*The Annals of Applied Probability*
2020, Vol. 30, No. 4, 1642–1668
https://doi.org/10.1214/19-AAP1539


# HIGH-DIMENSIONAL LIMITS OF EIGENVALUE DISTRIBUTIONS FOR GENERAL WISHART PROCESS

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In this article, we obtain an equation for the high-dimensional limit measure of eigenvalues of generalized Wishart processes, and the results are extended to random particle systems that generalize SDEs of eigenvalues. We also introduce a new set of conditions on the coefficient matrices for the existence and uniqueness of a strong solution for the SDEs of eigenvalues. The equation of the limit measure is further discussed assuming self-similarity on the eigenvalues.

**1. Introduction.** While the theory of stochastic differential equations (SDEs) with values in a Euclidean space is quite well developed in stochastic analysis, the study of SDEs on general manifolds is more recent. In this paper, we consider the eigenvalue process of the solution of a special class of matrix-valued SDEs as well as a more general class of particle systems introduced in Graczyk and Małecki (2014). For ease of notation, let $\mathcal{S}_N$ be the group of $N \times N$ symmetric matrices. For $X \in \mathcal{S}_N$ and $f$ a real-valued function, $f(X) \in \mathcal{S}_N$ denotes the matrix obtained from $X$ by acting $f$ on the spectrum of $X$. Namely, if $X$ has the spectral decomposition $X = \sum_{j=1}^{p} \alpha_j u_j u_j^\intercal$ with eigenvalues $(\alpha_j)$ and eigenvectors $(u_j)$, then $f(X) = \sum_{j=1}^{p} f(\alpha_j) u_j u_j^\intercal$. Here $A^\intercal$ denotes the transpose of a matrix or vector $A$.

There is not much work in the literature on SDEs with matrix state space $\mathcal{S}_N$. We consider the class of so-called *generalized Wishart process* which satisfies the following SDE on $\mathcal{S}_N$:

$$dX_t^N = g_N(X_t^N)\, dB_t h_N(X_t^N) + h_N(X_t^N)\, dB_t^\intercal g_N(X_t^N) + b_N(X_t^N)\, dt, \quad t \geq 0. \tag{1.1}$$

Here $B_t$ is a Brownian matrix of dimension $N \times N$, and the functions $g_N, h_N, b_N : \mathbb{R} \to \mathbb{R}$ act on the spectrum of $X_t^N$. Let

$$G_N(x, y) = g_N^2(x) h_N^2(y) + g_N^2(y) h_N^2(x), \tag{1.2}$$

which is symmetric with respect to $x$ and $y$. Let $\lambda_1^N(t) \leq \lambda_2^N(t) \leq \cdots \leq \lambda_N^N(t)$ be the eigenvalues of $X_t^N$. According to Theorem 3 in Graczyk and Małecki (2013), if $\lambda_1^N(0) < \lambda_2^N(0) < \cdots < \lambda_N^N(0)$, then before the first collision time

$$\tau_N = \inf\{t > 0 : \exists i \neq j, \lambda_i(t) = \lambda_j(t)\},$$

the eigenvalues satisfy the following SDEs: for $1 \leq i \leq N$,

$$\begin{aligned} d\lambda_i^N(t) &= 2 g_N(\lambda_i^N(t)) h_N(\lambda_i^N(t))\, dW_i(t) \\ &\quad + \left( b_N(\lambda_i^N(t)) + \sum_{j : j \neq i} \frac{G_N(\lambda_i^N(t), \lambda_j^N(t))}{\lambda_i^N(t) - \lambda_j^N(t)} \right) dt. \end{aligned} \tag{1.3}$$

Received January 2019; revised September 2019.
*MSC2020 subject classifications.* 60H15, 60F05.
*Key words and phrases.* Dyson Brownian motion, eigenvalue distribution, generalized Wishart process, high-dimensional limit, squared Bessel particle system, Wishart process.

Here, $\{W_i, i = 1, 2, \dots, N\}$ are independent Brownian motions. In Graczyk and Małecki (2013, 2014), some other conditions on the functions were imposed to ensure that (1.3) has a unique strong solution and the collision time is infinity almost surely.

The generalized Wishart process (1.1) extends the celebrated symmetric Brownian motion and Wishart process introduced respectively in Dyson (1962) and Bru (1989), as follows.

- If we take $g_N(x) = (2N)^{-1/2}$, $h_N(x) = 1$ and $b_N(x) = 0$ in (1.1), the random matrix $X_t^N$ becomes the symmetric Brownian motion with elements:

$$X_t^N(i,j) = \frac{1}{\sqrt{N}} B_t(i,j) 1_{\{i<j\}} + \frac{\sqrt{2}}{\sqrt{N}} B_t(i,i) 1_{\{i=j\}}, \quad 1 \le i \le j \le N, \tag{1.4}$$

  where $\{B_t(i,j), i \le j\}$ are independent Brownian motions.
- If we take $g_N(x) = \sqrt{x}$, $h_N(x) = 1/\sqrt{N}$, and $b_N(x) = p/N$ with $p > N-1$ in (1.1), then the random matrix $Y_t^N = N X_t^N$ is the Wishart process $B_t^\intercal B_t$, where $B_t$ is a $p \times N$ Brownian matrix.

Symmetric matrices appear in many scientific fields. Historically, Dyson (1962) used symmetric Brownian motions to analyse the Hamiltonian of a complex nucleic system in particle physics. Bru (1989) introduced her Wishart process to perform principal component analysis on a set of resistance data of Escherichia Coli to certain antibiotics. More recently, time series of positive definite matrices are particularly important in the following fields.

1. Financial data analysis: multivariate volatility/co-volatility between stock returns or interest rates from different markets have been studied recently through Wishart processes, see Gourieroux (2006), Gourieroux and Sufana (2010), Da Fonseca, Grasselli and Tebaldi (2008), Da Fonseca, Grasselli and Ielpo (2014), Gnoatto (2012), Gnoatto and Grasselli (2014) and Wu et al. (2018).
2. Machine learning: an important task in machine learning using kernel functions is the determination of a suitable kernel matrix for a given data analysis problem (Schölkopf and Smola (2002)). Such determination is referred as the *kernel matrix learning problem*. A kernel matrix is in fact a positive definite Gram-matrix of size $N \times N$ where $N$, the sample size of the data, is usually large. An innovative method for kernel learning is proposed by Zhang, Kwok and Yeung (2006) where unknown kernel matrix is modeled by a Wishart process prior. This approach has been followed in Kondor and Jebara (2007) and Li, Zhang and Yeung (2009).
3. Computer vision: real-time computer vision often involves tracking of objects of interest. At each time $t$, a target is encoded into a $N$-dimensional vector $a_t \in \mathbb{R}^N$ (feature vector). It is therefore clear that measuring "distance" between these vectors, say $a_t$ and $a_{t+dt}$ at two consecutive time spots $t$ and $t+dt$, is of crucial importance for object tracking. Because the standard Euclidean distance $\|a_{t+dt} - a_t\|^2$ is rarely optimal, it is more satisfactory to identify a better metric of the form $(a_{t+dt} - a_t)^\intercal M_t (a_{t+dt} - a_t)$ using a suitable positive definite matrix $M_t$. Again, the sequence of metric matrices $(M_t)$ is time varying; it should be data-adaptive, estimable from data available at time $t$. An innovative solution is proposed in Li et al. (2016) where $M_t$ follows a Wishart process.

Motivated by these recent applications where the dimension $N$ of a matrix process is usually large, we study in this paper *high-dimensional limits* of eigenvalue distributions of the generalized Wishart process (1.1) as $N$ tends to infinity. To the best of our knowledge, such high-dimensional limits are known in the literature only for some simple cases. An early result is the derivation of the Wigner semi-circle law from the eigenvalue empirical measure process in Chan (1992) where the symmetric matrix process has independent Ornstein–Uhlenbeck processes as its entries. The results were later generalized in Rogers and Shi

(1993) to the following SDEs:

$$dX_j = \sqrt{\frac{2\alpha}{N}}\,dB_j + \left(-\theta X_j + \frac{\alpha}{N}\sum_{j:j\neq i}\frac{1}{X_i - X_j}\right)dt, \quad 1 \le i \le N, t \ge 0.$$

Cépa and Lépingle (1997) further generalised these SDEs to

$$dX_j = \sigma(X_j)\,dB_j + \left(b(X_j) + \sum_{j:j\neq i}\frac{\gamma}{X_i - X_j}\right)dt, \quad 1 \le i \le N, t \ge 0,$$

with some coefficient functions $b$, $\sigma$ and constant $\gamma$. Another important case is the Marčenko–Pastur law for the eigenvalue empirical measure process derived in Cabanal-Duvillard and Guionnet (2001). The eigenvalues SDEs (1.3) considered in the present paper generalises the eigenvalue SDEs in Chan (1992) and Cabanal-Duvillard and Guionnet (2001), as well as the particle system in Rogers and Shi (1993). Also the particle system (3.1) in Section 3 which is introduced in Graczyk and Małecki (2014) generalizes the particle system in Cépa and Lépingle (1997).

The rest of the paper is organized as follows. In Section 2, we study high-dimensional limits of eigenvalue distributions of the generalized Wishart process (1.1). In Section 3, our results are extended to a random particle system that generalizes the eigenvalue SDEs (1.3). These results from the two sections presuppose that these SDEs have a unique strong solution (before colliding/exploding time). In Section 4, we introduce a new set of conditions on the coefficient matrices in (1.3) and its generalization, the particle system (3.1) (here the dimension $N$ is fixed). These conditions are thus compared with the ones proposed in Graczyk and Małecki (2013, 2014). In Section 5, assuming self-similarity on the eigenvalues, we simplify the equation (2.15) of the limit measure and indicate its connection with the Hilbert transform operator.

**2. Limit point of empirical measure for eigenvalues.** We denote by $M_1(\mathbb{R})$ the set of probability measures on $\mathbb{R}$. Since a probability measure can be viewed as a continuous linear functional on the space $C_b(\mathbb{R})$ of bounded continuous functions, $M_1(\mathbb{R})$ is a subset of the dual space $C_b(\mathbb{R})^*$ of $C_b(\mathbb{R})$. Since the space $C_b(\mathbb{R})$ endowed with the sup norm is a normable space, its dual $C_b(\mathbb{R})^*$ is a Banach space with the dual norm. The space $M_1(\mathbb{R})$ with the norm inherited from the dual norm of $C_b(\mathbb{R})^*$ is complete. Besides, the space $C([0,T], M_1(\mathbb{R}))$ endowed with the metric

$$d_{C([0,T],M_1(\mathbb{R}))}(f_1, f_2) = \sup_{t\in[0,T]} d_{M_1(\mathbb{R})}\big(f_1(t), f_2(t)\big),$$

is complete.

Consider the empirical measure of the eigenvalues $\lambda_i^N(t)$ satisfying (1.3)

$$L_N(t) = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i^N(t)}. \tag{2.1}$$

We shall study the limit point of $L_N$ in the space $C([0,T], M_1(\mathbb{R}))$, as $N$ goes to infinity, and we assume the following conditions.

(A) There exists a positive function $\varphi(x) \in C^2(\mathbb{R})$ such that $\lim_{|x|\to+\infty}\varphi(x) = +\infty$, $\varphi'(x)b_N(x)$ is bounded with respect to $(x, N)$, and $\varphi'(x)g_N(x)h_N(x)$ satisfies

$$\sum_{N=1}^{\infty}\left(\frac{\|\varphi' g_N h_N\|^2_{L^\infty(dx)}}{N}\right)^{l_1} < \infty,$$

for some positive integer $l_1$.

(B) The function $NG_N(x,y)\frac{\varphi'(x)-\varphi'(y)}{x-y}$ is bounded with respect to $(x,y,N)$.

(C)

$$C_0=\sup_{N>0}\langle\varphi,L_N(0)\rangle=\sup_{N>0}\frac{1}{N}\sum_{i=1}^{N}\varphi(\lambda_i^N(0))<\infty. \tag{2.2}$$

(D) There exists a sequence $\{\tilde{f}_k\}_{k\in\mathbb{N}}$ of $C^2(\mathbb{R})$ functions such that it is dense in the space $C_0(\mathbb{R})$ of continuous functions vanishing at infinity and that $\tilde{f}_k'(x)g_N(x)h_N(x)$ satisfies

$$\psi(k)=\sum_{N=1}^{\infty}\left(\frac{\|\tilde{f}_k'g_Nh_N\|_{L^\infty(dx)}^2}{N}\right)^{l_2}<\infty \tag{2.3}$$

for some positive integer $l_2\geq 2$.

REMARK 2.1. When one chooses the function $\varphi(x)$ in condition (A), although $\varphi(x)$ goes to $\infty$ as $|x|$ goes to $\infty$, one should expect that the first and second derivatives of $\varphi$ vanish fast enough. One typical choice is $\varphi(x)=\ln(1+x^2)$.

Condition (B) implies that

$$NG_N(x,x)\varphi''(x)=\lim_{y\to x}NG_N(x,y)\frac{\varphi'(x)-\varphi'(y)}{x-y}$$

is uniformly bounded with respect to $(x,N)$, and so is $Ng_N^2(x)h_N^2(x)\varphi''(x)$.

REMARK 2.2. Suppose that $b_N(x)\leq c_b|x|$, $g_N^2(x)\leq c_g|x|N^{-\alpha}$ and $h_N^2(x)\leq c_h|x|N^{-\beta}$ for large $N$ and large $|x|$ with constants $c_b$, $c_g$, $c_h$ and $\alpha+\beta\geq 1$, then we can choose $\varphi(x)=\ln(1+x^2)$ to satisfy the above conditions (A), (B) and (D).

THEOREM 2.1. *Let $T>0$ be a fixed number. Suppose that* (1.3) *has a strong solution that is nonexploding and noncolliding for $t\in[0,T]$. Then under the conditions* (A), (B), (C) *and* (D), *the sequence $\{L_N(t),t\in[0,T]\}_{N\in\mathbb{N}}$ is relatively compact in $C([0,T],M_1(\mathbb{R}))$, that is*, *every subsequence has a further subsequence that converges in $C([0,T],M_1(\mathbb{R}))$ almost surely.*

PROOF. We split the proof into three steps for the reader's convenience.

*Step* 1. In this step, we apply Itô's formula to estimate $\langle f,L_N(t)\rangle$ for $f\in C^2(\mathbb{R})$.

Note that

$$\langle f,L_N(t)\rangle=\int f(x)L_N(t)(dx)=\frac{1}{N}\sum_{i=1}^{N}\int f(x)\delta_{\lambda_i^N(t)}(dx)=\frac{1}{N}\sum_{i=1}^{N}f(\lambda_i^N(t)).$$

By Itô's formula and (1.3),

$$\begin{aligned}
&f(\lambda_i^N(t))-f(\lambda_i^N(0))\\
&\quad=\int_0^t f'(\lambda_i^N(s))\,d\lambda_i^N(s)+\frac{1}{2}\int_0^t f''(\lambda_i^N(s))\,d\langle\lambda_i^N\rangle_s\\
&\quad=2\int_0^t f'(\lambda_i^N(s))g_N(\lambda_i^N(s))h_N(\lambda_i^N(s))\,dW_i(s)\\
&\qquad+\int_0^t f'(\lambda_i^N(s))b_N(\lambda_i^N(s))\,ds+\int_0^t f'(\lambda_i^N(s))\sum_{j:j\neq i}\frac{G_N(\lambda_i^N(s),\lambda_j^N(s))}{\lambda_i^N(s)-\lambda_j^N(s)}\,ds\\
&\qquad+2\int_0^t f''(\lambda_i^N(s))g_N^2(\lambda_i^N(s))h_N^2(\lambda_i^N(s))\,ds.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\langle f, L_N(t)\rangle - \langle f, L_N(0)\rangle \\
&\quad = \frac{2}{N}\sum_{i=1}^{N}\int_0^t f'(\lambda_i^N(s)) g_N(\lambda_i^N(s)) h_N(\lambda_i^N(s))\, dW_i(s) \\
&\qquad + \frac{1}{N}\sum_{i=1}^{N}\int_0^t f'(\lambda_i^N(s)) b_N(\lambda_i^N(s))\, ds \\
&\qquad + \frac{1}{N}\sum_{i\neq j}\int_0^t f'(\lambda_i^N(s)) \frac{G_N(\lambda_i^N(s), \lambda_j^N(s))}{\lambda_i^N(s) - \lambda_j^N(s)}\, ds \\
&\qquad + \frac{2}{N}\sum_{i=1}^{N}\int_0^t f''(\lambda_i^N(s)) g_N^2(\lambda_i^N(s)) h_N^2(\lambda_i^N(s))\, ds \\
&\quad = M_f^N(t) + \int_0^t \langle f' b_N, L_N(s)\rangle\, ds + 2\int_0^t \langle f'' g_N^2 h_N^2, L_N(s)\rangle\, ds \\
&\qquad + \frac{1}{N}\sum_{i\neq j}\int_0^t f'(\lambda_i^N(s)) \frac{G_N(\lambda_i^N(s), \lambda_j^N(s))}{\lambda_i^N(s) - \lambda_j^N(s)}\, ds,
\end{aligned}
\tag{2.4}
$$

where

$$
M_f^N(t) = \frac{2}{N}\sum_{i=1}^{N}\int_0^t f'(\lambda_i^N(s)) g_N(\lambda_i^N(s)) h_N(\lambda_i^N(s))\, dW_i(s)
\tag{2.5}
$$

is a local martingale.

In the following, we adopt the convention that $\frac{f'(x)-f'(y)}{x-y} = f''(x)$ on $\{x = y\}$. We omit the integral domain when it is $\mathbb{R}$. We also omit the domain of the double integral when it is $\mathbb{R}^2$.

By changing the index in the sum and using the symmetry, the last term in (2.4) can be simplified as follows:

$$
\begin{aligned}
&\frac{1}{N}\sum_{i\neq j}\int_0^t f'(\lambda_i^N(s)) \frac{G_N(\lambda_i^N(s), \lambda_j^N(s))}{\lambda_i^N(s) - \lambda_j^N(s)}\, ds \\
&\quad = \frac{1}{2N}\sum_{i\neq j}\int_0^t \frac{f'(\lambda_i^N(s)) - f'(\lambda_j^N(s))}{\lambda_i^N(s) - \lambda_j^N(s)} G_N(\lambda_i^N(s), \lambda_j^N(s))\, ds \\
&\quad = \frac{1}{2N}\sum_{i\neq j}\int_0^t \iint \frac{f'(x) - f'(y)}{x - y} G_N(x, y)\delta_{\lambda_i^N(s)}(dx)\delta_{\lambda_j^N(s)}(dy)\, ds \\
&\quad = \frac{N}{2}\int_0^t \iint \frac{f'(x) - f'(y)}{x - y} G_N(x, y) L_N(s)(dx) L_N(s)(dy)\, ds \\
&\qquad - \frac{1}{2N}\sum_{i=1}^{N}\int_0^t \iint \frac{f'(x) - f'(y)}{x - y} G_N(x, y)\delta_{\lambda_i^N(s)}(dx)\delta_{\lambda_i^N(s)}(dy)\, ds.
\end{aligned}
$$

Hence, the second term on the right-hand side of the above equation can be simplified as

$$\begin{aligned}
&\frac{1}{2N}\sum_{i=1}^{N}\int_0^t \iint \frac{f'(x)-f'(y)}{x-y} G_N(x,y)\delta_{\lambda_i^N(s)}(dx)\delta_{\lambda_i^N(s)}(dy)\,ds \\
&= \frac{1}{2N}\sum_{i=1}^{N}\int_0^t f''(\lambda_i^N(s)) G_N(\lambda_i^N(s),\lambda_i^N(s))\,ds \\
&= \frac{1}{N}\sum_{i=1}^{N}\int_0^t f''(\lambda_i^N(s)) g_N^2(\lambda_i^N(s)) h_N^2(\lambda_i^N(s))\,ds \\
&= \int_0^t \langle f'' g_N^2 h_N^2, L_N(s)\rangle\,ds.
\end{aligned}$$

Therefore, (2.4) becomes

$$\begin{aligned}
\langle f, L_N(t)\rangle = {} & \langle f, L_N(0)\rangle + M_f^N(t) + \int_0^t \langle f' b_N, L_N(s)\rangle\,ds \\
& + \int_0^t \langle f'' g_N^2 h_N^2, L_N(s)\rangle\,ds \\
& + \frac{N}{2}\int_0^t \iint \frac{f'(x)-f'(y)}{x-y} G_N(x,y) L_N(s)(dx) L_N(s)(dy)\,ds.
\end{aligned} \tag{2.6}$$

Now we assume the boundedness of the following terms:

$$\sup_N |\langle f, L_N(0)\rangle|, \qquad \sup_{x,N} |f'(x) b_N(x)|, \qquad \sup_x |f'(x) g_N(x) h_N(x)|,$$

$$\sup_{x,N} |f''(x) g_N^2(x) h_N^2(x)| \quad \text{and} \quad \sup_{x,y,N} \left| N G_N(x,y)\frac{f'(x)-f'(y)}{x-y}\right|.$$

Note that the above assumption is satisfied by the function $\varphi$ appearing in conditions (A), (B) and (C).

Now the quadratic variation of the local martingale $M_f^N(t)$ has the following estimation:

$$\begin{aligned}
\langle M_f^N\rangle_t &= \frac{4}{N^2}\sum_{i=1}^{N}\int_0^t |f'(\lambda_i^N(s)) g_N(\lambda_i^N(s)) h_N(\lambda_i^N(s))|^2\,ds \\
&= \frac{4}{N}\int_0^t \langle |f' g_N h_N|^2, L_N(s)\rangle\,ds \\
&\le \frac{4T}{N}\|f' g_N h_N\|^2_{L^\infty(dx)}.
\end{aligned} \tag{2.7}$$

Thus, $M_f^N(t)$ is a martingale.

By (2.6), we have

$$\sup_{t\in[0,T]} |\langle f, L_N(t)\rangle| \le \sup_{N>0} |\langle f, L_N(0)\rangle| + \sup_{t\in[0,T]} |M_f^N(t)| + D_0 T, \tag{2.8}$$

where

$$\begin{aligned}
D_0 = \sup_{N>0}\biggl\{ & \|f' b_N\|_{L^\infty(dx)} + \|f'' g_N^2 h_N^2\|_{L^\infty(dx)} \\
& + \frac{1}{2}\left\| N G_N(x,y)\frac{f'(x)-f'(y)}{x-y}\right\|_{L^\infty(dx\,dy)}\biggr\}.
\end{aligned} \tag{2.9}$$

Fix $l \in \mathbb{N}$. By Markov inequality, Burkholder–Davis–Gundy inequality and (2.7), there exists a positive constant $\Lambda_l$ depending on $l$ such that for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[0,T]}\big|M_f^N(t)\big| \geq \varepsilon\Big) &\leq \frac{1}{\varepsilon^{2l}}\mathbb{E}\Big[\sup_{t\in[0,T]}\big|M_f^N(t)\big|^{2l}\Big] \\ &\leq \frac{\Lambda_l}{\varepsilon^{2l}}\mathbb{E}\big[\langle M_f^N\rangle_T^l\big] \leq \frac{4^l T^l \Lambda_l}{N^l \varepsilon^{2l}}\|f' g_N h_N\|_{L^\infty(dx)}^{2l}.\end{aligned}\tag{2.10}$$

Hence, for $M > \sup_{N>0}|\langle f, L_N(0)\rangle| + D_0 T$, it follows from (2.8) and (2.10) that

$$\begin{aligned}&\mathbb{P}\Big(\sup_{t\in[0,T]}\big|\langle f, L_N(t)\rangle\big| \geq M\Big) \\ &\quad\leq \mathbb{P}\Big(\sup_{t\in[0,T]}\big|M_f^N(t)\big| \geq M - C_0 T - \sup_{N>0}\big|\langle f, L_N(0)\rangle\big|\Big) \\ &\quad\leq \frac{4^l T^l \Lambda_l}{N^l(M - D_0 T - \sup_{N>0}|\langle f, L_N(0)\rangle|)^{2l}}\|f' g_N h_N\|_{L^\infty(dx)}^{2l}.\end{aligned}\tag{2.11}$$

*Step* 2. Now we study the Hölder continuity of $\langle f, L_N(t)\rangle$. For $t \geq s$, (2.6) implies

$$\begin{aligned}&\langle f, L_N(t)\rangle - \langle f, L_N(s)\rangle \\ &\quad= M_f^N(t) - M_f^N(s) + \int_s^t \langle f' b_N, L_N(u)\rangle\,du + \int_s^t \langle f'' g_N^2 h_N^2, L_N(u)\rangle\,du \\ &\qquad+ \frac{N}{2}\int_s^t \iint \frac{f'(x) - f'(y)}{x - y} G_N(x,y) L_N(u)(dx) L_N(u)(dy)\,du.\end{aligned}$$

Hence,

$$\begin{aligned}&\big|\langle f, L_N(t)\rangle - \langle f, L_N(s)\rangle\big| \\ &\quad\leq \big|M_f^N(t) - M_f^N(s)\big| + (t-s)\|f' b_N\|_{L^\infty(dx)} + (t-s)\|f'' g_N^2 h_N^2\|_{L^\infty(dx)} \\ &\qquad+ \frac{t-s}{2}\Big\|N\frac{f'(x) - f'(y)}{x-y} G_N(x,y)\Big\|_{L^\infty(dx\,dy)} \\ &\quad\leq \big|M_f^N(t) - M_f^N(s)\big| + (t-s) D_0,\end{aligned}$$

where $D_0$ is given in (2.9). Note that $[0, T]$ can be partitioned into small intervals of length $\eta < D_0^{-8/7}$ and the number of the intervals are $J = [T\eta^{-1}]$. Then by Markov inequality, Burkholder–Davis–Gundy inequality and (2.7), we have

$$\begin{aligned}&\mathbb{P}\Big(\sup_{|t-s|\leq\eta}\big|M_f^N(t) - M_f^N(s)\big| \geq M\eta^{1/8}\Big) \\ &\quad\leq \sum_{k=0}^J \mathbb{P}\Big(\sup_{k\eta\leq t\leq(k+1)\eta}\big|M_f^N(t) - M_f^N(k\eta)\big| \geq \frac{M\eta^{1/8}}{3}\Big) \\ &\quad\leq \sum_{k=0}^J \frac{3^{2l}}{M^{2l}\eta^{l/4}}\mathbb{E}\Big[\sup_{k\eta\leq t\leq(k+1)\eta}\big|M_f^N(t) - M_f^N(k\eta)\big|^{2l}\Big] \\ &\quad\leq \sum_{k=0}^J \frac{3^{2l}\Lambda_l}{M^{2l}\eta^{l/4}}\mathbb{E}\big[\langle M_f^N(k\eta + \cdot) - M_f^N(k\eta)\rangle_\eta^l\big]\end{aligned}$$

$$\le \sum_{k=0}^{J} \frac{6^{2l}\Lambda_l \eta^{3l/4}}{M^{2l}N^l} \|f' g_N h_N\|^{2l}_{L^\infty(dx)}$$

$$\le \eta^{3l/4-1} \cdot \frac{6^{2l}\Lambda_l T}{M^{2l}N^l} \|f' g_N h_N\|^{2l}_{L^\infty(dx)}.$$

Hence, noting that $\eta D_0 < \eta^{1/8}$, we have

$$\begin{aligned} &\mathbb{P}\Big(\sup_{|t-s|\le\eta} |\langle f, L_N(t)\rangle - \langle f, L_N(s)\rangle| \ge (M+1)\eta^{1/8}\Big) \\ &\quad\le \mathbb{P}\Big(\sup_{|t-s|\le\eta} |M_f^N(t) - M_f^N(s)| \ge (M+1)\eta^{1/8} - \eta D_0\Big) \\ &\quad\le \mathbb{P}\Big(\sup_{|t-s|\le\eta} |M_f^N(t) - M_f^N(s)| \ge M\eta^{1/8}\Big) \\ &\quad\le \eta^{3l/4-1} \cdot \frac{6^{2l}\Lambda_l T}{M^{2l}N^l} \|f' g_N h_N\|^{2l}_{L^\infty(dx)}. \end{aligned} \tag{2.12}$$

*Step* 3. In this last step, we obtain the relative compactness of $\{L_N\}_{N\in\mathbb{N}^+}$ and conclude the proof.

Let $M$ denote a generic positive constant that may vary in different places. Recalling that $\varphi$ is given in condition (A), we set

$$K(\varphi, M) = \Big\{\mu \in M_1(\mathbb{R}) : \langle \varphi, \mu\rangle = \int \varphi(x)\mu(dx) \le M+1\Big\}.$$

Since $\varphi(x)$ is positive and tends to infinity as $|x| \to +\infty$, $K(\varphi, M)$ is tight, that is, it is (sequentially) compact in $M_1(\mathbb{R})$.

By Arzela–Ascoli Lemma, the set

$$\begin{aligned} &C_M(\{\varepsilon_n\}, \{\eta_n\}) \\ &\quad = \bigcap_{n=1}^{\infty} \Big\{g \in C([0,T], \mathbb{R}) : \sup_{|t-s|\le\eta_n} |g(t) - g(s)| \le \varepsilon_n, \sup_{t\in[0,T]} |g(t)| \le M\Big\}, \end{aligned}$$

where $\{\varepsilon_n\}$ and $\{\eta_n\}$ are two positive sequences converging to 0, is (sequentially) compact in $C([0,T], \mathbb{R})$. For $\varepsilon > 0$ and a bounded function $\tilde{f} \in C^2(\mathbb{R})$, we define

$$\begin{aligned} C_T(\tilde{f}, \varepsilon) &= \bigcap_{n=1}^{\infty} \Big\{\mu \in C([0,T], M_1(\mathbb{R})) : \sup_{|t-s|\le n^{-4}} |\mu_t(\tilde{f}) - \mu_s(\tilde{f})| \le \frac{1}{\varepsilon\sqrt{n}}\Big\} \\ &= \Big\{\mu \in C([0,T], M_1(\mathbb{R})) : \sup_{|t-s|\le n^{-4}} |\mu_t(\tilde{f}) - \mu_s(\tilde{f})| \le \frac{1}{\varepsilon\sqrt{n}}, \forall n \in \mathbb{N}\Big\} \\ &= \{\mu \in C([0,T], M_1(\mathbb{R})) : t \to \mu_t(\tilde{f}) \in C_M(\{(\varepsilon\sqrt{n})^{-1}\}, \{n^{-4}\})\}, \end{aligned}$$

where we can choose $M = \|\tilde{f}\|_\infty$. By Lemma 4.3.13 in Anderson, Guionnet and Zeitouni (2010), for a positive sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ which will be determined in the sequel, the set

$$\mathcal{H}_M = \{\mu \in C([0,T], M_1(\mathbb{R})) : \mu_t \in K(\varphi, M), \forall t \in [0,T]\} \cap \bigcap_{k=1}^{\infty} C_T(\tilde{f}_k, \varepsilon_k),$$

where $\{\tilde{f}_k\}_{k\geq 1}$ is given in Condition (D), is compact in $C([0,T], M_1(\mathbb{R}))$. We have

$$\sum_{N=1}^{\infty}\mathbb{P}\big(L_N\in\mathcal{H}_M^{\mathsf{c}}\big)\leq\sum_{N=1}^{\infty}\mathbb{P}\big(\exists t\in[0,T],\text{ s.t. } L_N(t)\notin K(\varphi,M)\big)+\sum_{N=1}^{\infty}\sum_{k\geq 1}\mathbb{P}\big(L_N\notin C_T(\tilde{f}_k,\varepsilon_k)\big).\tag{2.13}$$

By using (2.11) for the case $l=l_1$ and $f=\varphi$ with $l_1$ and $\varphi$ given in condition (A), the first term on the right-hand side can be simplified as

$$\begin{aligned}&\sum_{N=1}^{\infty}\mathbb{P}\big(\exists t\in[0,T],\text{ s.t. } L_N(t)\notin K(\varphi,M)\big)\\&=\sum_{N=1}^{\infty}\mathbb{P}\Big(\sup_{t\in[0,T]}\langle\varphi,L_N(t)\rangle>M+1\Big)\\&\leq\sum_{N=1}^{\infty}\frac{4^{l_1}T^{l_1}\Lambda_{l_1}}{N^{l_1}(M+1-D_0T-\sup_{N>0}|\langle\varphi,L_N(0)\rangle|)^{2l_1}}\|\varphi' g_N h_N\|_{L^\infty(dx)}^{2l_1}\\&=\frac{4^{l_1}T^{l_1}\Lambda_{l_1}}{(M+1-D_0T-C_0)^{2l_1}}\sum_{N=1}^{\infty}\frac{\|\varphi' g_N h_N\|_{L^\infty(dx)}^{2l_1}}{N^{l_1}}<\infty,\end{aligned}\tag{2.14}$$

where $C_0$ is given by (2.2), $D_0$ is given by (2.9), and $M=M_0$ is sufficiently large such that $M_0>D_0T+C_0$.

By using (2.12) with $l=l_2$, $f=\tilde{f}_k$, $\eta=n^{-4}$ and $M=\varepsilon_k^{-1}-1$, where $l_2$ and $\tilde{f}_k$ are given in condition (D), the second term on the right-hand side of (2.13) can be simplified as follows, recalling that $\psi(k)$ is given in (2.3),

$$\begin{aligned}&\sum_{N=1}^{\infty}\sum_{k\geq 1}\mathbb{P}\big(L_N\notin C_T(\tilde{f}_k,\varepsilon_k)\big)\\&\leq\sum_{N=1}^{\infty}\sum_{k\geq 1}\sum_{n=1}^{\infty}\mathbb{P}\Big(\sup_{|t-s|\leq n^{-4}}\big|L_N(t)(\tilde{f}_k)-L_N(s)(\tilde{f}_k)\big|>\frac{1}{\varepsilon_k\sqrt{n}}\Big)\\&\leq\sum_{N=1}^{\infty}\sum_{k\geq 1}\sum_{n=1}^{\infty}\frac{6^{2l_2}\Lambda_{l_2}Tn^{-3l_2+4}}{(\varepsilon_k^{-1}-1)^{2l_2}N^{l_2}}\|\tilde{f}_k' g_N h_N\|_{L^\infty(dx)}^{2l_2}\\&=6^{2l_2}\Lambda_{l_2}T\sum_{n=1}^{\infty}n^{-3l_2+4}\sum_{k\geq 1}\frac{1}{(\varepsilon_k^{-1}-1)^{2l_2}}\sum_{N=1}^{\infty}\frac{\|\tilde{f}_k' g_N h_N\|_{L^\infty(dx)}^{2l_2}}{N^{l_2}}\\&=6^{2l_2}\Lambda_{l_2}T\sum_{n=1}^{\infty}n^{-3l_2+4}\sum_{k\geq 1}\frac{\psi(k)}{(\varepsilon_k^{-1}-1)^{2l_2}},\end{aligned}$$

which is finite if we take $\varepsilon_k$ so that $\varepsilon_k^{-1}>1+k\psi(k)^{1/(2l_2)}$.

Thus, it follows from (2.13), (2.14) and the above estimate that

$$\sum_{N=1}^{\infty}\mathbb{P}\big(L_N\in\mathcal{H}_{M_0}^{\mathsf{c}}\big)<\infty,$$

and Borel–Cantelli Lemma implies

$$\mathbb{P}\Big(\liminf_{N\to\infty}\{L_N\in\mathcal{H}_{M_0}\}\Big)=1.$$

Finally, the relative compactness of the family $\{L_N\}_{N\in\mathbb{N}^+}$ follows from the compactness of $\mathcal{H}_{M_0}$, and the proof is concluded. $\square$

The Corollary 3 in Graczyk and Małecki (2013) provided the conditions under which the system of SDEs (1.3) has a unique nonexploding and noncolliding strong solution. As a consequence, we have the following corollary.

COROLLARY 2.1. *For the system of SDEs* (1.3), *suppose that the initial value satisfies* $\lambda_1^N(0) < \cdots < \lambda_N^N(0)$ *and the condition* (C) *holds. Assume that there exist positive constants* $L$, $\alpha$ *and* $\beta$ *with* $\alpha+\beta \ge 1$, *such that* $b_N(x)$, $N^\alpha g_N^2(x)$ *and* $N^\beta h_N^2(x)$ *are Lipschitz continuous with the Lipschitz constant* $L$ *for all* $N \in \mathbb{N}$, *and that*

$$\max_{N\in\mathbb{N}}\{|b_N(0)| + N^\alpha g_N^2(0) + N^\beta h_N^2(0)\} \le L.$$

*Besides, suppose that* $G_N(x,x)$ *is convex or in the Hölder space* $\mathcal{C}^{1,1}(\mathbb{R})$, *and that* $G_N(x,y)$ *is strictly positive on* $\{x \neq y\}$ *for all* $N \in \mathbb{N}$. *Then for any fixed number* $T > 0$, *the sequence* $\{L_N(t), t \in [0,T]\}_{N\in\mathbb{N}}$ *is relatively compact in* $C([0,T], M_1(\mathbb{R}))$.

PROOF. Under the conditions given in the Corollary, by Graczyk and Małecki (2013), Corollary 3, for each $N$, the system of SDEs (1.3) has a unique strong solutions that is nonexploding and noncolliding on $[0,\infty)$. Besides, we have the following estimation:

$$|b_N(x)| \le |b_N(0)| + |b_N(x) - b_N(0)| \le L(1 + |x|),$$

which is also satisfied by $N^\alpha g_N^2(x)$ and $N^\beta h_N^2(x)$. Thus, it is easy to check that the conditions (A), (B) and (D) are now satisfied (with $\varphi(x) = \ln(1+x^2)$), and the conclusion follows from Theorem 2.1. $\square$

Under proper conditions, the following theorem provides an equation for the Stieltjes transform of the limit point of $\{L_N\}_{N\in\mathbb{N}}$.

THEOREM 2.2. *Let* $T > 0$ *be a fixed number. Assume that* (1.3) *has a strong solution that is nonexploding and noncolliding for* $t \in [0,T]$. *Furthermore, we assume that there exist continuous functions* $b(x)$ *and* $G(x,y)$, *such that* $b_N(x)$ *converges to* $b(x)$ *and* $NG_N(x,y)$ *converges to* $G(x,y)$ *uniformly as* $N$ *tends to infinity, and that*

$$\left\|\frac{b(x)}{1+x^2}\right\|_{L^\infty(dx)} < \infty, \qquad \left\|\frac{G(x,y)}{(1+|x|)(1+y^2)}\right\|_{L^\infty(dx\,dy)} < \infty.$$

*If almost surely, the empirical measure* $L_N(0)$ *converges weakly to a measure* $\mu_0$ *as* $N$ *goes to infinity, and the sequence* $\{L_N\}_{N\in\mathbb{N}}$ *has a limit measure* $\mu$ *in* $C([0,T], M_1(\mathbb{R}))$, *then the measure* $\mu$ *satisfies the equation*

$$\begin{aligned}\int \frac{\mu_t(dx)}{z-x} &= \int \frac{\mu_0(dx)}{z-x} + \int_0^t \left[\int \frac{b(x)}{(z-x)^2}\mu_s(dx)\right] ds \\ &\quad + \int_0^t \left[\iint \frac{G(x,y)}{(z-x)(z-y)^2}\mu_s(dx)\mu_s(dy)\right] ds,\end{aligned} \tag{2.15}$$

*for* $z \in \mathbb{C}\setminus\mathbb{R}$.

REMARK 2.3. Taking $x = y$, the boundedness condition

$$\left\|\frac{G(x,y)}{(1+|x|)(1+y^2)}\right\|_{L^\infty(dx\,dy)} < \infty$$

becomes

$$\left\| \frac{G(x,x)}{(1+|x|)^3} \right\|_{L^\infty(dx)} < \infty.$$

Thus,

$$\frac{NG_N(x,x)}{(1+|x|)^4} \le C$$

for some constant $C$ and large $N$. Note that $G_N(x,x) = 2g_N^2(x)h_N^2(x)$, we have

$$\sum_{N=1}^\infty \frac{1}{N} \left\| \frac{g_N(x)h_N(x)}{(z-x)^2} \right\|^2_{L^\infty(dx)} = \sum_{N=1}^\infty \frac{1}{2N} \left\| \frac{G_N(x,x)}{(z-x)^4} \right\|_{L^\infty(dx)} \le \sum_{N=1}^\infty \frac{C}{2N^2} < \infty.$$

PROOF OF THEOREM 2.2. For any limit point $\mu = (\mu_t, t \in [0,T])$ of $L_N$, we can find a subsequence $\{N_i\}$, such that $L_{N_i}$ converges to $\mu$ in $C([0,T], M_1(\mathbb{R}))$ as $N_i$ tends to infinity. By using (2.6) for the case $N = N_i$ and $f(x) = (z-x)^{-1}$, and then letting $N_i$ tends to infinity, we have

$$\begin{aligned}
&\int \frac{\mu_t(dx)}{z-x} - \int \frac{\mu_0(dx)}{z-x} \\
&\quad = \lim_{N_i\to\infty} M_f^{N_i}(t) + \lim_{N_i\to\infty} \int_0^t \int \frac{b_{N_i}(x)}{(z-x)^2} L_{N_i}(s)(dx)\,ds \\
&\qquad + \lim_{N_i\to\infty} \int_0^t \int \frac{2g_{N_i}^2(x)h_{N_i}^2(x)}{(z-x)^3} L_{N_i}(s)(dx)\,ds \\
&\qquad + \lim_{N_i\to\infty} \frac{1}{2} \int_0^t \iint \frac{(z-x)^{-2} - (z-y)^{-2}}{x-y} N_i G_{N_i}(x,y) L_{N_i}(s)(dx) L_{N_i}(s)(dy)\,ds.
\end{aligned} \tag{2.16}$$

The second term of right-hand side of (2.16) vanishes almost surely. Indeed, by using (2.10) for the case $l = 1$ and $f(x) = (z-x)^{-1}$ for some $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\sum_{N=1}^\infty \mathbb{P}\Big( \sup_{t\in[0,T]} |M_f^N(t)| \ge \varepsilon \Big) \le \sum_{N=1}^\infty \frac{4T\Lambda_1}{N\varepsilon^2} \left\| \frac{g_N(x)h_N(x)}{(z-x)^2} \right\|^2_{L^\infty(dx)},$$

of which the right-hand side is finite due to Remark 2.3. By Borel–Cantelli Lemma,

$$\mathbb{P}\Big( \liminf_{N\to\infty} \Big\{ \sup_{t\in[0,T]} |M_f^N(t)| < \varepsilon \Big\} \Big) = 1,$$

that is, $M_f^N(t)$ converges to zero uniformly with respect to $t$ almost surely.

For the third term on the right-hand side of (2.16), noting that the boundedness of $b(x)(1+x^2)^{-1}$ implies the boundedness of $b(x)(z-x)^{-2}$ for $z \in \mathbb{C} \setminus \mathbb{R}$, which is continuous, we have

$$\begin{aligned}
&\left| \int \frac{b_{N_i}(x)}{(z-x)^2} L_{N_i}(s)(dx) - \int \frac{b(x)}{(z-x)^2} \mu_s(dx) \right| \\
&\quad \le \left| \int \frac{b_{N_i}(x) - b(x)}{(z-x)^2} L_{N_i}(s)(dx) \right| + \left| \int \frac{b(x)}{(z-x)^2} \big( L_{N_i}(s)(dx) - \mu_s(dx) \big) \right| \\
&\quad \le \frac{\sup_x |b_{N_i}(x) - b(x)|}{(\operatorname{Im}(z))^2} + \left| \int \frac{b(x)}{(z-x)^2} \big( L_{N_i}(s)(dx) - \mu_s(dx) \big) \right|,
\end{aligned}$$

the right-hand of which converges to 0 as $N_i \to \infty$ by the uniform convergence of $b_{N_i}(x)$ towards $b(x)$ and the weak convergence of the empirical measure $L_{N_i}(s)$ towards $\mu_s$. Besides, the boundedness of $b(x)/(1+x^2)$ and the uniform convergence of $b_N(x)$ to $b(x)$ imply the boundedness of $b_N(x)/(z-x)^2$. Then it follows from the dominated convergence theorem that

$$\lim_{N_i\to\infty}\int_0^t\int\frac{b_{N_i}(x)}{(z-x)^2}L_{N_i}(s)(dx)\,ds=\int_0^t\int\frac{b(x)}{(z-x)^2}\mu_s(dx)\,ds.$$

Similarly, for the fourth term on the right-hand side of (2.16), noting that $2N_i g_{N_i}^2(x)\times h_{N_i}^2(x)=N_iG_{N_i}(x,x)$, we have

$$\begin{aligned}
&\left|\int\frac{2g_{N_i}^2(x)h_{N_i}^2(x)}{(z-x)^3}L_{N_i}(s)(dx)\right|\\
&\quad=\frac{1}{N_i}\left|\int\frac{N_iG_{N_i}(x,x)}{(z-x)^3}L_{N_i}(s)(dx)\right|\\
&\quad\le\frac{1}{N_i}\left|\int\frac{N_iG_{N_i}(x,x)-G(x,x)}{(z-x)^3}L_{N_i}(s)(dx)\right|+\frac{1}{N_i}\left|\int\frac{G(x,x)}{(z-x)^3}L_{N_i}(s)(dx)\right|\\
&\quad\le\frac{\sup_{x,y}|N_iG_{N_i}(x,x)-G(x,x)|}{N_i(\operatorname{Im}(z))^3}+\frac{C_z}{N_i}\left\|\frac{G(x,x)}{(1+|x|^3)}\right\|_{L^\infty(dx)}
\end{aligned}$$

which tend to 0 as $N_i\to\infty$. Here, $C_z$ is a constant depending only on $z$.

Finally, using the identity

$$\begin{aligned}
\frac{(z-x)^{-2}-(z-y)^{-2}}{x-y}&=\frac{(z-y)^2-(z-x)^2}{(z-x)^2(z-y)^2(x-y)}\\
&=\frac{2z-x-y}{(z-x)^2(z-y)^2}=\frac{1}{(z-x)(z-y)^2}+\frac{1}{(z-x)^2(z-y)},
\end{aligned}$$

the last term on the right-hand side of (2.16) can be simplified as

$$\begin{aligned}
&\lim_{N_i\to\infty}\frac{1}{2}\int_0^t\iint\frac{(z-x)^{-2}-(z-y)^{-2}}{x-y}N_iG_{N_i}(x,y)L_{N_i}(s)(dx)L_{N_i}(s)(dy)\,ds\\
&\quad=\lim_{N_i\to\infty}\frac{1}{2}\int_0^t\iint\left[\frac{1}{(z-x)(z-y)^2}+\frac{1}{(z-x)^2(z-y)}\right]\\
&\qquad\times N_iG_{N_i}(x,y)L_{N_i}(s)(dx)L_{N_i}(s)(dy)\,ds\\
&\quad=\lim_{N_i\to\infty}\int_0^t\iint\frac{N_iG_{N_i}(x,y)}{(z-x)(z-y)^2}L_{N_i}(s)(dx)L_{N_i}(s)(dy)\,ds,
\end{aligned}$$

where the last equality follows from the symmetry of $G_{N_i}$. Now,

$$\begin{aligned}
&\left|\iint\frac{N_iG_{N_i}(x,y)}{(z-x)(z-y)^2}L_{N_i}(s)(dx)L_{N_i}(s)(dy)-\iint\frac{G(x,y)}{(z-x)(z-y)^2}\mu_s(dx)\mu_s(dy)\right|\\
&\quad\le\left|\iint\frac{N_iG_{N_i}(x,y)-G(x,y)}{(z-x)(z-y)^2}L_{N_i}(s)(dx)L_{N_i}(s)(dy)\right|\\
&\qquad+\left|\iint\frac{G(x,y)}{(z-x)(z-y)^2}\big(L_{N_i}(s)(dx)L_{N_i}(s)(dy)-\mu_s(dx)\mu_s(dy)\big)\right|\\
&\quad\le\frac{\sup_{x,y}|N_iG_{N_i}(x,y)-G(x,y)|}{|\operatorname{Im}(z)|^3}\\
&\qquad+\left|\iint\frac{G(x,y)}{(z-x)(z-y)^2}\big(L_{N_i}(s)(dx)L_{N_i}(s)(dy)-\mu_s(dx)\mu_s(dy)\big)\right|
\end{aligned}$$

converges to 0 as $N_i \to \infty$. Also note that the boundedness of $G(x,y)/(1+|x|)(1+y^2)$ and the uniform convergence of $NG_N(x,y)$ to $G(x,y)$ yield the boundedness of $NG_N(x,y)/(z-x)(z-y)^2$. Thus, by the dominated convergence theorem and the continuity of the function $G(x,y)$,

$$\begin{aligned}\lim_{N_i\to\infty}&\int_0^t \iint \frac{N_i G_{N_i}(x,y)}{(z-x)(z-y)^2} L_{N_i}(s)(dx) L_{N_i}(s)(dy)\,ds\\ &= \int_0^t \iint \frac{G(x,y)}{(z-x)(z-y)^2}\mu_s(dx)\mu_s(dy)\,ds.\end{aligned}$$

Therefore, (2.15) is obtained from (2.16). The proof is complete. □

Using the conditions in Corollary 3 of Graczyk and Małecki (2013) that guarantee the existence and uniqueness of the nonexploding and noncolliding strong solution to the system of SDEs (1.3), we have the following corollary.

COROLLARY 2.2. *For the system of SDEs* (1.3), *suppose* $\lambda_1^N(0) < \cdots < \lambda_N^N(0)$. *Assume that there exist positive constants $L$ and $\alpha$, such that* $b_N(x)$, $N^\alpha g_N^2(x)$, $N^{1-\alpha}h_N^2(x)$ *are Lipschitz continuous with the Lipschitz constant $L$ for all* $N \in \mathbb{N}$, *and*

$$\max_{N\in\mathbb{N}}\{|b_N(0)| + N^\alpha g_N^2(0) + N^{1-\alpha}h_N^2(0)\} \le L.$$

*Besides, suppose that $G_N(x,x)$ is convex or in the Hölder space $\mathcal{C}^{1,1}$, and that $G_N(x,y)$ is strictly positive on $\{x \neq y\}$. Moreover, assume that $b_N(x)$ converges to a continuous function $b(x)$ and $NG_N(x,y)$ converges to a continuous function $G(x,y)$ uniformly as $N$ tends to infinity.*

*If the empirical measure $L_N(0)$ converges weakly to a measure $\mu_0$ almost surely as $N$ goes to infinity, and the sequence $\{L_N\}_{N\in\mathbb{N}}$ has a limit measure $\mu$ in $C([0,T], M_1(\mathbb{R}))$ for any fixed number $T > 0$, then the measure $\mu$ satisfies the equation*

$$\begin{aligned}\int \frac{\mu_t(dx)}{z-x} &= \int \frac{\mu_0(dx)}{z-x} + \int_0^t \left[\int \frac{b(x)}{(z-x)^2}\mu_s(dx)\right]ds\\ &\quad + \int_0^t \left[\iint \frac{G(x,y)}{(z-x)(z-y)^2}\mu_s(dx)\mu_s(dy)\right]ds,\end{aligned} \tag{2.17}$$

*for* $z \in \mathbb{C}\setminus\mathbb{R}$, $t \in [0,T]$.

PROOF. By Graczyk and Małecki (2013), Corollary 3, we can conclude that for each $N$, the SDEs (1.3) has a unique strong solution that is nonexploding and noncolliding on $[0,\infty)$. Moreover, the estimation in the proof of Corollary 2.1 is still valid for $b_N(x)$, $N^\alpha g_N^2(x)$ and $N^{1-\alpha}h_N^2(x)$. Besides, we have

$$\begin{aligned}|NG_N(x,y)| &\le \left(|N^\alpha g_N^2(x) - N^\alpha g_N^2(0)| + |N^\alpha g_N^2(0)|\right)\\ &\quad\times\left(|N^{1-\alpha}h_N^2(y) - N^{1-\alpha}h_N^2(0)| + |N^{1-\alpha}h_N^2(0)|\right)\\ &\le L^2(1+|x|)(1+|y|).\end{aligned}$$

It can be easily checked that all the conditions in Theorem 2.2 are satisfied. □

REMARK 2.4 (The normalized case). Now we suppose that $Y_t^N$ satisfies the following equation:

$$dY_t^N = g(Y_t^N)\,dB_t h(Y_t^N) + h(Y_t^N)\,dB_t^\intercal g(Y_t^N) + a(Y_t^N)\,dt. \tag{2.18}$$

Then the equation for $X_t^N := \frac{1}{N} Y_t^N$ is

$$dX_t^N = \frac{1}{N} g(NX_t^N)\, dB_t h(NX_t^N) + \frac{1}{N} h(NX_t^N)\, dB_t^\intercal g(NX_t^N) + \frac{1}{N} a(NX_t^N)\, dt,$$

which coincides with (1.1) with

$$g_N(x)h_N(y) = \frac{1}{N} g(Nx)h(Ny) \quad \text{and} \quad b_N(x) = \frac{1}{N} a(Nx).$$

Therefore, under the conditions in Theorem 2.1 and Theorem 2.2, the equation (2.15) is still valid for a limit measure $\mu$ of the empirical measures of the eigenvalues of $X^N$ with

$$b(x) = \lim_{N\to\infty} \frac{1}{N} a(Nx) \quad \text{and} \quad G(x,y) = \lim_{N\to\infty} \frac{1}{N} [g^2(Nx)h^2(Ny) + h^2(Nx)g^2(Ny)].$$

**3. Limit point of empirical measure for particle systems.** In Graczyk and Małecki (2014), the following system of SDEs was introduced: for $1 \le i \le N$ and $t \ge 0$,

$$dx_i^N(t) = \sigma_i^N(x_i^N(t))\, dW_i(t) + \left( b_N(x_i^N(t)) + \sum_{j:j\neq i} \frac{H_N(x_i^N(t), x_j^N(t))}{x_i^N(t) - x_j^N(t)} \right) dt, \tag{3.1}$$

where $H_N(x,y)$ is a nonnegative symmetric function, and the existence and uniqueness of the noncolliding strong solution was studied. Clearly, this particle system generalizes the system (1.3) for eigenvalues of a generalized Wishart process studied in Section 2. There is a huge literature on related interacting particle systems, particularly on those related to the Bessel processes. For background information, we here refer to the survey papers Göing-Jaeschke and Yor (2003) and Zambotti (2017), and the recent book Katori (2016).

In this section, we extend the results established in Section 2 for the particle system. Here the corresponding empirical measures are

$$L_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N(t)}.$$

We assume the following conditions which are similar to those in Section 2.

(A′) There exists a positive function $\varphi(x) \in C^2(\mathbb{R})$ such that $\lim_{|x|\to+\infty} \varphi(x) = +\infty$, $\varphi'(x)b_N(x)$ is bounded with respect to $(x,N)$, and $\varphi''(x)\sigma_i^N(x)^2$ is bounded with respect to $(x,i,N)$, and $\varphi'(x)\sigma_i^N(x)$ satisfies

$$\sum_{N=1}^\infty \left( \frac{\max_{1\le i\le N} \|\varphi'\sigma_i^N\|^2_{L^\infty(dx)}}{N} \right)^{l_1} < \infty$$

for some positive integer $l_1$.

(B′) The function $NH_N(x,y)\frac{\varphi'(x)-\varphi'(y)}{x-y}$ is bounded with respect to $(x,y,N)$.

(C′)

$$C_0' := \sup_{N>0} \langle \varphi, L_N(0) \rangle = \sup_{N>0} \frac{1}{N} \sum_{i=1}^N \varphi(\lambda_i^N(0)) < \infty.$$

(D′) There exists a sequence $\{\tilde{f}_k\}_{k\ge1}$ of $C^2(\mathbb{R})$ functions such that it is dense in $C_0(\mathbb{R})$ and that $\tilde{f}_k'(x)\sigma_i^N(x)$ satisfies

$$\psi(k) = \sum_{N=1}^\infty \left( \frac{\max_{1\le i\le N} \|\tilde{f}_k'\sigma_i^N\|^2_{L^\infty(dx)}}{N} \right)^{l_2} < \infty$$

for some positive integer $l_2 \ge 2$.

REMARK 3.1. Similar to Remark 2.2, suppose that $b_N(x) \le c_b|x|$, $\sigma_i^N(x) \le c_0|x|$ and $H_N(x, y) \le c_h|xy|N^{-\gamma}$ for large $N$ and large $|x|$, $|y|$ with constants $c_b$, $c_0$, $c_h$ and $\gamma \ge 1$, then we can choose $\varphi(x) = \ln(1 + x^2)$ to satisfy the conditions.

THEOREM 3.1. *Let $T > 0$ be a fixed number. Suppose that (3.1) has a strong solution that is nonexploding and noncolliding for $t \in [0, T]$. Then under the conditions* (A′), (B′), (C′) *and* (D′), *the sequence $\{L_N(t), t \in [0, T]\}_{N\in\mathbb{N}}$ is relatively compact in $C([0, T], M_1(\mathbb{R}))$ almost surely.*

Analogous to Corollary 2.1, we have the following corollary of Theorem 3.1, based on the conditions given in Graczyk and Małecki (2014) which assures a unique nonexploding and noncolliding strong solution to (3.1).

COROLLARY 3.1. *For the system of SDEs (3.1), assume that the initial value satisfies $\lambda_1^N(0) < \cdots < \lambda_N^N(0)$ and condition* (C′). *Suppose that $\sigma_i^N(x)^2$ is Lipschitz continuous for each $1 \le i \le N$, and $H_N(x, y)$ is continuous for all $N \in \mathbb{N}$. Moreover, there exists a positive number $L > 0$, such that $b_N(x)$ is Lipschitz continuous with the Lipschitz constant $L$ for all $N \in \mathbb{N}$, and*

$$\sup_{N\in\mathbb{N}} \{|b_N(0)|\} \le L.$$

*Besides, suppose that there exist constant $c_2 \ge 0$ that does not depend on $N$, and constants $c_3(N)$, $c_4(N)$ that may depend on $N$, such that for $1 \le i \le N$,*

(a) $H_N(x, y) \le \frac{c_2}{N}(1 + |xy|)$, $\forall x, y \in \mathbb{R}$;
(b) $\frac{H_N(w,z)}{z-w} \le \frac{H_N(x,y)}{y-x}$, $\forall w < x < y < z$;
(c) $\sigma_i^N(x)^2 + \sigma_i^N(y)^2 \le c_3(N)(x - y)^2 + 4H_N(x, y)$, $\forall x, y \in \mathbb{R}$;
(d) $H_N(x, y)(y - x) + H_N(y, z)(z - y) \le c_4(N)(z - y)(z - x)(y - x) + H_N(x, z)(z - x)$, $\forall x < y < z$.

*Then for any fixed number $T > 0$, the sequence $\{L_N(t), t \in [0, T]\}_{N\in\mathbb{N}}$ is relatively compact in $C([0, T], M_1(\mathbb{R}))$ almost surely.*

PROOF. As in the proof of Corollary 2.1, the estimation $|b_N(x)| \le L(1 + |x|)$ holds. By (a) and (c), we have $\sigma_i^N(x)^2 \le 2H_N(x, x) \le 2c_2(1 + |x|^2)/N$. By Graczyk and Małecki (2014), the system (3.1) has a unique strong solution that is nonexploding and noncolliding on $[0, \infty)$, for each $N \in \mathbb{N}$. Besides, it can be easily checked that the conditions (A′) (B′) and (D′) are satisfied with $\varphi(x) = \ln(1 + x^2)$. Thus, the desired result comes from Theorem 3.1. □

A similar equation for the Stieltjes transform of the limit measure is given below.

THEOREM 3.2. *Let $T > 0$ be a fixed number. Assume that (3.1) has a strong solution that is nonexploding and noncolliding for $t \in [0, T]$. Suppose that*

$$\sum_{N=1}^{\infty} \left( \frac{1}{N} \max_{1\le i\le N} \left\| \frac{\sigma_i^N(x)}{1 + x^2} \right\|_{L^\infty(dx)}^2 \right)^{l_3} < \infty$$

*for some positive integer $l_3$, and that there exists a continuous function $\sigma(x)$ such that*

$$\lim_{N\to\infty} \max_{1\le i\le N} \left\| \frac{\sigma_i^N(x)^2 - \sigma(x)^2}{1 + x^3} \right\|_{L^\infty(dx)} = 0. \tag{3.2}$$

*Furthermore, assume that there exist continuous functions $b(x)$ and $H(x,y)$, such that $b_N(x)$ converges to $b(x)$ and $NH_N(x,y)$ converges to $H(x,y)$ uniformly as $N$ tends to infinity, and that*

$$\left\|\frac{b(x)}{1+x^2}\right\|_{L^\infty(dx)}<\infty,\qquad \left\|\frac{H(x,y)}{(1+|x|)(1+y^2)}\right\|_{L^\infty(dx\,dy)}<\infty,\qquad \left\|\frac{\sigma(x)^2}{1+x^3}\right\|_{L^\infty(dx)}<\infty.$$

*If the empirical measure $L_N(0)$ converges weakly as $N$ goes to infinity to a measure $\mu_0$ almost surely, and the sequence $L_N$ has a limit measure $\mu$ in $C([0,T],M_1(\mathbb{R}))$, then the measure $\mu$ satisfies the equation*

$$\begin{aligned}\int\frac{\mu_t(dx)}{z-x}={}&\int\frac{\mu_0(dx)}{z-x}+\int_0^t\left[\int\frac{b(x)}{(z-x)^2}\mu_s(dx)\right]ds+\int_0^t\left[\int\frac{\sigma(x)^2}{(z-x)^3}\mu_s(dx)\right]ds\\&+\int_0^t\left[\iint\frac{H(x,y)}{(z-x)(z-y)^2}\mu_s(dx)\mu_s(dy)\right]ds,\end{aligned}\tag{3.3}$$

*for $z\in\mathbb{C}\setminus\mathbb{R}$.*

Similar to Corollary 3.1, we have the following consequence of Theorem 3.2.

COROLLARY 3.2. *Assume that the initial value of* (3.1) *satisfies $\lambda_1^N(0)<\cdots<\lambda_N^N(0)$. Suppose that $\sigma_i^N(x)^2$ is Lipschitz continuous for all $1\le i\le N$ and $H_N(x,y)$ is continuous for all $N\in\mathbb{N}$. Moreover, assume that there exists a positive number $L>0$, such that, $b_N(x)$ is Lipschitz continuous with the Lipschitz constant $L$ for all $N\in\mathbb{N}$, and*

$$\sup_{N\in\mathbb{N}}\{|b_N(0)|\}\le L.$$

*Besides, suppose that the conditions* (a)–(d) *in Corollary* 3.1 *hold. Furthermore, assume that there exist continuous functions $b(x)$ and $H(x,y)$, such that $b_N(x)$ converges to $b(x)$ and $NH_N(x,y)$ converges to $H(x,y)$ uniformly as $N$ tends to infinity.*

*If almost surely, the empirical measure $L_N(0)$ converges weakly as $N$ goes to infinity to a measure $\mu_0$, and the sequence $L_N$ has a limit measure $\mu$ in $C([0,T],M_1(\mathbb{R}))$ for a fixed number $T>0$, then the measure $\mu$ satisfies the equation*

$$\begin{aligned}&\int\frac{\mu_t(dx)}{z-x}-\int\frac{\mu_0(dx)}{z-x}\\&\quad=\int_0^t\left[\int\frac{b(x)}{(z-x)^2}\mu_s(dx)\right]ds+\int_0^t\left[\iint\frac{H(x,y)}{(z-x)(z-y)^2}\mu_s(dx)\mu_s(dy)\right]ds,\end{aligned}\tag{3.4}$$

*for $z\in\mathbb{C}\setminus\mathbb{R}$, $t\in[0,T]$.*

PROOF. By the proof of Corollary 3.1, we have the following estimation:

$$|b_N(x)|\le L(1+|x|),\qquad \sigma_i^N(x)^2\le\frac{2c_2}{N}(1+|x|^2).$$

Hence, according to Graczyk and Małecki (2014), for each $N$, the system (3.1) has a unique strong solution that is nonexploding and noncolliding on $[0,\infty)$. It can be checked easily that all the conditions in Theorem 3.2 hold with $\sigma(x)=0$. □

The proofs of Theorems 3.1 and 3.2 are analogous to those of Theorems 2.1 and 2.2 in Section 2, respectively. They are thus omitted.

REMARK 3.2 (The normalized case). For the particle system

$$dy_i^N(t) = \sigma_i(y_i^N(t))\,dW_i(t) + \left(a(y_i^N(t)) + \sum_{j:j\neq i} \frac{G(y_i^N(t), y_j^N(t))}{y_i^N(t) - y_j^N(t)}\right)dt, \tag{3.5}$$

where $G(x, y)$ is a symmetric function, the normalized particle system

$$x_i^N(t) = \frac{1}{N} y_i^N(t), \quad 1 \le i \le N, t \ge 0,$$

satisfies (3.1) with

$$\sigma_i^N(x) = \frac{1}{N}\sigma_i(Nx), \qquad b_N(x) = \frac{1}{N}a(Nx) \quad \text{and} \quad H_N(x, y) = \frac{1}{N^2}G(Nx, Ny).$$

In this case, if the conditions in Theorem 3.1 and Theorem 3.2 hold, any limit point $\mu$ of the empirical measures of $\{x_i^N, 1 \le i \le N\}$ satisfies (3.3) with

$$\sigma(x)^2 = \lim_{N\to\infty} \sigma_i^N(x)^2, \qquad b(x) = \lim_{N\to\infty} \frac{1}{N}a(Nx) \quad \text{and}$$
$$H(x, y) = \lim_{N\to\infty} \frac{1}{N}G(Nx, Ny).$$

In the rest of this section, we apply the above general results to general noncolliding squared Bessel particle, general noncolliding squared $\beta$-Bessel particle system, and Dyson Brownian motion.

*General noncolliding squared Bessel particle system.* We choose the coefficient functions $g_N(x)$, $h_N(x)$ and $b_N(x)$ and the initial value in (1.3) such that they satisfy the conditions in Corollary 2.1 and 2.2, where $NG_N(x, y) = N(g_N(x)^2h_N(y)^2 + g_N(y)^2h_N(x)^2)$ converges to $G(x, y) = x + y$, and $b_N(x)$ converges to $b(x) = c$, uniformly as $N$ tends to infinity. Thus the equation (2.17) for the limit measure becomes

$$\begin{aligned}\int \frac{\mu_t(dx)}{z - x} = \int \frac{\mu_0(dx)}{z - x} &+ \int_0^t \left[\int \frac{c}{(z - x)^2}\mu_s(dx)\right]ds \\ &+ \int_0^t \left[\iint \frac{x + y}{(z - x)(z - y)^2}\mu_s(dx)\mu_s(dy)\right]ds.\end{aligned} \tag{3.6}$$

However, it is challenging to determine the limit measure $\{\mu_t, t \in [0, T]\}$ in general. If we assume that $\mu_0(dx) = \delta_0(dx)$ and that $\mu_t$ is supported on $[0, \infty)$ for all $t \ge 0$, then (3.6) has a unique solution as established in Cabanal-Duvillard and Guionnet (2001). The paper also determined the solution by iterating the equation of the associated characteristic function, for which Gronwall's lemma was employed to deduce the convergence.

Here we sketch an alternative approach to find this particular $\{\mu_t, t \in [0, T]\}$. Actually, $\mu_t$ can be considered as the limit of empirical measure of the eigenvalues of $X_t^N = \frac{1}{N}B_t^\intercal B_t$ where $B_t$ is a $p \times N$ Brownian matrix. Note that $X_t^N$ and its eigenvalues solve (1.1) and (1.3), respectively, with $g_N(x) = \sqrt{x}/\sqrt{N}$, $h_N(x) = 1$ and $b_N(x) = p/N$. Here, $p > N - 1$ and $p/N \to c \ge 1$.

Denoting the Stieltjes transform of $\mu_t$ by

$$G_t(z) = \int \frac{1}{z - x}\mu_t(dx), \tag{3.7}$$

the equation (3.6) becomes

$$G_t(z) = G_0(z) - (c - 1)\int_0^t \partial_z G_s(z)\,ds - \int_0^t \left(G_s(z)^2 + 2zG_s(z)\partial_z G_s(z)\right)ds. \tag{3.8}$$

Assume $X_0^N = 0$, and the key observation to solve (3.8) is the following scaling property:

$$G_t(z) = \frac{1}{t} G_1\left(\frac{z}{t}\right), \tag{3.9}$$

which follows easily from the self-similarity of the process $(B_t^\top B_t)_{t\geq 0}$. By (3.9), we have

$$\partial_z G_t(z) = \frac{1}{t^2} G_1'\left(\frac{z}{t}\right) = -\frac{1}{z}\frac{d}{dt}\left(G_1\left(\frac{z}{t}\right)\right),$$

and

$$G_t(z)^2 + 2zG_t(z)\partial_z G_t(z) = \frac{1}{t^2}G_1^2\left(\frac{z}{t}\right) + \frac{2z}{t^3}G_1\left(\frac{z}{t}\right)G_1'\left(\frac{z}{t}\right) = -\frac{d}{dt}\left(\frac{1}{t}G_1^2\left(\frac{z}{t}\right)\right).$$

The above two equations and (3.8) imply

$$G_t(z) = G_0(z) + \frac{c-1}{z}G_1\left(\frac{z}{t}\right) + \frac{1}{t}G_1^2\left(\frac{z}{t}\right). \tag{3.10}$$

Let $t = 1$ in (3.10) and we have

$$zG_1^2(z) + (c - 1 - z)G_1(z) + 1 = 0, \tag{3.11}$$

of which the solution is

$$G_1(z) = \frac{(z + 1 - c) - \sqrt{(c - 1 - z)^2 - 4z}}{2z}, \tag{3.12}$$

where the square root maps from $\mathbb{C}_+$ to $\mathbb{C}_+$. Thus by (3.9),

$$G_t(z) = \frac{(z + t(1 - c)) - \sqrt{(z + t(1 - c))^2 - 4tz}}{2tz}. \tag{3.13}$$

REMARK 3.3. The matrix process

$$\tilde{X}^N(t) = \frac{1}{p}B_t^\top B_t = \frac{N}{p}X^N(t)$$

often appears in the literature. We take the notation $\tilde{c} = \lim_{N\to\infty}\frac{N}{p} = \frac{1}{c} \leq 1$. Let $\tilde{\mu}_t$ be the limit of the empirical measure of $\tilde{X}^N(t)$, and denote its Stieltjes transform by

$$\tilde{G}_t(z) = \int \frac{1}{x - z}\tilde{\mu}_t(dx).$$

Noting that $\tilde{X}^N(t)$ and $X^N(t)$ only differ by a multiple of $N/p$, we also have $\tilde{\lambda}_i^N(t) = \frac{N}{p}\lambda_i^N(t)$ and it is easy to verify that

$$\tilde{G}_t(z) = -cG_t(cz).$$

Letting $t = 1$, we have by (3.12),

$$\tilde{G}_1(z) = -cG_1(cz) = \frac{1 - \tilde{c} - z + \sqrt{(1 - \tilde{c} - z)^2 - 4\tilde{c}^2 z}}{2\tilde{c}z}$$

which is the Stieltjes transform of the *standard* Marčenko–Pastur law with parameter $\tilde{c} \leq 1$ (see, e.g., equation (3.1.1) in Bai and Silverstein (2010)).

*General noncolliding squared $\beta$-Bessel particle system.* This process is a slight generalization of the noncolliding squared Bessel particle system. We choose the coefficient functions $\sigma_i^N(x)$, $b_N(x)$, $H_N(x,y)$ in (3.1) such that they satisfy the conditions in Corollary 3.1 and Corollary 3.2, where $b_N(x)$ converges to $b(x)=\beta c$, and $NH_N(x,y)$ converges to $H(x,y)=\beta(x+y)$, uniformly as $N$ tends to infinity, and $\sigma(x)=0$. Then the equation (3.4) now is

$$\int \frac{\mu_t(dx)}{z-x} = \int \frac{\mu_0(dx)}{z-x} + \beta\int_0^t \left[\int \frac{c}{(z-x)^2}\mu_s(dx)\right] ds + \beta\int_0^t \left[\iint \frac{x+y}{(z-x)(z-y)^2}\mu_s(dx)\mu_s(dy)\right] ds,$$

which is equivalent to

$$G_t(z) = G_0(z) - \beta(c-1)\int_0^t \partial_z G_s(z)\,ds - \beta\int_0^t \big(G_s(z)^2 + 2zG_s(z)\partial_z G_s(z)\big)\,ds, \tag{3.14}$$

where $G_t(z)$ is the Stieltjes transform defined in (3.7).

Similar to general noncolliding squared Bessel particle system case, we consider the system of SDEs (3.1) with $\mu_0(dx)=\delta_0(x)$, $\sigma_i^N(x)=\sqrt{x}/\sqrt{N}$, $H_N(x,y)=\beta(x+y)/N$ and $b_N(x)=b_N$, where $\{b_N, N\in\mathbb{N}\}$ is a sequence of positive numbers that converges to $\beta c$. By the uniqueness of the solution to (3.1) and the self-similarity of Brownian motion, we can still obtain the scaling property (3.9) for $G_t(z)$. Thus, similar to the transformation from (3.8) into (3.10), (3.14) now is transformed into

$$G_t(z) = G_0(z) + \frac{\beta(c-1)}{z} G_1\left(\frac{z}{t}\right) + \frac{\beta}{t} G_1^2\left(\frac{z}{t}\right).$$

Letting $t=1$, it is easy to get

$$G_1(z) = \frac{z-\beta(c-1)-\sqrt{[\beta(c-1)-z]^2-4\beta z}}{2\beta z}.$$

Hence, by (3.9),

$$G_t(z) = \frac{z-\beta t(c-1)-\sqrt{[\beta t(c-1)-z]^2-4\beta tz}}{2\beta tz}. \tag{3.15}$$

In other words, $\mu_t$ is the celebrated Marčenko–Pastur law with parameters $(1/c, c\beta t)$.

REMARK 3.4. If we take $\sigma_i(x)=2\sqrt{x}$, $a(x)=\beta\alpha$, $G(x,y)=\beta(x+y)$ in (3.5) with $\alpha/N\to c$, the equation becomes

$$dy_i^N(t) = 2\sqrt{y_i^N(t)}\,dW_i(t) + \beta\left(\alpha + \sum_{j:j\neq i}\frac{y_i^N(t)+y_j^N(t)}{y_i^N(t)-y_j^N(t)}\right)dt. \tag{3.16}$$

This is the eigenvalue process of the classical $\beta$-Laguerre processes that are studied in Demni (2007) and König and O'Connell (2001). As discussed in Remark 3.2, the corresponding normalized particle equation is (3.1) with coefficient functions $\sigma_i^N(x)=2\sqrt{x/N}$, $b_N(x)=\beta\alpha/N$ and $H_N(x,y)=\beta(x+y)/N$.

*General Dyson Brownian motion.* We choose the coefficient functions $g_N(x)$, $h_N(x)$ and $b_N(x)$ and initial value in (1.3) such that they satisfy the conditions in Corollary 2.1 and Corollary 2.2, where $NG_N(x,y) = N(g_N(x)^2h_N(y)^2 + g_N(y)^2h_N(x)^2)$ converges to $G(x,y) = 1$, and $b_N(x)$ converges to $b(x) = 0$, uniformly as $N$ tends to infinity.

Similar to the examples above, (2.17) can be simplified as

$$G_t(z) = G_0(z) - \int_0^t G_s(z)\partial_z G_s(z)\,ds, \tag{3.17}$$

which was shown in Anderson, Guionnet and Zeitouni (2010).

Now we consider the system of SDEs (1.3) with $\mu_0(dx) = \delta_0(dx)$, $g_N(x) = (2N)^{-1/2}$, $h_N(x) = 1$ and $b_N(x) = b_N$, where $\{b_N, N \in \mathbb{N}\}$ is a sequence of positive numbers that converges to 0. Thanks to the uniqueness of the solution to (1.3) and the self-similarity of Brownian motion, we can obtain the following scaling property:

$$G_t(z) = \frac{1}{\sqrt{t}} G_1\left(\frac{z}{\sqrt{t}}\right). \tag{3.18}$$

Thus, (3.17) can be transformed to

$$G_t(z) = G_0(z) + \frac{1}{z} G_1^2\left(\frac{z}{\sqrt{t}}\right).$$

When $t = 1$, we have

$$G_1(z) = \frac{z - \sqrt{z^2 - 4}}{2},$$

which is the Stieltjes transform of the semicircle law. Finally, it follows from the scaling property (3.18) that

$$G_t(z) = \frac{z - \sqrt{z^2 - 4t}}{2t}, \tag{3.19}$$

is the Stieltjes transform of a limit measure, which is also a solution to (3.17). This yields the uniqueness of the limit measure of $L_N$. Note that in Anderson, Guionnet and Zeitouni (2010), the uniqueness of the limit measure was obtained from the uniqueness of the solution to the equation (3.17).

REMARK 3.5. The symmetric Brownian motion is obtained by taking $g_N(x) = (2N)^{-1/2}$, $h_N(x) = 1$ and $b_N(x) = 0$ in (1.1) and the solution of the corresponding eigenvalue SDEs (1.3) is the classical Dyson Brownian motion.

**4. Conditions for existence and uniqueness of the solutions to particle systems.** We stress that the results of large-$N$ limit in Sections 2 and 3 were obtained under the assumption that the eigenvalue SDEs (1.3) and (3.1) have solutions (before colliding/exploding). Also note that Graczyk and Małecki (2013, 2014) imposed conditions to guarantee the existence and uniqueness of such solutions.

In this section, we provide a new set of conditions for the existence and uniqueness of strong solutions to (1.3) and (3.1). Throughout this section, the dimension $N$ is fixed and we remove $N$ in subscripts/superscritps.

As (1.3) is a special case of (3.1), we consider the latter only: for $1 \le i \le N$ and $t \ge 0$,

$$\begin{cases} dx_i = \sigma_i(x_i)\,dW_i(t) + \left(b_i(x_i) + \sum_{j:j\neq i} \frac{H_{ij}(x_i, x_j)}{x_i - x_j}\right) dt, \\ x_1(0) < \cdots < x_N(0), \end{cases} \tag{4.1}$$

where $(W_i)_{1\le i\le N}$ are independent Brownian motions. In Graczyk and Małecki (2014), the existence and strong uniqueness of the system (4.1) were established under the following conditions:

(G1) The functions $\sigma_i$ are continuous. Besides, there exists a function $\rho:\mathbb{R}_+\to\mathbb{R}_+$, such that for any $\varepsilon>0$

$$\int_0^{\varepsilon}\rho^{-1}(x)\,dx=\infty,$$

and for all $x,y\in\mathbb{R}$ and $1\le i\le N$,

$$\bigl|\sigma_i(x)-\sigma_i(y)\bigr|^2\le\rho\bigl(|x-y|\bigr).$$

(G2) The functions $b_i$ and $H_{ij}$ are continuous for all $1\le i,j\le N$ and $i\ne j$. The functions $H_{ij}$ are nonnegative and symmetric, that is, $H_{ij}(x,y)=H_{ji}(y,x)$.

Now, we define, for $n\in\mathbb{N}$, $-\infty\le A<B\le+\infty$,

$$D^n=\Bigl\{(x_1,\dots,x_N):-\infty<A_n<x_1<\cdots<x_N<B_n<\infty,\\ x_{i+1}-x_i>\frac{1}{n}\text{ for }1\le i\le N-1\Bigr\},$$

with $A_n\searrow A$, $B_n\nearrow B$ and define

$$D=\bigl\{(x_1,\dots,x_N):A<x_1<\cdots<x_N<B\bigr\}.$$

Then $\overline{D^n}\subseteq D^{n+1}$ and $\bigcup_n D^n=D$. We impose the following conditions on the coefficient functions:

(E) The functions $\sigma_i$ are in $C^1((A,B))$ and strictly positive on $(A,B)$;
(F) For each $n\in\mathbb{N}$, there exists a number $p=p(n)>N$ such that the functions $b_i(x)$ are in $L^p(A_n,B_n)$ for $1\le i\le N$ and $H_{jk}(x,y)$ belongs to $L^p(\{(x,y|A_n<x<y<B_n,y-x\ge\frac{1}{n})\})$ for $1\le j<k\le N$.

Note that condition (G1) is not a consequence of condition (E) (consider, e.g., $\sigma_i(x)=x^2+1$), and condition (G2) clearly implies condition (F).

THEOREM 4.1. *Suppose that the initial value $(x_1(0),\dots,x_N(0))\in D$. Under the conditions* (E) *and* (F), *the system of SDEs* (4.1) *has a unique strong solution up to the first exit time $\tau$ from D, which is defined as follows*:

$$\tau=\inf_{t\ge0}\bigl\{\bigl(x_1(t),\dots,x_N(t)\bigr)\notin D\bigr\}.$$

The proof of Theorem 4.1 relies on the following result due to Krylov and Röckner (2005).

THEOREM 4.2. *Consider the SDE*

$$x_t=x_0+\int_0^t b(s+r,x_r)\,dr+w_t,\qquad t\ge0,\tag{4.2}$$

*where $w_t$ is a Brownian motion and $b(t,x)$ a $\mathbb{R}^d$-valued Borel function on an open set $Q\subseteq\mathbb{R}\times\mathbb{R}^d$. Let $Q^n$, $n\ge1$ be bounded open subsets of $Q$, such that $\overline{Q^n}\subseteq Q^{n+1}$ and $\bigcup_n Q^n=Q$. Suppose that for each $n\in\mathbb{N}^+$, there exist $p=p(n)\ge2$ and $q=q(n)>2$ satisfying*

$$\frac{d}{p}+\frac{2}{q}<1,$$

*and*

$$\big\| \|b(t,x)I_{Q^n}(t,x)\|_{L^p(dx)}\big\|_{L^q(dt)} < \infty.$$

*Then there exists a unique strong solution up to the first exit time, say $\tau$, from $Q$. Moreover this solution satisfies*

$$\int_0^t |b(s+r,x_r)|^2\,dr < \infty$$

*for $t < \tau$ almost surely.*

PROOF OF THEOREM 4.1. By condition (E), for $1 \le i \le N$, there exist $f_i(x) \in C^2((A,B))$ satisfying $f_i'(x) = 1/\sigma_i(x)$. Besides, $f_i(x)$ is increasing so it is invertible and the inverse is in $C^2((f_i(A), f_i(B)))$. For $1 \le i \le N$, let $y_i = f_i(x_i)$. By Itô formula,

$$\begin{aligned}
dy_i &= f_i'(x_i)\,dx_i + \frac{1}{2}f_i''(x_i)\,d\langle x_i\rangle \\
&= f_i'(x_i)\sigma_i(x_i)\,dW_i + f_i'(x_i)\Bigg(b_i(x_i) + \sum_{j:j\neq i}\frac{H_{ij}(x_i,x_j)}{x_i-x_j}\Bigg)\,dt \\
&\quad + \frac{1}{2}f_i''(x_i)\sigma_i(x_i)^2\,dt \\
&= dW_i + \frac{1}{\sigma_i(x_i)}\Bigg(b_i(x_i) + \sum_{j:j\neq i}\frac{H_{ij}(x_i,x_j)}{x_i-x_j}\Bigg)\,dt - \frac{1}{2}\big(\sigma_i(x_i)\big)'\,dt \\
&= dW_i + \frac{1}{\sigma_i(f_i^{-1}(y_i))}\Bigg(b_i\big(f_i^{-1}(y_i)\big) + \sum_{j:j\neq i}\frac{H_{ij}(f_i^{-1}(y_i), f_j^{-1}(y_j))}{f_i^{-1}(y_i)-f_j^{-1}(y_j)}\Bigg)\,dt \\
&\quad - \frac{1}{2}\big(\sigma_i\big(f_i^{-1}(y_i)\big)\big)'\,dt.
\end{aligned} \tag{4.3}$$

Introduce the map

$$\begin{aligned}
F : (A,B)^N &\longrightarrow \big(f_1(A), f_1(B)\big)\times\cdots\times\big(f_N(A), f_N(B)\big), \\
(x_1,\dots,x_N) &\longmapsto \big(f_1(x_1),\dots,f_N(x_N)\big).
\end{aligned}$$

Then $F$ is bijective, both $F$ and $F^{-1}$ being twice continuously differentiable. Then the system of SDEs (4.3) on $F(D)$ is equivalent to the the system of SDEs (4.1) on $D$.

Let $Q = \mathbb{R}_+ \times F(D)$ and $Q^n = (0,n)\times F(D^n)$. In order to apply Theorem 4.2, we only need to verify that the following functions are in $L^p(Q^n)$ for some $p = p(n) > N$:

$$\frac{b_i(f_i^{-1}(y_i))}{\sigma_i(f_i^{-1}(y_i))}, \qquad \frac{1}{\sigma_i(f_i^{-1}(y_i))}\frac{H_{ij}(f_i^{-1}(y_i), f_j^{-1}(y_j))}{f_i^{-1}(y_i)-f_j^{-1}(y_j)} \quad\text{and}\quad \big(\sigma_i\big(f_i^{-1}(y_i)\big)\big)'.$$

By change of variables, it is equivalent to show that the functions

$$\left(\frac{b_i(x_i)}{\sigma_i(x_i)}\right)^p\frac{1}{\sigma_i(x_i)}, \qquad \left(\frac{1}{\sigma_i(x_i)}\frac{H_{ij}(x_i,x_j)}{x_i-x_j}\right)^p\frac{1}{\sigma_i(x_i)\sigma_j(x_j)} \quad\text{and}\quad \frac{((\sigma_i(x_i))')^p}{\sigma_i(x_i)}$$

belong to $L^1(D^n)$, which is a direct consequence of Conditions (E) and (F).

The proof is concluded. $\square$

REMARK 4.1. Note that theorem 4.1 is valid for Dyson Brownian motion, noncolliding square Bessel process and noncolliding squared $\beta$-Bessel particle system. Indeed, for the Dyson Brownian motion, $\sigma_i(x) = (2N)^{-1/2}$, $b_i(x) = 0$ and $H_{ij}(x, y) = 1/N$, which satisfy the conditions (E) and (F) with $A = -\infty$ and $B = +\infty$. For the noncolliding square Bessel process, $\sigma_i(x) = 2\sqrt{x}/\sqrt{N}$, $b_i(x) = p/N$ and $H_{ij}(x, y) = (x + y)/N$, which satisfy the conditions (E) and (F) with $A = 0$ and $B = +\infty$. For the noncolliding squared $\beta$-Bessel particle system, $\sigma_i(x) = 2\sqrt{x/N}$, $b_i(x) = \beta\alpha/N$ and $H_{ij}(x, y) = \beta(x + y)/N$, which also satisfy the conditions (E) and (F) with $A = 0$ and $B = +\infty$. In the noncolliding square Bessel process case and the noncolliding squared $\beta$-Bessel particle system case, the first exit time $\tau$ is the first time the particles explode, collide or reach zero.

Furthermore, Theorem 4.1 also applies to the particle system (4.1) with discontinuous coefficient functions $b_i(x)$ and $H_{i,j}(x, y)$. For instance, it applies to the system with $\sigma_i(x) = (2N)^{-1/2}$, $b_i(x) = \frac{1}{N} f(x)$ and $H_{ij}(x, y) = \frac{1}{N} g(x, y)$ where $f$ and $g$ are bounded measurable functions.

Combining Theorem 4.1 with Theorem 3.1 and Theorem 3.2 which are obtained in Section 3, we have the following two corollaries for the particle system (3.1), in which now the continuity of the coefficient functions $b_N(x)$ and $H_N(x, y)$ is not required.

COROLLARY 4.1. *For the system of SDEs* (3.1), *assume that the initial value satisfies* $\lambda_1^N(0) < \cdots < \lambda_N^N(0)$ *and condition* (C′) *holds. Suppose that for each* $N \in \mathbb{N}$, $\sigma_i^N(x)$ *are in* $C^1(\mathbb{R})$ *and strictly positive for* $1 \le i \le N$ *and* $b_N(x)$ *is nondecreasing* (*or Lipschitz continuous*). *Moreover, we assume that there exist positive constants* $c_1$, $c_2$ *that does not depend on* $N$ *and positive constants* $c_3(N)$ *and* $c_4(N)$, *such that*

(a′) $|b_N(x)| \le c_1\sqrt{1 + |x|^2}$, $\forall x \in \mathbb{R}$;
(b′) $H_N(x, y) \le \frac{c_2}{N}(1 + |xy|)$, $\forall x, y \in \mathbb{R}$;
(c′) $\sigma_i^N(x)^2 + \sigma_i^N(y)^2 \le c_3(N)(x - y)^2 + 4H_N(x, y)$, $\forall x, y \in \mathbb{R}$, $\forall x, y \in \mathbb{R}$;
(d′) $H_N(x, y)(y - x) + H_N(y, z)(z - y) \le c_4(N)(z - y)(z - x)(y - x) + H_N(x, z)(z - x)$, $\forall x < y < z$.

*Then for any fixed number* $T > 0$, *the sequence* $\{L_N(t), t \in [0, T]\}_{N\in\mathbb{N}}$ *is relatively compact in* $C([0, T], M_1(\mathbb{R}))$ *almost surely.*

PROOF. It is obvious that conditions (a′) and (b′) imply condition (F). Thus, by Theorem 4.1, SDEs (3.1) has a unique strong solution. Conditions (a′), (b′) and (c′) allow to apply Graczyk and Małecki (2014), Proposition 3.4, and hence the solution is nonexploding. Moreover, conditions on $b_N$ and conditions (c′) and (d′) imply the noncollision of the solution by Graczyk and Małecki (2014), Proposition 4.2. Note that the continuity of the coefficient functions is not involved in the proofs of Graczyk and Małecki (2014), Proposition 3.4 and Proposition 4.2.

Finally, it is easy to check that conditions (A′)–(D′) in Section 3 are satisfied with $\varphi(x) = \ln(1 + x^2)$, and the conclusion follows from Theorem 3.1. □

The following result is a direct consequence of Corollary 4.1 and Theorem 3.2.

COROLLARY 4.2. *For the system of SDEs* (3.1), *assume that all the conditions in Corollary* 4.1 *hold. Besides*, *suppose there exist continuous functions* $b(x)$ *and* $H(x, y)$, *such that* $b_N(x)$ *converges to* $b(x)$ *and* $NH_N(x, y)$ *converges to* $H(x, y)$ *uniformly as* $N$ *tends to infinity. If the empirical measure* $L_N(0)$ *converges weakly as* $N$ *goes to infinity to a measure*

$\mu_0$ *almost surely, and the sequence* $L_N$ *has a limit measure* $\mu$ *in* $C([0,T], M_1(\mathbb{R}))$ *for a fixed number* $T > 0$, *then the measure* $\mu$ *satisfies the equation*

$$\begin{aligned}\int \frac{\mu_t(dx)}{z-x} &= \int \frac{\mu_0(dx)}{z-x} + \int_0^t \left[\int \frac{b(x)}{(z-x)^2}\mu_s(dx)\right] ds \\ &\quad + \int_0^t \left[\iint \frac{H(x,y)}{(z-x)(z-y)^2}\mu_s(dx)\mu_s(dy)\right] ds,\end{aligned} \tag{4.4}$$

*for* $z \in \mathbb{C}\setminus\mathbb{R}$, $t \in [0,T]$.

**5. Discussion on the equation (2.15) of the limit measure.** Consider the equation (2.15) of limit measure

$$\begin{aligned}\int \frac{\mu_t(dx)}{z-x} &= \int \frac{\mu_0(dx)}{z-x} + \int_0^t \left[\int \frac{b(x)}{(z-x)^2}\mu_s(dx)\right] ds \\ &\quad + \int_0^t \left[\iint \frac{G(x,y)}{(z-x)(z-y)^2}\mu_s(dx)\mu_s(dy)\right] ds \quad \text{for } z \in \mathbb{C}\setminus\mathbb{R}.\end{aligned}$$

The uniqueness of the limit measure $\mu_t(dx)$ is obtained so far only for some special cases in Section 3 by solving (2.15) directly with the help of the scaling property (3.9). For general cases, the uniqueness is still unknown.

In this section, we further explore equation (2.15) assuming self-similarity on the eigenvalues $\lambda_i^N(t)$, which hopefully may shed some light on solving the issue of the uniqueness of the limit measure.

Recalling that $G(x,y)$ is the limit of $NG_N(x,y)$ where $G_N(x,y)$ takes the form of (1.2), we assume that $G(x,y) = g^2(x)h^2(y) + g^2(y)h^2(x)$, and then (2.15) becomes

$$\begin{aligned}\partial_t \int \frac{\mu_t(dx)}{z-x} &= \int \frac{b(x)}{(z-x)^2}\mu_t(dx) + \int \frac{g^2(x)}{z-x}\mu_t(dx) \int \frac{h^2(x)}{(z-x)^2}\mu_t(dx) \\ &\quad + \int \frac{h^2(x)}{z-x}\mu_t(dx) \int \frac{g^2(x)}{(z-x)^2}\mu_t(dx).\end{aligned} \tag{5.1}$$

Suppose that the self-similarity $\lambda_i^N(t) \overset{d}{=} t^\alpha \lambda_i^N(1)$ holds for some constant $\alpha$, then for any $\varphi \in C_b(\mathbb{R})$

$$\begin{aligned}\int \varphi(x)\mu_t(dx) &= \lim_{N_j\to\infty} \frac{1}{N_j}\sum_{i=1}^{N_j} \varphi(\lambda_i^{N_j}(t)) \\ &= \lim_{N_j\to\infty} \frac{1}{N_j}\sum_{i=1}^{N_j} \varphi(t^\alpha \lambda_i^{N_j}(1)) = \int \varphi(t^\alpha x)\mu_1(dx).\end{aligned} \tag{5.2}$$

Hence, applying (5.2) to $\varphi(x) = (z-x)^{-1}$ and $\varphi(x) = x(z-x)^{-2}$, we have

$$\begin{aligned}\partial_t \int \frac{\mu_t(dx)}{z-x} &= \partial_t \int \frac{\mu_1(dx)}{z-t^\alpha x} = \int \frac{\alpha t^{\alpha-1}x}{(z-t^\alpha x)^2}\mu_1(dx) \\ &= \frac{\alpha}{t}\int \frac{t^\alpha x}{(z-t^\alpha x)^2}\mu_1(dx) = \frac{\alpha}{t}\int \frac{x}{(z-x)^2}\mu_t(dx) \\ &= -\frac{\alpha}{t}\partial_z \int \frac{x}{z-x}\mu_t(dx).\end{aligned}$$

Furthermore, we also have

$$\begin{aligned}
&\int \frac{g^2(x)}{z-x}\mu_t(dx)\int \frac{h^2(x)}{(z-x)^2}\mu_t(dx)+\int \frac{h^2(x)}{z-x}\mu_t(dx)\int \frac{g^2(x)}{(z-x)^2}\mu_t(dx)\\
&\quad=-\int \frac{g^2(x)}{z-x}\mu_t(dx)\partial_z\int \frac{h^2(x)}{z-x}\mu_t(dx)-\int \frac{h^2(x)}{z-x}\mu_t(dx)\partial_z\int \frac{g^2(x)}{z-x}\mu_t(dx)\\
&\quad=-\partial_z\biggl[\int \frac{g^2(x)}{z-x}\mu_t(dx)\int \frac{h^2(x)}{z-x}\mu_t(dx)\biggr],
\end{aligned}$$

and

$$\int \frac{b(x)}{(z-x)^2}\mu_t(dx)=-\partial_z\int \frac{b(x)}{z-x}\mu_t(dx).$$

Thus, (5.1) can be simplified as

$$\frac{\alpha}{t}\int \frac{x}{z-x}\mu_t(dx)=\int \frac{b(x)}{z-x}\mu_t(dx)+\int \frac{g^2(x)}{z-x}\mu_t(dx)\int \frac{h^2(x)}{z-x}\mu_t(dx)+C(t),$$

where $C(t)$ is a complex constant independent of $z$. Let $|z|\to\infty$. By dominated convergence theorem, we can see that $C(t)\equiv 0$.

Thus, for $G(x,y)=g^2(x)h^2(y)+g^2(y)h^2(x)$, assuming self-similarity on $\lambda_i^N(t)$, the equation (2.15) for limit measure $\mu_t(dx)$ becomes

$$\frac{\alpha}{t}\int \frac{x}{z-x}\mu_t(dx)=\int \frac{b(x)}{z-x}\mu_t(dx)+\int \frac{g^2(x)}{z-x}\mu_t(dx)\int \frac{h^2(x)}{z-x}\mu_t(dx). \tag{5.3}$$

In particular, when $b(x)$, $g^2(x)$ and $h^2(x)$ are polynomial functions (consider, for example, Bru's Wishart process, $\beta$-Wishart process, and Dyson Brownian motion), the above equation can be simplified to a polynomial equation only involving the variable $z$ and the Stieltjes transform $\int \frac{1}{z-x}\mu_t(dx)$ of the limit measure $\mu_t(dx)$.

We also would like to point out that equation (5.3) can be represented via the Hilbert transform, in light of the following lemma (see, e.g., Section 3.1 in Stein and Shakarchi (2011)).

LEMMA 5.1. *For $\varphi\in L^2(\mathbb{R})$, in the $L^2(\mathbb{R})$-norm we have*

$$\lim_{v\to 0^+}\int \frac{\varphi(x)}{z-x}\,dx=-2\pi i P(\varphi)(u),$$

*where $z=u+iv$, and the projective operator $P=(I+iH)/2$ with $H$ being the Hilbert transform operator.*

Assume that $\mu_t(dx)=p_t(x)\,dx$ is absolutely continuous with respect to the Lebesgue measure. Applying Lemma 5.1 to (5.3), we have the following equation for the density function $p_t(x)$:

$$\begin{aligned}
&\frac{\alpha}{t}(I+iH)\bigl(xp_t(x)\bigr)\\
&\quad=(I+iH)\bigl(b(x)p_t(x)\bigr)-\pi i(I+iH)\bigl(g^2(x)p_t(x)\bigr)(I+iH)\bigl(h^2(x)p_t(x)\bigr).
\end{aligned}$$

The imaginary part of the equation is

$$H\biggl(\biggl(\frac{\alpha}{t}x-b(x)\biggr)p_t(x)\biggr)=-\pi g^2(x)h^2(x)p_t^2(x)+\pi H\bigl(g^2(x)p_t(x)\bigr)H\bigl(h^2(x)p_t(x)\bigr),$$

which is equivalent to the real part, noting that $H^2=-I$,

$$\biggl(\frac{\alpha}{t}x-b(x)\biggr)p_t(x)=\pi g^2(x)p_t(x)H\bigl(h^2(x)p_t(x)\bigr)+\pi h^2(x)p_t(x)H\bigl(g^2(x)p_t(x)\bigr).$$

**Acknowledgments.** The authors are grateful to the Editor and a referee for numerous valuable and detailed comments. Addressing these comments has led to a significant improvement of the paper.

Part of this research was supported by the HKSAR-RGC-Grant GRF-17307319.

## REFERENCES

ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

BAI, Z. and SILVERSTEIN, J. W. (2010). *Spectral Analysis of Large Dimensional Random Matrices*, 2nd ed. *Springer Series in Statistics*. Springer, New York. MR2567175 https://doi.org/10.1007/978-1-4419-0661-8

BRU, M.-F. (1989). Diffusions of perturbed principal component analysis. *J. Multivariate Anal.* **29** 127–136. MR0991060 https://doi.org/10.1016/0047-259X(89)90080-8

CABANAL-DUVILLARD, T. and GUIONNET, A. (2001). Large deviations upper bounds for the laws of matrix-valued processes and non-communicative entropies. *Ann. Probab.* **29** 1205–1261. MR1872742 https://doi.org/10.1214/aop/1015345602

CÉPA, E. and LÉPINGLE, D. (1997). Diffusing particles with electrostatic repulsion. *Probab. Theory Related Fields* **107** 429–449. MR1440140 https://doi.org/10.1007/s004400050092

CHAN, T. (1992). The Wigner semi-circle law and eigenvalues of matrix-valued diffusions. *Probab. Theory Related Fields* **93** 249–272. MR1176727 https://doi.org/10.1007/BF01195231

DA FONSECA, J., GRASSELLI, M. and IELPO, F. (2014). Estimating the Wishart affine stochastic correlation model using the empirical characteristic function. *Stud. Nonlinear Dyn. Econom.* **18** 253–289. MR3258801 https://doi.org/10.1515/snde-2012-0009

DA FONSECA, J., GRASSELLI, M. and TEBALDI, C. (2008). Option pricing when correlations are stochastic: An analytical framework. *Rev. Deriv. Res.* **10** 151–180.

DEMNI, N. (2007). The Laguerre process and generalized Hartman–Watson law. *Bernoulli* **13** 556–580. MR2331264 https://doi.org/10.3150/07-BEJ6048

DYSON, F. J. (1962). A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.* **3** 1191–1198. MR0148397 https://doi.org/10.1063/1.1703862

GNOATTO, A. (2012). The Wishart short rate model. *Int. J. Theor. Appl. Finance* **15** 1250056, 24. MR3011743 https://doi.org/10.1142/S0219024912500562

GNOATTO, A. and GRASSELLI, M. (2014). An affine multicurrency model with stochastic volatility and stochastic interest rates. *SIAM J. Financial Math.* **5** 493–531. MR3252805 https://doi.org/10.1137/130922902

GÖING-JAESCHKE, A. and YOR, M. (2003). A survey and some generalizations of Bessel processes. *Bernoulli* **9** 313–349. MR1997032 https://doi.org/10.3150/bj/1068128980

GOURIEROUX, C. (2006). Continuous time Wishart process for stochastic risk. *Econometric Rev.* **25** 177–217. MR2256286 https://doi.org/10.1080/07474930600713234

GOURIEROUX, C. and SUFANA, R. (2010). Derivative pricing with Wishart multivariate stochastic volatility. *J. Bus. Econom. Statist.* **28** 438–451. MR2723611 https://doi.org/10.1198/jbes.2009.08105

GRACZYK, P. and MAŁECKI, J. (2013). Multidimensional Yamada–Watanabe theorem and its applications to particle systems. *J. Math. Phys.* **54** 021503, 15. MR3076363 https://doi.org/10.1063/1.4790507

GRACZYK, P. and MAŁECKI, J. (2014). Strong solutions of non-colliding particle systems. *Electron. J. Probab.* **19** no. 119, 21. MR3296535 https://doi.org/10.1214/EJP.v19-3842

KATORI, M. (2016). *Bessel Processes, Schramm–Loewner Evolution, and the Dyson Model*. *SpringerBriefs in Mathematical Physics* **11**. Springer, Tokyo. MR3445783

KONDOR, R. and JEBARA, T. (2007). Gaussian and Wishart hyperkernels. In *Advances in Neural Information Processing Systems* 729–736.

KÖNIG, W. and O'CONNELL, N. (2001). Eigenvalues of the Laguerre process as non-colliding squared Bessel processes. *Electron. Commun. Probab.* **6** 107–114. MR1871699 https://doi.org/10.1214/ECP.v6-1040

KRYLOV, N. V. and RÖCKNER, M. (2005). Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields* **131** 154–196. MR2117951 https://doi.org/10.1007/s00440-004-0361-z

LI, W.-J., ZHANG, Z. and YEUNG, D.-Y. (2009). Latent Wishart processes for relational kernel learning. *J. Mach. Learn. Res.* **5** 336–343.

LI, J., ZHAO, B., DENG, C. and XU, R. (2016). Time varying metric learning for visual tracking. *Pattern Recogn. Lett.* **80** 157–164.

ROGERS, L. C. G. and SHI, Z. (1993). Interacting Brownian particles and the Wigner law. *Probab. Theory Related Fields* **95** 555–570. MR1217451 https://doi.org/10.1007/BF01196734

SCHÖLKOPF, B. and SMOLA, A. J. (2002). *Learning with Kernels*. MIT Press, Cambridge.

STEIN, E. M. and SHAKARCHI, R. (2011). *Functional Analysis*: *Introduction to Further Topics in Analysis*. *Princeton Lectures in Analysis* **4**. Princeton Univ. Press, Princeton, NJ. MR2827930

WU, S.-J., GHOSH, S. K., KU, Y.-C. and BLOOMFIELD, P. (2018). Dynamic correlation multivariate stochastic volatility with latent factors. *Stat*. *Neerl*. **72** 48–69. MR3746787 https://doi.org/10.1111/stan.12115

ZAMBOTTI, L. (2017). Bessel processes. In *Random Obstacle Problems*. *Lecture Notes in Math*. **2181** 31–50. Springer, Cham. MR3616274 https://doi.org/10.1007/978-3-319-52096-4

ZHANG, Z., KWOK, J. and YEUNG, D.-Y. (2006). Model-based transductive learning of the kernel matrix. *Mach*. *Learn*. **63** 69–101.