MODELLING INFORMATION FLOWS BY MEYER-σ-FIELDS IN THE SINGULAR STOCHASTIC CONTROL PROBLEM OF IRREVERSIBLE INVESTMENT

BY PETER BANK[*] AND DAVID BESSLICH[†]

Institut für Mathematik, Technische Universität Berlin, [*]*bank@math.tu-berlin.de;* [†]*besslich@math.tu-berlin.de*

In stochastic control problems delicate issues arise when the controlled system can jump due to both exogenous shocks and endogenous controls. Here one has to specify what the controller knows when about the exogenous shocks and how and when she can act on this information. We propose to use Meyer-σ-fields as a flexible tool to model information flow in such situations. The possibilities of this approach are illustrated first in a very simple linear stochastic control problem and then in a fairly general formulation for the singular stochastic control problem of irreversible investment with inventory risk. For the latter, we illustrate in a first case study how different signals on exogenous jumps lead to different optimal controls, interpolating between the predictable and the optional case in a systematic manner.

1. Introduction. In stochastic control problems one seeks to influence a given system in an optimal way while taking into account the dynamically revealed information on this system. It is clear that this information flow is crucial for the determination of which controls can be used at all and for what an optimal control looks like. Of particular importance are moments in time where significant new information becomes available, for instance, on an impending exogenous jump. If the controller is restricted to predictable controls, she can only react after the jump has hit the system. In the case of optional controls she can react to jumps as they happen. Apart from these classical choices, it is perfectly conceivable though that the controller at times receives a signal on the upcoming jump that she can use for a proactive intervention and then still react after the jump is fully revealed.

We show how one can use Meyer-σ-fields Λ (introduced in Lenglart (1980)) embedded between the optional and predictable σ-field to model information in such situations. As a toy example, we consider a simple linear control problem, which we also use to introduce the basic tools from the *théorie générale des processus* which pertain to Meyer-σ-fields (e.g., Bismut and Skalli (1977), Dellacherie and Meyer (1978), Lenglart (1980), El Karoui (1981), Dellacherie and Meyer (1982)). For a more serious control problem, we discuss in depth an irreversible investment problem with inventory risk. Irreversible investment problems have been considered in great detail in the literature before (e.g., Arrow (1966), Dixit and Pindyck (1994), Bertola (1998), Merhi and Zervos (2007), Riedel and Su (2011), Federico and Pham (2014), Ferrari (2015), Al Motairi and Zervos (2017), De Angelis, Federico and Ferrari (2017)). This kind of problem can be formulated as the task to

$$(1) \qquad \text{Maximize} \quad \widetilde{V}(\tilde{C}) := \mathbb{E}\left[\int_{[0,\infty)} P_t \, d\tilde{C}_t - \int_{[0,\infty)} \rho_t(\tilde{C}_t) \, dR_t\right],$$

over Λ-measurable, increasing, right-continuous controls $\tilde{C}$ starting at a given level c_0. Here, the integral over ρ yields a convex risk assessment for a given control with the integrator R

---

serving as a risk clock; the process P describes the rewards accrued from an increase in the control $\tilde{C}$. It is conceivable and, in fact perfectly natural for the examples in this paper, that the reward process is not fully observable to the controller when she makes her decisions. Instead, she will work with the reward process's Λ-projection $^{\Lambda}P$ in (1), which is observable to her. To ensure existence of optimal controls we impose suitable mathematical semicontinuity assumptions on the reward process, but we also need to relax the set of controls to include also increasing controls C which are not right-continuous. We introduce a suitable integral $\int_{[0,\infty)} {}^{\Lambda}P \, {}^{*}\mathrm{d}C$ for such relaxed controls with possible double jumps (see Definition 3.5) and show that the ensuing relaxed optimization problem

$$(2) \qquad \text{Maximize} \quad V(C) := \mathbb{E}\left[\int_{[0,\infty)} {}^{\Lambda}P_t \, {}^{*}\mathrm{d}C_t - \int_{[0,\infty)} \rho_t(C_t) \, \mathrm{d}R_t \right]$$

over suitable increasing Λ-measurable C has the same value as the original problem (1). An optimal control for (2) is constructed in terms of the solution of a closely related stochastic representation problem, following an approach first studied in Bank and Riedel (2001). The general setting here requires a considerably refined argument though, which is based on a similarly refined version of the representation theorem from Bank and El Karoui (2004) which we study in the companion paper Bank and Besslich (2018a).

To illustrate this general result by a nontrivial explicit example, we focus on the special case $\rho_t(c) := \frac{1}{2}c^2$ and let P be a discounted compound Poisson process with initial value $\tilde{p}$, that is, $P_t := \mathrm{e}^{-rt}\tilde{P}_t := \mathrm{e}^{-rt}(\tilde{p} + \sum_{i=1}^{N_t} Y_i)$ with i.i.d. $Y_i \in \mathrm{L}^2(\mathbb{P})$, $i \in \mathbb{N}$, independent of the Poisson process N, that also drives our risk clock $R_t := \int_{(0,t]} \mathrm{e}^{-rs} \, \mathrm{d}N_s$ with discount rate $r > 0$. Apart from the classical choices of predictable and optional controls, we consider Λ^{η}-measurable controls where the Meyer-σ-field Λ^{η} is defined by

$$\Lambda^{\eta} := {}^{\mathbb{P}}\sigma(Z \text{ càdlàg and } \tilde{\mathcal{F}}^{\eta}\text{-adapted})$$

for a fixed sensor sensitivity $\eta \in [0, \infty]$, where $\tilde{\mathcal{F}}^{\eta}$ is generated by $\tilde{P}^{\eta} := \tilde{P}_{-} + \Delta\tilde{P}\mathbb{1}_{\{|\Delta\tilde{P}|\geq\eta\}}$ and $\mathbb{P}$ symbolizes that we consider the $\mathbb{P}$-completion of the σ-field at hand (see Definition and Theorem 2.2 below). A controller with information flow Λ^{η} may receive a warning about an impending jump, namely when the jump's absolute value is at least η. The case $p(\eta) := \mathbb{P}(|Y_1| \leq \eta) = 1$ corresponds to the predictable-σ-field while $p(\eta) = 0$ leads to the optional-σ-field. For any sensitivity η, we derive a closed-form solution to the stochastic representation problem associated with this example and thus obtain an explicit optimal control for the optimization problem. In general, this optimal control turns out to be neither left-continuous nor right-continuous; instead, it is merely làdlàg. Hence, optimal controls may exhibit "double jumps" which correspond to the controller's ability to proactively intervene to reduce the risk before the risk clock "rings" and to adjust her position afterwards in order to benefit from higher rewards available then. As one intuitively would expect, we indeed find a variety of optimal controls as we vary the sensitivity η of the considered jump sensor, allowing us to assess, for instance, how much in extra value for the controller can be generated by a further improvement in sensor sensitivity.

The article is structured in the following way. In Section 2 we consider a toy example illustrating the idea of using Meyer-σ-fields in optimal control problems and we introduce along the way the basic notions from the theory of Meyer-σ-fields. In Section 3 we formulate a general irreversible investment problem with inventory risk and we show how to reduce it to a suitable representation problem. In Section 4 we give an explicit example for the solution of this problem, where the reward process is given by a compound Poisson process. In the Appendix we collect some results concerning the special $^{*}\mathrm{d}C$-integral we have introduced for làdlàg controls C.

2. A general optimal control framework with Meyer-σ-fields. In this section we motivate and develop a continuous-time framework for the flow of information in optimal control problems by using Meyer-σ-fields. This framework is first illustrated using a toy example and will be fully exploited in Section 3, where we formulate a singular stochastic control problem, namely an irreversible investment problem with inventory risk.

Uncertainty is described by a filtered probability space $(\Omega, \mathbb{F}, \mathcal{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with $\mathbb{F} := \bigvee_t \mathcal{F}_t$ and $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions of right-continuity and completeness. The filtration $\mathcal{F}$ can be thought of as the information flow from observing the exogenous noise driving the controlled system. The immediacy with which this information can be acted upon by the controller is clearly crucial for the optimization problem to be studied, particularly in a setting with jumps.

To illustrate this, let us give a toy example and consider a compound Poisson process

$$\tilde{P}_t = \tilde{p} + \sum_{k=1}^{N_t} Y_k, \quad t \geq 0,$$

where $\tilde{p} \in \mathbb{R}$ and where the i.i.d. uniformly distributed jumps $Y_k \sim U[-1, 1]$, $k = 1, 2, \ldots$ are independent from the Poisson process N with intensity $\lambda > 0$. Let $\mathcal{F}$ be the augmented filtration generated by $\tilde{P}$. Let us study how to maximize

$$\mathbb{E}\left[\int_{[0,1]} C_s \, d\tilde{P}_s\right]$$

over controls C with $|C_s| \leq 1$, $0 \leq s \leq 1$. When restricted to $\mathcal{F}$-predictable controls C, $\int_{(0,\cdot]} C_s \, d\tilde{P}_s$ is obviously an $\mathbb{F}$-martingale and so

$$\mathbb{E}\left[\int_{[0,1]} C_s \, d\tilde{P}_s\right] = 0$$

for *any* such control. By contrast, when controls are allowed to be optional, we can estimate

$$\mathbb{E}\left[\int_{[0,1]} C_s \, d\tilde{P}_s\right] \leq \mathbb{E}\left[\sum_{0 \leq s \leq 1} |C_s||\Delta\tilde{P}_s|\right] \leq \mathbb{E}\left[\sum_{0 \leq s \leq 1} |\Delta\tilde{P}_s|\right] = \mathbb{E}[N_1]\mathbb{E}[|Y_1|] = \frac{\lambda}{2}$$

with equality holding true in all the above estimates for the (then optimal) choice

$$\hat{C}_s^{\mathcal{O}} := \text{sgn}(\Delta\tilde{P}_s), \quad s \in [0, 1],$$

with $\text{sgn}(0) = 0$. Of course, it is conceivable that, rather than being able to directly account for all jumps as in the optional case, the controller can, for lack of a perfect jump sensor, only react immediately to large enough jumps, say those of absolute value at least $\eta > 0$. This would suggest to consider

$$(3) \qquad \hat{C}_s^\eta := \text{sgn}(\Delta\tilde{P}_s)\mathbb{1}_{\{|\Delta\tilde{P}_s| \geq \eta\}}, \quad s \in [0, 1],$$

as the optimal choice—but among which controls exactly?

This question can be answered in a precise way by considering Meyer-σ-fields Λ (see Definition 2.1 below) satisfying $\mathcal{P}(\mathcal{F}) \subset \Lambda \subset \mathcal{O}(\mathcal{F})$, where $\mathcal{P}(\mathcal{F})$, $\mathcal{O}(\mathcal{F})$ denote, respectively, the predictable and the optional σ-field associated with $\mathcal{F}$. The theory of Meyer-σ-fields was initiated in Lenglart (1980). We review and extend some of this material in the companion paper Bank and Besslich (2018b). Let us recall here the basic concepts and results.

DEFINITION 2.1 (Meyer-σ-field, Lenglart (1980), Definition 2, page 502). A σ-field Λ on $\Omega \times [0, \infty)$ is called a *Meyer-σ-field*, if the following conditions hold:

(i) It is generated by some right-continuous, left-limited (RCLL or càdlàg for short) processes.

(ii) It contains $\{\varnothing, \Omega\} \times \mathcal{B}([0, \infty))$, where $\mathcal{B}([0, \infty))$ denotes the Borel-σ-field on $[0, \infty)$.

(iii) It is stable with respect to stopping at deterministic time points, that is, for a Λ-measurable process Z and for any $s \in [0, \infty)$, also the stopped process $(\omega, t) \mapsto Z_{t \wedge s}(\omega)$ is Λ-measurable.

Like for filtrations, also for Meyer-σ-fields there is a notion of completeness with respect to a probability measure $\mathbb{P}$:

DEFINITION AND THEOREM 2.2 ($\mathbb{P}$-complete Meyer-σ-field, see Lenglart (1980), pages 507–508). *A Meyer-σ-field $\Lambda \subset \mathbb{F} \otimes \mathcal{B}([0, \infty))$ is called $\mathbb{P}$-complete if any process $\tilde{Z}$ which is indistinguishable from a Λ-measurable process Z is itself already Λ-measurable. For any Meyer-σ-field $\tilde{\Lambda} \subset \mathbb{F} \otimes \mathcal{B}([0, \infty))$ there exists a smallest $\mathbb{P}$-complete Meyer-σ-field Λ containing $\tilde{\Lambda}$; it is called the $\mathbb{P}$-completion of $\tilde{\Lambda}$.*

EXAMPLE 2.3 (Lenglart (1980), Example, page 509). Let $\tilde{\mathcal{F}} := (\tilde{\mathcal{F}}_t)_{t \geq 0}$ be a filtration on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and denote by $\mathcal{F}$ the smallest filtration containing $\tilde{\mathcal{F}}$ that satisfies the usual conditions. Then the $\mathbb{P}$-completion of the $\tilde{\mathcal{F}}$-predictable σ-field is the $\mathcal{F}$-predictable σ-field. The $\mathbb{P}$-completion of the $\tilde{\mathcal{F}}$-optional σ-field is contained in the $\mathcal{F}$-optional σ-field; if $\tilde{\mathcal{F}}$ is right-continuous, then the $\mathbb{P}$-completion of the $\tilde{\mathcal{F}}$-optional σ-field is equal to the $\mathcal{F}$-optional σ-field.

In our example the $\mathbb{P}$-complete Meyer-σ-field encapsulating the jump information in a convenient manner is given by

$$(4) \qquad \Lambda^\eta := {}^{\mathbb{P}}\sigma(Z \text{ càdlàg and } \tilde{\mathcal{F}}^\eta\text{-adapted})$$

for a fixed sensor sensitivity $\eta \in [0, \infty]$, where $\tilde{\mathcal{F}}^\eta$ is generated by $\tilde{P}^\eta := \tilde{P}_- + \Delta \tilde{P} \mathbb{1}_{\{|\Delta \tilde{P}| \geq \eta\}}$ and $\mathbb{P}$ symbolizes that we consider the $\mathbb{P}$-completion of the σ-field at hand.

That Λ^η is indeed a Meyer-σ-field can be checked by the following result.

THEOREM 2.4 (Lenglart (1980), Theorem 5, page 509). *A σ-field on $\Omega \times [0, \infty)$ generated by càdlàg processes is a $\mathbb{P}$-complete Meyer-σ-field if and only if it lies between the predictable and the optional σ-field of a filtration satisfying the usual conditions.*

REMARK 2.5 (Meyer-σ-fields vs. Filtrations). The main advantages of a Meyer-σ-field Λ compared to a filtration are technical but powerful tools like the Meyer section theorem below which, for example, gives us uniqueness up to indistinguishability of two Λ-measurable processes once they coincide at every Λ-stopping time (cf. Definition 2.6 below for this notion). As one can see, for example, in Dellacherie and Meyer ((1978), Remark 91(b), page 144), adapted processes or even progressively measurable processes cannot in general be pinned down up to indistinguishability in this way.

Now we can make precise and corroborate our above optimality conjecture by showing that $\hat{C}^\eta$ from (3) satisfies

$$(5) \qquad \hat{C}^\eta \in \underset{|C| \leq 1, C \ \Lambda^\eta\text{-measurable}}{\operatorname{argmax}} \mathbb{E}\left[\int_{[0,1]} C_s \, dP_s\right].$$

To verify this we will need a generalization of classical stopping times to Meyer- or Λ-stopping times.

DEFINITION 2.6 (Following Lenglart (1980), Definition 1, page 502). A mapping S from Ω to $[0, \infty]$ is a Λ-*stopping time*, if

$$[\![S, \infty[\![:= \{(\omega, t) \in \Omega \times [0, \infty) \mid S(\omega) \leq t\} \in \Lambda.$$

The set of all Λ-stopping times is denoted by $\mathcal{S}^{\Lambda}$. Additionally, we define for each random mapping $S : \Omega \to [0, \infty]$ the σ-field

$$\mathcal{F}_S^{\Lambda} := \sigma(Z_S \mid Z \text{ is a } \Lambda\text{-measurable process}).$$

Next, we introduce some classification of random times with respect to a Meyer-σ-field:

DEFINITION AND THEOREM 2.7 (Compare Lenglart (1980), Definition and Theorem 6, page 510). *A random time* $T : \Omega \to [0, \infty]$ *is called* Λ-accessible *for a Meyer-σ-field* Λ *if there exists a sequence of* Λ-*stopping times* $(T_n)_{n \in \mathbb{N}}$ *such that*

$$\mathbb{P}\left(\bigcup_{n \in \mathbb{N}} \{T_n = T < \infty\}\right) = \mathbb{P}(T < \infty).$$

The random variable T *is called* totally Λ-inaccessible *if* $\mathbb{P}(S = T < \infty) = 0$ *for all* Λ-*stopping times* S.

Moreover, for $\Lambda \subset \mathcal{O}(\mathcal{F})$, *where* $\mathcal{F}$ *denotes some filtration satisfying the usual conditions, there exists for each* $\mathcal{F}$-*stopping time a partition of* $\{T < \infty\}$, *unique up to* $\mathbb{P}$-*null-sets, into two sets* $A, I \in \mathcal{F}_T$, *such that* T_A *is* Λ-*accessible and* T_I *is* Λ-*totally inaccessible. Here, for example,* T_A *denotes the restriction of* T *to* A *given by* $T_A = T$ *on* A *and* $T_A = \infty$ *on* A^c.

Having introduced totally Λ-inaccessible stopping times we can now state the following result, which will be crucial for proving that $\hat{C}^{\eta}$ from (3) is optimal.

PROPOSITION 2.8 (Compare Bank and Besslich (2018b), Proposition 2.21). *Let* Λ *be* $\mathbb{P}$-*complete and denote by* $\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}$ *a filtration satisfying the usual conditions such that* $\mathcal{P}(\mathcal{F}) \subset \Lambda \subset \mathcal{O}(\mathcal{F})$. *Then we have* $\mathbb{P}$-*almost surely for any* Λ-*measurable bounded process* C *at any totally* Λ-*inaccessible* $\mathcal{F}$-*stopping time* T,

$$C_T = (^{\mathcal{P}}C)_T.$$

To prove our optimality claim (5), take an arbitrary Λ^{η}-measurable control C and put $C_{\infty} := 0$ for notational convenience. Define furthermore

$$T_k := \inf\{t \in [0, 1] \mid N_t = k\}$$

(with $\inf \varnothing := \infty$) and observe that this $\mathcal{F}$-stopping time has the decomposition $T_k = (T_k)_{A_k^{\eta}} \wedge (T_k)_{(A_k^{\eta})^c}$ with $A_k^{\eta} := \{|Y_k| \geq \eta\}$ into a Λ^{η}-accessible $\mathcal{F}$-stopping time $(T_k)_{A_k^{\eta}}$ and a totally Λ^{η}-inaccessible $\mathcal{F}$-stopping time $(T_k)_{(A_k^{\eta})^c}$ (see Definition 2.7). By Proposition 2.8 it then follows that $C_{T_k} = {}^{\mathcal{P}}C_{T_k}$ on $(A_k^{\eta})^c$ almost surely. Therefore,

$$\mathbb{E}[C_{T_k} Y_k] = \mathbb{E}[C_{(T_k)_{\{|Y_k| \geq \eta\}}} Y_k] + \mathbb{E}[{}^{\mathcal{P}}C_{(T_k)_{\{|Y_k| < \eta\}}} Y_k].$$

We thus obtain for all Λ^{η}-measurable C with $0 \leq |C_s| \leq 1$,

$$\mathbb{E}\left[\int_{[0,1]} C_s \, d\tilde{P}_s\right] = \sum_{k=1}^{\infty} \mathbb{E}[C_{T_k} Y_k] = \sum_{k=1}^{\infty} \left(\mathbb{E}[C_{(T_k)_{\{|Y_k| \geq \eta\}}} Y_k] + \mathbb{E}[{}^{\mathcal{P}}C_{(T_k)_{\{|Y_k| < \eta\}}} Y_k]\right)$$

$$= \mathbb{E}\left[\sum_{0 \leq s \leq 1} C_s \Delta \tilde{P}_s \mathbb{1}_{\{|\Delta \tilde{P}_s| \geq \eta\}}\right] + \mathbb{E}\left[\sum_{0 \leq s \leq 1} {}^{\mathcal{P}}C_s \Delta \tilde{P}_s \mathbb{1}_{\{|\Delta \tilde{P}_s| < \eta\}}\right]$$

$$
\text{(6)} \qquad = \mathbb{E}\left[\sum_{0 \le s \le 1} C_s \Delta \tilde{P}_s \mathbb{1}_{\{|\Delta \tilde{P}_s| \ge \eta\}}\right] + 0
$$

$$
\text{(7)} \qquad \le \mathbb{E}\left[\sum_{0 \le s \le 1} |\Delta \tilde{P}_s| \mathbb{1}_{\{|\Delta \tilde{P}_s| \ge \eta\}}\right] = \mathbb{E}[N_1]\mathbb{E}\big[|Y_1|\mathbb{1}_{\{|Y_1| \ge \eta\}}\big] = \frac{\lambda}{2}(1 - \eta^2),
$$

where we have used in (6) that the process $\bar{P}_t := p + \sum_{k=1}^{N_t} Y_k \mathbb{1}_{\{|Y_k| < \eta\}}$ defines an $\mathcal{F}$-martingale and hence

$$
\mathbb{E}\left[\sum_{0 \le s \le 1} {}^{\mathcal{P}}C_s \Delta \tilde{P}_s \mathbb{1}_{\{|\Delta \tilde{P}_s| < \eta\}}\right] = \mathbb{E}\left[\int_{[0,1]} {}^{\mathcal{P}}C_s \, d\bar{P}_s\right] = 0.
$$

To finally conclude (5), we now just have to observe that we have equality in (7) for $C = \hat{C}^\eta$ from (3).

Before using Meyer-σ-fields in a more relevant setting than the above toy problem, let us conclude this section with another two important concepts from the general theory of stochastic processes that are Meyer-measurable: the Meyer section theorem and the Meyer projection theorem.

THEOREM 2.9 (Meyer dection theorem, Lenglart (1980), Section Theorem 1, page 506). *Let B be an element of a $\mathbb{P}$-complete Meyer-σ-field Λ. For every $\epsilon > 0$, there exists $S \in \mathcal{S}^\Lambda$ such that B contains the graph of S, that is,*

$$
B \supset [\![S]\!] := \mathrm{graph}(S) := \big\{(\omega, S(\omega)) \in \Omega \times [0, \infty) \mid S(\omega) < \infty\big\}
$$

and

$$
\mathbb{P}(S < \infty) > \mathbb{P}\big(\pi(B)\big) - \epsilon,
$$

where $\pi(B) := \{\omega \in \Omega \mid (\omega, t) \in B \text{ for some } t \in [0, \infty)\}$ is the projection of B onto Ω.

An important consequence is the following corollary:

COROLLARY 2.10 (Lenglart (1980), Corollary, page 507). *If Z and Z' are two Λ-measurable processes, such that for each bounded $T \in \mathcal{S}^\Lambda$ we have $Z_T \le Z'_T$ a.s. (resp. $Z_T = Z'_T$ a.s.), then the set $\{Z > Z'\}$ is evanescent (resp. Z and Z' are indistinguishable).*

Finally, we state an equivalent definition of Λ-projections. These projections were introduced in Lenglart ((1980), Definition, page 512), and they provide us with a generalization of the well-known optional and predictable projections.

DEFINITION AND THEOREM 2.11 (Compare Bank and Besslich (2018b), Theorem 2.14). *For any nonnegative $\mathbb{F} \otimes \mathcal{B}([0, \infty))$-measurable process Z, there exists a nonnegative Λ-measurable process ${}^\Lambda Z$, unique up to indistinguishability, such that*

$$
\mathbb{E}\left[\int_{[0,\infty)} Z_s \, dA_s\right] = \mathbb{E}\left[\int_{[0,\infty)} {}^\Lambda Z_s \, dA_s\right]
$$

for any càdlàg, Λ-measurable, increasing process A. This process ${}^\Lambda Z$ is called the Λ-projection of Z, and it is also uniquely determined by

$$
{}^\Lambda Z_S = \mathbb{E}[Z_S \mid \mathcal{F}_S^\Lambda], \quad S \in \mathcal{S}^\Lambda.
$$

3. Irreversible investment with inventory risk. Naturally, much more intricate information dynamics involving jumps can be considered than the simple sensor used in the previous section (see (4)). It is thus of interest to develop a general approach to optimal control with Meyer-σ-fields. It is the goal of the present paper to do so for the problem of irreversible investment, a nonlinear stochastic singular control problem which has been of considerable interest in the literature; see, for example, Arrow (1966), Dixit and Pindyck (1994), Bertola (1998), Merhi and Zervos (2007), Riedel and Su (2011), Federico and Pham (2014), Ferrari (2015), Al Motairi and Zervos (2017), De Angelis, Federico and Ferrari (2017).

Let us consider a controller who can choose her actions based on the information flow conveyed by a Meyer-σ-field Λ satisfying $\mathcal{P}(\mathcal{F}) \subset \Lambda \subset \mathcal{O}(\mathcal{F})$, where $\mathcal{F}$ is a complete, right-continuous filtration generated, for example, by the monitoring of exogenous random shocks hitting the controlled system. So, in our irreversible investment problem, *controls* are Λ-measurable, increasing and (for now) càdlàg processes C starting from a given value

$$C_{0-} := c_0 \in \mathbb{R}.$$

A control C will incur a risk described by

$$\mathbb{E}\left[\int_{[0,\infty)} \rho_t(C_t)\, dR_t\right],$$

where $\mathbb{E}$ corresponds to a given fixed probability measure $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t)$ and where ρ and R are as follows.

ASSUMPTION 3.1.

(i) dR is a random Borel measure on $[0,\infty)$ with $dR(\{\infty\}) := 0$.
(ii) The stochastic field $\rho : \Omega \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $(\omega, t, c) \mapsto \rho_t(\omega, c)$ satisfies:

(a) For $\omega \in \Omega$, $t \in [0,\infty)$, the function $\rho_t(\omega, \cdot)$ is strictly convex and continuously differentiable on $\mathbb{R}$ with

$$\lim_{c \downarrow -\infty} \frac{\partial}{\partial c} \rho_t(\omega, c) = -\infty, \qquad \lim_{c \uparrow \infty} \frac{\partial}{\partial c} \rho_t(\omega, c) = \infty.$$

(b) For $c \in \mathbb{R}$, the process $\rho_{\cdot}(\cdot, c) : \Omega \times [0,\infty) \to \mathbb{R}; (\omega, t) \mapsto \rho_t(\omega, c)$ is $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable.
(c) We have

(8)
$$\mathbb{E}\left[\int_{[0,\infty)} |\rho_t(c)|\, dR_t\right] < \infty, \quad c \in \mathbb{R},$$

and

(9)
$$\mathbb{E}\left[\int_{[0,\infty)} \inf_{c \in \mathbb{R}} \rho_t(c)\, dR_t\right] > -\infty.$$

REMARK 3.2. The process $R_t := dR([0,t])$, $t \geq 0$, can be viewed as a risk clock. Its jumps correspond to atoms of dR and indicate times of particular importance for a control's risk assessment. The random field $\rho = \rho_t(c)$ can be viewed as a description of, for example, inventory risk emerging from the inventory level c installed at time t.

The set $\widetilde{\mathcal{C}}(c_0)$ of *admissible controls* consist of all controls $\tilde{C}$ with $\tilde{C}_{0-} := c_0$, which exhibit limited risk in the sense that

(10)
$$\mathbb{E}\left[\int_{[0,\infty)} \rho_t(\tilde{C}_t)\, dR_t\right] < \infty$$

and which have reasonable expected rewards

$$\mathbb{E}\left[\int_{[0,\infty)} (P_t \wedge 0)\, d\tilde{C}_t\right] > -\infty,$$

where P denotes a given $\mathbb{F} \otimes \mathcal{B}([0,\infty))$-measurable reward process.

Admissible controls then yield the (possibly infinite) value

$$\tilde{V}(\tilde{C}) := \mathbb{E}\left[\int_{[0,\infty)} P_t\, d\tilde{C}_t - \int_{[0,\infty)} \rho_t(\tilde{C}_t)\, dR_t\right] \in (-\infty, \infty]$$

and we are led to consider the control problem:

(11) $\qquad\qquad$ Find $\hat{C} \in \tilde{\mathcal{C}}(c_0)$ attaining $\displaystyle\sup_{\tilde{C} \in \tilde{\mathcal{C}}(c_0)} \tilde{V}(\tilde{C})$.

In some applications the reward process P may not be fully observable to the controller at the time of decision making, that is, it may not be Λ-measurable. As a consequence, also the running reward $\int_{[0,t]} P_s\, dC_s$ may not be observable to the controller unless $(P_s)_{0 \le s \le t}$ is. Hence, the controller may not know immediately about the revenues generated from an intervention, but, of course, the controller can (and should) form an expectation about these revenues based on her information flow Λ. Mathematically, this is captured by the passage to the Λ-projection $^\Lambda P$ of P as introduced in Definition and Theorem 2.11. Thus, denoting by $^\Lambda P$ the Λ-projection of P, we can rewrite the value generated by a control $\tilde{C} \in \tilde{\mathcal{C}}(c_0)$ as

$$\tilde{V}(\tilde{C}) = \mathbb{E}\left[\int_{[0,\infty)} {}^\Lambda P_t\, d\tilde{C}_t - \int_{[0,\infty)} \rho_t(\tilde{C}_t)\, dR_t\right].$$

In order to ensure existence of a solution to the optimization problem (11), we impose the following mild regularity conditions on the reward process P which ensure just the right form of semicontinuity:

ASSUMPTION 3.3 (Assumptions on the reward process). The reward process P with $P_\infty := 0$ admits a Λ-projection $^\Lambda P$ and it satisfies the following conditions:

(i) $^\Lambda P$ is of class(D^Λ), that is, the family $\{^\Lambda P_T \mid T \in \mathcal{S}^\Lambda\}$ is uniformly integrable, and we have $^\Lambda P_S = 0$ for any Λ-stopping time S such that $dR([S,\infty)) = 0$ almost surely.

(ii) We have left-upper-semicontinuity in expectation at any predictable stopping time S in the sense that for any nondecreasing sequence $(S_n)_{n\in\mathbb{N}} \subset \mathcal{S}^\Lambda$ with $S_n < S$ on $\{S > 0\}$ and $\lim_{n\to\infty} S_n = S$ we have

$$\mathbb{E}[P_S] \ge \limsup_{n\to\infty} \mathbb{E}[P_{S_n}].$$

(iii) We have Λ-dR-right-upper-semicontinuity in expectation at every $S \in \mathcal{S}^\Lambda$ in the sense that for any sequence $(S_n)_{n\in\mathbb{N}} \subset \mathcal{S}^\Lambda$ with $S_n \ge S$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} dR([S, S_n)) = 0$ almost surely we have

$$\mathbb{E}[P_S] \ge \limsup_{n\to\infty} \mathbb{E}[P_{S_n}].$$

REMARK 3.4. Condition (i) ensures that rewards do not explode and no more rewards can be expected after all risk has evaporated from the system. Conditions (ii) and (iii) are needed to rule out obvious counterexamples for the existence of optimal controls.

Assumption 3.3 does not suffice to guarantee existence of an optimal control for (11) as also shown by the explicitly constructed optimal controls in our later examples. Indeed, we need to relax to increasing controls which are merely làdlàg and not necessarily right-continuous. For such controls we introduce the following *d-integral inspired by Czichowsky and Schachermayer (2016):

DEFINITION 3.5. For an increasing process $A = (A_t)_{t \geq 0}$ with $A_{0-} > -\infty$, we define the right-continuous, increasing processes for $t \in [0, \infty)$

$$A_t^r := \sum_{s \leq t} \Delta^+ A_s, \qquad A_{0-}^r := 0,$$

$$A_t^l := \sum_{s \leq t} \Delta^- A_s, \qquad A_{0-}^l := 0,$$

(12)

$$A_t^c := A_t - A_t^l - A_{t-}^r, \qquad A_{0-}^c := A_{0-},$$

$$A_t^{c,l} := A_t - A_{t-}^r = A_t^c + A_t^l, \qquad A_{0-}^{c,l} := A_{0-},$$

where $\Delta^+ A_t := A_{t+} - A_t$, $\Delta^- A_t := A_t - A_{t-}$ and $\Delta A_t := A_{t+} - A_{t-}$ for $t \geq 0$. Moreover, we set, for $t \geq 0$,

(13)

$$\int_{[0,t]} \phi_v \, {}^* \mathrm{d} A_v := \int_{[0,t]} \phi_v \, \mathrm{d} A_v^{c,l} + \int_{[0,t)} (\phi_v^*) \, \mathrm{d} A_v^r,$$

$$\int_{[t,\infty)} \phi_v \, {}^* \mathrm{d} A_v := \int_{[t,\infty)} \phi_v \, \mathrm{d} A_v^{c,l} + \int_{[t,\infty)} (\phi_v^*) \, \mathrm{d} A_v^r,$$

for any measurable process ϕ with

$$\int_{[0,\infty)} (\phi_v \wedge 0) \, \mathrm{d} A_v^{c,l} + \int_{[0,\infty)} (\phi_v^* \wedge 0) \, \mathrm{d} A_v^r > -\infty.$$

Here, integration with respect to A^r and $A^{c,l}$ is to be understood in the usual Lebesgue–Stieltjes sense; ϕ^* is the right-upper-semicontinuous envelope of ϕ defined by

(14) $$\phi_t^*(\omega) := \limsup_{s \downarrow t} \phi_s(\omega) := \lim_{n \to \infty} \sup_{s \in (t, t + \frac{1}{n})} \phi_s(\omega), \quad t \in [0, \infty).$$

Analogously, we define $_* \mathrm{d}$-integration by replacing in (13) the right-upper-semicontinuous envelope ϕ^* with the right-lower-semicontinuous envelope ϕ_* given by

(15) $$\phi_{t*} := \liminf_{s \downarrow t} \phi_s := \lim_{n \to \infty} \inf_{s \in (t, t + \frac{1}{n})} \phi_s, \quad t \in [0, \infty).$$

Some results concerning $^* \mathrm{d}$-integration and a comparison to similar integrals in the literature are collected in the Appendix.

Denote by $\mathcal{C}(c_0)$ the set of nondecreasing (and thus làdlàg), Λ-measurable controls C starting in $C_{0-} = c_0$ which incur limited risk in the sense that they satisfy (10) and which generate reasonable expected rewards in the sense that

(16) $$\mathbb{E}\left[\int_{[0,\infty)} ({}^\Lambda P_t \wedge 0) \, {}^* \mathrm{d} C_t \right] > -\infty.$$

Now we consider the following relaxed optimization problem:

(17) $$\text{Find } \hat{C} \in \mathcal{C}(c_0) \text{ attaining } \sup_{C \in \mathcal{C}(c_0)} V(C),$$

where

$$V(C) := \mathbb{E}\left[\int_{[0,\infty)} {}^\Lambda P_t \, {}^* \mathrm{d} C_t - \int_{[0,\infty)} \rho_t(C_t) \, \mathrm{d} R_t \right].$$

Theorem 3.9 below constructs a solution to this relaxed concave optimization problem, ensuring in particular existence in this relaxation. The $^* \mathrm{d}$-integral will turn out to be natural because we seek to maximize $V(C)$; $_* \mathrm{d}$-integration would be the choice in a minimization

problem. The explicit solutions computed in Section 4 reveal that làdlàg controls cannot be avoided, thus underlining the need to relax the original problem (11). The following proposition verifies that the relaxed optimization problem (17) admits the same value as our original problem (11):

PROPOSITION 3.6. *Assume P is an $\mathbb{F} \otimes \mathcal{B}([0, \infty))$-measurable process such that $^\Lambda P$ exists and suppose Assumption 3.1 is satisfied. Then for any làdlàg control $C \in \mathcal{C}(c_0)$ there are càdlàg controls $\tilde{C}^n \in \widetilde{\mathcal{C}}(c_0)$, $n \in \mathbb{N}$, such that*

(18) $$V(C) = \lim_{n \to \infty} \tilde{V}(\tilde{C}^n).$$

In particular, we have

$$\sup_{C \in \mathcal{C}(c_0)} V(C) = \sup_{\tilde{C} \in \widetilde{\mathcal{C}}(c_0)} \tilde{V}(\tilde{C}).$$

For the proof of Proposition 3.6 we will need the following result:

PROPOSITION 3.7. *Under the Assumptions of Proposition 3.6 for any $C \in \mathcal{C}(c_0)$ and $\tilde{C} \in \widetilde{\mathcal{C}}(c_0)$ we have*

$$V(C) = \lim_{n \to \infty} V(C \wedge n), \qquad \tilde{V}(\tilde{C}) = \lim_{n \to \infty} \tilde{V}(\tilde{C} \wedge n).$$

PROOF. This result follows via monotone convergence on the reward part and dominated convergence in the risk part of our target functionals V, $\tilde{V}$ as by convexity of $c \mapsto \rho_s(c)$ we have for $n \geq c_0$ the estimate

$$\left| \rho_s(C_s \wedge n) \right| \leq \max \left\{ \rho_s(C_s) \vee 0, \rho_s(c_0) \vee 0, -\left(\inf_{c \in \mathbb{R}} \rho_s(c) \wedge 0 \right) \right\}. \qquad \square$$

PROOF OF PROPOSITION 3.6. By Proposition 3.7, we can assume without loss of generality that $C \in \mathcal{C}(c_0)$ is bounded. Note, we will also construct bounded $\tilde{C}^n \in \widetilde{\mathcal{C}}(c_0)$, $n \in \mathbb{N}$ with (18). For such $\tilde{C}^n$, $n \in \mathbb{N}$, we get via Definition and Theorem 2.11 and admissibility of $\tilde{C}^n$ that $\tilde{V}(\tilde{C}^n) = V(\tilde{C}^n)$.

Case $V(C) = \infty$: In this case, one of the two right-continuous, admissible controls $C^{c,l}$, C^r (see (12)) generates an infinite value for $\tilde{V}$, which then shows (18).

Case $V(C) < \infty$: In this case, we can use Karatzas and Shreve ((1998), Proposition 2.26, page 10), to exhaust the jumps of C^r by a sequence of $\mathcal{F}$-stopping times $(T_n)_{n \in \mathbb{N}}$, where we can assume without loss of generality that the graphs of the stopping times are disjoint. Indeed, if this is not the case we can consider instead the sequence $(\tilde{T}_n)_{n \in \mathbb{N}}$ given by

$$\tilde{T}_1 := T_1, \qquad \tilde{T}_n := (T_n)_{\bigcup_{k=1}^{n-1} \{T_n \neq T_k\}}, \quad n = 2, 3, \ldots,$$

which exhibits the desired properties. Moreover, by Bank and Besslich ((2018b), Proposition 3.2(i)), there exists for each $n \in \mathbb{N}$ a sequence $(T_n^k)_{k \in \mathbb{N}} \subset \mathcal{S}^\Lambda$ such that $T_n^k \geq T_n$ with $\infty > T_n^k > T_n$ on $\{T_n < \infty\}$, $\lim_{k \to \infty} T_n^k = T_n$ and $(^\Lambda P)_{T_n}^* = \lim_{k \to \infty} {}^\Lambda P_{T_n^k}$ almost surely. Now define for $k \in \mathbb{N}$, $N \in \mathbb{N}$, $\tilde{C}^{k,N} \in \widetilde{\mathcal{C}}(c_0)$ by

$$\tilde{C}_t^{k,N} := C_t^{c,l} + \sum_{n=1}^{N} \Delta^+ C_{T_n} \mathbb{1}_{[\![T_n^k, \infty[\![}(t), \quad t \in [0, \infty),$$

where $C^{c,l}$ is defined in (12). One can easily see that for a.e. $\omega \in \Omega$, $t \in [0, \infty)$ we have $\lim_{N \to \infty} \lim_{k \to \infty} \tilde{C}_t^{k,N}(\omega) = C_t(\omega)$. Hence, using dominated convergence (which is applicable because of (8), (9) and admissibility of C), we obtain by continuity of ρ that

(19) $$\mathbb{E}\left[\int_{[0,\infty)} \rho_t(C_t) \, dR_t \right] = \lim_{N \to \infty} \lim_{k \to \infty} \mathbb{E}\left[\int_{[0,\infty)} \rho_t(\tilde{C}_t^{k,N}) \, dR_t \right].$$

Next,

$$\mathbb{E}\left[\int_{[0,\infty)}{}^{\Lambda}P_t\,\mathrm{d}\tilde{C}_t^{k,N}\right]=\mathbb{E}\left[\int_{[0,\infty)}{}^{\Lambda}P_t\,\mathrm{d}C_t^{c,l}\right]+\sum_{n=1}^{N}\mathbb{E}[{}^{\Lambda}P_{T_n^k}\Delta^{+}C_{T_n}]$$

and we have

$$\sum_{n=1}^{N}\mathbb{E}[{}^{\Lambda}P_{T_n^k}\Delta^{+}C_{T_n}]\xrightarrow{k\to\infty}\sum_{n=1}^{N}\mathbb{E}[({}^{\Lambda}P)_{T_n}^{*}\Delta^{+}C_{T_n}]\xrightarrow{N\to\infty}\sum_{n=1}^{\infty}\mathbb{E}[({}^{\Lambda}P)_{T_n}^{*}\Delta^{+}C_{T_n}],$$

where we have used that ${}^{\Lambda}P$ is of class(D^{Λ}) and that C is bounded to apply Lebesgue's theorem twice. Therefore, by $V(C)<\infty$,

$$(20)\qquad \lim_{N\to\infty}\lim_{k\to\infty}\mathbb{E}\left[\int_{[0,\infty)}{}^{\Lambda}P_t\,\mathrm{d}\tilde{C}_t^{k,N}\right]=\mathbb{E}\left[\int_{[0,\infty)}{}^{\Lambda}P_t\,{}^{*}\mathrm{d}C_t\right].$$

Combining (19) and (20) shows that the value $V(C)$ is attained as the limit of $\tilde{V}(\tilde{C}^{k,N})$, $k,N\in\mathbb{N}$, which finishes our proof. $\square$

An optimal control for (17) will be constructed in terms of a reference process L^{Λ} emerging from the following stochastic representation of the reward process, similar to an approach taken in Bank and Riedel (2001):

LEMMA 3.8. *Under Assumptions 3.1 and 3.3, there exists a Λ-measurable process L^{Λ} such that for any Λ-stopping time S we have*

$$(21)\qquad \mathbb{E}\left[\int_{[S,\infty)}\left|\frac{\partial}{\partial c}\rho_t\left(\sup_{v\in[S,t]}L_v^{\Lambda}\right)\right|\mathrm{d}R_t\right]<\infty,$$

$$(22)\qquad {}^{\Lambda}P_S=\mathbb{E}\left[\int_{[S,\infty)}\frac{\partial}{\partial c}\rho_t\left(\sup_{v\in[S,t]}L_v^{\Lambda}\right)\mathrm{d}R_t\,\bigg|\,\mathcal{F}_S^{\Lambda}\right].$$

Here, we can choose L^{Λ} to be maximal in the sense that for any other Λ-measurable solution $\tilde{L}$, satisfying mutatis mutandis the two properties (21), (22), we have $\tilde{L}_S\leq L_S^{\Lambda}$ at any Λ-stopping time S. The maximal solution L^{Λ} is unique up to indistinguishability and additionally satisfies

$$(23)\qquad L_S^{\Lambda}=\operatorname*{essinf}_{T\in\mathcal{S}^{\Lambda},T>S}\ell_{S,T}^{\Lambda},\quad S\in\mathcal{S}^{\Lambda},$$

where, for any Λ-stopping time T with $S<T$, $\ell_{S,T}^{\Lambda}$ is defined as the (up to a $\mathbb{P}$-nullset) unique $\mathcal{F}_S^{\Lambda}$-measurable random variable solving

$$(24)\qquad \mathbb{E}[P_S-P_T\mid\mathcal{F}_S^{\Lambda}]=\mathbb{E}\left[\int_{[S,T)}\frac{\partial}{\partial c}\rho_t(\ell_{S,T}^{\Lambda})\,\mathrm{d}R_t\,\bigg|\,\mathcal{F}_S^{\Lambda}\right]$$

on $\{\mathbb{P}(R_{T-}-R_{S-}>0\mid\mathcal{F}_S^{\Lambda})>0\}$ and $\ell_{S,T}^{\Lambda}:=\infty$ elsewhere.

PROOF. Set $X_t(\omega):={}^{\Lambda}P_t(\omega)$, $g_t(\omega,\ell):=\frac{\partial}{\partial c}\rho_t(\omega,\ell)$, $\mu(\omega,dt):=\mathrm{d}R_t(\omega)$. Then Assumptions 3.1 and 3.3 allow us to apply the representation theorem of Bank and Besslich (2018a), to obtain L^{Λ} with the desired properties. $\square$

The next theorem shows that under some additional integrability assumptions on L^{Λ} the value in (17) is finite and attained by a control explicitly constructed in terms of L^{Λ}:

THEOREM 3.9. *Let Assumptions* 3.1 *and* 3.3 *be satisfied and let* L^Λ *be a* Λ*-measurable process satisfying* (21), (22). *Suppose that the control* C^{L^Λ} *defined by*

$$(25) \qquad C_{0-}^{L^\Lambda} := c_0, \qquad C_t^{L^\Lambda} := c_0 \vee \sup_{v \in [0,t]} L_v^\Lambda, \quad t \in [0, \infty),$$

satisfies

$$(26) \qquad \mathbb{E}\left[\int_{[0,\infty)} \left\{ \frac{\partial}{\partial c} \rho_t(C_t^{L^\Lambda})(C_t^{L^\Lambda} - c_0) \right\} \vee 0 \, dR_t \right] < \infty.$$

Then C^{L^Λ} *is contained in* $\mathcal{C}(c_0)$ *and is optimal for the relaxed problem* (17):

$$C^{L^\Lambda} \in \underset{C \in \mathcal{C}(c_0)}{\operatorname{argmax}} V(C)$$

and

$$(27) \qquad V(C^{L^\Lambda}) = \mathbb{E}\left[\int_{[0,\infty)} \left\{ \frac{\partial}{\partial c} \rho_t(C_t^{L^\Lambda})(C_t^{L^\Lambda} - c_0) - \rho_t(C_t^{L^\Lambda}) \right\} dR_t \right] < \infty.$$

REMARK 3.10. One way to motivate the construction of an optimal control via (23) and (25) is the following. Due to concavity of the problem, first order conditions are necessary and sufficient for optimality of a control. For our problem, they essentially mean that a control $\widehat{C}$ will be optimal if, at any time S, it balances the (perceived) rewards of an intervention with its impact on a control's future risk assessment in such a way that, at any decision time $S \in \mathcal{S}^\Lambda$,

$$^\Lambda P_S \leq \mathbb{E}\left[\int_{[S,\infty)} \frac{\partial}{\partial c} \rho_t(\widehat{C}_t) \, dR_t \,\Big|\, \mathcal{F}_S^\Lambda \right]$$

with equality holding true if S is an optimal time to intervene. So, if $S \in \mathcal{S}^\Lambda$ is such a time, we get for any time $T \in \mathcal{S}^\Lambda$ with $T > S$ that

$$\mathbb{E}[^\Lambda P_S - {}^\Lambda P_T \mid \mathcal{F}_S^\Lambda] \geq \mathbb{E}\left[\int_{[S,T)} \frac{\partial}{\partial c} \rho_t(\widehat{C}_t) \, dR_t \,\Big|\, \mathcal{F}_S^\Lambda \right] \geq \mathbb{E}\left[\int_{[S,T)} \frac{\partial}{\partial c} \rho_t(\widehat{C}_S) \, dR_t \,\Big|\, \mathcal{F}_S^\Lambda \right].$$

Comparing this with (24) and recalling that T was arbitrary shows that at times of intervention we must have

$$(28) \qquad \widehat{C}_S \leq \underset{T}{\operatorname{essinf}} \, \ell_{S,T}^\Lambda = L_S^\Lambda,$$

where the last identity is just (23). In fact, we even get equality in the above estimates if $T = T_S$ is the next time of intervention after time S. This suggests that equality should hold true in (28) at times of intervention. Conversely, if S is not a time of intervention, similar considerations lead to

$$\mathbb{E}[^\Lambda P_S - {}^\Lambda P_{T_S} \mid \mathcal{F}_S^\Lambda] \leq \mathbb{E}\left[\int_{[S,T_S)} \frac{\partial}{\partial c} \rho_t(\widehat{C}_t) \, dR_t \,\Big|\, \mathcal{F}_S^\Lambda \right] = \mathbb{E}\left[\int_{[S,T_S)} \frac{\partial}{\partial c} \rho_t(\widehat{C}_S) \, dR_t \,\Big|\, \mathcal{F}_S^\Lambda \right],$$

and, so, $\widehat{C}_S \geq \ell_{S,T_S}^\Lambda \geq L_S^\Lambda$. Admissible controls being increasing above c_0, we thus expect $\widehat{C}_t = c_0 \vee \sup_{v \in [0,t]} L_v^\Lambda$ as in (25) to be optimal, a conjecture confirmed rigorously by our Theorem 3.9.

The rest of this section is devoted to the proof of Theorem 3.9. We start by showing a technical result:

LEMMA 3.11. *In the setting of Theorem 3.9, we have for any $\mathcal{F}$-stopping time T on* $\{\Delta^+ C_T^{L^\Lambda} > 0\}$ *that*

$$(29) \quad \left({}^\Lambda\left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t(C_t^{L^\Lambda})\,\mathrm{d}R_t\right)\right)_T^* = \left({}^\Lambda\left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\Big(\sup_{v\in[\cdot,t]} L_v^\Lambda\Big)\,\mathrm{d}R_t\right)\right)_T^*,$$

where $(\cdot)^*$ *is defined in* (14).

PROOF. For the sake of notational simplicity we will write in the following just L instead of L^Λ. By monotonicity of Λ-projections it is clear that "$\geq$" is satisfied and we only have to prove "$\leq$" in (29).

First, let $\tilde{T} := T_\Gamma$ for $\Gamma := \{\Delta^+ C_T^L > 0\}$ and consider a sequence of $\mathcal{F}$-stopping times $(T_n)_{n\in\mathbb{N}}$ such that $T_n \geq T_{n+1} \geq \tilde{T}$ with $\infty > T_n > \tilde{T}$ on $\{\tilde{T} < \infty\} = \Gamma$ for all $n \in \mathbb{N}$ and such that $\lim_{n\to\infty} T_n = \tilde{T}$. Then, for $n \in \mathbb{N}$ and $t \geq 0$,

$$0 \wedge \frac{\partial}{\partial c}\rho_t(c_0) \leq \mathbb{1}_{[T_n,\infty)}(t)\frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in[T_n,t]} L_v\Big) \leq \left|\frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in[0,t]} L_v\Big)\right|$$

and

$$\lim_{n\to\infty} \mathbb{1}_{[T_n,\infty)}(t)\frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in[T_n,t]} L_v\Big) = \mathbb{1}_{(T,\infty)}\frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in(\tilde{T},t]} L_v\Big).$$

By integrability assumption (21) on L and because $(T_n)_{n\in\mathbb{N}}$ was arbitrary, we can use dominated convergence to conclude

$$(30) \quad \left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in[\cdot,t]} L_v\Big)\,\mathrm{d}R_t\right)_{\tilde{T}+} = \int_{(\tilde{T},\infty)} \frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in(\tilde{T},t]} L_v\Big)\,\mathrm{d}R_t.$$

Moreover, we have for a.e. $\omega \in \Gamma$ and all $t > T(\omega)$ that

$$(31) \quad \sup_{v\in(T(\omega),t]} L_v(\omega) = \sup_{v\in[0,t]} L_v(\omega).$$

Let us now prove "$\leq$" in (29). By Bank and Besslich ((2018b), Proposition 3.7), we know

$$(32) \quad {}^\Lambda\left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in[0,t]} L_v\Big)\,\mathrm{d}R_t\right)_T^* \leq {}^\mathcal{O}\left(\int_{(\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in[0,t]} L_v\Big)\,\mathrm{d}R_t\right)_T.$$

Since $\Gamma \in \mathcal{F}_T$, we obtain

$$(33) \quad \begin{aligned} {}^\mathcal{O}&\left(\int_{(\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in[0,t]} L_v\Big)\,\mathrm{d}R_t\right)_T \mathbb{1}_\Gamma \\ &\overset{(31)}{=} \mathbb{E}\left[\left(\int_{(T,\infty)} \frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in(T,t]} L_v\Big)\,\mathrm{d}R_t\right)\mathbb{1}_\Gamma \,\Big|\, \mathcal{F}_T\right] \\ &\overset{(30)}{=} \mathbb{E}\left[\left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in[\cdot,t]} L_v\Big)\,\mathrm{d}R_t\right)_{T+} \mathbb{1}_\Gamma \,\Big|\, \mathcal{F}_T\right] \\ &= {}^\mathcal{O}\left(\left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in[\cdot,t]} L_v\Big)\,\mathrm{d}R_t\right)_+\right)_T \mathbb{1}_\Gamma. \end{aligned}$$

Now, we can apply again Bank and Besslich ((2018b), Proposition 3.7), to obtain on Γ that

$$(34) \quad {}^\mathcal{O}\left(\left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in[\cdot,t]} L_v\Big)\,\mathrm{d}R_t\right)_+\right)_T \leq {}^\Lambda\left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\Big(c_0 \vee \sup_{v\in[\cdot,t]} L_v\Big)\,\mathrm{d}R_t\right)_{T*},$$

where $(\cdot)_*$ denotes the right-lower-semicontinuous envelope defined in (15). Next, we will need the following claim, which we will prove at the end:

CLAIM. *There exists a sequence* $(T_n)_{n \in \mathbb{N}} \subset \mathcal{S}^\Lambda$ *such that for* $n \in \mathbb{N}$ *we have* $T_n \geq T_{n+1} \geq \tilde{T}$, $T_n > \tilde{T}$ *on* $\{\tilde{T} < \infty\}$, $L_{T_n} \geq c_0$ *on* $\{T_n < \infty\}$ *and* $\lim_{n \to \infty} T_n = \tilde{T}$.

Using the sequence of Λ-stopping times from the previous claim leads on Γ to

$$
\begin{aligned}
{}^\Lambda & \left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c} \rho_t \left(c_0 \vee \sup_{v \in [\cdot,t]} L_v \right) dR_t \right)_{T*} \\
&\leq \limsup_{n \to \infty} {}^\Lambda \left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c} \rho_t \left(c_0 \vee \sup_{v \in [\cdot,t]} L_v \right) dR_t \right)_{T_n} \\
&= \limsup_{n \to \infty} {}^\Lambda \left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c} \rho_t \left(\sup_{v \in [\cdot,t]} L_v \right) dR_t \right)_{T_n} \\
&\leq {}^\Lambda \left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c} \rho_t \left(\sup_{v \in [\cdot,t]} L_v \right) dR_t \right)_T^*.
\end{aligned}
$$

(35)

Combining (32), (33), (34) and (35) completes the proof of "$\leq$" in (29) once we have proven the above claim.

PROOF OF THE CLAIM. The proof is analogous to the proof of Bank and Besslich ((2018b), Proposition 3.2(i)). For $n \in \mathbb{N}$ we set $\epsilon_n := 2^{-n}$ and

$$
B_n := \rrbracket \tilde{T}, \infty \llbracket \cap \left[\!\left[0, \tilde{T} + \frac{1}{n} \right]\!\right[\cap \{L \geq c_0\} \in \Lambda.
$$

Then there exists by the Meyer section theorem (see Theorem 2.9) for each $n \in \mathbb{N}$ a Λ-stopping time S_n such that the graph of S_n is contained in B_n and $\mathbb{P}(S_n < \infty) > \mathbb{P}(\pi(B)) - \epsilon_n$. Now we set $T_n := \min_{k \in \{1,\ldots,n\}} S_k$, $n \in \mathbb{N}$. Using $\pi(B_n) = \Gamma$, a Borel–Cantelli argument shows that the sequence $(T_n)_{n \in \mathbb{N}} \subset \mathcal{S}^\Lambda$ will satisfy the desired properties. □

Now we have our tools at hand to prove the main result of this section:

PROOF OF THEOREM 3.9. For the sake of notational simplicity, we will again just write L instead of L^Λ. We start with the observation that convexity of $\rho = \rho_t(c)$ in c gives

$$
\inf_{c \in \mathbb{R}} \rho(c) - \rho_t(c_0) \leq \rho(C^L) - \rho(c_0) \leq \frac{\partial}{\partial c} \rho(C^L)(C^L - c_0).
$$

So, (26) in conjunction with (8) and (9) implies that $\frac{\partial}{\partial c} \rho(C^L)(C^L - c_0)$ is $\mathbb{P} \otimes dR$-integrable and that C^L of (25) has finite risk in the sense of (10). In particular, the expectation in (27) is finite.

As L solves (22) we obtain by monotonicity of Λ-projections

$$
{}^\Lambda P \leq {}^\Lambda \left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c} \rho_t(C_t^L) dR_t \right).
$$

Therefore, we get for any bounded $C \in \mathcal{C}(c_0)$ that

(36)
$$
\mathbb{E} \left[\int_{[0,\infty)} {}^\Lambda P_s \, {}^* dC_s \right] \leq \mathbb{E} \left[\int_{[0,\infty)} {}^\Lambda \left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c} \rho_t(C_t^L) dR_t \right)_s {}^* dC_s \right]
$$

$$
= \mathbb{E} \left[\int_{[0,\infty)} \int_{[s,\infty)} \frac{\partial}{\partial c} \rho_t(C_t^L) dR_t {}^* dC_s \right]
$$

(37)
$$
= \mathbb{E} \left[\int_{[0,\infty)} \frac{\partial}{\partial c} \rho_t(C_t^L)(C_t - c_0) dR_t \right].
$$

Here, by boundedness of C and integrability conditions (8), (9) and (21), we were allowed to use in the first equality the projection identity of Proposition A.4 and in the second equality the Fubini-like Proposition A.3. The preceding estimate together with convexity of ρ yields

$$V(C) \leq \mathbb{E}\left[\int_{[0,\infty)} \left\{\frac{\partial}{\partial c}\rho_t(C_t^L)(C_t - c_0) - \rho_t(C_t)\right\} dR_t\right]$$

$$\leq \mathbb{E}\left[\int_{[0,\infty)} \left\{\frac{\partial}{\partial c}\rho_t(C_t^L)(C_t^L - c_0) - \rho_t(C_t^L)\right\} dR_t\right].$$

Using Proposition 3.7, we thus have found an upper bound on the value $V(C)$ generated by an arbitrary admissible control $C \in \mathcal{C}(c_0)$.

It remains to prove that C^L satisfies the admissibility condition (16) and the identity in (27). For admissibility note first that

$$(38) \qquad \int_{[0,t]} \left|\frac{\partial}{\partial c}\rho_t(C_t^L)\right| {}^* dC_s^L = \left|\frac{\partial}{\partial c}\rho_t(C_t^L)\right|(C_t^L - c_0) \in L^1(\mathbb{P} \otimes dR).$$

Thus, we can apply Fubini's theorem for *d-Integrals (see Proposition A.3) to deduce that

$$(39) \qquad \mathbb{E}\left[\int_{[0,\infty)} \int_{[s,\infty)} \left|\frac{\partial}{\partial c}\rho_t(C_t^L)\right| dR_t \, {}^* dC_s^L\right] < \infty.$$

It follows that

$$\infty > \mathbb{E}\left[\int_{[0,\infty)} \int_{[s,\infty)} \frac{\partial}{\partial c}\rho_t(C_t^L) \vee 0 \, dR_t \, {}^* dC_s^L\right]$$

$$\geq \mathbb{E}\left[\int_{[0,\infty)} {}^\wedge\left(\int_{[s,\infty)} \frac{\partial}{\partial c}\rho_t(C_t^L) \vee 0 \, dR_t\right) {}^* dC_s^L\right]$$

$$\geq \mathbb{E}\left[\int_{[0,\infty)} {}^\wedge\left(\int_{[s,\infty)} \frac{\partial}{\partial c}\rho_t\left(\sup_{v\in[s,t]} L_v\right) dR_t\right) \vee 0 \, {}^* dC_s^L\right] \overset{(22)}{=} \mathbb{E}\left[\int_{[0,\infty)} {}^\wedge P_s \vee 0 \, {}^* dC_s^L\right],$$

where we have used Proposition A.4 in the first estimate. Hence,

$$\mathbb{E}\left[\int_{[0,\infty)} {}^\wedge P_s \vee 0 \, {}^* dC_s^L\right] < \infty,$$

and so we can use monotone convergence to obtain analogously to (37) that

$$\mathbb{E}\left[\int_{[0,\infty)} |{}^\wedge P_s| \, {}^* dC_s^L\right] = \lim_{n\to\infty} \mathbb{E}\left[\int_{[0,\infty)} |{}^\wedge P_s| \, {}^* d(C_s^L \wedge n)\right]$$

$$\leq \mathbb{E}\left[\int_{[0,\infty)} \left|\frac{\partial}{\partial c}\rho_t(C_t^L)(C_t^L - c_0)\right| dR_t\right] < \infty,$$

which shows that C^L is admissible.

Clearly, for the identity in (27), we now only have to show equality in (36) for $C = C^L$, that is,

$$(40) \qquad \mathbb{E}\left[\int_{[0,\infty)} {}^\wedge\left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\left(\sup_{v\in[\cdot,t]} L_v\right) dR_t\right)_s {}^* dC_s^L\right]$$

$$= \mathbb{E}\left[\int_{[0,\infty)} {}^\wedge\left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\left(c_0 \vee \sup_{v\in[0,t]} L_v\right) dR_t\right)_s {}^* dC_s^L\right].$$

To this end, note first that the latter expectation is well defined due to (38) and (39).

Let us split the *dC^L-integral in (40) into the contributions coming from $(C^L)^r$ and from $(C^L)^{c,l}$ (see (12)). For the $(C^L)^r$-contribution (see (12)) to (40) we obtain by Karatzas and

Shreve ((1998), Proposition 2.26, page 10), that there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of $\mathcal{F}$-stopping times with disjoint graphs (exactly as in the proof of Proposition 3.6) which exhaust the jumps of $(C^L)^{\mathrm{r}}$. Hence, we obtain

$$\int_{[0,\infty)} {}^{\Lambda}\left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\left(\sup_{v\in[\cdot,t]} L_v\right) dR_t\right)^*_s d(C^L)^{\mathrm{r}}_s$$

$$= \int_{[0,\infty)} {}^{\Lambda}\left(\int_{[\cdot,\infty)} \frac{\partial}{\partial c}\rho_t\left(c_0 \vee \sup_{v\in[0,t]} L_v\right) dR_t\right)^*_s d(C^L)^{\mathrm{r}}_s,$$

by application of Lemma 3.11 to T_n for $n \in \mathbb{N}$. We will argue at the end that the contribution from $(C^L)^{\mathrm{c,l}}$ satisfies

(41)
$$\int_{[0,\infty)} \int_{[s,\infty)} \frac{\partial}{\partial c}\rho_t\left(\sup_{v\in[s,t]} L_v\right) dR_t\, d(C^L)^{\mathrm{c,l}}_s$$

$$= \int_{[0,\infty)} \int_{[s,\infty)} \frac{\partial}{\partial c}\rho_t\left(c_0 \vee \sup_{v\in[0,t]} L_v\right) dR_t\, d(C^L)^{\mathrm{c,l}}_s.$$

Granted this identity, we get by Proposition A.4 and (39) that we can drop the projections in (40), which completes our proof of identity (40).

For the proof of (41), we fix $\omega \in \Omega$. We will show the result separately for the left jumps of C and for its continuous part C^c (see (12)). First, we get for $s \in [0,\infty)$ with $C^L_{s-}(\omega) < C^L_s(\omega)$ that $L_s(\omega) > c_0 \vee \sup_{v\in[0,s)} L_v(\omega)$ and therefore $\sup_{v\in[s,t]} L_v(\omega) = c_0 \vee \sup_{v\in[0,t]} L_v(\omega)$ for $t \geq s$. For the dC^c-contribution we can restrict to points $s \in [0,\infty)$ such that $\Delta R_s(\omega) = 0$ as for fixed ω the process R can have only countably many jumps. Moreover, the measure $dC^c(\omega)$ is supported by the set $\{s \in [0,\infty) \mid C^c_t(\omega) > C^c_s(\omega)$ for all $t > s\}$. So, let $s \in [0,\infty)$ such that $\Delta R_s(\omega) = 0$ and for all $t > s$ we have $C^c_s(\omega) < C^c_t(\omega)$. Then we get for any $t > s$ that $\sup_{v\in(s,t]} L_v(\omega) > c_0 \vee \sup_{v\in[0,s]} L_v(\omega)$ and therefore again $\sup_{v\in[s,t]} L_v(\omega) = c_0 \vee \sup_{v\in[0,t]} L_v(\omega)$. Hence,

$$\int_{(s,\infty)}\left|\frac{\partial}{\partial c}\rho_t\left(\omega, \sup_{v\in[s,t]} L_v(\omega)\right)\right| dR_t(\omega) = \int_{(s,\infty)}\left|\frac{\partial}{\partial c}\rho_t\left(\omega, C^L_t(\omega)\right)\right| dR_t(\omega)$$

and the rest follows because $\Delta R_s(\omega) = 0$. $\square$

4. Optimal irreversible investment in a compound Poisson setting with jump sensor. In this section we will illustrate in a compound Poisson process framework how different Meyer-σ-fields lead to different optimal controls in an irreversible investment problem that can be solved explicitly using Theorem 3.9. Specifically, let us fix a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with a compound Poisson process $\tilde{P}$ starting in $\tilde{p}$ of the form, that is,

$$\tilde{P}_t = \tilde{p} + \sum_{k=1}^{N_t} Y_k, \qquad P_t := e^{-rt}\tilde{P}_t, \quad t \in [0,\infty), \qquad P_\infty := \tilde{P}_\infty := 0,$$

where $\tilde{p} \in \mathbb{R}$, $r > 0$, N is a Poisson process with intensity $\lambda > 0$, independent of the i.i.d. sequence of $(Y_k)_{k\in\mathbb{N}} \subset L^2(\mathbb{P})$ with mean $m := \mathbb{E}[Y_1] \in \mathbb{R}$ and $\mathbb{P}(Y_1 = 0) = 0$. Let $\mathcal{F} := (\mathcal{F}_t)_{t\geq0}$ be the $\mathbb{P}$-augmented filtration generated by $\tilde{P}$. Define the risk clock

$$R_t := \int_{(0,t]} e^{-rs}\, dN_s, \quad t \in [0,\infty),$$

and, furthermore, choose $\rho_t(c) := \frac{1}{2}c^2$, so that obviously $\frac{\partial}{\partial c}\rho_t(c) = c$, $c \in \mathbb{R}$. One can readily check that Assumption 3.1 is satisfied.

We want to allow immediate reactions by our controller only for sufficiently large jumps and thus restrict controls to be Λ^η-measurable, where

$$(42) \qquad \Lambda^\eta := {}^{\mathbb{P}}\sigma(Z \text{ càdlàg and } \tilde{\mathcal{F}}^\eta\text{-adapted})$$

for a fixed sensitivity threshold $\eta \in [0, \infty]$, where $\tilde{\mathcal{F}}^\eta$ is generated by $\tilde{P}^\eta := \tilde{P}_- + \Delta\tilde{P}\mathbb{1}_{\{|\Delta\tilde{P}|\geq\eta\}}$ and $\mathbb{P}$ symbolizes that we consider the $\mathbb{P}$-completion of the σ-field at hand. The probability with which the controller's sensor fails to alert is thus

$$(43) \qquad p(\eta) := \mathbb{P}(|Y_1| < \eta) \in [0, 1].$$

The optimization problem (17) now takes the form

$$(44) \qquad \sup_{C\in\mathcal{C}(c_0)} \mathbb{E}\left[\int_{[0,\infty)} {}^{\Lambda^\eta}P_t{}^* dC_t - \frac{1}{2}\int_{[0,\infty)} C_t^2 dR_t\right].$$

In this problem, our controller is confronted with a reward process P that will jump due to external shocks $(Y_k)_{k\in\mathbb{N}}$ hitting at exponential times that also trigger the risk assessments in the clock R. For large enough shocks (when $|Y_k| \geq \eta$) the controller receives a warning signal from a sensor that affords her the opportunity to adjust the control C before the risk assessment is done; for smaller shocks, though, the controller receives no such signal and can only react after they have struck.

We want to construct an optimal control for problem (44) via Theorem 3.9. For that we need that the Λ^η-projection of P satisfies Assumption 3.3. This will not hold true in the optional case $p(\eta) = 0$ which we will cover separately in Section 4.3 below. For a fallible sensor, that is, in the case $p(\eta) > 0$, the following proposition characterizes the Λ^η-projection and shows that Theorem 3.9 will indeed lead to an optimal control. Moreover, it shows that the maximal solution L^{Λ^η} to (21) is a function of the reward process and the sensor:

LEMMA 4.1. (i) *For all $\eta \in [0, \infty]$ we have at every Λ^η-stopping time T that*

$$(45) \qquad \Delta N_T = \mathbb{1}_{\{|Y_{N_T}|\geq\eta\}\cap\{\Delta N_T>0\}}, \quad \mathbb{P}\text{-}a.s.,$$

$$(46) \qquad {}^{\Lambda^\eta}\tilde{P}_T = \tilde{P}_T, \quad \mathbb{P}\text{-}a.s.,$$

and we have $\mathbb{P}$-almost surely that

$$(47) \qquad {}^{\Lambda^\eta}(\Delta N)_t = \mathbb{1}_{\{|Y_{N_t}|\geq\eta\}\cap\{\Delta N_t>0\}},$$

$$(48) \qquad {}^{\Lambda^\eta}P_t = {}^{\Lambda^\eta}\tilde{P}_t e^{-rt} = \tilde{P}_t^\eta e^{-rt} := (\tilde{P}_{t-} + \Delta\tilde{P}_t\mathbb{1}_{\{|\Delta\tilde{P}_t|\geq\eta\}})e^{-rt}, \quad t \in [0, \infty).$$

In the case $p(\eta) = 0$ (resp. $p(\eta) = 1$), we have $\Lambda^\eta = \mathcal{O}(\mathcal{F})$ (resp. $\Lambda^\eta = \mathcal{P}(\mathcal{F})$).

(ii) *If the sensor is imperfect, that is, if $p(\eta) > 0$ (see (43)), then ${}^{\Lambda^\eta}P$ satisfies Assumption 3.3. The maximal solution L^{Λ^η} to (21) and (22) is given by*

$$(49) \qquad L_t^{\Lambda^\eta} = \ell^\eta({}^{\Lambda^\eta}\tilde{P}_t, {}^{\Lambda^\eta}(\Delta N)_t) = \begin{cases} \ell^\eta(\tilde{P}_{t-}, 0), & |\Delta P_t| \geq \eta, \\ \ell^\eta(\tilde{P}_t, 1), & |\Delta P_t| < \eta, \end{cases} \quad t \in [0, \infty),$$

where, for $p \in \mathbb{R}$, $\Delta \in \{0, 1\}$,

$$(50) \qquad \ell^\eta(p, \Delta) := \inf_{0<T\in\mathcal{S}^{\Lambda^\eta}} \ell_T(p, \Delta) > -\infty$$

and, for random times $T > 0$,

$$(51) \qquad \ell_T(p, \Delta) := \frac{(1 - \mathbb{E}[e^{-rT}])p - \mathbb{E}[e^{-rT}\sum_{k=1}^{N_T} Y_k]}{\mathbb{E}[R_{T-}] + \Delta},$$

with the convention $\frac{\cdot}{0} = \infty$. Moreover, the corresponding control $C^{L^{\Lambda^\eta}}$ from (25) satisfies (26) and it is optimal with a finite value $V(C^{L^{\Lambda^\eta}}) < \infty$.

PROOF.   *Proof of* (i). First, it is well known that $^{\mathcal{P}}N = N_-$ up to indistinguishability. Next for $T \in \mathcal{S}^{\Lambda^\eta}$ and $A := \{|\Delta \tilde{P}_T| \geq \eta\} = \{|Y_{N_T}| \geq \eta\} \cap \{\Delta N_T > 0\}$ one can see that the $\mathcal{F}$-stopping time T_A is a Λ^η-stopping time and T_{A^c} is Λ^η-totally inaccessible (see Definition 2.7). Hence, we get by Proposition 2.8 that

$$(52) \qquad ^{\Lambda^\eta}\tilde{P}_T = \tilde{P}_T \mathbb{1}_A + {}^{\mathcal{P}}\tilde{P}_T \mathbb{1}_{A^c} = \Delta \tilde{P}_T \mathbb{1}_{\{|\Delta \tilde{P}_T| \geq \eta\}} + \tilde{P}_{T-},$$

which implies (48) by Corollary 2.10 of the Meyer section theorem. Moreover, from $^{\mathcal{P}}N = N_-$ we get

$$^{\mathcal{P}}\tilde{P}_T \mathbb{1}_{A^c} = \tilde{P}_{T-}\mathbb{1}_{A^c} = \left(\tilde{p} + \sum_{k=1}^{N_{T-}} Y_k\right)\mathbb{1}_{A^c} = \left(\tilde{p} + \sum_{k=1}^{N_T} Y_k\right)\mathbb{1}_{A^c} = \tilde{P}_T \mathbb{1}_{A^c},$$

which shows in combination with (52) equation (46). The same argument also shows (45), which then implies again by Corollary 2.10 equation (47).

Case $p(\eta) = 0$. In this case we have for any $\mathcal{F}$-stopping time T that $\tilde{P}_T = \tilde{P}_T^\eta$, almost surely, which implies, by a corollary of the Meyer section theorem (see Corollary 2.10), that $\tilde{P}$ and $\tilde{P}^\eta$ are indistinguishable. Hence, by Definition and Theorem 2.2, the process $\tilde{P}$ is Λ^η-measurable. Moreover, using Dellacherie and Meyer ((1978), Theorem 97(a), page 147), we get

$$\mathcal{O}(\tilde{\mathcal{F}}) = \mathcal{P}(\tilde{\mathcal{F}}) \vee \sigma(\tilde{P}) \subset \Lambda^\eta \subset \mathcal{O}(\mathcal{F}).$$

This finishes our proof, as by Lenglart ((1980), Example 1°, page 509), the $\mathbb{P}$-completion of an optional σ-field with respect to a right-continuous filtration is the optional σ-field with respect to the augmented filtration.

Case $p(\eta) = 1$. In this case we have for any $\mathcal{F}$-stopping time T that $\tilde{P}_{T-} = \tilde{P}_T^\eta$, almost surely, which implies, by a corollary of the Meyer section theorem (see Corollary 2.10), that $\tilde{P}_-$ and $\tilde{P}^\eta$ are indistinguishable. Hence, by Definition and Theorem 2.2 the process $\tilde{P}^\eta$ is $\mathcal{P}(\mathcal{F})$-measurable. Next, one can derive analogously to Dellacherie and Meyer ((1978), Theorem 97(a), page 147), that $\tilde{\Lambda}^\eta = \mathcal{P}(\tilde{\mathcal{F}}) \vee \sigma(\tilde{P}^\eta)$. As $\tilde{P}^\eta$ is $\mathcal{P}(\mathcal{F})$-measurable we obtain

$$\mathcal{P}(\tilde{\mathcal{F}}) \subset \tilde{\Lambda}^\eta = \mathcal{P}(\tilde{\mathcal{F}}) \vee \sigma(\tilde{P}^\eta) \subset \mathcal{P}(\mathcal{F}).$$

The rest follows now by Lenglart ((1980), Example 2°, page 509), which gives us that the $\mathbb{P}$-completion of $\mathcal{P}(\tilde{\mathcal{F}})$ is given by $\mathcal{P}(\mathcal{F})$.

Proof of (ii). $^{\Lambda^\eta}P$ obviously satisfies (i) of Assumption 3.3. Property (ii) of this assumption holds by Fatou's lemma via $^{\mathcal{P}}N = N_-$. Finally, for (iii) of Assumption 3.3 note that for Λ^η-stopping times S, T we have $\mathbb{P}(dR([S, T)) > 0 | \mathcal{F}_S^{\Lambda^\eta}) > 0$ on $\{T > S\}$ by (43). Hence, any sequence of Λ^η-stopping times as considered in condition (iii) must decrease to S almost surely and therefore Assumption 3.3 is satisfied by right-continuity of P and Fatou's lemma. Now, the process L^{Λ^η} exists by Lemma 3.8 and $C^{L^{\Lambda^\eta}}$ is optimal by Theorem 3.9 since it satisfies the integrability conditions as verified next:

(26) is satisfied. As L^{Λ^η} satisfies (23) we see that

$$L_S^{\Lambda^\eta} \leq \ell_{S,\infty}^{\Lambda^\eta} = \frac{^{\Lambda^\eta}\tilde{P}_S}{\mathbb{E}[R_{\infty-}] + {}^{\Lambda^\eta}(\Delta N)_S} \leq \frac{|^{\Lambda^\eta}\tilde{P}_S|}{\mathbb{E}[R_{\infty-}]} = \frac{r}{\lambda}|\tilde{P}_S^\eta|, \quad S \in \mathcal{S}^{\Lambda^\eta},$$

and, hence, (26) follows by $Y_1 \in L^2(\mathbb{P})$ and $\mathbb{E}[R_{\infty-}] = \frac{\lambda}{r} < \infty$.

Equation (49). We claim that it is enough to show

$$(53) \qquad L_S^{\Lambda^\eta} = \inf_{0 < T \in \mathcal{S}^{\Lambda^\eta}} \frac{(1 - \mathbb{E}[e^{-rT}])^{\Lambda^\eta}\tilde{P}_S - \mathbb{E}[e^{-rT}\sum_{k=1}^{N_T} Y_k]}{\mathbb{E}[R_{T-}] + {}^{\Lambda^\eta}(\Delta N)_S}, \quad S \in \mathcal{S}^{\Lambda^\eta}.$$

Indeed, by (53) and Corollary 2.10 it then suffices to establish that the right hand side of (49) defines a Λ^η-measurable process or, equivalently, that ℓ^η is measurable. As Δ admits only two values we only have to argue why $p \mapsto \ell^\eta(p, \Delta)$ is measurable for fixed Δ, which is clear as it is concave. Moreover, if for fixed $\Delta \in \{0, 1\}$ we have $\ell^\eta(\bar{p}, \Delta) = -\infty$ for some $\bar{p} \in \mathbb{R}$, then also $\ell^\eta(p, \Delta) = -\infty$ for all $p \leq \bar{p}$. By (53) this would then contradict the representation property (22) of L^{Λ^η}. It follows that indeed $\ell^\eta(p, \Delta) > -\infty$ for all $\Delta \in \{0, 1\}$ and $p \in \mathbb{R}$.

Proof of (53). Let $S \in \mathcal{S}^{\Lambda^\eta}$. By (23), we know that

$$L_S^{\Lambda^\eta} = \operatorname*{essinf}_{T > S, T \in \mathcal{S}^{\Lambda^\eta}} \ell_{S,T}^{\Lambda^\eta}.$$

Due to our convention $\frac{\cdot}{0} = \infty$, we can write

$$(54) \qquad \ell_{S,T}^{\Lambda^\eta} = \frac{(1 - \mathbb{E}[e^{-r(T-S)} \mid \mathcal{F}_S^{\Lambda^\eta}])^{\Lambda^\eta} \tilde{P}_S - \mathbb{E}[(\tilde{P}_T - \tilde{P}_S)e^{-r(T-S)} \mid \mathcal{F}_S^{\Lambda^\eta}]}{e^{rS}\mathbb{E}[R_{T-} - R_S \mid \mathcal{F}_S^{\Lambda^\eta}] + {}^{\Lambda^\eta}(\Delta N)_S}.$$

We will argue next how to reduce the analysis to the special case $S = 0$ which will become possible by results of Dellacherie and Meyer ((1978), pages 145–149) and Courrège and Priouret (1965) on the general theory of processes when working on the canonical space with lifetime. For this, we now assume, without loss of generality, that Ω is the space of $\mathbb{R} \cup \{\dagger\}$-valued càdlàg paths with lifetime (see Dellacherie and Meyer ((1978), Definition 94, page 145). We let $\tilde{P}$ denote the canonical process $\tilde{P}_t(\omega) = \omega(t)$ with its natural filtration $\tilde{\mathcal{F}}$; $\mathbb{P}$ is the probability under which $\tilde{P}$ follows the same compound Poisson process dynamics considered above. We define, the process $X^\eta : \Omega \times [0, \infty) \to \mathbb{R}$ by

$$X_0^\eta(\omega) = \omega_0, \qquad X_t^\eta(\omega) := \omega_{t-} + \Delta\omega_t \mathbb{1}_{\{|\Delta\omega_t| \geq \eta\}}, \quad \omega \in \Omega, t \in (0, \infty).$$

Here, $\dagger$ denotes some point isolated from $\mathbb{R}$, $\omega_\dagger$ is the element of Ω with $X(\omega_\dagger) \equiv \dagger$ and $\mathcal{B}_\dagger$ the σ-field on $\mathbb{R} \cup \{\dagger\}$ generated by $\mathcal{B}(\mathbb{R})$. We have that $\bar{X}^\eta := (\Omega, \omega_\dagger, (X_t^\eta)_{t \in [0,\infty)}, \mathbb{R} \cup \{\dagger\}, \mathcal{B}_\dagger, \dagger)$ is a *stochastic function with cemetery taking values in* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (see Courrège and Priouret ((1965), Definition 0.3, pages 248–249), and Courrège and Priouret ((1965), A 1.3, pages 267–268). Moreover, one can check that $\bar{X}^\eta$ satisfies the *linking property* (see Courrège and Priouret ((1965), Definition 5.1, page 263). As a consequence, we can apply Lemma 4.3 and Theorem 5.3 in Courrège and Priouret (1965) for $\bar{X}^\eta$, which clarify the structure of $\tilde{\mathcal{F}}^\eta$-stopping times, where $\tilde{\mathcal{F}}^\eta$ is the filtration generated by X^η. To obtain results on the stopping times of the Meyer-σ-field

$$\tilde{\Lambda}^\eta := \sigma\{Z \text{ is càdlàg and } \tilde{\mathcal{F}}^\eta\text{-adapted}\}$$

we adapt Dellacherie and Meyer ((1978), Theorem 97, page 147), which is only stated for the predictable and optional-σ-field. This adaptation can be done via the mapping

$$\left(h_t^\eta(\omega)\right)_s := \left(\kappa_t(\omega)\right)_s \mathbb{1}_{\{|\Delta_t\omega| < \eta\}} + \left(\alpha_t(\omega)\right)_s \mathbb{1}_{\{|\Delta_t\omega| \geq \eta\}}, \quad \omega \in \Omega, s, t \in [0, \infty),$$

where κ denotes the killing operator and α the stopping operator of Dellacherie and Meyer ((1978), Definition 95, page 146). For this operator one can show that $(\omega, t) \mapsto h_t^\eta(\omega)$ is $\tilde{\Lambda}^\eta$-measurable and $\tilde{\mathcal{F}}_t^\eta = (h_t^\eta)^{-1}(\tilde{\mathcal{F}}_\infty)$, which then implies as in the proof of Dellacherie and Meyer ((1978), Theorem 97, page 147), that a process is $\tilde{\Lambda}^\eta$-measurable, if and only if it is $\tilde{\mathcal{F}}^\eta$-adapted. In particular, for $S : \Omega \to [0, \infty]$ we have

$$(55) \qquad S \text{ is a } \tilde{\Lambda}^\eta\text{-stopping time if and only if it is an } \tilde{\mathcal{F}}^\eta\text{-stopping time.}$$

Finally, recall that Λ^η (see (42)) is the $\mathbb{P}$-completion (see Definition and Theorem 2.2) of $\tilde{\Lambda}^\eta$, that is,

$$(56) \qquad \Lambda^\eta = {}^{\mathbb{P}}\tilde{\Lambda}^\eta = {}^{\mathbb{P}}(\sigma\{Z \text{ is càdlàg and } \tilde{\mathcal{F}}^\eta\text{-adapted}\}).$$

Proof of "$\leq$" in (53). Let $T \in \mathcal{S}^{\Lambda^\eta}$ with $T > 0$. By Lenglart ((1980), Theorem 3, page 508), and (56) we have that S and T are almost surely equal to $\tilde{\Lambda}^\eta$-stopping times $\tilde{S}$, $\tilde{T}$ and, hence, $\tilde{\mathcal{F}}^\eta$-stopping times (see (55)). Define $U(\omega) := \tilde{T}(\tilde{P}(\omega) - \tilde{P}_0(\omega) + \tilde{p})$, $\omega \in \Omega$. One can readily check (or use Courrège and Priouret (1965), Theorem 1.3, page 251 to see) that $U(\cdot)$ is an $\tilde{\mathcal{F}}^\eta$-stopping time. Hence,

$$\hat{T}(\omega) := \tilde{S}(\omega) + U(\theta_{\tilde{S}}(\omega)) > \tilde{S}(\omega), \quad \omega \in \Omega,$$

defines an $\tilde{\mathcal{F}}^\eta$-stopping time by Courrège and Priouret ((1965), Lemma 4.3, page 260). It is also a $\tilde{\Lambda}^\eta$-stopping time (see (55)) and thus in particular a Λ^η-stopping time. We will argue next that almost surely

$$(57) \qquad \ell^{\Lambda^\eta}_{S,\hat{T}} = \frac{(1 - \mathbb{E}[e^{-r\hat{T}}])^{\Lambda^\eta} \tilde{P}_S - \mathbb{E}[e^{-r\hat{T}} \sum_{k=1}^{N_{\hat{T}}} Y_k]}{\mathbb{E}[R_{\hat{T}-}] + {}^{\Lambda^\eta}(\Delta N)_S}$$

which readily yields "$\leq$" in (53). For (57) we need to manipulate the conditional expectations in (54). For the sake of brevity, we will do so only for the most complicated one:

$$\mathbb{E}[(\tilde{P}_{\hat{T}} - \tilde{P}_S)e^{-r(\hat{T}-S)} \mid \mathcal{F}^{\Lambda^\eta}_S] = \mathbb{E}[\mathbb{E}[(\tilde{P}_{\tilde{S}+U(\theta_{\tilde{S}})} - \tilde{P}_{\tilde{S}})e^{-rU(\theta_{\tilde{S}})} \mid \tilde{\mathcal{F}}_{\tilde{S}}] \mid \tilde{\mathcal{F}}^\eta_{\tilde{S}}]$$

$$= \mathbb{E}[\mathbb{E}_{\tilde{P}_{\tilde{S}}}[(\tilde{P}_U - \tilde{P}_0)e^{-rU}] \mid \tilde{\mathcal{F}}^\eta_{\tilde{S}}]$$

$$= \mathbb{E}[\mathbb{E}_{\tilde{p}}[(\tilde{P}_{\tilde{T}} - \tilde{p})e^{-r\tilde{T}}] \mid \tilde{\mathcal{F}}^\eta_{\tilde{S}}] = \mathbb{E}[(\tilde{P}_T - \tilde{p})e^{-rT}].$$

Here, we used for the first equality that, by (55), we can replace the $\mathcal{F}^{\Lambda^\eta}$-conditional expectation by an $\tilde{\mathcal{F}}^\eta_{\tilde{S}}$-conditional expectation and that $\tilde{\mathcal{F}}^\eta_{\tilde{S}} \subset \tilde{\mathcal{F}}_{\tilde{S}}$; the second equality is due to the strong Markov property of $\tilde{P}$ with respect to $\tilde{\mathcal{F}}$; the third equality is due to the Lévy property of $\tilde{P}$ and the choice of U; for the final identity we recall that under $\mathbb{P}$ the canonical process $\tilde{P}$ starts in $\tilde{p}$ almost surely and $T = \tilde{T}$ almost surely.

Proof of "$\geq$" in (53). Let $T \in \mathcal{S}^{\Lambda^\eta}$ with $T > S$. Again by Lenglart ((1980), Theorem 3, page 508), and (56), we have that S and T are almost surely equal to $\tilde{\Lambda}^\eta$-stopping times $\tilde{S}$, $\tilde{T}$ and, hence, $\tilde{\mathcal{F}}^\eta$-stopping times (see (55)). By Courrège and Priouret ((1965), Theorem 5.3, page 264), there thus exists an $\tilde{\mathcal{F}}^\eta_{\tilde{S}} \otimes \tilde{\mathcal{F}}^\eta_\infty$-measurable mapping $\tilde{U} : \Omega \times \Omega \to [0, \infty]$ such that, for every $\omega \in \Omega$, we have

$$\tilde{T}(\omega) = \tilde{S}(\omega) + \tilde{U}(\omega, \theta_{\tilde{S}}(\omega))$$

and also that $\omega' \mapsto \tilde{U}(\omega, \omega') > 0$ is an $\tilde{\mathcal{F}}^\eta$-stopping time. We have to mention here that Courrège and Priouret ((1965), Theorem 5.3, page 264) actually assumes that T is an $\tilde{\mathcal{F}}^\eta_+$-stopping time and then $\omega' \mapsto \tilde{U}(\omega, \omega')$ also would be an $\tilde{\mathcal{F}}^\eta_+$-stopping time. If T is even an $\tilde{\mathcal{F}}^\eta$-stopping time, one can prove along the same lines as given in Courrège and Priouret ((1965), Theorem 5.3, page 264), that $\tilde{U}$ can be found such that $\omega' \mapsto \tilde{U}(\omega, \omega') > 0$ is even an $\tilde{\mathcal{F}}^\eta$-stopping time. Now, we need again to manipulate the conditional expectations in (54). For instance, we get that, for $\omega \in \Omega$,

$$\mathbb{E}[e^{-r(T-S)} \mid \mathcal{F}^{\Lambda^\eta}_S](\omega) = \mathbb{E}[e^{-r(\tilde{T}-\tilde{S})} \mid \tilde{\mathcal{F}}^\eta_{\tilde{S}}](\omega)$$

$$= \mathbb{E}[e^{-r\tilde{U}(\cdot, \theta_{\tilde{S}}(\cdot))} \mid \tilde{\mathcal{F}}^\eta_{\tilde{S}}](\omega) = \mathbb{E}_{\tilde{P}_{\tilde{S}(\omega)}}[e^{-r\tilde{U}(\omega, \cdot)}],$$

where we have used that $\tilde{U}$ is $\tilde{\mathcal{F}}^\eta_{\tilde{S}}$-measurable in the first component and then the Markov property of $\tilde{P}$. Letting T_ω denote the $\tilde{\mathcal{F}}^\eta$-stopping time $T_\omega := \tilde{U}(\omega, \cdot - \tilde{p} + \tilde{P}_{\tilde{S}}(\omega))$, we can use the Lévy property of $\tilde{P}$ to write the last expectation as $\mathbb{E}[e^{-rT_\omega}]$. The other conditional

expectations from (54) can be treated similarly with the same stopping time T_ω, which by (55) is a Λ^η-stopping time. $\square$

The previous lemma shows that if (43) is satisfied, an optimal control to (44) exists. Additionally, one can see that for an explicit calculation of such an optimal control it suffices to find the maximal solution L^{Λ^η} to (21) and (22) or, equivalently, to find ℓ^η from (50). In Section 4.3 we show that also in the case $p(\eta) = 0$ an optimal control can be constructed, but via a slight change of P.

4.1. *Optimal predictable controls.* We start now with the simplest case and assume that $\mathbb{P}(|Y_1| < \eta) = p(\eta) = 1$. This corresponds to our controller operating without any sensor and using predictable controls: $\Lambda^\eta = \mathcal{P} = \Lambda^\infty$; see Lemma 4.1(i).

THEOREM 4.2 (Optimal predictable control). *In the case $p(\eta) = 1$, an optimal control for (44) is given by*

$$C_t^{\mathcal{P}} := c_0 \vee \sup_{v \in [0,t]} L_v^{\mathcal{P}}, \quad t \in [0, \infty),$$

with

(58) $$L_t^{\mathcal{P}} = a(\tilde{P}_{t-} - b), \quad t \in [0, \infty),$$

where the constants a, b are given by

$$a := \frac{1}{\mathbb{E}[R_{\infty-}]} = \frac{r}{\lambda}, \qquad b := \sup_{0 < T \in \mathcal{S}^{\mathcal{P}}} \frac{\mathbb{E}[e^{-rT} \sum_{k=1}^{N_T} Y_k]}{1 - \mathbb{E}[e^{-rT}]}.$$

In fact, the process $L^{\mathcal{P}}$ from (58) is the maximal solution to (21), (22).

PROOF. From (49) and $^{\mathcal{P}}\tilde{P} = \tilde{P}_-$, $^{\mathcal{P}}(\Delta N) \equiv 0$ (see Lemma 4.1(i)) we get that the maximal solution $L^{\mathcal{P}}$ to (21), (22) satisfies

$$L^{\mathcal{P}} = \inf_{0 < T \in \mathcal{S}^{\mathcal{P}}} \ell_T(\tilde{P}_-, 0) = \inf_{0 < T \in \mathcal{S}^{\mathcal{P}}} \frac{1 - \mathbb{E}[e^{-rT}]}{\mathbb{E}[R_{T-}]} \left(\tilde{P}_- - \frac{\mathbb{E}[e^{-rT} \sum_{k=1}^{N_T} Y_k]}{1 - \mathbb{E}[e^{-rT}]} \right)$$

$$= \inf_{0 < T \in \mathcal{S}^{\mathcal{P}}} \frac{r}{\lambda} \left(\tilde{P}_- - \frac{\mathbb{E}[e^{-rT} \sum_{k=1}^{N_T} Y_k]}{1 - \mathbb{E}[e^{-rT}]} \right) = \frac{r}{\lambda} \left(\tilde{P}_- - \sup_{0 < T \in \mathcal{S}^{\mathcal{P}}} \frac{\mathbb{E}[e^{-rT} \sum_{k=1}^{N_T} Y_k]}{1 - \mathbb{E}[e^{-rT}]} \right),$$

where the third equality holds because for any predictable stopping time T one has

$$\mathbb{E}[R_{T-}] = \mathbb{E}[R_T] = \lambda \mathbb{E}\left[\int_{[0,T]} e^{-rt} \, dt \right] = \frac{\lambda}{r}(1 - \mathbb{E}[e^{-rT}]). \qquad \square$$

PROPOSITION 4.3. (i) *The constant b from Theorem 4.2 can also be obtained by taking a supremum over all stopping times and it has an alternative representation in terms of the running supremum over $\tilde{P}_-$:*

(59) $$b = \sup_{0 < T \in \mathcal{S}} \frac{\mathbb{E}[e^{-rT} \sum_{k=1}^{N_T} Y_k]}{1 - \mathbb{E}[e^{-rT}]} = \frac{\mathbb{E}[\int_{[0,\infty)} (\sup_{v \in [0,t]} \tilde{P}_{v-} - \tilde{p}) \, dR_t]}{\mathbb{E}[R_{\infty-}]}.$$

(ii) *For $\eta \in [0, \infty]$ with $p(\eta) > 0$ and $\Delta \in \{0, 1\}$, we have $\ell^\eta(p, \Delta) < 0$ if and only if $p < b$.*

PROOF. (i) The first equation in (59) follows from Theorem 4.2 as any $T \in \mathcal{S}^{\Lambda^\eta}$ can be approximated by $(T_n)_{n \in \mathbb{N}} \subset \mathcal{S}^{\mathcal{P}}$ defined by $T_n := T + \frac{1}{n}$, $n \in \mathbb{N}$. It remains to show that b satisfies also the second equality. For that fix a predictable stopping time S and remember $L^{\mathcal{P}} = a(^{\mathcal{P}}\tilde{P} - b) = a(\tilde{P}_- - b)$. Then we obtain by the strong Markov property of Lévy processes, $^{\mathcal{P}}N_S = \mathbb{E}[N_S|\mathcal{F}_{S-}] = N_{S-}$ and (22) that

$$\tilde{P}_{S-}e^{-rS} = {}^{\mathcal{P}}P_S = \mathbb{E}\left[\int_{[S,\infty)} \sup_{v \in [S,t]} L_v^{\mathcal{P}} \, dR_t \,\Big|\, \mathcal{F}_{S-}\right] = \mathbb{E}\left[\int_{(S,\infty)} \sup_{v \in [S,t]} L_v^{\mathcal{P}} \, dR_t \,\Big|\, \mathcal{F}_{S-}\right]$$

$$= a\mathbb{E}\left[\int_{(S,\infty)} \sup_{v \in [S,t]} (\tilde{P}_{v-} - \tilde{P}_{S-}) \, dR_t \,\Big|\, \mathcal{F}_{S-}\right] + a\mathbb{E}[(\tilde{P}_{S-} - b)(R_{\infty-} - R_S) \,|\, \mathcal{F}_{S-}]$$

$$= a\mathbb{E}\left[\int_{[0,\infty)} \left(\sup_{v \in [0,t]} \sum_{k=1}^{N_{v-}} Y_k\right) dR_t\right]e^{-rS} + a(\tilde{P}_{S-} - b)e^{-rS}\mathbb{E}[R_{\infty-}].$$

Solving for b and recalling that $\mathbb{E}[R_{\infty-}] = \frac{\lambda}{r} = \frac{1}{a}$ gives the second equality for b.

(ii) By definition of ℓ^η (see (50)) we get

$$\ell^\eta(p, \Delta) = \inf_{0 < T \in \mathcal{S}^{\Lambda^\eta}} \frac{1 - \mathbb{E}[e^{-rT}]}{\mathbb{E}[R_{T-}] + \Delta}\left(p - \frac{\mathbb{E}[e^{-rT}\sum_{k=1}^{N_T} Y_k]}{1 - \mathbb{E}[e^{-rT}]}\right),$$

which shows (ii) by (i). $\quad\square$

4.2. *Optimal controls for an imperfect sensor.* Next we consider controls that can use an imperfect jump sensor, that is, a sensor with probability of failing to alert $p(\eta) = \mathbb{P}(|Y_1| < \eta) \in (0, 1)$. In this class, we find the following optimal control:

THEOREM 4.4 (Optimal control with η-sensor). *In the case $p(\eta) \in (0, 1)$, an optimal control for (44) is*

$$C_t^{\Lambda^\eta} := c_0 \vee \sup_{v \in [0,t]} L_v^{\Lambda^\eta}, \quad t \in [0, \infty),$$

with the maximal solution L^{Λ^η} to (21) and (22) given explicitly by

(60)
$$L_t^{\Lambda^\eta} = \begin{cases} 0, & \tilde{P}_t^\eta \geq b, |\Delta\tilde{P}_t^\eta| \geq \eta, \\ \frac{r}{\lambda}(\tilde{P}_t^\eta - b), & \tilde{P}_t^\eta \geq b, |\Delta\tilde{P}_t^\eta| < \eta, \\ \inf_{\gamma^0 \in (0, B_0^\eta \cdot (b - \tilde{P}_t^\eta))} f_1^\eta(\gamma^0, 0, \tilde{P}_t^\eta) < 0, & \tilde{P}_t^\eta < b, |\Delta\tilde{P}_t^\eta| \geq \eta, \\ \inf_{\gamma^1 \in (-B_1^\eta \cdot (b - \tilde{P}_t^\eta), 0)} f_0^\eta(0, \gamma^1, \tilde{P}_t^\eta) < 0, & \tilde{P}_t^\eta < b, |\Delta\tilde{P}_t^\eta| < \eta. \end{cases}$$

Here, b is as in Corollary 4.3,

(61) $$B_0^\eta := 1 - \frac{\lambda r}{\lambda + r} \cdot \frac{1 - p(\eta)}{r + \lambda(1 - \delta)} < 1, \qquad B_1^\eta := \frac{1 - \lambda p(\eta) + \frac{(\lambda+r)\lambda}{r}(1 - \delta)}{\lambda p(\eta)} > 0$$

with

(62) $$\delta := \mathbb{E}[e^{-rT^0}] \quad for \quad T^0 := \inf\left\{t \geq 0 \,\Big|\, \sum_{k=1}^{N_t} Y_k \geq 0, N_t \geq 1\right\};$$

moreover, for $\Delta \in \{0, 1\}$, the functions $f_\Delta^\eta : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given by

(63) $$f_\Delta^\eta(\gamma^0, \gamma^1, p) := \frac{(1 - \mathbb{E}[e^{-rT^\eta(\gamma^0, \gamma^1)}])p - \mathbb{E}[e^{-rT^\eta(\gamma^0, \gamma^1)}\sum_{k=1}^{N_{T^\eta(\gamma^0, \gamma^1)}} Y_k]}{\frac{\lambda}{r}(1 - \mathbb{E}[e^{-rT^\eta(\gamma^0, \gamma^1)}]) - \mathbb{E}[e^{-rT^\eta(\gamma^0, \gamma^1)}\mathbb{1}_{\{|\Delta\tilde{P}_{T^\eta(\gamma^0, \gamma^1)}| \geq \eta\}}] + \Delta},$$

with

(64)
$$T^\eta(\gamma^0, \gamma^1) := \inf\{t \in \{^{\Lambda^\eta} N > 0\} \mid (|\Delta \tilde{P}_t| < \eta \text{ and } \tilde{P}_{t-} - \tilde{p} \geq \gamma^0)$$
$$\text{or } (|\Delta \tilde{P}_t| \geq \eta \text{ and } \tilde{P}_t - \tilde{p} \geq \gamma^1)\}.$$

The proof of the previous theorem will be given at the end of this section. A separate result on the optional case $p(\eta) = 0$ is obtained in Section 4.3 and an illustration of all our findings is deferred to Section 4.4. Before proceeding to the proof of this theorem, let us note how the minimal storage level L^{Λ^η} approaches that of the predictable case $L^{\mathcal{P}}$ when the sensor becomes more and more useless as its failure probability tends to 1.

COROLLARY 4.5. *In the setting of Theorem* 4.4, *consider a sequence* $(\eta_n)_{n\in\mathbb{N}} \subset [0, \infty]$ *such that* $\lim_n p(\eta_n) = 1$. *Then the solution* $L^{\Lambda^{\eta_n}}$, $n \in \mathbb{N}$, *converges to* $L^{\mathcal{P}}$ (*see Theorem* 4.2) *for* $n \to \infty$:

$$\lim_{n\to\infty} L_t^{\Lambda^{\eta_n}}(\omega) = L_t^{\mathcal{P}}(\omega), \quad t \in [0, \infty), \omega \in \Omega.$$

PROOF. For $\omega \in \Omega$ and $t \in [0, \infty)$, we have for any $\eta \geq \Delta \tilde{P}_t(\omega)$ that $\tilde{P}_t^\eta(\omega) = \tilde{P}_{t-}(\omega)$ and $^{\Lambda^\eta}(\Delta N)_t(\omega) = 0$. Hence, due to (49), we only have to show $\lim_{n\to\infty} \ell^{\eta_n}(p, 0) = \ell^\infty(p, 0)$ for any $p \in \mathbb{R}$. In case $p \geq b$, we have by (58) and (60) that $\ell^\eta(p, 0) = \ell^\infty(p, 0)$ for any $\eta \in [0, \infty]$. Assume henceforth $p < b$. By definition of ℓ^η we have that $\eta \mapsto \ell^\eta(p, 0)$ is increasing. Hence, we infer that $\ell^{\eta_n}(p, 0) \leq \ell^\infty(p, 0)$ and it suffices to show $\lim_{n\to\infty} \ell^{\eta_n}(p, 0) \geq \ell^\infty(p, 0)$. We obtain by Theorem 4.2 and the characterization of a from Corollary 4.9 below that

(65)
$$\ell^\infty(p, 0) = \frac{1 - \mathbb{E}[e^{-rT^0}]}{\frac{\lambda}{r}(1 - \mathbb{E}[e^{-rT^0}])}(p - b).$$

In order to conclude our assertion, we will show next how to find a lower bound for $\ell^\eta(p, 0) = \inf_{\gamma_1 < 0} f_0^\eta(0, \gamma_1, p)$, $\eta \in [0, \infty]$, from (63) which converges to $\ell^\infty(p, 0)$ when $p(\eta) \to 1$. For $\gamma_1 < 0$, we have $T^0 \geq T^\eta(0, \gamma_1)$, $\eta \in [0, \infty]$, and therefore, using also

$$b \geq \frac{1}{1 - \mathbb{E}[e^{-rT^\eta(0,\gamma^1)}]}\mathbb{E}\left[e^{-rT^\eta(0,\gamma^1)} \sum_{k=1}^{N_{T^\eta(0,\gamma^1)}} Y_k\right],$$

establishes

(66)
$$f_0^\eta(0, \gamma^1, p) \geq \frac{1 - \mathbb{E}[e^{-rT^0}]}{\frac{\lambda}{r}(1 - \mathbb{E}[e^{-rT^\eta(0,\gamma^1)}]) - \mathbb{E}[e^{-rT^\eta(0,\gamma^1)}\mathbb{1}_{\{|\Delta \tilde{P}_{T(0,\gamma^1)}| \geq \eta\}}]}(p - b).$$

Comparing (65) and (66) shows that it remains to prove that the denominator in (66) converges to the denominator in (65) uniformly in $\gamma_1 < 0$ when $p(\eta) \to 1$:

$$\lim_{p(\eta)\to 1} \sup_{\gamma_1 < 0} \left|\frac{\lambda}{r}(\mathbb{E}[e^{-rT^0}] - \mathbb{E}[e^{-rT^\eta(0,\gamma^1)}]) - \mathbb{E}[e^{-rT^\eta(0,\gamma^1)}\mathbb{1}_{\{|\Delta \tilde{P}_{T(0,\gamma^1)}| \geq \eta\}}]\right| = 0.$$

This uniform convergence is a consequence of the estimates

$$0 \leq \mathbb{E}[e^{-rT(0,\gamma^1)}\mathbb{1}_{\{|\Delta \tilde{P}_{T(0,\gamma^1)}| \geq \eta\}}] \leq 1 - p(\eta), 0 \leq \mathbb{E}[e^{-rT(0,\gamma^1)} - e^{-rT^0}] \leq 1 - p(\eta),$$

which hold for any $\gamma_1 < 0$. $\square$

The rest of this section is devoted to the *proof of Theorem* 4.4. From Lemma 4.1 we know that we only have to determine ℓ^η from (50). Let us prepare this by some auxiliary results for ℓ^η.

PROPOSITION 4.6. *If $p(\eta) > 0$, the functions $\ell^\eta(\cdot, 0)$ and $\ell^\eta(\cdot, 1)$ are both continuous, increasing and concave on $\mathbb{R}$. Moreover, on $(-\infty, b)$, both functions are strictly increasing and satisfy $\ell^\eta(\cdot, 0) < \ell^\eta(p, 1) < 0$. On $[b, \infty)$, $\ell^\eta(\cdot, 0)$ and $\ell^\eta(\cdot, 1)$ are determined by*

$$\ell^\eta(p, 0) = a(p - b) \geq 0 = \ell^\eta(p, 1), \quad p \in [b, \infty),$$

where a, b are the constants from Theorem 4.2.

PROOF. For $\Delta \in \{0, 1\}$ we obtain easily from (50), (51) that $p \mapsto \ell^\eta(p, \Delta)$ is increasing. Moreover, concavity follows as the infimum of affine functions is concave. Moreover, as $\ell^\eta(\cdot, 0)$, $\ell^\eta(\cdot, 0)$ are concave functions taking real-values, we have that they are also continuous.

Next, we obtain from Corollary 4.3 that for $p \in \mathbb{R}$, $\Delta \in \{0, 1\}$, we have $\ell^\eta(p, \Delta) < 0$ if and only if $p < b$. Moreover, in the case $p < b$ we can restrict in (50) to $0 < T \in \mathcal{S}^{\Lambda^\eta}$ with $\ell_T(p, \Delta) < 0$. This immediately shows $\ell^\eta(p, 0) < \ell^\eta(p, 1) < 0$ for $p < b$.

$\ell^\eta(\cdot, 1)$ *for* $p \geq b$: We get with $T_n = \frac{1}{n}$ in (50) that $0 \leq \ell^\eta(p, 1) \leq \lim_{n \to \infty} \ell_{\frac{1}{n}}(p, 1) = 0$.

$\ell^\eta(\cdot, 0)$ *for* $p = b$: We have by Theorem 4.2 and Lemma 4.1(ii), that $\ell^\infty(p, 0) = 0$ and $0 \leq \ell^\eta(p, 0) \leq \ell^\infty(p, 0) = 0$.

$\ell^\eta(\cdot, 0)$ *for* $p > b$: If $p > b$, we get from Theorem 4.2, Lemma 4.1(ii) and (59) that

$$\ell^\infty(p, 0) \geq \ell^\eta(p, 0) \geq \inf_{0 < T \in \mathcal{S}^\Lambda} \frac{1 - \mathbb{E}[e^{-rT}]}{\mathbb{E}[R_{T-}]}(p - b)$$

$$\geq \inf_{0 < T \in \mathcal{S}^\Lambda} \frac{1 - \mathbb{E}[e^{-rT}]}{\mathbb{E}[R_T]}(p - b) = \frac{r}{\lambda}(p - b) = \ell^\infty(p, 0);$$

so, we must have equality everywhere and, in particular, $\ell^\eta(p, 0) = \ell^\infty(p, 0) > 0$.

Finally, we prove strict monotonicity on $(-\infty, b)$. For that fix $\Delta \in \{0, 1\}$ and assume by way of contradiction that $p_1 < p_2 < b$ with $\ell^\eta(p_1, \Delta) = \ell^\eta(p_2, \Delta)$. Now there exists a sequence $(T_n)_{n \in \mathbb{N}}$ such that $\ell_{T_n}(p_2, \Delta)$ (see (51)) decreases to $\ell^\eta(p_2, \Delta)$ with maximal distance $\frac{1}{n}$. Then

$$0 = \ell^\eta(p_1, \Delta) - \ell^\eta(p_2, \Delta) \leq \ell_{T_n}(p_1, \Delta) - \ell_{T_n}(p_2, \Delta) + \frac{1}{n}$$

$$= \frac{1 - \mathbb{E}[e^{-rT_n}]}{\mathbb{E}[R_{T_n-}] + \Delta}(p_1 - p_2) + \frac{1}{n} \leq 0 + \frac{1}{n}$$

and therefore, by $p_1 < p_2$,

$$\lim_{n \to \infty} \frac{1 - \mathbb{E}[e^{-rT_n}]}{\mathbb{E}[R_{T_n-}] + \Delta} = 0.$$

Hence, we get by (59) that

$$0 > \ell^\eta(p_2, \Delta) = \lim_{n \to \infty} \frac{1 - \mathbb{E}[e^{-rT_n}]}{\mathbb{E}[R_{T_n-}] + \Delta}\left(p_2 - \frac{\mathbb{E}[e^{-rT_n} \sum_{k=1}^{N_{T_n}} Y_k]}{1 - \mathbb{E}[e^{-rT_n}]}\right)$$

$$\geq \lim_{n \to \infty} \frac{1 - \mathbb{E}[e^{-rT_n}]}{\mathbb{E}[R_{T_n-}] + \Delta}(p_2 - b) = 0,$$

which is the desired contradiction. $\square$

The next lemma shows how the infimum in (50) is attained in a relaxed sense.

LEMMA 4.7. *In the case $p(\eta) > 0$, we have for $p \in \mathbb{R}$ and $\Delta \in \{0, 1\}$ that*

$$
(67) \qquad \ell^\eta(p, \Delta) = \frac{(1 - \mathbb{E}[e^{-rT^{\ell^\eta}(p,\Delta)}])p - \mathbb{E}[e^{-rT^{\ell^\eta}(p,\Delta)} \sum_{k=1}^{N_{T^{\ell^\eta}(p,\Delta)}} Y_k]}{\mathbb{E}[\Delta R_{T^{\ell^\eta}(p,\Delta)} \mathbb{1}_{(H^{\ell^\eta}(p,\Delta))^c} + R_{T^{\ell^\eta}(p,\Delta)-}] + \Delta},
$$

where

$$
(68) \qquad T^{\ell^\eta}(p, \Delta) := \inf\{t \in \{^{\Lambda^\eta} N > 0\} \mid \ell^\eta(^{\Lambda^\eta}\tilde{P}_t - \tilde{p} + p, {}^{\Lambda^\eta}(\Delta N)_t) \geq \ell^\eta(p, \Delta)\},
$$

$$
H^{\ell^\eta}(p, \Delta) := \{\ell^\eta(^{\Lambda^\eta} P_{T^{\ell^\eta}(p,\Delta)} - \tilde{p} + p, {}^{\Lambda^\eta}(\Delta N)_{T^{\ell^\eta}(p,\Delta)}) \geq \ell^\eta(p, \Delta)\}
$$
$$
(69) \qquad\qquad\qquad \cap \{^{\Lambda^\eta} N_{T^{\ell^\eta}(p,\Delta)} > 0\}.
$$

REMARK 4.8. (i) The quadruple

$$
\tau^{\ell^\eta}(p, \Delta) := (T^{\ell^\eta}(p, \Delta), \varnothing, H^{\ell^\eta}(p, \Delta), H^{\ell^\eta}(p, \Delta)^c)
$$

with $T^{\ell^\eta}(p, \Delta)$ and $H^{\ell^\eta}(p, \Delta)$ defined in the previous lemma is a so-called divided stopping time. The theory of divided stopping times was developed in El Karoui (1981) and we will give some more details in the Appendix.

(ii) Note that (67) does not by itself construct ℓ^η from (50) as ℓ^η is also contained on the right-hand side. It shows, however, that we should restrict attention to a certain class of stopping times (68) when computing the infimum in (50). This is the key to our explicit solution.

PROOF. To verify (70), we will just write L and Λ instead of L^{Λ^η} and Λ^η in the following.

Lemma 4.7 will follow from Lemma 4.1(ii) and from the fact that $\tilde{P}$ is a Lévy process, if we can show for any $S \in \mathcal{S}^\Lambda$ that

$$
(70) \qquad L_S^\Lambda = \frac{{}^\Lambda P_S - \mathbb{E}[P_{T_S^L} \mid \mathcal{F}_S^\Lambda]}{\mathbb{E}[\Delta R_{T_S^L} \mathbb{1}_{H_S^c} + R_{T_S^L-} - R_{S-} \mid \mathcal{F}_S^\Lambda]},
$$

where

$$
T_S^L := \inf\{t > S \mid L_t \geq L_S \text{ and } {}^\Lambda N_t > N_{S-}\},
$$
$$
H_S := \{L_{T_S^L} \geq L_S\} \cap \{^\Lambda N_{T_S^L} > N_{S-}\} \cap \{S < T_S^L\}.
$$

Let us consider the sequence

$$
(71) \qquad T_{S,n}^L := (T_S^L)_{H_S} \wedge \left((T_S^L)_{H_S^c} + \frac{1}{n}\right), \quad n \in \mathbb{N}:
$$

By Dellacherie and Meyer ((1978), Theorem 50, page 116), the random time T_S^L is an $\mathcal{F}$-stopping time, whence $(T_S^L)_{H_S^c} + \frac{1}{n}$ is a predictable $\mathcal{F}$-stopping time and, therefore, also a Λ-stopping time. Moreover we see that the graph of $(T_S^L)_{H_S}$ is contained in $\{t > S \mid L_t \geq L_S \text{ and } {}^\Lambda N_t > {}^\Lambda N_{S-}\}$ and therefore by Lenglart ((1980), Corollary 2, page 504), $(T_S^L)_{H_S}$ is also a Λ-stopping time. As the minimum of two Λ-stopping times is a Λ-stopping time we obtain that $T_{S,n}^L$ from (71) is a Λ-stopping time for every $n \in \mathbb{N}$. We define next $T_S^{L,1} := \inf\{t > T_S^L \mid N_t > N_{T_S^L}\}$. As N is piecewise constant, this implies $T_S^{L,1} > T_S^L$ almost surely on $\{T_S^L < \infty\}$. Now we define the sequence $(A_n)_{n \in \mathbb{N}} \subset \mathcal{F}_{T_{S,n}^L}^\Lambda$ via

$$
A_n := \{T_S^{L,1} > T_{S,n}^L\} \cap \{T_S^L < \infty\}
$$

and observe that $A_n \subset A_{n+1}$ for $n \in \mathbb{N}$ with $\bigcup_{n \in \mathbb{N}} A_n = \{T_S^L < \infty\}$. Note that

$$(72) \qquad\qquad \mathrm{d}R((T_S^L, T_{S,n}^L]) = 0 \quad \text{on } A_n.$$

By (53) we have on $A_n \cap H_S^c$ that $L_{T_S^L}^* = L_{T_S^L + \frac{1}{n}}$ implying

$$(73) \qquad \sup_{v \in (T_S^L, t]} L_v = \sup_{v \in [T_{S,n}^L, t]} L_v \quad \text{on } A_n \cap \{t \geq T_{S,n}^L\} \cap H_S^c \text{ for } t \in (0, \infty).$$

Next, we get by definition of A_n and (22) that

$$
\begin{aligned}
^{\Lambda}P_S &= \mathbb{E}\left[\int_{[S,\infty)} \sup_{v \in [S,t]} L_v \, \mathrm{d}R_t \,\Big|\, \mathcal{F}_S^{\Lambda} \right] \\
(74) \\
&= \lim_{n \to \infty} \mathbb{E}\left[\mathbb{1}_{A_n} \int_{[S,\infty)} \sup_{v \in [S,t]} L_v \, \mathrm{d}R_t \,\Big|\, \mathcal{F}_S^{\Lambda} \right] + L_S \mathbb{E}[\mathbb{1}_{\{T_S^L = \infty\}}(R_{\infty-} - R_{S-}) \mid \mathcal{F}_S^{\Lambda}].
\end{aligned}
$$

The first conditional expectation above is equal to

$$
\begin{aligned}
\mathbb{E}\Bigg[\mathbb{1}_{H_S^c \cap A_n} &\left(\int_{(T_S^L, \infty)} \sup_{v \in (T_S^L, t]} L_v \, \mathrm{d}R_t + \int_{[S, T_S^L]} L_S \, \mathrm{d}R_t \right) \\
&+ \mathbb{1}_{H_S \cap A_n} \left(\int_{[T_S^L, \infty)} \sup_{v \in [T_S^L, t]} L_v \, \mathrm{d}R_t + \int_{[S, T_S^L)} L_S \, \mathrm{d}R_t \right) \Bigg| \mathcal{F}_S^{\Lambda} \Bigg] \\
\overset{(72),\,(73)}{=} & L_S \mathbb{E}\big[\mathbb{1}_{A_n}(\Delta R_{T_S^L} \mathbb{1}_{H_S^c} + R_{T_S^L -} - R_{S-}) \mid \mathcal{F}_S^{\Lambda} \big] \\
&+ \mathbb{E}\left[\mathbb{1}_{A_n} \int_{[T_{S,n}^L, \infty)} \sup_{v \in [T_{S,n}^L, t]} L_v \, \mathrm{d}R_t \,\Big|\, \mathcal{F}_S^{\Lambda} \right] \\
\overset{(22)}{=} & L_S \mathbb{E}\big[\mathbb{1}_{A_n}(\Delta R_{T_S^L} \mathbb{1}_{H_S^c} + R_{T_S^L -} - R_{S-}) \mid \mathcal{F}_S^{\Lambda} \big] + \mathbb{E}\big[\mathbb{1}_{A_n} P_{T_{S,n}^L} \mid \mathcal{F}_S^{\Lambda} \big].
\end{aligned}
$$

Plugging this into (74) leads to

$$(75) \qquad ^{\Lambda}P_S = L_S \mathbb{E}\big[\Delta R_{T_S^L} \mathbb{1}_{H_S^c} + R_{T_S^L -} - R_{S-} \mid \mathcal{F}_S^{\Lambda} \big] + \mathbb{E}\big[P_{T_S^L} \mid \mathcal{F}_S^{\Lambda} \big],$$

where we used the right-continuity of P and its class(D^{Λ}) property to apply Lebesgue's theorem. Now, (75) is equivalent to the desired identity (70). $\quad\square$

COROLLARY 4.9. *The constants a, b defined in Theorem 4.2 have the alternative descriptions*

$$(76) \qquad a = \hat{a} := \frac{1 - \mathbb{E}[\mathrm{e}^{-rT^0}]}{\mathbb{E}[R_{T^0}]}, \qquad b = \hat{b} := \frac{\mathbb{E}[\mathrm{e}^{-rT^0} \sum_{k=1}^{N_{T^0}} Y_k]}{1 - \mathbb{E}[\mathrm{e}^{-rT^0}]},$$

where T^0 is defined in (62).

PROOF. For $\eta = \infty$ we have $p(\eta) = 1$ and $^{\Lambda^{\eta}}(\Delta N) \equiv 0$. Hence, by Lemma 4.1, Theorem 4.2 and Lemma 4.7, we obtain that $T^{\ell^{\eta}}(p, 0) = T^0$, $H^{\ell^{\eta}}(p, 0) = \varnothing$ giving (76). $\quad\square$

PROOF OF THEOREM 4.4. Lemma 4.1 shows that we only have to determine the function ℓ^{η} of (50). For $p \geq b$ and $\Delta \in \{0, 1\}$ this is already done by Proposition 4.6. So, assume henceforth that $p < b$. We will prove in the next lines that

$$(77) \qquad \ell^{\eta}(p, \Delta) = f_{\Delta}^{\eta}\big(\Gamma^0(p, \Delta) - p, \Gamma^1(p, \Delta) - p\big)$$

with f_Δ^η defined in (63) and

(78) $\Gamma^0(p, \Delta) := \min\{\hat{p} \in (-\infty, b] \mid \ell^\eta(\hat{p}, 0) \geq \ell^\eta(p, \Delta)\},$

(79) $\Gamma^1(p, \Delta) := \min\{\hat{p} \in (-\infty, b] \mid \ell^\eta(\hat{p}, 1) \geq \ell^\eta(p, \Delta)\}.$

We prove (77) by clarifying the structure of $T^{\ell^\eta}(p, \Delta)$ and $H^{\ell^\eta}(p, \Delta)$ from (68) and (69) respectively. As $p < b$ we obtain by Proposition 4.6 that the minima in (78) and (79) are well-defined and $\Gamma^0(p, \Delta) > \Gamma^1(p, \Delta)$. Moreover, as $\hat{p} \mapsto \ell^\eta(\hat{p}, \Delta)$ is strictly increasing on $(-\infty, b)$ (see again Proposition 4.6) we get

(80) $\Gamma^0(p, 0) = p \quad \text{and} \quad \Gamma^1(p, 1) = p.$

Now, using the definitions of Γ^0, Γ^1 we can see that

$$T^{\ell^\eta}(p, \Delta) = T^\eta(\Gamma^0(p, \Delta) - p, \Gamma^1(p, \Delta) - p),$$

and

$$H^{\ell^\eta}(p, \Delta) = \{|\Delta \tilde{P}_{T^{\ell^\eta}(p, \Delta)}| \geq \eta\},$$

where T^η, H^{ℓ^η} are defined in (64) and (69) respectively. Hence, plugging those observations into (67) proves (77).

Equation (77) implies immediately that

(81) $\ell^\eta(p, \Delta) = \inf_{\gamma^0, \gamma^1 \in (-\infty, b-p]} f_\Delta^\eta(\gamma^0, \gamma^1, p).$

Hence, to finish the proof of Theorem 4.4 we have to show that the infimum in (81) coincides with those in (60) for the respectively pertinent cases. This will be accomplished by characterizing further the optimal choices $\hat{\gamma}^0 := \Gamma^0(p, \Delta) - p$, $\hat{\gamma}^1 := \Gamma^1(p, \Delta) - p$.

Let $\Delta = 1$. By (80) we get $\hat{\gamma}^1 = 0$ and hence we can reduce the infimum in (81) to an infimum over $f_1^\eta(\gamma^0, 0, p)$. Moreover, by $\hat{\gamma}^0 > \hat{\gamma}^1 = 0$ we can restrict to $\gamma^0 > 0$. Next we want to obtain the stated upper bound for the relevant γ^0. Using $T^\eta(\gamma^0, 0) \geq T^1$, with T^1 the first jump of N, we obtain for any $\hat{p} \leq b$ that

(82)
$$\ell^\eta(\hat{p}, 0) \geq \frac{1 - \mathbb{E}[e^{-rT^0}]}{\frac{\lambda}{r}(1 - \mathbb{E}[e^{-rT^1}]) - \mathbb{E}[e^{-rT^1}\mathbb{1}_{\{|\Delta\tilde{P}_{T^1}|\geq\eta\}}]}(\hat{p} - b)$$
$$= \frac{(\lambda + r)(1 - \mathbb{E}[e^{-rT^0}])}{\lambda p(\eta)}(\hat{p} - b).$$

On the other hand, using $T_n := T^0 + \frac{1}{n}$, $n \in \mathbb{N}$, with T^0 from (62), we obtain due to (76) an upper bound on $\ell^\eta(p, 1)$:

(83) $\ell^\eta(p, 1) \leq \frac{1 - \mathbb{E}[e^{-rT^0}]}{\frac{\lambda}{r}(1 - \mathbb{E}[e^{-rT^0}]) + 1}(p - b).$

Observe that the right-hand side of (82) will be not smaller than the right-hand side of (83) if and only if

(84) $\hat{p} - p \geq \frac{r + \lambda(1 - \mathbb{E}[e^{-rT^0}]) - \frac{\lambda r}{\lambda + r}(1 - p(\eta))}{r + \lambda(1 - \mathbb{E}[e^{-rT^0}])}(b - p).$

For such $\hat{p}(\leq b)$ we will have $\ell^\eta(\hat{p}, 0) \geq \ell^\eta(p, 1)$ and thus $\Gamma^0(p, 1) \leq \hat{p}$, which is tantamount with $\hat{\gamma}^0$ being less than the right-hand side in (84). This shows that we can restrict to $\gamma_0 < B_0^\eta(b - p)$, which we wanted to show.

Let $\Delta = 0$. By (80) we get $\hat{\gamma}^0 = 0$. Hence, we can assume in (81) that $\gamma_0 = 0$ and by $\hat{\gamma}^0 > \hat{\gamma}^1 = 0$ we only have to consider $\gamma^1 < 0$. To obtain a lower bound, we can use the boundaries established in (82) and (83) to see that $\ell^\eta(p, 0) > \ell^\eta(\hat{p}, 1)$, if

$$\hat{p} - p < -\left(\frac{1 - \lambda p(\eta) + \frac{(\lambda+r)\lambda}{r}(1 - \mathbb{E}[e^{-rT^0}])}{\lambda p(\eta)}\right)(b - p).$$

This shows that γ_1 has to be chosen larger than $-B_1^\eta(b - p)$, which finishes our proof. $\square$

4.3. *Optimal optional controls.* Let us complete our analysis and determine the optimal control in the case of $p(\eta) = 0$ where $\Lambda = \mathcal{O}(\mathcal{F})$ (see Lemma 4.1(ii)). This case is special since Theorem 3.9 is not directly applicable as explained in the following proposition, which at the same time provides us with a remedy:

PROPOSITION 4.10. *Assume* $p(\eta) = 0$.

(i) *The process* P *is indistinguishable from* $^\Lambda P$.
(ii) *For* $\tilde{p} < m\frac{\lambda}{r}$ *the process* P *is* not dR-*right-upper-semicontinuous in expectation at time* 0 (*see Assumption* 3.3(iii)). *In particular,* P *does* not *satisfy Assumption* 3.3.
(iii) *The process* $\bar{P}$ *given by*

$$(85) \qquad \bar{P}_t := \begin{cases} \dfrac{\lambda}{\lambda + r}(\tilde{P}_t + m)e^{-rt} & \text{for } \tilde{P}_t < m\dfrac{\lambda}{r}, \Delta\tilde{P}_t = 0, \\ P_t & \text{else,} \end{cases} \qquad t \in [0, \infty),$$

with $\bar{P}_\infty := 0$, *satisfies Assumption* 3.3 *and, for any stopping time* $S \in \mathcal{S}$,

$$(86) \qquad \begin{aligned} \bar{P}_S &= \operatorname*{esssup}_{T \geq S, dR([S,T))=0 \text{ a.s.}} \mathbb{E}[P_T | \mathcal{F}_S] \\ &= P_S \vee \mathbb{E}[P_{T_S^1} | \mathcal{F}_S] = P_S \vee \frac{\lambda}{\lambda + r}(\tilde{P}_S + m)e^{-rS}, \end{aligned}$$

where

$$(87) \qquad T_S^1 := \inf\{t \geq S | N_t > N_S\}.$$

In particular, $\bar{P} \geq P$, *up to an evanescent set, and there exists a process* $\bar{L}$ *satisfying* (21) *and* (22) *with* P *replaced by* $\bar{P}$. *In fact,* $\bar{P}$ *is the smallest optional process larger than* P *satisfying Assumption* 3.3, *that is, for any other optional process* $\hat{P} \geq P$ *satisfying Assumption* 3.3 *we have that* $\{\hat{P} < \bar{P}\}$ *is evanescent.*

PROOF. (i) is an immediate consequence of $p(\eta) = 0$ and Corollary 2.10 of the Meyer section theorem.

(ii) With T_1 denoting the first jump time of the Poisson process N, we have $dR([0, T_1)) = 0$ and also

$$\mathbb{E}[P_0] = \tilde{p} < (\tilde{p} + m)\frac{\lambda}{\lambda + r} = \mathbb{E}[P_{T_1}],$$

where the inequality holds because $\tilde{p} < m\frac{\lambda}{r}$ by assumption. It follows that P is not dR-right-upper-semicontinuous in expectation at 0.

(iii) Let us first argue (86) for any stopping time S. We have on $\{\Delta\tilde{P}_S = 0\} \cap \{S < \infty\}$ that

$$\bar{P}_S = \operatorname*{esssup}_{T \geq S, dR([S,T))=0 \text{ a.s.}} \mathbb{E}[P_T | \mathcal{F}_S] = P_S \vee \mathbb{E}[P_{T_S^1} | \mathcal{F}_S] = P_S \vee \frac{\lambda}{\lambda + r}(\tilde{P}_S + m)e^{-rS},$$

where T_S^1 is defined in (87). As $P_S < \frac{\lambda}{\lambda+r}(\tilde{P}_S + m)\mathrm{e}^{-rS}$ is equivalent to $\tilde{P}_S < m\frac{\lambda}{r}$, we obtain

$$\underset{T \geq S, \mathrm{d}R([S,T))=0 \text{ a.s.}}{\mathrm{esssup}} \mathbb{E}[P_T|\mathcal{F}_S] = \begin{cases} \dfrac{\lambda}{\lambda+r}(\tilde{P}_S + m)\mathrm{e}^{-rS} & \text{if } \tilde{P}_S < m\dfrac{\lambda}{r}, \Delta\tilde{P}_S = 0, S < \infty, \\ P_S & \text{else.} \end{cases}$$

This proves (86). We will show next that $\bar{P}$ satisfies Assumption 3.3. Part (i) of Assumption 3.3 is clear. Part (ii) follows by Fatou's lemma and $\Delta\bar{P}_S = 0$ a.s. at every predictable $\mathcal{F}$-stopping times since for $\omega \in \{\Delta\tilde{P}_S = 0\}$ such that $\tilde{P}_S(\omega) = \tilde{P}_{S-}(\omega) < m\frac{\lambda}{r}$ we will also have $\tilde{P}_{S_n}(\omega) < m\frac{\lambda}{r}$ for n large enough. Hence, it remains to prove (iii) of Assumption 3.3. For that fix an $\mathcal{F}$-stopping time S and a sequence $(S_n)_{n\in\mathbb{N}}$ of $\mathcal{F}$-stopping times with $S_n \geq S$ for all $n \in \mathbb{N}$ such that we have $\lim_{n\to\infty} \mathrm{d}R([S, S_n)) = 0$ almost surely. Then for almost every $\omega \in \Omega$, we will have $S_n(\omega) \leq T_S^1(\omega)$ for sufficiently large n. Letting $\tilde{S}_n := (S_n)_{\{S_n \leq T_S^1\}}$, we thus obtain by definition of $\bar{P}$ and Fatou's lemma that

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{E}[\bar{P}_{S_n}] &\leq \limsup_{n\to\infty} \mathbb{E}[\bar{P}_{\tilde{S}_n}] + \mathbb{E}\Big[\limsup_{n\to\infty} \bar{P}_{S_n}\mathbb{1}_{\{S_n > T_S^1\}}\Big] \\
&\overset{(86)}{=} \limsup_{n\to\infty} \mathbb{E}\Big[\underset{T \geq \tilde{S}_n, \mathrm{d}R([\tilde{S}_n,T))=0 \text{ a.s.}}{\mathrm{esssup}} \mathbb{E}[P_T|\mathcal{F}_{\tilde{S}_n}]\Big] + 0 \\
&= \limsup_{n\to\infty} \mathbb{E}\Big[\underset{T \geq \tilde{S}_n, \mathrm{d}R([S,T))=0 \text{ a.s.}}{\mathrm{esssup}} \mathbb{E}[P_T|\mathcal{F}_{\tilde{S}_n}]\Big] \\
&= \limsup_{n\to\infty} \mathbb{E}\Big[\underset{T \geq \tilde{S}_n, \mathrm{d}R([S,T))=0 \text{ a.s.}}{\mathrm{esssup}} \mathbb{E}[P_T|\mathcal{F}_S]\Big] \\
&\leq \mathbb{E}\Big[\underset{T \geq S, \mathrm{d}R([S,T))=0 \text{ a.s.}}{\mathrm{esssup}} \mathbb{E}[P_T|\mathcal{F}_S]\Big] \overset{(86)}{=} \mathbb{E}[\bar{P}_S].
\end{aligned}$$

Here, we have used in the second equality that on $\{S_n \leq T_S^1\}$ also $\mathrm{d}R([S, S_n)) = 0$; in the third equality we used that the essential supremum is upwards directed and so dominated convergence allows us to interchange the esssup and the $\mathcal{F}_S$-conditional expectation. Hence, Assumption 3.3(iii) is satisfied. It now remains to show that $\bar{P}$ is the smallest optional process larger than P satisfying Assumption 3.3. For that assume there exists another process $\hat{P}$ with $\hat{P} \geq P$ satisfying Assumption 3.3. By the Meyer section theorem it is enough to prove $\bar{P}_S \leq \hat{P}_S$ at every $\mathcal{F}$-stopping time S. On the complement of $A := \{\tilde{P}_S < m\frac{\lambda}{r}\} \cap \{\Delta\tilde{P}_S = 0\} \cap \{S < \infty\} = \{P_S < \bar{P}_S\}$ we have by definition of $\bar{P}$ that $\bar{P}_S = P_S \leq \hat{P}_S$. Therefore, let us focus on A and assume by way of contradiction $\mathbb{P}(A) > 0$. Then we can define the (constant) sequence $S_n := (T_S^1)_A$, $n \in \mathbb{N}$, satisfying $S_n \geq S_A$ and $\lim_{n\to\infty} \mathrm{d}R([S_A, S_n)) = 0$. Hence, as $\hat{P}$ satisfies Assumption 3.3(iii), we get

$$\mathbb{E}[\hat{P}_{(T_S^1)_A}] \leq \mathbb{E}[\hat{P}_{S_A}] < \mathbb{E}[\bar{P}_{S_A}] \overset{(86)}{=} \mathbb{E}[\mathbb{E}[P_{(T_S^1)_A}|\mathcal{F}_{S_A}]] = \mathbb{E}[P_{(T_S^1)_A}] \leq \mathbb{E}[\hat{P}_{(T_S^1)_A}],$$

which is the desired contradiction. $\square$

Now we get the following analogue to Theorem 4.4 for the optional case:

THEOREM 4.11 (Optimal control in the optional case). *In the case $p(\eta) = 0$, the value of the optimization problem in* (44) *remains the same when we replace P by $\bar{P}$:*

$$(88) \qquad v := \sup_{C \in \mathcal{C}(c_0)} V(C) = \sup_{C \in \bar{\mathcal{C}}(c_0)} \bar{V}(C),$$

where $\bar{C}(c_0)$ and $\bar{V}(C)$ denote the set of admissible controls and the value when P is replaced by $\bar{P}$ from (85). *Moreover, an optimal optional control for both optimization problems is given by*

$$C_t^{\mathcal{O}} := c_0 \vee \sup_{v \in [0,t]} L_v^{\mathcal{O}}, \quad t \in [0, \infty),$$

with

$$(89) \qquad L_t^{\mathcal{O}} = \begin{cases} 0, & \tilde{P}_t \geq b, |\Delta \tilde{P}_t| > 0, \\ \dfrac{r}{\lambda}(\tilde{P}_t - b), & \tilde{P}_t \geq b, \Delta \tilde{P}_t = 0, \\ \dfrac{r}{\lambda + r}(b - \tilde{P}_t), & \tilde{P}_t < b, |\Delta \tilde{P}_t| > 0, \\ \inf_{\gamma \in (-\infty, 0)} f(\gamma, \tilde{P}_t) < 0, & m\dfrac{\lambda}{r} \leq \tilde{P}_t < b, \Delta \tilde{P}_t = 0, \\ -\infty, & \tilde{P}_t < m\dfrac{\lambda}{r}, \Delta \tilde{P}_t = 0. \end{cases}$$

Here, b is as in Theorem 4.2 *and the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by*

$$(90) \qquad f(\gamma, p) := \frac{(1 - \mathbb{E}[e^{-rT(\gamma)}])p - \mathbb{E}[e^{-rT(\gamma)} \sum_{k=1}^{N_{T(\gamma)}} Y_k]}{\frac{\lambda}{r}(1 - \mathbb{E}[e^{-rT(\gamma)}]) - \mathbb{E}[e^{-rT(\gamma)}]},$$

with

$$(91) \qquad T(\gamma) := \inf\{t \in \{N > 0\} \mid |\Delta \tilde{P}_t| > 0 \text{ and } \tilde{P}_t - \tilde{p} \geq \gamma\}.$$

REMARK 4.12. Notice, that the process $L^{\mathcal{O}}$ of (89) is a solution to (21) and (22) with P replaced by $\bar{P}$, but it is *not necessarily the maximal one* with (23). This is without harm for our claim of optimality because for an application of Theorem 3.9 we can use *any* solution, not necessarily the maximal one.

PROOF. Analogously to Lemma 4.1 one can show that we only have to establish the explicit form of $L^{\mathcal{O}}$ to prove that $C^{\mathcal{O}}$ attains the supremum over $\bar{V}$. Moreover, one can see that, with $L^{\mathcal{O}}$ of the given form, $C^{\mathcal{O}}$ only increases, when $\Delta \tilde{P} > 0$ or $\tilde{P} \geq m\frac{\lambda}{r}$. This shows $V(C^{\mathcal{O}}) = \bar{V}(C^{\mathcal{O}})$, which also establishes (88) since $P \leq \bar{P}$. It thus suffices to establish the stated characterization of $L^{\mathcal{O}}$. For this denote by $\bar{L}^{\mathcal{O}}$ the maximal solution to (21) and (22) (for $\bar{P}$ instead of P) which then satisfies (23), that is,

$$(92) \qquad \bar{L}_S^{\mathcal{O}} = \operatorname*{essinf}_{S < T \in \mathcal{S}} \frac{\bar{P}_S - \mathbb{E}[\bar{P}_T | \mathcal{F}_S]}{\mathbb{E}[dR([S, T)) | \mathcal{F}_S]}, \quad S \in \mathcal{S}.$$

Let $S \in \mathcal{S}$. We will argue now why for any $T \in \mathcal{S}^{\Lambda}$ with $T > S$ we can replace $\mathbb{E}[\bar{P}_T | \mathcal{F}_S]$ by $\mathbb{E}[P_T | \mathcal{F}_S]$ in (92). By definition of $\bar{P}$ we have, that $\mathbb{E}[\bar{P}_T | \mathcal{F}_S] \leq \mathbb{E}[\bar{P}_{\hat{T}} | \mathcal{F}_S]$ for $\hat{T}$ given by $\hat{T} := T_T^1$ (see (87)) on $\{\bar{P}_T > P_T\}$ and $\hat{T} := T$ else. We observe that $dR([S, T)) = dR([S, \hat{T}))$ and $\bar{P}_{\hat{T}} = P_{\hat{T}}$. Hence,

$$\frac{\bar{P}_S - \mathbb{E}[\bar{P}_T | \mathcal{F}_S]}{\mathbb{E}[dR([S, T)) | \mathcal{F}_S]} \geq \frac{\bar{P}_S - \mathbb{E}[\bar{P}_{\hat{T}} | \mathcal{F}_S]}{\mathbb{E}[dR([S, \hat{T})) | \mathcal{F}_S]} = \frac{\bar{P}_S - \mathbb{E}[P_{\hat{T}} | \mathcal{F}_S]}{\mathbb{E}[dR([S, \hat{T})) | \mathcal{F}_S]},$$

which shows our claim. Now, we can establish analogously to Lemma 4.1(ii) that

$$(93) \qquad \bar{L}_t^{\mathcal{O}} = \bar{\ell}(\tilde{P}_t, \Delta N_t), \quad t \in [0, \infty),$$

where $\bar{\ell}(p, \Delta) := \inf_{0 < T \in \mathcal{S}} \bar{\ell}_T(p, \Delta)$ and, for random times $T > 0$,

$$\bar{\ell}_T(p, \Delta) := \frac{\frac{\lambda}{\lambda+r}(p+m)\mathbb{1}_{\{p < m\frac{\lambda}{r}, \Delta=0\}} + p\mathbb{1}_{\{p < m\frac{\lambda}{r}, \Delta=0\}^c} - \mathbb{E}[e^{-rT}]p - \mathbb{E}[e^{-rT}\sum_{k=1}^{N_T} Y_k]}{\mathbb{E}[R_{T-}] + \Delta}$$

with the convention $\frac{\cdot}{0} = \infty$. For $p > m\frac{\lambda}{r}$ or $p < m\frac{\lambda}{r}$ and $\Delta = 1$ one can establish analogous results to Proposition 4.6, Lemma 4.7 and deduce that $\bar{L}^{\mathcal{O}}$ is equal to $L^{\mathcal{O}}$ given in (89) on the set $\{\bar{P} = P\}$. Note that for $\gamma_0 \geq 0$ we have $T(\gamma^0, 0) = T^0$, which explains the explicit form of $L^{\mathcal{O}}$ in the case $\tilde{P} < b$ and $\Delta N > 0$. On $\{\bar{P}_S > P_S\} = \{N_S = N_{S-}\} \cap \{P_S < b\}$ we have by (86) that $\bar{P}_S = \mathbb{E}[P_{T_S^1}|\mathcal{F}_S]$. As $\bar{L}^{\mathcal{O}}$ satisfies (22) we get with the definition of $\bar{P}$, that

$$\mathbb{E}\left[\int_{[T_S^1, \infty)} \frac{\partial}{\partial c}\rho_t\left(\sup_{v \in [S,t]} \bar{L}_v^{\mathcal{O}}\right) dR_t \,\Big|\, \mathcal{F}_S\right] = \mathbb{E}\left[\int_{[S,\infty)} \frac{\partial}{\partial c}\rho_t\left(\sup_{v \in [S,t]} \bar{L}_v^{\mathcal{O}}\right) dR_t \,\Big|\, \mathcal{F}_S\right]$$

$$= \bar{P}_S \overset{(86)}{=} \mathbb{E}[P_{T_S^1}|\mathcal{F}_S] = \mathbb{E}[\bar{P}_{T_S^1}|\mathcal{F}_S]$$

$$= \mathbb{E}\left[\int_{[T_S^1, \infty)} \frac{\partial}{\partial c}\rho_t\left(\sup_{v \in [T_S^1,t]} \bar{L}_v^{\mathcal{O}}\right) dR_t \,\Big|\, \mathcal{F}_S\right],$$

where we have used that $dR([S, T_S^1)) = 0$. Hence, as $dR(\{T_S^1\}) > 0$ on $\{S < \infty\}$ we obtain almost surely $\sup_{v \in [S, T_S^1]} \bar{L}_v^{\mathcal{O}} = \bar{L}_{T_S^1}^{\mathcal{O}}$ on $\{S < \infty\}$. Therefore we can replace $\bar{L}^{\mathcal{O}}$ by $-\infty$ on $\{\bar{P} > P\} = \{N = N_-\} \cap \{P < b\}$ to obtain another solution to (21) and (22), which is exactly $L^{\mathcal{O}}$ of (89). $\square$

In analogy to the predictable case, the next corollary shows how the minimal storage level L^{Λ^η} approaches the minimal storage level under full immediate information $L^{\mathcal{O}}$ when the sensor's probability to fail tends to 0:

COROLLARY 4.13. *In the setting of Theorem 4.4, consider a sequence $(\eta_n)_{n \in \mathbb{N}} \subset [0, \infty]$ such that $\lim_n p(\eta_n) = 0$. Then the solution $L^{\Lambda^{\eta_n}}, n \in \mathbb{N}$, converges to $L^{\mathcal{O}}$ (see Theorem 4.11) for $n \to \infty$:*

$$\lim_{n \to \infty} L_t^{\Lambda^{\eta_n}}(\omega) = L_t^{\mathcal{O}}(\omega), \quad t \in [0, \infty), \omega \in \Omega.$$

PROOF. In the case $\tilde{P}_t(\omega) \geq b$ we have nothing to show. Assume now $\tilde{P}_t(\omega) < b$ and $\Delta N_t(\omega) > 0$. Using (67) and (93) shows $L_t^{\mathcal{O}}(\omega) \leq L_t^{\Lambda^\eta}(\omega)$ for all $\eta \in [0, \infty]$ with $\eta \leq |\Delta \tilde{P}_t(\omega)|$. Moreover, for $n \in \mathbb{N}$ set $T_n^k := (T^0)_{\{|\Delta \tilde{P}_T^0| \geq \eta_n\}} \wedge (T^0 + \frac{1}{k}), k \in \mathbb{N}$, where T^0 is defined in (62). Then we obtain with δ as in (62) that

$$\lim_{n \to \infty} L_t^{\Lambda^{\eta_n}}(\omega) \leq \lim_{n \to \infty} \lim_{k \to \infty} \ell_{T_n^k}(\tilde{P}_t(\omega), 1)$$

$$= \lim_{n \to \infty} \frac{(1-\delta)(\tilde{P}_t(\omega) - b)}{\mathbb{E}[R_{T^0-}] + 1 + \mathbb{E}[e^{-rT^0}\mathbb{1}_{\{|\Delta \tilde{P}_{T_0}| < \eta_n\}}]}$$

$$= \frac{r}{r+\lambda}(\tilde{P}_t(\omega) - b) = L_t^{\mathcal{O}}(\omega).$$

Next, let $m\frac{\lambda}{r} \leq \tilde{P}_t(\omega) < b$ and $\Delta N_t(\omega) = 0$. First, $L_t^{\mathcal{O}}(\omega) \leq L_t^{\Lambda^\eta}(\omega)$ for all $\eta \in [0, \infty)$ follows as before and we claim that for any $\gamma < 0$ we have for f from (90) that

$$(94) \qquad\qquad f(\gamma, p) = \lim_{n \to \infty} f^{\eta_n}(0, \gamma, p).$$

Indeed, one can see that for $\gamma < 0$ we have $T(\gamma) \leq T^{\eta_n}(0, \gamma)$ for any $n \in \mathbb{N}$, where $T(\gamma)$ is given by (91). Moreover, as $\{T(\gamma) < T^{\eta_n}(0, \gamma)\} \subset \{|\Delta \tilde{P}_{T(\gamma)}| < \eta_n\}$ we see that $\lim_{n \to \infty} \mathbb{P}(T^{\eta_n}(0, \gamma) > T(\gamma)) = 0$. The rest follows now by Lebesgue's theorem. As $B_1^{\eta_n}$ of (61) decreases to $-\infty$ for $p(\eta_n)$ going to zero, we can choose η_n small enough to ensure that $\gamma > B_1^{\eta_n}(b - \tilde{P}_t(\omega))$. Therefore, we obtain with (94) that

$$f(\gamma, \tilde{P}_t(\omega)) = \lim_{n \to \infty} f^{\eta_n}(0, \gamma, \tilde{P}_t(\omega)) \geq \lim_{n \to \infty} L_t^{\eta_n}(\omega),$$

which shows $L_t^{\mathcal{O}} = \inf_{\gamma < 0} f(\gamma, \tilde{P}_t(\omega)) \geq \lim_{n \to \infty} L_t^{\eta_n}(\omega)$.

Finally, let $p := \tilde{P}_t(\omega) < m\frac{\lambda}{r}$ and $\Delta N_t(\omega) = 0$. One can see that B_1^η from (61) converges to $-\infty$. Moreover, for $(p(\eta_n))_{n \in \mathbb{N}}$ going to zero and $(\gamma_n)_{n \in \mathbb{N}}$ going to $-\infty$ we get that $(T^{\eta_n}(0, \gamma_n))_{n \in \mathbb{N}}$ is converging to T^1, which denotes the first jump of N. Hence,

$$\lim_{n \to \infty} f^{\eta_n}(0, \gamma_n, p) = \frac{(1 - \mathbb{E}[e^{-rT^1}])p - \mathbb{E}[e^{-rT^1}Y_1]}{\frac{\lambda}{r}(1 - \mathbb{E}[e^{-rT^1}]) - \mathbb{E}[e^{-rT^1}]} = \frac{\frac{r}{\lambda + r}(p - m\frac{\lambda}{r})}{0} = -\infty,$$

where the last equities are meant as a limiting procedure as the nominator converges to a strictly negative number and the denominator to zero. Hence, we obtain our result as $\lim_{n \to \infty} L_t^{\Lambda^{\eta_n}}(\omega) = \lim_{n \to \infty} \inf_{\gamma \in (B_1^{\eta_n}(b-p), 0)} f_0^{\eta_n}(0, \gamma, p) = -\infty$. $\square$

4.4. *Illustration.* For a detailed illustration of Theorem 4.2, Theorem 4.4 and Theorem 4.11, let us fix $\lambda = \frac{1}{2}$, let Y_1 be the bi-modal distribution $\frac{1}{2}(\mathcal{N}(-3, 2) + \mathcal{N}(6, 2))$ and choose $\tilde{p} = -10$ and $r = 1$, which gives $m = 1.5$ and $b \approx 1.37642$.

To obtain L^{Λ^η}, $\eta \in [0, \infty]$, we need to calculate numerically f_Δ^η, $\Delta \in \{0, 1\}$, from (63), which can be done via Monte Carlo (10^5 samples). Figure 1 plots a trajectory for the expected undiscounted reward process $\tilde{P}^\eta$ (gray) with its critical level b (dotted gray) along with the optimal Meyer-measurable controls for $\eta = 3$, $\eta = 6$, predictable ($\eta = \infty$) control, the optional ($\eta = 0$) one, all starting in $c_0 = -12$. Observe first of all that both of the Meyer controls fail to detect some jumps of $\tilde{P}$ (and P) immediately as they happen. The Meyer control with $\eta = 6$ also does not adjust its level after the jump number four, despite the higher rewards obtainable then. So the controller here gambles on her ability to detect future risk shocks in time to benefit then from even higher rewards and the risk reduction. In fact, after the sixth jump of the reward process in this scenario, the accumulated value of the undetected jumps of $\tilde{P}$ is finally high enough to make her adjust her position. The predictable controller with no warnings about jumps can only adopt her position after the reward process has changed. This leads to a left-continuous optimal control in the predictable case. Moreover, in this case the position is increased whenever $\tilde{P}$ reaches a new all-time high, the only exception being the first jump at time $t_1 \approx 0.5$, because the endogenously given starting position $c_0 = -12$ is higher than the minimal position a predictable trader would tolerate, which is approximately $L_{t_1}^{\mathcal{P}} \approx -19.12$. By contrast, the optional controller with perfect sensor can always intervene when rewards increase and she chooses to do so before the risk clock rings whenever the reward process $\tilde{P}$ is below the critical threshold b; with rewards beyond b, it is optimal for her and all the other controllers to only react to jumps after they have happened. One point of special interest is the moment T_b where $\tilde{P}^\eta$ passes the critical value b for the first time. The optional controller intervenes proactively to eliminate all risk ($C_{T_b} = 0$) and is reacting once more after the jump to profit from the newly available high rewards. Note that also the Meyer controllers with $\eta \in \{3, 6\}$ act here in a similar way because the jump in this scenario happens to be larger than their respective detection thresholds and so the controllers become aware of this jump in the moment when it occurs. Hence, one can see that neither the optimal control in the Meyer case nor the optimal control in the optional case is in general left- or right-continuous; they are both just làdlàg, which illustrates the necessity of
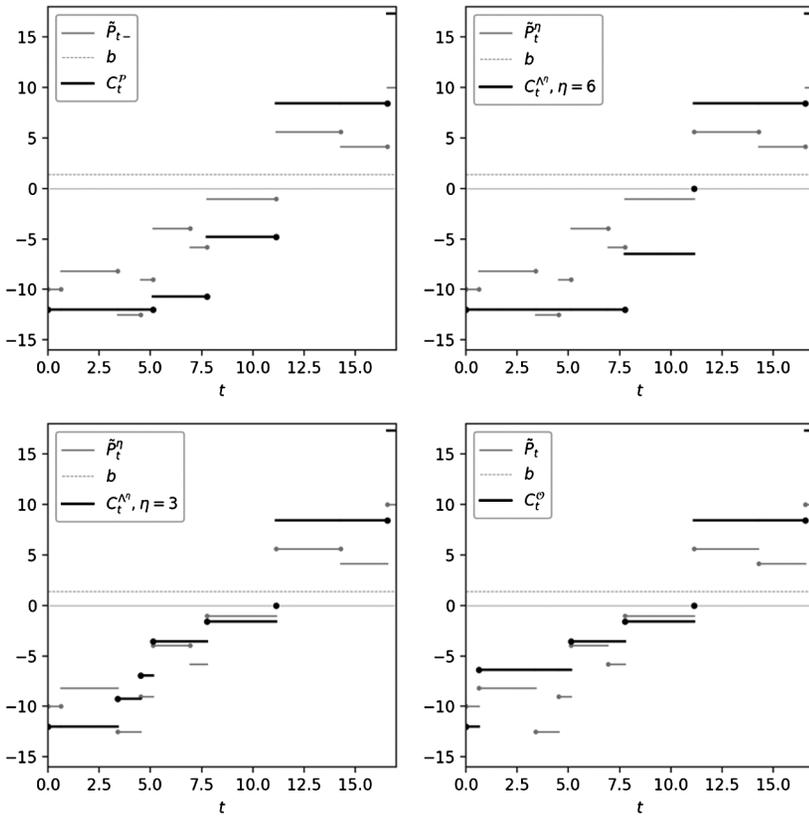
FIG. 1. $\tilde{P}^{\eta}$ (*gray*), b (*dotted gray*) *and optimal controls for* $\eta = 0$ (*optional, black*), $\eta = 3$, $\eta = 6$ (*black*) *and* $\eta = \infty$ (*predictable, black*). *The dots indicate the processes' value at their jump times.*

the general framework chosen in Section 3. Let us observe also that at the time of the fourth jump, both the predictable and the optional controller intervene while one of the Meyer controllers ($\eta = 6$) abstains; hence, the Meyer controls cannot be viewed as simple interpolations between those two extreme cases.

We can use Monte Carlo simulation also to compute the value $v(\eta)$ of an optimal control depending on η for the fixed initial values p and c_0 set above, see Figure 2. Obviously, $v(\eta)$ is decreasing in the detection threshold η, its maximum $v(0) \approx -22$ corresponding to the optional case with perfect sensor considerably exceeding its minimum $v(\infty) \approx -33$ without sensor. In between these extremes, we can see the value function to be concave for "small"
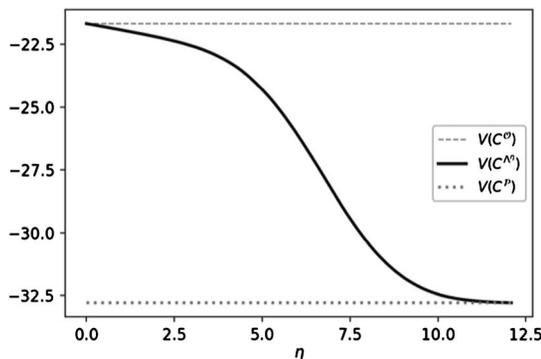

FIG. 2. *Value of an optimal control for* η *between* 0 *and* 12 *in our numerical example.*

and convex for "large" values of η. The switching point between these regimes is around $\eta \approx 6.5$ which marks the detection threshold where small improvements of the sensor will be most effective. By contrast, the same small improvements will have very little effect when η is small (as most jumps will be detected anyhow) and η is large (when only few jumps will be large enough to be detected).

APPENDIX: PROPERTIES OF THE INTEGRAL FOR LÀDLÀG INTEGRATORS

We will prove some results for the *d-integral defined in (13), which are well known for classical integrals and still valid for such *d-integrals. But before doing this we will start with a motivation for our definition of this integral and we close this chapter with a comparison to similar definitions in the literature.

For the rest of this section fix a filtered probability space $(\Omega, \mathbb{F}, \mathcal{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with $\mathbb{F} := \mathcal{F}_\infty := \bigvee_t \mathcal{F}_t$ and $\mathcal{F}$ fulfills the usual conditions. Furthermore, we consider a $\mathbb{P}$-complete Meyer-σ-field Λ (see Definition and Theorem 2.2), such that $\mathcal{P}(\mathcal{F}) \subset \Lambda \subset \mathcal{O}(\mathcal{F})$, where $\mathcal{P}(\mathcal{F})$ and $\mathcal{O}(\mathcal{F})$ denote, respectively, the predictable and the optional σ-field associated with $\mathcal{F}$.

Motivation of *d-integrals. Apart from the approximation argument for the *d-integral in the proof of Proposition 3.6, one can motivate our *d-integral by showing that optimizing our *d-integrals is equal to an optimal stopping problem over divided stopping times. This can be viewed as a version of a result of Bismut (1979) for divided stopping times instead of ordinary stopping times. Divided stopping times have been introduced in El Karoui (1981):

DEFINITION A.1 (El Karoui (1981), Definition 2.37, pages 136–137). A given quadruple $\tau := (T, H^-, H, H^+)$ is called a *divided stopping time*, if T is an $\mathcal{F}$-stopping time and H^-, H, H^+ build a partition of Ω such that:

(i) $H^- \in \mathcal{F}_{T-}$ and $H^- \cap \{T = 0\} = \varnothing$,
(ii) $H \in \mathcal{F}_T^\Lambda$,
(iii) $H^+ \in \mathcal{F}_T^\Lambda$ and $H^+ \cap \{T = \infty\} = \varnothing$,
(iv) T_{H^-} is an $\mathcal{F}$-predictable stopping time,
(v) T_H is a Λ-stopping time.

The set of all divided stopping times τ will be denoted as $\mathcal{S}^{\Lambda, \text{div}}$. For a Λ-measurable positive process Z we define the value attained at a divided stopping time $\tau = (T, H^-, H, H^+)$ as $Z_\tau := {}^*Z_T \mathbb{1}_{H^-} + Z_T \mathbb{1}_H + Z_T^* \mathbb{1}_{H^+}$, where $(\cdot)^*$ is defined in (14) and $^*(\cdot)$ is its analogue where we take the lim sup from the left (see, e.g., Bank and Besslich (2018b), equation (5)).

One key advantage of divided stopping times over ordinary stopping times is that under fairly mild conditions an optimal divided stopping time exists; see El Karoui ((1981), Theorem 2.39, page 138). The following result shows that *d-integrals yield a convex relaxation of optimal stopping over divided stopping times that is analogous to Bismut ((1979), equation (2.1), page 938).

THEOREM A.2. *Denote by $\mathcal{C}$ the set of increasing, Λ-measurable processes C satisfying $C_{0-} := 0$ and $C_t \leq 1, t \in [0, \infty)$. Furthermore, let Z be a Λ-measurable nonnegative process of* class(D^Λ) *with $Z_\infty = 0$. Then*

$$(95) \qquad \sup_{\tau \in \mathcal{S}^{\Lambda, \text{div}}} \mathbb{E}[Z_\tau] = \sup_{C \in \mathcal{C}} \mathbb{E}\left[\int_{[0, \infty)} Z_s \, {}^*dC_s \right],$$

and there exists a divided stopping time $\hat{\tau} := (\hat{T}, \hat{H}^-, \hat{H}, \hat{H}^+)$ attaining the value on the left hand side of (95).

If Z is additionally left-upper-semicontinuous in expectation at every predictable $\mathcal{F}$-stopping time (*see Assumption* 3.3(ii)) *then*

$$\hat{C} := \mathbb{1}_{[\![\hat{\tau},\infty[\![} := \begin{cases} \mathbb{1}_{[\![\hat{T},\infty[\![} & \text{on } \hat{H}^- \cup \hat{H}, \\ \mathbb{1}_{]\!]\hat{T},\infty[\![} & \text{on } (\hat{H}^- \cup \hat{H})^c, \end{cases}$$

will solve the optimal control problem, that is,

$$\sup_{C \in \mathcal{C}} \mathbb{E}\left[\int_{[0,\infty)} Z_s \,^*\mathrm{d}C_s\right] = \mathbb{E}\left[\int_{[0,\infty)} Z_s \,^*\mathrm{d}\hat{C}_s\right].$$

PROOF. By El Karoui ((1981), Theorem 2.39, page 138), there exists an optimal divided stopping time $\hat{\tau} = (\hat{T}, \hat{H}^-, \hat{H}, \hat{H}^+)$ for the left-hand side of (95). By definition $S := \hat{T}_{\hat{H}^-}$ is an $\mathcal{F}$-predictable stopping time. Note that $S > 0$ as $\{\hat{T} = 0\} \cap \hat{H}^- = \varnothing$ by definition of a divided stopping time. Hence, by Bank and Besslich ((2018b), Proposition 3.2(ii)), there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of Λ-stopping times such that $S_n < S$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} Z_{S_n} = {}^*Z_S$. Now define, for $n \in \mathbb{N}$,

$$\hat{C}^n := \big({}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n} \mathbb{1}_{[\![S_n,\hat{T}_{\hat{H}^-}[\![} + \mathbb{1}_{[\![\hat{T}_{\hat{H}^-},\infty[\![}$$
$$+ {}^{\Lambda}(\mathbb{1}_{(\hat{H}^-)^c})_{\hat{T}_{\hat{H}} \wedge S_n} \mathbb{1}_{[\![\hat{T}_{\hat{H}},\infty[\![} + \mathbb{1}_{(\hat{H}^-)^c}\mathbb{1}_{]\!]\hat{T}_{\hat{H}^+},\infty[\![}\big) \wedge 1.$$

Clearly, $\hat{C}^n$ is Λ-measurable taking values in $[0, 1]$; moreover, it is increasing which we verify separately on each part of the partition $\Omega = \hat{H}^- \cup \hat{H} \cup \hat{H}^+$: On the set $\hat{H}^-$, we have

$$(96) \qquad \hat{C}^n = {}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n} \mathbb{1}_{[\![S_n,\hat{T}[\![} + \mathbb{1}_{[\![\hat{T},\infty[\![} \leq 1;$$

on the set $\hat{H}$, we have

$$(97) \qquad \begin{aligned} \hat{C}^n &= {}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n} \mathbb{1}_{[\![S_n,\infty[\![} + {}^{\Lambda}(\mathbb{1}_{(\hat{H}^-)^c})_{\hat{T} \wedge S_n} \mathbb{1}_{[\![\hat{T},\infty[\![} \\ &= {}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n} \mathbb{1}_{[\![S_n,\hat{T}[\![} + {}^{\Lambda}(\mathbb{1}_{(\hat{H}^-)^c})_{\hat{T}} \mathbb{1}_{[\![\hat{T},S_n[\![} + \mathbb{1}_{[\![S_n \vee \hat{T},\infty[\![} \leq 1; \end{aligned}$$

finally, on the set $\hat{H}^+$, we have

$$(98) \qquad \begin{aligned} \hat{C}^n &= \big({}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n} \mathbb{1}_{[\![S_n,\infty[\![} + \mathbb{1}_{(\hat{H}^-)^c}\mathbb{1}_{]\!]\hat{T},\infty[\![}\big) \wedge 1 \\ &= \mathbb{1}_{[\![S_n,\infty[\![\cap]\!]\hat{T},\infty[\![} + {}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n} \mathbb{1}_{[\![S_n,\hat{T}]\!]} + \mathbb{1}_{]\!]\hat{T},S_n[\![}. \end{aligned}$$

It remains to show that

$$(99) \qquad \mathbb{E}[Z_{\hat{\tau}}] = \lim_{n\to\infty} \mathbb{E}\left[\int_{[0,\infty)} Z_s \,^*\mathrm{d}\hat{C}_s^n\right],$$

to conclude "$\leq$" in (95).

To this end, we first calculate with (96), (97) and (98) that

$$\begin{aligned} \int_{[0,\infty)} Z_s \,^*\mathrm{d}\hat{C}_s^n &= \mathbb{1}_{\{\hat{T}=\infty\}} {}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n} Z_{S_n} \\ &\quad + \mathbb{1}_{\{\hat{T}<\infty\}} \mathbb{1}_{\hat{H}^-} \big({}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n} Z_{S_n} + \big(1 - {}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n}\big)Z_{\hat{T}}\big) \\ &\quad + \mathbb{1}_{\{\hat{T}<\infty\}} \mathbb{1}_{\hat{H}\cap\{S_n\leq\hat{T}\}} \big({}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n} Z_{S_n} + \big(1 - {}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n}\big)Z_{\hat{T}}\big) \\ &\quad + \mathbb{1}_{\{\hat{T}<\infty\}} \mathbb{1}_{\hat{H}\cap\{S_n>\hat{T}\}} \big({}^{\Lambda}(\mathbb{1}_{(\hat{H}^-)^c})_{\hat{T}} Z_{\hat{T}} + \big(1 - {}^{\Lambda}(\mathbb{1}_{(\hat{H}^-)^c})_{\hat{T}}\big)Z_{S_n}\big) \\ &\quad + \mathbb{1}_{\hat{H}^+\cap\{S_n\leq\hat{T}\}} \big({}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n} Z_{S_n} + \big(1 - {}^{\Lambda}(\mathbb{1}_{\hat{H}^-})_{S_n}\big)Z_{\hat{T}}^*\big) \\ &\quad + \mathbb{1}_{\hat{H}^+\cap\{S_n>\hat{T}\}} Z_{\hat{T}}^*, \end{aligned}$$

where we have used that $\hat{H}^+ \cap \{\hat{T} = \infty\} = \varnothing$. Hence, from

$$\lim_{n\to\infty} {}^\Lambda(\mathbb{1}_{\hat{H}^-})_{S_n} = \mathbb{E}[\mathbb{1}_{\hat{H}^-} | \mathcal{F}_{S-}] = \mathbb{1}_{\hat{H}^-},$$

$$\lim_{n\to\infty} \mathbb{1}_{\{S_n \le \hat{T}\}} = \mathbb{1}_{\hat{H}^- \cup \{\hat{T} = \infty\}}, \lim_{n\to\infty} \mathbb{1}_{\{S_n > \hat{T}\}} = \mathbb{1}_{(\hat{H}^-)^c \cap \{\hat{T} < \infty\}},$$

we obtain, since $\hat{H}^-, \hat{H}, \hat{H}^+$ form a partition of Ω, that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,\infty)} Z_s \, {}^*\mathrm{d}\hat{C}_s^n\right] = \mathbb{E}[\mathbb{1}_{\hat{H}^-} {}^*Z_{\hat{T}} + \mathbb{1}_{\hat{H}^+} Z_{\hat{T}}^* + \mathbb{1}_{\hat{H}} Z_{\hat{T}}],$$

where we used $(\hat{H}^-)^c =\in \mathcal{F}_{\hat{T}_H}^\Lambda$ and $Z_\infty = 0$. This is our assertion (99).

We show next "$\ge$" in (95). Denote by $\bar{Z}$ El Karoui's Λ-Snell envelope of Z (see Bank and Besslich ((2018b), Definition 2.30), which is làdlàg (see Bank and Besslich ((2018b), Proposition 2.28(ii)), of class(D^Λ) (see Bank and Besslich ((2018b), Proposition 2.31) and by Bank and Besslich ((2018b), Theorem 2.39), we have

$$(100) \qquad \mathbb{E}[\bar{Z}_0] = \sup_{\tau \in \mathcal{S}^{\Lambda,\mathrm{div}}} \mathbb{E}[Z_\tau].$$

Furthermore, we can decompose $\bar{Z} = \bar{M} - \bar{A}$ into a Λ-martingale $\bar{M}$ of class(D^Λ) and an increasing Λ-measurable process $\bar{A}$ with $\bar{A}_0 = 0$ and $\mathbb{E}[\bar{A}_\infty] < \infty$ (see Bank and Besslich (2018b), Proposition 2.26(ii)). By $\bar{Z}_\infty = Z_\infty = 0$, or equivalently, by $\bar{A}_\infty = \bar{M}_\infty$ we obtain for any Λ-stopping time T that $({}^\Lambda\bar{A}_\infty)_T = \mathbb{E}[\bar{A}_\infty | \mathcal{F}_T^\Lambda] = \bar{M}_T$. Hence, for any $C \in \mathcal{C}$ we obtain by using $\bar{Z} = ({}^\Lambda\bar{A}_\infty) - \bar{A}$ and Proposition A.4 below that

$$\mathbb{E}\left[\int_{[0,\infty)} Z_s \, {}^*\mathrm{d}C_s\right] \le \mathbb{E}\left[\int_{[0,\infty)} \bar{Z}_s \, {}^*\mathrm{d}C_s\right] = \mathbb{E}\left[\int_{[0,\infty)} \bar{A}_\infty - \bar{A}_s \, {}^*\mathrm{d}C_s\right] \le \mathbb{E}[\bar{A}_\infty] = \mathbb{E}[\bar{Z}_0],$$

which shows with the help of (100) that also "$\ge$" is satisfied in (95).

If, additionally, Z is left-upper-semicontinuous in expectation at every predictable $\mathcal{F}$-stopping time, we obtain by Bank and Besslich ((2018b), Lemma 3.4(ii)), that ${}^*Z_S \le {}^P Z_S$ at any predictable $\mathcal{F}$-stopping time S. Hence, we can assume without loss of generality that the optimal stopping time $\hat{\tau}$ is of the form $\hat{\tau} = (\hat{T}, \varnothing, H, H^+)$. Then we have

$$\mathbb{E}[Z_{\hat{\tau}}] = \mathbb{E}\left[\int_{[0,\infty)} Z_s \, {}^*\mathrm{d}\hat{C}_s\right],$$

which shows the result by (95). $\quad\square$

*Two classical integration results for the **d*-integral.* Let us first verify that Fubini's theorem is still valid for a specific class of integrands.

PROPOSITION A.3 (Fubini's theorem for *d-Integrals). *Let $A, B : [0, \infty) \to \mathbb{R}$ be two increasing functions with B right-continuous and $A_{0-}, B_{0-} \in \mathbb{R}$. Additionally, let $\phi := \phi_{s,t} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a measurable function, which admits right limits in the first argument $s \in [0, \infty)$ for $t \in [0, \infty)$ fixed and which satisfies*

$$(101) \qquad \int_{[0,\infty)} \sup_{s\in[0,\infty)} |\phi_{s,t}| \, \mathrm{d}B_t < \infty.$$

(i) *If $\phi \ge 0$ or $\int_{[0,\infty)} \int_{[0,s]} |\phi_{s,t}| \, \mathrm{d}B_t \, {}^*\mathrm{d}A_s < \infty$ or $\int_{[0,\infty)} \int_{[t,\infty)} |\phi_{s,t}| \, {}^*\mathrm{d}A_s \, \mathrm{d}B_t < \infty$, then*

$$\int_{[0,\infty)} \int_{[0,s]} \phi_{s,t} \, \mathrm{d}B_t \, {}^*\mathrm{d}A_s = \int_{[0,\infty)} \int_{[t,\infty)} \phi_{s,t} \, {}^*\mathrm{d}A_s \, \mathrm{d}B_t.$$

(ii) *If $\phi \geq 0$ or $\int_{[0,\infty)} \int_{[s,\infty)} |\phi_{s,t}| \, dB_t \, {}^*dA_s < \infty$ or $\int_{[0,\infty)} \int_{[0,t]} |\phi_{s,t}| \, {}^*dA_s \, dB_t < \infty$, then*

$$\int_{[0,\infty)} \int_{[s,\infty)} \phi_{s,t} \, dB_t \, {}^*dA_s = \int_{[0,\infty)} \int_{[0,t]} \phi_{s,t} \, {}^*dA_s \, dB_t.$$

PROOF. Observe that because of (101) we have for any $s \in [0, \infty)$ that

$$(102) \quad \int_{[0,s]} \phi_{s+,t} \, dB_t = \lim_{u \downarrow s} \int_{[0,u]} \phi_{u,t} \, dB_t \quad \text{and} \quad \int_{(s,\infty)} \phi_{s+,t} \, dB_t = \lim_{u \downarrow s} \int_{[u,\infty)} \phi_{u,t} \, dB_t.$$

Now, recalling the notation $A^{c,l}$, A^r from (13), we get by (102) result (i) from the standard Fubini theorem:

$$\int_{[0,\infty)} \int_{[0,s]} \phi_{s,t} \, dB_t \, {}^*dA_s = \int_{[0,\infty)} \int_{[0,s]} \phi_{s,t} \, dB_t \, dA_s^{c,l} + \int_{[0,\infty)} \int_{[0,s]} \phi_{s+,t} \, dB_t \, dA_s^r$$

$$= \int_{[0,\infty)} \int_{[t,\infty)} \phi_{s,t} \, dA_s^{c,l} \, dB_t + \int_{[0,\infty)} \int_{[t,\infty)} \phi_{s+,t} \, dA_s^r \, dB_t$$

$$= \int_{[0,\infty)} \int_{[t,\infty)} \phi_{s,t} \, {}^*dA_s \, dB_t,$$

where we have used the definition (13) in the last step. Analogously we obtain by (102) result (ii). □

Next we show that we can replace a suitable process ϕ inside of the extended integral by the Meyer-projection of this process.

PROPOSITION A.4. *Fix a Λ-measurable increasing process $A : \Omega \times [0, \infty) \to \mathbb{R}$ and an $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable process $\phi : \Omega \times [0,\infty) \to \mathbb{R}$, such that*

$$\mathbb{E}\left[\int_{[0,\infty)} |\phi_t| \, {}^*dA_t \right] < \infty.$$

Then we have that

$$\mathbb{E}\left[\int_{[0,\infty)} \phi_t \, {}^*dA_t \right] \geq \mathbb{E}\left[\int_{[0,\infty)} {}^{\Lambda}\phi_t \, {}^*dA_t \right]$$

with equality if ϕ admits limits from the right or if A is right-continuous.

PROOF. By the integrability assumption we can assume without loss of generality that the process ϕ is bounded. Otherwise we could consider $\phi \wedge M$ and use dominated convergence for Lebesgue integrals to let M tend to ∞ afterwards.

Now we use the idea of the proof of Jacod and Shiryaev ((2003), Lemma 3.12, page 29), and introduce the inverses of $A^{c,l}$ and A^r by

$$T_t^{c,l} := \inf\{s \in [0,\infty) | A_s^{c,l} \geq t\}, \qquad T_t^r := \inf\{s \in [0,\infty) | A_s^r \geq t\}, \quad t \in [0,\infty).$$

As A is Λ-measurable, $T_t^{c,l}$ and T_t^r define $\mathcal{F}$-stopping times. Additionally

$$A_{T_t^{c,l}}^{c,l} \geq t \quad \text{on } \{T_t^{c,l} < \infty\}.$$

Hence, we get by Lenglart ((1980), Corollary 2, page 504), that $T^{c,l}$ is even a Λ-stopping time. Next, we have by definition

$$\mathbb{E}\left[\int_{[0,\infty)} \phi_t \, {}^*dA_t \right] = \mathbb{E}\left[\int_{[0,\infty)} \phi_t \, dA_t^{c,l} \right] + \mathbb{E}\left[\int_{[0,\infty)} \phi_t^* \, dA_t^r \right]$$

and we will show separately

(103)
$$\mathbb{E}\left[\int_{[0,\infty)}\phi_t\,\mathrm{d}A_t^{c,l}\right]=\mathbb{E}\left[\int_{[0,\infty)}{}^{\Lambda}\phi_t\,\mathrm{d}A_t^{c,l}\right],$$

(104)
$$\mathbb{E}\left[\int_{[0,\infty)}\phi_t^*\,\mathrm{d}A_t^{r}\right]\geq\mathbb{E}\left[\int_{[0,\infty)}({}^{\Lambda}\phi)_t^*\,\mathrm{d}A_t^{r}\right],$$

where we have equality in the latter inequality if ϕ admits limits from the right.

For (103) we follow the time-change argument of Jacod and Shiryaev ((2003), Lemma 3.12, page 29), and use Fubini's theorem and the definition of the Λ-projection (see Definition and Theorem 2.11) to obtain

$$\mathbb{E}\left[\int_{[0,\infty)}\phi_t\,\mathrm{d}A_t^{c,l}\right]=\int_{[0,\infty)}\mathbb{E}[\phi_{T_t^{c,l}}\mathbb{1}_{\{T_t^{c,l}<\infty\}}]\,\mathrm{d}t$$

$$=\int_{[0,\infty)}\mathbb{E}[{}^{\Lambda}\phi_{T_t^{c,l}}\mathbb{1}_{\{T_t^{c,l}<\infty\}}]\,\mathrm{d}t=\mathbb{E}\left[\int_{[0,\infty)}{}^{\Lambda}\phi_t\,\mathrm{d}A_t^{c,l}\right].$$

Similarly, we get (104) by

$$\mathbb{E}\left[\int_{[0,\infty)}\phi_t^*\,\mathrm{d}A_t^{r}\right]=\int_{[0,\infty)}\mathbb{E}[{}^{\mathcal{O}}(\phi^*)_{T_t^{r}}\mathbb{1}_{\{T_t^{r}<\infty\}}]\,\mathrm{d}t$$

$$\geq\int_{[0,\infty)}\mathbb{E}[({}^{\Lambda}\phi)_{T_t^{r}}^*\mathbb{1}_{\{T_t^{r}<\infty\}}]\,\mathrm{d}t=\mathbb{E}\left[\int_{[0,\infty)}({}^{\Lambda}\phi)_t^*\,\mathrm{d}A_t^{r}\right].$$

Here, the inequality is due to Bank and Besslich ((2018b), Proposition 3.7), which would give us equality if ϕ admits limits from the right. □

REMARK A.5. Proposition A.3 and Proposition A.4 remain valid with reversed inequality if we consider ${}_*\mathrm{d}$ instead of ${}^*\mathrm{d}$ (see the end of Definition 3.5).

Relation of our *d-integral to similar definitions in the literature. Let us compare our definition of an integral over làdàg integrators to other definitions proposed in Czichowsky and Schachermayer (2016) (following Campi and Schachermayer (2006)), Guasoni, Lépinette and Rásonyi (2012) and Lenglart (1980).

Comparison to Czichowsky and Schachermayer (2016) *and* Guasoni, Lépinette and Rásonyi (2012): In Czichowsky and Schachermayer ((2016), pages 4–5), the integral with respect to a làdlàg process ϕ and a process A of bounded variation with $A_0=A_{0-}$ is defined as

$$\int_{[0,t]}\phi_v\,{}^{cs}\mathrm{d}A_v:=\int_{[0,t]}\phi_v\,\mathrm{d}A_v^c+\sum_{v\in(0,t]}\phi_{v-}\Delta^-A_v+\sum_{v\in[0,t)}\phi_v\Delta^+A_v,\quad t\in[0,\infty),$$

with $\Delta^-A_v:=A_v-A_{v-}$ and $\Delta^+A_v:=A_{v+}-A_v$. Additionally they also define the integral with integrator ϕ in such a way that an integration by parts formula is satisfied. One can then calculate that

$$\int_{[0,t]}\phi_v\,{}^{cs}\mathrm{d}A_v=\int_{[0,t]}\phi_u\,\mathrm{d}A_{u-}-\sum_{0<u\leq t}\Delta^-\phi_u\Delta^-A_u$$

and the right-hand side is actually the definition used in Guasoni, Lépinette and Rásonyi ((2012), Definition A.6, page 766), apart from the fact that Guasoni, Lépinette and Rásonyi (2012) additionally assume that ϕ has to be right-continuous. For right-continuous ϕ, we get

$$\int_{[0,t]}\phi_v\,{}^{cs}\mathrm{d}A_v=\int_{[0,t]}\phi_{v-}\,{}^*\mathrm{d}A_v,$$

which also shows the main difference between the integral definition of Czichowsky and Schachermayer (2016), Guasoni, Lépinette and Rásonyi (2012) and our definition: In the definition of Czichowsky and Schachermayer (2016) the process A is integrated against the "previous" values of ϕ. Our definition differs to be suitable for the use in our irreversible investment problem with inventory risk and to connect the target functional $\widetilde{V}$ to the relaxed one V (see Proposition 3.6).

Comparison to Lenglart (1980): As Lenglart (1980) lays the foundation for Meyer-σ-fields we also want to mention the special integral proposed in that article: For a measurable and locally bounded process ϕ and a process A of bounded variation, Lenglart (1980) lets

$$\int_{[0,t]} \phi_v \,^{\mathrm{L}}\mathrm{d}A_v := \int_{[0,t]} \phi_v \,\mathrm{d}A_{v+} - \phi_t \Delta^+ A_t$$

and we have

$$\int_{[0,t]} \phi_v \,^{\mathrm{L}}\mathrm{d}A_v = \int_{[0,t]} \phi_v \,^*\mathrm{d}A_v - \sum_{v<t}(\phi_v^* - \phi_v)\Delta^+ A_v.$$

The latter equation shows that the main difference to our *d-integral results from a different treatment of the jumps $\Delta^+ A$.

REFERENCES

AL MOTAIRI, H. and ZERVOS, M. (2017). Irreversible capital accumulation with economic impact. *Appl. Math. Optim.* **75** 525–551. MR3641903 https://doi.org/10.1007/s00245-016-9341-9

ARROW, K. J. (1966). Optimal capital policy with irreversible investment. Technical report, Institute for Mathematical Studies in the Social Sciences, Stanford Univ.

BANK, P. and BESSLICH, D. (2018a). On a stochastic representation theorem for Meyer-measurable processes and its applications in stochastic optimal control and optimal stopping. Preprint. Available at arXiv:1810.08491.

BANK, P. and BESSLICH, D. (2018b). On Lenglart's theory of Meyer-sigma-fields and El Karoui's theory of optimal stopping. Preprint. Available at arXiv:1810.08485.

BANK, P. and EL KAROUI, N. (2004). A stochastic representation theorem with applications to optimization and obstacle problems. *Ann. Probab.* **32** 1030–1067. MR2044673 https://doi.org/10.1214/aop/1079021471

BANK, P. and RIEDEL, F. (2001). Optimal consumption choice with intertemporal substitution. *Ann. Appl. Probab.* **11** 750–788. MR1865023 https://doi.org/10.1214/aoap/1015345348

BERTOLA, G. (1998). Irreversible investment. *Res. Econ.* **52** 3–37. https://doi.org/10.1006/reec.1997.0153

BISMUT, J.-M. (1979). Temps d'arrêt optimal, quasi-temps d'arrêt et retournement du temps. *Ann. Probab.* **7** 933–964. MR0548890

BISMUT, J.-M. and SKALLI, B. (1977). Temps d'arrêt optimal, théorie générale des processus et processus de Markov. *Z. Wahrsch. Verw. Gebiete* **39** 301–313. MR0651550 https://doi.org/10.1007/BF01877497

CAMPI, L. and SCHACHERMAYER, W. (2006). A super-replication theorem in Kabanov's model of transaction costs. *Finance Stoch.* **10** 579–596. MR2276320 https://doi.org/10.1007/s00780-006-0022-4

COURRÈGE, P. and PRIOURET, P. (1965). Temps d'arrêt d'une fonction aléatoire: Relations d'équivalence associées et propriétés de décomposition. *Publ. Inst. Stat. Univ. Paris* **14** 245–274. MR0221588

CZICHOWSKY, C. and SCHACHERMAYER, W. (2016). Duality theory for portfolio optimisation under transaction costs. *Ann. Appl. Probab.* **26** 1888–1941. MR3513609 https://doi.org/10.1214/15-AAP1136

DE ANGELIS, T., FEDERICO, S. and FERRARI, G. (2017). Optimal boundary surface for irreversible investment with stochastic costs. *Math. Oper. Res.* **42** 1135–1161. MR3722429 https://doi.org/10.1287/moor.2016.0841

DELLACHERIE, C. and MEYER, P.-A. (1978). *Probabilities and Potential*. North-Holland Mathematics Studies **29**. North-Holland, Amsterdam. MR0521810

DELLACHERIE, C. and MEYER, P.-A. (1982). *Probabilities and Potential. B*: Theory of Martingales. North-Holland Mathematics Studies **72**. North-Holland, Amsterdam. MR0745449

DIXIT, A. K. and PINDYCK, R. S. (1994). *Investment Under Uncertainty*. Princeton Univ. Press, Princeton, NJ.

EL KAROUI, N. (1981). Les aspects probabilistes du contrôle stochastique. In *Ninth Saint Flour Probability Summer School—1979* (*Saint Flour*, 1979). *Lecture Notes in Math.* **876** 73–238. Springer, Berlin. MR0637471

FEDERICO, S. and PHAM, H. (2014). Characterization of the optimal boundaries in reversible investment problems. *SIAM J. Control Optim.* **52** 2180–2223. MR3231230 https://doi.org/10.1137/130927814

FERRARI, G. (2015). On an integral equation for the free-boundary of stochastic, irreversible investment problems. *Ann. Appl. Probab.* **25** 150–176. MR3297769 https://doi.org/10.1214/13-AAP991

GUASONI, P., LÉPINETTE, E. and RÁSONYI, M. (2012). The fundamental theorem of asset pricing under transaction costs. *Finance Stoch.* **16** 741–777. MR2972240 https://doi.org/10.1007/s00780-012-0185-0

JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877 https://doi.org/10.1007/978-3-662-05265-5

KARATZAS, I. and SHREVE, S. E. (1998). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

LENGLART, E. (1980). Tribus de Meyer et théorie des processus. In *Seminar on Probability, XIV* (*Paris*, 1978/1979) (*French*). *Lecture Notes in Math.* **784** 500–546. Springer, Berlin. MR0580151

MERHI, A. and ZERVOS, M. (2007). A model for reversible investment capacity expansion. *SIAM J. Control Optim.* **46** 839–876. MR2338430 https://doi.org/10.1137/050640758

RIEDEL, F. and SU, X. (2011). On irreversible investment. *Finance Stoch.* **15** 607–633. MR2863636 https://doi.org/10.1007/s00780-010-0131-y