\mathbf{X}\|^{a_{\mathbf{X}}}$ and $\|\mathbf{Y}\|^{a_{\mathbf{Y}}}$ have asymptotically equivalent tails. This is the content of the following condition.

CONDITION (T).

(T1) $\mathbf{X} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{X}})$, $\mathbf{Y} \in \text{RV}(\alpha_{\mathbf{Y}}, \mu^{\mathbf{Y}})$

(T2) $c_0 := \lim_{x \rightarrow \infty} \mathbb{P}(\|\mathbf{Y}\|^{a_{\mathbf{Y}}} > x) / \mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}} > x) \in [0, \infty)$

(T3) $\mathbb{E}[\|\mathbf{Y}\|^{a_{\mathbf{X}}a_{\mathbf{Y}}/a_{\mathbf{X}}} + \|\mathbf{X}\|^{a_{\mathbf{Y}}a_{\mathbf{X}}/a_{\mathbf{Y}}}] < \infty$ and

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}}\|\mathbf{Y}\|^{a_{\mathbf{Y}}} > x)}{\mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}} > x)} = \mathbb{E}[\|\mathbf{Y}\|^{a_{\mathbf{X}}a_{\mathbf{Y}}/a_{\mathbf{X}}}] + c_0\mathbb{E}[\|\mathbf{X}\|^{a_{\mathbf{Y}}a_{\mathbf{X}}/a_{\mathbf{Y}}}].$$

Whenever (H) or (T) hold a tail balance condition applies: the following finite limits exist:

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}} > t)}{\mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}} \cdot \|\mathbf{Y}\|^{a_{\mathbf{Y}}} > t)} = c_{\mathbf{X}} = \frac{1}{\mathbb{E}[\|\mathbf{Y}\|^{a_{\mathbf{X}}a_{\mathbf{Y}}/a_{\mathbf{X}}}] + c_0\mathbb{E}[\|\mathbf{X}\|^{a_{\mathbf{Y}}a_{\mathbf{X}}/a_{\mathbf{Y}}}]},$$

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{Y}\|^{a_{\mathbf{Y}}} > t)}{\mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}} \cdot \|\mathbf{Y}\|^{a_{\mathbf{Y}}} > t)} = c_{\mathbf{Y}} = c_0c_{\mathbf{X}}.$$

Finally, we introduce a condition that covers nonequivalent tails.

CONDITION (R).

(R1) $\mathbf{X} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{X}})$, $\mathbf{Y} \in \text{RV}(\alpha_{\mathbf{Y}}, \mu^{\mathbf{Y}})$

(R2) $\mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}} > t) + \mathbb{P}(\|\mathbf{Y}\|^{a_{\mathbf{Y}}} > t) = o(\mathbb{P}(\|\mathbf{X}\|^{a_{\mathbf{X}}} \cdot \|\mathbf{Y}\|^{a_{\mathbf{Y}}} > t))$.

If (R1) holds and $\mathbb{E}[\|\mathbf{Y}\|^{a_{\mathbf{X}}a_{\mathbf{Y}}/a_{\mathbf{X}}}] = \mathbb{E}[\|\mathbf{X}\|^{a_{\mathbf{Y}}a_{\mathbf{X}}/a_{\mathbf{Y}}}] = \infty$ then (R2) holds.

Now we are ready to formulate the main result of this paper.

THEOREM 2.1. *Assume that the $\mathbb{R}^{d_{\mathbf{X}}}$ -valued \mathbf{X} and the $\mathbb{R}^{d_{\mathbf{Y}}}$ -valued \mathbf{Y} random vectors are independent. Then (2.1) is satisfied for the $a_{\mathbf{X}}-a_{\mathbf{Y}}$ -homogeneous function $\mathbf{Z} = \psi(\mathbf{X}, \mathbf{Y})$ with the following Radon limit measures ξ on $\mathbb{R}_0^{d_{\mathbf{Z}}}$:*

(1) Under (R),

$$(2.2) \quad \xi(\cdot) = \mathbb{E}[\mu^{\mathbf{X}}(\{\mathbf{x} : \psi(\mathbf{x}, \Theta_{\mathbf{Y}}) \in \cdot\})].$$

(2) Under (T),

$$(2.3) \quad \xi(\cdot) = c_{\mathbf{X}}\mathbb{E}[\mu^{\mathbf{X}}(\{\mathbf{x} : \psi(\mathbf{x}, \mathbf{Y}) \in \cdot\})] + c_{\mathbf{Y}}\mathbb{E}[\mu^{\mathbf{Y}}(\{\mathbf{y} : \psi(\mathbf{X}, \mathbf{y}) \in \cdot\})].$$

(3) Under (H),

$$(2.4) \quad \xi(\cdot) = c_{\mathbf{X}}\mathbb{E}[\mu^{\mathbf{X}}(\{\mathbf{x} : \psi(\mathbf{x}, \mathbf{Y}) \in \cdot\})].$$

In particular, if ξ is nonnull then $\mathbf{Z} \in \text{RV}(\alpha_{\mathbf{Z}}, \mu^{\mathbf{Z}})$, where

$$\alpha_{\mathbf{Z}} = \begin{cases} \frac{\alpha_{\mathbf{X}}}{a_{\mathbf{X}}} \wedge \frac{\alpha_{\mathbf{Y}}}{a_{\mathbf{Y}}}, & \text{under (R) and (T),} \\ \frac{\alpha_{\mathbf{X}}}{a_{\mathbf{X}}}, & \text{under (H),} \end{cases}$$

$$\mu^{\mathbf{Z}}(\cdot) = \frac{\xi(\cdot)}{\xi(\{\mathbf{z} : \|\mathbf{z}\| > 1\})}.$$

From Theorem 2.1 we may derive some immediate consequences.

COROLLARY 2.2. *Assume that \mathbf{X}, \mathbf{Y} are independent.*

- (1) *If $\mathbf{X} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{X}})$, $\mathbf{Y} \in \text{RV}(\alpha_{\mathbf{Y}}, \mu^{\mathbf{Y}})$ and $\mathbb{E}[\|\mathbf{Y}\|^{\alpha_{\mathbf{X}}\alpha_{\mathbf{Y}}/\alpha_{\mathbf{X}}}] = \mathbb{E}[\|\mathbf{X}\|^{\alpha_{\mathbf{Y}}\alpha_{\mathbf{X}}/\alpha_{\mathbf{Y}}}] = \infty$ then (R) holds, hence (2.1) with ξ defined in (2.2).*
- (2) *If $\mathbf{X} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{X}})$, $\mathbf{Y} \in \text{RV}(\alpha_{\mathbf{Y}}, \mu^{\mathbf{Y}})$ and $\frac{\alpha_{\mathbf{X}}}{\alpha_{\mathbf{X}}} < \frac{\alpha_{\mathbf{Y}}}{\alpha_{\mathbf{Y}}}$ then (H) and (T) hold with $c_0 = 0$, hence (2.1) holds with ξ given in (2.3).*
- (3) *If $\mathbf{X} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{X}})$ and $\mathbb{E}[\|\mathbf{Y}\|^{\alpha_{\mathbf{Y}}\alpha_{\mathbf{X}}/\alpha_{\mathbf{X}}+\delta}] < \infty$ for some $\delta > 0$ then (H) holds, hence (2.1) with ξ given in (2.4).*

REMARK 2.3. As regards Theorem 2.1(1), one can verify that ξ is symmetric with respect to \mathbf{X} and \mathbf{Y} . In this case, necessarily $\frac{\alpha_{\mathbf{X}}}{\alpha_{\mathbf{X}}} = \frac{\alpha_{\mathbf{Y}}}{\alpha_{\mathbf{Y}}}$, and we can write

$$\begin{aligned} \mathbb{E}[\mu^{\mathbf{X}}(\{\mathbf{x} : \psi(\mathbf{x}, \Theta_{\mathbf{Y}}) \in \cdot\})] &= \int_0^\infty \alpha_{\mathbf{X}} r^{-\alpha_{\mathbf{X}}-1} \mathbb{P}(\psi(r\Theta_{\mathbf{X}}, \Theta_{\mathbf{Y}}) \in \cdot) dr \\ &= \int_0^\infty \alpha_{\mathbf{X}} r^{-\alpha_{\mathbf{X}}-1} \mathbb{P}(\psi(\Theta_{\mathbf{X}}, r^{\alpha_{\mathbf{X}}/\alpha_{\mathbf{Y}}}\Theta_{\mathbf{Y}}) \in \cdot) dr \\ &= \int_0^\infty \alpha_{\mathbf{Y}} r^{-\alpha_{\mathbf{Y}}-1} \mathbb{P}(\psi(\Theta_{\mathbf{X}}, r\Theta_{\mathbf{Y}}) \in \cdot) dr \\ &= \mathbb{E}[\mu^{\mathbf{Y}}(\{\mathbf{y} : \psi(\Theta_{\mathbf{X}}, \mathbf{y}) \in \cdot\})]. \end{aligned}$$

3. Applications.

3.1. *Products of regularly varying random matrices.* In what follows, we consider an i.i.d. sequence of $d \times d$ random matrices (\mathbf{A}_i) and we assume that a generic element $\mathbf{A} \in \text{RV}(\alpha, \mu^{\mathbf{A}})$. We apply Theorem 2.1 to the function $\psi(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ and an arbitrary matrix norm $\|\cdot\|$.

Next we formulate our findings for a general product $\mathbf{\Pi}_n = \mathbf{A}_1 \cdots \mathbf{A}_n$, $n \geq 1$. Here and in what follows, we also use the notation

$$\mathbf{\Pi}_{i,j} = \begin{cases} \prod_{s=i}^j \mathbf{A}_s, & i \leq j, \\ \text{Id}_d, & i > j, \end{cases}$$

where Id_d is the $d \times d$ identity matrix.

3.1.1. *The case of nonequivalent tails.* We first state the results in the case $\mathbb{P}(\|\mathbf{\Pi}_n\| > t) = o(\mathbb{P}(\|\mathbf{\Pi}_{n+1}\| > t))$ for all n . The complementary case is treated in Section 3.1.2.

COROLLARY 3.1. *Consider an i.i.d. sequence (\mathbf{A}_i) of $d \times d$ matrices with $\mathbf{A} \in \text{RV}(\alpha, \mu^{\mathbf{A}})$. Assume that*

$$(3.1) \quad \frac{\mathbb{P}(\|\mathbf{A}\| > t)}{\mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{A}_2\| > t)} \rightarrow 0, \quad t \rightarrow \infty.$$

Then for $n \geq 1$,

$$(3.2) \quad \frac{\mathbb{P}(\|\mathbf{\Pi}_n\| > t)}{\mathbb{P}(\|\mathbf{A}_1\| \cdots \|\mathbf{A}_n\| > t)} \rightarrow \mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_n}\|^\alpha], \quad t \rightarrow \infty.$$

If $\mathbb{P}(\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_n}\| > 0) > 0$ then Π_n is regularly varying and, as $t \rightarrow \infty$,

$$(3.3) \quad \mathbb{P}\left(\frac{\Pi_n}{\|\Pi_n\|} \in \cdot \mid \|\Pi_n\| > t\right) \xrightarrow{w} \mathbb{P}(\Theta_{\Pi_n} \in \cdot) = \mathbb{E}\left[\frac{\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_n}\|^\alpha}{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_n}\|^\alpha]} \mathbf{1}\left(\frac{\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_n}}{\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_n}\|} \in \cdot\right)\right].$$

In particular, if $\|\cdot\|$ is the operator norm corresponding to the Euclidean norm and \mathbf{A} is orthogonal, $\Theta_{\Pi_n} \stackrel{d}{=} \Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_n}$.

REMARK 3.2. In view of Lemma 1.2(2), (3.1) is satisfied if $\mathbb{E}[\|\mathbf{A}\|^\alpha] = \infty$.

PROOF. We proceed by induction. We will prove that for each n , (3.3), (3.2) and

$$(3.4) \quad \mathbb{P}(\|\mathbf{A}_1\| > t) + \mathbb{P}(\|\Pi_{2,n+1}\| > t) = o(\mathbb{P}(\|\mathbf{A}_1\| \cdot \|\Pi_{2,n+1}\| > t))$$

hold.

For $n = 1$, (3.3) follows from the regular variation of \mathbf{A} , (3.2) follows trivially since $\Pi_1 = \mathbf{A}_1$ and $\|\Theta_{\mathbf{A}}\| = 1$, and (3.4) is a consequence of (3.1).

Now suppose that it holds $n = k$ for some $k \geq 1$. Since (3.4) holds for $n = k$ the balance conditions

$$c_{\Pi_{2,k+1}} = \lim_{t \rightarrow \infty} \frac{\mathbb{P}(\|\Pi_k\| > t)}{\mathbb{P}(\|\mathbf{A}_1\| \|\Pi_{2,k+1}\| > t)} = 0,$$

$$c_{\mathbf{A}_1} = \lim_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{A}_1\| > t)}{\mathbb{P}(\|\mathbf{A}_1\| \|\Pi_{2,k+1}\| > t)} = 0$$

are satisfied. An application of Theorem 2.1(1) yields

$$\frac{\mathbb{P}(t^{-1}\mathbf{A}_1 \Pi_{2,k+1} \in \cdot)}{\mathbb{P}(\|\mathbf{A}_1\| \|\Pi_{2,k+1}\| > t)} \xrightarrow{v} \mathbb{E}[\mu^{\mathbf{A}}(\{\mathbf{x} : \mathbf{x}\Theta_{\Pi_k} \in \cdot\})].$$

An immediate consequence is

$$\begin{aligned} \frac{\mathbb{P}(\|\Pi_{k+1}\| > t)}{\mathbb{P}(\|\mathbf{A}_1\| \|\mathbf{A}_2 \cdots \mathbf{A}_{k+1}\| > t)} &\rightarrow \mathbb{E}[\mu^{\mathbf{A}}(\{\mathbf{x} : \|\mathbf{x}\Theta_{\Pi_k}\| > 1\})] = \mathbb{P}(Y \|\Theta_{\mathbf{A}_1} \Theta_{\Pi_{2,k+1}}\| > 1) \\ &= \mathbb{E}[\|\Theta_{\mathbf{A}_1} \Theta_{\Pi_{2,k+1}}\|^\alpha] = \frac{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}\|^\alpha]}{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_k}\|^\alpha]}, \end{aligned}$$

where the Pareto random variable Y , $\Theta_{\mathbf{A}_1}$ and $\Theta_{\Pi_{2,k+1}}$ are independent. Here we also used the induction assumption on the distribution of Π_k . Therefore,

$$\begin{aligned} \mathbb{P}\left(\frac{\Pi_{k+1}}{\|\Pi_{k+1}\|} \in \cdot \mid \|\Pi_{k+1}\| > t\right) &\xrightarrow{w} \frac{\mathbb{E}[\mu^{\mathbf{A}}(\{\mathbf{x} : \frac{\mathbf{x}\Theta_{\Pi_k}}{\|\mathbf{x}\Theta_{\Pi_k}\|} \in \cdot, \|\mathbf{x}\Theta_{\Pi_k}\| > 1\})]}{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}\|^\alpha] / \mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_k}\|^\alpha]} \\ &= \frac{\mathbb{P}(\frac{\Theta_{\mathbf{A}_1} \Theta_{\Pi_{2,k+1}}}{\|\Theta_{\mathbf{A}_1} \Theta_{\Pi_{2,k+1}}\|} \in \cdot, Y \|\Theta_{\mathbf{A}_1} \Theta_{\Pi_{2,k+1}}\| > 1)}{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}\|^\alpha] / \mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_k}\|^\alpha]} \\ &= \frac{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \Theta_{\Pi_{2,k+1}}\|^\alpha \mathbf{1}(\frac{\Theta_{\mathbf{A}_1} \Theta_{\Pi_{2,k+1}}}{\|\Theta_{\mathbf{A}_1} \Theta_{\Pi_{2,k+1}}\|} \in \cdot)]}{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}\|^\alpha] / \mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_k}\|^\alpha]} \\ &= \mathbb{E}\left[\frac{\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}\|^\alpha}{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}\|^\alpha]} \mathbf{1}\left(\frac{\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}}{\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}\|} \in \cdot\right)\right]. \end{aligned}$$

This proves (3.3) for $n = k + 1$. Finally, we turn to (3.2) for $n = k + 1$:

$$\begin{aligned} \frac{\mathbb{P}(\|\mathbf{\Pi}_{k+1}\| > t)}{\mathbb{P}(\|\mathbf{A}_1\| \cdots \|\mathbf{A}_{k+1}\| > t)} &= \frac{\mathbb{P}(\|\mathbf{\Pi}_{k+1}\| > t)}{\mathbb{P}(\|\mathbf{A}_1\| \|\mathbf{\Pi}_{2,k+1}\| > t)} \frac{\mathbb{P}(\|\mathbf{A}_1\| \|\mathbf{\Pi}_{2,k+1}\| > t)}{\mathbb{P}(\|\mathbf{A}_1\| (\|\mathbf{A}_2\| \cdots \|\mathbf{A}_{k+1}\|) > t)} \\ &\rightarrow \frac{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}\|^\alpha]}{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_k}\|^\alpha]} \mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_k}\|^\alpha] \\ &= \mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}\|^\alpha]. \end{aligned}$$

In the last step we used the induction assumption leading to tail equivalence of $\|\mathbf{A}_2\| \|\mathbf{A}_3 \cdots \mathbf{A}_{k+1}\|$ and $\|\mathbf{A}_2 \mathbf{A}_3 \cdots \mathbf{A}_{k+1}\|$ with factor $\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_k}\|^\alpha]$. To finish the proof we argue in favor of (3.4) for $n = k + 1$. We have shown that

$$\mathbb{P}(\|\mathbf{\Pi}_k\| > t) \sim \frac{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}\|^\alpha]}{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_k}\|^\alpha]} \mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{\Pi}_{2,k+1}\| > t)$$

which, in combination with (3.4) for $n = k$, gives $\mathbb{P}(\|\mathbf{\Pi}_k\| > t) = o(\mathbb{P}(\|\mathbf{\Pi}_{k+1}\| > t))$. Consequently for any $M > 0$ there exists t_0 sufficiently large such that

$$\mathbb{P}(\|\mathbf{\Pi}_{k+1}\| > t) \geq M \mathbb{P}(\|\mathbf{\Pi}_k\| > t), \quad t > t_0.$$

On the other hand, $\mathbb{P}(\|\mathbf{A}_1\| > t) = o(\mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{\Pi}_{2,k+1}\| > t))$ and

$$\mathbb{P}(\|\mathbf{\Pi}_{k+1}\| > t) \sim c_0 \mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{\Pi}_{2,k+1}\| > t), \quad c_0 = \frac{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_{k+1}}\|^\alpha]}{\mathbb{E}[\|\Theta_{\mathbf{A}_1} \cdots \Theta_{\mathbf{A}_k}\|^\alpha]}.$$

Take $\eta = t_0^{-1}$. We observe as $t \rightarrow \infty$ that

$$\begin{aligned} \mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{\Pi}_{2,k+2}\| > t) &\geq \mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{\Pi}_{2,k+2}\| > t, \|\mathbf{A}_1\| \leq \eta t) \\ &\geq M \mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{\Pi}_{2,k+1}\| > t, \|\mathbf{A}_1\| \leq \eta t) \\ &\geq M (\mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{\Pi}_{2,k+1}\| > t) - \mathbb{P}(\|\mathbf{A}_1\| > \eta t)) \\ &= M \mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{\Pi}_{2,k+1}\| > t) (1 + o(1)) \\ &= c_0 M \mathbb{P}(\|\mathbf{\Pi}_{k+1}\| > t) (1 + o(1)). \end{aligned}$$

This proves $\mathbb{P}(\|\mathbf{\Pi}_{k+1}\| > t) = o(\mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{\Pi}_{2,k+2}\| > t))$ and finishes the proof of the corollary. \square

3.1.2. *The case of tail-equivalent tails.* We also assume condition (1.5) which turns into

$$(3.5) \quad \lim_{M \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{A}_1\| \|\mathbf{A}_2\| > t, M < \|\mathbf{A}_1\| \leq t/M)}{\mathbb{P}(\|\mathbf{A}\| > x)} = 0$$

which is equivalent to

$$\frac{\mathbb{P}(\|\mathbf{A}_1\| > t)}{\mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{A}_2\| > t)} \rightarrow c_A = \frac{1}{2\mathbb{E}[\|\mathbf{A}\|^\alpha]}.$$

An appeal to the following corollary shows that this condition causes tail equivalence of all $\mathbf{\Pi}_n$.

COROLLARY 3.3. *Consider an i.i.d. sequence (\mathbf{A}_i) of $d \times d$ matrices such that $\mathbf{A} \in \text{RV}(\alpha, \mu^{\mathbf{A}})$ and (3.5) holds. Then for any $n \geq 2$,*

$$(3.6) \quad \frac{\mathbb{P}(\|\mathbf{\Pi}_n\| > t)}{\mathbb{P}(\|\mathbf{A}\| > t)} \rightarrow \sum_{k=1}^n \mathbb{E}[\|\mathbf{\Pi}_{k-1} \Theta_{\mathbf{A}_k} \mathbf{\Pi}_{k+1,n}\|^\alpha], \quad t \rightarrow \infty.$$

Additionally, if $\mathbb{P}(\|\mathbf{\Pi}_{k-1} \Theta_{\mathbf{A}_k} \mathbf{\Pi}_{k+1,n}\| > 0) > 0$ for some $k \leq n$ then $\mathbf{\Pi}_n$ is regularly varying and as $t \rightarrow \infty$,

$$\begin{aligned} &\mathbb{P}\left(\frac{\mathbf{\Pi}_n}{\|\mathbf{\Pi}_n\|} \in \cdot \mid \|\mathbf{\Pi}_n\| > t\right) \xrightarrow{w} \mathbb{P}(\Theta_{\mathbf{\Pi}_n} \in \cdot) \\ &= \sum_{k=1}^n p_k \mathbb{E}\left[\frac{\|\mathbf{\Pi}_{k-1} \Theta_{\mathbf{A}_k} \mathbf{\Pi}_{k+1,n}\|^\alpha}{\mathbb{E}[\|\mathbf{\Pi}_{k-1} \Theta_{\mathbf{A}_k} \mathbf{\Pi}_{k+1,n}\|^\alpha]} \mathbf{1}\left(\frac{\mathbf{\Pi}_{k-1} \Theta_{\mathbf{A}_k} \mathbf{\Pi}_{k+1,n}}{\|\mathbf{\Pi}_{k-1} \Theta_{\mathbf{A}_k} \mathbf{\Pi}_{k+1,n}\|} \in \cdot\right)\right], \end{aligned}$$

where

$$p_k = \frac{\mathbb{E}[\|\mathbf{\Pi}_{k-1} \Theta_{\mathbf{A}_k} \mathbf{\Pi}_{k+1,n}\|^\alpha]}{\sum_{k=1}^n \mathbb{E}[\|\mathbf{\Pi}_{k-1} \Theta_{\mathbf{A}_k} \mathbf{\Pi}_{k+1,n}\|^\alpha]}, \quad k = 1, \dots, n.$$

PROOF. We proceed by induction. We will prove (3.6) and

$$\mu^{\mathbf{\Pi}_n}(\cdot) = \frac{\sum_{k=1}^n \mathbb{E}[\mu^{\mathbf{A}}(\mathbf{a} : \mathbf{\Pi}_{k-1} \mathbf{a} \mathbf{\Pi}_{k+1,n} \in \cdot)]}{\sum_{k=1}^n \mathbb{E}[\|\mathbf{\Pi}_{k-1} \Theta_{\mathbf{A}_k} \mathbf{\Pi}_{k+1,n}\|^\alpha]}.$$

Since the claim is trivial for $n = 1$, suppose that it holds for some $n \geq 1$. Put $\tilde{c}_n = \sum_{k=1}^n \mathbb{E}[\|\mathbf{\Pi}_{k-1} \Theta_{\mathbf{A}_k} \mathbf{\Pi}_{k+1,n}\|^\alpha]$. Since $\|\mathbf{A}\|$ satisfies (3.5) and $\mathbb{P}(\|\mathbf{\Pi}_n\| > t) \sim \tilde{c}_n \mathbb{P}(\|\mathbf{A}\| > t)$ we infer that

$$\begin{aligned} \frac{\mathbb{P}(\|\mathbf{A}\| > t)}{\mathbb{P}(\|\mathbf{A}\| \cdot \|\mathbf{\Pi}_{2,n+1}\| > t)} &\rightarrow c_{n,\mathbf{A}} = \frac{1}{\mathbb{E}[\|\mathbf{\Pi}_{2,n+1}\|^\alpha] + \tilde{c}_n \mathbb{E}[\|\mathbf{A}\|^\alpha]}, \\ \frac{\mathbb{P}(\|\mathbf{\Pi}_{2,n+1}\| > t)}{\mathbb{P}(\|\mathbf{A}\| \cdot \|\mathbf{\Pi}_{2,n+1}\| > t)} &\rightarrow c_{n,\mathbf{\Pi}} = \frac{\tilde{c}_n}{\mathbb{E}[\|\mathbf{\Pi}_{2,n+1}\|^\alpha] + \tilde{c}_n \mathbb{E}[\|\mathbf{A}\|^\alpha]}. \end{aligned}$$

Theorem 2.1 yields

$$\frac{\mathbb{P}(t^{-1} \mathbf{\Pi}_{n+1} \in \cdot)}{\mathbb{P}(\|\mathbf{A}\| \cdot \|\mathbf{\Pi}_{2,n+1}\| > t)} \rightarrow \frac{\mathbb{E}[\mu^{\mathbf{A}}(\{\mathbf{a} : \mathbf{a} \mathbf{\Pi}_{2,n+1} \in \cdot\}) + \tilde{c}_n \mu^{\mathbf{\Pi}_n}(\{\boldsymbol{\pi} : \mathbf{A} \boldsymbol{\pi} \in \cdot\})]}{\mathbb{E}[\|\mathbf{\Pi}_{2,n+1}\|^\alpha] + \tilde{c}_n \mathbb{E}[\|\mathbf{A}\|^\alpha]}.$$

Consequently, by the induction hypothesis,

$$\begin{aligned} \frac{\mathbb{P}(t^{-1} \mathbf{\Pi}_{n+1} \in \cdot)}{\mathbb{P}(\|\mathbf{\Pi}_{n+1}\| > t)} &\xrightarrow{v} \frac{\mathbb{E}[\mu^{\mathbf{A}}(\{\mathbf{a} : \mathbf{a} \mathbf{\Pi}_{2,n+1} \in \cdot\}) + \tilde{c}_n \mu^{\mathbf{\Pi}_n}(\{\boldsymbol{\pi} : \mathbf{A} \boldsymbol{\pi} \in \cdot\})]}{\mathbb{E}[\mu^{\mathbf{A}}(\{\mathbf{a} : \|\mathbf{a} \mathbf{\Pi}_{2,n+1}\| > 1\}) + \tilde{c}_n \mu^{\mathbf{\Pi}_n}(\{\boldsymbol{\pi} : \|\mathbf{A} \boldsymbol{\pi}\| > 1\})]} \\ &= \frac{\mathbb{E}[\mu^{\mathbf{A}}(\{\mathbf{a} : \mathbf{a} \mathbf{\Pi}_{2,n+1} \in \cdot\}) + \sum_{k=1}^n \mu^{\mathbf{A}}(\{\mathbf{a} : \mathbf{\Pi}_k \mathbf{a} \mathbf{\Pi}_{k+2,n+1} \in \cdot\})]}{\mathbb{E}[\mu^{\mathbf{A}}(\{\mathbf{a} : \|\mathbf{a} \mathbf{\Pi}_{2,n+1}\| > 1\}) + \sum_{k=1}^n \mu^{\mathbf{A}}(\{\mathbf{a} : \|\mathbf{\Pi}_k \mathbf{a} \mathbf{\Pi}_{k+2,n+1}\| > 1\})]} \\ &= \frac{\sum_{k=1}^{n+1} \mathbb{E}[\mu^{\mathbf{A}}(\{\mathbf{a} : \mathbf{\Pi}_{k-1} \mathbf{a} \mathbf{\Pi}_{k+1,n+1} \in \cdot\})]}{\sum_{k=1}^{n+1} \mathbb{E}[\|\mathbf{\Pi}_{k-1} \Theta_{\mathbf{A}_k} \mathbf{\Pi}_{k+1,n+1}\|^\alpha]}. \end{aligned}$$

With this at hand, the convergence

$$\mathbb{P}\left(\frac{\mathbf{\Pi}_n}{\|\mathbf{\Pi}_n\|} \in \cdot \mid \|\mathbf{\Pi}_n\| > t\right) \xrightarrow{w} \mathbb{P}(\Theta_{\mathbf{\Pi}_n} \in \cdot)$$

follows. \square

3.2. Stochastic recurrence equations. We turn to the stochastic recurrence equation

$$(3.7) \quad \mathbf{R}_t = \mathbf{A}_t \mathbf{R}_{t-1} + \mathbf{B}_t, \quad t \in \mathbb{Z},$$

where $((\mathbf{A}_t, \mathbf{B}_t))_{t \in \mathbb{Z}}$ is an i.i.d. sequence with generic element (\mathbf{A}, \mathbf{B}) , \mathbf{A} is a $d \times d$ random matrix and \mathbf{B} an \mathbb{R}^d -valued random vector, possibly dependent on each other. A solution (\mathbf{R}_t) is causal if for every t , \mathbf{R}_t is a function only of values $((\mathbf{A}_s, \mathbf{B}_s))_{s \leq t}$, and then it constitutes a

Markov chain. If a stationary causal solution (\mathbf{R}_t) with generic element \mathbf{R} exists its marginal distribution satisfies the fixed point equation in law

$$(3.8) \quad \mathbf{R} \stackrel{d}{=} \mathbf{A}\mathbf{R} + \mathbf{B}, \quad \mathbf{R} \text{ independent of } (\mathbf{A}, \mathbf{B})$$

and \mathbf{R} has the representation in law

$$(3.9) \quad \mathbf{R} \stackrel{d}{=} \sum_{k=0}^{\infty} \Pi_k \mathbf{B}_{k+1} \quad \text{where } \Pi_k = \prod_{j=1}^k \mathbf{A}_j.$$

The latter infinite series converges under conditions on the distribution of (\mathbf{A}, \mathbf{B}) , for example $\mathbb{E}[\log \|\mathbf{A}\|] < 0$ and $\mathbb{E}[\log_+ \|\mathbf{B}\|] < \infty$. Under some mild integrability and nondegeneracy assumptions (3.9) is the unique solution to (3.8). Here and in what follows, we refer to the monograph Buraczewski et al. [5] for details concerning the existence, uniqueness and other properties of the solutions to (3.7) and (3.8).

The equations (3.7) and (3.8) have attracted a lot of attention since the seminal paper by Kesten [18] who proved that \mathbf{R} has some regular variation property with tail index $\alpha > 0$ given by

$$\lim_{n \rightarrow \infty} (\mathbb{E} \|\Pi_n\|^\alpha)^{1/n} = 1.$$

If $d = 1$, the latter equation reads as $\mathbb{E}[|A|^\alpha] = 1$. In the Kesten setting, it is typically assumed that $\mathbb{E}[\|\mathbf{B}\|^\alpha] < \infty$ and $\mathbb{E}[\|\mathbf{A}\|^\alpha \log_+ \|\mathbf{A}\|] < \infty$, implying the existence and uniqueness of the solution (\mathbf{R}_t) . Under these and further mild conditions on the distribution of (\mathbf{A}, \mathbf{B}) one has $\mathbf{R} \in \text{RV}(\alpha, \mu^{\mathbf{R}})$ and the tail asymptotics

$$\mathbb{P}(\|\mathbf{R}\| > t) \sim c_0 t^{-\alpha} \quad \text{for some } c_0 > 0.$$

Since $\mathbb{E}[\|\mathbf{R}\|^\alpha] = \infty$ we have $\mathbb{P}(\|\mathbf{B}\| > t) = o(\mathbb{P}(\|\mathbf{R}\| > t))$, and elementary calculations (Lemma C.3.1 in Buraczewski et al. [5]) show that for $\mu^{\mathbf{R}}$ -continuity sets C ,

$$t^\alpha \mathbb{P}(t^{-1}\mathbf{R} \in C) \sim t^\alpha \mathbb{P}(t^{-1}\mathbf{A}\mathbf{R} \in C),$$

and the multivariate Breiman result Lemma C.3.1 in [5] yields

$$\frac{\mathbb{P}(t^{-1}\mathbf{A}\mathbf{R} \in \cdot)}{\mathbb{P}(\|\mathbf{R}\| > t)} \xrightarrow{v} \mathbb{E}[\mu^{\mathbf{R}}(\{\mathbf{x} : \mathbf{A}\mathbf{x} \in \cdot\})].$$

Hence we have the identity

$$\mu^{\mathbf{R}}(\cdot) = \mathbb{E}[\mu^{\mathbf{R}}(\{\mathbf{x} : \mathbf{A}\mathbf{x} \in \cdot\})].$$

Using induction on the recursion (3.7) and similar arguments, we find that

$$\mu^{\mathbf{R}}(\cdot) = \mathbb{E}[\mu^{\mathbf{R}}(\{\mathbf{x} : \Pi_k \mathbf{x} \in \cdot\})], \quad k \geq 1.$$

This relation holds, in particular, if \mathbf{A} is regularly varying with index α but the additional moment condition $\mathbb{E}[\|\mathbf{A}\|^\alpha \log_+ \|\mathbf{A}\|] < \infty$ must be satisfied.

Regular variation of (\mathbf{R}_t) may also arise from regular variation of \mathbf{B} under the alternative conditions

$$(3.10) \quad \mathbf{B} \in \text{RV}(\alpha, \mu^{\mathbf{B}}), \quad \mathbb{E}[\|\mathbf{A}\|^\alpha] < 1 \quad \text{and} \quad \mathbb{E}[\|\mathbf{A}\|^{\alpha+\delta}] < \infty \quad \text{for some } \delta > 0.$$

Then \mathbf{R} is regularly varying with index α and

$$\frac{\mathbb{P}(t^{-1}\mathbf{R} \in \cdot)}{\mathbb{P}(\|\mathbf{B}\| > t)} \xrightarrow{v} \int \mu_{\mathbf{B}}(\{\mathbf{y} : \mathbf{z}\mathbf{y} \in \cdot\}) \nu_{\Pi}(d\mathbf{z}),$$

where $\nu_{\Pi}(\cdot) = \sum_{k=0}^{\infty} \mathbb{P}(\Pi_k \in \cdot)$ is a measure on $\mathbb{M}_{d \times d}$; see Theorem 4.4.24 in [5].

For our purposes we will treat (\mathbf{A}, \mathbf{B}) as a random element of $\mathbb{M}_{d \times d} \times \mathbb{R}^d$ equipped with the norm $\|(\mathbf{a}, \mathbf{b})\| = \|\mathbf{a}\| + \|\mathbf{b}\|$, where $\|\mathbf{a}\|$ stands for the operator norm of the matrix \mathbf{a} (with respect to the Euclidean distance) and $\|\mathbf{b}\|$ is the Euclidean norm of the vector \mathbf{b} . We assume that the following set of conditions on (\mathbf{A}, \mathbf{B}) holds.

CONDITION (C).

(C1) A regular variation condition holds for some nonnull Radon measure $\mu^{(\mathbf{A}, \mathbf{B})}$ on $\mathbb{M}_{d \times d} \times \mathbb{R}^d$:

$$(3.11) \quad \frac{\mathbb{P}(t^{-1}(\mathbf{A}, \mathbf{B}) \in \cdot)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} \xrightarrow{v} \mu^{(\mathbf{A}, \mathbf{B})}(\cdot), \quad t \rightarrow \infty.$$

(C2) $X = \|(\mathbf{A}_1, \mathbf{B}_1)\|$ and $Y = \|(\mathbf{A}_2, \mathbf{B}_2)\|$ satisfy (1.5).

(C3) $\mathbb{E}[\|\mathbf{A}\|^\alpha] < 1$ and $\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\| > 1\}) > 0$.

Some comments.

- Note that

$$\begin{aligned} 1 &= \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|(\mathbf{a}, \mathbf{b})\| > 1\}) \\ &\leq \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\| > 1/2\}) + \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{b}\| > 1/2\}). \end{aligned}$$

In particular, at least one of the quantities on the right-hand side must be strictly positive. Hence condition (C1) implies that \mathbf{A} or \mathbf{B} must be regularly varying. Condition (C3) ensures that \mathbf{A} is regularly varying.

- To the best of our knowledge, except for some univariate cases treated in Damek and Dyszewski [8] and Kevei [19], not much is known about regular variation of \mathbf{R} under regular variation of \mathbf{A} and (C3). Then (3.10) is violated since $\mathbb{E}[\|\mathbf{A}\|^{\alpha+\delta}] = \infty$ for any $\delta > 0$.
- In view of Lemma 1.2 condition (C2) implies

$$\frac{\mathbb{P}(\|(\mathbf{A}_1, \mathbf{B}_1)\| \cdot \|(\mathbf{A}_2, \mathbf{B}_2)\| > t)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} \rightarrow 2\mathbb{E}\|\mathbf{A}, \mathbf{B}\|^\alpha.$$

The following result is a multivariate counterpart of the results obtained in Damek and Dyszewski [8].

THEOREM 3.4. *Assume (C). Then \mathbf{R} given in (3.9) satisfies*

$$\frac{\mathbb{P}(t^{-1}\mathbf{R} \in \cdot)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} \xrightarrow{v} \nu(\cdot) = \sum_{n=0}^{\infty} \mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \Pi_n(\mathbf{a}\mathbf{R}_0 + \mathbf{b}) \in \cdot\})].$$

In particular, if the measure ν on \mathbb{R}_0^d is nonnull then $\mathbf{R} \in \text{RV}(\alpha, \mu^{\mathbf{R}})$ with

$$\mu^{\mathbf{R}}(\cdot) = \nu(\cdot) / \nu(\{\mathbf{r} : \|\mathbf{r}\| > 1\}).$$

The remainder of this section is devoted to the proof of the theorem. A main step in the proof is provided by the following lemma.

LEMMA 3.5. *Assume that the \mathbb{R}^d -valued random vector $\mathbf{X} \in \text{RV}(\alpha, \mu^{\mathbf{X}})$ is independent of (\mathbf{A}, \mathbf{B}) which satisfies (C) and there is a positive constant $\tilde{d}_{\mathbf{X}}$ such that*

$$(3.12) \quad \frac{\mathbb{P}(\|\mathbf{X}\| > t)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} \rightarrow \tilde{d}_{\mathbf{X}}, \quad t \rightarrow \infty.$$

Then, as $t \rightarrow \infty$,

$$\begin{aligned} \frac{\mathbb{P}(\|\mathbf{A}\mathbf{X} + \mathbf{B}\| > t)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} &\rightarrow \mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\mathbf{X} + \mathbf{b}\| > 1\})] \\ &\quad + \tilde{d}_{\mathbf{X}}\mathbb{E}[\mu^{\mathbf{X}}(\{\mathbf{x} : \|\mathbf{A}\mathbf{x}\| > 1\})] =: C_0, \\ \frac{\mathbb{P}(t^{-1}(\mathbf{A}\mathbf{X} + \mathbf{B}) \in \cdot)}{\mathbb{P}(\|\mathbf{A}\mathbf{X} + \mathbf{B}\| > t)} &\xrightarrow{v} C_0^{-1}(\mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \mathbf{a}\mathbf{X} + \mathbf{b} \in \cdot\})] \\ &\quad + \tilde{d}_{\mathbf{X}}\mathbb{E}[\mu^{\mathbf{X}}(\{\mathbf{x} : \mathbf{A}\mathbf{x} \in \cdot\})]). \end{aligned}$$

PROOF OF LEMMA 3.5. Write $\mathbf{1}_d = (1, \dots, 1)^\top \in \mathbb{R}^d$, Id_d and $\text{diag}(\mathbf{b})$, $\mathbf{b} \in \mathbb{R}^d$, in $\mathbb{M}_{d \times d}$ for the identity matrix and the diagonal matrix whose consecutive diagonal entries are the consecutive components of \mathbf{b} , respectively. Put

$$\widehat{\mathbf{X}} = \begin{pmatrix} \mathbf{X} \\ \mathbf{1}_d \end{pmatrix} \in \mathbb{R}^{2d} \quad \text{and} \quad \widehat{\mathbf{A}} = \begin{pmatrix} \mathbf{A} & \text{diag}(\mathbf{B}) \\ \mathbf{0} & \text{Id}_d \end{pmatrix} \in \mathbb{M}_{2d \times 2d},$$

then $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{A}}$ are both regularly varying. Indeed, for $\widehat{\mathbf{X}}$ we have

$$\frac{\mathbb{P}(t^{-1}\widehat{\mathbf{X}} \in \cdot)}{\mathbb{P}(\|\widehat{\mathbf{X}}\| > t)} \sim \frac{\mathbb{P}(t^{-1}\widehat{\mathbf{X}} \in \cdot)}{\mathbb{P}(\|\mathbf{X}\| > t)} \xrightarrow{v} \mu^{\widehat{\mathbf{X}}}(\cdot) = \mu^{\mathbf{X}}\left(\left\{\mathbf{x} \in \mathbb{R}_0^d : \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \cdot\right\}\right).$$

For $\widehat{\mathbf{A}}$, choosing the operator norm $\|\cdot\|$, we have

$$\frac{\mathbb{P}(\|\widehat{\mathbf{A}}\| > t)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} \rightarrow \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\| \vee \|\text{diag}(\mathbf{b})\| > 1\}) = \tilde{d}_{\widehat{\mathbf{A}}}$$

and thus

$$\frac{\mathbb{P}(t^{-1}\widehat{\mathbf{A}} \in \cdot)}{\mathbb{P}(\|\widehat{\mathbf{A}}\| > t)} \xrightarrow{v} \mu^{\widehat{\mathbf{A}}}(\cdot) = \frac{1}{\tilde{d}_{\widehat{\mathbf{A}}}}\mu^{(\mathbf{A}, \mathbf{B})}\left(\left\{(\mathbf{a}, \mathbf{b}) : \begin{pmatrix} \mathbf{a} & \text{diag}(\mathbf{b}) \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \cdot\right\}\right).$$

To prove the claim we intend to use the fact that

$$\widehat{\mathbf{A}}\widehat{\mathbf{X}} = \begin{pmatrix} \mathbf{A}\mathbf{X} + \mathbf{B} \\ \mathbf{1}_d \end{pmatrix}$$

in combination with Theorem 2.1. In view of the tail equivalence condition (3.12) we have

$$\begin{aligned} c_{\widehat{\mathbf{A}}} &= \lim_{t \rightarrow \infty} \frac{\mathbb{P}(\|\widehat{\mathbf{A}}\| > t)}{\mathbb{P}(\|\widehat{\mathbf{A}}\| \cdot \|\widehat{\mathbf{X}}\| > t)} = \frac{\tilde{d}_{\widehat{\mathbf{A}}}}{\tilde{d}_{\widehat{\mathbf{A}}}\mathbb{E}[\|\widehat{\mathbf{X}}\|^\alpha] + \tilde{d}_{\widehat{\mathbf{X}}}\mathbb{E}[\|\widehat{\mathbf{A}}\|^\alpha]}, \\ c_{\widehat{\mathbf{X}}} &= \lim_{t \rightarrow \infty} \frac{\mathbb{P}(\|\widehat{\mathbf{X}}\| > t)}{\mathbb{P}(\|\widehat{\mathbf{A}}\| \cdot \|\widehat{\mathbf{X}}\| > t)} = \frac{\tilde{d}_{\widehat{\mathbf{X}}}}{\tilde{d}_{\widehat{\mathbf{A}}}\mathbb{E}[\|\widehat{\mathbf{X}}\|^\alpha] + \tilde{d}_{\widehat{\mathbf{X}}}\mathbb{E}[\|\widehat{\mathbf{A}}\|^\alpha]}. \end{aligned}$$

Therefore, Theorem 2.1(2) yields

$$\begin{aligned} &\frac{\mathbb{P}(t^{-1}\widehat{\mathbf{A}}\widehat{\mathbf{X}} \in \cdot)}{\mathbb{P}(\|\widehat{\mathbf{A}}\| \cdot \|\widehat{\mathbf{X}}\| > t)} \\ &\quad \xrightarrow{v} c_{\widehat{\mathbf{A}}}\mathbb{E}[\mu^{\widehat{\mathbf{A}}}(\{\widehat{\mathbf{a}} : \widehat{\mathbf{a}}\widehat{\mathbf{X}} \in \cdot\})] + c_{\widehat{\mathbf{X}}}\mathbb{E}[\mu^{\widehat{\mathbf{X}}}(\{\widehat{\mathbf{x}} : \widehat{\mathbf{A}}\widehat{\mathbf{x}} \in \cdot\})] \\ &= c_{\widehat{\mathbf{A}}}\tilde{d}_{\widehat{\mathbf{A}}}^{-1}\mathbb{E}\left[\mu^{(\mathbf{A}, \mathbf{B})}\left(\left\{(\mathbf{a}, \mathbf{b}) : \begin{pmatrix} \mathbf{a}\mathbf{X} + \mathbf{b} \\ \mathbf{0} \end{pmatrix} \in \cdot\right\}\right)\right] + c_{\widehat{\mathbf{X}}}\mathbb{E}\left[\mu^{\mathbf{X}}\left(\left\{\mathbf{x} : \begin{pmatrix} \mathbf{A}\mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \cdot\right\}\right)\right] \end{aligned}$$

which implies

$$\begin{aligned} & \frac{\mathbb{P}(t^{-1}(\mathbf{A}\mathbf{X} + \mathbf{B}) \in \cdot)}{\mathbb{P}(\|\widehat{\mathbf{A}}\| \cdot \|\widehat{\mathbf{X}}\| > t)} \\ & \xrightarrow{v} c_{\widehat{\mathbf{A}}} \tilde{d}_{\widehat{\mathbf{A}}}^{-1} \mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \mathbf{a}\mathbf{X} + \mathbf{b} \in \cdot\})] + c_{\widehat{\mathbf{X}}} \mathbb{E}[\mu^{\mathbf{X}}(\{\mathbf{x} : \mathbf{A}\mathbf{x} \in \cdot\})] \\ & = \frac{\mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \mathbf{a}\mathbf{X} + \mathbf{b} \in \cdot\})] + \tilde{d}_{\widehat{\mathbf{X}}} \mathbb{E}[\mu^{\mathbf{X}}(\{\mathbf{x} : \mathbf{A}\mathbf{x} \in \cdot\})]}{\tilde{d}_{\widehat{\mathbf{A}}} \mathbb{E}[\|\widehat{\mathbf{X}}\|^\alpha] + \tilde{d}_{\widehat{\mathbf{X}}} \mathbb{E}[\|\widehat{\mathbf{A}}\|^\alpha]}. \end{aligned}$$

Both claims now follow since

$$\begin{aligned} \frac{\mathbb{P}(\|\widehat{\mathbf{A}}\| \cdot \|\widehat{\mathbf{X}}\| > t)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} &= \frac{\mathbb{P}(\|\widehat{\mathbf{A}}\| \cdot \|\widehat{\mathbf{X}}\| > t)}{\mathbb{P}(\|\widehat{\mathbf{A}}\| > t)} \cdot \frac{\mathbb{P}(\|\widehat{\mathbf{A}}\| > t)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} \\ &\rightarrow \frac{\tilde{d}_{\widehat{\mathbf{A}}}}{c_{\widehat{\mathbf{A}}}} = \tilde{d}_{\widehat{\mathbf{A}}} \mathbb{E}[\|\widehat{\mathbf{X}}\|^\alpha] + \tilde{d}_{\widehat{\mathbf{X}}} \mathbb{E}[\|\widehat{\mathbf{A}}\|^\alpha]. \quad \square \end{aligned}$$

Consider the Markov chain $(\mathbf{R}_n^0)_{n \geq 0}$ given by the recursion (3.7) with $\mathbf{R}_0^0 = 0$. Then

$$\mathbf{R}_n^0 \stackrel{d}{=} \sum_{k=0}^{n-1} \mathbf{\Pi}_k \mathbf{B}_{k+1} \xrightarrow{d} \mathbf{R}.$$

By Lemma 3.5,

$$\frac{\mathbb{P}(t^{-1} \mathbf{R}_n^0 \in \cdot)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} \xrightarrow{v} \nu_n(\cdot),$$

and the sequence $(\nu_n)_{n \geq 0}$ of measures on \mathbb{R}_0^d satisfies the recursive relation for $n \geq 0$,

$$(3.13) \quad \nu_{n+1}(\cdot) = \mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \mathbf{a}\mathbf{R}_n^0 + \mathbf{b} \in \cdot\})] + \mathbb{E}[\nu_n(\{\mathbf{x} : \mathbf{A}\mathbf{x} \in \cdot\})],$$

and $\nu_0 = o$ is the null measure. We have

$$\left\| \sum_{k=0}^{\infty} \mathbf{\Pi}_k \mathbf{B}_{k+1} \right\| \leq R = \sum_{k=0}^{\infty} \|\mathbf{B}_{k+1}\| \prod_{j=1}^k \|\mathbf{A}_j\|,$$

where R solves the equation in law

$$R \stackrel{d}{=} \|\mathbf{A}\| R + \|\mathbf{B}\|, \quad R \text{ independent of } (\mathbf{A}, \mathbf{B}).$$

From the main result in Damek and Dyszewski [8] (see Lemma A.1 in the Appendix) we also have under (C),

$$(3.14) \quad \begin{aligned} \limsup_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{R}\| > t)}{\mathbb{P}(\|\mathbf{A}\| > t)} &\leq \limsup_{t \rightarrow \infty} \frac{\mathbb{P}(R > t)}{\mathbb{P}(\|\mathbf{A}\| > t)} < \infty, \\ \sup_n \mathbb{E}[\|\mathbf{R}_n^0\|^\alpha] &\leq \mathbb{E}[R^\alpha] < \infty. \end{aligned}$$

LEMMA 3.6. Assume (C). Then

$$\nu_n(\cdot) \xrightarrow{v} \nu(\cdot) = \sum_{k=0}^{\infty} \underbrace{\mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \mathbf{\Pi}_k(\mathbf{a}\mathbf{R}_0 + \mathbf{b}) \in \cdot\})]}_{= \eta_k},$$

where ν is a Radon measure on \mathbb{R}_0^d .

PROOF OF LEMMA 3.6. For $k \leq n$ write $\Pi_{n,k}^\downarrow = \mathbf{A}_n \mathbf{A}_{n-1} \cdots \mathbf{A}_k$. We have by (3.13),

$$v_n(\cdot) = \sum_{k=1}^n \mathbb{E} \left[\underbrace{\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \Pi_{n,k+1}^\downarrow(\mathbf{a}\mathbf{R}_{k-1}^0 + \mathbf{b}) \in \cdot\})}_{=\eta_{n,k}} \right].$$

We intend to show $v_n \xrightarrow{v} \nu$ or, equivalently, $\int f d\nu_n \rightarrow \int f d\nu$ for any $f \in C_c^+(\mathbb{R}_0^d)$. There are $c_f, M_f > 0$ such that

$$(3.15) \quad \text{supp}(f) \subseteq \{\mathbf{z} \in \mathbb{R}^d : c_f^{-1} \leq \|\mathbf{z}\| \leq c_f\} \quad \text{and} \quad \sup_{\mathbf{z} \in \mathbb{R}^d} f(\mathbf{z}) \leq M_f.$$

Our strategy is to use the following approximations:

$$\int f d\nu_n \stackrel{(1)}{\approx} \int f d\left(\sum_{n/2 < k \leq n} \eta_{n,k}\right) \stackrel{(2)}{\approx} \int f d\left(\sum_{0 < k \leq n/2} \eta_k\right) \stackrel{(3)}{\approx} \int f d\nu.$$

In what follows, we will make these approximations precise.

Approximations (1) and (3). For (1), we will show that

$$(3.16) \quad \lim_{n \rightarrow \infty} \int f d\left(\sum_{k \leq n/2} \eta_{n,k}\right) = \lim_{n \rightarrow \infty} \sum_{k=1}^{[n/2]} \mathbb{E} \left[\int f(\Pi_{n,k+1}^\downarrow(\mathbf{a}\mathbf{R}_{k-1}^0 + \mathbf{b})) \mu^{(\mathbf{A}, \mathbf{B})}(d(\mathbf{a}, \mathbf{b})) \right] = 0.$$

For $c = c_f^{-1}$ and $k \leq [n/2]$ we have

$$\begin{aligned} & \mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\Pi_{n,k+1}^\downarrow(\mathbf{a}\mathbf{R}_{k-1}^0 + \mathbf{b})\| > c\})] \\ & \leq \mathbb{E}[\|\Pi_{n,k+1}^\downarrow\|^\alpha] \mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\mathbf{R}_{k-1}^0 + \mathbf{b}\| > c\})] \\ & \leq (\mathbb{E}[\|\mathbf{A}\|^\alpha])^{n-k} (\mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\mathbf{R}_{k-1}^0\| > c/2\})] \\ & \quad + \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{b}\| > c/2\})) \\ & \leq (\mathbb{E}[\|\mathbf{A}\|^\alpha])^{n-k} (\mathbb{E}[\|\mathbf{R}_{k-1}^0\|^\alpha] \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\| > c/2\}) \\ & \quad + \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{b}\| > c/2\})). \end{aligned}$$

Now, by (3.14) we can take a constant const big enough such that

$$\mathbb{E}[\|\mathbf{R}_{k-1}^0\|^\alpha] \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\| > c/2\}) + \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{b}\| > c/2\}) \leq \text{const}$$

for all k to obtain

$$\mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\Pi_{n,k+1}^\downarrow(\mathbf{a}\mathbf{R}_{k-1}^0 + \mathbf{b})\| > c\})] \leq \text{const} (\mathbb{E}[\|\mathbf{A}\|^\alpha])^{n-k}.$$

Now (3.16) is immediate in view of condition $\mathbb{E}[\|\mathbf{A}\|^\alpha] < 1$ and since $f \leq M_f$. The proof of

$$\lim_{n \rightarrow \infty} \int f d\left(\sum_{k > n/2} \eta_k\right) = \lim_{n \rightarrow \infty} \sum_{k=[n/2]+1}^\infty \mathbb{E} \left[\int f(\Pi_k(\mathbf{a}\mathbf{R}_0 + \mathbf{b})) \mu^{(\mathbf{A}, \mathbf{B})}(d(\mathbf{a}, \mathbf{b})) \right] = 0,$$

goes along similar lines. We have

$$\begin{aligned} & \mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\Pi^k(\mathbf{a}\mathbf{R}_0 + \mathbf{b})\| > c\})] \\ & \leq \mathbb{E}[\|\Pi^k\|^\alpha] \mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\mathbf{R}_0 + \mathbf{b}\| > c\})] \\ & \leq (\mathbb{E}[\|\mathbf{A}\|^\alpha])^k (\mathbb{E}[\mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\mathbf{R}_0\| > c/2\})] + \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{b}\| > c/2\})) \\ & \leq \text{const} \cdot (\mathbb{E}[\|\mathbf{A}\|^\alpha])^k. \end{aligned}$$

The fact that ν is a Radon measure is proved by using the above estimates.

Approximation (2). We have

$$\begin{aligned}
 & \left| \int f d \left(\sum_{n/2 < k \leq n} \eta_{n,k} - \sum_{0 < k \leq n/2} \eta_k \right) \right| \\
 &= \left| \int f d \left(\sum_{n/2 < k \leq n} (\eta_{n,k} - \eta_{n-k}) \right) \right| \\
 (3.17) \quad &= \left| \sum_{k=[n/2]+1}^n \left(\mathbb{E} \left[\int f(\mathbf{\Pi}_{n,k+1}^\downarrow(\mathbf{aR}_{k-1}^0 + \mathbf{b})) \mu^{(\mathbf{A}, \mathbf{B})}(d(\mathbf{a}, \mathbf{b})) \right] \right. \right. \\
 & \quad \left. \left. - \mathbb{E} \left[\int f(\mathbf{\Pi}_{n-k}(\mathbf{aR}_0 + \mathbf{b})) \mu^{(\mathbf{A}, \mathbf{B})}(d(\mathbf{a}, \mathbf{b})) \right] \right) \right|,
 \end{aligned}$$

and we will show that the right-hand side converges to zero as $n \rightarrow \infty$. By uniform continuity of f ,

$$(3.18) \quad \text{for any } \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that } \|\mathbf{s} - \mathbf{r}\| \leq \delta \implies |f(\mathbf{r}) - f(\mathbf{s})| \leq \varepsilon.$$

Let $(\mathbf{\Pi}'_i)$ be an independent copy of $(\mathbf{\Pi}_i)$. For $[n/2] < k \leq n$ and fixed $\mathbf{a} \in \mathbb{M}_{d \times d}$ write

$$A_{k,\delta}(\mathbf{a}) = \left\{ \left\| \mathbf{\Pi}'_{n-k} \mathbf{a} \sum_{j=k-1}^{\infty} \mathbf{\Pi}_j \mathbf{B}_{j+1} \right\| > \delta \right\}.$$

Since $\mathbf{\Pi}_{n,k+1}^\downarrow \stackrel{d}{=} \mathbf{\Pi}_{n-k}$ we have

$$\begin{aligned}
 & \left| \mathbb{E} \left[\int f(\mathbf{\Pi}_{n,k+1}^\downarrow(\mathbf{aR}_{k-1}^0 + \mathbf{b})) \mu^{(\mathbf{A}, \mathbf{B})}(d(\mathbf{a}, \mathbf{b})) \right] \right. \\
 & \quad \left. - \mathbb{E} \left[\int f(\mathbf{\Pi}_{n-k}(\mathbf{aR}_0 + \mathbf{b})) \mu^{(\mathbf{A}, \mathbf{B})}(d(\mathbf{a}, \mathbf{b})) \right] \right| \\
 & \leq \int \mathbb{E} \left[(\mathbf{1}_{A_{k,\delta}(\mathbf{a})} + \mathbf{1}_{A_{k,\delta}^c(\mathbf{a})}) \right. \\
 & \quad \times \left| f \left(\mathbf{\Pi}'_{n-k} \left(\mathbf{a} \sum_{j=0}^{k-2} \mathbf{\Pi}_j \mathbf{B}_{j+1} + \mathbf{b} \right) \right) - f \left(\mathbf{\Pi}'_{n-k} \left(\mathbf{a} \sum_{j=0}^{\infty} \mathbf{\Pi}_j \mathbf{B}_{j+1} + \mathbf{b} \right) \right) \right| \\
 & \quad \times \mu^{(\mathbf{A}, \mathbf{B})}(d(\mathbf{a}, \mathbf{b})) \left. \right] \\
 & = H_k^{(1)} + H_k^{(2)}.
 \end{aligned}$$

We will first treat $H_k^{(1)}$. Using $f \in [0, M_f]$ for the first inequality and the homogeneity of $\mu^{(\mathbf{A}, \mathbf{B})}$ for the equality, we have

$$\begin{aligned}
 H_k^{(1)} & \leq M_f \mathbb{E} \left[\mu^{(\mathbf{A}, \mathbf{B})} \left(\left\{ (\mathbf{a}, \mathbf{b}) : \|\mathbf{\Pi}'_{n-k}\| \|\mathbf{a}\| \sum_{j=k-1}^{\infty} \|\mathbf{\Pi}_j \mathbf{B}_{j+1}\| > \delta \right\} \right) \right] \\
 & = M_f \mathbb{E} [\|\mathbf{\Pi}_{n-k}\|^\alpha] \cdot \mathbb{E} \left[\left(\sum_{j=k-1}^{\infty} \|\mathbf{\Pi}_j \mathbf{B}_{j+1}\| \right)^\alpha \right] \delta^{-\alpha} \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\| > 1\}).
 \end{aligned}$$

Note that $\mathbb{E}[\|\mathbf{\Pi}_{n-k}\|^\alpha] \leq (\mathbb{E}[\|\mathbf{A}\|^\alpha])^{n-k}$ and

$$\sum_{j=k-1}^{\infty} \|\mathbf{\Pi}_j \mathbf{B}_{j+1}\| \leq \|\mathbf{\Pi}_{k-1}\| \cdot \sum_{j=k-1}^{\infty} \|\mathbf{\Pi}_{k,j} \mathbf{B}_{j+1}\| \leq \|\mathbf{\Pi}_{k-1}\| \cdot \sum_{j=k-1}^{\infty} \|\mathbf{B}_{j+1}\| \prod_{i=k}^j \|\mathbf{A}_i\|,$$

where the series appearing on the right-hand side is distributed as R . The last two observations yield

$$\mathbb{E} \left[\left(\sum_{j=k-1}^{\infty} \|\Pi_j \mathbf{B}_{j+1}\| \right)^\alpha \right] \leq (\mathbb{E}[\|\mathbf{A}\|^\alpha])^{k-1} \cdot \mathbb{E}[R^\alpha].$$

This constitutes an upper bound on $H_k^{(1)}$ of the form

$$H_k^{(1)} \leq \text{const}(\mathbb{E}[\|\mathbf{A}\|^\alpha])^{n-1} \delta^{-\alpha},$$

for a sufficiently large constant const . Turning our attention to $H_k^{(2)}$, we first note that

$$\left\| \Pi'_{n-k} \left(\mathbf{a} \sum_{j=0}^{k-2} \Pi_j \mathbf{B}_{j+1} + \mathbf{b} \right) \right\| \vee \left\| \Pi'_{n-k} \left(\mathbf{a} \sum_{j=0}^{\infty} \Pi_j \mathbf{B}_{j+1} + \mathbf{b} \right) \right\| \leq \|\Pi'_{n-k}\| (\|\mathbf{a}\| R + \|\mathbf{b}\|).$$

Recall (3.15) and put $c = c_f^{-1}$. On the event $\{\|\Pi'_{n-k}\|(\|\mathbf{a}\| R + \|\mathbf{b}\|) < c\}$,

$$f \left(\Pi'_{n-k} \left(\mathbf{a} \sum_{j=0}^{k-2} \Pi_j \mathbf{B}_{j+1} + \mathbf{b} \right) \right) = f \left(\Pi'_{n-k} \left(\mathbf{a} \sum_{j=0}^{\infty} \Pi_j \mathbf{B}_{j+1} + \mathbf{b} \right) \right) = 0$$

which justifies

$$H_k^{(2)} = \int \mathbb{E}[|\dots| \mathbf{1}(A_{k,\delta}^c(\mathbf{a}) \cap \{\|\Pi'_{n-k}\| \|\mathbf{a}\| R > c\})] \mu^{(\mathbf{A}, \mathbf{B})}(d(\mathbf{a}, \mathbf{b})).$$

Now use (3.18) and the homogeneity of $\mu^{(\mathbf{A}, \mathbf{B})}$ to get

$$H_k^{(2)} \leq \varepsilon (\mathbb{E}[\|\mathbf{A}\|^\alpha])^{n-k} \mu^{(\mathbf{A}, \mathbf{B})}(\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\| R + \|\mathbf{b}\| > c\}) \leq \varepsilon \text{const}(\mathbb{E}[\|\mathbf{A}\|^\alpha])^{n-k}$$

for a sufficiently large constant const . These computations yield

$$\sum_{k=\lfloor n/2 \rfloor + 1}^n (H_k^{(1)} + H_k^{(2)}) \leq \text{const} n (\mathbb{E}[\|\mathbf{A}\|^\alpha])^{n-1} \delta^{-\alpha} + \varepsilon \text{const}.$$

This bound shows that the right-hand side of (3.17) converges to zero by first letting $n \rightarrow \infty$ and then $\varepsilon \rightarrow 0$. \square

FINAL STEPS IN THE PROOF OF THEOREM 3.4. Choose $f \in C_c^+(\mathbb{R}_0^d)$ and fix constants $c_f, M_f > 0$ such that (3.15) holds. By uniform continuity of f , we can choose $\varepsilon, \delta > 0$ such that (3.18) holds. Write

$$A_{n,t} = \left\{ \left\| \sum_{j=n}^{\infty} \Pi_j \mathbf{B}_{j+1} \right\| > \delta t \right\}.$$

We have

$$\begin{aligned} |\mathbb{E}[f(t^{-1}\mathbf{R}) - f(t^{-1}\mathbf{R}_n^0)]| &\leq \mathbb{E}[|f(t^{-1}\mathbf{R}) - f(t^{-1}\mathbf{R}_n^0)|] \\ &= \mathbb{E} \left[\left| f(t^{-1}\mathbf{R}) - f \left(t^{-1} \sum_{j=0}^{n-1} \Pi_j \mathbf{B}_{j+1} \right) \right| (\mathbf{1}(A_{n,t}) + \mathbf{1}(A_{n,t}^c)) \right] \\ &= \tilde{H}_1(t) + \tilde{H}_2(t). \end{aligned}$$

Both terms are asymptotically negligible. Indeed, for the first one,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\tilde{H}_1(t)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} &\leq M_f \limsup_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{\Pi}'_n\| R > \delta t)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} \\ &\leq \text{const}(\mathbb{E}[\|\mathbf{A}\|^\alpha])^n \delta^{-\alpha}. \end{aligned}$$

The right-hand side converges to zero as $n \rightarrow \infty$, since $\mathbb{E}[\|\mathbf{A}\|^\alpha] < 1$. As regards $\tilde{H}_2(t)$ first note that

$$\left\| \sum_{j=0}^{n-1} \mathbf{\Pi}_j \mathbf{B}_{j+1} \right\| \vee \|\mathbf{R}\| \leq R$$

and so with $c = c_f^{-1}$, where c_f is given in (3.15), we have on the event $\{t^{-1}R < c\}$,

$$f\left(t^{-1} \sum_{j=0}^{n-1} \mathbf{\Pi}_j \mathbf{B}_{j+1}\right) = f(t^{-1}\mathbf{R}) = 0.$$

Using (3.18), we can write

$$\begin{aligned} \frac{\tilde{H}_2(t)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} &= \frac{\mathbb{E}[f(t^{-1}\mathbf{R}) - f(t^{-1} \sum_{j=0}^{n-1} \mathbf{\Pi}_j \mathbf{B}_{j+1}) | \mathbf{1}(A_{n,t}^c) \mathbf{1}(R > ct)]}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} \\ &\leq \varepsilon \frac{\mathbb{P}(R > ct)}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} \leq \text{const } \varepsilon. \end{aligned}$$

In view of Lemma 3.6, first letting $t \rightarrow \infty$, then $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$, we may conclude that

$$\frac{\mathbb{E}[f(t^{-1}\mathbf{R})]}{\mathbb{P}(\|(\mathbf{A}, \mathbf{B})\| > t)} \rightarrow \int f(\mathbf{r}) \nu(d\mathbf{r}).$$

Since f is arbitrary the theorem follows. \square

4. Proof of Theorem 2.1. Throughout this section we consider an \mathbb{R}^{d_X} -valued $\mathbf{X} \in \text{RV}(\alpha_X, \mu^{\mathbf{X}})$ random vector independent of an \mathbb{R}^{d_Y} -valued \mathbf{Y} and we will write for shorthand

$$\alpha_{\mathbf{X}} = \alpha, \quad \alpha_{\mathbf{X}} = a, \quad \alpha_{\mathbf{Y}} = \beta, \quad \alpha_{\mathbf{Y}} = b.$$

Recall that $\mathbf{Z} = \psi(\mathbf{X}, \mathbf{Y}) \in \mathbb{R}^{d_Z}$ and the definition of ξ_t from (2.1). Then (2.1) can be reformulated as

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[f(t^{-1}\mathbf{Z})]}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} = \lim_{t \rightarrow \infty} \int f(\mathbf{z}) \xi_t(d\mathbf{z}) = \int f(\mathbf{z}) \xi(d\mathbf{z}), \quad f \in C_c^+(\mathbb{R}_0^{d_Z}).$$

Since ψ is continuous

$$M_\psi = \sup\{\|\psi(\mathbf{x}, \mathbf{y})\| : \|\mathbf{x}\| = 1, \|\mathbf{y}\| = 1\} < \infty.$$

It is also a - b -homogeneous and therefore

$$\|\psi(\mathbf{x}, \mathbf{y})\| \leq M_\psi \|\mathbf{x}\|^a \|\mathbf{y}\|^b.$$

Then we also have for any set $A_r = \{\mathbf{z} : \|\mathbf{z}\| > r\}$, $r > 0$, in view of regular variation of $\|\mathbf{X}\|^a \|\mathbf{Y}\|^b$,

$$\sup_{t > 0} \xi_t(A_r) \leq \frac{\mathbb{P}(M_\psi \|\mathbf{X}\|^a \|\mathbf{Y}\|^b > rt)}{\mathbb{P}(\|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t)} < \infty.$$

It follows from Resnick [21], Proposition 3.16, that (ξ_t) is vaguely relatively compact. Hence (ξ_{t_k}) converges vaguely along sequences $t_k \rightarrow \infty$ as $k \rightarrow \infty$, and it remains to show that these limits coincide with ξ .

The proof of the theorem is given through several auxiliary results which we provide first. The main steps of the proof are given at the end of this section.

Limits of $\mathbb{E}[f(t^{-1}\psi(\mathbf{X}, \mathbf{Y}) \mid \mathbf{Y})]$. By regular variation of \mathbf{X} we have

$$(4.1) \quad \mu_t^{\mathbf{X}}(\cdot) = \frac{\mathbb{P}(t^{-1}\mathbf{X} \in \cdot)}{\mathbb{P}(\|\mathbf{X}\| > t)} \xrightarrow{v} \mu^{\mathbf{X}}(\cdot), \quad t \rightarrow \infty.$$

Define

$$(4.2) \quad g_t(\mathbf{y}) = \frac{\mathbb{E}[f(t^{-1}\psi(\mathbf{X}, \mathbf{y}))]}{\mathbb{P}(\|\mathbf{X}\|^a > t)} = \int f(\psi(\mathbf{x}, \mathbf{y}))\mu_{t^{1/a}}^{\mathbf{X}}(d\mathbf{x}), \quad \mathbf{y} \in \mathbb{R}^{d_Y}, t > 0.$$

In view of (4.1) we expect that the right-hand side converges as $t \rightarrow \infty$ to

$$(4.3) \quad g(\mathbf{y}) = \int f(\psi(\mathbf{x}, \mathbf{y}))\mu^{\mathbf{X}}(d\mathbf{x}) < \infty, \quad \mathbf{y} \in \mathbb{R}^{d_Y}.$$

However, for some choices of $\mathbf{y} \in \mathbb{R}^{d_Y}$, the function $\mathbf{x} \mapsto f(\psi(\mathbf{x}, \mathbf{y}))$ may not have compact support and therefore some additional argument is needed.

LEMMA 4.1. *Relation (4.3) holds for any $f \in C_c^+(\mathbb{R}_0^{d_Z})$.*

PROOF OF LEMMA 4.1. Fix $\mathbf{y} \in \mathbb{R}^{d_Y}$. Since f is compactly supported there are constants $M_f, c_f > 0$ such that

$$(4.4) \quad \text{supp}(f) \subseteq \{\mathbf{z} \in \mathbb{R}^{d_Z} : c_f^{-1} \leq \|\mathbf{z}\| \leq c_f\} \quad \text{and} \quad \sup_{\mathbf{z} \in \mathbb{R}^{d_Z}} f(\mathbf{z}) \leq M_f.$$

For $r \geq 1$ choose any continuous function $\varphi_r : \mathbb{R}^{d_X} \rightarrow [0, 1]$ such that

$$\varphi_r(\mathbf{x}) = \begin{cases} 1, & \|\mathbf{x}\| \leq r, \\ 0, & \|\mathbf{x}\| \geq 2r. \end{cases}$$

We have

$$g_t(\mathbf{y}) = \int f(\psi(\mathbf{x}, \mathbf{y}))\varphi_r(\mathbf{x})\mu_{t^{1/a}}^{\mathbf{X}}(d\mathbf{x}) + \int f(\psi(\mathbf{x}, \mathbf{y}))(1 - \varphi_r(\mathbf{x}))\mu_{t^{1/a}}^{\mathbf{X}}(d\mathbf{x}) = I_1 + I_2.$$

The contribution of the second term is negligible since in view of (4.1),

$$\begin{aligned} 0 &\leq \lim_{r \rightarrow \infty} \limsup_{t \rightarrow \infty} I_2 \\ &\leq M_f \lim_{r \rightarrow \infty} \lim_{t \rightarrow \infty} \mu_{t^{1/a}}^{\mathbf{X}}(\{\mathbf{x} : \|\mathbf{x}\| > r\}) \\ &= M_f \lim_{r \rightarrow \infty} \mu^{\mathbf{X}}(\{\mathbf{x} : \|\mathbf{x}\| > r\}) = 0. \end{aligned}$$

Thus it suffices to prove $\lim_{r \rightarrow \infty} \lim_{t \rightarrow \infty} I_1 = g(\mathbf{y})$. The function $\mathbf{x} \mapsto f(\psi(\mathbf{x}, \mathbf{y}))\varphi_r(\mathbf{x})$ is continuous and nonnegative for any choice of $\mathbf{y} \in \mathbb{R}^{d_Y}$ and $r > 1$, and its support is contained in $\{\mathbf{x} \in \mathbb{R}^{d_X} : (M_\psi \|\mathbf{y}\|^b c_f)^{-1/a} \leq \|\mathbf{x}\| \leq 2r\}$ which is a compact subset of $\mathbb{R}_0^{d_X}$. Regular variation of \mathbf{X} and monotone convergence allow one to take the successive limits

$$\begin{aligned} \lim_{r \rightarrow \infty} \lim_{t \rightarrow \infty} I_1 &= \lim_{r \rightarrow \infty} \int f(\psi(\mathbf{x}, \mathbf{y}))\varphi_r(\mathbf{x})\mu^{\mathbf{X}}(d\mathbf{x}) = g(\mathbf{y}) \\ &= \int_{\|\mathbf{x}\| \geq (M_\psi \|\mathbf{y}\|^b c_f)^{-1/a}} f(\psi(\mathbf{x}, \mathbf{y}))\mu^{\mathbf{X}}(d\mathbf{x}) \leq M_f (M_\psi \|\mathbf{y}\|^b c_f)^{\alpha/a} < \infty. \quad \square \end{aligned}$$

The next result presents a continuity bound for g_t .

LEMMA 4.2. *Let $f \in C_c^+(\mathbb{R}_0^{dz})$. For any $\varepsilon > 0$ one can choose $\delta > 0$ and $t_0 > 0$ such that, for any $\mathbf{s}, \mathbf{r} \in \mathbb{S}^{d_Y-1}$ with $\|\mathbf{s} - \mathbf{r}\| \leq \delta$ and any $t > t_0$,*

$$(4.5) \quad |g_t(\mathbf{r}) - g_t(\mathbf{s})| \leq \varepsilon.$$

PROOF OF LEMMA 4.2. Fix $\varepsilon_1 > 0$. Choose $M_f, c_f > 0$ from (4.4). By uniform continuity of f we can choose $\eta \in (0, \varepsilon_1)$ such that $\|\mathbf{z}_1 - \mathbf{z}_2\| \leq \eta$ implies $\|f(\mathbf{z}_1) - f(\mathbf{z}_2)\| \leq \varepsilon_1$. Since ψ is uniformly continuous on $\mathbb{S}^{d_X-1} \times \mathbb{S}^{d_Y-1}$ we can find $\delta > 0$ such that for $\mathbf{r}, \mathbf{s} \in \mathbb{S}^{d_Y-1}$ with $\|\mathbf{r} - \mathbf{s}\| < \delta$,

$$\|\psi(\mathbf{x}, \mathbf{r}) - \psi(\mathbf{x}, \mathbf{s})\| < \eta^2, \quad \|\mathbf{x}\| = 1.$$

Then by homogeneity of ψ ,

$$\|\psi(\mathbf{x}, \mathbf{r}) - \psi(\mathbf{x}, \mathbf{s})\| < \|\mathbf{x}\|^a \eta^2, \quad \mathbf{x} \in \mathbb{R}^{d_X},$$

and we can write for $t > 0$ and $\|\mathbf{r} - \mathbf{s}\| < \delta$,

$$\begin{aligned} |g_t(\mathbf{s}) - g_t(\mathbf{r})| &\leq \int |f(\psi(\mathbf{x}, \mathbf{s})) - f(\psi(\mathbf{x}, \mathbf{r}))| \mu_{t^{1/a}}^{\mathbf{X}}(d\mathbf{x}) \\ &= \int_{\|\mathbf{x}\| \geq (M_\psi c_f)^{-1/a}} |f(\psi(\mathbf{x}, \mathbf{s})) - f(\psi(\mathbf{x}, \mathbf{r}))| \mu_{t^{1/a}}^{\mathbf{X}}(d\mathbf{x}) \\ &\leq \int_{\|\mathbf{x}\| > \varepsilon_1^{-1/a}} |f(\psi(\mathbf{x}, \mathbf{s})) - f(\psi(\mathbf{x}, \mathbf{r}))| \mu_{t^{1/a}}^{\mathbf{X}}(d\mathbf{x}) \\ &\quad + \int_{\|\mathbf{x}\| \leq \eta^{-1/a}, \|\mathbf{x}\| \geq (M_\psi c_f)^{-1/a}} |f(\psi(\mathbf{x}, \mathbf{s})) - f(\psi(\mathbf{x}, \mathbf{r}))| \mu_{t^{1/a}}^{\mathbf{X}}(d\mathbf{x}) \\ &\leq 2M_f \mu_{t^{1/a}}^{\mathbf{X}}(\{\mathbf{x} : \|\mathbf{x}\| > \varepsilon_1^{-1/a}\}) + \varepsilon_1 \mu_{t^{1/a}}^{\mathbf{X}}(\{\mathbf{x} : \|\mathbf{x}\| \geq (M_\psi c_f)^{-1/a}\}) \\ &= J_1 + J_2. \end{aligned}$$

Choose $t_0 = t_0(\varepsilon)$ sufficiently large such that, for $t > t_0$,

$$\begin{aligned} J_1 &\leq 4M_f \mu_{t^{1/a}}^{\mathbf{X}}(\{\mathbf{x} : \|\mathbf{x}\| > \varepsilon_1^{-1/a}\}) \\ &= 4M_f \varepsilon_1^a \mu_{t^{1/a}}^{\mathbf{X}}(\{\mathbf{x} : \|\mathbf{x}\| > \mathbf{1}\}) = 4M_f \varepsilon_1^a, \\ J_2 &\leq 2\varepsilon_1 \mu_{t^{1/a}}^{\mathbf{X}}(\{\mathbf{x} : \|\mathbf{x}\| \geq (M_\psi c_f)^{-1/a}\}) = 2\varepsilon_1 M_\psi c_f. \end{aligned}$$

Then, for given $\varepsilon > 0$ and ε_1 sufficiently small,

$$|g_t(\mathbf{s}) - g_t(\mathbf{r})| \leq 4M_f \varepsilon_1^a + 2\varepsilon_1 M_\psi c_f \leq \varepsilon. \quad \square$$

Note that by continuity of f and ψ , g is also continuous on \mathbb{R}^{d_Y} , hence also uniformly continuous on the unit sphere \mathbb{S}^{d_Y-1} . We will use this comment in the proof of the next lemma.

LEMMA 4.3. *Let $f \in C_c^+(\mathbb{R}_0^{dz})$. Then $g_t \rightarrow g$ as $t \rightarrow \infty$ uniformly on \mathbb{S}^{d_Y-1} .*

PROOF OF LEMMA 4.3. Fix $\varepsilon > 0$ and choose $\delta > 0, t_0 > 0$ as in the formulation of Lemma 4.2 and such that

$$\|\mathbf{s} - \mathbf{r}\| \leq \delta \quad \Rightarrow \quad |g(\mathbf{r}) - g(\mathbf{s})| \leq \varepsilon.$$

There exist $N = N(\delta)$ and a collection of points $\{\mathbf{r}_k\}_{k=1}^N \subset \mathbb{S}^{d_Y-1}$ such that $\mathbb{S}^{d_Y-1} \subseteq \bigcup_{k=1}^N \{\mathbf{y} : \|\mathbf{r}_k - \mathbf{y}\| \leq \delta\}$. Take $t_1 > 0$ so large that

$$\max_{1 \leq k \leq N} |g_t(\mathbf{r}_k) - g(\mathbf{r}_k)| \leq \varepsilon, \quad t > t_1.$$

Then for any $\mathbf{s} \in \mathbb{S}^{d_{\mathbf{Y}}-1}$ we have $\|\mathbf{s} - \mathbf{r}_k\| \leq \delta$ for some k , and for $t > t_0 \vee t_1$ we have

$$|g_t(\mathbf{s}) - g(\mathbf{s})| \leq |g_t(\mathbf{s}) - g_t(\mathbf{r}_k)| + |g_t(\mathbf{r}_k) - g(\mathbf{r}_k)| + |g(\mathbf{r}_k) - g(\mathbf{s})| \leq 3\varepsilon.$$

This finishes the proof of the lemma. \square

Before we proceed with the final steps in the proof of Theorem 2.1 we observe that homogeneity of $\mu^{\mathbf{X}}$ and ψ implies for any $r > 0$ and $\mathbf{y} \in \mathbb{R}^{d_{\mathbf{Y}}}$,

$$(4.6) \quad g(r\mathbf{y}) = r^{\frac{ab}{a}} g(\mathbf{y}).$$

We will first prove Theorem 2.1 in the case when only one of the vectors is assumed to be regularly varying.

PROOF OF THEOREM 2.1 UNDER (H). Take an arbitrary $f \in C_c^+(\mathbb{R}_0^{d_{\mathbf{Z}}})$ and fix $c_f, M_f > 0$ such that (4.4) holds. Note that

$$f(t^{-1}\psi(\mathbf{X}, \mathbf{Y})) = f(t^{-1}\psi(\mathbf{X}, \mathbf{Y}))\mathbf{1}_{\{M_\psi c_f \|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t\}}.$$

Take $\varphi: \mathbb{R}^{d_{\mathbf{X}}} \rightarrow [0, 1]$ such that $\varphi(\mathbf{x}) = 1$ if $\|\mathbf{x}\|^a > 1$ and $\varphi(\mathbf{x}) = 0$ if $\|\mathbf{x}\|^a < 1/2$. For $\varepsilon > 0$ write

$$\begin{aligned} \int f(\mathbf{z})\xi_t(d\mathbf{z}) &= \frac{\mathbb{E}[f(t^{-1}\psi(\mathbf{X}, \mathbf{Y}))(1 - \varphi((t\varepsilon)^{-1/a}\mathbf{X}))]}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} + \frac{\mathbb{E}[f(t^{-1}\psi(\mathbf{X}, \mathbf{Y}))\varphi((t\varepsilon)^{-1/a}\mathbf{X})]}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} \\ &= K_1 + K_2. \end{aligned}$$

Recall (H3):

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}\|^a > t)}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} = c_{\mathbf{X}} = \frac{1}{\mathbb{E}\|\mathbf{Y}\|^{\alpha b/a}}$$

which, by Lemma 1.2(3), is equivalent to

$$\lim_{\varepsilon \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t, \|\mathbf{X}\|^a \leq \varepsilon t)}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} = 0.$$

Then

$$\lim_{\varepsilon \rightarrow 0} \limsup_{t \rightarrow \infty} K_1 \leq \lim_{\varepsilon \rightarrow 0} \limsup_{t \rightarrow \infty} M_f \frac{\mathbb{P}(M_\psi c_f \|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t, \|\mathbf{X}\|^a \leq \varepsilon t)}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} = 0.$$

As regards K_2 , we observe that

$$\begin{aligned} K_2 &= c_{\mathbf{X}}(1 + o(1)) \int \frac{\mathbb{E}[f(t^{-1}\psi(\mathbf{X}, \mathbf{y}))\varphi((t\varepsilon)^{-1/a}\mathbf{X})]}{\mathbb{P}(\|\mathbf{X}\|^a > t)} \mathbb{P}(\mathbf{Y} \in d\mathbf{y}) \\ &= c_{\mathbf{X}}(1 + o(1)) \int \int f(\psi(\mathbf{x}, \mathbf{y}))\varphi(\varepsilon^{-1/a}\mathbf{x})\mu_{t^{1/a}}^{\mathbf{X}}(d\mathbf{x})\mathbb{P}(\mathbf{Y} \in d\mathbf{y}). \end{aligned}$$

The expression appearing under the first integral is bounded by $M_f \frac{\mathbb{P}(\|\mathbf{X}\|^a > \varepsilon t/2)}{\mathbb{P}(\|\mathbf{X}\|^a > t)}$. Hence we can let $t \rightarrow \infty$, pass with it under the integral and use regular variation of \mathbf{X} to obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} K_2 &= c_{\mathbf{X}} \int \int f(\psi(\mathbf{x}, \mathbf{y}))\varphi(\varepsilon^{-1/a}\mathbf{x})\mu^{\mathbf{X}}(d\mathbf{x})\mathbb{P}(\mathbf{Y} \in d\mathbf{y}) \\ &= c_{\mathbf{X}} \mathbb{E} \left[\int f(\psi(\mathbf{x}, \mathbf{Y}))\varphi(\varepsilon^{-1/a}\mathbf{x})\mu^{\mathbf{X}}(d\mathbf{x}) \right]. \end{aligned}$$

Finally, letting $\varepsilon \rightarrow 0$, monotone convergence yields

$$\lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} K_2 = c_{\mathbf{X}} \mathbb{E} \left[\int f(\psi(\mathbf{x}, \mathbf{Y})) \mu^{\mathbf{X}}(d\mathbf{x}) \right],$$

which concludes the proof. \square

In what follows, we assume that $\mathbf{X} \in \text{RV}(\alpha_{\mathbf{X}}, \mu^{\mathbf{X}})$, $\mathbf{Y} \in \text{RV}(\alpha_{\mathbf{Y}}, \mu^{\mathbf{Y}})$ and define functions $h_t: \mathbb{R}^{d_{\mathbf{X}}} \rightarrow [0, +\infty)$ by

$$h_t(\mathbf{x}) = \int f(\psi(\mathbf{x}, \mathbf{y})) \mu_{t^{1/b}}^{\mathbf{Y}}(d\mathbf{y}) = \frac{\mathbb{E}[f(t^{-1}\psi(\mathbf{x}, \mathbf{Y}))]}{\mathbb{P}(\|\mathbf{Y}\|^b > t)}, \quad \mathbf{x} \in \mathbb{R}^{d_{\mathbf{X}}}, t > 0.$$

By a symmetry argument, interchanging the roles of \mathbf{Y} and \mathbf{X} , we conclude that $h_t \rightarrow h$ as $t \rightarrow \infty$ point-wise in $\mathbb{R}^{d_{\mathbf{X}}}$ and uniformly on $\mathbb{S}^{d_{\mathbf{X}}-1}$ where

$$(4.7) \quad h(\mathbf{x}) = \int f(\psi(\mathbf{x}, \mathbf{y})) \mu^{\mathbf{Y}}(d\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^{d_{\mathbf{X}}}.$$

The limiting function is also homogeneous, that is, for $r > 0$ and $\mathbf{x} \in \mathbb{R}^{d_{\mathbf{X}}}$,

$$(4.8) \quad h(r\mathbf{x}) = r^{\frac{\beta a}{b}} h(\mathbf{x}).$$

We will now treat the cases (T) and (R) of Theorem 2.1. Using the conditions (T3) and (R2) it is possible to give two separate, but shorter proofs. However, since both cases will use the same decomposition of $\{\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t\}$, we prefer the simultaneous approach.

PROPOSITION 4.4. *Assume that the $\mathbb{R}^{d_{\mathbf{X}}}$ -valued $\mathbf{X} \in \text{RV}(\alpha, \mu^{\mathbf{X}})$ and the $\mathbb{R}^{d_{\mathbf{Y}}}$ -valued $\mathbf{Y} \in \text{RV}(\beta, \mu^{\mathbf{Y}})$ random vectors are independent and the following balance condition is satisfied for positive a, b :*

$$(4.9) \quad \lim_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}\|^a > t)}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} = c_{\mathbf{X}}, \quad \lim_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{Y}\|^b > t)}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} = c_{\mathbf{Y}}.$$

Then the following relation holds for any $f \in C_c^+(\mathbb{R}_0^{d_{\mathbf{Z}}})$:

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\mathbb{E}[f(t^{-1}\psi(\mathbf{X}, \mathbf{Y}))]}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} &= (1 - c_{\mathbf{X}} \mathbb{E}[\|\mathbf{Y}\|^{ab/a}] - c_{\mathbf{Y}} \mathbb{E}[\|\mathbf{X}\|^{ab/b}]) \mathbb{E}[g(\Theta_{\mathbf{Y}})] \\ &\quad + c_{\mathbf{X}} \mathbb{E}[g(\mathbf{Y})] + c_{\mathbf{Y}} \mathbb{E}[h(\mathbf{X})], \end{aligned}$$

where $g: \mathbb{R}^{d_{\mathbf{Y}}} \rightarrow \mathbb{R}$ and $h: \mathbb{R}^{d_{\mathbf{X}}} \rightarrow \mathbb{R}$ are given in (4.3) and (4.7), respectively.

PROOF. Choose $M_f > 0$ from (4.4) and consider the following decomposition, for $\eta \in (0, 1)$:

$$\begin{aligned} \mathbb{E}[f(t^{-1}\psi(\mathbf{X}, \mathbf{Y}))] &= \mathbb{E}[f(t^{-1}\psi(\mathbf{X}, \mathbf{Y}))\mathbf{1}(\|\mathbf{Y}\|^b \leq \eta t)] \\ &\quad + \mathbb{E}[f(t^{-1}\psi(\mathbf{X}, \mathbf{Y}))\mathbf{1}(\|\mathbf{X}\|^a \leq \eta t, \|\mathbf{Y}\|^b > \eta t)] \\ &\quad + \mathbb{E}[f(t^{-1}\psi(\mathbf{X}, \mathbf{Y}))\mathbf{1}(\|\mathbf{X}\|^a > \eta t, \|\mathbf{Y}\|^b > \eta t)] \\ &= J_1(t) + J_2(t) + J_3(t). \end{aligned}$$

Since f is bounded and \mathbf{X}, \mathbf{Y} are independent we have $J_3(t) = o(\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t))$. Thus it remains to investigate J_1 and J_2 . We begin with the analysis of the first term, since it requires more work.

Analysis of J_1 . We claim that

$$\begin{aligned} & \lim_{\eta \rightarrow 0} \liminf_{t \rightarrow \infty} \frac{J_1(t)}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} \\ &= \lim_{\eta \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{J_1(t)}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} \\ &= (1 - c_{\mathbf{X}}\mathbb{E}\|\mathbf{Y}\|^{ab/a} - c_{\mathbf{Y}}\mathbb{E}\|\mathbf{X}\|^{\beta a/b})\mathbb{E}[g(\Theta_{\mathbf{Y}})] + c_{\mathbf{X}}\mathbb{E}[g(\mathbf{Y})]. \end{aligned}$$

Below we will present a detailed argument for

$$(4.10) \quad \begin{aligned} & \lim_{\eta \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{J_1(t)}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} \\ & \leq (1 - c_{\mathbf{X}}\mathbb{E}\|\mathbf{Y}\|^{ab/a} - c_{\mathbf{Y}}\mathbb{E}\|\mathbf{X}\|^{\beta a/b})\mathbb{E}[g(\Theta_{\mathbf{Y}})] + c_{\mathbf{X}}\mathbb{E}[g(\mathbf{Y})]. \end{aligned}$$

The lower bound can be established in a similar fashion. Write for $\mathbf{y} \neq \mathbf{0}$, $\tilde{\mathbf{y}} = \mathbf{y}/\|\mathbf{y}\|$, and

$$\begin{aligned} J_1(t) &= \int_{\|\mathbf{y}\|^b \leq \eta t} \mathbb{E}[f(t^{-1}\psi(\mathbf{X}, \mathbf{y}))]\mathbb{P}(\mathbf{Y} \in d\mathbf{y}) \\ &= \int_{\|\mathbf{y}\|^b \leq \eta t} \mathbb{E}\left[f\left(\frac{\|\mathbf{y}\|^b}{t}\psi(\mathbf{X}, \tilde{\mathbf{y}})\right)\right] \cdot \frac{\mathbb{P}(\|\mathbf{X}\|^a > t/\|\mathbf{y}\|^b)}{\mathbb{P}(\|\mathbf{X}\|^a > t/\|\mathbf{y}\|^b)}\mathbb{P}(\mathbf{Y} \in d\mathbf{y}) \\ &= \int_{\|\mathbf{y}\|^b \leq \eta t} g_{\frac{t}{\|\mathbf{y}\|^b}}(\tilde{\mathbf{y}})\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{y}\|^b > t)\mathbb{P}(\mathbf{Y} \in d\mathbf{y}), \end{aligned}$$

where g_t is given via (4.2). By virtue of Lemma 4.3, for any $\varepsilon > 0$ there is a sufficiently small $\eta > 0$ such that

$$\begin{aligned} & \left| J_1(t) - \int_{\|\mathbf{y}\|^b \leq \eta t} g(\tilde{\mathbf{y}})\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{y}\|^b > t)\mathbb{P}(\mathbf{Y} \in d\mathbf{y}) \right| \\ & \leq \int_{\|\mathbf{y}\|^b \leq \eta t} |g_{\frac{t}{\|\mathbf{y}\|^b}}(\tilde{\mathbf{y}}) - g(\tilde{\mathbf{y}})|\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{y}\|^b > t)\mathbb{P}(\mathbf{Y} \in d\mathbf{y}) \\ & \leq \varepsilon\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t). \end{aligned}$$

Thus, since ε is arbitrary, we only need to investigate the expectation

$$I(t) = \mathbb{E}[g(\tilde{\mathbf{Y}})\mathbf{1}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t, \|\mathbf{Y}\|^b \leq \eta t)].$$

If $\mathbb{E}[g(\tilde{\mathbf{Y}})] = 0$ then by homogeneity of g , $g(\mathbf{Y}) = 0$ a.s. which implies $\mathbb{E}[g(\mathbf{Y})] = 0$ and $\mathbb{E}[g(\Theta_{\mathbf{Y}})] = 0$, so the claim follows trivially. Now assume $\mathbb{E}[g(\tilde{\mathbf{Y}})] > 0$. Let Y' be a random variable independent of \mathbf{X} and \mathbf{Y} with distribution given by

$$\mathbb{P}(Y' \in \cdot) = \mathbb{E}\left[\frac{g(\tilde{\mathbf{Y}})}{\mathbb{E}[g(\tilde{\mathbf{Y}})]}\mathbf{1}(\|\mathbf{Y}\|^b \in \cdot)\right].$$

Then, by regular variation of \mathbf{Y} , as $t \rightarrow \infty$,

$$\frac{\mathbb{P}(Y' > t)}{\mathbb{P}(\|\mathbf{Y}\|^b > t)} = \mathbb{E}\left[\frac{g(\tilde{\mathbf{Y}})}{\mathbb{E}[g(\tilde{\mathbf{Y}})]}\middle|\|\mathbf{Y}\|^b > t\right] \rightarrow \frac{\mathbb{E}[g(\Theta_{\mathbf{Y}})]}{\mathbb{E}[g(\tilde{\mathbf{Y}})]}.$$

Therefore, for any $\delta > 0$ there exists $T = T(\delta)$ such that

$$(4.11) \quad (1 - \delta)\frac{\mathbb{E}[g(\Theta_{\mathbf{Y}})]}{\mathbb{E}[g(\tilde{\mathbf{Y}})]} \leq \frac{\mathbb{P}(Y' > t)}{\mathbb{P}(\|\mathbf{Y}\|^b > t)} \leq (1 + \delta)\frac{\mathbb{E}[g(\Theta_{\mathbf{Y}})]}{\mathbb{E}[g(\tilde{\mathbf{Y}})]}, \quad t \geq T.$$

Without loss of generality we may assume that $T \uparrow \infty$ when $\delta \downarrow 0$. Consider the following decomposition:

$$\begin{aligned} \frac{I(t)}{\mathbb{E}[g(\tilde{\mathbf{Y}})]} &= \mathbb{P}(\|\mathbf{X}\|^a Y' > t, Y' \leq \eta t) \\ &= \mathbb{P}(\|\mathbf{X}\|^a Y' > t, Y' > T) + \mathbb{P}(\|\mathbf{X}\|^a Y' > t, Y' \leq T) - \mathbb{P}(\|\mathbf{X}\|^a Y' > t, Y' > \eta t) \\ &= I_1(t) + I_2(t) - I_3(t). \end{aligned}$$

By Breiman’s lemma 1.1 and definition of $c_{\mathbf{X}}$ we have

$$\begin{aligned} \lim_{T \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{\mathbb{E}[g(\tilde{\mathbf{Y}})]I_2(t)}{\mathbb{P}(\|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t)} &= \lim_{T \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{\mathbb{E}[g(\tilde{\mathbf{Y}})]I_2(t)}{\mathbb{P}(\|\mathbf{X}\|^a > t)} \frac{\mathbb{P}(\|\mathbf{X}\|^a > t)}{\mathbb{P}(\|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t)} \\ &= \lim_{T \rightarrow \infty} c_{\mathbf{X}} \mathbb{E}[g(\tilde{\mathbf{Y}})] \mathbb{E}[(Y')^{\alpha/a} \mathbf{1}(Y' \leq T)] \\ &= c_{\mathbf{X}} \mathbb{E}[g(\tilde{\mathbf{Y}}) \|\mathbf{Y}\|^{ab/a}] \\ &= c_{\mathbf{X}} \mathbb{E}[g(\mathbf{Y})]. \end{aligned}$$

For the first term we have, by (4.11),

$$\begin{aligned} \mathbb{E}[g(\tilde{\mathbf{Y}})]I_1(t) &= \mathbb{E}[g(\tilde{\mathbf{Y}})] \int \mathbb{P}(Y' > T \vee (t/\|\mathbf{x}\|^a)) \mathbb{P}(\mathbf{X} \in d\mathbf{x}) \\ &\leq (1 + \delta) \mathbb{E}[g(\Theta_{\mathbf{Y}})] \int \mathbb{P}(\|\mathbf{Y}\|^b > T \vee (t/\|\mathbf{x}\|^a)) \mathbb{P}(\mathbf{X} \in d\mathbf{x}) \\ &= (1 + \delta) \mathbb{E}[g(\Theta_{\mathbf{Y}})] \mathbb{P}(\|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t, \|\mathbf{Y}\|^b > T) \\ &= (1 + \delta) \mathbb{E}[g(\Theta_{\mathbf{Y}})] [\mathbb{P}(\|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t) \\ &\quad - \mathbb{P}(\|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t, \|\mathbf{Y}\|^b \leq T)] \\ &\sim (1 + \delta) \mathbb{E}[g(\Theta_{\mathbf{Y}})] \mathbb{P}(\|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t) \\ &\quad \times \left[1 - \mathbb{E}[\|\mathbf{Y}\|^{ab/a} \mathbf{1}(\|\mathbf{Y}\|^a \leq T)] \frac{\mathbb{P}(\|\mathbf{X}\|^a > t)}{\mathbb{P}(\|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t)} \right]. \end{aligned}$$

In the last step we used Breiman’s result as $t \rightarrow \infty$. Now, recalling the definition of $c_{\mathbf{Y}}$, we conclude that

$$\begin{aligned} \lim_{T \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{\mathbb{E}[g(\tilde{\mathbf{Y}})]I_1(t)}{\mathbb{P}(\|\mathbf{X}\|^a \|\mathbf{Y}\|^b > t)} \\ \leq (1 + \delta) \mathbb{E}[g(\Theta_{\mathbf{Y}})] [1 - c_{\mathbf{X}} \mathbb{E}[\|\mathbf{Y}\|^{ab/a}]], \end{aligned}$$

and the corresponding lower bound can be derived in an analogous way for any small $\delta > 0$.

Finally, we deal with the third term. First we observe that, by regular variation,

$$\begin{aligned} \lim_{\eta \downarrow 0} \lim_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{X}\|^a > \eta^{-1}, \|\mathbf{Y}\|^b > \eta t)}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} \\ (4.12) \quad &= \lim_{\eta \downarrow 0} \mathbb{P}(\|\mathbf{X}\|^a > \eta^{-1}) \eta^{-\beta/b} \lim_{t \rightarrow \infty} \frac{\mathbb{P}(\|\mathbf{Y}\|^b > t)}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} \\ &= c_{\mathbf{Y}} \lim_{\eta \downarrow 0} \mathbb{P}(\|\mathbf{X}\|^a > \eta^{-1}) \eta^{-\beta/b} = 0. \end{aligned}$$

Indeed, if $\mathbb{E}[\|\mathbf{X}\|^{\beta a/b}] = \infty$ then $c_{\mathbf{Y}} = 0$ and therefore the right-hand side is zero; see Lemma 1.2(2). On the other hand, if $\mathbb{E}[\|\mathbf{X}\|^{\beta a/b}] < \infty$ then

$$\mathbb{P}(\|\mathbf{X}\|^a > \eta^{-1}) = \mathbb{P}(\|\mathbf{X}\|^{a\beta/b} > \eta^{-\beta/b}) = o(\eta^{\beta/b}), \quad \eta \downarrow 0,$$

and therefore the right-hand side in (4.12) is zero.

With (4.11) and Breiman’s result at hand, we have as $t \rightarrow \infty$,

$$\begin{aligned} & \mathbb{E}[g(\tilde{\mathbf{Y}})]I_3(t) \\ & \leq (1 + \delta)\mathbb{E}[g(\Theta_{\mathbf{Y}})]\mathbb{P}(\|\mathbf{X}\|^a\|\mathbf{Y}\|^b > t, \|\mathbf{Y}\|^b > \eta t) \\ & = (1 + \delta)\mathbb{E}[g(\Theta_{\mathbf{Y}})] \\ & \quad \times [\mathbb{P}(\|\mathbf{X}\|^a > \eta^{-1}, \|\mathbf{Y}\|^b > \eta t) + \mathbb{P}(\|\mathbf{X}\|^a \leq \eta^{-1}, \|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)] \\ & \sim (1 + \delta)\mathbb{E}[g(\Theta_{\mathbf{Y}})][\mathbb{P}(\|\mathbf{X}\|^a > \eta^{-1}, \|\mathbf{Y}\|^b > \eta t) \\ & \quad + \mathbb{E}[\|\mathbf{X}\|^{a\beta/b}\mathbf{1}(\|\mathbf{X}\|^a \leq \eta^{-1})]]\mathbb{P}(\|\mathbf{Y}\|^b > t), \quad t \rightarrow \infty. \end{aligned}$$

Now an application of (4.12) and the definition of $c_{\mathbf{Y}}$ yield

$$\lim_{\eta \downarrow 0} \limsup_{t \rightarrow \infty} \frac{\mathbb{E}[g(\tilde{\mathbf{Y}})]I_3(t)}{\mathbb{P}(\|\mathbf{X}\|^a \cdot \|\mathbf{Y}\|^b > t)} \leq c_{\mathbf{Y}}(1 + \delta)\mathbb{E}[g(\Theta_{\mathbf{Y}})]\mathbb{E}[\|\mathbf{X}\|^{a\beta/b}].$$

This establishes an upper bound; the corresponding lower bound is completely analogous. This proves (4.10).

Analysis of J_2 . This term can be handled in a significantly simpler way. Similarly to J_1 we have

$$J_2(t) = \int_{\|\mathbf{x}\|^a \leq \eta t} h_{\frac{t}{\|\mathbf{x}\|^a}}(\tilde{\mathbf{x}})\mathbb{P}\left(\|\mathbf{Y}\|^b \cdot \left(\frac{1}{\eta} \wedge \|\mathbf{x}\|^a\right) > t\right)\mathbb{P}(\mathbf{X} \in d\mathbf{x}),$$

where $\tilde{\mathbf{x}} = \mathbf{x}/\|\mathbf{x}\|$ for $\mathbf{x} \neq \mathbf{0}$. Appealing to the dominated convergence theorem, we obtain

$$\lim_{t \rightarrow \infty} \frac{J_2(t)}{\mathbb{P}(\|\mathbf{X}\|^a\|\mathbf{Y}\|^b > t)} = \int_{\mathbb{R}^d} h(\tilde{\mathbf{x}})c_{\mathbf{Y}}\left(\frac{1}{\eta} \wedge \|\mathbf{x}\|^a\right)^{\beta/b} \mathbb{P}(\mathbf{X} \in d\mathbf{x}).$$

Now monotone convergence yields

$$\lim_{\eta \rightarrow 0} \lim_{t \rightarrow \infty} \frac{J_2(t)}{\mathbb{P}(\|\mathbf{X}\|^a\|\mathbf{Y}\|^b > t)} = c_{\mathbf{Y}} \int_{\mathbb{R}^d} h(\tilde{\mathbf{x}})\|\mathbf{x}\|^{a\beta/b} \mathbb{P}(\mathbf{X} \in d\mathbf{x}) = c_{\mathbf{Y}}\mathbb{E}h(\mathbf{X}),$$

where the last equality employs the homogeneity of h stated in (4.8). \square

APPENDIX

A.1. Proof of Lemma 1.2. (1) was proved in Embrechts and Goldie ([12], page 245). We start with (2). Observe that for any $M > 0$, by the uniform convergence theorem for regularly varying functions,

$$\frac{\mathbb{P}(XY > x)}{\mathbb{P}(X > x)} \geq \int_0^M \frac{\mathbb{P}(X > x/y)}{\mathbb{P}(X > x)}\mathbb{P}(Y \in dy) \rightarrow \int_0^M y^\alpha \mathbb{P}(Y \in dy), \quad x \rightarrow \infty.$$

If $\mathbb{E}[Y^\alpha] = \infty$ we can make the right-hand side arbitrarily large by letting $M \rightarrow \infty$.

Since (3) and (4) can be proven with the same arguments we only present a proof for the latter. We follow the lines of the proof of Proposition 3.1 in Davis and Resnick [10] who consider the case of i.i.d. X, Y . Choose any $M > 1$. Then

$$\begin{aligned} & \mathbb{P}(XY > t) \\ & = \mathbb{P}(XY > t, X \leq M) + \mathbb{P}(XY > t, M < X \leq t/M) + \mathbb{P}(XY > t, X > t/M) \\ & = I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

In view of Breiman’s lemma 1.1 we have, as $t \rightarrow \infty$,

$$\frac{I_3(t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(X(Y \wedge M) > t)}{\mathbb{P}(X > t)} \rightarrow \mathbb{E}[(Y \wedge M)^\alpha],$$

$$\frac{I_1(t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(YX\mathbf{1}(X \leq M) > t)}{\mathbb{P}(Y > t)} \frac{\mathbb{P}(Y > t)}{\mathbb{P}(X > t)} \rightarrow c_0\mathbb{E}[X^\alpha\mathbf{1}_{\{X \leq M\}}],$$

where $c_0 = \lim_{t \rightarrow \infty} \mathbb{P}(Y > t)/\mathbb{P}(X > t)$ is assumed finite. By an appeal to the monotone convergence theorem

$$\lim_{M \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{I_3(t)}{\mathbb{P}(X > t)} = \mathbb{E}[Y^\alpha],$$

$$\lim_{M \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{I_1(t)}{\mathbb{P}(X > t)} = c_0\mathbb{E}[X^\alpha],$$

and thus

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}(XY > t)}{\mathbb{P}(x > t)} = \mathbb{E}[Y^\alpha] + c_0\mathbb{E}[X^\alpha]$$

if and only if

$$\lim_{M \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{I_2(t)}{\mathbb{P}(X > t)} = 0.$$

We continue with (5). Denote

$$\underline{c} = \liminf_{t \rightarrow \infty} \frac{\mathbb{P}(X_1X_2 > t)}{\mathbb{P}(X > t)}, \quad \bar{c} = \limsup_{t \rightarrow \infty} \frac{\mathbb{P}(X_1X_2 > t)}{\mathbb{P}(X > t)},$$

where X_1 and X_2 are independent copies of X . Fix $\varepsilon > 0$ and take $M > 1$ such that

$$(A.1) \quad c_0(1 - \varepsilon) \leq \frac{\mathbb{P}(Y > t)}{\mathbb{P}(X > t)} \leq c_0(1 + \varepsilon) \quad \text{for } t > M,$$

where $c_0 = \lim_{t \rightarrow \infty} \mathbb{P}(Y > t)/\mathbb{P}(X > t)$ is assumed finite and positive. Then

$$\mathbb{P}(XY > t) = \mathbb{P}(XY > t, Y \leq M) + \mathbb{P}(XY > t, M < Y) = \tilde{I}_1(t) + \tilde{I}_2(t).$$

In view of Breiman’s lemma 1.1 we have, as $t \rightarrow \infty$,

$$\frac{\tilde{I}_1(t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(YX\mathbf{1}(Y \leq M) > t)}{\mathbb{P}(X > t)} \rightarrow \mathbb{E}[Y^\alpha\mathbf{1}_{\{Y \leq M\}}].$$

The second term can be bounded from above in the following fashion, using (A.1):

$$\begin{aligned} \frac{\tilde{I}_2(t)}{\mathbb{P}(X > t)} &= \frac{\int_0^\infty \mathbb{P}(Y > M \vee tx^{-1})\mathbb{P}(X \in dx)}{\mathbb{P}(X > t)} \\ &\leq c_0(1 + \varepsilon) \frac{\int_0^\infty \mathbb{P}(X > M \vee tx^{-1})\mathbb{P}(X \in dx)}{\mathbb{P}(X > t)} \\ &= c_0(1 + \varepsilon) \frac{\mathbb{P}(X_1X_2 > t, X_1 > M)}{\mathbb{P}(X > t)} \\ &= c_0(1 + \varepsilon) \left(\frac{\mathbb{P}(X_1X_2 > t)}{\mathbb{P}(X > t)} - \frac{\mathbb{P}(X_1X_2 > t, X_1 \leq M)}{\mathbb{P}(X > t)} \right). \end{aligned}$$

Yet another appeal to Breiman’s lemma 1.1 yields

$$\frac{\mathbb{P}(X_1X_2 > t, X_1 \leq M)}{\mathbb{P}(X > t)} \rightarrow \mathbb{E}[X^\alpha\mathbf{1}_{\{X \leq M\}}].$$

If we put everything together we get the upper bound

$$\begin{aligned} c &= \lim_{t \rightarrow \infty} \frac{\mathbb{P}(XY > t)}{\mathbb{P}(X > t)} \leq \lim_{\varepsilon \rightarrow 0} \lim_{M \rightarrow \infty} (\mathbb{E}[Y^\alpha \mathbf{1}_{\{Y \leq M\}}] + c_0(1 + \varepsilon)(\underline{c} - \mathbb{E}[X^\alpha \mathbf{1}_{\{X \leq M\}}])) \\ &= \mathbb{E}[Y^\alpha] + c_0(\underline{c} - \mathbb{E}[X^\alpha]). \end{aligned}$$

If one goes back to the analysis of $\tilde{I}_2(t)$ and uses the lower bound in (A.1), similar arguments as above yield the bound

$$c = \lim_{t \rightarrow \infty} \frac{\mathbb{P}(XY > t)}{\mathbb{P}(X > t)} \geq \mathbb{E}[Y^\alpha] + c_0(\bar{c} - \mathbb{E}[X^\alpha]).$$

Since $c_0 > 0$ then the above in particular implies that $\bar{c} < \infty$ and

$$\mathbb{E}[Y^\alpha] + c_0(\underline{c} - \mathbb{E}[X^\alpha]) = c = \mathbb{E}[Y^\alpha] + c_0(\bar{c} - \mathbb{E}[X^\alpha]).$$

This implies that

$$\bar{c} = \underline{c} = \frac{c - \mathbb{E}Y^\alpha}{c_0} + \mathbb{E}[X^\alpha]$$

which means in particular that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}(X_1 X_2 > t)}{\mathbb{P}(X > t)} = \frac{c - \mathbb{E}Y^\alpha}{c_0} + \mathbb{E}[X^\alpha] < \infty.$$

An appeal to [15], Theorem 3, yields that the only possible value of the limit above is $2\mathbb{E}X^\alpha$ which in turn gives

$$c = c_0\mathbb{E}X^\alpha + \mathbb{E}Y^\alpha$$

as claimed.

A.2. A result from [8]. We state some part of Theorem 3.1 from [8] applied to the Lipschitz function $\Psi(t) = \|\mathbf{A}\|t + \|\mathbf{B}\|$.

LEMMA A.1. *Assume that $\|\mathbf{A}\|$ is regularly varying with index $\alpha > 0$, $\mathbb{E}[\|\mathbf{A}\|^\alpha] < 1$, $\mathbb{P}(\|\mathbf{B}\| > t) = O(\mathbb{P}(\|\mathbf{A}\| > t))$, and*

$$\frac{\mathbb{P}(\|\mathbf{A}_1\| \cdot \|\mathbf{A}_2\| > t)}{\mathbb{P}(\|\mathbf{A}\| > t)} \rightarrow 2\mathbb{E}[\|\mathbf{A}\|^\alpha], \quad t \rightarrow \infty.$$

Then $R = \sum_{k=0}^\infty \|\mathbf{B}_{k+1}\| \prod_{j=1}^k \|\mathbf{A}_j\|$ is finite and satisfies $\mathbb{P}(R > t) = O(\mathbb{P}(\|\mathbf{A}\| > t))$ as $t \rightarrow \infty$. In particular, $\mathbb{E}[R^\alpha] < \infty$.

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