The Annals of Applied Probability
2020, Vol. 30, No. 4, 1884–1909
https://doi.org/10.1214/19-AAP1546

PARTICLES SYSTEMS AND NUMERICAL SCHEMES FOR MEAN REFLECTED STOCHASTIC DIFFERENTIAL EQUATIONS

BY PHILIPPE BRIAND[1,*], PAUL-ÉRIC CHAUDRU DE RAYNAL[1,†], ARNAUD GUILLIN[2] AND CÉLINE LABART[1,‡]

[1] *Université Savoie Mont Blanc, CNRS, LAMA,* [*] *philippe.briand@univ-smb.fr;* [†] *paul-eric.chaudru-de-raynal@univ-smb.fr;* [‡] *celine.labart@univ-smb.fr*

[2] *LM UMR 6620, Univ. Blaise Pascal & CNRS, arnaud.guillin@math.univ-bpclermont.fr*

This paper is devoted to the study of reflected Stochastic Differential Equations when the constraint is not on the paths of the solution but acts on its law. These reflected equations have been introduced recently in a backward form by Briand, Elie and Hu (*Ann. Appl. Probab.* **28** (2018) 482–510) in the context of risk measures. We here focus on the forward version of such reflected equations. Our main objective is to provide an approximation of the solutions with the help of interacting particles systems. This approximation allows to design a numerical scheme for this kind of equations.

1. Introduction. In this paper, we are concerned with a special type of reflected stochastic differentials equations (SDEs for short in the sequel) in which the constraint is not directly on the paths of the solution to the SDE as in the usual case but on the law of the solution. Typically, the integral of a given function, say h, with respect to the law of the solution to the SDE is asked to be nonnegative. We call mean reflected stochastic differential equation (MR-SDE) this kind of reflected SDEs which are described by the following system:

$$\begin{cases} X_t = \xi + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s + K_t, & t \geq 0, \\ \mathbb{E}\big[h(X_t)\big] \geq 0, \qquad \int_0^t \mathbb{E}\big[h(X_s)\big]\,dK_s = 0, & t \geq 0, \end{cases} \tag{1.1}$$

where b, σ and h are given Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$ and where $(B_t, t \geq 0)$ stands for a standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will always assume that h is nondecreasing and that the law of ξ is such that $\mathbb{E}[h(\xi)] \geq 0$. The solution to (1.1) is the couple of continuous processes (X, K), K being needed to ensure that the constraint is satisfied, in a minimal way according to the last condition namely the Skorokhod condition.

Reflected stochastic differential equations have been widely studied in the literature and we refer to the works [8, 14] for an overview of this theory. As said before, the main particularity comes here from the fact that the constraint acts on the law of the process X rather than on its paths. To the best of our knowledge, Skorokhod problem on the law of type (1.1) have been introduced by Briand, Elie and Hu in their backward forms in [3]. In that work, they show that mean reflected backward stochastic processes exist and are uniquely defined by the associated system of equations of the form of (1.1) under the same assumptions we use below. The main requirement for uniqueness is to ask the process K to be a deterministic function. They then show that such system is useful for super hedging of claims under running risk management constraint.

In this paper, our main objectives are twofold. Due to the fact that the reflection process K depends on the law of the position, it is hence nonlinear in the McKean–Vlasov's terminology (see [12] for an overview). It thus seems natural to investigate whether such a system can

Received February 2017; revised October 2019.
MSC2020 subject classifications. 60H10.
Key words and phrases. McKean–Vlasov SDE, mean-reflected, particle system.

be seen as the asymptotic counterpart of a mean field Skorokhod problem that is, as the asymptotic dynamic of particles system reflected in mean field. Having this counterpart at hand, we then aim at designing a numerical scheme for computing solutions to (1.1). This algorithm allows to solve asset management problems involving risk measure constraints.

Let us indeed consider an asset manager holding a number of stocks $S = (S^1, \dots, S^d)$ in his portfolio and let $\pi = (\pi^1, \dots, \pi^d)$ denotes his investment strategy. When the investor is allowed to choose his strategy as he wants, the value of his portfolio at time t, $X_t = X_0 + \int_0^t \pi_r \cdot dS_r$, can become very negative. For regulation purpose, one can allow only strategies such that X_t remains nonnegative or above a certain level. We are interested here in a weaker regulation constraint: the asset manager is allowed to hold its position X_t at time t only if it remains acceptable for a given risk measure ρ that is, it is such that $\rho(X_t) \leq 0$. Such a risk measure (see e.g., [1] or [5] for partial account on risk measures) could be for example, a utility function $u : \mathbb{R} \to \mathbb{R}$ so that $\rho(X) = \inf\{m : \mathbb{E}[u(m + X)] \geq p\}$, which means that a minimal profit is guaranteed.

In order to satisfy this constraint, the asset manager thus has to add some cash in the portfolio through the time and the dynamic of its position becomes

$$dX_t = \pi_t \, dS_t + dK_t,$$

where K_t is the amount of cash added up to time t in the portfolio to balance the "risk" associated to X_t. Of course, the agent wants to cover the risk in a minimal way, adding cash only when needed: this leads to the Skorokhod condition $\mathbb{E}[h(X_t)] \, dK_t = 0$.

Let us now consider the Black & Scholes model and suppose that the investment strategy depends on the wealth of the portfolio. We end up with the following SDE:

$$dX_t = b(X_t) \, dt + \sigma(X_t) \, dB_t + dK_t, \quad t \geq 0.$$

Putting together all conditions on X, we end up with a dynamic of the form (1.1) for the portfolio. This financial issue is close to the one studied in [3] but differs from the fact that, here, the terminal value is not fixed. Indeed, in [3], the authors look for the initial amount to invest and for the replicating strategy $(\pi^1, \dots, \pi^d)$ for a given terminal value, namely the payoff, when the portfolio is also submitted to a risk measure constraint. From a practical point a view, using a particle system (this seems to be a more involved problem in the backward case) to approximate the solution to (1.1) leads to solve numerically a N-dimensional stochastic equation, where N is the number of particles. Since solving numerically backward SDEs in high-dimension remains a difficult problem, to address the difficulties by increasing order, we focus in this paper on mean reflected forward SDEs.

Let us eventually point out that, in his work [7], Jabir studies diffusion processes whose time-marginal densities are constrained to belong to a given set at all time. By using a penalization approach, he builds weak solutions and obtains only partial uniqueness results, leaving the Skorokhod minimality condition and the approximation by particle systems open. Weak conditioning involving path-distribution constraints of the form

$$\mathrm{Law}(X_t; 0 \leq t \leq T) \in \mathcal{K} \quad \text{for } \mathcal{K} \text{ a given subset of } \mathcal{P}\big(\mathcal{C}([0, T]; \mathbb{R}^d)\big)$$

appears in various theoretical and applied situations such as stochastic mechanics (see [4]), diffusion processes with conditioned initial-terminal distribution (see [2, 10] and [13]) and for the modeling of crowd motion with congestion phenomenon (see [9]).

Organization of this paper. The paper is organized as follows. In Section 2, by letting the coefficients satisfy the usual smoothness assumptions (say Lipschitz continuity) and adding a structural assumption on the function h (say h bi-Lipschitz), we show that the system admits a unique strong solution that is, there exists a unique pair of process (X, K) satisfying system (1.1) almost surely, the process K being an increasing and deterministic process. Then, we show that, by adding some regularity on the function h, the Stieljes measure dK is absolutely continuous with respect to the Lebesgue measure and we give the explicit expression of its density.

Having in mind the analogy with McKean–Vlasov processes, we also show in Section 3 that system (1.1) can be seen as the limit of an interacting particles system with oblique reflection of mean field type. This could reflect a system of a large number of players whose positions are constrained by the mean of the positions of the other players. If all the players have the same dynamic, and if the interaction between the players is of mean field type, we show that there is a propagation of chaos phenomenon so that when the number of players tends to the infinity, the reflection no more depends on the other positions, but only on their statistical distribution. This obviously comes from the law of large number and is what, in fact, exactly happened in the classical McKean–Vlasov setting.

As an application, this result allows to define in Section 4 an algorithm based on this interacting particle system together with a classical Euler scheme which gives a strong approximation of the solution of (1.1). This leads to an approximation error proportional, up to a log factor, to the number of points of the discretization grid of the time interval (namely of $(\log n/n)^{1/2}$, where n is the number of points of the discretization grid) and on the number of particles (namely $N^{-1/4}$ when the function h is only bi-Lipschitz and $N^{-1/2}$ when the function h is smooth, N standing for the number of particles). Finally, we illustrate in Section 5 these results numerically.

2. Existence, uniqueness and properties of the solution. Throughout this paper, we consider the following set of assumptions.

ASSUMPTION 2.1.

(i) The functions $b : \mathbb{R} \longmapsto \mathbb{R}$ and $\sigma : \mathbb{R} \longmapsto \mathbb{R}$ are Lipschitz continuous.
(ii) The random variable ξ is square integrable.

ASSUMPTION 2.2.

(i) The function $h : \mathbb{R} \longmapsto \mathbb{R}$ is an increasing function and there exist $0 < m \leq M$ such that

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad m|x-y| \leq \left|h(x)-h(y)\right| \leq M|x-y|.$$

(ii) The initial condition ξ satisfies: $\mathbb{E}[h(\xi)] \geq 0$.

ASSUMPTION 2.3. $\exists p > 4$ such that ξ belongs to $\mathbb{L}^p$: $\mathbb{E}[|\xi|^p] < \infty$.

ASSUMPTION 2.4. The mapping h is a twice continuously differentiable function with bounded derivatives.

In the following, we make an intensive use of this representation formula of the process K. Define the function

$$H : \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \ni (x, \nu) \mapsto H(x, \nu) = \int h(x+z)\nu(dz), \tag{2.1}$$

where $\mathcal{P}_1(\mathbb{R})$ denotes the set of probability measures on $\mathbb{R}$ with finite first moment. We will need also the inverse function in space of H evaluated at 0 namely:

$$\bar{G}_0 : \mathcal{P}_1(\mathbb{R}) \ni \nu \mapsto \inf\{x \in \mathbb{R} : H(x, \nu) \geq 0\}, \tag{2.2}$$

as well as G_0, the positive part of G_0:

$$G_0 : \mathcal{P}_1(\mathbb{R}) \ni \nu \mapsto \inf\{x \geq 0 : H(x, \nu) \geq 0\}. \tag{2.3}$$

We start by studying some properties of H and G_0.

LEMMA 2.1. *Under Assumption 2.2 we have*:

(i) *For all* ν *in* $\mathcal{P}_1(\mathbb{R})$, *the mapping* $H(\cdot, \nu) : \mathbb{R} \ni x \mapsto H(x, \nu)$ *is a bi-Lipschitz function, namely*:

$$\forall x, y \in \mathbb{R}, \quad m|x-y| \leq \big|H(x,\nu) - H(y,\nu)\big| \leq M|x-y|. \tag{2.4}$$

(ii) *For all* x *in* $\mathbb{R}$, *the mapping* $H(x, \cdot) : \mathcal{P}_1(\mathbb{R}) \ni \nu \mapsto H(x, \nu)$ *satisfies the following Lipschitz estimate*:

$$\forall \nu, \nu' \in \mathcal{P}_1(\mathbb{R}), \quad \big|H(x,\nu) - H\big(x,\nu'\big)\big| \leq \left|\int h(x + \cdot)\big(d\nu - d\nu'\big)\right|. \tag{2.5}$$

PROOF. The proof is straightforward from the definition of H see (2.1). $\square$

Note that thanks to the Monge–Kantorovitch theorem, assertion (2.5) implies that for all x in $\mathbb{R}$, the function $H(x, \cdot)$ is Lipschitz continuous w.r.t. the Wasserstein-1 distance. Indeed, for two probability measures ν and ν', the Wasserstein-1 distance between ν and ν' is defined by:

$$W_1\big(\nu, \nu'\big) = \sup_{\varphi 1-\text{Lipschitz}} \left|\int \varphi\big(d\nu - d\nu'\big)\right| = \inf_{X\sim\nu;\, Y\sim\nu'} \mathbb{E}\big[|X - Y|\big].$$

Therefore

$$\forall \nu, \nu' \in \mathcal{P}_1(\mathbb{R}), \quad \big|H(x,\nu) - H\big(x,\nu'\big)\big| \leq M W_1\big(\nu, \nu'\big). \tag{2.6}$$

Then, we have the following result about the regularity of G_0:

LEMMA 2.2. *Under Assumption 2.2, the mapping* $G_0 : \mathcal{P}_1(\mathbb{R}) \ni \nu \mapsto G_0(\nu)$ *is Lipschitz-continuous in the following sense*:

$$\big|G_0(\nu) - G_0\big(\nu'\big)\big| \leq \frac{1}{m}\left|\int h\big(\bar{G}_0(\nu) + \cdot\big)\big(d\nu - d\nu'\big)\right|,$$

where we recall that $\bar{G}_0(\nu)$ *is the inverse of* $H(\cdot, \nu)$ *at point* 0. *In particular,*

$$\big|G_0(\nu) - G_0\big(\nu'\big)\big| \leq \frac{M}{m} W_1\big(\nu, \nu'\big). \tag{2.7}$$

PROOF. Let ν and ν' be two probability measures on $\mathbb{R}$. From Lipschitz regularity of the positive part, we have

$$\big|G_0(\nu) - G_0\big(\nu'\big)\big| \leq \big|\bar{G}_0(\nu) - \bar{G}_0\big(\nu'\big)\big|.$$

Next, using the bi-Lipschitz in space property of H, we get that for any η in $\mathcal{P}(\mathbb{R})$:

$$\big|\bar{G}_0(\nu) - \bar{G}_0\big(\nu'\big)\big| \leq \frac{1}{m}\big|H\big(\bar{G}_0(\nu), \eta\big) - H\big(\bar{G}_0\big(\nu'\big), \eta\big)\big|.$$

By definitions of H and G_0 we have, for all η in $\mathcal{P}_1(\mathbb{R})$: $H(\bar{G}_0(\eta), \eta) = 0$. Hence, by choosing $\eta = \nu'$:

$$\begin{aligned} |G_0(\nu) - G_0(\nu')| &\le \frac{1}{m}\big|H(\bar{G}_0(\nu), \nu') - H(\bar{G}_0(\nu'), \nu')\big| \\ &= \frac{1}{m}\big|H(\bar{G}_0(\nu), \nu') - H(\bar{G}_0(\nu), \nu)\big| \\ &= \frac{1}{m}\left|\int h(\bar{G}_0(\nu) + \cdot)(d\nu - d\nu')\right|. \end{aligned}$$

The last assertion immediately follows from (2.6). $\square$

We emphasize that existence and uniqueness results hold only under Assumption 2.1 which is the standard assumption for SDEs and Assumption 2.2 which is the assumption used in [3]. The convergence of particle systems require only an additional integrability assumption on the initial condition, namely Assumption 2.3. We sometimes add the smoothness assumption (Assumption 2.4) on h in order to improve some of the results.

We first recall the existence and uniqueness result of [3] in the case of SDEs.

DEFINITION 2.3. A couple of continuous processes (X, K) is said to be a flat deterministic solution to (1.1) if (X, K) satisfy (1.1) with X such that $\mathbb{E}(\sup_{t\in[0,T]} |X_t|^p) < \infty$ for some $p \ge 2$ and K being a nondecreasing deterministic function with $K_0 = 0$.

Given this definition, we have the following result.

THEOREM 2.4 (Briand, Elie and Hu, [3]). *Under Assumptions 2.1 and 2.2, the mean reflected SDE (1.1) has a unique deterministic flat solution (X, K). Moreover,*

$$\forall t \ge 0, \qquad K_t = \sup_{s\le t} \inf\{x \ge 0 : \mathbb{E}[h(x + U_s)] \ge 0\},$$

where $(U_t)_{0\le t\le T}$ is the process defined by:

$$U_t = \xi + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s. \tag{2.8}$$

With these notations, denoting by $(\mu_t)_{0\le t\le T}$ the family of marginal laws of $(U_t)_{0\le t\le T}$ we have

$$K_t = \sup_{s\le t} G_0(\mu_s). \tag{2.9}$$

PROOF OF THEOREM 2.4. The proof for the case of backward SDEs is given in [3]. For the ease of the reader, we sketch the proof for the forward case.

Let $\tilde{X}$ be a given continuous process such that, for all $t > 0$, $\mathbb{E}[\sup_{s\le t} |\tilde{X}_s|^2] < +\infty$. We set

$$\tilde{U}_t = \xi + \int_0^t b(\tilde{X}_s)\,ds + \int_0^t \sigma(\tilde{X}_s)\,dB_s, \qquad \mathrm{Law}(\tilde{U}_s) =: \tilde{\mu}_s,$$

and define the function K by setting

$$K_t = \sup_{s\le t} \inf\{x \ge 0 : \mathbb{E}[h(x + \tilde{U}_s)] \ge 0\} = G_0(\tilde{\mu}_s). \tag{2.10}$$

The function K being given, let us define the process X by the formula

$$X_t = \xi + \int_0^t b(\tilde{X}_s)\,ds + \int_0^t \sigma(\tilde{X}_s)\,dB_s + K_t.$$

Let us check that (X, K) is the solution to (1.1). By definition of K, $\mathbb{E}[h(X_t)] \geq 0$ and we have, dK almost everywhere,

$$K_t = \sup_{s \leq t} \inf\{x \geq 0 : \mathbb{E}[h(x + \tilde{U}_s)] \geq 0\} > 0,$$

so that $\mathbb{E}[h(X_t)] = \mathbb{E}[h(\tilde{U}_t + K_t)] = 0$ dK-a.e. since h is continuous and nondecreasing.

Next, consider the map Ξ which associates to $\tilde{X}$ the solution X of (1.1). Let us show that Ξ is a contraction. Let $\tilde{X}$ and $\tilde{X}'$ be given, and define $\tilde{U}$, K and $\tilde{U}'$, K' as above, using the same Brownian motion. We have from Cauchy–Schwarz and Doob inequality

$$\mathbb{E}\Big[\sup_{t\leq T}|X_t - X_t'|^2\Big] \leq (T\|b\|_{\mathrm{Lip}} + 2\|\sigma\|_{\mathrm{Lip}})\mathbb{E}\Big[\int_0^T |\tilde{X}_s - \tilde{X}_s'|^2\,ds\Big] + 2\sup_{t\leq T}|K_t - K_t'|^2.$$

From the representation (2.10) of the process K we have that

$$\sup_{t\leq T}|K_t - K_t'|^2 = \sup_{t\leq T}\Big|\sup_{s\leq t} G_0(\tilde{\mu}_s) - \sup_{s\leq t} G_0(\tilde{\mu}_s')\Big| \leq \sup_{t\leq T}|G_0(\tilde{\mu}_t) - G_0(\tilde{\mu}_t')|.$$

Then Lemma 2.2 gives

$$\sup_{t\leq T}|K_t - K_t'|^2 \leq \frac{M}{m}\mathbb{E}\Big[\sup_{t\leq T}|\tilde{U}_t - \tilde{U}_t'|^2\Big] \leq C(T\|b\|_{\mathrm{Lip}} + 2\|\sigma\|_{\mathrm{Lip}})\mathbb{E}\Big[\int_0^T |\tilde{X}_s - \tilde{X}_s'|^2\,ds\Big].$$

Therefore,

$$\mathbb{E}\Big[\sup_{t\leq T}|X_t - X_t'|^2\Big] \leq C(1+T)\mathbb{E}\Big[\int_0^T |\tilde{X}_s - \tilde{X}_s'|^2\,ds\Big] \leq C(1+T)T\mathbb{E}\Big[\sup_{t\leq T}|\tilde{X}_t - \tilde{X}_t'|^2\Big].$$

Hence, there exists a positive $\mathcal{T}$, depending on b, σ and h only, such that for all $T < \mathcal{T}$, the map Ξ is a contraction. We first deduce the existence and uniqueness of the solution on $[0, \mathcal{T}]$ and then on $\mathbb{R}^+$ by iterating the construction. □

REMARK 2.5. Note that from this construction, we deduce that for all positive r:

$$K_{t+r} - K_t = \sup_{0\leq s\leq r} \inf\Big\{x \geq 0 : \mathbb{E}\Big[h\Big(x + X_t + \int_t^{t+s} b(X_u)\,du + \int_t^{t+s} \sigma(X_u)\,dB_u\Big)\Big] \geq 0\Big\}.$$

It then follows that the unique solution of (1.1) is a Markov process on the space $\mathbb{R} \times \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ denotes the space of probability measures on $\mathbb{R}$.

PROPOSITION 2.6. *Suppose that Assumptions 2.1 and 2.2 hold. Then, for all $p \geq 1$, there exists a positive constant C_p, depending on T, b, σ and h such that*

$$\mathbb{E}\Big[\sup_{t\leq T}|X_t|^p\Big] \leq C_p(1 + \mathbb{E}[|\xi|^p]).$$

Moreover, there exists a positive constant C, depending on p, T, b, σ and h such that

$$\forall 0 \leq s \leq t \leq T, \quad |K_t - K_s| \leq C|t-s|^{1/2}$$

and for all $p \geq 1$ such that $\xi \in \mathbb{L}^p$ we have

$$\forall 0 \leq s \leq t \leq T, \quad \mathbb{E}[|X_t - X_s|^p] \leq C|t-s|^{p/2}.$$

PROOF. We have

$$\mathbb{E}\Big[\sup_{t\leq T}|X_t|^p\Big] \leq 4^{p-1}\Big\{\mathbb{E}[|\xi|^p] + \mathbb{E}\Big[\sup_{t\leq T}\Big(\int_0^t |b(X_s)|\,ds\Big)^p\Big] + \mathbb{E}\Big[\sup_{t\leq T}\Big|\int_0^t \sigma(X_s)\,dB_s\Big|^p\Big] + K_T^p\Big\}.$$

Let us first consider the last term $K_T = \sup_{t\le T} |G_0(\mu_s)|^p$. From the Lipschitz property of Lemma 2.2 of G_0, and the definition of the Wasserstein metric we have

$$\forall t \ge 0, \quad |G_0(\mu_t)| \le \frac{M}{m}\mathbb{E}[|U_t - U_0|],$$

since $G_0(\mu_0) = 0$ as $\mathbb{E}[h(\xi)] \ge 0$ and where U is defined by (2.8). Therefore

$$\Big|\sup_{t\le T} G_0(\mu_s)\Big|^p \le 2^{p-1}\left(\frac{M}{m}\right)^p \left\{\mathbb{E}\left[\sup_{t\le T}\left(\int_0^t |b(X_s)|\,ds\right)^p\right] + \mathbb{E}\left[\sup_{t\le T}\left|\int_0^t \sigma(X_s)\,dB_s\right|^p\right]\right\},$$

so that there exists $C(p, M, m) > 0$ such that

$$\mathbb{E}\Big[\sup_{t\le T} |X_t|^p\Big] \le C(p, M, m)\mathbb{E}\left[|\xi|^p + \sup_{t\le T}\left(\int_0^t |b(X_s)|\,ds\right)^p + \sup_{t\le T}\left|\int_0^t \sigma(X_s)\,dB_s\right|^p\right].$$

The first assertion of the result follows from standard computations since b and σ are Lipschitz continuous.

For the second assertion, note that from the Lipschitz property of $\nu \mapsto G_0(\nu)$ (Lemma 2.2) and of the supremum of a Lipschitz fonction, we have

$$K_t - K_s = \sup_{r\le t} G_0(\mu_r) - \sup_{r\le s} G_0(\mu_r) \le \frac{M}{m} W_1(\mu_r, \mu_s) \le \frac{M}{m}\mathbb{E}[|U_r - U_s|],$$

so that the result follows from standard computations. The third assertion ensues from the following decomposition: $X_t - X_s = U_t - U_s + K_t - K_s$. $\square$

We close this section by giving some additional properties of the solution (X, K) of (1.1) when h is smooth.

Let $\mathcal{L}$ be the linear partial operator of second order defined by

$$\mathcal{L}f(x) := b(x)\frac{\partial}{\partial x} f(x) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f(x) \tag{2.11}$$

for any twice continuously differentiable function f.

PROPOSITION 2.7. *Suppose that Assumptions 2.1, 2.2 and 2.4 hold and let (X, K) denotes the unique deterministic flat solution to (1.1). Then the process K is Lipschitz continuous and the Stieljes measure dK has the following density w.r.t. the Lebesgue measure*:

$$k : t \ni \mathbb{R}^+ \longmapsto \frac{(\mathbb{E}[\mathcal{L}h(X_t)])^-}{\mathbb{E}[h'(X_t)]}\mathbf{1}_{\mathbb{E}[h(X_t)]=0}. \tag{2.12}$$

PROOF. Let us first prove that K is Lipschitz continuous. To do so, we prove that $s \longmapsto \bar{G}_0(\mu_s)$ is Lipschitz continuous on $[0, T]$. Indeed, since by definition $H(\bar{G}_0(\mu_t), \mu_t) = 0$, if $s < t$, using (2.4),

$$\begin{aligned}
|\bar{G}_0(\mu_s) - \bar{G}_0(\mu_t)| &\le \frac{1}{m}|H(\bar{G}_0(\mu_s), \mu_t)| \\
&= \frac{1}{m}|\mathbb{E}[h(\bar{G}_0(\mu_s) + U_t)]| \\
&= \frac{1}{m}\left|\mathbb{E}\left[h\left(\bar{G}_0(\mu_s) + U_s + \int_s^t b(X_r)\,dr + \int_s^t \sigma(X_r)\,dB_r\right)\right]\right|.
\end{aligned}$$

We get from Itô's formula, setting

$$\bar{\mathcal{L}}_y := b(y)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2},$$

$$\begin{aligned}\mathbb{E}\big[h\big(\bar{G}_0(\mu_s)+U_t\big)\big] &= \mathbb{E}\big[h\big(\bar{G}_0(\mu_s)+U_s\big)\big] + \int_s^t \mathbb{E}\big[\bar{\mathcal{L}}_{X_r} h\big(\bar{G}_0(\mu_s)+U_r\big)\big]\,dr \\ &= H\big(\bar{G}_0(\mu_s),\mu_s\big) + \int_s^t \mathbb{E}\big[\bar{\mathcal{L}}_{X_r} h\big(\bar{G}_0(\mu_s)+U_r\big)\big]\,dr \\ &= \int_s^t \mathbb{E}\big[\bar{\mathcal{L}}_{X_r} h\big(\bar{G}_0(\mu_s)+U_r\big)\big]\,dr.\end{aligned}$$

Since h has bounded derivatives and $\sup_{s\le T}|X_s|$ is a square integrable random variable for each $T>0$ (see Corollary 2.6), the result follows easily.

The same is true for $s \longmapsto G_0(\mu_s) = (\bar{G}_0(\mu_s))^+$; let L be the Lipschitz constant of this last function. For $0\le s\le t\le T$, we have

$$K_t = \sup_{r\le t} G_0(\mu_r) = \max\Big(\sup_{r\le s} G_0(\mu_r), \sup_{s\le r\le t} G_0(\mu_r)\Big) = \max\Big(K_s, \sup_{s\le r\le t} G_0(\mu_r)\Big),$$

and, for any $s\le r\le t$, $G_0(\mu_r) - G_0(\mu_s) \le |G_0(\mu_r) - G_0(\mu_s)| \le L(r-s)$ so that

$$G_0(\mu_r) \le G_0(\mu_s) + L(r-s) \le K_s + L(t-s), \qquad \sup_{s\le r\le t} G_0(\mu_r) \le K_s + L(t-s).$$

It follows that, for $0\le s\le t\le T$,

$$0 \le K_s \le K_t \le K_s + L(t-s).$$

Therefore K is Lipschitz continuous and thus has a bounded density on $[0,T]$ for each $T>0$.

Let us fix $T>0$ and prove that (2.12) holds almost everywhere on $[0,T]$. Let us consider the compact set

$$\mathcal{K} := \big\{t\in[0,T] : \mathbb{E}\big[h(X_t)\big] = 0\big\}.$$

Due to the Skorokhod condition, $k_t = 0$ a.e. on $\mathcal{K}^c = [0,T]\setminus\mathcal{K}$.

Since k belongs to $\mathbb{L}^1(0,T)$ and $0<m\le\mathbb{E}[h'(X_t)]\le M$, Lebesgue's differentiation theorem implies that the function

$$u(t) = \int_0^t \mathbb{E}\big[h'(X_s)\big]k_s\,ds$$

is differentiable almost everywhere on $[0,T]$, say differentiable for every t in the measurable set $E\subset[0,T]$ with $\lambda(E^c)=0$ (on which k is nonnegative). Without loss of generality, we can assume that $\{0,T\}\subset E^c$ and we have

$$\forall t\in E,\quad u'(t) = \mathbb{E}\big[h'(X_t)\big]k_t.$$

Since h'' is continuous and bounded and X is a continuous process, the function v given by

$$v(t) = \int_0^t \mathbb{E}\big[\mathcal{L}h(X_s)\big]\,ds$$

is $\mathcal{C}^1$ with $v'(t) = \mathbb{E}[\mathcal{L}h(X_t)]$.

Let us observe, that if $\mathbb{E}[\mathcal{L}h(X_t)]>0$ for some $t>0$, then $\mathbb{E}[h(X_t)]>0$. Indeed, by continuity, there exists $0<s<t$ such that $\mathbb{E}[\mathcal{L}h(X_r)]>0$ on $[s,t]$ and

$$\mathbb{E}\big[h(X_t)\big] = \mathbb{E}\big[h(X_s)\big] + \int_s^t \mathbb{E}\big[\mathcal{L}h(X_r)\big]\,dr + \int_s^t \mathbb{E}\big[h'(X_r)\big]k_r\,dr \ge \int_s^t \mathbb{E}\big[\mathcal{L}h(X_r)\big]\,dr > 0.$$

Thus, $\mathcal{K}\cap E$ is a subset of the compact set $F = \{t\in[0,T] : \mathbb{E}[\mathcal{L}h(X_t)]\le 0\}$.

Let $t \in \mathcal{K} \cap E$. Since $\mathcal{K}$ is a closed set, t is either an accumulation point or an isolated point of $\mathcal{K}$. In the case of an accumulation point, there exists a sequence $(t_n)_{n\geq 0}$ of points of $\mathcal{K}$ such that $t_n \neq t$ and $\lim_{n\to\infty} t_n = t$. Since $\mathbb{N} = \{n \in \mathbb{N} : t_n < t\} \cup \{n \in \mathbb{N} : t_n > t\}$, one at least of the two previous sets of integers is infinite and, as a byproduct, $(t_n)_{n\geq 0}$ as a subsequence with $t_n > t$ for all n or with $t_n < t$ for all n. We keep the notation $(t_n)_{n\geq 0}$ for this subsequence and deal with the case $t_n > t$ for all n (the other case can be treated in the same way). For each n, the equality

$$\mathbb{E}[h(X_{t_n})] = \mathbb{E}[h(X_t)] + \int_t^{t_n} \mathbb{E}[\mathcal{L}h(X_r)]\,dr + \int_t^{t_n} \mathbb{E}[h'(X_r)]k_r\,dr$$

rewrites, since t and t_n belong to $\mathcal{K}$,

$$\int_t^{t_n} \mathbb{E}[\mathcal{L}h(X_r)]\,dr + \int_t^{t_n} \mathbb{E}[h'(X_r)]k_r\,dr = 0.$$

Thus, dividing by $t_n - t$ and sending n to ∞ we get, since $t \in E$ and $t \in F$,

$$\mathbb{E}[h'(X_t)]k_t = -\mathbb{E}[\mathcal{L}h(X_t)] = \mathbb{E}[\mathcal{L}h(X_t)]^-.$$

We have proved that, for all $t \in \mathcal{K} \cap E$ such that t is an accumulation point of $\mathcal{K}$,

$$k_t = \frac{\mathbb{E}[\mathcal{L}h(X_t)]^-}{\mathbb{E}[h'(X_t)]}.$$

To conclude, let us recall that the set of isolated points of $\mathcal{K}$ is at most countable and thus of 0 Lebesgue-measure so that, as $\lambda(E^c) = 0$,

$$k_t = \frac{\mathbb{E}[\mathcal{L}h(X_t)]^-}{\mathbb{E}[h'(X_t)]}, \quad \text{a.e. on } \mathcal{K}.$$

□

REMARK 2.8. This justifies, at least under the smoothness assumption (Assumption 2.4) on the constraint function h, the nonnegative hypothesis imposed on h'.

3. Mean reflected SDE as the limit of an interacting reflected particles system. Having in mind the notations defined in the beginning of Section 2 and especially equation (2.9), we can write the unique solution of the SDE (1.1) as:

$$X_t = \xi + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s + \sup_{s\leq t} G_0(\mu_s), \tag{3.1}$$

where μ_t stands for the law of

$$U_t = \xi + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s.$$

We are here interested in the particle approximation of such a system. Our candidates are the particles

$$X_t^i = \hat{\xi}^i + \int_0^t b(X_s^i)\,ds + \int_0^t \sigma(X_s^i)\,dB_s^i + \sup_{s\leq t} G_0(\mu_s^N), \quad 1 \leq i \leq N, \tag{3.2}$$

where B^i are independent Brownian motions, $\hat{\xi}^i = \xi^i + G_0(\mu^{\xi,N})$, $(\xi^i)_i$ are independent copies of ξ, μ_s^N denotes the empirical distribution at time s of the particles

$$U_s^i = \hat{\xi}^i + \int_0^s b(X_r^i)\,dr + \int_0^s \sigma(X_r^i)\,dB_r^i, \quad 1 \leq i \leq N, \qquad \mu_s^N = \frac{1}{N}\sum_{i=1}^N \delta_{U_s^i}$$

and $\mu^{\xi,N} = \frac{1}{N}\sum_{i=1}^N \delta_{\xi^i}$. It is worth noticing that

$$G_0(\mu_s^N) = \inf\left\{x \geq 0 : \frac{1}{N}\sum_{i=1}^N h(x + U_s^i) \geq 0\right\}.$$

We also denote, for all $t \in [0, T]$, $K_t^N := \sup_{s \leq t} G_0(\mu_s^N)$.

REMARK 3.1. At first glance, we can ask why we need to introduce $\hat{\xi}^i$ in equation (3.2) instead of using ξ^i. If this had been the case, K_0^N would not be equal to 0. With this choice of starting point we have

$$\begin{aligned} K_0^N &= \inf\left\{x \geq 0 : \frac{1}{N}\sum_{i=1}^N h(x + U_0^i) \geq 0\right\} \\ &= \inf\left\{x \geq 0 : \frac{1}{N}\sum_{i=1}^N h(x + \hat{\xi}^i) \geq 0\right\} \\ &= \inf\left\{x \geq 0 : \frac{1}{N}\sum_{i=1}^N h(x + \xi^i + G_0(\mu^{\xi,N})) \geq 0\right\} = 0, \end{aligned}$$

by definition of $G_0(\mu^{\xi,N})$.

REMARK 3.2. Let us emphasize that the previous system of interacting particles can be seen as a multidimensional reflected SDE with oblique reflection. Indeed, if h is concave, the set

$$\mathcal{S} := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : h(x_1) + \cdots + h(x_N) \geq 0\}$$

is convex and the system

$$\begin{cases} X_t^i = \hat{\xi}^i + \int_0^t b(X_s^i)\,ds + \int_0^t \sigma(X_s^i)\,dB_s^i + K_t^N, \quad 1 \leq i \leq N, \\ \frac{1}{N}\sum_{i=1}^N h(X_t^i) \geq 0, \qquad \frac{1}{N}\sum_{i=1}^N \int_0^t h(X_t^i)\,dK_s^N = 0, \end{cases}$$

is nothing else but the SDE driven by b and σ reflected in the convex $\mathcal{S}$ with oblique reflexion in the direction $(1, \ldots, 1)$. We refer to [8].

In order to prove that there is indeed a propagation of chaos effect, we introduce the following independent copies of X

$$\bar{X}_t^i = \xi^i + \int_0^t b(\bar{X}_s^i)\,ds + \int_0^t \sigma(\bar{X}_s^i)\,dB_s^i + \sup_{s \leq t} G_0(\mu_s), \quad 1 \leq i \leq N,$$

and we couple these particles with the previous ones by choosing the same Brownian motion.

We also introduce the decoupled particles $\bar{U}^i$, $1 \leq i \leq N$:

$$\bar{U}_t^i = \xi^i + \int_0^t b(\bar{X}_s^i)\,ds + \int_0^t \sigma(\bar{X}_s^i)\,dB_s^i, \quad t \geq 0.$$

Note for instance that the particles $\bar{U}^i$ are i.i.d. and let us denote by $\bar{\mu}^N$ the empirical measure associated with this system of particles.

We have the following result concerning the approximation (1.1) by the interacting particles system (3.2).

THEOREM 3.3. *Let Assumptions 2.1 and 2.2 hold and $T > 0$.*

(i) *Under Assumption 2.3, there exists a constant C depending on b and σ such that, for each $j \in \{1, \dots, N\}$,*

$$\mathbb{E}\Big[\sup_{s\le T}|X_s^j - \bar{X}_s^j|^2\Big] \le C\exp\Big(C\Big(1+\frac{M^2}{m^2}\Big)(1+T^2)\Big)\frac{M^2}{m^2}N^{-1/2}.$$

(ii) *If Assumption 2.4 is in force, then there exists a constant C depending on b and σ such that, for each $j \in \{1, \dots, N\}$,*

$$\mathbb{E}\Big[\sup_{s\le T}|X_s^j - \bar{X}_s^j|^2\Big] \le C\exp\Big(C\Big(1+\frac{M^2}{m^2}\Big)(1+T^2)\Big)\frac{1+T^2}{m^2}\Big(1+\mathbb{E}\Big[\sup_{s\le t}|X_T|^2\Big]\Big)N^{-1}.$$

REMARK 3.4. As shown in [6], the rate in case (i) is optimal in full generality for the control of the Wasserstein distance of the empirical measure of an i.i.d. sample of random variables towards its own law. It is interesting to note that the supremum over time implies no loss here, as the "propagation of chaos" is mainly herited from the reflection term through $\sup_{s\le t} W_1^2(\bar{\mu}_s^N, \mu_s)$.

PROOF OF THEOREM 3.3. Let $t > 0$. We have, for $r \le t$,

$$\begin{aligned}|X_r^j - \bar{X}_r^j| \le{}& |\hat{\xi}^j - \xi^j| \\ &+ \int_0^r |b(X_s^j) - b(\bar{X}_s^j)|\,ds + \Big|\int_0^r (\sigma(X_s^j) - \sigma(\bar{X}_s^j))\,dB_s^j\Big| \\ &+ \Big|\sup_{s\le r} G_0(\mu_s^N) - \sup_{s\le r} G_0(\mu_s)\Big|.\end{aligned}$$

Let us study the first term:

$$|\hat{\xi}^j - \xi^j| = |G_0(\mu^{\xi,N})| = |G_0(\mu_0^N) - G_0(\mu_0)| \le \sup_{s\le r}|G_0(\mu_s^N) - G_0(\mu_s)|.$$

Taking into account the fact that

$$\begin{aligned}\Big|\sup_{s\le r} G_0(\mu_s^N) - \sup_{s\le r} G_0(\mu_s)\Big| &\le \sup_{s\le r}|G_0(\mu_s^N) - G_0(\mu_s)| \le \sup_{s\le t}|G_0(\mu_s^N) - G_0(\mu_s)| \\ &\le \sup_{s\le t}|G_0(\mu_s^N) - G_0(\bar{\mu}_s^N)| + \sup_{s\le t}|G_0(\bar{\mu}_s^N) - G_0(\mu_s)|,\end{aligned}$$

we get the inequality

$$\sup_{r\le t}|X_r^j - \bar{X}_r^j| \le I_1^j(t) + 2\sup_{s\le t}|G_0(\mu_s^N) - G_0(\bar{\mu}_s^N)| + 2\sup_{s\le t}|G_0(\bar{\mu}_s^N) - G_0(\mu_s)|, \tag{3.3}$$

where we have set

$$I_1^j(t) := \int_0^t |b(X_s^j) - b(\bar{X}_s^j)|\,ds + \sup_{r\le t}\Big|\int_0^r (\sigma(X_s^j) - \sigma(\bar{X}_s^j))\,dB_s^j\Big|.$$

On the one hand we have, using Doob and Cauchy–Schwarz inequalities

$$\mathbb{E}[|I_1^j(t)|^2] \le C(1+t)\int_0^t \mathbb{E}[|X_s^j - \bar{X}_s^j|^2]\,ds,$$

where C depends only on b and σ.

On the other hand, by using (2.7),

$$\sup_{s\le t}|G_0(\mu_s^N) - G_0(\bar{\mu}_s^N)| \le \frac{M}{m}\sup_{s\le t}\frac{1}{N}\sum_{i=1}^N |U_s^i - \bar{U}_s^i| \le \frac{M}{m}\frac{1}{N}\sum_{i=1}^N \sup_{s\le t}|U_s^i - \bar{U}_s^i|,$$

and Cauchy–Schwarz inequality gives, since the variables are exchangeable,

$$\mathbb{E}\Big[\sup_{s\le t}\big|G_0(\mu_s^N)-G_0(\bar\mu_s^N)\big|^2\Big]\le\frac{M^2}{m^2}\frac{1}{N}\sum_{i=1}^N\mathbb{E}\Big[\sup_{s\le t}|U_s^i-\bar U_s^i|^2\Big]=\frac{M^2}{m^2}\mathbb{E}\Big[\sup_{s\le t}|U_s^j-\bar U_s^j|^2\Big].$$

Since,

$$U_s^j-\bar U_s^j=\int_0^s\big(b(X_r^j)-b(\bar X_r^j)\big)\,dr+\int_0^s\big(\sigma(X_r^j)-\sigma(\bar X_r^j)\big)\,dB_r^j$$

the same computations as we did above lead to

$$\mathbb{E}\Big[\sup_{s\le t}\big|G_0(\mu_s^N)-G_0(\bar\mu_s^N)\big|^2\Big]\le C\frac{M^2}{m^2}(1+t)\int_0^t\mathbb{E}\big[|X_s^j-\bar X_s^j|^2\big]\,ds,\quad C:=C(\sigma,b)>0.$$

Hence, with the previous estimates we get, coming back to (3.3),

$$\begin{aligned}\mathbb{E}\Big[\sup_{r\le t}|X_r^j-\bar X_r^j|^2\Big]&\le\bar C\int_0^t\mathbb{E}\big[|X_s^j-\bar X_s^j|^2\big]\,ds+6\mathbb{E}\Big[\sup_{s\le t}\big|G_0(\bar\mu_s^N)-G_0(\mu_s)\big|^2\Big]\\&\le\bar C\int_0^t\mathbb{E}\Big[\sup_{r\le s}|X_s^j-\bar X_s^j|^2\Big]\,ds+6\mathbb{E}\Big[\sup_{s\le t}\big|G_0(\bar\mu_s^N)-G_0(\mu_s)\big|^2\Big],\end{aligned}$$

where $\bar C=C(1+t)(1+M^2/m^2)$. Thanks to Gronwall's lemma,

$$\mathbb{E}\Big[\sup_{r\le t}|X_r^j-\bar X_r^j|^2\Big]\le Ce^{\bar Ct}\mathbb{E}\Big[\sup_{s\le t}\big|G_0(\bar\mu_s^N)-G_0(\mu_s)\big|^2\Big].$$

By Lemma 2.2 we know that

$$\mathbb{E}\Big[\sup_{s\le t}\big|G_0(\bar\mu_s^N)-G_0(\mu_s)\big|^2\Big]\le\frac{1}{m^2}\mathbb{E}\bigg[\sup_{s\le t}\bigg|\int h\big(\bar G_0(\mu_s)+\cdot\big)\big(d\bar\mu_s^N-d\mu_s\big)\bigg|^2\bigg],$$

from which we deduce that

$$\mathbb{E}\Big[\sup_{r\le t}|X_r^j-\bar X_r^j|^2\Big]\le Ce^{\bar Ct}\frac{1}{m^2}\mathbb{E}\bigg[\sup_{s\le t}\bigg|\int h\big(\bar G_0(\mu_s)+\cdot\big)\big(d\bar\mu_s^N-d\mu_s\big)\bigg|^2\bigg].\tag{3.4}$$

Since the function h is, at least, a Lipschitz function, we understand that the rate of convergence follows from the convergence of empirical measure of i.i.d. diffusion processes.

Proof of (i). In full generality (i.e., if we only suppose that Assumption 2.2 holds) we get that

$$\frac{1}{m^2}\mathbb{E}\bigg[\sup_{s\le t}\bigg|\int h\big(\bar G_0(\mu_s)+\cdot\big)\big(d\bar\mu_s^N-d\mu_s\big)\bigg|^2\bigg]\le\frac{M^2}{m^2}\mathbb{E}\Big[\sup_{s\le t}W_1^2(\bar\mu_s^N,\mu_s)\Big].$$

The crucial point here is that we consider a uniform (in time) convergence, which may possibly damage the usual rate of convergence. We however succeeded in preserving this optimal rate.

Thanks to the additional Assumption 2.3 and to Proposition 2.6, we will adapt and simplify the proof of Theorem 10.2.7 of [11] using recent results about the control of Wasserstein distance of empirical measures of i.i.d. sample to the true law by [6], to obtain

$$\mathbb{E}\Big[\sup_{s\le T}W_1^2(\bar\mu_s^N,\mu_s)\Big]\le CN^{-1/2}.$$

Indeed, let n be a positive integer and set $t_k=kT/n$, $0\le k\le n$. As in [11], denote

$$Z_k=\sup_{t_k\le s\le t_{k+1}}\big\{W_1^2(\bar\mu_s^N,\bar\mu_{t_k}^N)\wedge W_1^2(\bar\mu_s^N,\bar\mu_{t_{k+1}}^N)\big\}.$$

Then

$$\sup_{s\le T} W_1^2(\bar{\mu}_s^N, \mu_s) \le 3\Big[\max_k Z_k + \max_k W_1^2(\bar{\mu}_{t_k}^N, \mu_{t_k}) + \max_k \sup_{t_k\le t\le t_{k+1}} W_1^2(\mu_{t_k}, \mu_t)\Big].$$

Now, using the regularity properties of Proposition 2.6 and proceeding exactly as in [11], Theorem 10.2.7, we have that there exists $C > 0$ such that

$$W_1^2(\mu_{t_k}, \mu_t) \le \frac{C}{n}, \qquad \mathbb{E}[\max Z_k] \le \frac{C}{\sqrt{n}}.$$

We are led to control $\mathbb{E}[\max_k W_1^2(\bar{\mu}_{t_k}^N, \mu_{t_k})]$: first remark that

$$\mathbb{E}\Big[\max_k W_1^2(\bar{\mu}_{t_k}^N, \mu_{t_k})\Big] \le \sqrt{n}\sqrt{\max_k \mathbb{E}[W_1^4(\bar{\mu}_{t_k}^N, \mu_{t_k})]}.$$

Use now Assumption 2.3 and Theorem 2 (case (3)) in [6] to get

$$\mathbb{E}[W_1^4(\bar{\mu}_{t_k}^N, \mu_{t_k})] \le \frac{C}{N^2},$$

from which we deduce that

$$\mathbb{E}\Big[\sup_{s\le 1} W_1^2(\bar{\mu}_s^N, \mu_s)\Big] \le C\left[\frac{\sqrt{n}}{N} + \frac{1}{\sqrt{n}}\right]$$

and optimization procedure on n finishes the proof. Let us emphasize that this result does not care of the fact that $\bar{\mu}^N$ is an empirical measure associated to i.i.d. copies of a *diffusion process*.

Proof of (ii). In the case where h is a twice continuously differentiable function with bounded derivatives (i.e., under Assumption 2.4), we succeed in taking benefit from the fact that $\bar{\mu}^N$ is an empirical measure associated to i.i.d. copies of diffusion process, in particular we can get rid of the supremum in time. In view of (3.4), we need a sharp estimate of

$$\mathbb{E}\Big[\sup_{s\le t}\Big|\int h(\bar{G}_0(\mu_s) + \cdot)(d\bar{\mu}_s^N - d\mu_s)\Big|^2\Big].$$

Let us denote by ψ the Radon–Nikodym derivative of $\bar{G}_0(\mu_\cdot)$ (we have proved in Proposition 2.7 that $s \longmapsto \bar{G}_0(\mu_s)$ is Lipschitz continuous). By definition, we have, denoting by V^i the semimartingale $s \longmapsto \bar{G}_0(\mu_s) + \bar{U}_s^i$, since $\bar{U}^i$ are independent copies of U,

$$\begin{aligned}
R_N(s) &:= \int h(\bar{G}_0(\mu_s) + \cdot)(d\bar{\mu}_s^N - d\mu_s) \\
&= \frac{1}{N}\sum_{i=1}^N h(\bar{G}_0(\mu_s) + \bar{U}_s^i) - \mathbb{E}[h(\bar{G}_0(\mu_s) + U_s)] \\
&= \frac{1}{N}\sum_{i=1}^N \{h(\bar{G}_0(\mu_s) + \bar{U}_s^i) - \mathbb{E}[h(\bar{G}_0(\mu_s) + \bar{U}_s^i)]\} \\
&= \frac{1}{N}\sum_{i=1}^N \{h(V_s^i) - \mathbb{E}[h(V_s^i)]\}.
\end{aligned}$$

From Itô's formula we have

$$\begin{aligned}
h(V_s^i) &= h(V_0^i) + \int_0^s h'(V_r^i)\psi_r\,dr + \int_0^s \bar{\mathcal{L}}_{\bar{X}_r} h(V_r^i)\,dr + \int_0^s h'(V_r^i)\sigma(\bar{X}_r^i)\,dB_r^i \\
&= h(V_0^i) + \int_0^s \{h'(V_r^i)\psi_r + \bar{\mathcal{L}}_{\bar{X}_r} h(V_r^i)\}\,dr + \int_0^s h'(V_r^i)\sigma(\bar{X}_r^i)\,dB_r^i,
\end{aligned}$$

so that

$$\mathbb{E}[h(V_s^i)] = \mathbb{E}[h(V_0^i)] + \int_0^s \mathbb{E}[h'(V_r^i)\psi_r + \bar{\mathcal{L}}_{\bar{X}_r} h(V_r^i)]\,dr$$
$$= 0 + \int_0^s \mathbb{E}[h'(V_r^i)\psi_r + \bar{\mathcal{L}}_{\bar{X}_r} h(V_r^i)]\,dr.$$

We thus deduce immediately that

$$R_N(s) = \frac{1}{N}\sum_{i=1}^N h(V_0^i) + \frac{1}{N}\sum_{i=1}^N \int_0^s Y^i(r)\,dr + M_N(s)$$
$$= \frac{1}{N}\sum_{i=1}^N h(V_0^i) + \int_0^s \left(\frac{1}{N}\sum_{i=1}^N Y^i(r)\right)dr + M_N(s),$$

where we have set

$$Y^i(r) = h'(V_r^i)\psi_r + \bar{\mathcal{L}}_{\bar{X}_r} h(V_r^i) - \mathbb{E}[h'(V_r^i)\psi_r + \bar{\mathcal{L}}_{\bar{X}_r} h(V_r^i)],$$
$$M_N(s) = \frac{1}{N}\sum_{i=1}^N \int_0^s h'(V_r^i)\sigma(\bar{X}_r^i)\,dB_r^i.$$

As a byproduct,

$$\sup_{s\le t}|R_N(s)| \le \left|\frac{1}{N}\sum_{i=1}^N h(V_0^i)\right| + \sup_{s\le t}\int_0^s \left|\frac{1}{N}\sum_{i=1}^N Y^i(r)\right| dr + \sup_{s\le t}|M_N(s)|$$
$$\le \left|\frac{1}{N}\sum_{i=1}^N h(V_0^i)\right| + \int_0^t \left|\frac{1}{N}\sum_{i=1}^N Y^i(r)\right| dr + \sup_{s\le t}|M_N(s)|.$$

We get, using Cauchy–Schwarz inequality, since U^i and $\bar{X}^i$ are i.i.d,

$$\mathbb{E}\Big[\sup_{s\le t}|R_N(s)|^2\Big]$$
$$\le 3\left\{\mathrm{Var}\left[\frac{1}{N}\sum_{i=1}^N h(V_0^i)\right] + \mathbb{E}\left[\left(\int_0^t \left|\frac{1}{N}\sum_{i=1}^N Y^i(r)\right| dr\right)^2\right] + \mathbb{E}\Big[\sup_{s\le t}|M_N(s)|^2\Big]\right\}$$
$$\le 3\left\{\mathrm{Var}\left[\frac{1}{N}\sum_{i=1}^N h(V_0^i)\right] + t\mathbb{E}\left[\int_0^t \left|\frac{1}{N}\sum_{i=1}^N Y^i(r)\right|^2 dr\right] + \mathbb{E}\Big[\sup_{s\le t}|M_N(s)|^2\Big]\right\}$$
$$= 3\left\{\mathrm{Var}\left[\frac{1}{N}\sum_{i=1}^N h(V_0^i)\right] + t\int_0^t \mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^N Y^i(r)\right) dr + \mathbb{E}\Big[\sup_{s\le t}|M_N(s)|^2\Big]\right\}.$$

Thus, we obtain defining accordingly to V^i (resp. C^i) the semimartingale V (resp. centered r.v. Y)

$$\mathbb{E}\Big[\sup_{s\le t}|R_N(s)|^2\Big]$$
$$\le \frac{3}{N}\mathrm{Var}[h(V_0)] + \frac{3t}{N}\int_0^t \mathrm{Var}(Y(r))\,dr + 3\mathbb{E}\Big[\sup_{s\le t}|M_N(s)|^2\Big]$$
$$= \frac{3}{N}\mathrm{Var}[h(V_0)] + \frac{3t}{N}\int_0^t \mathrm{Var}(h'(V_r)\psi_r + \bar{\mathcal{L}}_{X_r} h(V_r))\,dr + 3\mathbb{E}\Big[\sup_{s\le t}|M_N(s)|^2\Big].$$

Since M_N is a martingale with

$$\langle M_N\rangle_t = \frac{1}{N^2}\sum_{i=1}^N \int_0^t \big(h'(V_r^i)\sigma(\bar{X}_r^i)\big)^2\,dr,$$

Doob's inequality leads to

$$\begin{aligned}\mathbb{E}\Big[\sup_{s\le t}|M_N(s)|^2\Big] &\le 4\mathbb{E}\big[|M_N(t)|^2\big]\\ &= \frac{4}{N^2}\sum_{i=1}^N\int_0^t \mathbb{E}\big[\big(h'(V_r^i)\sigma(\bar{X}_r^i)\big)^2\big]\,dr\\ &= \frac{4}{N}\int_0^t \mathbb{E}\big[\big(h'(V_r)\sigma(X_r)\big)^2\big]\,dr.\end{aligned}$$

Finally, using the fact that h has bounded derivatives, b and σ are Lipschitz, we get

$$\mathbb{E}\Big[\sup_{s\le t}|R_N(s)|^2\Big] \le C(1+t^2)\Big(1+\mathbb{E}\Big[\sup_{s\le t}|X_s|^2\Big]\Big)N^{-1}.$$

This gives the result coming back to (3.4). $\square$

4. A numerical scheme for MRSDE. We are interested in the numerical approximation of the SDE (1.1) on $[0, T]$. Here are the main steps of the scheme. Let $0 = T_0 < T_1 < \cdots < T_n = T$ be a subdivision of $[0, T]$. Given this subdivision, we denote by "_" the mapping $s \mapsto \underline{s} = T_k$ if $s \in [T_k, T_{k+1})$, $k \in \{0, \dots, n-1\}$. For simplicity, we consider only the case of regular subdivisions: for a given integer n, $T_k = kT/n$, $k = 0, \dots, n$.

Let us recall that we proved in the previous section that particles system

$$X_t^i = \hat{\xi}^i + \int_0^t b(X_s^i)\,ds + \int_0^t \sigma(X_s^i)\,dB_s^i + \sup_{s\le t} G_0(\mu_s^N), \quad 1\le i\le N,$$

where we have set

$$U_s^i = \hat{\xi}^i + \int_0^s b(X_r^i)\,dr + \int_0^s \sigma(X_r^i)\,dB_r^i, \quad 1\le i\le N, \qquad \mu_s^N = \frac{1}{N}\sum_{i=1}^N \delta_{U_s^i}.$$

B^i being independent Brownian motions and $\hat{\xi}^i = \xi^i + G_0(\mu^{\xi,N})$, $(\xi^i)_i$ being independent copies of ξ, converges toward the solution of (1.1). Thus, the numerical approximation is obtained by an Euler scheme applied to this particles system. We introduce the following discrete version of the particles system

$$\tilde{X}_t^i = \hat{\xi}^i + \int_0^t b(\tilde{X}_{\underline{s}}^i)\,ds + \int_0^t \sigma(\tilde{X}_{\underline{s}}^i)\,dB_s^i + \sup_{s\le t} G_0(\tilde{\mu}_{\underline{s}}^N), \quad 1\le i\le N,$$

with the notation $\tilde{K}_t^N = \sup_{s\le t} G_0(\tilde{\mu}_{\underline{s}}^N)$ and

$$\tilde{U}_t^i = \hat{\xi}^i + \int_0^t b(\tilde{X}_{\underline{s}}^i)\,ds + \int_0^t \sigma(\tilde{X}_{\underline{s}}^i)\,dB_s^i, \quad 1\le i\le N, \qquad \tilde{\mu}_s^N = \frac{1}{N}\sum_{i=1}^N \delta_{\tilde{U}_s^i}.$$

Algorithm 1 Particle approximation

for $1 \le j \le N$ **do**
  Simulate $(\xi^1, \dots, \xi^N)$, compute $G_0(\mu^{\xi,N})$ and $(\hat{\xi}^1, \dots, \hat{\xi}^N)$.
  $\tilde{X}_0^i = \tilde{U}_0^i = \hat{\xi}^i$, $\tilde{\mu}_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{\hat{\xi}^i}$
end for
for $1 \le k \le n$ **do**
  for $1 \le j \le N$ **do**
    $G^j \sim \mathcal{N}(0,1)$
    $(\tilde{U}_{T_k})^j = (\tilde{U}_{T_{k-1}})^j + (T/n) b((\tilde{X}_{T_{k-1}})^j) + \sqrt{(T/n)}\sigma((\tilde{X}_{T_{k-1}})^j)G^j$
  end for
  $\tilde{\mu}_{T_k}^N = N^{-1}\sum_{j=1}^N \delta_{(\tilde{U}_{T_k})^j}$
  $\Delta_k \tilde{K}^N = \sup_{l \le k} G_0(\tilde{\mu}_{T_l}^N) - \sup_{l \le k-1} G_0(\tilde{\mu}_{T_l}^N)$
  for $1 \le j \le N$ **do**
    $(\tilde{X}_{T_k})^j = (\tilde{X}_{T_{k-1}})^j + (T/n) b((\tilde{X}_{T_{k-1}})^j) + \sqrt{(T/n)}\sigma((\tilde{X}_{T_{k-1}})^j)G^j + \Delta_k \tilde{K}^N$
  end for
end for

4.1. *Scheme*. Using the notations given above, the result on the interacting system of mean reflected particles of the MR-SDE of Section 3 and Remark 2.5, we deduce Algorithm 1 for the numerical approximation of the MR-SDE.

REMARK 4.1. We emphasize that, at each step k of the algorithm, we approximate the increment of the reflection process K by the increment of this approximation:

$$\Delta_k \tilde{K}^N := \sup_{l \le k} G_0(\tilde{\mu}_{T_l}^N) - \sup_{l \le k-1} G_0(\tilde{\mu}_{T_l}^N).$$

As suggested in Remark 2.5, this increment can be approached by:

$$\widetilde{\Delta_k K}^N := \inf\left\{x \ge 0 : \frac{1}{N}\sum_{i=1}^N h\left(x + (\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N})^i + \frac{T}{n} b((\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N})^i) + \frac{\sqrt{T}}{\sqrt{n}}\sigma((\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N})^i)G^i\right) \ge 0\right\}.$$

Indeed, using the same kind of arguments as in the sketch of the proof of Theorem 2.4, one can show that the increments of the approximated reflection process are equals to the approximation of the increments:

$$\forall k \in \{1, \dots, n\}: \quad \widetilde{\Delta_k K}^N = \Delta_k \tilde{K}^N.$$

4.2. *Scheme error.* We recall that $\bar{X}^i$ denotes an i.i.d. copie of X:

$$\bar{X}_t^i = \xi^i + \int_0^t b(\bar{X}_s^i)\,ds + \int_0^t \sigma(\bar{X}_s^i)\,dB_s^i + K_t,$$

we have the following result.

THEOREM 4.2. *Let $T > 0$, N and n be two nonnegative integers. Let Assumptions 2.1, 2.2 and 2.3 hold.*

(i) *There exists a constant C depending on T, b, σ, h and ξ such that: for all $i \in \{1, \dots, N\}$,*

$$\mathbb{E}\Big[\sup_{t \le T}|\bar{X}_t^i - \tilde{X}_t^i|^2\Big] + \mathbb{E}\Big[\sup_{t \le T}|K_t - \tilde{K}_t^N|^2\Big] \le C\left(\frac{\log n}{n} + N^{-1/2}\right).$$

(ii) *If in addition 2.4 hold, there exists a positive constant C depending on T, b, σ, h and ξ such that: for all $i \in \{1, \dots, N\}$,*

$$\mathbb{E}\Big[\sup_{t\le T}|\bar{X}_t^i - \tilde{X}_t^i|^2\Big] + \mathbb{E}\Big[\sup_{t\le T}|K_t - \tilde{K}_t^N|^2\Big] \le C\left(\frac{\log n}{n} + N^{-1}\right).$$

PROOF. Concerning X, the proof is straightforward. Write

$$|\bar{X}_t^i - \tilde{X}_t^i| \le |\bar{X}_t^i - X_t^i| + |X_t^i - \tilde{X}_t^i|.$$

We bound the first term in the above right-hand side by using the proof of Theorem 3.3 and the second term with the following proposition, which gives the error approximation of the Euler discretization.

PROPOSITION 4.3. *Let $T > 0$, N and n be two nonnegative integers and let Assumptions 2.1, 2.2 and 2.3 hold. There exists a constant C, depending on T, b, σ, h and ξ but independent of N, such that: for all $i = 1, \dots, N$,*

$$\mathbb{E}\Big[\sup_{s\le T}|X_t^i - \tilde{X}_t^i|^2\Big] \le C\frac{\log n}{n}.$$

Concerning K, we write

$$\begin{aligned} K_t - \tilde{K}_t^N &= \sup_{s\le t} G_0(\mu_s) - \sup_{s\le t} G_0(\tilde{\mu}_{\underline{s}}^N) \\ &= \sup_{s\le t} G_0(\mu_s) - \sup_{s\le t} G_0(\mu_s^N) + \sup_{s\le t} G_0(\mu_s^N) - \sup_{s\le t} G_0(\tilde{\mu}_{\underline{s}}^N). \end{aligned}$$

Then

$$E\Big[\sup_{t\le T}|K_t - \tilde{K}_t^N|^2\Big] \le \mathbb{E}\Big[\sup_{t\le T}|G_0(\mu_t) - G_0(\mu_t^N)|^2\Big] + \mathbb{E}\Big[\sup_{t\le T}|G_0(\mu_t^N) - G_0(\tilde{\mu}_t^N)|^2\Big].$$

We bound the first term by using the proof of Theorem 3.3 and the second term by using the proof of Proposition 4.3. □

PROOF OF PROPOSITION 4.3. Let us admit for the moment the following result that will be useful for our analysis and whose proof is postponed at the end of the current one.

LEMMA 4.4. *There exists a constant C such that*

$$\mathbb{E}\Big[\sup_{s\le T}|B_s - B_{\underline{s}}|^4\Big] \le C\left(\frac{\log n}{n}\right)^2.$$

We may now proceed to the proof of Proposition 4.3. Let us fix $i \in \{1, \dots, N\}$ and $T > 0$. We have, for $t \le T$,

$$\begin{aligned} |X_t^i - \tilde{X}_t^i| \le \int_0^t |b(X_s^i) - b(\tilde{X}_{\underline{s}}^i)|\, ds + \left|\int_0^t (\sigma(X_s^i) - \sigma(\tilde{X}_{\underline{s}}^i))\, dB_s\right| \\ + \sup_{s\le t}|G_0(\mu_s^N) - G_0(\tilde{\mu}_{\underline{s}}^N)|. \end{aligned}$$

Hence, using Cauchy–Schwarz and Doob inequality we get

$$\mathbb{E}\Big[\sup_{s\le t}|X_s^i - \tilde{X}_s^i|^2\Big] \le C\int_0^t \mathbb{E}[|X_s^i - \tilde{X}_{\underline{s}}^i|^2]\, ds + \mathbb{E}\Big[\sup_{s\le t}|G_0(\mu_s^N) - G_0(\tilde{\mu}_{\underline{s}}^N)|^2\Big]. \tag{4.1}$$

We now deal with the last term in the above right-hand side: from Lemma 2.2 we have

$$\mathbb{E}\Big[\sup_{s\le t}|G_0(\mu_s^N)-G_0(\tilde{\mu}_{\underline{s}}^N)|^2\Big]\le\left(\frac{M}{m}\right)^2\mathbb{E}\left[\sup_{s\le t}\frac{1}{N}\sum_{j=1}^N|U_s^j-\tilde{U}_{\underline{s}}^j|^2\right]$$
$$\le 2\left(\frac{M}{m}\right)^2\mathbb{E}\left[\sup_{s\le t}\frac{1}{N}\sum_{j=1}^N\{|U_s^j-\tilde{U}_s^j|^2+|\tilde{U}_s^j-\tilde{U}_{\underline{s}}^j|^2\}\right],$$

and, by exchangeability of the particles,

$$\mathbb{E}\Big[\sup_{s\le t}|G_0(\mu_s^N)-G_0(\tilde{\mu}_{\underline{s}}^N)|^2\Big]\le 2\left(\frac{M}{m}\right)^2\mathbb{E}\Big[\sup_{s\le t}|U_s^i-\tilde{U}_s^i|^2+\sup_{s\le t}|\tilde{U}_s^i-\tilde{U}_{\underline{s}}^i|^2\Big]. \tag{4.2}$$

For the first term of the right-hand side, let us observe that, by exchangeability again,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\le t}|U_s^i-\tilde{U}_s^i|^2\Big]\\ &\le\mathbb{E}\left[\sup_{s\le t}\left\{\left(\int_0^s|b(X_r^i)-b(\tilde{X}_{\underline{r}}^i)|\,dr\right)^2+\left|\int_0^s(\sigma(X_r^i)-\sigma(\tilde{X}_{\underline{r}}^i))\,dB_r\right|^2\right\}\right]\\ &\le t\|b\|_{\mathrm{Lip}}^2\int_0^t\mathbb{E}[|X_s^i-\tilde{X}_{\underline{s}}^i|^2]\,ds+2\|\sigma\|_{\mathrm{Lip}}^2\int_0^t\mathbb{E}[|X_s^i-\tilde{X}_{\underline{s}}^i|^2]\,ds.\end{aligned} \tag{4.3}$$

We have for the second term

$$\mathbb{E}\Big[\sup_{s\le t}|\tilde{U}_s^i-\tilde{U}_{\underline{s}}^i|^2\Big]\le\mathbb{E}\Big[\sup_{s\le t}\{|b(\tilde{X}_{\underline{s}}^i)|^2(s-\underline{s})^2+|\sigma(\tilde{X}_{\underline{s}}^i)|^2(B_s-B_{\underline{s}})^2\}\Big].$$

Since

$$\mathbb{E}\Big[\sup_{s\le t}|b(\tilde{X}_{\underline{s}}^i)|^2(s-\underline{s})^2\Big]\le C\Big(1+\mathbb{E}\Big[\sup_{s\le T}|\tilde{X}_s^i|^2\Big]\Big)\left(\frac{T}{n}\right)^2$$

and

$$\mathbb{E}\Big[\sup_{s\le t}|\sigma(\tilde{X}_{\underline{s}}^i)|^2(B_s-B_{\underline{s}})^2\Big]\le\mathbb{E}\Big[\sup_{s\le T}|\sigma(\tilde{X}_{\underline{s}}^i)|^4\Big]^{1/2}\mathbb{E}\Big[\sup_{s\le T}|B_s-B_{\underline{s}}|^4\Big]^{1/2}$$
$$\le C\Big(1+\mathbb{E}\Big[\sup_{s\le T}|\tilde{X}_s^i|^4\Big]^{1/2}\Big)\mathbb{E}\Big[\sup_{s\le T}|B_s-B_{\underline{s}}|^4\Big]^{1/2},$$

we thus obtain

$$\mathbb{E}\Big[\sup_{s\le t}|\tilde{U}_s^i-\tilde{U}_{\underline{s}}^i|^2\Big]\le C\frac{\log n}{n}. \tag{4.4}$$

Using (4.4) and (4.3) with (4.2) we get

$$\mathbb{E}\Big[\sup_{s\le t}|G_0(\mu_s^N)-G_0(\tilde{\mu}_{\underline{s}}^N)|^2\Big]\le C\left\{\left(\frac{\log n}{n}\right)+\int_0^t\mathbb{E}[|X_s^i-\tilde{X}_{\underline{s}}^i|^2]\,ds\right\}. \tag{4.5}$$

Plugging this estimate in (4.1) gives

$$\mathbb{E}[|X_t^i-\tilde{X}_t^i|^2]\le C\left\{\left(\frac{\log n}{n}\right)+\int_0^t\mathbb{E}[|X_s^i-\tilde{X}_{\underline{s}}^i|^2]\,ds\right\}. \tag{4.6}$$

Since

$$\mathbb{E}[|X_s^i-\tilde{X}_{\underline{s}}^i|^2]\le 2\mathbb{E}[|X_s^i-\tilde{X}_s^i|^2]+2\mathbb{E}[|\tilde{X}_s^i-\tilde{X}_{\underline{s}}^i|^2]$$
$$=2\mathbb{E}[|X_s^i-\tilde{X}_s^i|^2]+2\mathbb{E}[|\tilde{U}_s^i-\tilde{U}_{\underline{s}}^i|^2],$$

it follows from (4.4) and (4.6) that

$$\mathbb{E}[|X_t^i - \tilde{X}_t^i|^2] \le C\left\{\left(\frac{\log n}{n}\right) + \int_0^t \mathbb{E}[|X_s^i - \tilde{X}_s^i|^2]\,ds\right\},$$

and we conclude the proof with Gronwall's lemma. □

PROOF OF LEMMA 4.4. Let us start by observing that

$$\begin{aligned}\sup_{s\le T}|B_s - B_{\underline{s}}| &= \max_{k=0,\dots,n-1}\sup_{T_k\le s\le T_{k+1}}|B_s - B_{T_k}|\\ &= \max_{k=0,\dots,n-1}\sup_{T_k\le s\le T_{k+1}}\max\bigl(B_s - B_{T_k}, -(B_s - B_{T_k})\bigr)\\ &\le \max_{k=0,\dots,n-1}\sup_{T_k\le s\le T_{k+1}}(B_s - B_{T_k}) + \max_{k=0,\dots,n-1}\sup_{T_k\le s\le T_{k+1}}\bigl(-(B_s - B_{T_k})\bigr).\end{aligned}$$

Since the random variables $\sup_{T_k\le s\le T_{k+1}}(B_s - B_{T_k})$, $k=0,\dots,n-1$, as well as the variables $\sup_{T_k\le s\le T_{k+1}}(-(B_s - B_{T_k}))$, $k=0,\dots,n-1$, are independent and have the same law as $|B_{T_1}|$,

$$\mathbb{E}\Bigl[\sup_{s\le T}|B_s - B_{\underline{s}}|^4\Bigr] \le 8\frac{T^2}{n^2}\mathbb{E}[\max(|G_1|^4,\dots,|G_n|^4)],$$

where G_k, $k=1,\dots,n$ are independent normal Gaussian random variables.

Let f be the function defined on $\mathbb{R}_+$ by $f(x) = e^{\sqrt{1+x/16}}$; f is convex, increasing with values in $[e,+\infty[$. The inverse of f is concave on $[e,+\infty[$, increasing and $f^{-1}(y) = 16[(\log y)^2 - 1]$. We have, by Jensen inequality,

$$\mathbb{E}\Bigl[\max_{k=1,\dots,n}|G_k|^4\Bigr] = \mathbb{E}\Bigl[\max_{k=1,\dots,n} f^{-1}\circ f\bigl(|G_k|^4\bigr)\Bigr] \le f^{-1}\Bigl(\mathbb{E}\Bigl[\max_{k=1,\dots,n} f\bigl(|G_k|^4\bigr)\Bigr]\Bigr),$$

from which we deduce that

$$\mathbb{E}\Bigl[\max_{k=1,\dots,n}|G_k|^4\Bigr] \le f^{-1}\left(\mathbb{E}\left[\sum_{k=1}^n f\bigl(|G_k|^4\bigr)\right]\right) = f^{-1}\bigl(n\mathbb{E}[f(|G_1|^4)]\bigr) \le f^{-1}\bigl(ne\mathbb{E}[e^{|G_1|^2/4}]\bigr).$$

Finally, we have

$$f^{-1}\bigl(ne\mathbb{E}[e^{|G_1|^2/4}]\bigr) = f^{-1}(ne\sqrt{2}) = 16\bigl((1+\log n + \log(2)/2)^2 - 1\bigr).$$

This concludes the proof of the lemma. □

5. Numerical illustrations. Throughout this section, we consider, on $[0,T]$, the following type of processes:

$$\begin{cases} X_t = \xi - \int_0^t (\beta_s + a_s X_s)\,ds + \int_0^t (\sigma_s + \gamma_s X_s)\,dB_s + K_t,\\ \mathbb{E}[h(X_t)] \ge 0, \qquad \int_0^t \mathbb{E}[h(X_s)]\,dK_s = 0,\end{cases} \tag{5.1}$$

where $(\beta_t)_{t\ge0}$, $(a_t)_{t\ge0}$, $(\sigma_t)_{t\ge0}$ and $(\gamma_t)_{t\ge0}$ are bounded adapted processes. This kind of processes allow us to make some explicit computations which, in turn, allow to illustrate the algorithm. Our results are then presented for different processes of the form of (5.1) and functions h. Associated computations are postponed to Section 5.2 below.

5.1. *Illustrations.* Let $0 = T_0 < T_1 < \cdots < T_n = T$ be a subdivision of $[0, T]$ of step size $1/n$, n being a positive integer. Using Algorithm 1 we draw L independent copies of $((\tilde{X}^i_{T_k})_{1\le i\le N}, \tilde{K}^N_{T_k})_{0\le k\le n}$, with $\tilde{K}^N_{T_k} = \sum_{\ell=1}^k \Delta_\ell \tilde{K}^N$ and N is the number of path of particles over the time interval $[0, T]$. We denote it by $(\tilde{X}^{i,l}, \tilde{K}^{N,l})_{1\le i\le N, 1\le l\le L}$. We show in the plots below the approximation of the reflection process K by the empirical mean of the i.i.d. copies of our estimator $\tilde{K}^N_{T_k}$ obtained by Algorithm 1. In order to illustrate the variance of the estimation of the reflection process K, we plot the boxplots associated with $(\tilde{K}^{N,l}_t)_{1\le l\le L}$ for five arbitrary times t in $[0, T]$. Indeed, apart from one example (case (iii) below), it appears that the amplitude the empirical 95% confidence interval obtained from the $(\tilde{K}^{N,l}_t)_{1\le l\le L}$ is too small to deserve the numerical illustrations.

When possible approximate the $\mathbb{L}^2$-error in Theorem 4.2 by:

$$\hat{E} = \left(\frac{1}{L}\sum_{l=1}^L \max_{0\le k\le n} |X^{1,l}_{T_k} - \tilde{X}^{1,l}_{T_k}|^2\right)^{1/2} \tag{5.2}$$

and

$$\hat{E}^K = \left(\frac{1}{L}\sum_{l=1}^L \max_{0\le k\le n} |K_{T_k} - \tilde{K}^{N,l}_{T_k}|^2\right)^{1/2}. \tag{5.3}$$

Every simulation are launched on Matlab and the minimisation used to compute the increment of the reflection process $\Delta_k \tilde{K}^N$, $k = 1, \ldots, n$ is done through the Matlab function *fsolve*.

Linear constraint. We first consider cases where $h : \mathbb{R} \ni x \longmapsto x - p \in \mathbb{R}$.

Case (i) Drifted Brownian motion: $\beta_t = \beta > 0$, $a_t = \gamma_t = 0$, $\sigma_t = \sigma > 0$, $\xi = x_0 \ge p$. We have

$$K_t = (p + \beta t - x_0)^+.$$

Figure 1 represents the evolution of the empirical mean of the $(\tilde{K}^{N,l})_{1\le l\le L}$ (circles) and K (full line) with respect to time. We notice that the approximation of K is very close to the exact solution with small variance. Figure 2 represents the evolution of $\log(\hat{E})$ and $\log(\hat{E}^K)$ w.r.t. $\log(N)$. We get a slope of $1/2$, which is consistent with Theorem 4.2.

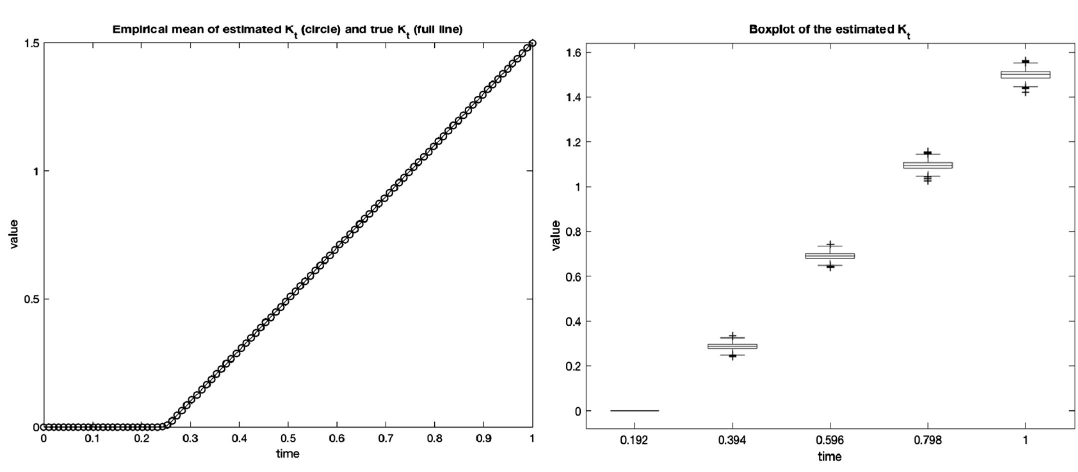

FIG. 1. *Case* (i). *Left*: *value of the empirical mean of* $\tilde{K}^{N,l}$, $1 \le l \le L$ *and* K_t *w.r.t. time. Right*: *boxplots of estimated* K_t *for five arbitrary times* t. *Data*: $n = 100$, $N = 2000$, $L = 1000$, $T = 1$, $\beta = 2$, $\sigma = 1$, $x_0 = 1$, $p = 1/2$.

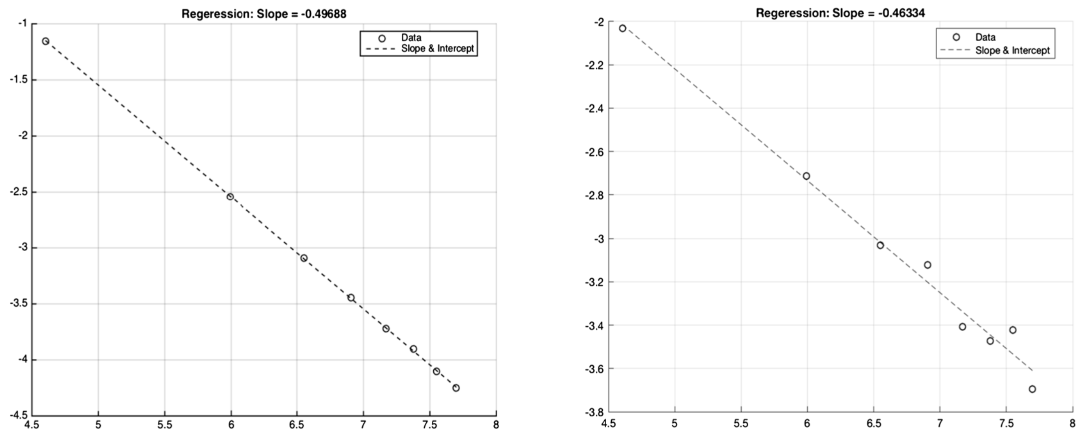

FIG. 2. *Case* (i). *Regression of* $\log(\hat{E})$ *and* $\log(\hat{E}^K)$ *w.r.t.* $\log(N)$. *Data*: $\hat{E}$ *when* N *varies from* 100 *to* 2200 *with step size* 300. *Parameters*: $n = 100$, $T = 1$, $\beta = 2$, $\sigma = 1$, $x_0 = 1$, $p = 1/2$, $L = 1000$.

Case (ii) Ornstein Uhlenbeck process: $\beta_t = \beta > 0$, $a_t = a > 0$, $\gamma_t = 0$, $\sigma_t = \sigma > 0$, $\xi = x_0$ with $x_0 \geq p > -\beta/a$. We have

$$K_t = (ap + \beta)(t - t^\star)1_{t \geq t^\star} \quad \text{where } t^\star = \frac{1}{a}\big(\ln(x_0 + \beta/a) - \ln(p + \beta/a)\big).$$

Figure 3 represents the evolution of of the empirical mean of the $(\tilde{K}^{N,l})_{1 \leq l \leq L}$ (circles) and K (full line) with respect to time. As in the previous example, the approximation of K is very close to the exact solution with small variance. Figure 4 represents the evolution of $\log(\hat{E})$ and $\log(\hat{E}^K)$ w.r.t. $\log(N)$. We get a slope of $1/2$, which is consistent with Theorem 4.2.

Case (iii) Ornstein–Uhlenbeck process with stochastic mean parameter: $\beta_t = \beta > 0$, $a_t = -\epsilon B_t$, $\epsilon > 0$, $\gamma_t = 0$, $\sigma_t = \sigma > 0$, $\xi = x_0$, $x_0 > p$. When $\epsilon \to 0^+$

$$K_t = \left(p - x_0 + \beta t - \sigma\epsilon\frac{t^2}{2}\right)1_{[t^\star, \bar{t}[}(t) + \left(-(x_0 - p) + \frac{\beta^2}{2\epsilon\sigma}\right)1_{t \geq \bar{t}} + o(\epsilon),$$

where $\bar{t} = \beta/(\epsilon\sigma)$ and $t^\star = (\beta - \sqrt{\beta^2 - 2(x_0 - p)\sigma\epsilon})/(\epsilon\sigma)$.

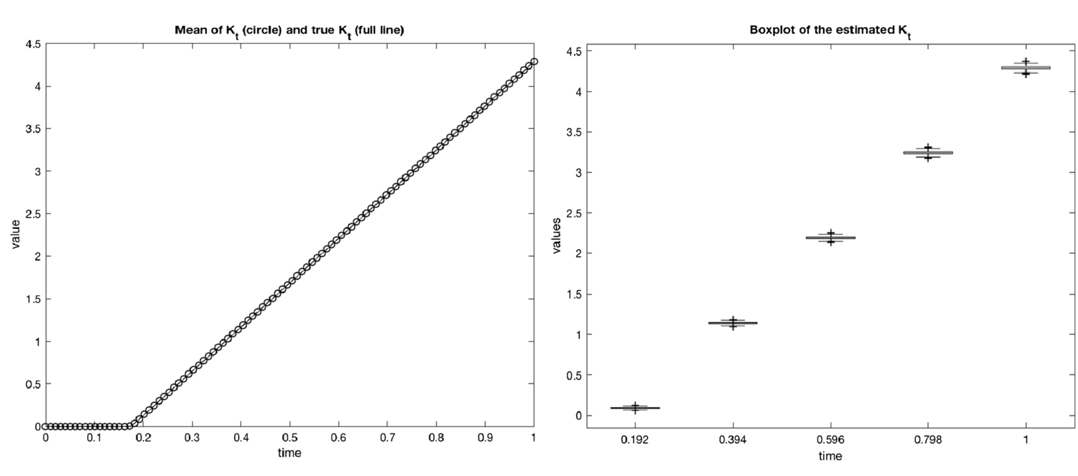

FIG. 3. *Case* (ii). *Left*: *value of the empirical mean of* $\tilde{K}^{N,l}$, $1 \leq l \leq L$ *and* K_t *w.r.t. time. Right*: *boxplots of estimated* K_t *for five arbitrary times* t. *Data*: $n = 100$, $N = 2000$, $L = 1000$, $T = 1$, $\beta = 2.1$, $a = 1$, $\sigma = 1$, $p = 3.1$, $x_0 = p + 1$.

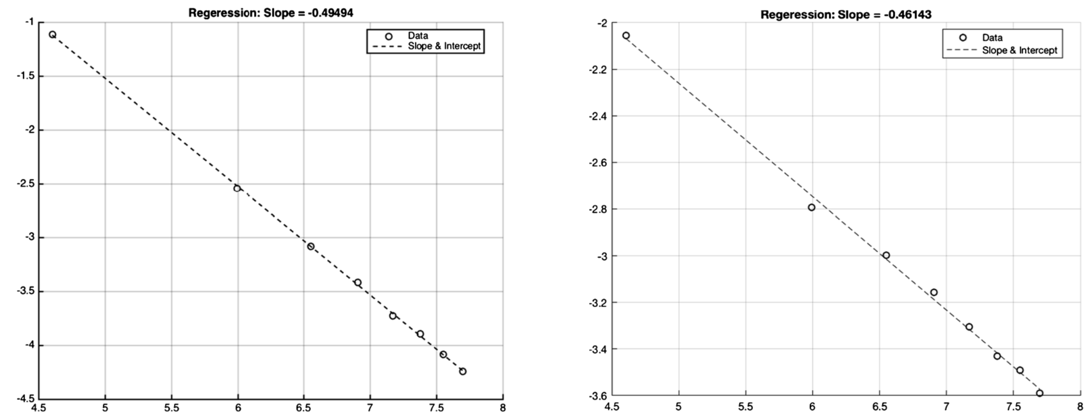

FIG. 4. *Case* (ii). *Regression of* $\log(\hat{E})$ *and* $\log(\hat{E}^K)$ *w.r.t.* $\log(N)$. *Data*: $\hat{E}$ *when* N *varies from* 100 *to* 2200 *with step size* 300. *Parameters*: $n = 100$, $T = 1$, $\beta = 2$, $\sigma = 1$, $x_0 = 1$, $p = 3.1$, $L = 1000$.

Figure 5 represents the of the empirical mean of the $(\tilde{K}^{N,l})_{1\le l\le L}$ (circles) and K (full line) with respect to time. The times $t^\star$ and $\bar{t}$ are also ploted. We notice that the approximation of K is biased on $[\bar{t}, T]$, this may comes from the fact that we plot the first order approximation of K in ϵ.

REMARK 5.1. The reader may object that case (iii) is out of the scope of our theoretical results, which is true. We nevertheless choose to give this example in order to illustrate, numerically, the robustness of the method.

Case (iv) Black and Sholes process: $\beta_t = \beta > 0$, $a_t = a > 0$, $\sigma_t = 0$, $\gamma_t = \gamma > 0$. Then

$$K_t = (ap + \beta)(t - t^\star)1_{t \ge t^\star} \quad \text{where } t^\star = \frac{1}{a}\big(\ln(x_0 + \beta/a) - \ln(p + \beta/a)\big).$$

Figure 6 represents the evolution of of the empirical mean of the $(\tilde{K}^{N,l})_{1\le l\le L}$ (circles) and K (full line) with respect to time. We notice that the approximation of K is quite precise with small variance. As in cases (i) and (ii) the coefficients a, β, σ and γ are constants and the constraint is linear, the numerical scheme is closer to the exact solution than in case (iii).

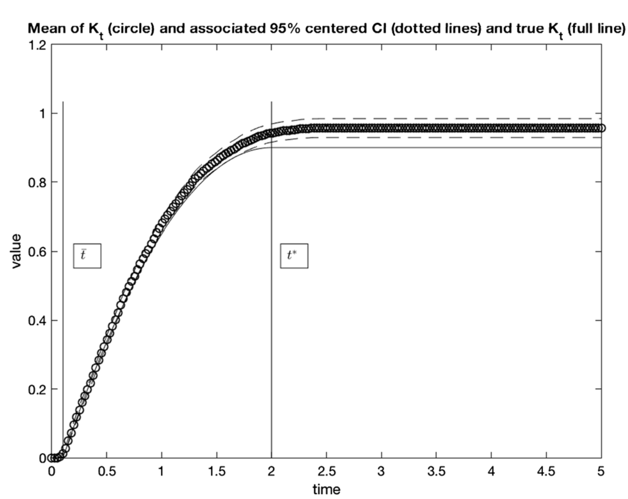

FIG. 5. *Case* (iii). *Value of the empirical mean of* $\tilde{K}^{N,l}$, $1 \le l \le L$ *and associated* 95% *approximated centered confidence interval and* K_t *w.r.t. time. Data*: $n = 200$, $N = 10{,}000$, $L = 100$, $T = 5$, $\beta = 1$, $\epsilon = 5/100$, $\sigma = 1/(2\epsilon)$, $x_0 = 1$, $p = 0.9$.

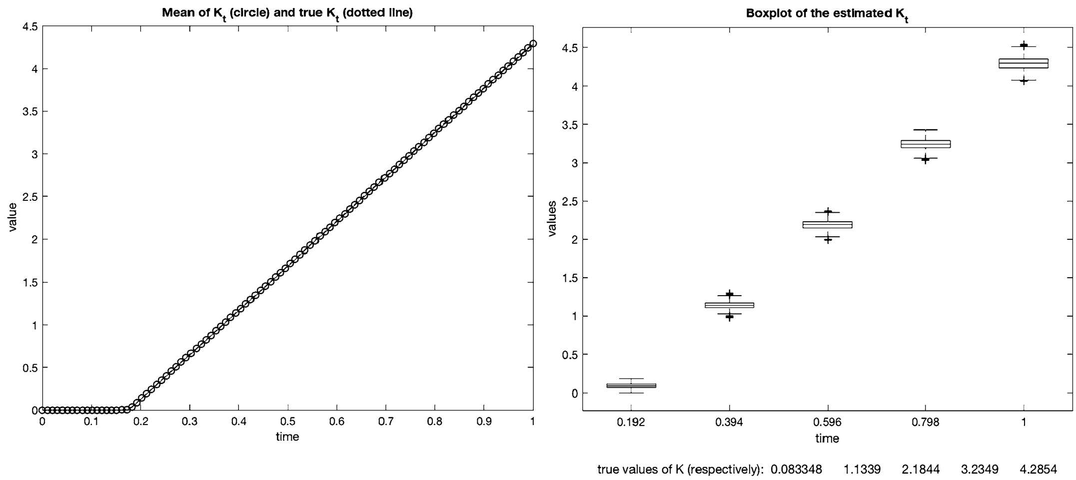

FIG. 6. *Case* (iv). *Left*: *value of the empirical mean of* $\tilde{K}^{N,l}, 1 \le l \le L$ *and* K_t *w.r.t. time*. *Right*: *boxplots of estimated* K_t *for five arbitrary times* t. *Data*: $n = 100$, $N = 2000$, $L = 1000$, $T = 1$, $a = 1$, $\gamma = 1$, $x_0 = 4$, $p = 1$.

Nonlinear constraint. Second, we illustrate the case of nonlinear function h:

$$h : \mathbb{R} \ni x \mapsto x + \alpha \sin(x) - p \in \mathbb{R}, -1 < \alpha < 1.$$

Case (v) Ornstein–Uhlenbeck process: $a_t = a > 0$, $\beta_t = \beta > 0$, $\gamma_t = 0$, $\sigma_t = \sigma > 0$, $\xi = x_0$ with $x_0 > |\alpha| + p$. We obtain

$$dK_t = e^{-at}\, d \sup_{s \le t} \big(F_s^{-1}(0)\big)^+,$$

where for all t in $[0, T]$,

$$F_t : \mathbb{R} \ni x \mapsto \bigg\{ e^{-at}\bigg(x_0 - \beta\bigg(\frac{e^{at}-1}{a}\bigg) + x\bigg) \\ + \alpha \exp\bigg(-e^{-at}\frac{\sigma^2}{2a}\sinh(at)\bigg) \sin\bigg(e^{-at}\bigg(x_0 - \beta\bigg(\frac{e^{at}-1}{a}\bigg) + x\bigg)\bigg) - p\bigg\}.$$

Figure 7 represents the evolution of of the empirical mean of the $(\tilde{K}^{N,l})_{1 \le l \le L}$ (circles) and K (full line) with respect to time. We notice that the approximation of K is precise with a small variance, although the latter is higher than most of the previous examples. This is obviously due to the nonlinearity of the h constraint function.

5.2. *Proofs of the numerical illustrations*. In order to have closed, or almost closed, expression for the compensator K we introduce the process Y solution to the nonreflected SDE

$$Y_t = \xi - \int_0^t (\beta_s + a_s Y_s)\, ds + \int_0^t (\sigma_s + \gamma_s Y_s)\, dB_s.$$

Letting $A_t = \int_0^t a_s\, ds$ and applying Itô's formula on $e^{A_t}(X_t - Y_t)$, we get

$$X_t = Y_t + e^{-A_t} \int_0^t e^{A_s}\, dK_s + e^{-A_t} \int_0^t e^{A_s} \gamma_s (X_s - Y_s)\, dB_s.$$

Hence, the constraint $\mathbb{E}[h(X_t)] \ge 0$ rewrites

$$\mathbb{E}\bigg[h\bigg(Y_t + e^{-A_t} \int_0^t e^{A_s}\, dK_s + e^{-A_t} \int_0^t e^{A_s} \gamma_s (X_s - Y_s)\, dB_s\bigg)\bigg] \ge 0. \tag{5.4}$$

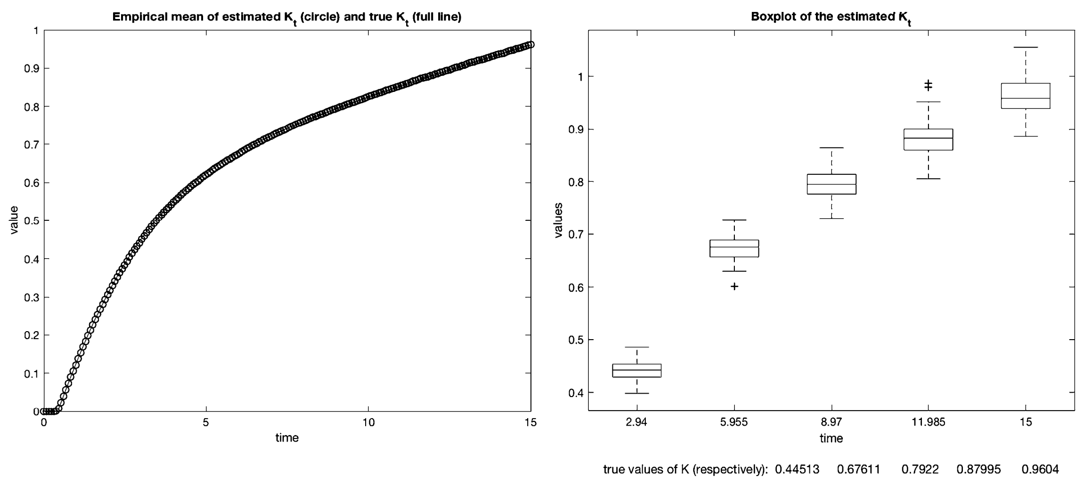

FIG. 7. *Case* (v). *Left*: *value of true and empirical mean of estimated* K_t *w.r.t. time*. *Right*: *boxplots of estimated* K_t *for five arbitrary time* t. *Data*: $n = 200$, $N = 10{,}000$, $L = 100$, $T = 15$, $\beta = 10^{-2}$, $\sigma = 1$, $p = \pi/2$, $\alpha = 0.9$, x_0 *is the unique solution of* $x + \alpha \sin(x) - p = 0$ *plus* 10^{-1}.

PROOF OF ASSERTIONS (i), (ii) AND (iv). The formula for K comes from the expression of its density given in Corollary 2.7 and the fact that in all these cases

$$\mathbb{E}[Y_t] - p = e^{-at}\left(x_0 + \frac{\beta}{a}\right) - \left(p + \frac{\beta}{a}\right). \qquad \square$$

PROOF OF (iii). Recall that we supposed $h : \mathbb{R} \ni x \mapsto x - p \in \mathbb{R}$. In that case, since $\gamma \equiv 0$, the constraint (5.4) becomes

$$\mathbb{E}\left[e^{-A_t}\int_0^t e^{A_s}\,dK_s\right] \geq p - \mathbb{E}[Y_t], \tag{5.5}$$

so that K is nondecreasing with $K_0 = 0$ and, for all t in $[0, T]$,

$$\mathbb{E}\left[e^{-A_t}\int_0^t e^{A_s}\,dK_s\right] \geq p - \mathbb{E}[Y_t], \qquad \int_0^t \big(\mathbb{E}[X_s] - p\big)\,dK_s = 0. \tag{5.6}$$

Note first that since

$$\mathbb{E}[Y_s] = \mathbb{E}[e^{-A_s}]x_0 - \mathbb{E}\left[\int_0^s e^{-(A_s - A_r)}\beta\,dr\right] + \mathbb{E}\left[e^{-A_s}\int_0^s e^{A_r}\sigma\,dB_r\right],$$

and using the integration by parts formula we have

$$\mathbb{E}\left[e^{-A_s}\int_0^s e^{A_r}\sigma\,dB_r\right] = \sigma\mathbb{E}\left[\int_0^s D_r\big(e^{-A_s}\big)e^{A_r}\,dr\right] = -\sigma\mathbb{E}\left[\int_0^s \int_r^s (D_r a_u)\,du\,e^{-(A_s - A_r)}\,dr\right].$$

Remember that, in this case, we supposed that $T \leq 1$, $\beta_t = \beta > 0$ and $a_t = -\epsilon B_t$ for $\epsilon > 0$ supposed to be small enough. We here illustrate the dependence of the processes w.r.t. the parameter ϵ by adding a superscript ϵ on Y and K. Since $\int_s^t B_r\,dr$ is a centered gaussian random variable with variance $(t-s)^3/3$, we have

$$\mathbb{E}[Y_t^\epsilon] = x_0 e^{\epsilon^2 t^3/6} - \beta\int_0^t e^{\epsilon^2(t-s)^3/6}\,ds + \sigma\epsilon\int_0^t (t-s)e^{\epsilon^2(t-s)^3/6}\,ds,$$

$$\mathbb{E}[X_t^\epsilon] = \mathbb{E}[Y_t^\epsilon] + \int_0^t e^{\epsilon^2(t-s)^3/6}\,dK_s^\epsilon.$$

Therefore, for all $s \le t$,

$$\mathbb{E}[X_s^\epsilon] - p = x_0 - p - \beta s + \epsilon\sigma \frac{s^2}{2} + o(\epsilon) + K_t^\epsilon\big(1 + o(\epsilon)\big).$$

It follows that, up to $o(\epsilon)$,

$$K_t^\epsilon = \sup_{s\le t}\left(-(x_0 - p) + \beta s - \sigma\epsilon\frac{s^2}{2}\right)^+.$$

Since $\epsilon \to 0^+$, we assume that $\beta^2 > 2\epsilon\sigma(x_0 - p)$ and we obtain $K_t^\epsilon = 0$ if $t < t^\star$,

$$K_t^\epsilon = -(x_0 - p) + \beta t - \sigma\epsilon\frac{t^2}{2} \quad \text{if } t^\star \le t < \frac{\beta}{\epsilon\sigma},$$
$$K_t^\epsilon = -(x_0 - p) + \frac{\beta^2}{2\epsilon\sigma} \quad \text{for } t \ge \frac{\beta}{\epsilon\sigma},$$

where $t^\star = (\beta - \sqrt{\beta^2 - 2\epsilon\sigma(x_0 - p)})/(\epsilon\sigma)$. $\square$

PROOF OF ASSERTION (v). In that case, we have

$$Y_t = e^{-at}\left(x_0 - \beta\left(\frac{e^{at} - 1}{a}\right)\right) + \sigma e^{-at}\int_0^t e^{as}\,dB_s := f_t + G_t$$

and

$$X_t = Y_t + e^{-at}\bar{K}_t, \quad \bar{K}_t = \int_0^t e^{as}\,dK_s.$$

Hence

$$\begin{aligned} h(X_t) &= Y_t + e^{-at}\bar{K}_t + \alpha\sin\big(Y_t + e^{-at}\bar{K}_t\big) - p \\ &= Y_t + e^{-at}\bar{K}_t + \alpha\big(\sin(Y_t)\cos\big(e^{-at}\bar{K}_t\big) + \cos(Y_t)\sin\big(e^{-at}\bar{K}_t\big)\big) - p \\ &= Y_t + e^{-at}\bar{K}_t + \alpha\big[\cos\big(e^{-at}\bar{K}_t\big)\{\sin(f_t)\cos(G_t) + \sin(G_t)\cos(f_t)\} \\ &\quad + \sin\big(e^{-at}\bar{K}_t\big)\{\cos(f_t)\cos(G_t) - \sin(f_t)\sin(G_t)\}\big] - p. \end{aligned}$$

Since G_t is a centered gaussian random variable with variance $\sigma^2\frac{1-e^{-2at}}{2a} = \sigma^2 e^{-at}\frac{\sinh(at)}{a}$,

$$\mathbb{E}\big[\sin(G_t)\big] = 0 \quad \text{and} \quad \mathbb{E}\big[\cos(G_t)\big] = \exp\left(-e^{-at}\frac{\sigma^2}{a}\sinh(at)\right) =: g(t),$$

we obtain that

$$\begin{aligned} \mathbb{E}\big[h(X_t)\big] &= f(t) + e^{-at}\bar{K}_t + \alpha g(t)\sin\big(f_t + e^{-at}\bar{K}_t\big) - p \\ &:= F_t(\bar{K}_t). \end{aligned}$$

Therefore,

$$\bar{K}_t = \sup_{s\le t}\big(F_s^{-1}(0)\big)^+ \quad \text{and} \quad dK_t = e^{-at}\,d\sup_{s\le t}\big(F_s^{-1}(0)\big)^+. \qquad \square$$

Acknowledgments. The authors would like to thank very much Hélène Hibon for her careful reading of a previous version of this work and for her valuable comments and remarks.

REFERENCES

[1] ARTZNER, P., DELBAEN, F., EBER, J.-M. and HEATH, D. (1999). Coherent measures of risk. *Math. Finance* **9** 203–228. MR1850791 https://doi.org/10.1111/1467-9965.00068

[2] BAUDOIN, F. (2002). Conditioned stochastic differential equations: Theory, examples and application to finance. *Stochastic Process. Appl.* **100** 109–145. MR1919610 https://doi.org/10.1016/S0304-4149(02)00109-6

[3] BRIAND, P., ELIE, R. and HU, Y. (2018). BSDEs with mean reflection. *Ann. Appl. Probab.* **28** 482–510. MR3770882 https://doi.org/10.1214/17-AAP1310

[4] CATTIAUX, P. and LÉONARD, C. (1995). Large deviations and Nelson processes. *Forum Math.* **7** 95–115. MR1307957 https://doi.org/10.1515/form.1995.7.95

[5] FÖLLMER, H. and SCHIED, A. (2002). Convex measures of risk and trading constraints. *Finance Stoch.* **6** 429–447. MR1932379 https://doi.org/10.1007/s007800200072

[6] FOURNIER, N. and GUILLIN, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields* **162** 707–738. MR3383341 https://doi.org/10.1007/s00440-014-0583-7

[7] JABIR, J.-F. (2017). Diffusion processes with weak constraint through penalization approximation. Available at arXiv:1704.01505v2.

[8] LIONS, P.-L. and SZNITMAN, A.-S. (1984). Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37** 511–537. MR0745330 https://doi.org/10.1002/cpa.3160370408

[9] MAURY, B., ROUDNEFF-CHUPIN, A. and SANTAMBROGIO, F. (2010). A macroscopic crowd motion model of gradient flow type. *Math. Models Methods Appl. Sci.* **20** 1787–1821. MR2735914 https://doi.org/10.1142/S0218202510004799

[10] MIKAMI, T. and THIEULLEN, M. (2008). Optimal transportation problem by stochastic optimal control. *SIAM J. Control Optim.* **47** 1127–1139. MR2407010 https://doi.org/10.1137/050631264

[11] RACHEV, S. T. and RÜSCHENDORF, L. (1998). *Mass Transportation Problems, Vol. II: Applications. Probability and Its Applications (New York)*. Springer, New York. MR1619171

[12] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX—1989. Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[13] TAN, X. and TOUZI, N. (2013). Optimal transportation under controlled stochastic dynamics. *Ann. Probab.* **41** 3201–3240. MR3127880 https://doi.org/10.1214/12-AOP797

[14] TANAKA, H. (1979). Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.* **9** 163–177. MR0529332