

SAMPLE PATH LARGE DEVIATIONS FOR LÉVY PROCESSES AND RANDOM WALKS WITH WEIBULL INCREMENTS

BY MIHAIL BAZHBA^{1,*}, JOSE BLANCHET², CHANG-HAN RHEE³ AND BERT ZWART^{1,†}

¹Centrum Wiskunde & Informatica, *bazhba@cwi.nl; †bert.zwart@cwi.nl

²Department of Management Science and Engineering, Stanford University, jblanche@stanford.edu

³Department of Industrial Engineering and Management Sciences, Northwestern University,
chang-han.rhee@northwestern.edu

We study sample path large deviations for Lévy processes and random walks with heavy-tailed jump-size distributions that are of Weibull type. The sharpness and applicability of these results are illustrated by a counterexample proving the nonexistence of a full LDP in the J_1 topology, and by an application to a first passage problem.

1. Introduction. In this paper, we develop sample path large deviations for Lévy processes and random walks, assuming the jump sizes have a semiexponential distribution. Specifically, let $X(t), t \geq 0$, be a centered Lévy process with positive jumps and Lévy measure ν which has nonnegative support. Assume that $-\log \nu[x, \infty)$ is regularly varying of index $\alpha \in (0, 1)$ and define $\bar{X}_n = \{\bar{X}_n(t), t \in [0, 1]\}$, with $\bar{X}_n(t) = X(nt)/n$. We are interested in large deviations of \bar{X}_n .

The investigation of tail estimates of the one-dimensional distributions of \bar{X}_n (or random walks with heavy-tailed step size distribution) was initiated in Nagaev (1969, 1977). The state of the art of such results is well summarized in Borovkov and Borovkov (2008), Denisov, Dieker and Shneer (2008), Embrechts, Klüppelberg and Mikosch (1997), Foss, Korshunov and Zachary (2011). In particular, Denisov, Dieker and Shneer (2008) describe in detail how fast x needs to grow with n for the asymptotic relation

$$(1.1) \quad \mathbf{P}(X(n) > x) = n\mathbf{P}(X(1) > x)(1 + o(1))$$

to hold, as $n \rightarrow \infty$. If (1.1) is valid, the so-called *principle of one big jump* is said to hold. It turns out that, if x increases linearly with n , this principle holds if $\alpha < 1/2$ and does not hold if $\alpha > 1/2$, and the asymptotic behavior of $\mathbf{P}(X(n) > x)$ becomes more complicated. When studying more general functionals of X it becomes natural to consider logarithmic asymptotics, as is common in large deviations theory; cf. Dembo and Zeitouni (2010), Gantert (1998), Gantert, Ramanan and Rembart (2014).

The study of large deviations of sample paths of processes with Weibullian increments is relatively limited. The only paper we are aware of is Gantert (1998), where the inverse contraction principle is applied to obtain a large deviation principle in the L_1 topology. As noted in Duffy and Sapozhnikov (2008) this topology is not suitable for many applications—ideally one would like to work with the J_1 topology; see equation (2.2).

Let us now describe precisely our results. We first develop an extended LDP (large deviations principle) in the J_1 topology, that is, there exists a rate function $I(\cdot)$ such that

$$(1.2) \quad \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{L(n)n^\alpha} \geq - \inf_{x \in A} I(x),$$

Received November 2017; revised November 2019.

MSC2020 subject classifications. Primary 60F10; secondary 60G17.

Key words and phrases. Sample path large deviations, Lévy processes, random walks, heavy tails.

if A is open, and

$$(1.3) \quad \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{L(n)n^\alpha} \leq -\liminf_{\epsilon \downarrow 0} \inf_{x \in A^\epsilon} I(x),$$

if A is closed. Here $A^\epsilon = \{x : d(x, A) \leq \epsilon\}$. The rate function I is given by

$$I(\xi) = \begin{cases} \sum_{t:\xi(t) \neq \xi(t^-)} (\xi(t) - \xi(t^-))^\alpha & \text{if } \xi \in \mathbb{D}_{\leq \infty}[0, 1], \\ \infty & \text{otherwise,} \end{cases}$$

where $\mathbb{D}_{\leq \infty}[0, 1]$ is the subspace of $\mathbb{D}[0, 1]$ consisting of nondecreasing pure jump functions vanishing at the origin and continuous at 1. (As usual, $\mathbb{D}[0, 1]$ is the space of càdlàg functions from $[0, 1]$ to \mathbb{R} .)

The notion of an extended large deviations principle has been introduced by Borovkov and Mogul’skiĭ (2010). We derive this result as follows: we use a suitable representation for the Lévy process in terms of Poisson random measures, allowing us to decompose the process into the contribution generated by the k largest jumps, and the remainder. The contribution generated by the k largest jumps is a step function for which we obtain the large deviations behavior by Bryc’s inverse Varadhan lemma (see, e.g., Theorem 4.4.13 of Dembo and Zeitouni (2010)). The remainder term is tamed by modifying a concentration bound due to Jelenković and Momčilović (2003).

To combine both estimates we need to consider the ϵ -fattening A^ϵ of the set A , which precludes us from obtaining a full LDP. To show that our approach cannot be improved, we construct a set A that is closed in the Skorokhod J_1 topology for which the large deviation upper bound does not hold. In this sense, our extended large deviations principle can be seen as optimal. This is in line with the observation made for the regularly varying Lévy processes and random walks Rhee, Blanchet and Zwart (2019), for which the full LDP w.r.t. J_1 topology in a classical sense is shown to be unobtainable as well.

Following a similar proof strategy, we also derive an extended sample path LDP for random walks in $\mathbb{D}[0, 1]$. There are however also some differences. The distributional representation of our random walk is different from the continuous-time case. More importantly, the resulting rate function differs at time 1, since the rescaled random walk always has a jump at time 1. We present an exact formulation of this result in Theorem 2.2.

We derive several implications of our extended LDP that facilitate its use in applications. First of all, if a Lipschitz functional ϕ of \bar{X}_n is chosen for which the function $I_\phi(y) = \inf_{x:\phi(x)=y} I(x)$ is a good rate function, then $\phi(X_n)$ satisfies an LDP. We illustrate this procedure by considering an example concerning the probability of ruin for an insurance company where large claims are reinsured.

A second implication is a sample path LDP in the M'_1 topology. We show that the rate function I is good in this topology, allowing us to conclude $\lim_{\epsilon \downarrow 0} \inf_{x \in A^\epsilon} I(x) = \inf_{x \in A} I(x)$, if A is closed in the M'_1 topology. An overview of the M'_1 topology on $\mathbb{D}[0, 1]$ can be seen in Appendix A; for related results on the M'_1 topology in $\mathbb{D}[0, \infty)$ we refer to Puhalskii and Whitt (1997). The LDP for the M'_1 topology in this paper is applied in Bazhba et al. (2019) to obtain tail asymptotics of the queue length in the multiple server queue with heavy tailed Weibull service times. In that work, we show that the most likely number of big jobs that cause a large queue length is determined by the solution of an L^α -norm minimization problem for $\alpha \in (0, 1)$. This result essentially answers a question posed by Sergey Foss at the Erlang Centennial conference in 2009. For earlier conjectures on this problem we refer to Whitt (2000).

We note that both implications of our extended LDP constitute two complementary tools, and that the two aforementioned applications are dealt with using precisely one of these tools.

In particular, the functional in the reinsurance example is not continuous in the M'_1 topology, and the most likely paths in the queueing application are discontinuous at time 0, rendering the J_1 or M_1 topologies useless.

Another application of our results, which will be pursued in detail elsewhere, arises in the large deviations analysis of Markov random walks. More precisely, set $\bar{Z}_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} f(Y_k)/n$, where $Y_k, k \geq 0$ is a geometrically ergodic Markov chain and $f(\cdot)$ is a given measurable function. Classical large deviations results pioneered by Donsker and Varadhan on this topic (see, e.g., Donsker and Varadhan (1976)) and the more recent treatment in Kontoyiannis and Meyn (2005), impose certain Lyapunov-type assumptions involving the underlying function $f(\cdot)$.

These assumptions are not merely technical requirements, but are needed for a large deviations theory with a linear (in n) speed function (as opposed to sublinear as we obtain here). Even in simple cases (e.g., Blanchet, Glynn and Meyn (2013), Duffy and Meyn (2014)) the case of unbounded $f(\cdot)$ can result in a sublinear large deviations scaling of the type considered here. For Harris chains, this can be seen by splitting $\bar{Z}_n(\cdot)$ into cycles. Each term corresponding to a cycle can be seen as the area under a curve generated by $f(Y)$. For linear f , this results in a contribution towards $\bar{Z}_n(\cdot)$ which often is roughly proportional to the square of the cycle. Hence, the behavior of $\bar{Z}_n(1)$ is close to that of a sum of i.i.d. Weibull-type random variables. To summarize, the main results of this paper can be applied to significantly extend the classical Donsker–Varadhan theory to unbounded functionals of Markov chains.

This paper is organized as follows. Section 2 introduces notation and presents extended LDP’s. These are complemented in Section 3 by LDP’s of Lipschitz functionals, LDP’s in the M'_1 topology, and a counterexample. Section 4 is considering an application to boundary crossing probabilities with moderate jumps. Additional proofs are presented in Section 5. The Appendix develops further details about the M'_1 topology that are needed in the body of the paper.

2. Sample path LDPs. In this section, we discuss sample path large deviations for Lévy processes and random walks. Before presenting our main results, we start with general background. Let (\mathbb{S}, d) be a metric space, and \mathcal{T} denote the topology induced by the metric d . Let X_n be a sequence of \mathbb{S} -valued random variables. Let $A^\epsilon \triangleq \{\xi \in \mathbb{S} : d(\xi, A) \leq \epsilon\}$ where $d(\xi, A) = \inf_{\zeta \in A} d(\xi, \zeta)$, and $A^{-\epsilon} \triangleq \{\xi \in \mathbb{S} : d(\xi, \zeta) \leq \epsilon \text{ implies } \zeta \in A\}$. Let I be a nonnegative lower semicontinuous function on \mathbb{S} , and (a_n) be a sequence of positive real numbers that tends to infinity as $n \rightarrow \infty$. We say that X_n satisfies the LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and the rate function I if

$$-\inf_{x \in A^\circ} I(x) \leq \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq -\inf_{x \in A^-} I(x)$$

for any measurable set A . Here, A° and A^- are respectively the interior and the closure of the set A . We say that X_n satisfies the *extended* LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and rate function I if

$$-\inf_{x \in A^\circ} I(x) \leq \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq -\lim_{\epsilon \rightarrow 0} \inf_{x \in A^\epsilon} I(x)$$

for any measurable set A . The next proposition provides the key framework for proving our main results.

PROPOSITION 2.1. *Let I and $I_k, k \geq 1$ be rate functions. Suppose that for each n, X_n has a sequence of approximations $\{Y_n^k\}_{k=1, \dots}$ such that:*

- (i) *For each k, Y_n^k satisfies the extended LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and rate function I_k .*

(ii) For each closed set F ,

$$\lim_{k \rightarrow \infty} \inf_{x \in F} I_k(x) \geq \inf_{x \in F} I(x).$$

(iii) For each $\delta > 0$ and each open set G , there exist $\epsilon > 0$ and $K \geq 0$ such that $k \geq K$ implies

$$\inf_{x \in G^{-\epsilon}} I_k(x) \leq \inf_{x \in G} I(x) + \delta.$$

(iv) For every $\epsilon > 0$ it holds that

$$(2.1) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{a_n} \log \mathbf{P}(d(X_n, Y_n^k) > \epsilon) = -\infty.$$

Then, X_n satisfies the extended LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and rate function I .

The proof of this proposition is provided in Section 5. We conclude this section with an immediate implication of Proposition 2.1.

COROLLARY 2.1. *Suppose that Y_n satisfies the extended LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and rate function I . If for each $\epsilon > 0$,*

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n} \log \mathbf{P}(d(X_n, Y_n) > \epsilon) = -\infty,$$

then X_n satisfies the extended LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and rate function I .

PROOF. Let $Y_n^k \triangleq Y_n$ and $I_k \triangleq I$ for $k = 1, 2, \dots$. Then, (i) and (ii) of Proposition 2.1 are trivially satisfied. For (iii), we note that by the definition of $G^{-\epsilon}$, for each $\delta > 0$ and G an open set, there exists $\epsilon > 0$ such that

$$\inf_{x \in G^{-\epsilon}} I(x) \leq \inf_{x \in G} I(x) + \delta,$$

and hence, (iii) are satisfied for $I_k = I$. Since (iv) is also satisfied by the assumption, all the conditions of Proposition 2.1 are satisfied and the conclusion of the corollary follows. \square

2.1. Extended sample path LDP for Lévy processes. Throughout the rest of this paper we assume that:

A1. X is a real-valued Lévy process with Lévy measure ν which has nonnegative support satisfying $\nu[x, \infty) = \exp(-L(x)x^\alpha)$ where $\alpha \in (0, 1)$ and $L(\cdot)$ is slowly varying at infinity.

A2. The mapping $x \mapsto L(x)x^{\alpha-1}$ is nonincreasing for sufficiently large x .

Let $\bar{X}_n(t), t \in [0, 1]$, denote the centered and scaled process:

$$\bar{X}_n(t) \triangleq \frac{1}{n} X(nt) - t \mathbf{E}X(1).$$

Let $\mathbb{D}[0, 1]$ denote the Skorokhod space—space of càdlàg functions from $[0, 1]$ to \mathbb{R} —and \mathcal{T}_{J_1} denote the J_1 Skorokhod topology on $\mathbb{D}[0, 1]$. That is, $\mathbb{D}[0, 1]$ is metrized by the usual Skorokhod metric

$$(2.2) \quad d_{J_1}(\xi, \zeta) \triangleq \inf_{\lambda \in \Lambda} \{ \max \{ \|\xi(t) - \zeta(\lambda(t))\|_\infty, \|\lambda(t) - e(t)\|_\infty \} \},$$

where $\xi, \zeta \in \mathbb{D}[0, 1]$, λ is a nondecreasing homeomorphism of $[0, 1]$ onto itself, Λ is the set of such homeomorphisms, $e(t) = t$ is the identity map, and $\|\xi\|_\infty = \sup_{t \in [0, 1]} |\xi(t)|$ is the supremum norm. We say that $\xi \in \mathbb{D}[0, 1]$ is a pure jump function if $\xi = \sum_{i=1}^\infty x_i \mathbb{1}_{[u_i, 1]}$ for

some x_i 's and u_i 's such that $x_i \in \mathbb{R}$ and $u_i \in (0, 1)$ for each i and u_i 's are all distinct. Let $\mathbb{D}_{\leq\infty}[0, 1]$ denote the subspace of $\mathbb{D}[0, 1]$ consisting of nondecreasing pure jump functions vanishing at the origin and continuous at 1. For the rest of the paper, if there is no confusion regarding the domain of a function space, we will omit the domain and simply write, for example, $\mathbb{D}_{\leq\infty}$ instead of $\mathbb{D}_{\leq\infty}[0, 1]$. The next theorem is the main result of this paper.

THEOREM 2.1. \bar{X}_n satisfies the extended large deviation principle in $(\mathbb{D}, \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and rate function

$$(2.3) \quad I(\xi) = \begin{cases} \sum_{t:\xi(t) \neq \xi(t-)} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \mathbb{D}_{\leq\infty}, \\ \infty & \text{otherwise.} \end{cases}$$

That is, for any measurable A ,

$$(2.4) \quad - \inf_{\xi \in A^\epsilon} I(\xi) \leq \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{L(n)n^\alpha} \leq \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{L(n)n^\alpha} \leq - \lim_{\epsilon \rightarrow 0} \inf_{\xi \in A^\epsilon} I(\xi),$$

where $A^\epsilon \triangleq \{\xi \in \mathbb{D} : d_{J_1}(\xi, \zeta) \leq \epsilon \text{ for some } \zeta \in A\}$.

REMARK 2.1. Note that it is straightforward to extend Theorem 2.1 to spectrally two-sided Lévy processes. For instance, suppose that the Lévy measure ν of the process X has Weibull tail $\nu[x, \infty) = \exp(-L(x)x^\alpha)$ where $\alpha \in (0, 1)$, $L(x)x^{\alpha-1}$ satisfies Assumption A2, and $\nu(-\infty, x]$ is light-tailed. We can decompose \bar{X}_n into a centered spectrally positive part \bar{Y}_n and a centered spectrally negative part $\bar{X}_n - \bar{Y}_n$. Then, \bar{Y}_n satisfies the extended LDP in Theorem 2.1. On the other hand, observe that

$$\mathbf{P}(d(\bar{X}_n, \bar{Y}_n) > \epsilon) \leq \mathbf{P}(\|\bar{X}_n - \bar{Y}_n\|_\infty > \epsilon) \leq 3\mathbf{P}(|\bar{X}_n(1) - \bar{Y}_n(1)| > \epsilon/3),$$

where we used Etemadi's inequality for Lévy processes (see, e.g., Rhee, Blanchet and Zwart (2019), Lemma A.4) in the last step. Since $\bar{X}_n - \bar{Y}_n$ is light-tailed, the latter probability vanishes at exponential rate due to Cramèrs theorem. This allows one to apply Corollary 2.1 with Y_n and conclude that \bar{X}_n satisfies the same LDP as the one in Theorem 2.1.

Recall that $X_n(\cdot) \triangleq X(n\cdot)$ has Itô representation

$$(2.5) \quad X_n(s) = ns a + B(ns) + \int_{x < 1} x \hat{N}([0, ns] \times dx) - ns \nu(dx) + \int_{x \geq 1} x \hat{N}([0, ns] \times dx),$$

with a a drift parameter, B a Brownian motion and \hat{N} a Poisson random measure with mean measure $\text{Leb} \times \nu$ on $[0, n] \times (0, \infty)$; Leb here denotes the Lebesgue measure. All terms in (2.5) are independent. We will see that the large deviation behavior is dominated by the last term of (2.5). It turns out to be convenient to consider the following distributional representation of the centered and scaled version of the last term:

$$\bar{Y}_n(\cdot) \triangleq \frac{1}{n} \sum_{l=1}^{N(n\cdot)} (Z_l - \mathbf{E}Z) \stackrel{\mathcal{D}}{=} \frac{1}{n} \int_{x \geq 1} x \hat{N}([0, n\cdot] \times dx) - \frac{1}{n} (\mathbf{E}Z) \hat{N}([0, n\cdot] \times [1, \infty)),$$

where $N(t) \triangleq \hat{N}([0, t] \times [1, \infty))$ is a Poisson process with arrival rate $\nu_1 \triangleq \nu[1, \infty)$, and the Z_i 's are i.i.d. copies of Z such that $\mathbf{P}(Z \geq x) = \nu[x \vee 1, \infty)/\nu_1$, independent of N . To facilitate the proof of Theorem 2.1, we consider a further decomposition of \bar{Y}_n into two pieces, one of which consists of the big increments, and the other one keeps the residual fluctuations. To be more specific, we introduce an extra notation for the rank of the increments. Given $N(n)$, define $\mathbf{S}_{N(n)}$ to be the set of all permutations of $\{1, \dots, N(n)\}$. Let

$R_n : \{1, \dots, N(n)\} \rightarrow \{1, \dots, N(n)\}$ be a random permutation of $\{1, \dots, N(n)\}$ sampled uniformly from $\Sigma_n \triangleq \{\sigma \in \mathbf{S}_{N(n)} : Z_{\sigma^{-1}(1)} \geq \dots \geq Z_{\sigma^{-1}(N(n))}\}$. In words, $R_n(i)$ is the rank of Z_i among $\{Z_1, \dots, Z_{N(n)}\}$ when sorted in decreasing order with the ties broken uniformly. Now, we see that

$$\bar{Y}_n(t) = \underbrace{\frac{1}{n} \sum_{i=1}^{N(nt)} Z_i \mathbb{1}_{\{R_n(i) \leq k\}}}_{\triangleq \bar{J}_n^k(t)} + \underbrace{\frac{1}{n} \sum_{i=1}^{N(nt)} (Z_i \mathbb{1}_{\{R_n(i) > k\}} - \mathbf{E}Z)}_{\triangleq \bar{H}_n^k(t)}.$$

The proof of Theorem 2.1 is straightforward once the following technical lemmas are in our hands; their proofs are provided in Section 5. Let $\mathbb{D}_{\leq k}$ denote the subspace of $\mathbb{D}_{\leq \infty}$ consisting of paths that have less than or equal to k discontinuities.

LEMMA 2.1. *For each fixed k , \bar{J}_n^k satisfies the LDP in $(\mathbb{D}, \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and rate function*

$$(2.6) \quad I_k(\xi) = \begin{cases} \sum_{t \in [0,1]} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \mathbb{D}_{\leq k}, \\ \infty & \text{otherwise.} \end{cases}$$

Recall that $A^{-\epsilon} \triangleq \{\xi \in \mathbb{D} : d_{J_1}(\xi, \zeta) \leq \epsilon \text{ implies } \zeta \in A\}$.

LEMMA 2.2. *For each $\delta > 0$ and each open set G , there exist $\epsilon > 0$ and $K \geq 0$ such that for any $k \geq K$*

$$(2.7) \quad \inf_{\xi \in G^{-\epsilon}} I_k(\xi) \leq \inf_{\xi \in G} I(\xi) + \delta.$$

Let $B_{J_1}(\xi, \epsilon)$ be the open ball w.r.t. the J_1 Skorokhod metric centered at ξ with radius ϵ and $B_\epsilon \triangleq B_{J_1}(0, \epsilon)$.

LEMMA 2.3. *For every $\epsilon > 0$ it holds that*

$$(2.8) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\|\bar{H}_n^k\|_\infty > \epsilon) = -\infty.$$

PROOF OF THEOREM 2.1. For this proof, we use the following representation of \bar{X}_n :

$$(2.9) \quad \bar{X}_n \stackrel{\mathcal{D}}{=} \bar{Y}_n + \bar{R}_n = \bar{J}_n^k + \bar{H}_n^k + \bar{R}_n,$$

where $\bar{R}_n(s) = \frac{1}{n}B(ns) + \frac{1}{n} \int_{|x| \leq 1} x[N([0, ns] \times dx) - ns\nu(dx)] + \frac{1}{n}(\mathbf{E}Z)\hat{N}([0, n\cdot] \times [1, \infty)) - \nu_1 s$. Next, we verify the conditions of Proposition 2.1. Lemma 5.1 confirms that I is lower semicontinuous. Lemma 2.1 verifies (i). To see that (ii) is satisfied, note that $I_k(\xi) \geq I(\xi)$ for any $\xi \in \mathbb{D}$. Lemma 2.2 verifies (iii). Since $d_{J_1}(\bar{X}_n, \bar{J}_n^k) \leq \|\bar{H}_n^k\|_\infty + \|\bar{R}_n\|_\infty$, Lemma 2.3 and $\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\|\bar{R}_n\|_\infty > \epsilon) = -\infty$ implies (iv). Note that \bar{R}_n is a Lévy process whose moment generating function is finite everywhere, and hence, the LDP upper bound in Theorem 2.5 of Mogul'skii (1993) applies (in particular, his statement about the space D_c) to $\mathbf{P}(d_{J_1}(0, \bar{R}_n) > \epsilon)$. This, in turn, implies that $\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\|\bar{R}_n\|_\infty > \epsilon) = -\infty$. Now, the conclusion of the theorem follows from Proposition 2.1. \square

2.2. *Extended LDP for random walks.* Let $S_n \triangleq Z_1 + \dots + Z_n$ where the Z_i 's are non-negative random variables. Consider the centered and scaled process $\bar{S}_n = \{\bar{S}_n(t), t \in [0, 1]\}$ where $\bar{S}_n(t) \triangleq \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} (Z_i - \mathbf{E}Z)$, $t \in [0, 1]$. We assume that $\mathbf{P}(Z \geq x) = \exp(-L(x)x^\alpha)$ where $\alpha \in (0, 1)$ and $L(\cdot)$ is a slowly varying function. We also assume A2 is in force as in Section 2.1. The goal of this section is to prove an extended LDP for \bar{S}_n . Let $\tilde{\mathbb{D}}_{\leq \infty}$ denote the subspace of \mathbb{D} consisting of nondecreasing pure jump functions vanishing at the origin (not necessarily continuous at 1, though). Let $\tilde{\mathbb{D}}_{\leq k}$ denote the subspace of $\tilde{\mathbb{D}}_{\leq \infty}$ consisting of paths that have less than or equal to k discontinuities. Define \tilde{I} as follows:

$$\tilde{I}(\xi) = \begin{cases} \sum_{t:\xi(t) \neq \xi(t-)} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \tilde{\mathbb{D}}_{\leq \infty}, \\ \infty & \text{otherwise.} \end{cases}$$

THEOREM 2.2. \bar{S}_n satisfies the extended large deviation principle in $(\mathbb{D}, \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and rate function \tilde{I} .

Similarly to the case of Lévy processes, the proof of Theorem 2.2 is facilitated by Lemma 2.4, Lemma 2.5 and Lemma 2.6; their proofs are deferred to Section 5.6. Let $\tilde{Q}^\leftarrow(x) = \inf\{y \geq 0 : \mathbf{P}(Z \geq y) < x\}$. Set

$$\tilde{J}_n^k(t) \triangleq \frac{1}{n} \sum_{i=1}^k \tilde{Q}^\leftarrow(V_{(i)}) \mathbb{1}_{[U_i, 1]}(t) + \frac{1}{n} Z \mathbb{1}_{\{1\}}(t)$$

and

$$\tilde{H}_n^k(t) \triangleq \frac{1}{n} \sum_{i=k+1}^{n-1} \tilde{Q}^\leftarrow(V_{(i)}) \mathbb{1}_{[U_i, 1]}(t) - \frac{1}{n} \mathbf{E}Z \sum_{i=1}^{n-1} \mathbb{1}_{[U_i, 1]}(t) - \frac{1}{n} \mathbf{E}Z \mathbb{1}_{\{1\}}(t),$$

where $V_{(1)}, V_{(2)}, \dots, V_{(n-1)}$ are the order statistics (in ascending order) of V_1, V_2, \dots, V_{n-1} , which are i.i.d. Uniform[0, 1] and independent of Z . (In the proof of Theorem 2.2, we will show that $\tilde{S}_n \triangleq \tilde{J}_n^k + \tilde{H}_n^k$ satisfies the extended LDP with speed $L(n)n^\alpha$ and is exponentially equivalent to \bar{S}_n so that Corollary 2.1 applies, and hence, in turn, \bar{S}_n satisfies the same extended LDP.) Let \tilde{I}_k be defined as follows:

$$(2.10) \quad \tilde{I}_k(\xi) = \begin{cases} \sum_{t:\xi(t) \neq \xi(t-)} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \tilde{\mathbb{D}}_{\leq k}, \\ \infty & \text{otherwise.} \end{cases}$$

LEMMA 2.4. For each fixed k , \tilde{J}_n^k satisfies the LDP in $(\mathbb{D}, \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and rate function \tilde{I}_k .

LEMMA 2.5. For each $\delta > 0$ and each open set G , there exist $\epsilon > 0$ and $K \geq 0$ such that for any $k \geq K$

$$\inf_{\xi \in G^{-\epsilon}} \tilde{I}_k(\xi) \leq \inf_{\xi \in G} \tilde{I}(\xi) + \delta.$$

LEMMA 2.6. For every $\epsilon > 0$ it holds that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\|\tilde{H}_n^k\|_\infty > \epsilon) = -\infty.$$

With these lemmas in our hands, we are ready to prove Theorem 2.2.

PROOF OF THEOREM 2.2. Let $\tilde{R}_i \triangleq \{j \in \mathbb{N} : U_j \leq U_i, 1 \leq j \leq n - 1\}$. Then, we claim that

$$\tilde{S}_n \stackrel{\mathcal{D}}{=} \frac{1}{n} \sum_{i=1}^{n-1} (\tilde{Q}^{\leftarrow}(V_{(i)}) - \mathbf{E}Z) \mathbb{1}_{[\tilde{R}_i/n, 1]} + \frac{1}{n} (Z - \mathbf{E}Z) \mathbb{1}_{\{1\}}.$$

To see why this distributional equality holds, note that $\{\tilde{R}_1, \dots, \tilde{R}_{n-1}\}$ is a uniformly random permutation of $\{1, \dots, n - 1\}$, and $\{\tilde{Q}^{\leftarrow}(V_{(1)}), \dots, \tilde{Q}^{\leftarrow}(V_{(n-1)})\}$ has the same distribution as the order statistics (in descending order) of Z_1, \dots, Z_{n-1} since $\tilde{Q}^{\leftarrow}(V_i)$ has the same distribution as Z for each i . Now, we move on to showing that \tilde{S}_n is close to \tilde{S}_n —that is, $\mathbf{P}(d_{J_1}(\tilde{S}_n, \tilde{S}_n) > \epsilon)$ is asymptotically negligible. Recall that

$$\tilde{S}_n = \frac{1}{n} \sum_{i=1}^{n-1} (\tilde{Q}^{\leftarrow}(V_{(i)}) - \mathbf{E}Z) \mathbb{1}_{[U_i, 1]} + \frac{1}{n} (Z - \mathbf{E}Z) \mathbb{1}_{\{1\}}.$$

First, observe that \tilde{R}_i is the rank of U_i among U_1, \dots, U_{n-1} , and hence, the \tilde{R}_i th earliest jump of both \tilde{S}_n and \tilde{S}_n equals $\tilde{Q}^{\leftarrow}(V_{(i)})$. Therefore, the jumps associated with $\tilde{Q}^{\leftarrow}(V_{(1)}), \dots, \tilde{Q}^{\leftarrow}(V_{(n-1)})$, Z are arranged in the same order for \tilde{S}_n and \tilde{S}_n with probability 1. Moreover, the jump times of \tilde{S}_n and \tilde{S}_n are $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}$ and $U_{(1)}, U_{(2)}, \dots, U_{(n-1)}, 1$, respectively. Since $0 < U_{(1)} < \dots < U_{(n-1)} < 1$ with probability 1, the piecewise linear time change $\lambda : [0, 1] \rightarrow [0, 1]$ defined by the linear interpolation of $\lambda(0) = 0$, $\lambda(1) = 1$, and $\lambda(i/n) = U_{(i)}$ for $i = 1, \dots, n - 1$ is a homeomorphism with probability 1. Therefore, the J_1 distance between \tilde{S}_n and \tilde{S}_n is bounded by

$$\sup_{1 \leq i \leq n-1} |i/n - U_{(i)}|$$

with probability 1. The latter supremum can be bounded in terms of the Kolmogorov–Smirnov statistic, and from the inequality (1.5) in Corollary 1 of Massart (1990), we obtain

$$\mathbf{P}\left(\sup_{1 \leq i \leq n-1} |i/n - U_{(i)}| > \epsilon\right) \leq \mathbf{P}\left(\sqrt{n} \sup_{x \in [0, 1]} \left| \frac{1}{n} \sum_{i=1}^n I(U_i \leq x) - x \right| > \epsilon \sqrt{n}\right) \leq 2e^{-2\epsilon^2 n}.$$

Hence,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(d_{J_1}(\tilde{S}_n, \tilde{S}_n) > \epsilon) &\leq \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\sup_{1 \leq i \leq n-1} |i/n - U_{(i)}| > \epsilon\right) \\ &= -\infty. \end{aligned}$$

In view of Corollary 2.1, the proof is done if we show that \tilde{S}_n satisfies the extended LDP with speed $L(n)n^\alpha$ and rate function \tilde{I} . To do so, we apply Proposition 2.1. Note that Lemma 2.4 verifies condition (i) of Proposition 2.1; (ii) is trivially satisfied since $\tilde{J}_k \geq \tilde{I}$; Lemma 2.5 verifies (iii); Lemma 2.6 verifies (iv). Therefore Proposition 2.1 applies to $\tilde{J}_n^k + \tilde{H}_n^k$, and the proof of Theorem 2.2 is complete. \square

2.3. Multidimensional processes. Let $X^{(1)}, \dots, X^{(d)}$ be independent processes satisfying Assumptions A1 and A2. Let $\bar{X}_n^{(i)}(t)$ denote the centered and scaled processes:

$$\bar{X}_n^{(i)}(t) \triangleq \frac{1}{n} X^{(i)}(nt) - t \mathbf{E}X^{(i)}(1).$$

The next theorem establishes the extended LDP for $(\bar{X}_n^{(1)}, \dots, \bar{X}_n^{(d)})$.

THEOREM 2.3. $(\bar{X}_n^{(1)}, \bar{X}_n^{(2)}, \dots, \bar{X}_n^{(d)})$ satisfies the extended LDP in $(\prod_{i=1}^d \mathbb{D}([0, 1], \mathbb{R}_+), \prod_{i=1}^d \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and rate function

$$(2.11) \quad I^d(\xi_1, \dots, \xi_d) = \begin{cases} \sum_{j=1}^d \sum_{t \in [0,1]} (\xi_j(t) - \xi_j(t-))^\alpha & \text{if } \xi_j \in \mathbb{D}_{\leq \infty} \text{ for each } j = 1, \dots, d, \\ \infty & \text{otherwise.} \end{cases}$$

For each i , we consider the same distributional decomposition of $\bar{X}_n^{(i)}$ as in Section 2.1:

$$\bar{X}_n^{(i)} \stackrel{\mathcal{D}}{=} \bar{J}_n^{k(i)} + \bar{H}_n^{k(i)} + \bar{R}_n^{(i)}.$$

The proof of the theorem follows the same lines as in the one-dimensional case, from Proposition 2.1, Lemma 2.3 and the following lemmas that parallel Lemma 2.1 and Lemma 2.2.

LEMMA 2.7. $(\bar{J}_n^{k(1)}, \dots, \bar{J}_n^{k(d)})$ satisfies the LDP in $(\prod_{i=1}^d \mathbb{D}, \prod_{i=1}^d \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and rate function $I_k^d : \prod_{i=1}^d \mathbb{D} \rightarrow [0, \infty]$

$$(2.12) \quad I_k^d(\xi_1, \dots, \xi_d) \triangleq \begin{cases} \sum_{i=1}^d \sum_{t \in [0,1]} (\xi_i(t) - \xi_i(t-))^\alpha & \text{if } \xi_i \in \mathbb{D}_{\leq k} \text{ for } i = 1, \dots, d, \\ \infty & \text{otherwise.} \end{cases}$$

LEMMA 2.8. For each $\delta > 0$ and each open set G , there exists $\epsilon > 0$ and $K \geq 0$ such that for any $k \geq K$

$$(2.13) \quad \inf_{(\xi_1, \dots, \xi_d) \in G^{-\epsilon}} I_k^d(\xi_1, \dots, \xi_d) \leq \inf_{(\xi_1, \dots, \xi_d) \in G} I^d(\xi_1, \dots, \xi_d) + \delta.$$

We conclude this section with the extended LDP for multidimensional random walks. Let $S_n^{(i)} = Z_1^{(i)} + \dots + Z_n^{(i)}$ be a random walk with nonnegative increments for each $i = 1, \dots, d$. Consider $\bar{S}_n^{(i)} = \{\bar{S}_n^{(i)}(t), t \in [0, 1]\}$ where $\bar{S}_n^{(i)}(t) = \frac{1}{n} \sum_{j=1}^{[nt]} (Z_j^{(i)} - \mathbf{E}Z^{(i)})$. We assume that $\mathbf{P}(Z_j^{(i)} \geq x) = \exp(-L(x)x^\alpha)$ where $\alpha \in (0, 1)$ and $L(\cdot)$ is a slowly varying function, and A2 is in force. The following theorem can be obtained by “lifting” Lemma 2.4 and Lemma 2.5 to the multidimensional context—in the same way Lemma 2.1 and Lemma 2.2 were lifted to the multidimensional counterparts in the proof of Theorem 2.3—and then applying Proposition 2.1. Let

$$(2.14) \quad \tilde{I}^d(\xi_1, \dots, \xi_d) = \begin{cases} \sum_{j=1}^d \sum_{t \in [0,1]} (\xi_j(t) - \xi_j(t-))^\alpha & \text{if } \xi_j \in \tilde{\mathbb{D}}_{\leq \infty} \text{ for each } j = 1, \dots, d, \\ \infty & \text{otherwise.} \end{cases}$$

THEOREM 2.4. $(\bar{S}_n^{(1)}, \bar{S}_n^{(2)}, \dots, \bar{S}_n^{(d)})$ satisfies the extended LDP in $(\prod_{i=1}^d \mathbb{D}, \prod_{i=1}^d \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and rate function \tilde{I}^d .

REMARK 2.2. Note that Theorem 2.3 and Theorem 2.4 can be extended to heterogeneous processes. For example, if the Lévy measure $\nu^{(i)}$ of the process $X^{(i)}$ has Weibull tail distribution $\nu^{(i)}[x, \infty) = \exp(-c_i L(x)x^\alpha)$ where $c_i \in (0, \infty)$ for each $i \leq d_0 \leq d$, and all the other processes have lighter tails—that is, $L(x)x^\alpha = o(L_i(x)x^{\alpha_i})$ for $i > d_0$ —then it is

straightforward to check that $(\bar{X}_n^{(1)}, \dots, \bar{X}_n^{(d)})$ satisfies the extended LDP with rate function

$$I^d(\xi_1, \dots, \xi_d) = \begin{cases} \sum_{j=1}^{d_0} c_j \sum_{t \in [0,1]} (\xi_j(t) - \xi_j(t-))^\alpha & \text{if } \xi_j \in \mathbb{D}_{\leq \infty}[0, 1] \text{ for } j = 1, \dots, d_0 \\ & \text{and } \xi_j \equiv 0 \text{ for } j > d_0, \\ \infty & \text{otherwise.} \end{cases}$$

Similarly, $(\bar{S}_n^{(1)}, \dots, \bar{S}_n^{(d)})$ satisfies the extended LDP with rate function \tilde{I}^d defined by replacing $\mathbb{D}_{\leq \infty}$ with $\bar{\mathbb{D}}_{\leq \infty}$ in the definition of I^d above under the corresponding conditions on the tail distribution of the $Z_1^{(i)}$'s.

3. Implications and further discussions.

3.1. *LDP for Lipschitz functions of Lévy processes.* Let $\bar{\mathbf{X}}_n$ denote the scaled Lévy processes $(\bar{X}_n^{(1)}, \dots, \bar{X}_n^{(d)})$, and let $\bar{\mathbf{S}}_n$ denote the scaled random walks $(\bar{S}_n^{(1)}, \dots, \bar{S}_n^{(d)})$ as defined in Section 2. Recall also the rate functions I^d defined in (2.11) and \tilde{I}^d defined in (2.14).

COROLLARY 3.1. *Let (\mathbb{S}, d) be a metric space and $\phi : \prod_{i=1}^d \mathbb{D} \rightarrow \mathbb{S}$ be a Lipschitz continuous mapping w.r.t. the J_1 Skorokhod metric. Set*

$$I'(x) \triangleq \inf_{\phi(\xi)=x} I^d(\xi) \quad \text{and} \quad \tilde{I}'(x) \triangleq \inf_{\phi(\xi)=x} \tilde{I}^d(\xi)$$

and suppose that I' (or \tilde{I}') is a good rate function—that is, $\Psi_{I'}(a) \triangleq \{x \in \mathbb{S} : I'(s) \leq a\}$ (or $\Psi_{\tilde{I}'}(a) \triangleq \{x \in \mathbb{S} : \tilde{I}'(s) \leq a\}$) is compact for each $a \in [0, \infty)$. Then, $\phi(\bar{\mathbf{X}}_n)$ (or $\phi(\bar{\mathbf{S}}_n)$) satisfies the large deviation principle in (\mathbb{S}, d) with speed $L(n)n^\alpha$ and good rate function I' (or \tilde{I}').

PROOF. Since the argument for $\phi(\bar{\mathbf{S}}_n)$ is very similar, we only prove the LDP for $\phi(\bar{\mathbf{X}}_n)$. We start with the upper bound. Suppose that the Lipschitz constant of ϕ is $\|\phi\|_{\text{Lip}}$. Note that since the J_1 distance is dominated by the supremum distance, $\|\bar{\mathbf{H}}_n^k\|_\infty \leq \epsilon$ and $\|\bar{\mathbf{R}}_n\|_\infty \leq \epsilon$ implies $d_{J_1}(\phi(\bar{\mathbf{J}}_n^k), \phi(\bar{\mathbf{X}}_n)) \leq 2\epsilon\|\phi\|_{\text{Lip}}$, where $\bar{\mathbf{J}}_n^k \triangleq (\bar{J}_n^{k(1)}, \dots, \bar{J}_n^{k(d)})$, $\bar{\mathbf{H}}_n^k \triangleq (\bar{H}_n^{k(1)}, \dots, \bar{H}_n^{k(d)})$ and $\bar{\mathbf{R}}_n \triangleq (\bar{R}_n^{(1)}, \dots, \bar{R}_n^{(d)})$. Therefore, for any closed set F ,

$$\begin{aligned} & \mathbf{P}(\phi(\bar{\mathbf{X}}_n) \in F) \\ & \leq \mathbf{P}(\phi(\bar{\mathbf{X}}_n) \in F, d_{J_1}(\phi(\bar{\mathbf{J}}_n^k), \phi(\bar{\mathbf{X}}_n)) \leq 2\epsilon\|\phi\|_{\text{Lip}}) + \mathbf{P}(d_{J_1}(\phi(\bar{\mathbf{J}}_n^k), \phi(\bar{\mathbf{X}}_n)) > 2\epsilon\|\phi\|_{\text{Lip}}) \\ & \leq \mathbf{P}(\phi(\bar{\mathbf{J}}_n^k) \in F^{2\epsilon\|\phi\|_{\text{Lip}}}) + \mathbf{P}(d_{J_1}(\phi(\bar{\mathbf{J}}_n^k), \phi(\bar{\mathbf{X}}_n)) > 2\epsilon\|\phi\|_{\text{Lip}}) \\ & \leq \mathbf{P}(\bar{\mathbf{J}}_n^k \in \phi^{-1}(F^{2\epsilon\|\phi\|_{\text{Lip}}})) + \mathbf{P}(\|\bar{\mathbf{H}}_n^k\|_\infty > \epsilon) + \mathbf{P}(\|\bar{\mathbf{R}}_n\|_\infty > \epsilon). \end{aligned}$$

Since $\mathbf{P}(\|\bar{\mathbf{R}}_n\|_\infty > \epsilon)$ decays at an exponential rate and

$$\mathbf{P}(\|\bar{\mathbf{H}}_n^k\|_\infty > \epsilon) \leq \sum_{i=1}^d \mathbf{P}(\|\bar{H}_n^{k(i)}\|_\infty > \epsilon/d),$$

we get the following bound by applying the principle of the maximum term and Theorem 2.3:

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\phi(\bar{\mathbf{X}}_n) \in F) \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \{ \mathbf{P}(\bar{\mathbf{J}}_n^k \in \phi^{-1}(F^{2\epsilon\|\phi\|_{\text{Lip}}})) + \mathbf{P}(\|\bar{\mathbf{H}}_n^k\|_\infty > \epsilon) + \mathbf{P}(\|\bar{\mathbf{R}}_n\|_\infty > \epsilon) \} \end{aligned}$$

$$\begin{aligned}
 &= \max \left\{ \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{\mathbf{J}}_n^k \in \phi^{-1}(F^{2\epsilon\|\phi\|_{\text{Lip}}}))}{L(n)n^\alpha}, \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\|\bar{\mathbf{H}}_n^k\|_\infty > \epsilon)}{L(n)n^\alpha} \right\} \\
 &\leq \max \left\{ - \inf_{(\xi_1, \dots, \xi_d) \in \phi^{-1}(F^{2\epsilon\|\phi\|_{\text{Lip}}})} I_k^d(\xi_1, \dots, \xi_d), \max_{i=1, \dots, d} \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\|\bar{H}_n^{k(i)}\|_\infty > \epsilon/d)}{L(n)n^\alpha} \right\} \\
 &\leq \max \left\{ - \inf_{(\xi_1, \dots, \xi_d) \in \phi^{-1}(F^{2\epsilon\|\phi\|_{\text{Lip}}})} I^d(\xi_1, \dots, \xi_d), \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\|\bar{H}_n^{k(1)}\|_\infty > \epsilon/d)}{L(n)n^\alpha} \right\}.
 \end{aligned}$$

From Lemma 2.3, we can take $k \rightarrow \infty$ to get

$$\begin{aligned}
 (3.1) \quad \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\phi(\bar{\mathbf{X}}_n) \in F)}{L(n)n^\alpha} &\leq - \inf_{(\xi_1, \dots, \xi_d) \in \phi^{-1}(F^{2\epsilon\|\phi\|_{\text{Lip}}})} I^d(\xi_1, \dots, \xi_d) \\
 &= - \inf_{x \in F^{2\epsilon\|\phi\|_{\text{Lip}}}} I'(x).
 \end{aligned}$$

From Lemma 4.1.6 of Dembo and Zeitouni (2010), $\lim_{\epsilon \rightarrow 0} \inf_{x \in F^{\epsilon\|\phi\|_{\text{Lip}}}} I'(x) = \inf_{x \in F} I'(x)$. Letting $\epsilon \rightarrow 0$ in (3.1), we arrive at the desired large deviation upper bound.

Turning to the lower bound, consider an open set G . Since $\phi^{-1}(G)$ is open, from Theorem 2.3,

$$\begin{aligned}
 \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\phi(\bar{\mathbf{X}}_n) \in G) &= \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\bar{\mathbf{X}}_n \in \phi^{-1}(G)) \\
 &\geq - \inf_{(\xi_1, \dots, \xi_d) \in \phi^{-1}(G)} I(\xi) = - \inf_{x \in G} I'(x). \quad \square
 \end{aligned}$$

3.2. *Sample path LDP w.r.t. M'_1 topology.* Recall that $\bar{X}_n(t) \triangleq \frac{1}{n}X(nt) - t\mathbf{E}X(1)$ and $\bar{S}_n(t) = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} (Z_i - \mathbf{E}Z)$. In this section, we establish the full LDP for \bar{X}_n and \bar{S}_n w.r.t. the M'_1 topology. For the definition of the M'_1 topology, see Appendix A.

COROLLARY 3.2. \bar{X}_n and \bar{S}_n satisfy the LDP in $(\mathbb{D}, \mathcal{T}_{M'_1})$ with speed $L(n)n^\alpha$ and good rate function $I_{M'_1}$.

$$I_{M'_1}(\xi) \triangleq \begin{cases} \sum_{t \in [0,1]} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \text{ is a nondecreasing pure jump function} \\ & \text{with } \xi(0) \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

PROOF. Since the proof for \bar{S}_n is nearly identical, we only provide the proof for \bar{X}_n . From Proposition A.3 we know that $I_{M'_1}$ is a good rate function. For the LDP upper bound, suppose that F is a closed set w.r.t. the M'_1 topology. Then, it is also closed w.r.t. the J_1 topology. From the upper bound of Theorem 2.1 and the fact that $I_{M'_1}(\xi) \leq I(\xi)$ for any $\xi \in \mathbb{D}$,

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in F)}{L(n)n^\alpha} \leq - \lim_{\epsilon \rightarrow 0} \inf_{\xi \in F^\epsilon} I(\xi) \leq - \lim_{\epsilon \rightarrow 0} \inf_{\xi \in F^\epsilon} I_{M'_1}(\xi) = - \inf_{\xi \in F} I_{M'_1}(\xi).$$

Turning to the lower bound, suppose that G is an open set w.r.t. the M'_1 topology. We claim that

$$\inf_{\xi \in G} I_{M'_1}(\xi) = \inf_{\xi \in G} I(\xi).$$

To show this, we only have to show that the RHS is not strictly larger than the LHS. Suppose that $I_{M'_1}(\xi) < I(\xi)$ for some $\xi \in G$. Since I and $I_{M'_1}$ differ only if the path has a jump at either 0 or 1, this means that ξ is a nonnegative pure jump function of the following form:

$$\xi = \sum_{i=1}^{\infty} z_i \mathbb{1}_{[u_i, 1]},$$

where $u_1 = 0, u_2 = 1, u_i$'s are all distinct in $(0, 1)$ for $i \geq 3$ and $z_i \geq 0$ for all i 's. Note that one can pick an arbitrarily small ϵ so that $\sum_{i \in \{n: u_n < \epsilon\}} z_i < \epsilon, \sum_{i \in \{n: u_n > 1-\epsilon\}} z_i < \epsilon, \epsilon \neq u_i$ for all $i \geq 2$, and $1 - \epsilon \neq u_i$ for all $i \geq 2$. For such ϵ 's, if we set

$$\xi_\epsilon \triangleq z_1 \mathbb{1}_{[\epsilon, 1]} + z_2 \mathbb{1}_{[1-\epsilon, 1]} + \sum_{i=3}^{\infty} z_i \mathbb{1}_{[u_i, 1]},$$

then $d_{M'_1}(\xi, \xi_\epsilon) \leq \epsilon$ while $I(\xi_\epsilon) = I_{M'_1}(\xi)$. That is, we can find an arbitrarily close element ξ_ϵ from ξ w.r.t. the M'_1 metric by pushing the jump times at 0 and 1 slightly to the inside of $(0, 1)$; at such an element, I assumes the same value as $I_{M'_1}(\xi)$. Since G is open w.r.t. M'_1 , one can choose ϵ small enough so that $\xi_\epsilon \in G$. This proves the claim. Now, the desired LDP lower bound is immediate from the LDP lower bound in Theorem 2.1 since G is also an open set in the J_1 topology. \square

3.3. *Nonexistence of large deviation principle in the J_1 topology.* Consider a compound Poisson process with arrival rate equal to 1 whose jump distribution is Weibull with shape parameter $1/2$. More specifically, $\bar{X}_n(t) \triangleq \frac{1}{n} \sum_{i=1}^{N(nt)} Z_i - t$ with $\mathbf{P}(Z_i \geq x) \sim \exp(-x^\alpha)$, $\mathbf{E}Z_i = 1$ and $\alpha = 1/2$. If \bar{X}_n satisfies a full LDP w.r.t. the J_1 topology, the rate function that controls the LDP (with speed n^α) associated with \bar{X}_n should be of the same form as the one that controls the extended LDP:

$$I(\xi) = \begin{cases} \sum_{t \in [0, 1]} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \mathbb{D}_{\leq \infty}, \\ \infty & \text{otherwise.} \end{cases}$$

To show that such a LDP is fundamentally impossible, we construct a closed set A for which

$$(3.2) \quad \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{n^\alpha} > - \inf_{\xi \in A} I(\xi).$$

Let

$$\varphi_{s,t}(\xi) \triangleq \lim_{\epsilon \rightarrow 0} \sup_{0 \vee (s-\epsilon) \leq u \leq v \leq 1 \wedge (t+\epsilon)} (\xi(v) - \xi(u)).$$

Let $A_{c;s,t} \triangleq \{\xi : \varphi_{s,t}(\xi) \geq c\}$ be (roughly speaking) the set of paths which increase at least by c between time s and t . Then $A_{c;s,t}$ is a closed set for each c, s and t . Next, define

$$A_m \triangleq (A_{1; \frac{m+1}{m+2}, \frac{m+1}{m+2} + mh_m}) \cap (A_{1; \frac{m}{m+2}, \frac{m}{m+2} + mh_m}) \cap \left(\bigcap_{j=0}^{m-1} A_{\frac{1}{m^2}; \frac{j}{m+2}, \frac{j}{m+2} + mh_m} \right),$$

where $h_m = \frac{1}{(m+1)(m+2)}$, and let

$$A \triangleq \bigcup_{m=1}^{\infty} A_m.$$

To see that A is closed, we first claim that $\zeta \in \mathbb{D} \setminus A$ implies the existence of $\epsilon > 0$ and $N \geq 0$ such that $B(\zeta; \epsilon) \cap A_m = \emptyset$ for all $m \geq N$. To prove this claim, suppose not. It is straightforward to check that for each n , there has to be $s_n, t_n \in [1 - 1/n, 1)$ such that

$s_n \leq t_n$ and $\zeta(t_n) - \zeta(s_n) \geq 1/2$, which in turn implies that ζ must possess infinite number of increases of size at least $1/2$ in $[1 - \delta, 1)$ for any $\delta > 0$. This implies that ζ cannot possess a left limit, which is contradictory to the assumption that $\zeta \in \mathbb{D} \setminus A$. On the other hand, since each A_m is closed, $\bigcup_{i=1}^N A_i$ is also closed, and hence, there exists $\epsilon' > 0$ such that $B(\zeta; \epsilon') \cap A_m = \emptyset$ for $m = 1, \dots, N$. Now, from the construction of ϵ and ϵ' , $B(\zeta, \epsilon \vee \epsilon') \cap A = \emptyset$, proving that A is closed.

Next, we show that A satisfies (3.2). First note that if ξ is a pure jump function that belongs to A_m , ξ has to possess m upward jumps of size $1/m^2$ and 2 upward jumps of size 1, and hence,

$$(3.3) \quad \inf_{\xi \in A} I(\xi) \geq \inf_m (1^{1/2} + 1^{1/2} + m(1/m^2)^{1/2}) = 3.$$

On the other hand, letting $\Delta\xi(t) \triangleq \xi(t) - \xi(t-)$,

$$\mathbf{P}(\bar{X}_{(n+1)(n+2)} \in A_n)$$

$$\begin{aligned} &\geq \prod_{j=0}^{n-1} \mathbf{P}\left(\sup_{t \in [0,1]} \left\{ \bar{X}_{(n+1)(n+2)}\left(\frac{(n+1)j + nt}{(n+1)(n+2)}\right) - \bar{X}_{(n+1)(n+2)}\left(\frac{(n+1)j}{(n+1)(n+2)}\right) \right\} \geq \frac{1}{n^2}\right) \\ &\quad \cdot \mathbf{P}\left(\sup_{t \in [0,1]} \left\{ \Delta \bar{X}_{(n+1)(n+2)}\left(\frac{(n+1)n + nt}{(n+1)(n+2)}\right) \right\} \geq 1\right) \\ &\quad \cdot \mathbf{P}\left(\sup_{t \in [0,1]} \left\{ \Delta \bar{X}_{(n+1)(n+2)}\left(\frac{(n+1)(n+1) + nt}{(n+1)(n+2)}\right) \right\} \geq 1\right) \\ &= \mathbf{P}\left(\sup_{t \in [0,1]} \left\{ \bar{X}_{(n+1)(n+2)}\left(\frac{nt}{(n+1)(n+2)}\right) \right\} \geq \frac{1}{n^2}\right)^n \\ &\quad \cdot \mathbf{P}\left(\sup_{t \in [0,1]} \left\{ \Delta \bar{X}_{(n+1)(n+2)}\left(\frac{nt}{(n+1)(n+2)}\right) \right\} \geq 1\right)^2 \\ &= \mathbf{P}\left(\sup_{t \in [0,1]} \{X(nt)\} \geq \frac{(n+1)(n+2)}{n^2}\right)^n \cdot \mathbf{P}\left(\sup_{t \in [0,1]} \{\Delta X(nt)\} \geq (n+1)(n+2)\right)^2 \\ &\geq \mathbf{P}\left(\sup_{t \in [0,1]} \{X(nt)\} \geq 6\right)^n \cdot \mathbf{P}\left(\sup_{t \in [0,1]} \{\Delta X(nt)\} \geq (n+1)(n+2)\right)^2, \end{aligned}$$

and hence,

$$\begin{aligned} (3.4) \quad \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{n^\alpha} &\geq \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_{(n+1)(n+2)} \in A_n)}{((n+1)(n+2))^\alpha} \\ &\geq \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\sup_{t \in [0,1]} \{X(nt)\} \geq 6)^n}{((n+1)(n+2))^\alpha} \\ &\quad + 2 \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\sup_{t \in [0,1]} \{\Delta X(nt)\} \geq (n+1)(n+2))}{((n+1)(n+2))^\alpha} \\ &= \text{(I)} + \text{(II)}. \end{aligned}$$

Letting $p_n \triangleq \mathbf{P}(\sup_{t \in [0,n]} \{X(t)\} < 6)$,

$$\begin{aligned} (3.5) \quad \text{(I)} &= \limsup_{n \rightarrow \infty} \frac{\log(1 - p_n)^n}{((n+1)(n+2))^\alpha} = \limsup_{n \rightarrow \infty} \frac{np_n \log(1 - p_n)^{1/p_n}}{((n+1)(n+2))^\alpha} \\ &= \limsup_{n \rightarrow \infty} \frac{-np_n}{((n+1)(n+2))^\alpha} = 0 \end{aligned}$$

since $p_n \rightarrow 0$ as $n \rightarrow \infty$.

Next, utilize $\{Z_1 \geq (n + 1)(n + 2)\} \implies \{\sup_{t \in [0,1]} \{\Delta X(nt)\} \geq (n + 1)(n + 2)\}$ to conclude

$$\begin{aligned}
 \text{(II)} &= 2 \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\sup_{t \in [0,1]} \{\Delta X(nt)\} \geq (n + 1)(n + 2))}{((n + 1)(n + 2))^\alpha} \\
 (3.6) \quad &\geq 2 \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(Z_1 \geq (n + 1)(n + 2))}{((n + 1)(n + 2))^\alpha} = 2 \limsup_{n \rightarrow \infty} \frac{\log e^{-((n+1)(n+2))^\alpha}}{((n + 1)(n + 2))^\alpha} \\
 &= -2.
 \end{aligned}$$

From (3.4), (3.5), (3.6),

$$(3.7) \quad \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{n^\alpha} \geq -2.$$

This along with (3.3),

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{n^\alpha} \geq -2 > -3 \geq - \inf_{\xi \in A} I(\xi),$$

which means that A indeed is a counterexample for the desired LDP.

Note that a simpler counterexample can be constructed using the peculiarity of the J_1 topology at the boundary of the domain; that is, jumps (of size 1, say) at time 0 are bounded away from the jumps at arbitrarily close jump times. For example, if $\bar{Y}_n(t) \triangleq \frac{1}{n} \sum_{i=1}^{N(nt)} Z_i + t$ where the right tail of Z is Weibull and $\mathbf{E}Z = -1$, then the set $B = \{x : x(t) \geq t/2 \text{ for all } t \in [0, 1]\}$ gives a counterexample for the LDP. Note that the M'_1 topology we used in Section 3.2 is a treatment for the same peculiarity of the M_1 topology at time 0. However, it should be clear from the above counterexample \bar{X}_n and A that the LDP is fundamentally impossible w.r.t. other J_1 -like topologies, that is, topologies that do not allow merging two or more jumps to approximate a single jump at any time.

4. Boundary crossing with moderate jumps. In this section, we illustrate the value of Corollary 3.1; for an application of Corollary 3.2 we refer to [Bazhba et al. \(2019\)](#). We consider level crossing probabilities of Lévy processes where the jump sizes are conditioned to be moderate. Specifically, we apply Corollary 3.1 in order to provide large deviations estimates for

$$(4.1) \quad \mathbf{P}\left(\sup_{t \in [0,1]} \bar{X}_n(t) \geq c, \sup_{t \in [0,1]} |\bar{X}_n(t) - \bar{X}_n(t-)| \leq b\right).$$

We emphasize that these types of rare events are difficult to analyze with the tools developed previously. In particular, the sample path LDP proved in [Gantert \(1998\)](#) is w.r.t. the L_1 topology. Since the closure of the sets in (4.1) w.r.t. the L_1 topology contains the zero function, the LDP upper bound would not provide any information.

Functionals like (4.1) appear in actuarial models, in case excessively large insurance claims are reinsured and therefore do not play a role in the ruin of an insurance company. [Asmussen and Pihlsgård \(2005\)](#), for example, studied various estimates of infinite-time ruin probabilities with analytic methods. [Rhee, Blanchet and Zwart \(2019\)](#) studied the finite-time ruin probability, using probabilistic techniques in case of regularly varying Lévy measures and confirmed that the conventional wisdom “the principle of a single big jump” can be extended to “the principle of the minimal number of big jumps” in such a context. Here we show that a similar result—with subtle differences—can be obtained in case the Lévy measure has a Weibull tail.

Define the function $\phi : \mathbb{D} \rightarrow \mathbb{R}^2$ as

$$\phi(\xi) = (\phi_1(\xi), \phi_2(\xi)) \triangleq \left(\sup_{t \in [0,1]} \xi(t), \sup_{t \in [0,1]} |\xi(t) - \xi(t-)| \right).$$

In order to apply Corollary 3.1, we will validate that ϕ is Lipschitz continuous and that $I'(x, y) \triangleq \inf_{\{\xi \in \mathbb{D} : \phi(\xi) = (x, y)\}} I(\xi)$ is a good rate function.

For the Lipschitz continuity of ϕ , we claim that each component of ϕ is Lipschitz continuous. We first examine ϕ_1 . Let $\xi, \zeta \in \mathbb{D}$ and suppose w.l.o.g. that $\sup_{t \in [0,1]} \xi(t) > \sup_{t \in [0,1]} \zeta(t)$. For an arbitrary nondecreasing homeomorphism $\lambda : [0, 1] \rightarrow [0, 1]$,

$$\begin{aligned} |\phi_1(\xi) - \phi_1(\zeta)| &= \left| \sup_{t \in [0,1]} \xi(t) - \sup_{t \in [0,1]} \zeta(t) \right| = \left| \sup_{t \in [0,1]} \xi(t) - \sup_{t \in [0,1]} \zeta \circ \lambda(t) \right| \\ (4.2) \qquad &\leq \sup_{t \in [0,1]} |\xi(t) - \zeta \circ \lambda(t)| \leq \sup_{t \in [0,1]} |\xi(t) - \zeta \circ \lambda(t)| \vee |\lambda(t) - t|. \end{aligned}$$

Taking the infimum over λ , we conclude that

$$|\phi_1(\xi) - \phi_1(\zeta)| \leq \inf_{\lambda \in \Lambda} \sup_{t \in [0,1]} \{ |\xi(t) - \zeta(\lambda(t))| \vee |\lambda(t) - t| \} = d_{J_1}(\xi, \zeta).$$

Therefore, ϕ_1 is Lipschitz with the Lipschitz constant 1.

Now, in order to prove that $\phi_2(\xi)$ is Lipschitz, fix two distinct paths $\xi, \zeta \in \mathbb{D}$ and assume w.l.o.g. that $\phi_2(\zeta) > \phi_2(\xi)$. Let $c \triangleq \phi_2(\zeta) - \phi_2(\xi) > 0$, and let $t^* \in [0, 1]$ be the maximum jump time of ξ , that is, $\phi_2(\xi) = |\xi(t^*) - \xi(t^*-)|$. For any $\epsilon > 0$ there exists λ^* so that

$$\begin{aligned} d_{J_1}(\xi, \zeta) &\triangleq \inf_{\lambda \in \Lambda} \{ \|\xi - \zeta \circ \lambda\|_\infty \vee \|\lambda - e\|_\infty \} \geq \|\xi - \zeta \circ \lambda^*\|_\infty \vee \|\lambda^* - e\|_\infty - \epsilon. \\ (4.3) \qquad &\geq |\xi(t^*) - \zeta \circ \lambda^*(t^*)| \vee |\xi(t^*-) - \zeta \circ \lambda^*(t^*-)| - \epsilon. \end{aligned}$$

From the general inequality $|a - b| \vee |c - d| \geq \frac{1}{2}(|a - c| - |b - d|)$,

$$\begin{aligned} &|\xi(t^*) - \zeta \circ \lambda^*(t^*)| \vee |\xi(t^*-) - \zeta \circ \lambda^*(t^*-)| \\ (4.4) \qquad &\geq \frac{1}{2} (|\xi(t^*) - \xi(t^*-)| - |\zeta \circ \lambda^*(t^*) - \zeta \circ \lambda^*(t^*-)|) \\ &= \frac{1}{2} (\phi_2(\xi) - \phi_2(\zeta)) = c/2. \end{aligned}$$

In view of (4.3) and (4.4), $d_{J_1}(\xi, \zeta) \geq \frac{1}{2}|\phi(\xi) - \phi(\zeta)| - \epsilon$. Since ϵ is arbitrary, we get the desired Lipschitz bound with Lipschitz constant 2. Therefore, $|\phi(\xi) - \phi(\zeta)| = |\phi_1(\xi) - \phi_1(\zeta)| \vee |\phi_2(\xi) - \phi_2(\zeta)| \leq 2d_{J_1}(\xi, \zeta)$ and hence, ϕ is Lipschitz with Lipschitz constant 2.

Now, we claim that I' is of the following form:

$$(4.5) \qquad I'(c, b) = \begin{cases} \left\lfloor \frac{c}{b} \right\rfloor b^\alpha + \left(c - \left\lfloor \frac{c}{b} \right\rfloor b \right)^\alpha & \text{if } 0 < b \leq c, \\ 0 & \text{if } b = c = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Note first that (4.5) is obvious except for the first case, and hence, we will assume that $0 < b \leq c$ from now on. Note also that $I'(c, b) = \inf \{ I(\xi) : \xi \in \mathbb{D}_{\leq \infty}, \phi(\xi) = (c, b) \}$ since $I(\xi) = \infty$ if $\xi \notin \mathbb{D}_{\leq \infty}$. Set $\mathcal{C} \triangleq \{ \xi \in \mathbb{D}_{\leq \infty}, (c, b) = \phi(\xi) \}$ and remember that any $\xi \in \mathbb{D}_{\leq \infty}$ admits the following representation:

$$(4.6) \qquad \xi = \sum_{i=1}^{\infty} x_i \mathbb{1}_{[u_i, 1]},$$

where u_i 's are distinct in $(0, 1)$ and x_i 's are nonnegative and sorted in a decreasing order. Consider a step function $\xi_0 \in \mathcal{C}$, with $\lfloor \frac{c}{b} \rfloor$ jumps of size b and one jump of size $c - \lfloor \frac{c}{b} \rfloor b$, so that $\xi_0 = \sum_{i=1}^{\lfloor \frac{c}{b} \rfloor} b \mathbb{1}_{[u_i, 1]} + (c - \lfloor \frac{c}{b} \rfloor b) \mathbb{1}_{[u_{\lfloor \frac{c}{b} \rfloor + 1}, 1]}$. Clearly, $\phi(\xi_0) = (c, b)$ and $I(\xi_0) = \lfloor \frac{c}{b} \rfloor b^\alpha + (c - \lfloor \frac{c}{b} \rfloor b)^\alpha$. Since $\xi_0 \in \mathcal{C}$, the infimum of I over \mathcal{C} should be at most $I(\xi_0)$ that is, $I(\xi_0) \geq I'(c, b)$.

To prove that ξ_0 is the minimizer of I over \mathcal{C} , we will show that $I(\xi) \geq I(\xi_0)$ for any $\xi = \sum_{i=1}^\infty x_i \mathbb{1}_{[u_i, 1]} \in \mathcal{C}$ by constructing ξ' such that $I(\xi) \geq I(\xi')$ while $I(\xi') = I(\xi_0)$. There has to be an integer k such that $x'_k \triangleq \sum_{i=k}^\infty x_i \leq b$. Let $\xi^1 \triangleq \sum_{i=1}^k x_i^1 \mathbb{1}_{[u_i, 1]}$ where x_i^1 is the i th largest element of $\{x_1, \dots, x_{k-1}, x'_k\}$. Then, $\xi^1 \in \mathcal{C}$ and $I(\xi^1) \leq I(\xi)$ due to the sub-additivity of $x \mapsto x^\alpha$. Now, given $\xi^j = \sum_{i=1}^k x_i^j \mathbb{1}_{[u_i, 1]}$, we construct ξ^{j+1} as follows. Find the first l such that $x_l^j < b$. If $x_l^j = 0$ or $x_{l+1}^j = 0$, set $\xi^{j+1} \triangleq \xi^j$. Otherwise, find the first m such that $x_{m+1}^j = 0$ and merge the l th jump and the m th jump. More specifically, set $x_l^{j+1} \triangleq x_l^j + x_m^j \wedge (b - x_l^j)$, $x_m^{j+1} \triangleq x_m^j - x_m^j \wedge (b - x_l^j)$, $x_i^{j+1} \triangleq x_i^j$ for $i \neq l, m$ and $\xi^{j+1} \triangleq \sum_{i=1}^k x_i^{j+1} \mathbb{1}_{[u_i, 1]}$. Note that $x_l^{j+1} + x_m^{j+1} = x_l^j + x_m^j$ while either $x_l^{j+1} = b$ or $x_m^{j+1} = 0$. That is, compared to ξ^j , ξ^{j+1} has either one less jump or one more jump with size b , while the total sum of the jump sizes and the maximum jump size remain the same. From this observation and the concavity of $x \mapsto x^\alpha$, it is straightforward to check that $I(\xi^{j+1}) \leq I(\xi^j)$. By iterating this procedure k times, we arrive at $\xi^k \triangleq \xi^k$ such that all the jump sizes of ξ^k are b , or there is only one jump whose size is not b . From this, we see that ξ^k has to coincide with ξ_0 . We conclude that $I(\xi) \geq I(\xi^1) \geq \dots \geq I(\xi^k) = I(\xi') = I(\xi_0)$, proving that ξ_0 is indeed a minimizer.

Now we check that $\Psi_{I'}(\gamma) \triangleq \{(c, b) : I'(c, b) \leq \gamma\}$ is compact for each $\gamma \in [0, \infty)$ so that I' is a good rate function. It is clear that $\Psi_{I'}(\gamma)$ is bounded. To see that $\Psi_{I'}(\gamma)$ is closed, suppose that $(c_1, b_1) \notin \Psi_{I'}(\gamma)$. In case $0 < b_1 < c_1$, note that I' can be written as $I'(c, b) = b^\alpha \{(c/b - \lfloor c/b \rfloor)^\alpha + \lfloor c/b \rfloor\}$, from which it is easy to see that I' is continuous at such (c_1, b_1) 's. Therefore, one can find an open ball around (c_1, b_1) in such a way that it doesn't intersect with $\Psi_{I'}(\gamma)$. By considering the cases $c_1 < b_1$, $b_1 = 0$, $b_1 = c_1$ separately, the rest of the cases are straightforward to check. We thus conclude that I' is a good rate function. Now we can apply Corollary 3.1. Note that

$$\inf_{(x, y) \in (c, \infty) \times [0, b]} I'(x, y) = \inf_{(x, y) \in (c, \infty) \times [0, b]} I'(x, y) = I'(c, b).$$

That is, the large deviation lower and upper bound coincide and hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log \mathbf{P}(\sup_{t \in [0, 1]} \bar{X}_n(t) \geq c, \sup_{t \in [0, 1]} |\bar{X}_n(t) - \bar{X}_n(t-)| \leq b)}{L(n)n^\alpha} \\ = \begin{cases} \left\lfloor \frac{c}{b} \right\rfloor b^\alpha + \left(c - \left\lfloor \frac{c}{b} \right\rfloor b \right)^\alpha & \text{if } 0 < b \leq c, \\ 0 & \text{if } b = c = 0, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

From the expression of the rate function, it can be inferred that the most likely way level c is reached is due to $\lfloor \frac{c}{b} \rfloor$ jumps of size b and one jump of size $(c - \lfloor \frac{c}{b} \rfloor b)$. If we compare this with the insights obtained from the case of truncated regularly varying tails in Rhee, Blanchet and Zwart (2019), we see that the total number of jumps is identical, but the size of the jumps are deterministic and nonidentical, while in the regularly varying case, they are random and identically distributed.

5. Proofs. This section provides technical proofs.

5.1. *Lower semicontinuity of I and I^d .* Recall the definition of I in (2.3) and I^d in (2.11).

LEMMA 5.1. *I and I^d are lower semicontinuous, and hence, rate functions.*

PROOF. We start with I . To show that the sub-level sets $\Psi_I(\gamma)$ are closed for each $\gamma < \infty$, let ξ be any given path that does not belong to $\Psi_I(\gamma)$. We will show that there exists an $\epsilon > 0$ such that $d_{J_1}(\xi, \Psi_I(\gamma)) \geq \epsilon$. Note that $\Psi_I(\gamma)^c = (A \cap B \cap C \cap D)^c = (A^c) \cup (A \cap B^c) \cup (A \cap B \cap C^c) \cup (A \cap B \cap C \cap D^c)$ where

$$A = \{\xi \in \mathbb{D} : \xi(0) = 0 \text{ and } \xi(1) = \xi(1-)\}, \quad B = \{\xi \in \mathbb{D} : \xi \text{ is nondecreasing}\},$$

$$C = \{\xi \in \mathbb{D} : \xi \text{ is a pure jump function}\}, \quad D = \left\{ \xi \in \mathbb{D} : \sum_{t \in [0,1]} (\xi(t) - \xi(t-))^\alpha \leq \gamma \right\}.$$

For $\xi \in A^c$, we will show that $d_{J_1}(\xi, \Psi_I(\gamma)) \geq \delta$ where $\delta = \frac{1}{2} \max\{|\xi(0)|, |\xi(1) - \xi(1-)|\}$. Suppose not so that there exists $\zeta \in \Psi_I(\gamma)$ such that $d_{J_1}(\xi, \zeta) < \delta$. Then $|\zeta(0)| > |\xi(0)| - 2\delta$ and $|\zeta(1) - \zeta(1-)| > |\xi(1) - \xi(1-)| - 2\delta$. That is, $\max\{|\zeta(0)|, |\zeta(1) - \zeta(1-)|\} > \max\{|\xi(0)| - 2\delta, |\xi(1) - \xi(1-)| - 2\delta\} = 0$. Therefore, $\zeta \in A^c$, and hence, $I(\zeta) = \infty$, which contradicts the assumption that $\zeta \in \Psi_I(\gamma)$.

If $\xi \in A \cap B^c$, there are $T_s < T_t$ such that $c \triangleq \xi(T_s) - \xi(T_t) > 0$. We claim that $d_{J_1}(\xi, \zeta) \geq c/2$ if $\zeta \in \Psi_I(\gamma)$. Suppose that this is not the case and there exists $\zeta \in \Psi_I(\gamma)$ such that $d_{J_1}(\xi, \zeta) < c/2$. Let λ be a nondecreasing homeomorphism $\lambda : [0, 1] \rightarrow [0, 1]$ such that $\|\zeta \circ \lambda - \xi\|_\infty < c/2$, in particular, $\zeta \circ \lambda(T_s) > \xi(T_s) - c/2$ and $\zeta \circ \lambda(T_t) < \xi(T_t) + c/2$. Subtracting the latter inequality from the former, we get $\zeta \circ \lambda(T_s) - \zeta \circ \lambda(T_t) > \xi(T_s) - \xi(T_t) - c = 0$. That is, ζ is not nondecreasing, which is contradictory to the assumption $\zeta \in \Psi_I(\gamma)$. Therefore, the claim has to be the case.

If $\xi \in A \cap B \cap C^c$, there exists an interval $[T_s, T_t]$ so that ξ is continuous and $c \triangleq \xi(T_t) - \xi(T_s) > 0$. Pick δ small enough so that $(c - 2\delta)(2\delta)^{\alpha-1} > \gamma$. We will show that $d_{J_1}(\xi, \Psi_I(\gamma)) \geq \delta$. Suppose that $\zeta \in \Psi_I(\gamma)$ and $d_{J_1}(\zeta, \xi) < \delta$, and let λ be a nondecreasing homeomorphism such that $\|\zeta \circ \lambda - \xi\|_\infty < \delta$. Note that this implies that each of the jump sizes of $\zeta \circ \lambda$ in $[T_s, T_t]$ has to be less than 2δ . On the other hand, $\zeta \circ \lambda(T_t) \geq \xi(T_t) - \delta$ and $\zeta \circ \lambda(T_s) \leq \xi(T_s) + \delta$, which in turn implies that $\zeta \circ \lambda(T_t) - \zeta \circ \lambda(T_s) \geq c - 2\delta$. Since $\zeta \circ \lambda$ is a nondecreasing pure jump function,

$$\begin{aligned} c - 2\delta &\leq \zeta \circ \lambda(T_t) - \zeta \circ \lambda(T_s) \\ &= \sum_{t \in (T_s, T_t]} (\zeta \circ \lambda(t) - \zeta \circ \lambda(t-)) \\ &= \sum_{t \in (T_s, T_t]} (\zeta \circ \lambda(t) - \zeta \circ \lambda(t-))^\alpha (\zeta \circ \lambda(t) - \zeta \circ \lambda(t-))^{1-\alpha} \\ &\leq \sum_{t \in (T_s, T_t]} (\zeta \circ \lambda(t) - \zeta \circ \lambda(t-))^\alpha (2\delta)^{1-\alpha}. \end{aligned}$$

That is, $\sum_{t \in (T_s, T_t]} (\zeta \circ \lambda(t) - \zeta \circ \lambda(t-))^\alpha \geq (2\delta)^{\alpha-1} (c - 2\delta) > \gamma$, which is contradictory to our assumption that $\zeta \in \Psi_I(\gamma)$. Therefore, $d_{J_1}(\xi, \Psi_I(\gamma)) \geq \delta$.

Finally, let $\xi \in A \cap B \cap C \cap D^c$. This implies that ξ admits the following representation: $\xi = \sum_{i=1}^\infty x_i \mathbb{1}_{[u_i, 1]}$ where u_i 's are all distinct in $(0, 1)$ and $\sum_{i=1}^\infty x_i^\alpha > \gamma$. Choose k large enough and δ small enough so that $\sum_{i=1}^k (x_i - 2\delta)^\alpha > \gamma$. We will show that $d_{J_1}(\xi, \Psi_I(\gamma)) \geq \delta$. Suppose that this is not the case. That is, there exists $\zeta \in \Psi_I(\gamma)$ so that

$d_{J_1}(\xi, \zeta) < \delta$. Let λ be a nondecreasing homeomorphism such that $\|\zeta \circ \lambda - \xi\|_\infty < \delta$. Thus for each $i \in \{1, \dots, k\}$, $\zeta \circ \lambda(u_i) - \zeta \circ \lambda(u_{i-}) \geq \xi(u_i) - \xi(u_{i-}) - 2\delta = x_i - 2\delta$, and hence,

$$I(\zeta) = \sum_{t \in [0,1]} (\zeta \circ \lambda(t_i) - \zeta \circ \lambda(t_{i-}))^\alpha \geq \sum_{i=1}^k (\zeta \circ \lambda(u_i) - \zeta \circ \lambda(u_{i-})) \geq \sum_{i=1}^k (x_i - 2\delta)^\alpha > \gamma,$$

which contradicts the assumption that $\zeta \in \Psi_I(\gamma)$.

I^d is lower semicontinuous since it is a sum of lower semicontinuous functions. \square

5.2. *Proof of Proposition 2.1.* PROOF OF PROPOSITION 2.1. We start with the extended large deviation upper bound. For any measurable set A ,

$$\begin{aligned} \mathbf{P}(X_n \in A) &= \mathbf{P}(X_n \in A, d(X_n, Y_n^k) \leq \epsilon) + \mathbf{P}(X_n \in A, d(X_n, Y_n^k) > \epsilon) \\ (5.1) \quad &\leq \underbrace{\mathbf{P}(Y_n^k \in A^\epsilon)}_{\triangleq(I)} + \underbrace{\mathbf{P}(d(X_n, Y_n^k) > \epsilon)}_{\triangleq(II)}. \end{aligned}$$

From the principle of the largest term and (i),

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq \max \left\{ - \inf_{x \in A^{2\epsilon}} I_k(x), \limsup_{n \rightarrow \infty} \frac{1}{a_n} \log \mathbf{P}(d(X_n, Y_n^k) > \epsilon) \right\}.$$

Now letting $k \rightarrow \infty$ and then $\epsilon \rightarrow 0$, (ii) and (iv) lead to

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n} \log \mathbf{P}(X_n \in A) \leq - \lim_{\epsilon \rightarrow 0} \inf_{x \in A^\epsilon} I(x),$$

which is the upper bound of the extended LDP.

Turning to the lower bound, note that the lower bound is trivial if $\inf_{x \in A^\circ} I(x) = \infty$. Therefore, we focus on the case $\inf_{x \in A^\circ} I(x) < \infty$. Consider an arbitrary but fixed $\delta \in (0, 1)$. In view of (iii) and (iv), one can pick $\epsilon > 0$ and $k \geq 1$ in such a way that

$$\begin{aligned} - \inf_{x \in A^\circ} I(x) &\leq - \inf_{x \in A^{-\epsilon}} I_k(x) + \delta \quad \text{and} \\ (5.2) \quad \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(d(X_n, Y_n^k) > \epsilon)}{a_n} &\leq - \inf_{x \in A^\circ} I(x) - 1. \end{aligned}$$

Hence

$$(5.3) \quad \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(d(X_n, Y_n^k) > \epsilon)}{a_n} \leq - \inf_{x \in A^{-\epsilon}} I_k(x) + \delta - 1.$$

Now, from (5.3) and the lower bound of the assumed extended LDP for Y_n^k , one can easily verify that

$$(5.4) \quad \frac{\mathbf{P}(d(X_n, Y_n^k) > \epsilon)}{\mathbf{P}(Y_n^k \in A^{-\epsilon})} \rightarrow 0$$

as $n \rightarrow \infty$. Using (5.4) and the first inequality of (5.2),

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{a_n} \log \mathbf{P}(X_n \in A) &\geq \liminf_{n \rightarrow \infty} \frac{1}{a_n} \log \mathbf{P}(Y_n^k \in A^{-\epsilon}, d(X_n, Y_n^k) \leq \epsilon) \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{a_n} \log(\mathbf{P}(Y_n^k \in A^{-\epsilon}) - \mathbf{P}(d(X_n, Y_n^k) > \epsilon)) \\ &= \liminf_{n \rightarrow \infty} \frac{1}{a_n} \log \left(\mathbf{P}(Y_n^k \in A^{-\epsilon}) \left(1 - \frac{\mathbf{P}(d(X_n, Y_n^k) > \epsilon)}{\mathbf{P}(Y_n^k \in A^{-\epsilon})} \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= \liminf_{n \rightarrow \infty} \frac{1}{a_n} \log \mathbf{P}(Y_n^k \in A^{-\epsilon}) \\
 &\geq - \inf_{x \in A^{-\epsilon}} I_k(x) \geq - \inf_{x \in A} I(x) - \delta.
 \end{aligned}$$

Since δ was arbitrary in $(0, 1)$, the lower bound is proved by letting $\delta \rightarrow 0$. \square

5.3. *Proof of Lemma 2.1.* We prove Lemma 2.1 in several steps. Before we proceed, we introduce some notation and a distributional representation of the compound Poisson processes Y_n . The following representation for the time-scaled compound Poisson process is a straightforward modification of the distributional representation on page 305 of [Lindskog, Resnick and Roy \(2014\)](#); see also exercise 5.4 on page 163 of [Resnick \(2007\)](#):

$$\int_{x \geq 1} x N([0, n \cdot] \times dx) \stackrel{\mathcal{D}}{=} \sum_{l=1}^{\tilde{N}_n} Q_n^{\leftarrow}(\Gamma_l) \mathbb{1}_{[U_l, 1]}(\cdot),$$

where $\Gamma_l = E_1 + E_2 + \dots + E_l$; E_i 's are i.i.d. and standard exponential random variables; U_l 's are i.i.d. and uniform variables in $[0, 1]$; $\tilde{N}_n = N_n([0, 1] \times [1, \infty))$; $N_n = \sum_{l=1}^{\infty} \delta_{(U_l, Q_n^{\leftarrow}(\Gamma_l))}$, where $\delta_{(x,y)}$ is the Dirac measure concentrated on (x, y) ; $Q_n(x) \triangleq n\nu[x, \infty)$, and $Q_n^{\leftarrow}(y) \triangleq \inf\{s > 0 : n\nu[s, \infty) < y\}$. It should be noted that \tilde{N}_n is the largest l such that $\Gamma_l \leq n\nu_1$, where $\nu_1 \triangleq \nu[1, \infty)$, and hence, $\tilde{N}_n \sim \text{Poisson}(n\nu_1)$. Recall the definition of \bar{J}_n^k —the process which keeps (up to) the k biggest Z_i 's among $Z_1, \dots, Z_{N(n)}$. From this and the observation that $Q_n^{\leftarrow}(\Gamma_l)$ is decreasing in l , we conclude that \bar{J}_n^k has the following distributional representation:

$$\bar{J}_n^k(\cdot) \stackrel{\mathcal{D}}{=} \underbrace{\frac{1}{n} \sum_{i=1}^k Q_n^{\leftarrow}(\Gamma_i) \mathbb{1}_{[U_i, 1]}(\cdot)}_{\triangleq \hat{J}_n^{\leq k}(\cdot)} - \frac{1}{n} \mathbb{1}_{\{\tilde{N}_n < k\}} \underbrace{\sum_{i=\tilde{N}_n+1}^k Q_n^{\leftarrow}(\Gamma_i) \mathbb{1}_{[U_i, 1]}(\cdot)}_{\triangleq \check{J}_n^{\leq k}(\cdot)}.$$

Roughly speaking, $(Q_n^{\leftarrow}(\Gamma_1)/n, \dots, Q_n^{\leftarrow}(\Gamma_k)/n)$ represents the k largest jump sizes of \bar{Y}_n , and $\hat{J}_n^{\leq k}$ can be regarded as the process obtained by keeping only the k largest jumps of \bar{Y}_n while disregarding the rest. Lemma 5.2 and Corollary 5.1 prove an LDP for $(Q_n^{\leftarrow}(\Gamma_1)/n, \dots, Q_n^{\leftarrow}(\Gamma_k)/n, U_1, \dots, U_k)$. Subsequently, Lemma 5.3 yields a sample path LDP for $\hat{J}_n^{\leq k}$. Finally, Lemma 2.1 is proved by showing that \bar{J}_n^k satisfies the same LDP as the one satisfied by $\hat{J}_n^{\leq k}$.

LEMMA 5.2. $(Q_n^{\leftarrow}(\Gamma_1)/n, Q_n^{\leftarrow}(\Gamma_2)/n, \dots, Q_n^{\leftarrow}(\Gamma_k)/n)$ satisfies a large deviation principle in \mathbb{R}_+^k with normalization $L(n)n^\alpha$, and with good rate function

$$(5.5) \quad \check{I}_k(x_1, \dots, x_k) = \begin{cases} \sum_{i=1}^k x_i^\alpha & \text{if } x_1 \geq x_2 \geq \dots \geq x_k \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

PROOF. It is straightforward to check that \check{I}_k is a good rate function. For each $f \in \mathcal{C}_b(\mathbb{R}_+^k)$, let

$$(5.6) \quad \Lambda_f^* \triangleq \lim_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log(\mathbf{E} e^{L(n)n^\alpha f(Q_n^{\leftarrow}(\Gamma_1)/n, Q_n^{\leftarrow}(\Gamma_2)/n, \dots, Q_n^{\leftarrow}(\Gamma_k)/n)}).$$

Applying Bryc's inverse Varadhan lemma (see, e.g., Theorem 4.4.13 of [Dembo and Zeitouni \(2010\)](#)), we can show that $(Q_n^{\leftarrow}(\Gamma_1)/n, \dots, Q_n^{\leftarrow}(\Gamma_k)/n)$ satisfies a large deviation principle

with speed $L(n)n^\alpha$ and good rate function $\check{I}_k(x)$ if

$$(5.7) \quad \Lambda_f^* = \sup_{x \in \mathbb{R}_+^k} \{f(x) - \check{I}_k(x)\}$$

for every $f \in C_b(\mathbb{R}_+^k)$.

To prove (5.7), fix $f \in C_b(\mathbb{R}_+^k)$ and let M be a constant such that $|f(x)| \leq M$ for all $x \in \mathbb{R}_+^k$. We first claim that the supremum of $\Lambda_f \triangleq f - \check{I}_k$ is finite and attained. Pick a constant R so that $R^\alpha > 2M$. Since Λ_f is upper semicontinuous on $[0, R]^k$, which is compact, there exists a maximizer $\hat{x} \triangleq (\hat{x}_1, \dots, \hat{x}_k)$ of Λ_f on $[0, R]^k$. Since

$$\sup_{x \in [0, R]^k} \{f(x) - \check{I}_k(x)\} \geq \sup_{x \in [0, R]^k} f(x) \geq -M$$

and

$$\sup_{x \in \mathbb{R}_+^k \setminus [0, R]^k} \{f(x) - \check{I}_k(x)\} < \sup_{x \in \mathbb{R}_+^k \setminus [0, R]^k} \{f(x) - 2M\} \leq -M,$$

\hat{x} is, in fact, a global maximizer. Now, it is enough to prove that

$$(5.8) \quad \Lambda_f(\hat{x}) \leq \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \Upsilon_f(n) \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \Upsilon_f(n) \leq \Lambda_f(\hat{x}),$$

where

$$\Upsilon_f(n) \triangleq \int_{\mathbb{R}_+^k} e^{L(n)n^\alpha f(Q_n^{\leftarrow}(y_1)/n, \dots, Q_n^{\leftarrow}(y_1 + \dots + y_k)/n)} e^{-\sum_{i=1}^k y_i} dy_1 \dots dy_k.$$

We start with the lower bound—that is, the first inequality of (5.8). Fix an arbitrary $\epsilon > 0$. Since Λ_f is continuous on $A_\infty \triangleq \{(x_1, \dots, x_k) \in \mathbb{R}_+^k : x_1 \geq \dots \geq x_k\}$, there exists $\delta > 0$ such that $x \in B(\hat{x}; 2\sqrt{k}\delta) \cap A_\infty$ implies $\Lambda_f(x) \geq \Lambda_f(\hat{x}) - \epsilon$. Since $\prod_{j=1}^k [\hat{x}_j + \delta, \hat{x}_j + 2\delta] \subseteq B(\hat{x}; 2\sqrt{k}\delta)$ and $Q_n^{\leftarrow}(\cdot)$ is nonincreasing, $Q_n^{\leftarrow}(\sum_{i=1}^j y_i)/n \in [\hat{x}_j + \delta, \hat{x}_j + 2\delta]$ for all $j = 1, \dots, k$ implies $(Q_n^{\leftarrow}(y_1)/n, \dots, Q_n^{\leftarrow}(y_1 + \dots + y_k)/n) \in B(\hat{x}; 2\sqrt{k}\delta)$, and hence,

$$(5.9) \quad \Lambda_f(Q_n^{\leftarrow}(y_1)/n, \dots, Q_n^{\leftarrow}(y_1 + \dots + y_k)/n) \geq \Lambda_f(\hat{x}) - \epsilon.$$

That is, if we define $D_n^j (= D_n^{y_1, \dots, y_{j-1}})$ as

$$D_n^j \triangleq \left\{ y_j \in \mathbb{R}_+ : Q_n^{\leftarrow}\left(\sum_{i=1}^j y_i\right)/n \in [\hat{x}_j + \delta, \hat{x}_j + 2\delta] \right\},$$

then (5.9) holds for (y_1, \dots, y_k) 's such that $y_j \in D_n^j$ for $j = 1, \dots, k$. Therefore,

$$(5.10) \quad \begin{aligned} \Upsilon_f(n) &= \int_{\mathbb{R}_+^k} e^{L(n)n^\alpha \Lambda_f(Q_n^{\leftarrow}(y_1)/n, \dots, Q_n^{\leftarrow}(y_1 + \dots + y_k)/n) + L(n) \sum_{i=1}^k Q_n^{\leftarrow}(\sum_{j=1}^i y_j)^\alpha - \sum_{i=1}^k y_i} dy_1 \dots dy_k \\ &\geq \int_{D_n^1} \dots \int_{D_n^k} e^{L(n)n^\alpha \Lambda_f(Q_n^{\leftarrow}(y_1)/n, \dots, Q_n^{\leftarrow}(y_1 + \dots + y_k)/n) + L(n) \sum_{i=1}^k Q_n^{\leftarrow}(\sum_{j=1}^i y_j)^\alpha - \sum_{i=1}^k y_i} dy_k \dots dy_1 \\ &\geq \int_{D_n^1} \dots \int_{D_n^k} e^{L(n)n^\alpha (\Lambda_f(\hat{x}_1, \dots, \hat{x}_k) - \epsilon)} e^{L(n) \sum_{i=1}^k Q_n^{\leftarrow}(\sum_{j=1}^i y_j)^\alpha - \sum_{i=1}^k y_i} dy_k \dots dy_1 \\ &\geq \int_{D_n^1} \dots \int_{D_n^k} e^{L(n)n^\alpha (\Lambda_f(\hat{x}_1, \dots, \hat{x}_k) - \epsilon)} e^{L(n) \sum_{i=1}^k (n(\hat{x}_i + \delta))^\alpha - \sum_{i=1}^k y_i} dy_k \dots dy_1 \\ &= \underbrace{e^{L(n)n^\alpha (\Lambda_f(\hat{x}_1, \dots, \hat{x}_k) - \epsilon)}}_{\triangleq (I)_n} \underbrace{e^{L(n) \sum_{i=1}^k (n(\hat{x}_i + \delta))^\alpha}}_{\triangleq (II)_n} \underbrace{\int_{D_n^1} \dots \int_{D_n^k} e^{-\sum_{i=1}^k y_i} dy_k \dots dy_1}_{\triangleq (III)_n} \end{aligned}$$

where the first equality is obtained by adding and subtracting $L(n) \sum_{i=1}^k Q_n^{\leftarrow}(\sum_{j=1}^i y_j)^\alpha$ to the exponent of the integrand. Note that by the construction of the D_n^j 's,

$$Q_n(n(\hat{x}_j + 2\delta)) \leq y_1 + \dots + y_j \leq Q_n(n(\hat{x}_j + \delta))$$

on the domain of the integral in $(III)_n$, and hence,

$$(5.11) \quad (III)_n \geq e^{-Q_n(n(\hat{x}_k + \delta))} \prod_{i=1}^k (Q_n(n(\hat{x}_i + \delta)) - Q_n(n(\hat{x}_i + 2\delta))).$$

Since $Q_n(n(\hat{x}_k + \delta)) \rightarrow 0$ and $L(n(\hat{x}_i + \delta))n^\alpha(\hat{x}_i + \delta)^\alpha - L(n(\hat{x}_i + 2\delta))n^\alpha(\hat{x}_i + 2\delta)^\alpha \rightarrow -\infty$ for each i ,

$$(5.12) \quad \begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log (III)_n \\ & \geq \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} (-Q_n(n(\hat{x}_k + \delta))) \\ & \quad + \sum_{i=1}^k \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log(Q_n(n(\hat{x}_i + \delta)) - Q_n(n(\hat{x}_i + 2\delta))) \\ & = \sum_{i=1}^k \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \\ & \quad \times \log(ne^{-L(n(\hat{x}_i + \delta))n^\alpha(\hat{x}_i + \delta)^\alpha} (1 - e^{L(n(\hat{x}_i + \delta))n^\alpha(\hat{x}_i + \delta)^\alpha - L(n(\hat{x}_i + 2\delta))n^\alpha(\hat{x}_i + 2\delta)^\alpha})) \\ & = \sum_{i=1}^k \liminf_{n \rightarrow \infty} \left(\frac{-L(n(\hat{x}_i + \delta))n^\alpha(\hat{x}_i + \delta)^\alpha}{L(n)n^\alpha} \right. \\ & \quad \left. + \frac{\log(1 - e^{L(n(\hat{x}_i + \delta))n^\alpha(\hat{x}_i + \delta)^\alpha - L(n(\hat{x}_i + 2\delta))n^\alpha(\hat{x}_i + 2\delta)^\alpha})}{L(n)n^\alpha} \right) \\ & = - \sum_{i=1}^k (\hat{x}_i + \delta)^\alpha. \end{aligned}$$

From this, along with

$$\liminf_{n \rightarrow \infty} \frac{1}{n^\alpha L(n)} \log (I)_n = \liminf_{n \rightarrow \infty} \frac{1}{n^\alpha L(n)} \log(e^{n^\alpha L(n)(\Lambda_f(\hat{x}_1, \dots, \hat{x}_k) - \epsilon)}) = \Lambda_f(\hat{x}_1, \dots, \hat{x}_k) - \epsilon$$

and

$$\liminf_{n \rightarrow \infty} \frac{1}{n^\alpha L(n)} \log (II)_n = \liminf_{n \rightarrow \infty} \frac{1}{n^\alpha L(n)} \log(e^{L(n) \sum_{i=1}^k (n(\hat{x}_i + \delta))^\alpha}) = \sum_{i=1}^k (\hat{x}_i + \delta)^\alpha,$$

we arrive at

$$(5.13) \quad \Lambda_f(\hat{x}) - \epsilon \leq \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \Upsilon_f(n).$$

Letting $\epsilon \rightarrow 0$, we obtain the lower bound of (5.8).

Turning to the upper bound, consider

$$D_{R,n} \triangleq \{(y_1, y_2, \dots, y_k) : Q_n^{\leftarrow}(y_1)/n \leq R\},$$

and decompose $\Upsilon_f(n)$ into two parts:

$$\begin{aligned} \Upsilon_f(n) &= \int_{D_{R,n}} e^{L(n)n^\alpha f(Q_n^\leftarrow(x_1)/n, \dots, Q_n^\leftarrow(x_1+\dots+x_k)/n)} e^{-\sum_{i=1}^k x_i} dx_1 \dots dx_k \\ &\quad + \int_{D_{R,n}^c} e^{L(n)n^\alpha f(Q_n^\leftarrow(x_1)/n, \dots, Q_n^\leftarrow(x_1+\dots+x_k)/n)} e^{-\sum_{i=1}^k x_i} dx_1 \dots dx_k. \end{aligned}$$

We first evaluate the integral over $D_{R,n}^c$. Since $|f| \leq M$,

$$\begin{aligned} &\int_{D_{R,n}^c} e^{L(n)n^\alpha f(Q_n^\leftarrow(x_1)/n, \dots, Q_n^\leftarrow(x_1+\dots+x_k)/n)} e^{-\sum_{i=1}^k x_i} dx_1 \dots dx_k \\ &= \int e^{L(n)n^\alpha f(Q_n^\leftarrow(x_1)/n, \dots, Q_n^\leftarrow(x_1+\dots+x_k)/n)} e^{-\sum_{i=1}^k x_i} \mathbb{1}_{\{Q_n^\leftarrow(x_1)/n > R\}} dx_1 \dots dx_k \\ (5.14) \quad &= \int e^{L(n)n^\alpha f(Q_n^\leftarrow(x_1)/n, \dots, Q_n^\leftarrow(x_1+\dots+x_k)/n)} e^{-\sum_{i=1}^k x_i} \mathbb{1}_{\{x_1 \leq Q_n(nR)\}} dx_1 \dots dx_k \\ &\leq \int e^{L(n)n^\alpha M} e^{-\sum_{i=1}^k x_i} \mathbb{1}_{\{x_1 \leq Q_n(nR)\}} dx_1 \dots dx_k \leq e^{L(n)n^\alpha M} (1 - e^{-Q_n(nR)}) \\ &\leq e^{L(n)n^\alpha M} Q_n(nR). \end{aligned}$$

Turning to the integral over $D_{R,n}$, fix $\epsilon > 0$ and pick $\{\check{x}^{(1)}, \dots, \check{x}^{(m)}\} \subset \mathbb{R}_+^k$ in such a way that $\{\prod_{j=1}^k [\check{x}_j^{(l)} - \epsilon, \check{x}_j^{(l)} + \epsilon]\}_{l=1, \dots, m}$ covers A_R . Set

$$\begin{aligned} H_{R,n,l} &\triangleq \{(y_1, \dots, y_k) \in \mathbb{R}_+^k : \\ &Q_n^\leftarrow(y_1)/n \in [\check{x}_1^{(l)} - \epsilon, \check{x}_1^{(l)} + \epsilon], \dots, Q_n^\leftarrow(y_1 + \dots + y_k)/n \in [\check{x}_k^{(l)} - \epsilon, \check{x}_k^{(l)} + \epsilon]\}. \end{aligned}$$

We see that $D_{R,n} \subseteq \bigcup_{l=1}^m H_{R,n,l}$, and hence,

$$\begin{aligned} &\int_{D_{R,n}} e^{L(n)n^\alpha f(Q_n^\leftarrow(y_1)/n, \dots, Q_n^\leftarrow(y_1+\dots+y_k)/n)} e^{-\sum_{i=1}^k y_i} dy_1 \dots dy_k \\ &\leq \sum_{l=1}^m \int_{H_{R,n,l}} e^{L(n)n^\alpha f(Q_n^\leftarrow(y_1)/n, \dots, Q_n^\leftarrow(y_1+\dots+y_k)/n)} e^{-\sum_{i=1}^k y_i} dy_1 \dots dy_k \\ &= \sum_{l=1}^m \int_{H_{R,n,l}} e^{L(n)n^\alpha \Lambda_f(Q_n^\leftarrow(y_1)/n, \dots, Q_n^\leftarrow(y_1+\dots+y_k)/n)} \\ (5.15) \quad &\times e^{L(n) \sum_{i=1}^k Q_n^\leftarrow(\sum_{j=1}^i y_j)^\alpha - \sum_{i=1}^k y_i} dy_1 \dots dy_k \\ &\leq \sum_{l=1}^m \int_{H_{R,n,l}} e^{L(n)n^\alpha \Lambda_f(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k)} e^{L(n) \sum_{i=1}^k Q_n^\leftarrow(\sum_{j=1}^i y_j)^\alpha - \sum_{i=1}^k y_i} dy_1 \dots dy_k \\ &= \sum_{l=1}^m \underbrace{e^{L(n)n^\alpha \Lambda_f(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k)} \int_{H_{R,n,l}} e^{L(n) \sum_{i=1}^k Q_n^\leftarrow(\sum_{j=1}^i y_j)^\alpha - \sum_{i=1}^k y_i} dy_1 \dots dy_k}_{\triangleq H(R,n,l)}, \end{aligned}$$

where the first equality is obtained by adding and subtracting $L(n) \sum_{i=1}^k Q_n^\leftarrow(\sum_{j=1}^i y_j)^\alpha$ to the exponent of the integrand. Since

$$Q_n^\leftarrow\left(\sum_{j=1}^i y_j\right)/n \in [\check{x}_i^{(l)} - \epsilon, \check{x}_i^{(l)} + \epsilon] \implies Q_n(n(\check{x}_i^{(l)} + \epsilon)) \leq \sum_{j=1}^i y_j \leq Q_n(n(\check{x}_i^{(l)} - \epsilon)),$$

we can bound the integral in (5.15) as follows:

$$\begin{aligned}
 & \int_{H_{R,n,l}} e^{L(n) \sum_{i=1}^k Q_n^{\leftarrow}(\sum_{j=1}^i y_j)^\alpha - \sum_{i=1}^k y_i} dy_1 \dots dy_k \\
 & \leq \int_{H_{R,n,l}} e^{L(n) \sum_{i=1}^k (n(\check{x}_i^{(l)} - \epsilon))^\alpha - \sum_{i=1}^k y_i} dy_1 \dots dy_k \\
 (5.16) \quad & \leq \int_{H_{R,n,l}} e^{L(n) \sum_{i=1}^k (n(\check{x}_i^{(l)} - \epsilon))^\alpha - Q_n(n(\check{x}_k^{(l)} + \epsilon))} dy_1 \dots dy_k \\
 & = e^{L(n) \sum_{i=1}^k (n(\check{x}_i^{(l)} - \epsilon))^\alpha - Q_n(n(\check{x}_k^{(l)} + \epsilon))} \int_{H_{R,n,l}} dy_1 \dots dy_k \\
 & = e^{L(n)n^\alpha \sum_{i=1}^k (\check{x}_i^{(l)} - \epsilon)^\alpha - Q_n(n(\check{x}_k^{(l)} + \epsilon))} \prod_{i=1}^k (Q_n(n(\check{x}_i^{(l)} - \epsilon)) - Q_n(n(\check{x}_i^{(l)} + \epsilon))).
 \end{aligned}$$

With (5.15) and (5.16), a straightforward calculation as in the lower bound leads to

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log H(R, n, l) \\
 & \leq \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log(e^{L(n)n^\alpha \Lambda_f(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k)}) \\
 & \quad + \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log(e^{L(n)n^\alpha \sum_{i=1}^k (\hat{x}_i^{(l)} - \epsilon)^\alpha - Q_n(n(\hat{x}_k^{(l)} + \epsilon))}) \\
 & \quad + \sum_{i=1}^k \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log(Q_n(n(\hat{x}_i^{(l)} - \epsilon)) - Q_n(n(\hat{x}_i^{(l)} + \epsilon))) \\
 & = \Lambda_f(\hat{x}_1, \dots, \hat{x}_k).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \Upsilon_f(n) \\
 & = \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log(e^{L(n)n^\alpha M} Q_n(nR)) \vee \max \left\{ \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log H(R, n, l) \right\} \\
 & \leq (M - R^\alpha) \vee \Lambda_f(\hat{x}_1, \dots, \hat{x}_k) = (M - R^\alpha) \vee \sup_{x \in \mathbb{R}_+^k} \{f(x) - \check{I}_k(x)\}.
 \end{aligned}$$

Since R was arbitrary, we can send $R \rightarrow \infty$ to arrive at the desired upper bound of (5.8). \square

The following corollary is immediate from Lemma 5.2 and Theorem 4.14 of Ganesh, O’Connell and Wischik (2004).

COROLLARY 5.1. $(Q_n^{\leftarrow}(\Gamma_1)/n, \dots, Q_n^{\leftarrow}(\Gamma_k)/n, U_1, \dots, U_k)$ satisfies a large deviation principle in $\mathbb{R}_+^k \times [0, 1]^k$ with speed $L(n)n^\alpha$ and good rate function

$$(5.17) \quad \hat{I}_k(x_1, \dots, x_k, u_1, \dots, u_k) \triangleq \begin{cases} \sum_{i=1}^k x_i^\alpha & \text{if } x_1 \geq x_2 \geq \dots \geq x_k \text{ and } u_1, \dots, u_k \in [0, 1], \\ \infty & \text{otherwise.} \end{cases}$$

Recall that $\hat{J}_n^{\leq k} = \frac{1}{n} \sum_{i=1}^k Q_n^{\leftarrow}(\Gamma_i) \mathbb{1}_{[U_i, 1]}$ and rate function I_k defined in (2.6). We next prove a sample path LDP for $\hat{J}_n^{\leq k}$.

LEMMA 5.3. $\hat{J}_n^{\leq k}$ satisfies the LDP in $(\mathbb{D}, \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and rate function I_k .

PROOF. First, we note that I_k is indeed a rate function since the sublevel sets of I_k equal the intersection between the sublevel sets of I and a closed set $\mathbb{D}_{\leq k}$, and I is a rate function (Lemma 5.1).

Next, we prove the LDP in $\mathbb{D}_{\leq k}$ w.r.t. the relative topology induced by \mathcal{T}_{J_1} . (Note that I_k is a rate function in $\mathbb{D}_{\leq k}$ as well.) Set $T_k(x, u) \triangleq \sum_{i=1}^k x_i \mathbb{1}_{[u_i, 1]}$. Since

$$\inf_{(x,u) \in T_k^{-1}(\xi)} \hat{I}_k(x, u) = I_k(\xi)$$

for $\xi \in \mathbb{D}_{\leq k}$, the LDP in $\mathbb{D}_{\leq k}$ is established once we show that for any closed set $F \subseteq \mathbb{D}_{\leq k}$,

$$(5.18) \quad \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\hat{J}_n^{\leq k} \in F) \leq - \inf_{(x,u) \in T_k^{-1}(F)} \hat{I}_k(x, u),$$

and for any open set $G \subseteq \mathbb{D}_{\leq k}$,

$$(5.19) \quad - \inf_{(x,u) \in T_k^{-1}(G)} \hat{I}_k(x, u) \leq \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\hat{J}_n^{\leq k} \in G).$$

We start with the upper bound. Note that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\hat{J}_n^{\leq k} \in F) \\ &= \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}((Q_n^{\leftarrow}(\Gamma_1), \dots, Q_n^{\leftarrow}(\Gamma_k), U_1, \dots, U_k) \in T_k^{-1}(F)) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}((Q_n^{\leftarrow}(\Gamma_1), \dots, Q_n^{\leftarrow}(\Gamma_k), U_1, \dots, U_k) \in T_k^{-1}(F)^-) \\ &\leq - \inf_{(x_1, \dots, x_k, u_1, \dots, u_k) \in T_k^{-1}(F)^-} \hat{I}_k(x_1, \dots, x_k, u_1, \dots, u_k). \end{aligned}$$

In view of (5.18), it is therefore enough for the upper bound to show that

$$\inf_{(x,u) \in T_k^{-1}(F)} \hat{I}_k(x, u) \leq \inf_{(x,u) \in T_k^{-1}(F)^-} \hat{I}_k(x, u).$$

To prove this, we proceed with proof by contradiction. Suppose that

$$(5.20) \quad c \triangleq \inf_{(x,u) \in T_k^{-1}(F)} \hat{I}_k(x, u) > \inf_{(x,u) \in T_k^{-1}(F)^-} \hat{I}_k(x, u).$$

Pick an $\epsilon > 0$ in such a way that $\inf_{(x,u) \in T_k^{-1}(F)^-} \hat{I}_k(x, u) < c - 2\epsilon$. Then there exists $(x^*, u^*) \in T_k^{-1}(F)^-$ such that $\hat{I}_k(x^*, u^*) < c - 2\epsilon$. Let $\bar{I}_k(x_1, \dots, x_k, u_1, \dots, u_k) \triangleq \sum_{i=1}^k x_i^\alpha$. Since \bar{I}_k is continuous, one can find $(x', u') = (x'_1, \dots, x'_k, u'_1, \dots, u'_k) \in T_k^{-1}(F)$ sufficiently close to (x^*, u^*) so that $\bar{I}_k(x', u') < c - \epsilon$. Note that for any permutation $p : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$, $(x'', u'') \triangleq (x'_{p(1)}, \dots, x'_{p(k)}, u'_{p(1)}, \dots, u'_{p(k)})$ also belongs to $T_k^{-1}(F)$ and $\bar{I}_k(x'', u'') = \bar{I}_k(x', u')$ due to the symmetric structure of T_k and \bar{I}_k . If we pick p so that $x'_{p(1)} \geq \dots \geq x'_{p(k)}$, then $\hat{I}_k(x'', u'') = \bar{I}_k(x', u') < c - \epsilon \leq \inf_{(x,u) \in T_k^{-1}(F)} \hat{I}_k(x, u)$, which contradicts to $(x'', u'') \in T_k^{-1}(F)$. Therefore, (5.20) cannot be the case, which proves the upper bound.

Turning to the lower bound, consider an open set $G \subseteq \mathbb{D}_{\leq k}$.

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\hat{J}_n^{\leq k} \in G) \\ &= \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}((Q_n^\leftarrow(\Gamma_1), \dots, Q_n^\leftarrow(\Gamma_k), U_1, \dots, U_k) \in T_k^{-1}(G)) \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}((Q_n^\leftarrow(\Gamma_1), \dots, Q_n^\leftarrow(\Gamma_k), U_1, \dots, U_k) \in T_k^{-1}(G)^\circ) \\ &\geq - \inf_{(x_1, \dots, x_k, u_1, \dots, u_k) \in T_k^{-1}(G)^\circ} \hat{I}_k(x_1, \dots, x_k, u_1, \dots, u_k). \end{aligned}$$

In view of (5.19), we are done if we prove that

$$(5.21) \quad \inf_{(x, u) \in T_k^{-1}(G)^\circ} \hat{I}_k(x, u) \leq \inf_{(x, u) \in T_k^{-1}(G)} \hat{I}_k(x, u).$$

Let (x, u) be an arbitrary point in $T_k^{-1}(G)$ so that $T_k(x, u) \in G$. We will show that there exists $(x^*, u^*) \in T_k^{-1}(G)^\circ$ such that $I_k(x^*, u^*) \leq I_k(x, u)$. Note first that if $u_i \in \{0, 1\}$ for some i , then x_i has to be 0 since $G \subseteq \mathbb{D}_{\leq k}$. This means that we can replace u_i with an arbitrary number in $(0, 1)$ without changing the value of I_k and T_k . Therefore, we assume w.l.o.g. that $u_i > 0$ for each $i = 1, \dots, k$. Now, suppose that $u_i = u_j$ for some $i \neq j$. Then one can find (x', u') such that $T_k(x', u') = T_k(x, u)$ by setting

$$(x', u') \triangleq (x_1, \dots, \underbrace{x_i + x_j}_{i \text{th coordinate}}, \dots, \underbrace{0_j}_{j \text{th coordinate}}, \dots, x_k, u_1, \dots, \underbrace{u_{ij}}_{k+i \text{th coordinate}}, \dots, \underbrace{u'_j}_{k+j \text{th coordinate}}, \dots, u_k),$$

where u'_j is an arbitrary number in $(0, 1)$; in particular, we can choose u'_j so that $u'_j \neq u_l$ for $l = 1, \dots, k$. It is easy to see that $\bar{I}_k(x', u') \leq \hat{I}_k(x, u)$. Now one can permute the coordinates of (x', u') as in the upper bound to find (x'', u'') such that $T_k(x'', u'') = T_k(x, u)$ and $\hat{I}_k(x'', u'') \leq \hat{I}_k(x, u)$. Iterating this procedure until there is no $i \neq j$ for which $u_i = u_j$, we can find (x^*, u^*) such that $T_k(x^*, u^*) = T_k(x, u)$, u_i^* 's are all distinct in $(0, 1)$, and $I_k(x^*, u^*) \leq I_k(x, u)$. Note that since T_k is continuous at (x^*, u^*) , $T_k(x^*, u^*) \in G$, and G is open, we conclude that $(x^*, u^*) \in T_k^{-1}(G)^\circ$. Therefore,

$$\inf_{(x, u) \in T_k^{-1}(G)^\circ} I_k(x, u) \leq I_k(x, u).$$

Since (x, u) was arbitrarily chosen in $T_k^{-1}(G)$, (5.21) is proved. Along with the upper bound, this proves the LDP in $\mathbb{D}_{\leq k}$. Finally, since $\mathbb{D}_{\leq k}$ is a closed subset of \mathbb{D} , $\mathbf{P}(\hat{J}_n^{\leq k} \notin \mathbb{D}_{\leq k}) = 0$, and $I_k = \infty$ on $\mathbb{D} \setminus \mathbb{D}_{\leq k}$, Lemma 4.1.5 of Dembo and Zeitouni (2010) applies, proving the desired LDP in \mathbb{D} . \square

Now we are ready to prove Lemma 2.1.

PROOF OF LEMMA 2.1. Recall that

$$\bar{J}_n^k \stackrel{\mathcal{D}}{=} \underbrace{\frac{1}{n} \sum_{i=1}^k Q_n^\leftarrow(\Gamma_i) \mathbb{1}_{[U_i, 1]}}_{= \hat{J}_n^{\leq k}} - \frac{1}{n} \mathbb{1}_{\{\tilde{N}_n < k\}} \underbrace{\sum_{i=\tilde{N}_n+1}^k Q_n^\leftarrow(\Gamma_i) \mathbb{1}_{[U_i, 1]}}_{= \hat{J}_n^{\leq k}}.$$

Let F be a closed set and note that

$$\begin{aligned} \mathbf{P}(\bar{J}_n^k \in F) &= \mathbf{P}(\hat{J}_n^{\leq k} - \check{J}_n^{\leq k} \in F) \\ &\leq \mathbf{P}(\hat{J}_n^{\leq k} - \check{J}_n^{\leq k} \in F, \mathbb{1}\{N(n) < k\} = 0) + \mathbf{P}(\mathbb{1}\{N(n) < k\} \neq 0) \\ &\leq \mathbf{P}(\hat{J}_n^{\leq k} \in F) + \mathbf{P}(N(n) < k). \end{aligned}$$

From Lemma 5.3,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{J}_n^k \in F)}{L(n)n^\alpha} &\leq \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\hat{J}_n^{\leq k} \in F)}{L(n)n^\alpha} \vee \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(N(n) < k)}{L(n)n^\alpha} \\ &\leq - \inf_{\xi \in F} I_k(\xi), \end{aligned}$$

since $\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(N(n) < k) = -\infty$.

Turning to the lower bound, let G be an open set. Since the lower bound is trivial in case $\inf_{x \in G} I_k(x) = \infty$, we focus on the case $\inf_{x \in G} I_k(x) < \infty$. In this case,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{J}_n^k \in G)}{L(n)n^\alpha} &\geq \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{J}_n^k \in G, N(n) \geq k)}{L(n)n^\alpha} \\ &= \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}(\hat{J}_n^{\leq k} \in G, N(n) \geq k)}{L(n)n^\alpha} \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log(\mathbf{P}(\hat{J}_n^{\leq k} \in G) - \mathbf{P}(N(n) < k)) \\ &= \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log\left(\mathbf{P}(\hat{J}_n^{\leq k} \in G) \left(1 - \frac{\mathbf{P}(N(n) < k)}{\mathbf{P}(\hat{J}_n^{\leq k} \in G)}\right)\right) \\ &= \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \left\{ \log(\mathbf{P}(\hat{J}_n^{\leq k} \in G)) + \log\left(1 - \frac{\mathbf{P}(N(n) < k)}{\mathbf{P}(\hat{J}_n^{\leq k} \in G)}\right) \right\} \\ &= \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\hat{J}_n^{\leq k} \in G) \geq - \inf_{\xi \in G} I_k(\xi). \end{aligned}$$

The last equality holds since

$$(5.22) \quad \lim_{n \rightarrow \infty} \frac{\mathbf{P}(N(n) < k)}{\mathbf{P}(\hat{J}_n^{\leq k} \in G)} = \lim_{n \rightarrow \infty} \left\{ \exp\left(\frac{\log \mathbf{P}(N(n) < k)}{L(n)n^\alpha} - \frac{\log \mathbf{P}(\hat{J}_n^{\leq k} \in G)}{L(n)n^\alpha}\right) \right\}^{L(n)n^\alpha} = 0,$$

which in turn follows from

$$\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(N(n) < k) = -\infty$$

and

$$\limsup_{n \rightarrow \infty} \frac{-1}{L(n)n^\alpha} \log \mathbf{P}(\hat{J}_n^{\leq k} \in G) \leq \inf_{x \in G} I_k(x) < \infty. \quad \square$$

5.4. *Proof of Lemma 2.2.* PROOF OF LEMMA 2.2. Since the inequality is obvious if $\inf_{\xi \in G} I(\xi) = \infty$, we assume that $\inf_{\xi \in G} I(\xi) < \infty$. Then, there exists a $\xi_0 \in G$ such that $I(\xi_0) \leq \inf_{\xi \in G} I(\xi) + \delta$. Since G is open, we can pick $\epsilon > 0$ such that $B_{J_1}(\xi_0; 2\epsilon) \subseteq G$ so that $B_{J_1}(\xi_0; \epsilon) \subseteq G^{-\epsilon}$. Note that since $I(\xi_0) < \infty$, ξ_0 has the representation $\xi_0 = \sum_{i=1}^\infty x_i \mathbb{1}_{[u_i, 1]}$ where $x_i \geq 0$ for all $i = 1, 2, \dots$, and the u_i 's all distinct in $(0, 1)$. Note also that since $I(\xi_0) =$

$\sum_{i=1}^{\infty} x_i^\alpha < \infty$ with $\alpha < 1$, $\sum_{i=1}^{\infty} x_i$ has to be finite as well. Thus, there exists K such that $k \geq K$ implies $\sum_{i=k+1}^{\infty} x_i < \epsilon$. For these ϵ and K , we claim that (2.7) holds. For any given $k \geq K$, let $\xi_1 \triangleq \sum_{i=1}^k x_i \mathbb{1}_{[u_i, 1]}$, then $I_k(\xi_1) \leq I(\xi_0)$ while $d_{J_1}(\xi_0, \xi_1) \leq \|\xi_0 - \xi_1\|_\infty \leq \sum_{i=k+1}^{\infty} x_i < \epsilon$. That is, $\xi_1 \in B_{J_1}(\xi_0; \epsilon) \subseteq G^{-\epsilon}$. Therefore, $\inf_{\xi \in G^{-\epsilon}} I_k(\xi) \leq I(\xi_1) \leq I(\xi_0) \leq \inf_{\xi \in G} I(\xi) + \delta$. \square

5.5. Proof of Lemma 2.3. In our proof of Lemma 2.3, the following lemmas (Lemma 5.4 and Lemma 5.5) play key roles.

LEMMA 5.4. For each $\epsilon > \delta > 0$,

$$(5.23) \quad \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P} \left(\max_{1 \leq j \leq 2n} \sum_{i=1}^j (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > n\epsilon \right) \leq -(\epsilon/3)^\alpha (\epsilon/\delta)^{1-\alpha}.$$

PROOF. We refine an argument developed in Jelenković and Momčilović (2003). Note that for any $s > 0$ such that $1/s \leq n\delta$,

$$(5.24) \quad \mathbf{E}e^{sZ \mathbb{1}_{\{Z \leq n\delta\}}} = \mathbf{E}e^{sZ \mathbb{1}_{\{Z \leq n\delta\}}} \mathbb{1}_{\{Z \geq \frac{1}{s}\}} + \mathbf{E}e^{sZ \mathbb{1}_{\{Z \leq n\delta\}}} \mathbb{1}_{\{Z < \frac{1}{s}\}} = (I) + (II),$$

and

$$\begin{aligned} (I) &= \int_{[1/s, n\delta]} e^{sy} d\mathbf{P}(Z \leq y) + \int_{(n\delta, \infty)} d\mathbf{P}(Z \leq y) \\ &= [e^{sy} \mathbf{P}(Z \leq y)]_{(1/s)-}^{(n\delta)+} - s \int_{[1/s, n\delta]} e^{sy} \mathbf{P}(Z \leq y) dy + \mathbf{P}(Z > n\delta) \\ &= e^{sn\delta} \mathbf{P}(Z \leq n\delta) - e \mathbf{P}(Z < 1/s) \\ &\quad - s \int_{[1/s, n\delta]} e^{sy} dy + s \int_{[1/s, n\delta]} e^{sy} \mathbf{P}(Z > y) dy + \mathbf{P}(Z > n\delta) \\ (5.25) \quad &= e^{sn\delta} \mathbf{P}(Z \leq n\delta) - e \mathbf{P}(Z < 1/s) - e^{sn\delta} + e \\ &\quad + s \int_{[1/s, n\delta]} e^{sy} \mathbf{P}(Z > y) dy + \mathbf{P}(Z > n\delta) \\ &= -e^{sn\delta} \mathbf{P}(Z > n\delta) + e \mathbf{P}(Z \geq 1/s) + s \int_{[1/s, n\delta]} e^{sy} \mathbf{P}(Z > y) dy + \mathbf{P}(Z > n\delta) \\ &\leq s \int_{[1/s, n\delta]} e^{sy} \mathbf{P}(Z > y) dy + e \mathbf{P}(Z \geq 1/s) + \mathbf{P}(Z > n\delta) \\ &\leq s \int_{[1/s, n\delta]} e^{sy} \mathbf{P}(Z > y) dy + s^2(e + 1) \mathbf{E}Z^2, \end{aligned}$$

where the last inequality is from $\mathbf{P}(Z \geq n\delta) \leq \mathbf{P}(Z \geq 1/s) \leq s^2 \mathbf{E}Z^2$; while

$$(5.26) \quad (II) \leq \int_0^{1/s} e^{sy} d\mathbf{P}(Z \leq y) \leq \int_0^{1/s} (1 + sy + (sy)^2) d\mathbf{P}(Z \leq y) \leq 1 + s \mathbf{E}Z + s^2 \mathbf{E}Z^2.$$

Therefore, from (5.24), (5.25) and (5.26), if $1/s \leq n\delta$ and s is sufficiently small,

$$\begin{aligned} \mathbf{E}e^{sZ \mathbb{1}_{\{Z \leq n\delta\}}} &\leq s \int_{\frac{1}{s}}^{n\delta} e^{sy} \mathbf{P}(Z > y) dy + 1 + s \mathbf{E}Z + s^2(e + 2) \mathbf{E}Z^2 \\ (5.27) \quad &= s \int_{\frac{1}{s}}^{n\delta} e^{sy - q(y)} dy + 1 + s \mathbf{E}Z + s^2(e + 2) \mathbf{E}Z^2 \\ &\leq sn\delta e^{sn\delta - q(n\delta)} + 1 + s \mathbf{E}Z + s^2(e + 2) \mathbf{E}Z^2, \end{aligned}$$

where $q(x) \triangleq -\log \mathbf{P}(X > x) = L(x)x^\alpha$, and the last inequality is from the fact that $e^{sy-q(y)}$ is increasing over $[1/s, n\delta]$ due to the assumption that $L(y)y^{\alpha-1}$ is nonincreasing for sufficiently large y 's. Now, from the Markov inequality,

$$\begin{aligned}
 & \mathbf{P}\left(\sum_{i=1}^j (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > n\epsilon\right) \\
 & \leq \mathbf{P}\left(\exp\left(s \sum_{i=1}^j Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}\right) > \exp(s(n\epsilon + j\mathbf{E}Z))\right) \\
 (5.28) \quad & \leq \exp\{-s(n\epsilon + j\mathbf{E}Z) + j \log(\mathbf{E}e^{sZ \mathbb{1}_{\{Z \leq n\delta\}}})\} \\
 & \leq \exp\{-s(n\epsilon + j\mathbf{E}Z) + j(sn\delta e^{sn\delta-q(n\delta)} + s\mathbf{E}Z + s^2(e+2)\mathbf{E}Z^2)\} \\
 & = \exp\{-sn\epsilon + jsn\delta e^{sn\delta-q(n\delta)} + js^2(e+2)\mathbf{E}Z^2\} \\
 & \leq \exp\{-sn\epsilon + 2n^2s\delta e^{sn\delta-q(n\delta)} + 2ns^2(e+2)\mathbf{E}Z^2\}
 \end{aligned}$$

for $j \leq 2n$, where the third inequality is from (5.27) and the generic inequality $\log(x+1) \leq x$. Fix $\gamma \in (0, (\epsilon/\delta)^{1-\alpha})$ and set $s = \frac{\gamma q(n\epsilon)}{n\epsilon}$. Note that $1/s \rightarrow \infty$ as $n \rightarrow \infty$, while $1/s \leq n\delta$ for sufficiently large n . From now on, we only consider sufficiently large n 's such that $1/s < n\delta$ and s is sufficiently small so that (5.27) and (5.28) are valid. To establish an upper bound for (5.28), we next examine $e^{sn\delta-q(n\delta)}$. Note that $q(n\epsilon) = q(n\delta) \frac{L(n\epsilon)}{L(n\delta)} (\delta/\epsilon)^{-\alpha}$, and hence,

$$sn\delta - q(n\delta) = \frac{\gamma q(n\epsilon)}{n\epsilon} n\delta - q(n\delta) = -q(n\delta) \left(1 - \gamma \frac{L(n\epsilon)}{L(n\delta)} (\delta/\epsilon)^{1-\alpha}\right)$$

and

$$(5.29) \quad e^{sn\delta-q(n\delta)} \leq e^{-q(n\delta)(1-\gamma \frac{L(n\epsilon)}{L(n\delta)} (\delta/\epsilon)^{1-\alpha})}.$$

Plugging this $s(= \frac{\gamma q(n\epsilon)}{n\epsilon})$ into (5.28) along with (5.29),

$$\begin{aligned}
 & \max_{0 \leq j \leq 2n} \mathbf{P}\left(\sum_{i=1}^j (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > n\epsilon\right) \\
 & \leq \exp\left\{-\gamma q(n\epsilon) + \frac{2\gamma \delta n q(n\epsilon)}{\epsilon} e^{-q(n\delta)(1-\gamma \frac{L(n\epsilon)}{L(n\delta)} (\delta/\epsilon)^{1-\alpha})} + \frac{2\gamma^2(e+2)\mathbf{E}Z^2 q(n\epsilon)^2}{\epsilon^2 n}\right\}.
 \end{aligned}$$

Since

$$\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \frac{2\gamma \delta n q(n\epsilon)}{\epsilon} e^{-q(n\delta)(1-\gamma \frac{L(n\epsilon)}{L(n\delta)} (\delta/\epsilon)^{1-\alpha})} = 0$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \frac{2\gamma^2(e+2)\mathbf{E}Z^2 q(n\epsilon)^2}{\epsilon^2 n} = 0,$$

we conclude that

$$\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \max_{0 \leq j \leq 2n} \mathbf{P}\left(\sum_{i=1}^j (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > n\epsilon\right) = \limsup_{n \rightarrow \infty} \frac{-\gamma q(n\epsilon)}{L(n)n^\alpha} = -\epsilon^\alpha \gamma.$$

From Etemadi’s inequality,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P} \left(\max_{0 \leq j \leq 2n} \sum_{i=1}^j (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > 3n\epsilon \right) \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \left\{ 3 \max_{0 \leq j \leq 2n} \mathbf{P} \left(\sum_{i=1}^j (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > n\epsilon \right) \right\} = -\epsilon^\alpha \gamma. \end{aligned}$$

Since this is true for any γ such that $\gamma \in (0, (\epsilon/\delta)^{1-\alpha})$, we arrive at the conclusion of the lemma. \square

LEMMA 5.5. For every $\epsilon, \delta > 0$,

$$\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P} \left(\sup_{1 \leq j \leq 2n} \sum_{i=1}^j (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > n\epsilon \right) = -\infty.$$

PROOF. Note first that there is n_0 such that $\mathbf{E}(Z_i \mathbb{1}_{\{Z_i > n\delta\}}) \leq \frac{\epsilon}{3}$ for $n \geq n_0$. For $n \geq n_0$ and $j \leq 2n$,

$$\begin{aligned} \mathbf{P} \left(\sum_{i=1}^j (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > n\epsilon \right) &= \mathbf{P} \left(\sum_{i=1}^j (\mathbf{E}Z \mathbb{1}_{\{Z \leq n\delta\}} - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > n\epsilon - j\mathbf{E}Z \mathbb{1}_{\{Z > n\delta\}} \right) \\ &\leq \mathbf{P} \left(\sum_{i=1}^j (\mathbf{E}Z \mathbb{1}_{\{Z \leq n\delta\}} - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > n\epsilon - j\epsilon/3 \right) \\ &\leq \mathbf{P} \left(\sum_{i=1}^j (\mathbf{E}Z \mathbb{1}_{\{Z \leq n\delta\}} - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > \frac{n\epsilon}{3} \right). \end{aligned}$$

Let $Y_i^{(n)} \triangleq \mathbf{E}(Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}$. Recall the definition of Z in Section 2.1 and note that it is bounded from below. Furthermore, $\mathbf{E}Y_i^{(n)} = 0$, $\mathbf{var} Y_i^{(n)} \leq \mathbf{E}Z^2$, and $Y_i^{(n)} \leq \mathbf{E}Z$ almost surely. From Bennet’s inequality,

$$\begin{aligned} & \mathbf{P} \left(\sum_{i=1}^j (\mathbf{E}Z \mathbb{1}_{\{Z_i \leq n\delta\}} - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > \frac{n\epsilon}{3} \right) \\ & \leq \exp \left[-\frac{j \mathbf{var} Y^{(n)}}{(\mathbf{E}Z)^2} \left\{ \left(1 + \frac{n\epsilon \mathbf{E}Z}{3j \mathbf{var} Y^{(n)}} \right) \log \left(1 + \frac{n\epsilon \mathbf{E}Z}{3j \mathbf{var} Y^{(n)}} \right) - \left(\frac{n\epsilon \mathbf{E}Z}{3j \mathbf{var} Y^{(n)}} \right) \right\} \right] \\ & \leq \exp \left[-\frac{j \mathbf{var} Y^{(n)}}{(\mathbf{E}Z)^2} \left\{ \left(\frac{n\epsilon \mathbf{E}Z}{3j \mathbf{var} Y^{(n)}} \right) \log \left(1 + \frac{n\epsilon \mathbf{E}Z}{3j \mathbf{var} Y^{(n)}} \right) - \left(\frac{n\epsilon \mathbf{E}Z}{3j \mathbf{var} Y^{(n)}} \right) \right\} \right] \\ & \leq \exp \left[-\left\{ \left(\frac{n\epsilon}{3\mathbf{E}Z} \right) \log \left(1 + \frac{n\epsilon \mathbf{E}Z}{3j \mathbf{var} Y^{(n)}} \right) - \left(\frac{n\epsilon}{3\mathbf{E}Z} \right) \right\} \right] \\ & \leq \exp \left[-n \left\{ \left(\frac{\epsilon}{3\mathbf{E}Z} \right) \log \left(1 + \frac{\epsilon \mathbf{E}Z}{6\mathbf{E}Z^2} \right) - \left(\frac{\epsilon}{3\mathbf{E}Z} \right) \right\} \right] \end{aligned}$$

for $j \leq 2n$. Therefore, for $n \geq n_0$ and $j \leq 2n$,

$$\mathbf{P} \left(\sum_{i=1}^j (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > n\epsilon \right) \leq \exp \left[-n \left\{ \left(\frac{\epsilon}{3\mathbf{E}Z} \right) \log \left(1 + \frac{\epsilon \mathbf{E}Z}{6\mathbf{E}Z^2} \right) - \left(\frac{\epsilon}{3\mathbf{E}Z} \right) \right\} \right].$$

Now, from Etemadi’s inequality,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P} \left(\sup_{1 \leq j \leq 2n} \sum_{i=1}^j (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > 3n\epsilon \right) \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \left\{ 3 \max_{1 \leq j \leq 2n} \mathbf{P} \left(\sum_{i=1}^j (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > n\epsilon \right) \right\} \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \left\{ 3 \exp \left[-n \left\{ \left(\frac{\epsilon}{3\mathbf{E}Z} \right) \log \left(1 + \frac{\epsilon \mathbf{E}Z}{6\mathbf{E}Z^2} \right) - \left(\frac{\epsilon}{3\mathbf{E}Z} \right) \right\} \right] \right\} = -\infty. \end{aligned}$$

Replacing ϵ with $\epsilon/3$, we arrive at the conclusion of the lemma. \square

Now we are ready to prove Lemma 2.3.

PROOF OF LEMMA 2.3.

$$\begin{aligned} & \mathbf{P}(\|\bar{H}_n^k\|_\infty > \epsilon) \\ & \leq \mathbf{P}(\|\bar{H}_n^k\|_\infty > \epsilon, N(nt) \geq k) + \mathbf{P}(\|\bar{H}_n^k\|_\infty > \epsilon, N(nt) < k) \\ (5.30) \quad & \leq \mathbf{P}(\|\bar{H}_n^k\|_\infty > \epsilon, N(nt) \geq k, Z_{R_n^{-1}(k)} \leq n\delta) \\ & \quad + \mathbf{P}(\|\bar{H}_n^k\|_\infty > \epsilon, N(nt) \geq k, Z_{R_n^{-1}(k)} > n\delta) + \mathbf{P}(N(nt) < k) \end{aligned}$$

$$\begin{aligned} (5.31) \quad & \leq \mathbf{P}(\|\bar{H}_n^k\|_\infty > \epsilon, N(nt) \geq k, Z_{R_n^{-1}(k)} \leq n\delta) \\ & \quad + \mathbf{P}(N(nt) \geq k, Z_{R_n^{-1}(k)} > n\delta) + \mathbf{P}(N(nt) < k). \end{aligned}$$

An explicit upper bound for the second term can be obtained:

$$\begin{aligned} & \mathbf{P}(N(nt) \geq k, Z_{R_n^{-1}(k)} > n\delta) \\ & \leq \mathbf{P}(Q_n^{\leftarrow}(\Gamma_k) > n\delta) \leq \mathbf{P}(Q_n^{\leftarrow}(\Gamma_k) \geq n\delta) = \mathbf{P}(\Gamma_k \leq Q_n(n\delta)) \\ & = \int_0^{Q_n(n\delta)} \frac{1}{k!} t^{k-1} e^{-t} dt = \int_0^{nv[n\delta, \infty)} \frac{1}{k!} t^{k-1} e^{-t} dt \leq \int_0^{nv(n\delta, \infty)} t^{k-1} dt \\ & = \frac{1}{k} n^k e^{-kL(n\delta)n^\alpha \delta^\alpha}. \end{aligned}$$

Therefore,

$$(5.32) \quad \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(Q_n^{\leftarrow}(\Gamma_k) \geq n\delta) \leq -k\delta^\alpha.$$

Turning to the first term of (5.30), we consider the following decomposition:

$$\begin{aligned} & \mathbf{P}(\|\bar{H}_n^k\|_\infty > \epsilon, N(nt) \geq k, Z_{R_n^{-1}(k)} \leq n\delta) \\ & = \underbrace{\mathbf{P}\left(N(nt) \geq k, Z_{R_n^{-1}(k)} \leq n\delta, \sup_{t \in [0,1]} \bar{H}_n^k(t) > \epsilon\right)}_{\triangleq (i)} \\ & \quad + \underbrace{\mathbf{P}\left(N(nt) \geq k, Z_{R_n^{-1}(k)} \leq n\delta, \sup_{t \in [0,1]} -\bar{H}_n^k(t) > \epsilon\right)}_{\triangleq (ii)}. \end{aligned}$$

Since $Z_{R_n^{-1}(k)} \leq n\delta$ implies $\mathbb{1}_{\{R_n(t) > k\}} \leq \mathbb{1}_{\{Z_i \leq n\delta\}}$,

$$\begin{aligned}
 \text{(i)} &\leq \mathbf{P}\left(\sup_{t \in [0,1]} \sum_{i=1}^{N(nt)} (Z_i \mathbb{1}_{\{R_n(i) > k\}} - \mathbf{E}Z) > n\epsilon, N(nt) \geq k, Z_{R_n^{-1}(k)} \leq n\delta\right) \\
 &\leq \mathbf{P}\left(\sup_{t \in [0,1]} \sum_{i=1}^{N(nt)} (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > n\epsilon\right) = \mathbf{P}\left(\sup_{0 \leq j \leq N(n)} \sum_{i=1}^j (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > n\epsilon\right) \\
 &\leq \mathbf{P}\left(\sup_{0 \leq j \leq 2n} \sum_{i=1}^j (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > n\epsilon, N(n) < 2n\right) + \mathbf{P}(N(n) \geq 2n) \\
 &\leq \mathbf{P}\left(\sup_{0 \leq j \leq 2n} \sum_{i=1}^j (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > n\epsilon\right) + \mathbf{P}(N(n) \geq 2n).
 \end{aligned}$$

From Lemma 5.4 and the fact that the second term decays at an exponential rate,

$$(5.33) \quad \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \mathbf{P}\left(Z_{R_n^{-1}(k)} \leq n\delta, \sup_{t \in [0,1]} \bar{H}_n^k(t) > \epsilon\right) \leq -(\epsilon/3)^\alpha (\epsilon/\delta)^{1-\alpha}.$$

Turning to (ii),

$$\begin{aligned}
 \text{(ii)} &\leq \mathbf{P}\left(\sup_{t \in [0,1]} \sum_{i=1}^{N(nt)} (\mathbf{E}Z - Z_i \mathbb{1}_{\{R_n(i) > k\}}) > n\epsilon\right) \\
 &= \mathbf{P}\left(\sup_{t \in [0,1]} \sum_{i=1}^{N(nt)} (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} + Z_i (\mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbb{1}_{\{R_n(i) > k\}})) > n\epsilon\right) \\
 &\leq \mathbf{P}\left(\sup_{t \in [0,1]} \sum_{i=1}^{N(nt)} (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} + Z_i \mathbb{1}_{\{Z_i \leq n\delta\} \cap \{R_n(i) \leq k\}}) > n\epsilon\right) \\
 &\leq \mathbf{P}\left(\sup_{t \in [0,1]} \sum_{i=1}^{N(nt)} (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) + kn\delta > n\epsilon\right) \\
 &= \mathbf{P}\left(\sup_{t \in [0,1]} \sum_{i=1}^{N(nt)} (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > n(\epsilon - k\delta)\right) \\
 &\leq \mathbf{P}\left(\sup_{0 \leq j \leq 2n} \sum_{i=1}^j (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > n(\epsilon - k\delta), N(nt) < 2n\right) + \mathbf{P}(N(nt) \geq 2n) \\
 &\leq \mathbf{P}\left(\sup_{0 \leq j \leq 2n} \sum_{i=1}^j (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) > n(\epsilon - k\delta)\right) + \mathbf{P}(N(nt) \geq 2n).
 \end{aligned}$$

Applying Lemma 5.5 to the first term and noticing that the second term vanishes at an exponential rate, we conclude that for δ and k such that $k\delta < \epsilon$

$$(5.34) \quad \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(Z_{R_n^{-1}(k)} \leq n\delta, \sup_{t \in [0,1]} \bar{H}_n^k(t) > \epsilon\right) = -\infty.$$

From (5.33) and (5.34),

$$(5.35) \quad \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(Z_{R_n^{-1}(k)} \leq n\delta, \|\bar{H}_n^k\|_\infty > \epsilon\right) \leq -(\epsilon/3)^\alpha (\epsilon/\delta)^{1-\alpha}.$$

This, together with (5.30) and (5.32),

$$\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \mathbf{P}(\|\bar{H}_n^k\|_\infty > \epsilon) \leq \max\{-(\epsilon/3)^\alpha (\epsilon/\delta)^{1-\alpha}, -k\delta^\alpha\}$$

for any δ and k such that $k\delta < \epsilon$. Choosing, for example, $\delta = \frac{\epsilon}{2k}$ and letting $k \rightarrow \infty$, we arrive at the conclusion of the lemma. \square

5.6. *Proof of Lemmas 2.4, 2.5 and 2.6.* PROOF OF LEMMA 2.4. We follow a similar program as in the proof of Lemma 2.1. Recall that $\tilde{Q}^\leftarrow(x) = \inf\{s > 0 : \mathbf{P}(Z \geq s) < y\}$ and $V_{(1)}, \dots, V_{(n-1)}$ are the order statistics of $n - 1$ i.i.d. Uniform(0,1) random variables V_1, \dots, V_{n-1} . We first claim that $(\tilde{Q}^\leftarrow(V_{(1)})/n, \dots, \tilde{Q}^\leftarrow(V_{(k)})/n)$ satisfies the LDP with speed $L(n)n^\alpha$ and good rate function \check{I}_k defined in (5.5). Let f be a bounded continuous function such that $|f(x)| < M, x \in \mathbb{R}_+^k$ for some $M \in \mathbb{R}$. We want to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^\leftarrow(V_{(1)})/n, \dots, \tilde{Q}^\leftarrow(V_{(k)})/n)\} = \sup_x \Lambda_f(x),$$

where $\Lambda_f = f - \check{I}_k$, to invoke inverse Varadhan lemma and establish the LDP for $(\tilde{Q}^\leftarrow(V_{(1)})/n, \dots, \tilde{Q}^\leftarrow(V_{(k)})/n)$. Recall that in the proof of Lemma 5.2, we have shown that the supremum of $f(x) - \check{I}_k(x)$ over \mathbb{R}_+^k is attained. Let \hat{x} denote one of the optimizers that attain the supremum. Then, due to the form of \check{I}_k , for any given $\epsilon > 0$, we can find $\delta > 0$ and $\check{x} = (\check{x}_1, \dots, \check{x}_k)$ such that $\check{x}_i \geq \check{x}_{i+1} + \delta$ for $i = 1, \dots, k - 1$ and $x \in \prod_{i=1}^k [\check{x}_i, \check{x}_i + \delta]$ implies

$$\check{I}_k(x) \geq \check{I}_k(\hat{x}) - \epsilon \quad \text{and} \quad f(x) - \check{I}_k(x) \geq f(\hat{x}) - \check{I}_k(\hat{x}) - \epsilon.$$

Therefore, if we set $A_n(\delta) \triangleq \{(y_1, \dots, y_k) : \tilde{Q}^\leftarrow(y_i)/n \in [\check{x}_i, \check{x}_i + \delta], i = 1, \dots, k + 1\}$, then $y \in A_n(\delta)$ implies

$$\check{I}_k(\tilde{Q}^\leftarrow(y_1)/n, \dots, \tilde{Q}^\leftarrow(y_k)/n) \geq \check{I}_k(\hat{x}) - \epsilon$$

and

$$f(\tilde{Q}^\leftarrow(y_1)/n, \dots, \tilde{Q}^\leftarrow(y_k)/n) - \check{I}_k(\tilde{Q}^\leftarrow(y_1)/n, \dots, \tilde{Q}^\leftarrow(y_k)/n) \geq f(\hat{x}) - \check{I}_k(\hat{x}) - \epsilon,$$

and hence,

$$f(\tilde{Q}^\leftarrow(y_1)/n, \dots, \tilde{Q}^\leftarrow(y_k)/n) \geq f(\hat{x}) - 2\epsilon.$$

Note also that $\tilde{Q}^\leftarrow(y) < x$ if and only if $\mathbf{P}(Z \geq x) < y$, and hence, $\mathbf{P}(Z \geq n(\check{x}_i + \delta)) < y_i \leq \mathbf{P}(Z \geq n\check{x}_i)$ implies $\tilde{Q}^\leftarrow(y_i)/n \in [\check{x}_i, \check{x}_i + \delta]$. We have that

$$\mathbb{1}_{\{\tilde{Q}^\leftarrow(y_i)/n \in [\check{x}_i, \check{x}_i + \delta], i=1, \dots, k\}} \geq \mathbb{1}_{\{\mathbf{P}(Z \geq n(\check{x}_i + \delta)) < y_i \leq \mathbf{P}(Z \geq n\check{x}_i), i=1, \dots, k\}},$$

and hence, for $y_{k+1} \geq \mathbf{P}(Z \geq n\check{x}_i)$,

$$\begin{aligned} & \int_0^{y_{k+1}} \cdots \int_0^{y_2} \mathbb{1}_{\{\tilde{Q}^\leftarrow(y_i)/n \in [\check{x}_i, \check{x}_i + \delta], i=1, \dots, k\}} dy_1 \cdots dy_k \\ & \geq \int_0^{y_{k+1}} \cdots \int_0^{y_2} \mathbb{1}_{\{\mathbf{P}(Z \geq n(\check{x}_i + \delta)) < y_i \leq \mathbf{P}(Z \geq n\check{x}_i), i=1, \dots, k\}} dy_1 \cdots dy_k \\ & = \int_0^1 \cdots \int_0^1 \mathbb{1}_{\{\mathbf{P}(Z \geq n(\check{x}_i + \delta)) < y_i \leq \mathbf{P}(Z \geq n\check{x}_i), i=1, \dots, k\}} dy_1 \cdots dy_k \\ & = \prod_{i=1}^k (\mathbf{P}(Z \geq n\check{x}_i) - \mathbf{P}(Z \geq n\check{x}_i + n\delta)). \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \\
 & \geq \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \mathbb{1}_{\{(V_{(1)}, \dots, V_{(k)}) \in A_n(\delta)\}} \\
 & = \int_0^1 \int_0^{y_{n-1}} \dots \int_0^{y_2} e^{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(y_1)/n, \dots, \tilde{Q}^{\leftarrow}(y_k)/n)} (n-1)! \\
 & \quad \times \mathbb{1}_{\{(y_1, \dots, y_k) \in A_n(\delta)\}} dy_1 \cdots dy_{n-2} dy_{n-1} \\
 & \geq (n-1)! e^{L(n)n^\alpha (f(\hat{x}) - 2\epsilon)} \\
 & \quad \times \int_0^1 \int_0^{y_{n-1}} \dots \int_0^{y_2} \mathbb{1}_{\{\tilde{Q}^{\leftarrow}(y_i)/n \in [\check{x}_i, \check{x}_i + \delta], i=1, \dots, k\}} dy_1 \cdots dy_{n-2} dy_{n-1} \\
 & \geq (n-1)! e^{L(n)n^\alpha (f(\hat{x}) - 2\epsilon)} \\
 & \quad \times \prod_{i=1}^k (\mathbf{P}(Z \geq n\check{x}_i) - \mathbf{P}(Z \geq n\check{x}_i + n\delta)) \int_0^1 \int_0^{y_{n-1}} \dots \int_{\mathbf{P}(Z \geq n\check{x}_i)}^{y_{k+2}} dy_{k+1} \cdots dy_{n-2} dy_{n-1} \\
 & = (n-1)! e^{L(n)n^\alpha (f(\hat{x}) - 2\epsilon)} \\
 & \quad \times \prod_{i=1}^k (\mathbf{P}(Z \geq n\check{x}_i) - \mathbf{P}(Z \geq n\check{x}_i + n\delta)) \frac{1}{(n-k-1)!} (1 - \mathbf{P}(Z \geq n\check{x}_i))^{n-k-1}.
 \end{aligned}$$

Since

$$\liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \prod_{i=1}^k (\mathbf{P}(Z \geq n\check{x}_i) - \mathbf{P}(Z \geq n\check{x}_i + n\delta)) = - \sum_{i=1}^k \hat{x}^\alpha = -\check{I}_k(\hat{x})$$

and

$$\liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log (1 - \mathbf{P}(Z \geq n\check{x}_i))^{n-k-1} = 0,$$

we get

$$\begin{aligned}
 \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{E} e^{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)} & \geq f(\hat{x}) - 2\epsilon - \check{I}_k(\hat{x}) \\
 & = \sup_{x \in \mathbb{R}_+^k} \{f(x) - \check{I}_k(x)\} - 2\epsilon.
 \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we arrive at the lower bound.

Turning to the upper bound,

$$\begin{aligned}
 & \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \\
 & = \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \mathbb{1}_{\{\tilde{Q}^{\leftarrow}(V_{(1)})/n > R\}} \\
 & \quad + \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \mathbb{1}_{\{\tilde{Q}^{\leftarrow}(V_{(1)})/n \leq R\}}.
 \end{aligned}$$

For the first term, note that

$$\begin{aligned}
 & \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \mathbb{1}_{\{\tilde{Q}^{\leftarrow}(V_{(1)})/n > R\}} \\
 & \leq \mathbf{E} \exp\{L(n)n^\alpha M\} \mathbb{1}_{\{\tilde{Q}^{\leftarrow}(V_{(1)})/n > R\}} \\
 (5.36) \quad & \leq \exp\{L(n)n^\alpha M\} \mathbf{P}(V_{(1)} \leq \mathbf{P}(Z \geq nR))
 \end{aligned}$$

$$\begin{aligned} &= \exp\{L(n)n^\alpha M\}(1 - (1 - \mathbf{P}(Z \geq nR))^{n-1}) \\ &= \exp\{L(n)n^\alpha M\}(1 - (1 - \exp\{-L(nR)(nR)^\alpha\})^{n-1}). \end{aligned}$$

Also, from the generic inequality $1 - \exp(-z) \leq z$,

$$\begin{aligned} 1 - (1 - 1/x)^y &= 1 - \{(1 - 1/x)^x\}^{y/x} = 1 - \exp\log\{(1 - 1/x)^x\}^{y/x} \\ &= 1 - \exp\{(y/x) \log(1 - 1/x)^x\} \leq (y/x) \log(1 - 1/x)^{-x} \end{aligned}$$

for any $x, y > 0$. Setting $x = \exp(L(nR)(nR)^\alpha)$ and $y = n - 1$, we get

$$\begin{aligned} &1 - (1 - \exp\{-L(nR)(nR)^\alpha\})^{n-1} \\ &\leq (n - 1) \exp\{-L(nR)(nR)^\alpha\} \log(1 - 1/\exp\{L(nR)(nR)^\alpha\})^{-\exp\{L(nR)(nR)^\alpha\}}. \end{aligned}$$

Substituting this into (5.36), we arrive at the upper bound for the first term:

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \mathbb{1}_{\{\tilde{Q}^{\leftarrow}(V_{(1)})/n > R\}} \\ &\leq M - R^\alpha. \end{aligned}$$

For the second term, fix $\epsilon > 0$ and pick $\{\check{x}^{(1)}, \dots, \check{x}^{(m)}\} \subset \mathbb{R}_+^k$ in such a way that

$$\left\{ \prod_{j=1}^k [\check{x}_j^{(l)} - \epsilon, \check{x}_j^{(l)} + \epsilon] \right\}_{l=1, \dots, m}$$

covers $\{(x_1, \dots, x_k) : R \geq x_1 \geq x_2 \geq \dots \geq x_k \geq 0\}$ and $\check{x}_1^{(l)} \geq \check{x}_2^{(l)} \geq \dots \geq \check{x}_k^{(l)} \geq 0$ for $l = 1, \dots, m$. Set

$$A_{n,l}(R) \triangleq \{(y_1, \dots, y_k) \in \mathbb{R}_+^k : y_1 \leq \dots \leq y_k, Q^{\leftarrow}(y_j)/n \in [\check{x}_j^{(l)} - \epsilon, \check{x}_j^{(l)} + \epsilon], j = 1, \dots, k\}.$$

Note that $y_1 \leq \dots \leq y_k$ & $Q^{\leftarrow}(y_1)/n \leq R$ implies $R \geq Q^{\leftarrow}(y_1)/n \geq Q^{\leftarrow}(y_2)/n \geq \dots \geq Q^{\leftarrow}(y_k)/n$, which, in turn, implies $Q^{\leftarrow}(y_j)/n \in [\check{x}_j^{(l)} - \epsilon, \check{x}_j^{(l)} + \epsilon], j = 1, \dots, k$ for some $l \in \{1, \dots, m\}$. Therefore, $\{(y_1, \dots, y_k) \in \mathbb{R}_+^k : y_1 \leq \dots \leq y_k, Q^{\leftarrow}(y_1)/n \leq R\} \subseteq \bigcup_{l=1}^m A_{n,l}(R)$, and hence,

$$\begin{aligned} &\mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \mathbb{1}_{\{\tilde{Q}^{\leftarrow}(V_{(1)})/n \leq R\}} \\ &\leq \sum_{i=1}^m \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \mathbb{1}_{\{(V_{(1)}, \dots, V_{(k)}) \in A_{n,i}(R)\}}. \end{aligned}$$

Note that

$$\begin{aligned} &\mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \mathbb{1}_{\{(V_{(1)}, \dots, V_{(k)}) \in A_{n,i}(R)\}} \\ &= \mathbf{E} e^{L(n)n^\alpha \Lambda_f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)} \\ &\quad \times e^{L(n)n^\alpha \check{I}_k(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)} \mathbb{1}_{\{(V_{(1)}, \dots, V_{(k)}) \in A_{n,i}(R)\}} \\ &\leq e^{L(n)n^\alpha \Lambda_f(\hat{x}_1, \dots, \hat{x}_k)} \mathbf{E} e^{L(n)n^\alpha \check{I}_k(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)} \mathbb{1}_{\{(V_{(1)}, \dots, V_{(k)}) \in A_{n,i}(R)\}} \\ &\leq e^{L(n)n^\alpha \Lambda_f(\hat{x}_1, \dots, \hat{x}_k)} e^{L(n)n^\alpha \check{I}_k(\check{x}_1^{(i)} + \epsilon, \dots, \check{x}_k^{(i)} + \epsilon)} \mathbf{E} \mathbb{1}_{\{(V_{(1)}, \dots, V_{(k)}) \in A_{n,i}(R)\}} \end{aligned}$$

and

$$\begin{aligned}
 & \mathbf{E} \mathbb{1}_{\{(V_{(1)}, \dots, V_{(k)}) \in A_{n,l}(R)\}} \\
 &= \int_0^1 \int_0^{y_{n-1}} \cdots \int_0^{y_2} (n-1)! \mathbb{1}_{\{(y_1, \dots, y_k) \in A_{n,l}(R)\}} dy_1 \cdots dy_{n-2} dy_{n-1} \\
 &\leq (n-1)! \int_0^1 \int_0^{y_{n-1}} \cdots \int_0^{y_2} \mathbb{1}_{\{\tilde{Q}^{\leftarrow}(y_i)/n \in [\check{x}_i^{(l)} - \epsilon, \check{x}_i^{(l)} + \epsilon], i=1, \dots, k\}} dy_1 \cdots dy_{n-2} dy_{n-1} \\
 &\leq (n-1)! \prod_{i=1}^k (\mathbf{P}(Z \geq n\check{x}_i^{(l)} - n\epsilon) - \mathbf{P}(Z \geq n\check{x}_i^{(l)} + n\epsilon)) \\
 &\quad \times \int_0^1 \int_0^{y_{n-1}} \cdots \int_0^{y_{k+2}} dy_{k+1} \cdots dy_{n-2} dy_{n-1} \\
 &= (n-1)! \prod_{i=1}^k (\mathbf{P}(Z \geq n\check{x}_i^{(l)} - n\epsilon) - \mathbf{P}(Z \geq n\check{x}_i^{(l)} + n\epsilon)) \frac{1}{(n-k-1)!} \\
 &\leq n^k \prod_{i=1}^k (\mathbf{P}(Z \geq n\check{x}_i^{(l)} - n\epsilon) - \mathbf{P}(Z \geq n\check{x}_i^{(l)} + n\epsilon)).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \mathbb{1}_{\{\tilde{Q}^{\leftarrow}(V_{(1)}) \leq R\}} \\
 &\leq n^k e^{L(n)n^\alpha \Lambda_f(\hat{x}_1, \dots, \hat{x}_k)} \sum_{i=1}^l e^{L(n)n^\alpha \check{I}_k(\check{x}_1^{(l)} + \epsilon, \dots, \check{x}_k^{(l)} + \epsilon)} \\
 &\quad \times \prod_{i=1}^k (\mathbf{P}(Z \geq n\check{x}_i^{(l)} - n\epsilon) - \mathbf{P}(Z \geq n\check{x}_i^{(l)} + n\epsilon)).
 \end{aligned}$$

Note that

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \prod_{i=1}^k (\mathbf{P}(Z \geq n\check{x}_i^{(l)} - n\epsilon) - \mathbf{P}(Z \geq n\check{x}_i^{(l)} + n\epsilon)) \\
 &= - \sum_{i=1}^k (\check{x}_i^{(l)} - \epsilon)_+^\alpha = -\check{I}_k((\check{x}_1^{(l)} - \epsilon, \dots, \check{x}_k^{(l)} - \epsilon)_+),
 \end{aligned}$$

where $(y)_+$ denotes $\max\{y, 0\}$ and $(y_1, \dots, y_k)_+$ denotes $((y_1)_+, \dots, (y_k)_+)$. This, along with the principle of the largest term,

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \mathbb{1}_{\{\tilde{Q}^{\leftarrow}(V_{(1)}) \leq R\}} \\
 &\leq \max_{l=1, \dots, m} (\Lambda_f(\hat{x}_1, \dots, \hat{x}_k) + \check{I}_k(\check{x}_1^{(l)} + \epsilon, \dots, \check{x}_k^{(l)} + \epsilon) - \check{I}_k((\check{x}_1^{(l)} - \epsilon, \dots, \check{x}_k^{(l)} - \epsilon)_+)) \\
 &\leq \max_{l=1, \dots, m} (\Lambda_f(\hat{x}_1, \dots, \hat{x}_k) + k\epsilon^\alpha).
 \end{aligned}$$

Sending $\epsilon \rightarrow 0$, we get

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \mathbb{1}_{\{\tilde{Q}^{\leftarrow}(V_{(1)}) \leq R\}} \\
 &\leq \Lambda_f(\hat{x}_1, \dots, \hat{x}_k).
 \end{aligned}$$

Now, combining with the bound for the first term, and sending $R \rightarrow \infty$, we get the upper bound:

$$\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} \\ \leq \max\{\Lambda_f(\hat{x}_1, \dots, \hat{x}_k), M - R^\alpha\} \rightarrow \Lambda_f(\hat{x}_1, \dots, \hat{x}_k).$$

Together with the lower bound, we get

$$\lim_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{E} \exp\{L(n)n^\alpha f(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)\} = \Lambda_f(\hat{x}_1, \dots, \hat{x}_k),$$

which in turn allows us to apply Bryc’s inverse Varadhan Lemma to prove that $(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n)$ satisfies the LDP with rate function \check{I}_k . From Theorem 4.14 of Ganesh, O’Connell and Wischik (2004), we see that $(\tilde{Q}^{\leftarrow}(V_{(1)})/n, \dots, \tilde{Q}^{\leftarrow}(V_{(k)})/n, Z/n)$ satisfies the LDP with rate function \check{I}'_k given by

$$(5.37) \quad \check{I}'_k(x_1, \dots, x_{k+1}) = \begin{cases} \sum_{i=1}^{k+1} x_i^\alpha & \text{if } x_1 \geq x_2 \geq \dots \geq x_k \geq 0 \text{ and } x_{k+1} \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

Proceeding with essentially the same argument as in Corollary 5.1 and Lemma 5.3—except for considering a mapping $\tilde{T}_k : (x_1, \dots, x_{k+1}, u_1, \dots, u_k) \mapsto \sum_{i=1}^k x_i \mathbb{1}_{[u_i, 1]} + x_{k+1} \mathbb{1}_{\{1\}}$ instead of the mapping $T_k : (x_1, \dots, x_k, u_1, \dots, u_k) \mapsto \sum_{i=1}^k x_i \mathbb{1}_{[u_i, 1]}$ and $\tilde{\mathbb{D}}_{\leq k}$ instead of $\mathbb{D}_{\leq k}$ —we conclude that $\check{J}_n^k(t) = \frac{1}{n} \sum_{i=1}^k \tilde{Q}^{\leftarrow}(V_{(i)}) \mathbb{1}_{[u_i, 1]}(t) + \frac{1}{n} Z \mathbb{1}_{\{1\}}(t)$ satisfies the LDP with speed $L(n)n^\alpha$ and rate function \check{I}_k in (2.10). \square

PROOF OF LEMMA 2.5. The proof is essentially identical to Lemma 2.2, and hence, omitted. \square

PROOF OF LEMMA 2.6. Let

$$\check{H}_n^k(t) \triangleq \tilde{H}_n^k(t) + \frac{1}{n} \mathbf{E} Z \mathbb{1}_{\{1\}}(t) = \frac{1}{n} \sum_{i=k+1}^{n-1} \tilde{Q}^{\leftarrow}(V_{(i)}) \mathbb{1}_{[u_i, 1]}(t) - \frac{1}{n} \sum_{i=1}^{n-1} \mathbf{E} Z \mathbb{1}_{[u_i, 1]}(t).$$

Since $\mathbf{P}(\|\tilde{H}_n^k\|_\infty \geq \epsilon) \leq \mathbf{P}(\|\tilde{H}_n^k\|_\infty \geq \epsilon/2) + \mathbf{P}(\|\frac{1}{n} \mathbf{E} Z \sum_{i=1}^{n-1} \mathbb{1}_{[u_i, 1]} + \frac{1}{n} \mathbf{E} Z \mathbb{1}_{\{1\}}\|_\infty \geq \epsilon/2)$ and $\mathbf{P}(\|\frac{1}{n} \mathbf{E} Z \sum_{i=1}^{n-1} \mathbb{1}_{[u_i, 1]}(t) + \frac{1}{n} \mathbf{E} Z \mathbb{1}_{\{1\}}\|_\infty > \epsilon/2) = 0$ for large enough n , we only need to prove that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\|\check{H}_n^k\|_\infty > \epsilon) = -\infty.$$

To show this, we fix an arbitrary $\delta \in (0, \epsilon/k)$ and consider the following decomposition:

$$\mathbf{P}(\|\check{H}_n^k\|_\infty > \epsilon) \leq \mathbf{P}(\|\tilde{H}_n^k\|_\infty > \epsilon, \tilde{Q}^{\leftarrow}(V_{(k)}) < n\delta) + \mathbf{P}(\tilde{Q}^{\leftarrow}(V_{(k)}) \geq n\delta).$$

We first bound the second term. Since the density of the k th order statistic of the uniform distribution on $[0, 1]$ is $n \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k}$,

$$\mathbf{P}(\tilde{Q}^{\leftarrow}(V_{(k)}) \geq n\delta) = \mathbf{P}(V_{(k)} \leq \mathbf{P}(Z \geq n\delta)) \leq \int_0^{\mathbf{P}(Z \geq n\delta)} n \binom{n-1}{k-1} x^{k-1} dx \\ = \binom{n}{k} (\mathbf{P}(Z \geq n\delta))^k = \binom{n}{k} \exp(-kL(n\delta)(n\delta)^\alpha)$$

and hence, $\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\tilde{Q}_n^{\leftarrow}(V_{(k)}) > n\delta) \leq -k\delta^\alpha$. For the first term,

$$\begin{aligned} & \mathbf{P}(\|\tilde{H}_n^k\|_\infty > \epsilon, \tilde{Q}^{\leftarrow}(V_{(k)}) < n\delta) \\ &= \mathbf{P}\left(\sup_{t \in [0,1]} \check{H}_n^k(t) > \epsilon, \tilde{Q}^{\leftarrow}(V_{(k)}) \leq n\delta\right) + \mathbf{P}\left(\sup_{t \in [0,1]} -\check{H}_n^k > \epsilon, \tilde{Q}^{\leftarrow}(V_{(k)}) \leq n\delta\right) \\ &\leq \mathbf{P}\left(\max_{1 \leq j \leq n-1} \sum_{i=1}^j (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > n\epsilon\right) \\ &\quad + \mathbf{P}\left(\max_{1 \leq j \leq n-1} \sum_{i=1}^j (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) + kn\delta > n\epsilon\right). \end{aligned}$$

Note that from Lemma 5.4,

$$\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \mathbf{P}\left(\max_{1 \leq j \leq n-1} \sum_{i=1}^j (Z_i \mathbb{1}_{\{Z_i \leq n\delta\}} - \mathbf{E}Z) > n\epsilon\right) \leq -(\epsilon/3)^\alpha (\epsilon/\delta)^{1-\alpha}$$

and from 5.5, since $\delta < \epsilon/k$,

$$\limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \mathbf{P}\left(\max_{1 \leq j \leq n-1} \sum_{i=1}^j (\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \leq n\delta\}}) + kn\delta > n\epsilon\right) = -\infty.$$

Therefore,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \mathbf{P}(\|\tilde{H}_n^k\|_\infty > \epsilon, Q_n^{\leftarrow}(V_{(k)}) \leq n\delta) &\leq \max\{-(\epsilon/3)^\alpha (\epsilon/\delta)^{1-\alpha}, -\infty\} \\ &= -(\epsilon/3)^\alpha (\epsilon/\delta)^{1-\alpha}. \end{aligned}$$

Applying the principle of the maximum term once again,

$$\begin{aligned} \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \mathbf{P}(\|\tilde{H}_n^k\|_\infty > \epsilon) &\leq \lim_{k \rightarrow \infty} \max\{-(\epsilon/3)^\alpha (\epsilon/\delta)^{1-\alpha}, -k\delta^\alpha\} \\ &= -(\epsilon/3)^\alpha (\epsilon/\delta)^{1-\alpha}. \end{aligned}$$

Since δ can be chosen arbitrarily small,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \mathbf{P}(\|\tilde{H}_n^k\|_\infty > \epsilon) = -\infty. \quad \square$$

5.7. *Proof of Theorem 2.3.* We follow a similar program as in Section 2.1 and the earlier subsections of this section. Let $\bar{Q}_n^{(i)}(j) \triangleq Q_n^{\leftarrow}(\Gamma_j^{(i)})/n$ where $Q_n^{\leftarrow}(t) = \inf\{s > 0 : nv[s, \infty) < t\}$ and $\Gamma_l^{(i)} = E_1^{(i)} + \dots + E_l^{(i)}$ where $E_j^{(i)}$'s are independent standard exponential random variables. Let $U_j^{(i)}$ be independent uniform random variables in $[0,1]$ and $Z_n^{(i)} \triangleq (\bar{Q}_n^{(i)}(1), \dots, \bar{Q}_n^{(i)}(k), U_1^{(i)}, \dots, U_k^{(i)})$. The following corollary is an immediate consequence of Corollary 5.1 and Theorem 4.14 of Ganesh, O'Connell and Wischik (2004).

COROLLARY 5.2. $(Z_n^{(1)}, \dots, Z_n^{(d)})$ satisfies the LDP in $\prod_{i=1}^d (\mathbb{R}_+^k \times [0, 1]^k)$ with rate function $\hat{I}_k^d(z^{(1)}, \dots, z^{(d)}) \triangleq \sum_{j=1}^d \hat{I}_k(z^{(j)})$ where $z^{(j)} = (x_1^{(j)}, \dots, x_k^{(j)}, u_1^{(j)}, \dots, u_k^{(j)})$ for each $j \in \{1, \dots, d\}$.

$$\text{Let } \hat{J}_n^{\leq k(i)} \triangleq \sum_{j=1}^k \bar{Q}_n^{(i)}(j) \mathbb{1}_{[U_j^{(i)}, 1]}.$$

LEMMA 5.6. $(\hat{J}_n^{\leq k(1)}, \dots, \hat{J}_n^{\leq k(d)})$ satisfies the LDP in $\prod_{i=1}^d \mathbb{D}([0, 1], \mathbb{R})$ with speed $L(n)n^\alpha$ and rate function

$$I_k^d(\xi_1, \dots, \xi_d) \triangleq \sum_{i=1}^d I_k(\xi_i) = \begin{cases} \sum_{i=1}^d \sum_{t: \xi_i(t) \neq \xi_i(t-)} (\xi_i(t) - \xi_i(t-))^\alpha & \text{if } \xi_i \in \mathbb{D}_{\leq k} \text{ for } i = 1, \dots, d, \\ \infty & \text{otherwise.} \end{cases}$$

PROOF. Since I_{k_i} is lower semicontinuous in $\prod_{i=1}^d \mathbb{D}([0, 1], \mathbb{R})$ for each i , I_{k_1, \dots, k_d} is a sum of lower semicontinuous functions, and hence, is lower semicontinuous itself. The rest of the proof for the LDP upper bound and the lower bounds mirrors that of the one-dimensional case (Lemma 5.3) closely, and hence, omitted. \square

PROOF OF LEMMA 2.7. Again, we consider the same distributional relation for each coordinate as in the one-dimensional case:

$$\bar{J}_n^{k(i)} \stackrel{\mathcal{D}}{=} \underbrace{\frac{1}{n} \sum_{j=1}^k Q_n^{(i)}(j) \mathbb{1}_{[U_j, 1]}}_{=\hat{J}_n^{\leq k(i)}} - \frac{1}{n} \mathbb{1}_{\{\tilde{N}_n^{(i)} < k\}} \underbrace{\sum_{j=\tilde{N}_n^{(i)}+1}^k Q_n^{(i)}(j) \mathbb{1}_{[U_j^{(i)}, 1]}}_{=\check{J}_n^{\leq k(i)}}.$$

Note that this distributional equality holds jointly w.r.t. $i = 1, \dots, d$ due to the assumed independence. Let F be a closed set and write

$$\begin{aligned} & \mathbf{P}((\bar{J}_n^{k(1)}, \dots, \bar{J}_n^{k(d)}) \in F) \\ & \leq \mathbf{P}\left((\hat{J}_n^{\leq k(1)}, \dots, \hat{J}_n^{\leq k(d)}) \in F, \sum_{i=1}^d \mathbb{1}_{\{\tilde{N}_n^{(i)} < k\}} = 0\right) + \sum_{i=1}^d \mathbf{P}(\mathbb{1}_{\{N_n^{(i)} < k\}} \neq 0) \\ & \leq \mathbf{P}((\hat{J}_n^{\leq k(1)}, \dots, \hat{J}_n^{\leq k(d)}) \in F) + \sum_{i=1}^d \mathbf{P}(\mathbb{1}_{\{N_n^{(i)} < k\}} \neq 0). \end{aligned}$$

From Lemma 5.6 and the principle of the largest term,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}((\bar{J}_n^{k(1)}, \dots, \bar{J}_n^{k(d)}) \in F)}{L(n)n^\alpha} \\ & \leq \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}((\hat{J}_n^{\leq k(1)}, \dots, \hat{J}_n^{\leq k(d)}) \in F)}{L(n)n^\alpha} \vee \max_{i=1, \dots, d} \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\tilde{N}_n^{(i)} < k)}{L(n)n^\alpha} \\ & \leq - \inf_{(\xi_1, \dots, \xi_d) \in F} I_k^d(\xi_1, \dots, \xi_d). \end{aligned}$$

Turning to the lower bound, let G be an open set. Since the lower bound is trivial in case $\inf_{x \in G} I_k(x) = \infty$, we focus on the case $\inf_{x \in G} I_k(x) < \infty$. In this case, using a reasoning similar to the one leading to (5.22),

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}((\bar{J}_n^{k(1)}, \dots, \bar{J}_n^{k(d)}) \in G)}{L(n)n^\alpha} \\ & \geq \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}((\bar{J}_n^{k(1)}, \dots, \bar{J}_n^{k(d)}) \in G, \sum_{i=1}^d \mathbb{1}_{\{\tilde{N}_n^{(i)} \geq k\}} = 0)}{L(n)n^\alpha} \end{aligned}$$

$$\begin{aligned}
 &= \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}((\hat{J}_n^{\leq k(1)}, \dots, \hat{J}_n^{\leq k(d)}) \in G, \sum_{i=1}^d \mathbb{1}\{\tilde{N}_n^{(i)} \geq k\} = 0)}{L(n)n^\alpha} \\
 &\geq \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log(\mathbf{P}((\hat{J}_n^{\leq k(1)}, \dots, \hat{J}_n^{\leq k(d)}) \in G) - d\mathbf{P}(\tilde{N}_n^{(1)} < k)) \\
 &= \liminf_{n \rightarrow \infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}((\hat{J}_n^{\leq k(1)}, \dots, \hat{J}_n^{\leq k(d)}) \in G) \\
 &\geq - \inf_{(\xi_1, \dots, \xi_d) \in G} I_k^d(\xi_1, \dots, \xi_d). \quad \square
 \end{aligned}$$

The proof of Lemma 2.8 is completely analogous to the one-dimensional case, and therefore omitted.

APPENDIX A: M'_1 TOPOLOGY AND GOODNESS OF THE RATE FUNCTION

Let $\tilde{\mathbb{D}}[0, 1]$ be the space of functions from $[0, 1]$ to \mathbb{R} such that the left limit exists at each $t \in (0, 1]$, the right limit exists at each $t \in [0, 1)$, and

$$(A.1) \quad \xi(t) \in [\xi(t-) \wedge \xi(t+), \xi(t-) \vee \xi(t+)]$$

for each $t \in [0, 1]$ where we interpret $\xi(0-)$ as 0 and $\xi(1+)$ as $\xi(1)$.

DEFINITION 1. For $\xi \in \tilde{\mathbb{D}}$, define the extended completed graph $\Gamma'(\xi)$ of ξ as

$$\Gamma'(\xi) \triangleq \{(u, t) \in \mathbb{R} \times [0, 1] : u \in [\xi(t-) \wedge \xi(t+), \xi(t-) \vee \xi(t+)]\},$$

where $\xi(0-) \triangleq 0$ and $\xi(1+) \triangleq \xi(1)$. Define an order on the graph $\Gamma'(\xi)$ by setting $(u_1, t_1) < (u_2, t_2)$ if either:

- $t_1 < t_2$; or
- $t_1 = t_2$ and $|\xi(t_1-) - u_1| < |\xi(t_2-) - u_2|$.

We call a continuous nondecreasing function $(u, t) = ((u(s), t(s)), s \in [0, 1])$ from $[0, 1]$ to $\mathbb{R} \times [0, 1]$ a parametrization of $\Gamma'(\xi)$ —or a parametrization of ξ —if $\Gamma'(\xi) = \{(u(s), t(s)) : s \in [0, 1]\}$.

DEFINITION 2. Define the M'_1 metric on \mathbb{D} as follows:

$$d_{M'_1}(\xi, \zeta) \triangleq \inf_{\substack{(u,t) \in \Gamma'(\xi) \\ (v,r) \in \Gamma'(\zeta)}} \{\|u - v\|_\infty + \|t - r\|_\infty\}.$$

Let $\mathbb{D}^\uparrow \triangleq \{\xi \in \mathbb{D} : \xi \text{ is nondecreasing and } \xi(0) \geq 0\}$.

PROPOSITION A.1. Suppose that $\hat{\xi}_0 \in \tilde{\mathbb{D}}$ with $\hat{\xi}_0(0) \geq 0$ and $\xi_n \in \mathbb{D}^\uparrow$ for each $n \geq 1$. If $T \triangleq \{t \in [0, 1] : \xi_n(t) \rightarrow \hat{\xi}_0(t)\}$ is dense on $[0, 1]$ and $1 \in T$, then $\xi_n \xrightarrow{M'_1} \xi_0 \in \mathbb{D}^\uparrow$ where $\xi_0(t) \triangleq \lim_{s \downarrow t} \hat{\xi}_0(s)$ for $t \in [0, 1)$ and $\xi_0(1) \triangleq \hat{\xi}_0(1)$.

PROOF. It is easy to check that $\hat{\xi}_0$ has to be nonnegative and nondecreasing, and for such $\hat{\xi}_0$, ξ_0 should be in \mathbb{D}^\uparrow . Let (x, t) be a parametrization of $\Gamma'(\hat{\xi}_0)$, and let $\epsilon > 0$ be given. Note that $\Gamma'(\xi_0)$ and $\Gamma'(\hat{\xi}_0)$ coincide. Therefore, the proposition is proved if we show that there exists an integer N_0 such that for each $n \geq N_0$, $\Gamma'(\xi_n)$ can be parametrized by some (y, r) such that

$$(A.2) \quad \|x - y\|_\infty + \|t - r\|_\infty \leq \epsilon.$$

We start with making an observation that one can always construct a finite number of points $S = \{s_i\}_{i=0,1,\dots,m} \subseteq [0, 1]$ such that:

- (S1) $0 = s_0 < s_1 < \dots < s_m = 1$;
- (S2) $t(s_i) - t(s_{i-1}) < \epsilon/4$ for $i = 1, \dots, m$;
- (S3) $x(s_i) - x(s_{i-1}) < \epsilon/8$ for $i = 1, \dots, m$;
- (S4) if $t(s_{k-1}) < t(s_k) < t(s_{k+1})$ then $t(s_k) \in T$;
- (S5) if $t(s_{k-1}) < t(s_k) = t(s_{k+1})$, then $t(s_{k-1}) \in T$; if, in addition, $k - 1 > 0$, then $t(s_{k-2}) < t(s_{k-1})$;
- (S6) if $t(s_{k-1}) = t(s_k) < t(s_{k+1})$, then $t(s_{k+1}) \in T$; if, in addition, $k + 1 < m$, then $t(s_{k+1}) < t(s_{k+2})$.

One way to construct such a set is to start with S such that (S1), (S2) and (S3) are satisfied. This is always possible because x and t are continuous and nondecreasing. Suppose that (S4) is violated for some three consecutive points in S , say s_{k-1}, s_k, s_{k+1} . We argue that it is always possible to eliminate this violation by either adding an additional point \hat{s}_k or moving s_k slightly. More specifically, if there exists $\hat{s}_k \in (s_{k-1}, s_{k+1}) \setminus \{s_k\}$ such that $t(\hat{s}_k) = t(s_k)$, add \hat{s}_k to S . If there is no such \hat{s}_k , $t(\cdot)$ has to be strictly increasing at s_k , and hence, from the continuity of x and t along with the fact that T is dense, we can deduce that there has to be $\tilde{s}_k \in (s_{k-1}, s_{k+1})$ such that $t(\tilde{s}_k) \in T$ and $|t(\tilde{s}_k) - t(s_k)|$ and $|x(\tilde{s}_k) - x(s_k)|$ are small enough so that (S2) and (S3) are still satisfied when we replace s_k with \tilde{s}_k in S . Iterating this procedure, we can construct S so that (S1)–(S4) are satisfied. Now turning to (S5), suppose that it is violated for three consecutive points s_{k-1}, s_k, s_{k+1} in S . Since T is dense and t is continuous, one can find \hat{s}_k between s_{k-1} and s_k such that $t(s_{k-1}) < t(\hat{s}_k) < t(s_k)$ and $t(\hat{s}_k) \in T$. Note that after adding \hat{s}_k to S , (S2), (S3) and (S4) should still hold while the number of triplets that violate (S5) is reduced by one. Repeating this procedure for each triplet that violates (S5), one can construct a new S which satisfies (S1)–(S5). One can also check that the same procedure for the triplets that violate (S6) can reduce the number of triplets that violate (S6) while not introducing any new violation for (S2), (S3), (S4) and (S5). Therefore, S can be augmented so that the resulting finite set satisfies (S6) as well. Set $\hat{S} \triangleq \{s_i \in S : t(s_i) \in T, t(s_{i-1}) < t(s_i) \text{ in case } i > 0, t(s_i) < t(s_{i+1}) \text{ in case } i < m\}$ and let N_0 be such that $n \geq N_0$ implies $|\xi_n(t(s_i)) - \hat{\xi}_0(t(s_i))| < \epsilon/8$ for all $s_i \in \hat{S}$. Now we will fix $n \geq N_0$ and proceed to showing that we can re-parametrize an arbitrary parametrization (y', r') of $\Gamma(\xi_n)$ to obtain a new parametrization (y, r) such that (A.2) is satisfied. Let (y', r') be an arbitrary parametrization of $\Gamma(\xi_n)$. For each i such that $s_i \in \hat{S}$, let $s'_i \triangleq \max\{s \geq 0 : r'(s) = t(s_i)\}$ so that $r'(s'_i) = t(s_i)$ and $\xi_n(r'(s'_i)) = y'(s'_i)$. For i 's such that $s_i \in S \setminus \hat{S}$, note that there are three possible cases: $t(s_i) \in (0, 1)$, $t(s_i) = 0$ and $t(s_i) = 1$. Since the other cases can be handled in similar (but simpler) manners, we focus on the case $t(s_i) \in (0, 1)$. In this case, one can check that there exist k and j such that $k \leq i \leq k + j$, $t(s_{k-1}) < t(s_k) = t(s_{k+j}) < t(s_{k+j+1})$ and $s_{k-1}, s_{k+j+1} \in \hat{S}$. Here we assume that $k > 1$; the case $k = 1$ is essentially identical but simpler—hence omitted. Note that from the monotonicity of $\hat{\xi}_0$ and (A.1),

$$\begin{aligned} x(s_{k-2}) &\leq \hat{\xi}_0(t(s_{k-2})+) \leq \hat{\xi}_0(t(s_{k-1})-) \leq \hat{\xi}_0(t(s_{k-1})) \leq \hat{\xi}_0(t(s_{k-1})+) \\ &\leq \hat{\xi}_0(t(s_k)-) \leq x(s_k), \end{aligned}$$

that is, $\hat{\xi}_0(t(s_{k-1})) \in [x(s_{k-2}), x(s_k)]$, which along with (S3) implies $|\hat{\xi}_0(t(s_{k-1})) - x(s_{k-1})| < \epsilon/8$. From this, (S5) and the constructions of s'_{k-1} and N_0 ,

$$\begin{aligned} |y'(s'_{k-1}) - x(s_{k-1})| &= |\xi_n(r'(s'_{k-1})) - x(s_{k-1})| \\ &= |\xi_n(r'(s'_{k-1})) - \hat{\xi}_0(t(s_{k-1}))| + |\hat{\xi}_0(t(s_{k-1})) - x(s_{k-1})| \\ &= |\xi_n(t(s_{k-1})) - \hat{\xi}_0(t(s_{k-1}))| + |\hat{\xi}_0(t(s_{k-1})) - x(s_{k-1})| < \epsilon/4. \end{aligned}$$

Following the same line of reasoning, we can show that $|y'(s'_{k+j+1}) - x(s_{k+j+1})| < \epsilon/4$. Noting that both x and y' are nondecreasing, there have to exist $s'_k, s'_{k+1}, \dots, s'_{k+j}$ such that $s'_{k-1} < s'_k < \dots < s'_{k+j} < s'_{k+j+1}$ and $|y'(s'_l) - x(s_l)| < \epsilon/4$ for $l = k, k + 1, \dots, k + j$. Note also that from (S2)

$$\begin{aligned} t(s_l) - \epsilon/4 = t(s_k) - \epsilon/4 < t(s_{k-1}) = r'(s'_{k-1}) \leq r'(s'_l) \leq r'(s'_{k+j+1}) \\ = t(s_{k+j+1}) < t(s_{k+j}) + \epsilon/4 = t(s_l) + \epsilon/4, \end{aligned}$$

and hence, $|r'(s'_l) - t(s_l)| < \epsilon/4$ for $l = k, \dots, k + j$ as well. Repeating this procedure for the i 's for which s'_i is not designated until there is no such i 's are left, we can construct s'_1, \dots, s'_m in such a way that

$$|y'(s'_i) - x(s_i)| < \epsilon/4 \quad \text{and} \quad |r'(s'_i) - t(s_i)| < \epsilon/4$$

for all i 's. Now, define a (piecewise linear) map $\lambda : [0, 1] \rightarrow [0, 1]$ by setting $\lambda(s_i) = s'_i$ at each s_i 's and interpolating (s_i, s'_i) 's in between. Then, $y \triangleq y' \circ \lambda$ and $r \triangleq r' \circ \lambda$ consist a parametrization (y, r) of $\Gamma(\xi_n)$ such that $|x(s_i) - y(s_i)| < \epsilon/4$ and $|t(s_i) - r(s_i)| < \epsilon/4$ for each $i = 1, \dots, m$. Due to the monotonicity of x, y, t and r along with (S2) and (S3), we conclude that $\|y - x\|_\infty < \epsilon/2$ and $\|t - r\|_\infty < \epsilon/2$, proving (A.2). \square

PROPOSITION A.2. *Let K be a subset of \mathbb{D}^\uparrow . If $M \triangleq \sup_{\xi \in K} \|\xi\|_\infty < \infty$ then K is relatively compact w.r.t. the M'_1 topology.*

PROOF. Let $\{\xi_n\}_{n=1,2,\dots}$ be a sequence in K . We prove that there exists a subsequence $\{\xi_{n_k}\}_{k=1,2,\dots}$ and $\xi_0 \in \mathbb{D}$ such that $\xi_{n_k} \xrightarrow{M'_1} \xi_0$ as $k \rightarrow \infty$. Let $T \triangleq \{t_n\}_{n=1,2,\dots}$ be a dense subset of $[0, 1]$ such that $1 \in T$. By the assumption, $\sup_{n=1,2,\dots} |\xi_n(t_1)| < M$, and hence there is a subsequence $\{n_k^{(1)}\}_{k=1,2,\dots}$ of $\{1, 2, \dots\}$ such that $\xi_{n_k^{(1)}}(t_1)$ converges to a real number $x_1 \in [-M, M]$. For each $i \geq 1$, given $\{n_k^{(i)}\}$, one can find a further subsequence $\{n_k^{(i+1)}\}_{k=1,2,\dots}$ of $\{n_k^{(i)}\}_{k=1,2,\dots}$ in such a way that $\xi_{n_k^{(i+1)}}(t_{i+1})$ converges to a real number x_{i+1} . Let $n_k \triangleq n_k^{(k)}$ for each $k = 1, 2, \dots$. Then, $\xi_{n_k}(t_i) \rightarrow x_i$ as $k \rightarrow \infty$ for each $i = 1, 2, \dots$. Define a function $\hat{\xi}_0 : T \rightarrow \mathbb{R}$ on T so that $\hat{\xi}_0(t_i) = x_i$. We claim that $\hat{\xi}_0$ has left limit everywhere; more precisely, we claim that for each $s \in (0, 1]$, if a sequence $\{s_n\} \subseteq T \cap [0, s)$ is such that $s_n \rightarrow s$ as $n \rightarrow \infty$, then $\hat{\xi}_0(s_n)$ converges as $n \rightarrow \infty$. (With a similar argument, one can show that $\hat{\xi}_0$ has right limit everywhere—that is, for each $s \in [0, 1)$, if a sequence $\{s_n\} \subseteq T \cap (s, 1]$ is such that $s_n \rightarrow s$ as $n \rightarrow \infty$, then $\hat{\xi}_0(s_n)$ converges as $n \rightarrow \infty$.) To prove this claim, we proceed with proof by contradiction; suppose that the conclusion of the claim is not true—that is, $\hat{\xi}_0(s_n)$ is not convergent. Then, there exist a $\epsilon > 0$ and a subsequence r_n of s_n such that

$$(A.3) \quad |\hat{\xi}_0(r_{n+1}) - \hat{\xi}_0(r_n)| > \epsilon.$$

Note that since $\hat{\xi}_0$ is a pointwise limit of nondecreasing functions $\{\xi_{n_k}\}$ (restricted on T):

- $\hat{\xi}_0$ is also nondecreasing on T , (monotonicity)
- $\sup_{t \in T} |\hat{\xi}_0(t)| < M$. (boundedness)

However, these two are contradictory to each other since the monotonicity together with (A.3) implies $\hat{\xi}_0(r_{N_0+j}) > \hat{\xi}_0(r_{N_0}) + j\epsilon$, which leads to the contradiction to the boundedness for a large enough j . This proves the claim.

Note that the above claim means that $\hat{\xi}_0$ has both left and right limit at each point of $T \cap (0, 1)$, and due to the monotonicity, the function value has to be between the left limit

and the right limit. Since T is dense in $[0, 1]$, we can extend $\hat{\xi}$ from T to $[0, 1]$ by setting $\hat{\xi}_0(t) \triangleq \lim_{t_i \rightarrow t} \hat{\xi}_0(t_i)$ for $t \in [0, 1] \setminus T$. Note that such $\hat{\xi}_0$ is an element of $\tilde{\mathbb{D}}$ and satisfies the conditions of Proposition A.1. We therefore conclude that $\xi_{n_k} \rightarrow \xi_0 \in \mathbb{D}^\uparrow$ in M'_1 as $k \rightarrow \infty$, where $\xi_0(t) \triangleq \lim_{s \downarrow t} \hat{\xi}_0(s)$ for $t \in [0, 1)$ and $\xi_0(1) \triangleq \hat{\xi}_0(1)$. This proves that K is indeed relatively compact. \square

Recall that our rate function for one-sided compound Poisson processes is as follows:

$$I_{M'_1}(\xi) = \begin{cases} \sum_{t \in [0, 1]} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \text{ is a nondecreasing pure jump function} \\ & \text{with } \xi(0) \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

PROPOSITION A.3. $I_{M'_1}$ is a good rate function w.r.t. the M'_1 topology.

PROOF. In view of Proposition A.2, it is enough to show that the sublevel sets of $I_{M'_1}$ are closed. Let a be an arbitrary finite constant, and consider the sublevel set $\Psi_{I_{M'_1}}(a) \triangleq \{\xi \in \mathbb{D} : I_{M'_1}(\xi) \leq a\}$. Let $\xi^c \in \mathbb{D}$ be any given path that does not belong to $\Psi_{I_{M'_1}}(a)$. We will show that there exists $\epsilon > 0$ such that $d_{M'_1}(\xi^c, \Psi_{I_{M'_1}}(a)) \geq \epsilon$. Note that $\Psi_{I_{M'_1}}(a)^c = A \cup B \cup C \cup D$ where

$$A = \{\xi \in \mathbb{D} : \xi(0) < 0\},$$

$$B = \{\xi \in \mathbb{D} : \xi \text{ is not a nondecreasing function}\},$$

$$C = \{\xi \in \mathbb{D} : \xi \text{ is nondecreasing but not a pure jump function}\},$$

$$D = \left\{ \xi \in \mathbb{D} : \xi \text{ is a nondecreasing pure jump function with } \xi(0) \geq 0 \right.$$

$$\left. \text{and } \sum_{t \in [0, 1]} (\xi(t) - \xi(t-))^\alpha > a \right\}.$$

In each case, we will show that ξ^c is bounded away from $\Psi_{I_{M'_1}}(a)$. In case $\xi^c \in A$, note that for any parametrization (x, t) of ξ^c , there has to be $s^* \in [0, 1]$ such that $x(s^*) = \xi^c(0) < 0$. On the other hand, $y(s) \geq 0$ for all $s \in [0, 1]$ for any parametrization (y, r) of ζ such that $\zeta \in \Psi_I(a)$, and hence, $\|x - y\|_\infty \geq \xi^c(0)$. Therefore,

$$d_{M'_1}(\xi^c, \zeta) \geq \inf_{\substack{(x, t) \in \Gamma(\xi^c) \\ (y, r) \in \Gamma(\zeta)}} \|x - y\|_\infty \geq |\xi^c(0)|.$$

Since ζ was an arbitrary element of $\Psi_{I_{M'_1}}(a)$, we conclude that $d_{M'_1}(\xi^c, \Psi_I(a)) \geq \epsilon$ with $\epsilon = |\xi^c(0)|$.

Using a similar argument, it is straightforward to show that any $\xi^c \in B$ is bounded away from $\Psi_I(a)^c$.

If $\xi^c \in C$, there has to be T_s and T_t such that $0 \leq T_s < T_t \leq 1$, ξ^c is continuous on $[T_s, T_t]$ and $c \triangleq \xi^c(T_t) - \xi^c(T_s) > 0$. Pick a small enough $\epsilon \in (0, 1)$ so that

$$(A.4) \quad (4\epsilon)^{\alpha-1}(c - 5\epsilon) > a.$$

Note that since ξ^c is uniformly continuous on $[T_s, T_t]$, there exists $\delta > 0$ such that $|\xi^c(t) - \xi^c(s)| < \epsilon$ if $|t - s| \leq \delta$. In particular, we pick δ so that $\delta < \epsilon$ and $T_s + \delta < T_t - \delta$. We claim that

$$d_{M'_1}(\Psi_{I_{M'_1}}(a), \xi^c) \geq \delta.$$

Suppose not. That is, there exists $\zeta \in \Psi_{I_{M'_1}}(a)$ such that $d_{M'_1}(\zeta, \xi^c) < \delta$. Let $(x, t) \in \Gamma(\xi^c)$ and $(y, r) \in \Gamma(\zeta)$ be the parametrizations of ξ^c and ζ , respectively, such that $\|x - y\|_\infty + \|t - r\|_\infty < \delta$. Since $I_{M'_1}(\zeta) \leq a < \infty$, one can find a finite set $K \subseteq \{t \in [0, 1] : \zeta(t) - \zeta(t-) > 0\}$ of jump times of ζ in such a way that $\sum_{t \notin K} (\zeta(t) - \zeta(t-))^\alpha < \epsilon$. Note that since $\epsilon \in (0, 1)$, this implies that $\sum_{t \notin K} (\zeta(t) - \zeta(t-)) < \epsilon$. Let T_1, \dots, T_k denote (the totality of) the jump times of ζ in $K \cap (T_s + \delta, T_t - \delta]$, and let $T_0 \triangleq T_s + \delta$ and $T_{k+1} \triangleq T_t - \delta$. That is, $\{T_1, \dots, T_k\} = K \cap (T_s + \delta, T_t - \delta] = K \cap (T_0, T_{k+1}]$. Note that:

- There exist s_0 and s_{k+1} in $[0, 1]$ such that

$$y(s_0) = \zeta(T_0), \quad r(s_0) = T_0, \quad y(s_{k+1}) = \zeta(T_{k+1}), \quad r(s_{k+1}) = T_{k+1}.$$

- For each $i = 1, \dots, k$, there exists s_i^+ and s_i^- such that

$$r(s_i^+) = r(s_i^-) = T_i, \quad y(s_i^+) = \zeta(T_i), \quad y(s_i^-) = \zeta(T_i-).$$

Since $t(s_{k+1}) \in [r(s_{k+1}) - \delta, r(s_{k+1}) + \delta] \subseteq [T_s, T_t]$, and ξ^c is continuous on $[T_s, T_t]$ and nondecreasing,

$$\begin{aligned} y(s_{k+1}) &\geq x(s_{k+1}) - \delta = \xi^c(t(s_{k+1})) - \delta \geq \xi^c(r(s_{k+1}) - \delta) - \delta = \xi^c(T_{k+1} - \delta) - \delta \\ &\geq \xi^c(T_{k+1}) - \epsilon - \delta \geq \xi^c(T_{k+1}) - 2\epsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} y(s_0) &\leq x(s_0) + \delta = \xi^c(t(s_0)) + \delta \leq \xi^c(r(s_0) + \delta) + \delta \\ &= \xi^c(T_0 + \delta) + \delta \leq \xi^c(T_0) + \epsilon + \delta \leq \xi^c(T_0) + 2\epsilon. \end{aligned}$$

Subtracting the two equations,

$$y(s_{k+1}) - y(s_0) \geq \xi^c(T_{k+1}) - \xi^c(T_0) - 4\epsilon = c - 4\epsilon.$$

Note that

$$\begin{aligned} \sum_{i=1}^k (\zeta(T_i) - \zeta(T_i-)) &= \zeta(T_{k+1}) - \zeta(T_0) - \sum_{t \in (T_0, T_{k+1}] \cap K^c} (\zeta(t) - \zeta(t-)) \\ \text{(A.5)} \qquad \qquad \qquad &\geq \zeta(T_{k+1}) - \zeta(T_0) - \epsilon \\ &= y(s_{k+1}) - y(s_0) - \epsilon \geq c - 5\epsilon. \end{aligned}$$

On the other hand,

$$\begin{aligned} y(s_i^+) - y(s_i^-) &\leq (x(s_i^+) + \delta) - (x(s_i^-) - \delta) = x(s_i^+) - x(s_i^-) + 2\delta \\ &\leq \xi^c(t(s_i^+)) - \xi^c(t(s_i^-)) + 2\delta \\ &\leq \xi^c(r(s_i^+) + \delta) - \xi^c(r(s_i^-) - \delta) + 2\delta \\ &\leq \xi^c(T_i + \delta) - \xi^c(T_i - \delta) + 2\delta \leq 2\epsilon + 2\delta \leq 4\epsilon. \end{aligned}$$

That is, $(\zeta(T_i) - \zeta(T_i-))^{\alpha-1} = (y(s_i^+) - y(s_i^-))^{\alpha-1} \geq (4\epsilon)^{\alpha-1}$. Combining this with (A.5),

$$\begin{aligned} I_{M'_1}(\zeta) &\geq \sum_{i=1}^k (\zeta(T_i) - \zeta(T_i-))^\alpha = \sum_{i=1}^k (\zeta(T_i) - \zeta(T_i-)) (\zeta(T_i) - \zeta(T_i-))^{\alpha-1} \\ &\geq (c - 5\epsilon)(4\epsilon)^{\alpha-1} > a, \end{aligned}$$

which is contradictory to the assumption that $\zeta \in \Psi_{I_{M'_1}}(a)$. Therefore, the claim that ξ^c is bounded away from $\Psi_{I_{M'_1}}(a)$ by δ is proved.

Finally, suppose that $\xi^c \in D$. That is, there exists $\{(z_i, u_i) \in \mathbb{R}_+ \times [0, 1]\}_{i=1, \dots}$ such that $\xi^c = \sum_{i=1}^{\infty} z_i \mathbb{1}_{[u_i, 1]}$ where u_i 's are all distinct and $\sum_{i=1}^{\infty} z_i^\alpha > a$. Pick sufficiently large k and sufficiently small $\delta > 0$ such that $\sum_{i=1}^k (z_i - 2\delta)^\alpha > a$ and $u_{i+1} - u_i > 2\delta$ for $i = 1, \dots, k-1$. We claim that $d_{M_1'}(\zeta, \xi^c) \geq \delta$ for any $\zeta \in \Psi_{I_{M_1'}}(a)$. Suppose not and there is $\zeta \in \Psi_{I_{M_1'}}(a)$ such that $\|x - y\|_\infty + \|t - r\|_\infty < \delta$ for some parametrizations $(x, t) \in \Gamma(\xi^c)$ and $(y, r) \in \Gamma(\zeta)$. Note first that there are s_i^+ 's and s_i^- 's for each $i = 1, \dots, k$ such that $t(s_i^-) = t(s_i^+) = u_i$, $x(s_i^-) = \xi^c(u_i^-)$ and $x(s_i^+) = \xi^c(u_i)$. Since $y(s_i^+) \geq x(s_i^+) - \delta = \xi^c(u_i) - \delta$ and $y(s_i^-) \leq x(s_i^-) + \delta = \xi^c(u_i^-) + \delta$,

$$\zeta(r(s_i^+)) - \zeta(r(s_i^-)) \geq y(s_i^+) - y(s_i^-) \geq \xi^c(u_i) - \xi^c(u_i^-) - 2\delta = z_i - 2\delta.$$

Note that by construction, $r(s_i^+) < t(s_i^+) + \delta = u_i + \delta < u_{i+1} - \delta = t(s_{i+1}^-) - \delta < r(s_{i+1}^-)$ for each $i = 1, \dots, k-1$, and hence, along with the subadditivity of $x \mapsto x^\alpha$,

$$I_{M_1'}(\zeta) = \sum_{t \in [0, 1]} (\zeta(t) - \zeta(t-))^\alpha \geq \sum_{i=1}^k [\zeta(r(s_i^+)) - \zeta(r(s_i^-))]^\alpha \geq \sum_{i=1}^k (z_i - 2\delta)^\alpha > a,$$

which is contradictory to the assumption $\zeta \in \Psi_{I_{M_1'}}(a)$. \square

Acknowledgements. The research of BZ and MB is supported by NWO Grant 639.033.413. The research of JB is supported by NSF Grants 1915967, 1820942, 1838576 as well as DARPA Award N660011824028.

REFERENCES

- ASMUSSEN, S. and PIHLGÅRD, M. (2005). Performance analysis with truncated heavy-tailed distributions. *Methodol. Comput. Appl. Probab.* **7** 439–457. MR2235154 <https://doi.org/10.1007/s11009-005-5002-1>
- BAZHBA, M., BLANCHET, J., RHEE, C.-H. and ZWART, B. (2019). Queue length asymptotics for the multiple-server queue with heavy-tailed Weibull service times. *Queueing Syst.* **93** 195–226. MR4032925 <https://doi.org/10.1007/s11134-019-09640-z>
- BLANCHET, J., GLYNN, P. and MEYN, S. (2013). Large deviations for the empirical mean of an $M/M/1$ queue. *Queueing Syst.* **73** 425–446. MR3036010 <https://doi.org/10.1007/s11134-013-9349-7>
- BOROVKOV, A. A. and BOROVKOV, K. A. (2008). *Asymptotic Analysis of Random Walks: Heavy-Tailed Distributions. Encyclopedia of Mathematics and Its Applications* **118**. Cambridge Univ. Press, Cambridge. MR2424161 <https://doi.org/10.1017/CBO9780511721397>
- BOROVKOV, A. A. and MOGUL'SKIĬ, A. A. (2010). On large deviation principles in metric spaces. *Sib. Math. J.* **51** 989–1003.
- DEMBO, A. and ZEITOUNI, O. (2010). *Large Deviations Techniques and Applications. Stochastic Modelling and Applied Probability* **38**. Springer, Berlin. MR2571413 <https://doi.org/10.1007/978-3-642-03311-7>
- DENISOV, D., DIEKER, A. B. and SHNEER, V. (2008). Large deviations for random walks under subexponentiality: The big-jump domain. *Ann. Probab.* **36** 1946–1991. MR2440928 <https://doi.org/10.1214/07-AOP382>
- DONSKER, M. D. and VARADHAN, S. R. S. (1976). Asymptotic evaluation of certain Markov process expectations for large time. III. *Comm. Pure Appl. Math.* **29** 389–461. MR0428471 <https://doi.org/10.1002/cpa.3160290405>
- DUFFY, K. R. and MEYN, S. P. (2014). Large deviation asymptotics for busy periods. *Stoch. Syst.* **4** 300–319. MR3353220 <https://doi.org/10.1214/13-SSY098>
- DUFFY, K. R. and SAPOZHNIKOV, A. (2008). The large deviation principle for the on–off Weibull sojourn process. *J. Appl. Probab.* **45** 107–117. MR2409314 <https://doi.org/10.1239/jap/1208358955>
- EMBRECHTS, P., KLÜPPELBERG, C. and MIKOSCH, T. (1997). *Modelling Extremal Events: For Insurance and Finance. Applications of Mathematics (New York)* **33**. Springer, Berlin. MR1458613 <https://doi.org/10.1007/978-3-642-33483-2>
- FOSS, S., KORSHUNOV, D. and ZACHARY, S. (2011). *An Introduction to Heavy-Tailed and Subexponential Distributions. Springer Series in Operations Research and Financial Engineering*. Springer, New York. MR2810144 <https://doi.org/10.1007/978-1-4419-9473-8>
- GANESH, A., O'CONNELL, N. and WISCHIK, D. (2004). *Big Queues. Lecture Notes in Math.* **1838**. Springer, Berlin. MR2045489 <https://doi.org/10.1007/b95197>

- GANTERT, N. (1998). Functional Erdős–Rényi laws for semiexponential random variables. *Ann. Probab.* **26** 1356–1369. [MR1640348 https://doi.org/10.1214/aop/1022855755](https://doi.org/10.1214/aop/1022855755)
- GANTERT, N., RAMANAN, K. and REMBART, F. (2014). Large deviations for weighted sums of stretched exponential random variables. *Electron. Commun. Probab.* **19** Art. ID 41. [MR3233203 https://doi.org/10.1214/ECP.v19-3266](https://doi.org/10.1214/ECP.v19-3266)
- JELENKOVIĆ, P. and MOMČILOVIĆ, P. (2003). Large deviation analysis of subexponential waiting times in a processor-sharing queue. *Math. Oper. Res.* **28** 587–608. [MR1997251 https://doi.org/10.1287/moor.28.3.587.16396](https://doi.org/10.1287/moor.28.3.587.16396)
- KONTOYIANNIS, I. and MEYN, S. P. (2005). Large deviations asymptotics and the spectral theory of multiplicatively regular Markov processes. *Electron. J. Probab.* **10** 61–123. [MR2120240 https://doi.org/10.1214/EJP.v10-231](https://doi.org/10.1214/EJP.v10-231)
- LINDSKOG, F., RESNICK, S. I. and ROY, J. (2014). Regularly varying measures on metric spaces: Hidden regular variation and hidden jumps. *Probab. Surv.* **11** 270–314. [MR3271332 https://doi.org/10.1214/14-PS231](https://doi.org/10.1214/14-PS231)
- MASSART, P. (1990). The tight constant in the Dvoretzky–Kiefer–Wolfowitz inequality. *Ann. Probab.* **18** 1269–1283. [MR1062069](https://doi.org/10.1214/aop/1022855755)
- MOGUL'SKIĬ, A. A. (1993). Large deviations for processes with independent increments. *Ann. Probab.* **21** 202–215. [MR1207223](https://doi.org/10.1214/aop/1022855755)
- NAGAEV, A. V. (1969). Limit theorems that take into account large deviations when Cramér's condition is violated. *Izv. Akad. Nauk Uzb. SSR, Ser. Fiz.-Mat. Nauk* **13** 17–22. [MR0282396](https://doi.org/10.1214/aop/1022855755)
- NAGAEV, A. V. (1977). A property of sums of independent random variables. *Teor. Veroyatn. Primen.* **22** 335–346. [MR0438438](https://doi.org/10.1214/aop/1022855755)
- PUHALSKII, A. A. and WHITT, W. (1997). Functional large deviation principles for first-passage-time processes. *Ann. Appl. Probab.* **7** 362–381. [MR1442318 https://doi.org/10.1214/aoap/1034625336](https://doi.org/10.1214/aoap/1034625336)
- RESNICK, S. I. (2007). *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*. Springer Series in Operations Research and Financial Engineering. Springer, New York. [MR2271424](https://doi.org/10.1214/aop/1022855755)
- RHEE, C.-H., BLANCHET, J. and ZWART, B. (2019). Sample path large deviations for Lévy processes and random walks with regularly varying increments. *Ann. Probab.* **47** 3551–3605. [MR4038038 https://doi.org/10.1214/18-aop1319](https://doi.org/10.1214/18-aop1319)
- WHITT, W. (2000). The impact of a heavy-tailed service-time distribution upon the $M/GI/s$ waiting-time distribution. *Queueing Syst.* **36** 71–87. [MR1806961 https://doi.org/10.1023/A:1019143505968](https://doi.org/10.1023/A:1019143505968)