SEMI-IMPLICIT EULER–MARUYAMA APPROXIMATION FOR NONCOLLIDING PARTICLE SYSTEMS

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We introduce a semi-implicit Euler–Maruyama approximation which preserves the noncolliding property for some class of noncolliding particle systems such as Dyson–Brownian motions, Dyson–Ornstein–Uhlenbeck processes and Brownian particle systems with nearest neighbor repulsion, and study its rates of convergence in both L^p-norm and pathwise sense.

1. Introduction. Let $X = (X(t) = (X_1(t), \ldots, X_d(t))^*)_{t \geq 0}$ be a solution of the following system of stochastic differential equations (SDEs):

(1.1)
$$dX_i(t) = \left\{ \sum_{j \neq i} \frac{\gamma_{i,j}}{X_i(t) - X_j(t)} + b_i(X(t)) \right\} dt$$
$$+ \sum_{j=1}^d \sigma_{i,j}(X(t)) dW_j(t), \quad i = 1, \ldots, d,$$

with $X(0) \in \Delta_d = \{\mathbf{x} = (x_1, \ldots, x_d)^* \in \mathbb{R}^d : x_1 < x_2 < \cdots < x_d\}$, $\gamma_{i,j} = \gamma_{j,i} \geq 0$, and $W = (W(t) = (W_1(t), \ldots, W_d(t))^*)_{t \geq 0}$ a d-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions.

The systems of SDEs (1.1) are used to model the stochastic evolution of d particles with electrostatic repulsion and restoring force. An interesting feature of these systems is their deep connection with the theory of eigenvalue distribution of randomly diffusing symmetric matrices and Jack symmetric polynomials (see [2, 3, 6, 14, 20]). The existence and uniqueness of a strong noncolliding solution to such kind of systems have been studied intensively by many authors (see [4, 7, 12, 21] and the references therein). However, there are still few results on the numerical approximation for these systems, in spite of their practical importance. To the best of our knowledge, the paper of Li and Menon [16] is the only work in this direction. These authors introduced an explicit tamed Euler–Maruyama approximation for Dyson–Brownian motion and studied its consistency via a couple of numerical experiments. However, their scheme, unfortunately, does not preserve the noncolliding property of solution, which is an important characteristic of the Dyson–Brownian motion.

Many authors have studied the numerical approximation for one-dimensional SDEs with boundary (Bessel process and Cox–Ingersoll–Ross (CIR) process). Dereich, Neuenkirch and Szpruch [5] introduced an implicit Euler–Maruyama scheme for the CIR process and showed that the rate of convergence is 1/2. That result was extended to one-dimensional SDEs with boundary condition by Alfonsi [1] and Neuenkirch and Szpruch [18]. It was proved that if the drift coefficient is one-sided Lipschitz and smooth, then the implicit Euler–Maruyama scheme is well defined and converges to the unique solution in L^p sense with convergence rate 1/2 or 1 provided that the boundaries are not accessible. In the case of CIR and Bessel

processes with accessible boundaries, the rates of strong convergence of discrete approximation schemes may be very slow (see Hutzenthaler et al. [11] and Hefter and Jentzen [10]). It should be noted that if we consider $d = 2$, $b_i = 0$ and $(\sigma_{i,j})_{1 \le i, j \le d}$ is a diagonal and constant matrix, then $X_2 - X_1$ is a Bessel process. The numerical approximation for multidimensional SDEs with boundary has been studied by Gyöngy [8] and Jentzen et al. [13]. These authors introduced various explicit and implicit Euler–Maruyama schemes and studied their convergence in the pathwise sense. The numerical approximation for multidimensional reflected SDEs has also been addressed by many authors (see [19, 23, 24] and the references therein). They proposed various methods such as Euler–Maruyama approximation and recursive projection scheme and studied their convergence in L^p-norm. Note that the reflected SDEs have an additional local time factor which forces the solution to stay inside the underlying domain.

The main aims of this paper are to introduce a numerical approximation method which preserves the noncolliding property of the solutions to system (1.1) and to study its strong rate of convergence both in L^p-norm and in pathwise sense. To the best of our knowledge, this is among the first papers to discuss the strong rate of approximation for a multidimensional stochastic differential equation whose solution stays in a domain (see [24]). Note that the singular coefficients $\frac{1}{X_i - X_j}$ make the system difficult to deal with. In order to overcome this obstacle, we need an upper bound for both moments and inverse moments of $X_i - X_j$.

The remainder of this paper is organised as follows. In the next section, we introduce a semi-implicit Euler–Maruyama approximation $X^{(n)}$ for equation (1.1) and study its consistency. More precisely, we first show that the rate at which $X^{(n)}$ converges to X is of order $1/2$ in the pathwise sense. Then under some key conditions on the integrability of X, we show that the rate is of order $1/2$ in the L^p-norm. Finally, under further conditions on the regularity of b_i, we show that the rate is of order 1 in the L^p-norm. In Section 3, we study some generalized classes of interacting Brownian particle systems and Brownian particles with nearest neighbor repulsion. We first show the existence and uniqueness of the solutions to these systems and then we show that the solution satisfies the key integrability condition which allows us to obtain the rates of convergence of $X^{(n)}$. In the Appendix A.1, we discuss how to compute the implicit scheme in some particular cases.

2. Approximation for noncolliding processes.
Throughout this paper, we suppose that the following assumptions hold.

ASSUMPTION 2.1. (A1) $X(0) \in \Delta_d$ almost surely.

(A2) The parameters $\gamma_{i,j}$ are nonnegative constants satisfying $\gamma_{i,j} = \gamma_{j,i}$ for $i, j = 1, \ldots, d$ with $i \ne j$ and $\gamma_{i,i+1} > 0$ for $i = 1, \ldots, d - 1$.

(A3) The coefficients $b_i : \mathbb{R}^d \to \mathbb{R}$, $i = 1, \ldots, d$ are globally Lipschitz continuous, that is,

$$\|b\|_{\text{Lip}} := \sup_{i=1,\ldots,d} \sup_{x \ne y} \frac{|b_i(x) - b_i(y)|}{|x - y|} < \infty.$$

(A4) The coefficients $\sigma_{i,j} : \mathbb{R}^d \to \mathbb{R}$, $i, j = 1, \ldots, d$ are globally Lipschitz continuous and bounded, that is,

$$\|\sigma\|_{\text{Lip}} := \sup_{i,j=1,\ldots,d} \sup_{x \ne y} \frac{|\sigma_{i,j}(x) - \sigma_{i,j}(y)|}{|x - y|} < \infty,$$

$$\sigma_d^2 := \sup_{i=1,\ldots,d} \sup_{x \in \mathbb{R}^d} \sum_{k=1}^{d} \sigma_{i,k}(x)^2 < \infty.$$

2.1. *Explicit Euler–Maruyama scheme.* Let us first consider the explicit Euler–Maruyama approximation for noncolliding particle system (1.1) which is defined by $\widetilde{X}^{(n)}(0) = X(0)$ and for $t \in (0, T]$ and $i = 1, \ldots, d$,

$$d\widetilde{X}_i^{(n)}(t) = \left\{ \sum_{j \neq i} \frac{\gamma_{i,j}}{\widetilde{X}_i^{(n)}(\eta_n(t)) - \widetilde{X}_j^{(n)}(\eta_n(t))} + b_i\big(\widetilde{X}^{(n)}(\eta_n(t))\big) \right\} dt$$

$$+ \sum_{j=1}^d \sigma_{i,j}\big(\widetilde{X}^{(n)}(\eta_n(t))\big) dW_j(t),$$

where $\eta_n(s) = kT/n =: t_k^{(n)}$ if $s \in [kT/n, (k+1)T/n)$. For $X(0) \in \Delta_d$, the explicit Euler–Maruyama scheme is well defined. Since, for each $i = 1, \ldots, d-1$, the quantity

$$\widetilde{X}_{i+1}^{(n)}\big(t_1^{(n)}\big) - \widetilde{X}_i^{(n)}\big(t_1^{(n)}\big)$$

$$= X_{i+1}(0) - X_i(0)$$

$$+ \left\{ \sum_{k \neq i+1} \frac{\gamma_{i+1,k}}{X_{i+1}(0) - X_k(0)} - \sum_{k \neq i} \frac{\gamma_{i,k}}{X_i(0) - X_k(0)} \right.$$

$$+ b_{i+1}(X(0)) - b_i(X(0)) \bigg\} \frac{T}{n}$$

$$+ \sum_{j=1}^d \{\sigma_{i+1,j}(X(0)) - \sigma_{i,j}(X(0))\} W_j\big(t_1^{(n)}\big)$$

is normally distributed conditional on the state of the initial condition provided that $\sigma_{i+1,j}(X(0)) \neq \sigma_{i,j}(X(0))$. This implies

$$\mathbb{P}\big(\widetilde{X}^{(n)}\big(t_1^{(n)}\big) \in \Delta_d\big) < 1.$$

Therefore, the explicit Euler–Maruyama scheme is not suitable for approximating the noncolliding process (1.1).

2.2. *Semi-implicit Euler–Maruyama scheme.* In the following, we propose a semi-implicit Euler–Maruyama scheme for (1.1), which preserves the noncolliding property of the solution. The construction of the semi-implicit scheme is based on the following result.

PROPOSITION 2.2. *Let* $a = (a_1, \ldots, a_d)^* \in \mathbb{R}^d$, $c_{i,j} = c_{j,i} \geq 0$, *for any* $1 \leq i < j \leq d$ *and* $c_{i,i+1} > 0$. *The following system of equations has a unique solution*:

$$(2.1) \qquad \xi_i = a_i + \sum_{j \neq i} \frac{c_{i,j}}{\xi_i - \xi_j}, \quad i = 1, \ldots, d,$$

which satisfies $\xi_1 < \xi_2 < \cdots < \xi_d$.

PROOF. The following proof is based on a homotopy argument presented in [9], page 230. Denote $\mathbf{J} = (1, 2, \ldots, d)^* \in \Delta_d$ and

$$g_i(\mathbf{x}) = g_i(x_1, \ldots, x_d) = a_i - i + \sum_{j \neq i} \frac{c_{i,j}}{x_i - x_j}, \quad i = 1, \ldots, d.$$

Note that $\mathbf{g} = (g_1, \ldots, g_d)^* \in C^\infty(\Delta_d; \mathbb{R}^d)$. We consider the following differential equation:

$$(2.2) \qquad \begin{cases} \dfrac{d\mathbf{x}}{dt} = \dfrac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} \dfrac{d\mathbf{x}}{dt} + \mathbf{g}(\mathbf{J}), & t > 0, \\ \mathbf{x}(0) = \mathbf{J}, \end{cases}$$

where $\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} = (\frac{\partial g_i(\mathbf{x})}{\partial x_j})_{i,j}$. Note that for any $\mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d$, since $c_{i,j} = c_{j,i}$,

$$\left\langle \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{y}, \mathbf{y} \right\rangle = \sum_{i,j} \frac{\partial g_i(\mathbf{x})}{\partial x_j} y_j y_i$$

$$= \sum_{i \neq j} \frac{c_{i,j}}{(x_i - x_j)^2} y_j y_i - \sum_{i \neq j} \frac{c_{i,j}}{(x_i - x_j)^2} y_i^2$$

$$= -\frac{1}{2} \sum_{i \neq j} \frac{c_{i,j}}{(x_i - x_j)^2} (y_i - y_j)^2 \leq 0.$$

Therefore, $I_d - \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}$ is a strictly positive definite matrix. Since $\mathbf{g} \in C^\infty(\Delta_d; \mathbb{R}^d)$, equation (2.2) has a local unique solution which can be continued up to the boundary of Δ_d. Denote $t^* = \inf\{t > 0 : \mathbf{x}(t) \notin \Delta_d\}$. For $t < t^*$, thanks to the initial condition $\mathbf{x}(0) = \mathbf{J}$, we have

$$\mathbf{x}(t) = \mathbf{g}(\mathbf{x}(t)) + \mathbf{J} + (t - 1)\mathbf{g}(\mathbf{J}).$$

Moreover,

$$\left| \frac{d\mathbf{x}}{dt} \right|^2 = \left\langle \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt}, \frac{d\mathbf{x}}{dt} \right\rangle + \left\langle \mathbf{g}(\mathbf{J}), \frac{d\mathbf{x}}{dt} \right\rangle \leq \left\langle \mathbf{g}(\mathbf{J}), \frac{d\mathbf{x}}{dt} \right\rangle \leq \left| \frac{d\mathbf{x}}{dt} \right| |\mathbf{g}(\mathbf{J})|.$$

Thus

$$\left| \frac{d\mathbf{x}}{dt} \right| \leq |\mathbf{g}(\mathbf{J})|.$$

This estimation together with mean-value theorem yields $|\mathbf{g}(\mathbf{x}(t))| \leq (2t + 1)|\mathbf{g}(\mathbf{J})|$ for $t \in [0, t^*)$. Hence the fact that $\mathbf{g}(\mathbf{x})$ blows up at the boundary of Δ_d implies that $t^* = \infty$. Let $t = 1$, and we get

$$\mathbf{x}(1) = \mathbf{g}(\mathbf{x}(1)) + \mathbf{J} \in \Delta_d,$$

which means that $\xi = \mathbf{x}(1)$ is a solution to equation (2.1).

Now we consider the uniqueness of solution to equation (2.1) in Δ_d. Let $\xi = (\xi_1, \ldots, \xi_d)$, $\mu = (\mu_1, \ldots, \mu_d) \in \Delta_d$ be solutions of the equation (2.1). Then, since $c_{i,j} = c_{j,i} \geq 0$, it follows from the identity

$$(2.3) \qquad \sum_{i=1}^d A_i \sum_{j \neq i} B_{i,j} = \sum_{i < j} \{A_i B_{i,j} + A_j B_{j,i}\}, \quad A_i, B_{i,j} \in \mathbb{R},$$

that

$$|\xi - \mu|^2 = \langle \xi - \mu, \xi - \mu \rangle$$

$$= \sum_{i=1}^d (\xi_i - \mu_i) \sum_{j \neq i} c_{i,j} \left\{ \frac{1}{\xi_i - \xi_j} - \frac{1}{\mu_i - \mu_j} \right\}$$

$$= \sum_{i < j} c_{i,j} \{(\xi_i - \mu_i) - (\xi_j - \mu_j)\} \left\{ \frac{1}{\xi_i - \xi_j} - \frac{1}{\mu_i - \mu_j} \right\}$$

$$= \sum_{i < j} c_{i,j} \{(\xi_i - \xi_j) - (\mu_i - \mu_j)\} \left\{ \frac{1}{\xi_i - \xi_j} - \frac{1}{\mu_i - \mu_j} \right\} \leq 0.$$

This concludes $\xi = \mu$. $\square$

REMARK 2.3. An interesting consequence of Proposition 2.2 is that the nonlinear system of equations (2.1) has exactly $d!$ solutions on $\mathbb{R}^d$.

REMARK 2.4. The system of equations (2.1) does not have a closed-form solution in general. In Section A.1, we will construct an approximation scheme for its solution in some particular cases.

Based on Proposition 2.2, a semi-implicit Euler–Maruyama scheme for noncolliding process (1.1) is defined as follows: $X^{(n)}(0) := X(0)$ and for each $k = 0, \ldots, n-1$, $X^{(n)}(t_{k+1}^{(n)})$ is the unique solution in Δ_d of the following equation:

$$X_i^{(n)}(t_{k+1}^{(n)}) = X_i^{(n)}(t_k^{(n)}) + \left\{ \sum_{j \neq i} \frac{\gamma_{i,j}}{X_i^{(n)}(t_{k+1}^{(n)}) - X_j^{(n)}(t_{k+1}^{(n)})} + b_i(X^{(n)}(t_k^{(n)})) \right\} \frac{T}{n}$$
$$+ \sum_{j=1}^{d} \sigma_{i,j}(X^{(n)}(t_k^{(n)}))\{W_j(t_{k+1}^{(n)}) - W_j(t_k^{(n)})\}.$$

We then define for $t \in (0, T] \setminus \{t_1^{(n)}, \ldots, t_n^{(n)}\}$,

$$X_i^{(n)}(t) = X_i^{(n)}(\eta_n(t)) + \left\{ \sum_{j \neq i} \frac{\gamma_{i,j}}{X_i^{(n)}(\kappa_n(t)) - X_j^{(n)}(\kappa_n(t))} + b_i(X^{(n)}(\eta_n(t))) \right\}$$
$$\times (t - \eta_n(t)) + \sum_{j=1}^{d} \sigma_{i,j}(X^{(n)}(\eta_n(t)))\{W_j(t) - W_j(\eta_n(t))\},$$

where $\kappa_n(s) = (k+1)T/n = t_{k+1}^{(n)}$ if $s \in [kT/n, (k+1)T/n)$. Hence $X^{(n)}(t)$ satisfies

$$X_i^{(n)}(t) = X_i(0) + \int_0^t \left\{ \sum_{j \neq i} \frac{\gamma_{i,j}}{X_i^{(n)}(\kappa_n(s)) - X_j^{(n)}(\kappa_n(s))} + b_i(X^{(n)}(\eta_n(s))) \right\} ds$$
$$+ \sum_{j=1}^{d} \int_0^t \sigma_{i,j}(X^{(n)}(\eta_n(s))) dW_j(s).$$

We denote $X_{i,j}(t) = X_i(t) - X_j(t)$ and $X_{i,j}^{(n)}(t) := X_i^{(n)}(t) - X_j^{(n)}(t)$.

We also repeatedly use the following representation of the estimation error, $e_i(t) := X_i(t) - X_i^{(n)}(t)$ and $e(t) := (e_1(t), \ldots, e_d(t))^*$. Then for $k = 0, \ldots, n-1$, we have

(2.4)
$$e_i(t_{k+1}^{(n)}) = e_i(t_k^{(n)}) + \sum_{j \neq i} \left\{ \frac{\gamma_{i,j}}{X_{i,j}(t_{k+1}^{(n)})} - \frac{\gamma_{i,j}}{X_{i,j}^{(n)}(t_{k+1}^{(n)})} \right\} \frac{T}{n}$$
$$+ \{b_i(X(t_k^{(n)})) - b_i(X^{(n)}(t_k^{(n)}))\} \frac{T}{n}$$
$$+ \sum_{j=1}^{d} \{\sigma_{i,j}(X(t_k^{(n)})) - \sigma_{i,j}(X^{(n)}(t_k^{(n)}))\}\{W_j(t_{k+1}^{(n)}) - W_j(t_k^{(n)})\}$$
$$+ r_i(k),$$

where

$$
\begin{aligned}
r_i(k) := {} & \sum_{j\neq i} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \left\{ \frac{\gamma_{i,j}}{X_{i,j}(s)} - \frac{\gamma_{i,j}}{X_{i,j}(t_{k+1}^{(n)})} \right\} ds \\
& + \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \{ b_i(X(s)) - b_i(X(t_k^{(n)})) \} ds \\
& + \sum_{j=1}^{d} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \{ \sigma_{i,j}(X(s)) - \sigma_{i,j}(X(t_k^{(n)})) \} dW_j(s).
\end{aligned}
\tag{2.5}
$$

2.3. *The case of constant diffusion coefficient.* In this subsection, we consider the convergence of $X^{(n)}$ where diffusion coefficient is a constant.

The following result states that $X^{(n)}$ converges to X at the rate of almost $1/2$ in the pathwise sense provided that the system (1.1) has a strong solution in Δ_d on $[0, T]$. In Section 3, we will show the existence and uniqueness for some classes of SDEs of the form (1.1).

THEOREM 2.5. *Assume that $\sigma_{i,j}(x) \equiv \sigma_{i,j}$ and system of equations* (1.1) *has a unique strong solution in Δ_d on $[0, T]$. Then there exists a finite random variable η which does not depend on n such that*

$$
\sup_{k=1,\ldots,n} |X(t_k^{(n)}) - X^{(n)}(t_k^{(n)})| \leq \frac{\sqrt{\log n}}{\sqrt{n}} \eta \quad a.s.
$$

PROOF. Using the identity (2.3) and the fact that $e_j - e_i = X_{j,i} - X_{j,i}^{(n)}$, we get

$$
\begin{aligned}
& |e(t_{k+1}^{(n)})|^2 \\
& = \sum_{i=1}^{d} |e_i(t_{k+1}^{(n)})|^2 \\
& = \sum_{i=1}^{d} e_i(t_k^{(n)}) e_i(t_{k+1}^{(n)}) + \sum_{i=1}^{d} e_i(t_{k+1}^{(n)}) \sum_{j\neq i} \left\{ \frac{\gamma_{i,j}}{X_{i,j}(t_{k+1}^{(n)})} - \frac{\gamma_{i,j}}{X_{i,j}^{(n)}(t_{k+1}^{(n)})} \right\} \frac{T}{n} \\
& \quad + \sum_{i=1}^{d} e_i(t_{k+1}^{(n)}) \{ b_i(X(t_k^{(n)})) - b_i(X^{(n)}(t_k^{(n)})) \} \frac{T}{n} + \sum_{i=1}^{d} e_i(t_{k+1}^{(n)}) r_i(k) \\
& = \sum_{i=1}^{d} e_i(t_k^{(n)}) e_i(t_{k+1}^{(n)}) \\
& \quad + \sum_{i<j} \{ X_{j,i}(t_{k+1}^{(n)}) - X_{j,i}^{(n)}(t_{k+1}^{(n)}) \} \left\{ \frac{\gamma_{i,j}}{X_{j,i}(t_{k+1}^{(n)})} - \frac{\gamma_{i,j}}{X_{j,i}^{(n)}(t_{k+1}^{(n)})} \right\} \frac{T}{n} \\
& \quad + \sum_{i=1}^{d} e_i(t_{k+1}^{(n)}) \{ b_i(X(t_k^{(n)})) - b_i(X^{(n)}(t_k^{(n)})) \} \frac{T}{n} + \sum_{i=1}^{d} e_i(t_{k+1}^{(n)}) r_i(k).
\end{aligned}
$$

Using the fact that $(x - y)(\frac{1}{x} - \frac{1}{y}) \leq 0$ and $xy \leq x^2/2 + y^2/2 \; x, y > 0$, we have

$$
\begin{aligned}
|e(t_{k+1}^{(n)})|^2 \leq {} & \frac{1}{2} |e(t_k^{(n)})|^2 + \frac{1}{2} |e(t_{k+1}^{(n)})|^2 \\
& + |e(t_k^{(n)})| \sum_{i=1}^{d} |e_i(t_{k+1}^{(n)})| \frac{T \|b\|_{\text{Lip}}}{n} + \sum_{i=1}^{d} e_i(t_{k+1}^{(n)}) r_i(k).
\end{aligned}
$$

Hence we have, for any $k = 0, \ldots, n-1$,

$$|e(t_{k+1}^{(n)})|^2 \le |e(t_k^{(n)})|^2 + |e(t_k^{(n)})| \sum_{i=1}^{d} |e_i(t_{k+1}^{(n)})| \frac{C_1}{n} + 2 \sum_{i=1}^{d} |e_i(t_{k+1}^{(n)})||r_i(k)|,$$

$$\le \sum_{\ell=0}^{k} |e(t_\ell^{(n)})| \sum_{i=1}^{d} |e_i(t_{\ell+1}^{(n)})| \frac{C_1}{n} + 2 \sum_{\ell=0}^{k} \sum_{i=1}^{d} |e_i(t_{\ell+1}^{(n)})||r_i(\ell)|,$$

where $C_1 := 2T \|b\|_{\mathrm{Lip}}$. By taking the supremum with respect to k, we obtain for any $m = 1, \ldots, n$,

$$\sup_{k=1,\ldots,m} |e(t_k^{(n)})|^2 \le \sum_{\ell=0}^{m-1} |e(t_\ell^{(n)})| \sum_{i=1}^{d} |e_i(t_{\ell+1}^{(n)})| \frac{C_1}{n} + 2 \sum_{\ell=0}^{m-1} \sum_{i=1}^{d} |e_i(t_{\ell+1}^{(n)})||r_i(\ell)|$$

$$\le \sup_{k=1,\ldots,m} |e(t_k^{(n)})| \sum_{\ell=0}^{m-1} \sup_{k=1,\ldots,\ell} |e(t_k^{(n)})| \frac{dC_1}{n}$$

$$+ 2 \sup_{k=1,\ldots,m} |e(t_k^{(n)})| \sum_{\ell=0}^{m-1} \sum_{i=1}^{d} |r_i(\ell)|,$$

and thus,

$$\sup_{k=1,\ldots,m} |e(t_k^{(n)})| \le \sum_{\ell=0}^{m-1} \sup_{k=1,\ldots,\ell} |e(t_k^{(n)})| \frac{dC_1}{n} + 2 \sum_{\ell=0}^{m-1} \sum_{i=1}^{d} |r_i(\ell)|.$$

By using discrete Gronwall's inequality (e.g., Chapter XIV, Theorem 1 and Remarks 1 and 2 in [17], pages 436–437), we obtain

(2.6)
$$\sup_{k=1,\ldots,m} |e(t_k^{(n)})| \le 2 \left\{ 1 + \sum_{\ell=0}^{m-1} \frac{dC_1}{n} \exp\left(\sum_{\ell=0}^{m-1} \frac{dC_1}{n} \right) \right\} \sum_{\ell=0}^{m-1} \sum_{i=1}^{d} |r_i(\ell)|$$

$$\le 2 \{ 1 + dC_1 \exp(dC_1) \} \sum_{\ell=0}^{m-1} \sum_{i=1}^{d} |r_i(\ell)|.$$

Therefore,

$$\frac{\sup_{k=1,\ldots,n} |e(t_k^{(n)})|}{2\{1 + dC_1 \exp(dC_1)\}}$$

$$\le \sum_{i=1}^{d} \sum_{j \ne i} \int_0^T \left| \frac{\gamma_{i,j}}{X_{i,j}(s)} - \frac{\gamma_{i,j}}{X_{i,j}(\kappa_n(s))} \right| ds$$

$$+ \sum_{i=1}^{d} \int_0^T \left| b_i(X(s)) - b_i(X(\eta_n(s))) \right| ds$$

$$\le \sum_{i=1}^{d} \sum_{j \ne i} \int_0^T \gamma_{i,j} \left| \frac{X_{i,j}(s) - X_{i,j}(\kappa_n(s))}{\inf_{0 \le s \le T} X_{i,j}(s)^2} \right| ds$$

$$+ d\|b\|_{\mathrm{Lip}} \int_0^T \left| X(s) - X(\eta_n(s)) \right| ds.$$

Moreover, for any $0 \leq s \leq t \leq T$,

$$
\left|X_i(t) - X_i(s)\right| \leq \int_s^t \left\{ \sum_{j \neq i} \frac{\gamma_{i,j}}{|X_i(u) - X_j(u)|} + \|b\|_{\mathrm{Lip}} |X(u)| + |b_i(0)| \right\} \mathrm{d}u
$$

$$
+ \sum_{j=1}^d |\sigma_{i,j}| |W_j(t) - W_j(s)|.
$$

Since $X_t \in \Delta_d$ for $t \in [0, T]$, we have

$$
\sup_{u \in [0,T]} \left\{ \sum_{j \neq i} \frac{\gamma_{i,j}}{|X_i(u) - X_j(u)|} + \|b\|_{\mathrm{Lip}} |X(u)| + |b_i(0)| \right\} < \infty,
$$

and

$$
\inf_{0 \leq s \leq T} \inf_{i \neq j} X_{i,j}(s)^2 > 0.
$$

These estimates together with Lévy's modulus of continuity theorem yield the desired result.
□

REMARK 2.6.   The class of SDEs (1.1) with $\sigma_{i,j}(x) \equiv \sigma_{i,j}$ contains both Dyson–Brownian motions (e.g., [1, 6]) and Dyson–Ornstein–Uhlenbeck processes (e.g., [20]).

In order to show the convergence of the semi-implicit Euler–Maruyama scheme in L^p-norm, we need the following hypothesis on the integrability and Kolmogorov-type condition of X.

HYPOTHESIS 2.7.   *There exist constants $\hat{p} > 0$ and $0 < \hat{C} < \infty$ such that*

$$
\sup_{t \in [0,T]} \mathbb{E}\left[|X(t)|^{\hat{p}} \right] + \max_{0 \leq i < d} \sup_{t \in [0,T]} \mathbb{E}\left[|X_{i,i+1}(t)|^{-\hat{p}} \right] < \hat{C}
$$

and

$$
\mathbb{E}\left[|X(t) - X(s)|^{\hat{p}} \right] \leq \hat{C} |t - s|^{\hat{p}/2} \quad \text{for all } 0 \leq s < t \leq T.
$$

In Section 3, we will introduce some conditions on $\gamma_{i,j}$, b_i and $\sigma_{i,j}$, which guarantee that Hypothesis 2.7 holds.

THEOREM 2.8.   *Suppose that the assumptions of Theorem 2.5 hold. Moreover, suppose that Hypothesis 2.7 holds for some $\hat{p} = 3p \geq 3$. Then there exists $C > 0$ which depends on d such that for any $n \in \mathbb{N}$,*

$$
\mathbb{E}\left[\sup_{k=1,\ldots,n} |X(t_k^{(n)}) - X^{(n)}(t_k^{(n)})|^p \right]^{1/p} \leq \frac{C}{n^{1/2}}.
$$

PROOF.   We will use the estimate (2.6) to show the desired result. Note that from (2.5) we get

(2.7)
$$
\sum_{\ell=0}^{m-1} \sum_{i=1}^d |r_i(\ell)| \leq \sum_{i=1}^d \sum_{j \neq i} \int_0^{t_m^{(n)}} \left| \frac{\gamma_{i,j}}{X_{i,j}(s)} - \frac{\gamma_{i,j}}{X_{i,j}(\kappa_n(s))} \right| \mathrm{d}s
$$

$$
+ \sum_{i=1}^d \int_0^{t_m^{(n)}} \left| b_i(X_i(s)) - b_i(X_i(\eta_n(s))) \right| \mathrm{d}s.
$$

It follows from Hölder's inequality that

$$\mathbb{E}\left[\left(\int_0^{t_m^{(n)}}\left|\frac{1}{X_{i,j}(s)} - \frac{1}{X_{i,j}(\kappa_n(s))}\right|ds\right)^p\right]$$

$$\leq T^{p-1}\int_0^T \mathbb{E}\left[\frac{|X_{i,j}(s) - X_{i,j}(\kappa_n(s))|^p}{|X_{i,j}(s)|^p|X_{i,j}(\kappa_n(s))|^p}\right]ds$$

$$\leq T^{p-1}\int_0^T \left(\mathbb{E}[|X_{i,j}(s) - X_{i,j}(\kappa_n(s))|^{3p}]\right)^{\frac{1}{3}}\left(\mathbb{E}[|X_{i,j}(s)|^{-3p}]\right)^{1/3}$$

$$\times \left(\mathbb{E}[|X_{i,j}(\kappa_n(s))|^{-3p}]\right)^{1/3}ds.$$

This estimate together with Hypothesis 2.7 implies

$$\mathbb{E}\left[\left(\int_0^{t_m^{(n)}}\left|\frac{1}{X_{i,j}(s)} - \frac{1}{X_{i,j}(\kappa_n(s))}\right|ds\right)^p\right] \leq \frac{C}{n^{p/2}},$$

for some constant $C > 0$. Since each b_i is Lipschitz continuous for $i = 1, \ldots, d$, by using Hypothesis 2.7, we have

$$\mathbb{E}\left[\left(\int_0^{t_m^{(n)}}|b_i(X(s)) - b_i(X(\eta_n(s)))|ds\right)^p\right]$$

$$\leq T^{p-1}\|b\|_{\mathrm{Lip}}^p\int_0^T \mathbb{E}[|X(s) - X(\eta_n(s))|^p]ds \leq \frac{C}{n^{p/2}},$$

for some constant $C > 0$. It then follows from (2.7) that

$$\mathbb{E}\left[\left(\sum_{\ell=0}^{m-1}\sum_{i=1}^d|r_i(\ell)|\right)^p\right] \leq \frac{C}{n^{p/2}},$$

for some constant $C > 0$. This estimate together with (2.6) yields the desired result. $\square$

Now we prove that if the drift coefficients b_i are smooth, then the semi-implicit Euler–Maruyama scheme converges at the strong rate of order 1.

THEOREM 2.9. *Suppose that the assumptions of Theorem 2.5 hold. Moreover, suppose that Hypothesis 2.7 holds for some $\hat{p} = 4p \geq 16$ and $b_i \in C_b^2(\mathbb{R}^d; \mathbb{R})$. Then there exists $C > 0$ which depends on d such that, for any $n \in \mathbb{N}$ with $T/n \leq 1$,*

$$\mathbb{E}\left[\sup_{k=1,\ldots,n}|X(t_k^{(n)}) - X^{(n)}(t_k^{(n)})|^p\right]^{1/p} \leq \frac{C}{n}.$$

PROOF. For $\mathbf{x} = (x_1, \ldots, x_d) \in \Delta_d$, we denote

$$f_i(\mathbf{x}) := \sum_{j \neq i}\frac{\gamma_{i,j}}{x_i - x_j}.$$

The first- and second-order derivatives of f are given as follows:

$$\partial_m f_i(\mathbf{x}) = \frac{\partial f_i}{\partial x_m} := \begin{cases} -\sum_{j \neq i}\dfrac{\gamma_{i,j}}{(x_i - x_j)^2} & \text{if } m = i, \\ \dfrac{\gamma_{i,m}}{(x_i - x_m)^2} & \text{if } m \neq i, \end{cases}$$

and

$$
\partial_\ell \partial_m f_i(\mathbf{x}) := \frac{\partial^2 f_i}{\partial x_\ell \partial x_m} =
\begin{cases}
\displaystyle \sum_{j \neq i} \frac{2\gamma_{i,j}}{(x_i - x_j)^3} & \text{if } m = \ell = i, \\[4mm]
-\dfrac{2\gamma_{i,\ell}}{(x_i - x_\ell)^3} & \text{if } m = i, \ell \neq i, \\[4mm]
-\dfrac{2\gamma_{i,m}}{(x_i - x_m)^3} & \text{if } m \neq i, \ell = i, \\[4mm]
\dfrac{2\gamma_{i,m}}{(x_i - x_m)^3} & \text{if } m \neq i, \ell = m, \\[4mm]
0 & \text{if } m \neq i, \ell \neq i, m \neq \ell.
\end{cases}
$$

Recall that for $k = 0, \ldots, n-1$, we have

$$
\begin{aligned}
(2.8) \qquad e_i\big(t_{k+1}^{(n)}\big) = {}& e_i\big(t_k^{(n)}\big) + \big\{ f_i\big(X\big(t_{k+1}^{(n)}\big)\big) - f_i\big(X^{(n)}\big(t_{k+1}^{(n)}\big)\big) \big\} \frac{T}{n} \\
& + \big\{ b_i\big(X\big(t_k^{(n)}\big)\big) - b_i\big(X^{(n)}\big(t_k^{(n)}\big)\big) \big\} \frac{T}{n} + r_i(k),
\end{aligned}
$$

and by using Itô's formula, we have

$$
r_i(k) = r_i^{(1)}(k) + r_i^{(2)}(k) + r_i^{(3)}(k) + r_i^{(4)}(k),
$$

where

$$
r_i^{(1)}(k) := \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \int_t^{t_{k+1}^{(n)}} h_i^{(1)}(X(s)) \, ds \, dt,
$$

$$
r_i^{(2)}(k) := \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \int_{t_k^{(n)}}^t h_i^{(2)}(X(s)) \, ds \, dt,
$$

$$
r_i^{(3)}(k) := \sum_{j=1}^d \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \int_t^{t_{k+1}^{(n)}} h_{i,j}^{(3)}(X(s)) \, dW_j(s) \, dt,
$$

$$
r_i^{(4)}(k) := \sum_{j=1}^d \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \int_{t_k^{(n)}}^t h_{i,j}^{(4)}(X(s)) \, dW_j(s) \, dt,
$$

and for $\mathbf{x} = (x_1, \ldots, x_d) \in \Delta_d$,

$$
h_i^{(1)}(\mathbf{x}) := -\sum_{m=1}^d \partial_m f_i(\mathbf{x}) \big(f_m(\mathbf{x}) + b_m(\mathbf{x}) \big) - \sum_{m,k,k'=1}^d \frac{\sigma_{k,m}\sigma_{k',m}}{2} \partial_k \partial_{k'} f_i(\mathbf{x}),
$$

$$
h_i^{(2)}(\mathbf{x}) := \sum_{m=1}^d \partial_m b_i(\mathbf{x}) \big(f_m(\mathbf{x}) + b_m(\mathbf{x}) \big) + \sum_{m,k,k'=1}^d \frac{\sigma_{k,m}\sigma_{k',m}}{2} \partial_k \partial_{k'} b_i(\mathbf{x}),
$$

$$
h_{i,j}^{(3)}(\mathbf{x}) = -\sigma_{i,j} \sum_{m=1}^d \partial_m f_i(\mathbf{x}) \quad \text{and} \quad h_{i,j}^{(4)}(\mathbf{x}) = \sigma_{i,j} \sum_{m=1}^d \partial_m b_i(\mathbf{x}).
$$

From (2.8), we have

$$
\begin{aligned}
(2.9) \qquad & \left| e_i\big(t_{k+1}^{(n)}\big) - \big\{ f_i\big(X\big(t_{k+1}^{(n)}\big)\big) - f_i\big(X^{(n)}\big(t_{k+1}^{(n)}\big)\big) \big\} \frac{T}{n} \right|^2 \\
& = \left| e_i\big(t_k^{(n)}\big) + \big\{ b_i\big(X\big(t_k^{(n)}\big)\big) - b_i\big(X^{(n)}\big(t_k^{(n)}\big)\big) \big\} \frac{T}{n} + r_i(k) \right|^2
\end{aligned}
$$

and thus

$$
\begin{aligned}
|e_i(t_{k+1}^{(n)})|^2 = |e_i(t_k^{(n)})|^2 &- \left|\{f_i(X(t_{k+1}^{(n)})) - f_i(X^{(n)}(t_{k+1}^{(n)}))\}\frac{T}{n}\right|^2 \\
&+ \left|\{b_i(X(t_k^{(n)})) - b_i(X^{(n)}(t_k^{(n)}))\}\frac{T}{n}\right|^2 + r_i(k)^2 \\
&+ 2e_i(t_{k+1}^{(n)})\{f_i(X(t_{k+1})) - f_i(X^{(n)}(t_{k+1}))\}\frac{T}{n} \\
&+ 2e_i(t_k^{(n)})\{b_i(X(t_k^{(n)})) - b_i(X^{(n)}(t_k^{(n)}))\}\frac{T}{n} + 2e_i(t_k^{(n)})r_i(k) \\
&+ 2\{b_i(X(t_k^{(n)})) - b_i(X^{(n)}(t_k^{(n)}))\}\frac{T}{n}r_i(k).
\end{aligned}
$$

Using the identity (2.3), the fact that $e_j - e_i = X_{j,i} - X_{j,i}^{(n)}$, the Lipschitz continuity of b_i, the inequality $xy \le x^2/2 + y^2/2$ and the fact that $T/n \le 1$, we get

$$
\begin{aligned}
|e(t_{k+1}^{(n)})|^2 \le |e(t_k^{(n)})|^2 &+ d\|b\|_{\mathrm{Lip}}^2 |e(t_k^{(n)})|^2\left(\frac{T}{n}\right)^2 + 2d\|b\|_{\mathrm{Lip}}|e(t_k^{(n)})|^2\frac{T}{n} \\
&+ 2\sum_{i=1}^{d} e_i(t_k^{(n)})r_i(k) + 2\|b\|_{\mathrm{Lip}}|e(t_k^{(n)})|\frac{T}{n}\sum_{i=1}^{d} r_i(k) + \sum_{i=1}^{d} r_i(k)^2 \\
\le |e(t_k^{(n)})|^2 &+ C_2|e(t_k^{(n)})|^2\frac{T}{n} + 2\sum_{i=1}^{d} e_i(t_k^{(n)})r_i(k) + \frac{3}{2}\sum_{i=1}^{d} r_i(k)^2,
\end{aligned}
$$

where $C_2 := d\{3\|b\|_{\mathrm{Lip}}^2 + 2\|b\|_{\mathrm{Lip}}\}$. Thus, we obtain

$$
|e(t_k^{(n)})|^2 \le \sum_{j=0}^{k-1}\left\{C_2|e(t_j^{(n)})|^2\frac{T}{n} + 2\sum_{i=1}^{d} e_i(t_j^{(n)})r_i(j) + \frac{3}{2}\sum_{i=1}^{d} r_i(j)^2\right\}.
$$

Hence for $p = 2q \ge 4$, we have

$$
\begin{aligned}
(2.10) \qquad \sup_{k=0,\dots,\ell}|e(t_k^{(n)})|^{2q} \le\; &3^{q-1}C_2^q \sup_{k=0,\dots,\ell}\left|\sum_{j=0}^{k-1}|e(t_j^{(n)})|^2\frac{T}{n}\right|^q \\
&+ 3^{q-1}2^q \sup_{k=0,\dots,\ell}\left|\sum_{j=0}^{k-1}\sum_{i=1}^{d} e_i(t_j^{(n)})r_i(j)\right|^q \\
&+ 3^{2q-1}2^{-q} \sup_{k=0,\dots,\ell}\left|\sum_{j=0}^{k-1}\sum_{i=1}^{d} r_i(j)^2\right|^q.
\end{aligned}
$$

Now for $r \in [1, 2q]$, we consider the upper bound of the rth moment of $r_i^{(1)}(j)$, $r_i^{(2)}(j)$, $r_i^{(3)}(j)$ and $r_i^{(4)}(j)$. Using Jensen's inequality and the inequality $xy \le x^2/2 + y^2/2$, there exist $K_r^{(1)}$, $K_r^{(2)}$, $K_r^{(3)}$ and $K_r^{(4)}$ such that

$$
\begin{aligned}
|h_i^{(1)}(\mathbf{x})|^r \le (2d + d^3)^{r-1}\Bigg\{&\sum_{m=1}^{d}|\partial_m f_i(\mathbf{x})|^r(|f_m(\mathbf{x})|^r + |b_m(\mathbf{x})|^r) \\
&+ \sum_{m,k,k'=1}^{d}|\sigma_{k,m}|^r|\sigma_{k',m}|^r|\partial_k\partial_{k'} f_i(\mathbf{x})|^r\Bigg\}
\end{aligned}
$$

$$\leq (2d + d^3)^{r-1} \left\{ \sum_{m=1}^d \left(|\partial_m f_i(\mathbf{x})|^{2r} + |f_m(\mathbf{x})|^{2r} + \|b_m\|_\infty^r |\partial_m f_i(\mathbf{x})|^r \right) \right.$$

$$\left. + \sum_{m,k,k'=1}^d |\sigma_{k,m}|^r |\sigma_{k',m}|^r |\partial_k \partial_{k'} f_i(\mathbf{x})|^r \right\}$$

$$\leq K_r^{(1)} \left\{ \sum_{j \neq i} \left| \frac{1}{x_i - x_j} \right|^{4r} + \sum_{j \neq i} \left| \frac{1}{x_i - x_j} \right|^{2r} + \sum_{j \neq i} \left| \frac{1}{x_i - x_j} \right|^{3r} \right\}$$

and

$$|h_i^{(2)}(\mathbf{x})|^r \leq (d + d^3)^{r-1} \left\{ 2^{r-1} \sum_{m=1}^d \|\partial_m b_i\|_\infty^r (|f_i(\mathbf{x})|^r + \|b_i\|_\infty^r) \right.$$

$$\left. + \sum_{m,k,k'=1}^d \left| \frac{\sigma_{k,m} \sigma_{k',m}}{2} \right|^r \|\partial_k \partial_{k'} b_i\|_\infty^r \right\}$$

$$\leq K_r^{(2)} \left\{ \sum_{m=1}^d \sum_{k \neq m} \left| \frac{1}{x_k - x_m} \right|^r + 1 \right\}$$

and

$$|h_{i,j}^{(3)}(\mathbf{x})|^r \leq d^{r-1} |\sigma_{i,j}|^r \sum_{m=1}^d |\partial_m f_i(\mathbf{x})|^r \leq K_r^{(3)} \sum_{j \neq i} \left| \frac{1}{x_i - x_j} \right|^{2r}$$

and

$$|h_{i,j}^{(4)}(\mathbf{x})|^r \leq d^{r-1} |\sigma_{i,j}|^r \sum_{m=1}^d \|\partial_m b_i\|_\infty^r \leq K_r^{(4)}.$$

Thus, from Hypothesis 2.7, there exist $K_r^{(1,2)}$ and $K_r^{(3,4)}$ such that

$$\mathbb{E}[|r_i^{(1)}(j)|^r] + \mathbb{E}[|r_i^{(2)}(j)|^r]$$

$$\leq \left(\frac{T}{n} \right)^{2(r-1)} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} dt \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} ds \, \mathbb{E}[|h_i^{(1)}(X(s))|^r + |h_i^{(2)}(X(s))|^r]$$

$$\leq K_r^{(1,2)} \left(\frac{T}{n} \right)^{2r},$$

and by using Burkholder–Davis–Gundy's inequality,

$$\mathbb{E}[|r_i^{(3)}(j)|^r] + \mathbb{E}[|r_i^{(4)}(j)|^r]$$

$$\leq d^{r-1} \sum_{j=1}^d \left(\frac{T}{n} \right)^{r-1}$$

$$\times \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E}\left[\left| \int_t^{t_{k+1}^{(n)}} h_{i,j}^{(3)}(X(s)) \, dW_j(s) \right|^r + \left| \int_{t_k^{(n)}}^t h_{i,j}^{(4)}(X(s)) \, dW_j(s) \right|^r \right] dt$$

$$\leq c_r d^{r-1} \sum_{j=1}^{d} \sum_{m'=3}^{4} \left(\frac{T}{n}\right)^{\frac{3r}{2}-2} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} dt \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} ds \, \mathbb{E}\big[|h_{i,j}^{(m')}(X(s))|^r\big]$$

$$\leq K_r^{(3,4)} \left(\frac{T}{n}\right)^{\frac{3r}{2}}.$$

Let $M_k := \sum_{j=0}^{k-1} \sum_{i=1}^{d} e_i(t_j^{(n)})\{r_i^{(3)}(j) + r_i^{(4)}(j)\}$. Then it follows from Hypothesis 2.7 and the upper bound of $h_{i,m}^{(3)}(x)$ and $h_{i,j}^{(4)}(x)$ that

$$\mathbb{E}[M_k|\mathcal{F}_{t_{k-1}^{(n)}}] = M_{k-1} + \sum_{i=1}^{d} e_i(t_{k-1})\mathbb{E}[r_i^{(3)}(k-1) + r_i^{(4)}(k-1)|\mathcal{F}_{t_{k-1}^{(n)}}]$$

$$= M_{k-1}.$$

Hence $(M_k)_{k=1,\dots,n}$ is a $(\mathcal{F}_{t_k^{(n)}})_{k=1,\dots,n}$-martingale. By using Burkholder–Davis–Gundy's inequality, we have

$$\mathbb{E}\Big[\sup_{k=0,\dots,\ell} |M_k|^q\Big]$$

(2.11)
$$\leq c_q \mathbb{E}\bigg[\Big\{\sum_{j=0}^{\ell-1} \sum_{i=1}^{d} |e_i(t_j^{(n)})|^2 |r_i^{(3)}(j) + r_i^{(4)}(j)|^2\Big\}^{q/2}\bigg]$$

$$\leq 2^{q-1} d^{\frac{q}{2}-1} c_q \sum_{j=0}^{\ell-1} \sum_{i=1}^{d} n^{\frac{q}{2}-1} \mathbb{E}[|e_i(t_j^{(n)})|^q \{|r_i^{(3)}(j)|^q + |r_i^{(4)}(j)|^q\}].$$

Therefore, by taking the expectation on (2.10), we obtain

$$\mathbb{E}\Big[\sup_{k=0,\dots,\ell} |e(t_k^{(n)})|^{2q}\Big]$$

$$\leq 3^{q-1} C_2^q \mathbb{E}\bigg[\sup_{k=0,\dots,\ell} \Big|\sum_{j=0}^{k-1} |e(t_j^{(n)})|^2 \frac{T}{n}\Big|^q\bigg]$$

$$+ 3^{q-1} 2^{2q-1} \mathbb{E}\bigg[\sup_{k=0,\dots,\ell} \Big|\sum_{j=0}^{k-1} \sum_{i=1}^{d} e_i(t_j^{(n)})\{r_i^{(1)}(j) + r_i^{(2)}(j)\}\Big|^q\bigg]$$

$$+ 3^{q-1} 2^{2q-1} \mathbb{E}\Big[\sup_{k=0,\dots,\ell} |M_k|^q\Big] + 3^{2q-1} 2^{-q} \mathbb{E}\bigg[\sup_{k=0,\dots,\ell} \Big|\sum_{j=0}^{k-1} \sum_{i=1}^{d} r_i(j)^2\Big|^q\bigg]$$

$$\leq 3^{q-1} C_2^q \mathbb{E}\bigg[\sup_{k=0,\dots,\ell} \Big|\sum_{j=0}^{k-1} |e(t_j^{(n)})|^2 \frac{T}{n}\Big|^q\bigg]$$

$$+ 3^{q-1} 2^{3q-2} d^{q-1} \sum_{j=0}^{\ell-1} \sum_{i=1}^{d} \sum_{m=1}^{2} n^{q-1} \mathbb{E}[|e_i(t_j^{(n)})|^q |r_i^{(m)}(j)|^q]$$

$$+ 3^{q-1} 2^{3q-2} d^{\frac{q}{2}-1} c_q \sum_{j=0}^{\ell-1} \sum_{i=1}^{d} \sum_{m=3}^{4} n^{\frac{q}{2}-1} \mathbb{E}[|e_i(t_j^{(n)})|^q |r_i^{(m)}(j)|^q]$$

$$+ 3^{2q-1}2^{-q}d^{q-1}\sum_{j=0}^{\ell-1}\sum_{i=1}^{d}n^{q-1}\mathbb{E}[|r_i(j)|^{2q}]$$

$$= 3^{q-1}C_2^q\mathbb{E}\left[\sup_{k=0,\dots,\ell}\left|\sum_{j=0}^{k-1}|e(t_j^{(n)})|^2\frac{T}{n}\right|^q\right] + I_\ell^{(1,2)} + I_\ell^{(3,4)} + J_\ell.$$

From Hölder's inequality and the inequality $xy \le x^2/2 + y^2/2$, we have

$$I_\ell^{(1,2)} \le 3^{q-1}2^{3q-2}d^{q-1}\sqrt{K_{2q}^{(1,2)}}\sum_{j=0}^{\ell-1}\sum_{i=1}^{d}n^{q-1}(\mathbb{E}[|e_i(t_j^{(n)})|^{2q}])^{1/2}\left(\frac{T}{n}\right)^{2q}$$

$$= (3T)^{q-1}2^{3q-2}d^{q-1}\sqrt{K_{2q}^{(1,2)}}\sum_{j=0}^{\ell-1}\sum_{i=1}^{d}(\mathbb{E}[|e_i(t_j^{(n)})|^{2q}])^{1/2}\left(\frac{T}{n}\right)^{q+1}$$

$$\le (3T)^{q-1}2^{3q-3}d^{q-1}\sqrt{K_{2q}^{(1,2)}}\sum_{j=0}^{\ell-1}\sum_{i=1}^{d}\left\{\mathbb{E}[|e_i(t_j^{(n)})|^{2q}]\frac{T}{n} + \left(\frac{T}{n}\right)^{2q+1}\right\}$$

$$\le \widetilde{K}_{2q}^{(1,2)}\left\{\sum_{j=0}^{\ell-1}\sum_{i=1}^{d}\mathbb{E}[|e_i(t_j^{(n)})|^{2q}]\frac{T}{n} + \frac{1}{n^{2q}}\right\},$$

for some constant $\widetilde{K}_{2q}^{(1,2)}$ and

$$I_\ell^{(3,4)} \le 3^{q-1}2^{3q-2}d^{\frac{q}{2}-1}c_q\sqrt{K_{2q}^{(3,4)}}\sum_{j=0}^{\ell-1}\sum_{i=1}^{d}n^{\frac{q}{2}-1}(\mathbb{E}[|e_i(t_j^{(n)})|^{2q}])^{1/2}\left(\frac{T}{n}\right)^{3q/2}$$

$$= 3^{q-1}2^{3q-2}(dT)^{\frac{q}{2}-1}c_q\sqrt{K_{2q}^{(3,4)}}\sum_{j=0}^{\ell-1}\sum_{i=1}^{d}(\mathbb{E}[|e_i(t_j^{(n)})|^{2q}])^{1/2}\left(\frac{T}{n}\right)^{q+1}$$

$$\le 3^{q-1}2^{3q-3}d^{\frac{q}{2}-1}c_q\sqrt{K_{2q}^{(3,4)}}\sum_{j=0}^{\ell-1}\sum_{i=1}^{d}\left\{\mathbb{E}[|e_i(t_j^{(n)})|^{2q}]\frac{T}{n} + \left(\frac{T}{n}\right)^{2q+1}\right\}$$

$$\le \widetilde{K}_{2q}^{(3,4)}\left\{\sum_{j=0}^{\ell-1}\sum_{i=1}^{d}\mathbb{E}[|e_i(t_j^{(n)})|^{2q}]\frac{T}{n} + \frac{1}{n^{2q}}\right\},$$

for some constant $\widetilde{K}_{2q}^{(3,4)}$. Finally, we have

$$J_\ell \le 3^{2q-1}2^{3q-2}d^{q-1}\sum_{j=0}^{\ell-1}\sum_{i=1}^{d}\sum_{m=1}^{4}n^{q-1}\mathbb{E}[|r_i^{(m)}(j)|^{2q}]$$

$$\le 3^{2q-1}2^{3q-2}(dT)^{q-1}\sum_{j=0}^{\ell-1}\sum_{i=1}^{d}\left\{\widetilde{K}_{2q}^{(1,2)}\left(\frac{T}{n}\right)^{3q+1} + \widetilde{K}_{2q}^{(3,4)}\left(\frac{T}{n}\right)^{2q+1}\right\}$$

$$\le \frac{\widetilde{K}_{2q}}{n^{2q}},$$

for some $\widetilde{K}_{2q}$.

Therefore, we obtain for some $C > 0$,

$$\mathbb{E}\left[\sup_{k=0,\dots,\ell}|e(t_k^{(n)})|^{2q}\right] \le C\sum_{j=0}^{\ell-1}\mathbb{E}[|e(t_j^{(n)})|^{2q}]\frac{T}{n} + \frac{C}{n^{2q}}.$$

By Gronwall's inequality, we conclude the proof. $\square$

2.4. *The case of general diffusion coefficient.* We obtain the following results on the rate of convergence for the semi-implicit Euler–Maruyama scheme in L^2 norm, under the nonconstant diffusion coefficient.

THEOREM 2.10. *Suppose that Hypothesis 2.7 holds for $\widehat{p} = 6$. Then there exists $C > 0$ which depends on d such that for any $n \in \mathbb{N}$ with $T/n \leq 1$,*

$$\sup_{k=1,\ldots,n} \mathbb{E}\big[\big|X(t_k^{(n)}) - X^{(n)}(t_k^{(n)})\big|^2\big]^{1/2} \leq \frac{C}{n^{1/2}}.$$

REMARK 2.11. We note that the proof of Theorem 2.10 is also based on the identity (2.9). However, since the coefficients b and σ are Lipschitz continuous, it is difficult to combine the proof of the theorems together.

PROOF OF THEOREM 2.10. We first recall that $f_i(\mathbf{x}) := \sum_{j \neq i} \frac{\gamma_{i,j}}{x_i - x_j}$. It follows from (2.4) that

$$\left| e_i(t_{k+1}^{(n)}) - \{f_i(X(t_{k+1}^{(n)})) - f_i(X^{(n)}(t_{k+1}^{(n)}))\}\frac{T}{n}\right|^2 = |e_i(t_k^{(n)}) + R_i(k)|^2,$$

where $R_i(k) := R_{b,i}(k) + R_{\sigma,i}(k) + r_i(k)$ and

$$R_{b,i}(k) := \{b_i(X(t_k^{(n)})) - b_i(X^{(n)}(t_k^{(n)}))\}\frac{T}{n},$$

$$R_{\sigma,i}(k) := \sum_{j=1}^{d}\{\sigma_{i,j}(X(t_k)) - \sigma_{i,j}(X^{(n)}(t_k))\}\{W_j(t_{k+1}^{(n)}) - W_j(t_k^{(n)})\}.$$

Thus we have

$$|e_i(t_{k+1}^{(n)})|^2 = |e_i(t_k^{(n)})|^2 - \left|\{f_i(X(t_{k+1}^{(n)})) - f_i(X^{(n)}(t_{k+1}^{(n)}))\}\frac{T}{n}\right|^2$$
$$+ 2e_i(t_{k+1}^{(n)})\{f_i(X(t_{k+1})) - f_i(X^{(n)}(t_{k+1}))\}\frac{T}{n}$$
$$+ 2e_i(t_k^{(n)})R_i(k) + R_i(k)^2.$$

Using the identity (2.3), the fact that $e_j - e_i = X_{j,i} - X_{j,i}^{(n)}$, Lipschitz continuity of b_i and $T/n \leq 1$, we get

$$|e(t_{k+1}^{(n)})|^2 \leq |e(t_k^{(n)})|^2 + 2\sum_{i=1}^{d}e_i(t_k^{(n)})R_i(k) + \sum_{i=1}^{d}R_i(k)^2$$

$$\leq |e(t_k^{(n)})|^2 + |e(t_k^{(n)})|\sum_{i=1}^{d}e_i(t_k^{(n)})\frac{2\|b\|_{\text{Lip}}T}{n}$$

$$+ 2\sum_{i=1}^{d}e_i(t_k^{(n)})R_{\sigma,i}(k) + 2\sum_{i=1}^{d}e_i(t_k^{(n)})r_i(k)$$

(2.12)

$$+ |e(t_k^{(n)})|^2\frac{3\|b\|_{\text{Lip}}^2 T^2}{n^2} + 3\sum_{i=1}^{d}R_{\sigma,i}(k)^2 + 3\sum_{i=1}^{d}r_i(k)^2$$

$$\leq |e(t_k^{(n)})|^2 + C_3 |e(t_k^{(n)})|^2 \frac{T}{n} + 2\sum_{i=1}^{d} e_i(t_k^{(n)}) R_{\sigma,i}(k)$$

$$+ 2\sum_{i=1}^{d} e_i(t_k^{(n)}) r_i(k) + 3\sum_{i=1}^{d} R_{\sigma,i}(k)^2 + 3\sum_{i=1}^{d} r_i(k)^2,$$

where $C_3 := 2\sqrt{d}\|b\|_{\text{Lip}} + 3\|b\|_{\text{Lip}}^2$. By the independence of the increments of the Brownian motion W, the expectation of $e_i(t_k^{(n)}) R_{\sigma,i}(k)$ is equal to zero. Therefore, by taking the expectation in (2.12) and using the Lipschitz continuity of $\sigma_{i,j}$ conditional independence, we obtain

$$\mathbb{E}[|e(t_{k+1}^{(n)})|^2] \leq \mathbb{E}[|e(t_k^{(n)})|^2] + C_4 \mathbb{E}[|e(t_k^{(n)})|^2]\frac{T}{n}$$

$$+ 2\sum_{i=1}^{d} \mathbb{E}[e_i(t_k^{(n)}) r_i(k)] + 3\mathbb{E}[|r(k)|^2],$$

where $C_4 := C_3 + 3d^2\|\sigma\|_{\text{Lip}}^2$. Thus we have for any $k = 1, \ldots, n$,

$$
\begin{aligned}
\mathbb{E}[|e(t_k^{(n)})|^2] \leq\ & C_4 \sum_{\ell=0}^{k-1} \mathbb{E}[|e(t_\ell^{(n)})|^2]\frac{T}{n} \\
& + 2\sum_{\ell=0}^{k-1}\sum_{i=1}^{d} \mathbb{E}[e_i(t_\ell^{(n)}) r_i(\ell)] \\
& + 3\sum_{\ell=0}^{k-1} \mathbb{E}[|r(\ell)|^2].
\end{aligned}
$$
(2.13)

Recall that $r_i(k) = r_{f,i}(k) + r_{b,i}(k) + r_{\sigma,i}(k)$, where

$$r_{f,i}(k) := \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \{f_i(X(s)) - f_i(X(t_{k+1}^{(n)}))\}\, ds,$$

$$r_{b,i}(k) := \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \{b_i(X(s)) - b_i(X(t_k^{(n)}))\}\, ds,$$

$$r_{\sigma,i}(k) := \sum_{j=1}^{d} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \{\sigma_{i,j}(X(s)) - \sigma_{i,j}(X(t_k^{(n)}))\}\, dW_j(s).$$

We now estimate the expectation of $|r(\ell)|^2$. By using Hölder's inequality and Hypothesis 2.7 with $\widehat{p} = 6$, we have

$$
\begin{aligned}
\mathbb{E}[|r_f(\ell)|^2] &= \sum_{i=1}^{d} \mathbb{E}[|r_{f,i}(\ell)|^2] \\
&\leq \sum_{i=1}^{d} \frac{T}{n} \int_{t_\ell^{(n)}}^{t_{\ell+1}^{(n)}} \mathbb{E}[|f_i(X(s)) - f_i(X(t_{\ell+1}^{(n)}))|^2]\, ds \\
&\leq \sum_{i=1}^{d}\sum_{j\neq i} \frac{(d-1)\gamma_{i,j}^2 T}{n} \int_{t_\ell^{(n)}}^{t_{\ell+1}^{(n)}} \mathbb{E}\left[\frac{|X_{i,j}(s) - X_{i,j}(t_{\ell+1}^{(n)})|^2}{|X_{i,j}(s)|^2 |X_{i,j}(t_{\ell+1}^{(n)})|^2}\right] ds
\end{aligned}
$$
(2.14)

$$\leq \sum_{i=1}^{d} \sum_{j \neq i} \frac{(d-1)\gamma_{i,j}^2 T}{n} \int_{t_\ell^{(n)}}^{t_{\ell+1}^{(n)}} \mathbb{E}\big[|X_{i,j}(s) - X_{i,j}(t_{\ell+1}^{(n)})|^6\big]^{\frac{1}{3}}$$

$$\times \mathbb{E}\big[|X_{i,j}(s)|^{-6}\big]^{\frac{1}{3}} \mathbb{E}\big[|X_{i,j}(t_{\ell+1}^{(n)})|^{-6}\big]^{\frac{1}{3}} \, ds$$

$$\leq \sum_{i=1}^{d} \sum_{j \neq i} 4\hat{C}(d-1)\gamma_{i,j}^2 \Big(\frac{T}{n}\Big)^3 = C_f \Big(\frac{T}{n}\Big)^3.$$

From the Lipschitz continuity of b_i for each $i = 1, \ldots, d$ and Jensen's inequality, we have

(2.15)
$$\mathbb{E}\big[|r_b(\ell)|^2\big] = \sum_{i=1}^{d} \mathbb{E}\big[|r_{b,i}(\ell)|^2\big]$$
$$\leq \frac{d\|b\|_{\mathrm{Lip}}^2 T}{n} \int_{t_\ell^{(n)}}^{t_{\ell+1}^{(n)}} \mathbb{E}\big[|X(s) - X(t_\ell^{(n)})|^2\big] \, ds$$
$$\leq d\widehat{C}^{1/3}\|b\|_{\mathrm{Lip}}^2 \Big(\frac{T}{n}\Big)^3 = C_b \Big(\frac{T}{n}\Big)^3.$$

From Itô's isometry and the Lipschitz continuity of $\sigma_{i,j}$, we have

(2.16)
$$\mathbb{E}\big[|r_\sigma(\ell)|^2\big] = \sum_{i=1}^{d} \mathbb{E}\big[|r_{\sigma,i}(\ell)|^2\big]$$
$$= \sum_{i,j=1}^{d} \int_{t_\ell^{(n)}}^{t_{\ell+1}^{(n)}} \mathbb{E}\big[|\sigma_{i,j}(X(s)) - \sigma_{i,j}(X(t_\ell^{(n)}))|^2\big] \, ds$$
$$\leq d^2\|\sigma\|_{\mathrm{Lip}}^2 \int_{t_\ell^{(n)}}^{t_{\ell+1}^{(n)}} \mathbb{E}\big[|X(s) - X(t_\ell^{(n)})|^2\big] \, ds$$
$$\leq d^2\hat{C}^{1/3}\|\sigma\|_{\mathrm{Lip}}^2 \Big(\frac{T}{n}\Big)^2 = C_\sigma \Big(\frac{T}{n}\Big)^2.$$

Next, we consider $\sum_{i=1}^{d} \mathbb{E}[e_i(t_\ell^{(n)})r_i(\ell)]$. Since $e_i(t_\ell^{(n)})$ is $\mathcal{F}_{t_\ell^{(n)}}$-measurable and the conditional expectation $\mathbb{E}[r_{\sigma,i}(\ell)|\mathcal{F}_{t_\ell^{(n)}}]$ is equal to zero for each $i = 1, \ldots, n$, we obtain

$$\sum_{i=1}^{d} \mathbb{E}[e_i(t_\ell^{(n)})r_{\sigma,i}(\ell)] = \sum_{i=1}^{d} \mathbb{E}[e_i(t_\ell^{(n)})\mathbb{E}[r_{\sigma,i}(\ell)|\mathcal{F}_{t_\ell^{(n)}}]] = 0.$$

Hence, from (2.14) and (2.15) and the inequality $xy \leq x^2/2 + y^2/2$, we have

(2.17)
$$\sum_{i=1}^{d} \mathbb{E}[e_i(t_\ell^{(n)})r_i(\ell)] = \sum_{i=1}^{d} \mathbb{E}[e_i(t_\ell^{(n)})(r_{f,i}(\ell) + r_{b,i}(\ell))]$$
$$\leq \frac{1}{2}\mathbb{E}\big[|e(t_\ell^{(n)})|^2\big]\frac{T}{n} + \frac{1}{2}\sum_{i=1}^{d} \mathbb{E}\big[|r_{f,i}(\ell) + r_{b,i}(\ell)|^2\big]\frac{n}{T}$$
$$\leq \frac{1}{2}\mathbb{E}\big[|e(t_\ell^{(n)})|^2\big]\frac{T}{n} + (C_f + C_b)\Big(\frac{T}{n}\Big)^2.$$

Therefore, it follows from (2.13), (2.14), (2.15), (2.16), (2.17) and the fact $T/n \leq 1$ that, for each $k = 1, \ldots, n$,

$$\mathbb{E}[|e(t_k^{(n)})|^2] \leq (C_4 + 1) \sum_{\ell=0}^{k-1} \mathbb{E}[|e(t_\ell^{(n)})|^2] \frac{T}{n}$$

$$+ \frac{2(C_f + C_b)T^2}{n} + \frac{6(C_f + C_b + C_\sigma)T^2}{n}.$$

From discrete type Gronwall's inequality, we conclude the proof. $\square$

We obtain the following results on the rate of convergence in L^2-sup norm.

COROLLARY 2.12. *Suppose that Hypothesis 2.7 holds for $\widehat{p} = 6$. Then there exists $C > 0$ which depends on d such that for any $n \in \mathbb{N}$ with $T/n \leq 1$,*

$$(2.18) \qquad \mathbb{E}\Big[\sup_{k=1,\ldots,n} |X(t_k^{(n)}) - X^{(n)}(t_k^{(n)})|^2\Big]^{1/2} \leq \frac{C}{n^{1/4}}.$$

REMARK 2.13. In the proof of Theorem 2.10, we use the fact that the expectations of $e_i(t_k^{(n)})R_{\sigma,i}(k)$ and $e_i(t_k^{(n)})r_{\sigma,i}(k)$ are zero. In order to show (2.18), it is needed to bound the expectation of absolute values of these random variables. This reduces the rate of convergence from $1/2$ to $1/4$.

PROOF OF COROLLARY 2.12. It follows from (2.12), Lipschitz continuity of $\sigma_{i,j}$ and Schwarz's inequality that

$$\sup_{k=1,\ldots,n} |e(t_k^{(n)})|^2 \leq C_3 \sum_{\ell=1}^{n-1} |e(t_\ell^{(n)})|^2 \frac{T}{n}$$

$$+ 2\|\sigma\|_{\mathrm{Lip}} \sum_{\ell=1}^{n-1} \sum_{j=1}^{d} |e(t_\ell^{(n)})|^2 |W_j(t_{\ell+1}^{(n)}) - W_j(t_\ell^{(n)})|$$

$$+ 3d\|\sigma\|_{\mathrm{Lip}}^2 \sum_{\ell=1}^{n-1} \sum_{j=1}^{d} |e(t_\ell^{(n)})|^2 |W_j(t_{\ell+1}^{(n)}) - W_j(t_\ell^{(n)})|^2$$

$$+ 2 \sum_{\ell=1}^{n-1} |e(t_\ell^{(n)})||r(\ell)| + 3 \sum_{\ell=1}^{n-1} |r(\ell)|^2.$$

Therefore, since the random variables $e(t_\ell^{(n)})$ and $W_j(t_{\ell+1}^{(n)}) - W_j(t_\ell^{(n)})$ are independent, by taking the expectation and by using Hölder inequality, $T/n \leq 1$, (2.14), (2.15) and (2.16), we have

$$\mathbb{E}\Big[\sup_{k=1,\ldots,n} |e(t_k^{(n)})|^2\Big]$$

$$\leq \Big\{ C_3 T + 2d\|\sigma\|_{\mathrm{Lip}}(nT)^{1/2} + 3d\|\sigma\|_{\mathrm{Lip}}^2 \frac{T}{n} \Big\} \sup_{k=1,\ldots,n} \mathbb{E}[|e(t_k^{(n)})|^2]$$

$$+ 2 \cdot 3^{1/2}(C_f + C_b + C_\sigma)^{1/2} T \sup_{k=1,\ldots,n} \mathbb{E}[|e(t_k^{(n)})|^2]^{1/2}$$

$$+ \frac{9(C_f + C_b + C_\sigma)T^2}{n}.$$

Hence, Theorem 2.10 implies the proof of statement. $\square$

3. Examples. In this section, we will study some classes of SDEs (1.1) which have a unique noncolliding strong solution satisfying Hypothesis 2.7. Note that under Assumptions (A1)–(A4), the coefficients of equation (1.1) are locally Lipschitz continuous on Δ_d. Therefore, given $X(0) \in \Delta_d$, equation (1.1) has a unique strong local solution up to the stopping time

$$(3.1) \qquad \tau = \inf\left\{t > 0 : \min_{1 \le i \le d-1} |X_{i+1}(t) - X_i(t)| = 0 \text{ or } \max_{1 \le i \le d} |X_i(t)| = \infty\right\}.$$

In order to show the existence and uniqueness of global solution to equation (1.1), it is sufficient to prove that $\tau = \infty$ almost surely.

3.1. *Interacting Brownian particles.* We consider the following interacting Brownian particle systems:

$$
\begin{aligned}
dX_i(t) = &\left\{\sum_{j \ne i} \frac{\gamma}{X_i(t) - X_j(t)} + b_i(X_i(t))\right\} dt \\
&+ \sum_{j=1}^{d} \sigma_{i,j}(X(t)) \, dW_j(t), \quad i = 1, \ldots, d,
\end{aligned}
$$

(3.2)

with $X(0) \in \Delta_d = \{(x_1, \ldots, x_d)^* \in \mathbb{R}^d : x_1 < x_2 < \cdots < x_d\}$.

ASSUMPTION 3.1. Suppose that the domain of the drift coefficient b is $\mathbb{R}$ and it holds that $b_i(x) \le b_{i+1}(x)$ for any $x \in \mathbb{R}$.

These systems contain several classes of well-known particle systems such as the Dyson–Brownian particle systems, Dyson–Ornstein–Uhlenbeck process, and the systems considered by Cépa and Lépingle [4]. Graczyk and Malecki [7] studied a class of noncolliding particle systems satisfying condition $\sigma_{i,j}(x) = \delta_{i,j}\sigma_i(x_i)$, where $\delta_{i,j}$ is the Dirac delta function. In particular, they obtained the following result.

PROPOSITION 3.2 ([7], Corollary 6.2). *Suppose that*:

- $\sigma_{i,j}(x) = \delta_{i,j}\sigma_i(x_i)$ *where σ_i is at least $1/2$-Hölder and $\sigma_i^2(x) \le 2\gamma$;*
- b_i *be Lipschitz and $b_i(x) \le b_{i+1}(x)$, $b_i(x)x \le c(1 + |x|^2)$.*

Then the system (3.2) has a unique strong solution in Δ_d for all $t > 0$.

In the following, we will establish a sufficient condition for the existence and uniqueness of a solution to equation (3.2). Moreover, we show that Hypothesis 2.7 holds under a certain condition on γ, σ and d.

We need the following elementary inequality whose proof is given in Appendix A.2.

LEMMA 3.3. *For any $d \ge 2$, $p \ge 0$ and $(x_1, \ldots, x_d) \in \Delta_d$, it holds that*

$$
\sum_{i=1}^{d-1} \sum_{k \ne i, i+1} \frac{1}{(x_{i+1} - x_i)^p (x_{i+1} - x_k)(x_i - x_k)} < \left(2 - \frac{3}{d}\right) \sum_{i=1}^{d-1} \frac{1}{(x_{i+1} - x_i)^{p+2}}.
$$

Recall that $X_{i,j}(t) = X_i(t) - X_j(t)$. Then, for $i > j$, we have

$$
X_{i,j}(t) = X_{i,j}(0) + \int_0^t \frac{2\gamma}{X_{i,j}(s)} \, ds - \int_0^t \sum_{k \ne i, j} \frac{\gamma X_{i,j}(s)}{X_{i,k}(s) X_{j,k}(s)} \, ds
$$

$$+ \int_0^t \{b_i(X_i(s)) - b_j(X_j(s))\} \, ds$$

$$+ \sum_{k=1}^d \int_0^t \{\sigma_{i,k}(X(s)) - \sigma_{j,k}(X(s))\} \, dW_k(s).$$

For each $N > 0$, we define the stopping time

(3.3) $$\tau_N := \inf\Big\{s > 0 : \inf_{1 \le i \le d-1} X_{i+1,i}(s) \le 1/N \text{ or } \sup_{i=1,\dots,d} |X_i(s)| \ge N\Big\}.$$

It is clear that $\tau_N \uparrow \tau$ as $N \to \infty$.

LEMMA 3.4.   *We recall that σ_d^2 is defined in (A4) of Assumption 2.1. Suppose that Assumption 3.1 holds. Assume that $\frac{3\gamma}{d\sigma_d^2} \ge 1$, $p \in [0, \frac{3\gamma}{d\sigma_d^2} - 1]$, $T > 0$ and $\mathbb{E}[X_{i+1,i}(0)^{-p}] < \infty$ for each $i = 1, \dots, d-1$. Then it holds that*

$$\sum_{i=1}^{d-1} \sup_{0 \le t \le T} \mathbb{E}\big[X_{i+1,i}(t \wedge \tau)^{-p}\big] \le \Big(\sum_{i=1}^{d-1} \mathbb{E}[X_{i+1,i}(0)^{-p}]\Big) e^{pT\|b\|_{\mathrm{Lip}}}.$$

In particular,

$$\sup_{i \ne j} \sup_{0 \le t \le T} \mathbb{E}\big[X_{i,j}(t \wedge \tau)^{-p}\big] \le \Big(\sum_{i=1}^{d-1} \mathbb{E}[X_{i+1,i}(0)^{-p}]\Big) e^{pT\|b\|_{\mathrm{Lip}}}.$$

PROOF.   By using Itô's formula, we have

$$X_{i+1,i}(t \wedge \tau_N)^{-p}$$

$$= X_{i+1,i}(0)^{-p}$$

$$+ \int_0^{t \wedge \tau_N} \Big\{ \frac{-2p\gamma}{X_{i+1,i}(s)^{p+2}} + \frac{p\gamma}{X_{i+1,i}(s)^p} \sum_{k \ne i, i+1} \frac{1}{X_{i,k}(s) X_{i+1,k}(s)} \Big\} \, ds$$

$$- \int_0^{t \wedge \tau_N} \frac{p\{b_{i+1}(X_{i+1}(s)) - b_i(X_i(s))\}}{X_{i+1,i}(s)^{p+1}} \, ds$$

$$+ \sum_{k=1}^d \int_0^{t \wedge \tau_N} \frac{p(p+1)|\sigma_{i+1,k}(X(s)) - \sigma_{i,k}(X(s))|^2}{2 X_{i+1,i}(s)^{p+2}} \, ds$$

$$- \sum_{k=1}^d \int_0^{t \wedge \tau_N} \frac{p\{\sigma_{i+1,k}(X(s)) - \sigma_{i,k}(X(s))\}}{X_{i+1,i}(s)^{p+1}} \, dW_k(s).$$

Since for each $i = 1, \dots, d$,

$$\int_0^t \Big| \frac{\{\sigma_{i+1,k}(X(s)) - \sigma_{i,k}(X(s))\}}{X_{i+1,i}(s)^{p+1}} \mathbf{1}_{\{s \le \tau_N\}} \Big|^2 \, ds \le 2\sigma_d^2 N^{2(p+1)} t,$$

thus the expectations of the above stochastic integrals are zero. Moreover, since $(p+1)d\sigma_d^2 < 3\gamma$, by applying Lemma 3.3 we obtain

$$\sum_{i=1}^{d-1} \mathbb{E}\big[X_{i+1,i}(t \wedge \tau_N)^{-p}\big]$$

$$\le \sum_{i=1}^{d-1} \mathbb{E}[X_{i+1,i}(0)^{-p}] - \sum_{i=1}^{d-1} \mathbb{E}\Big[\int_0^{t \wedge \tau_N} \frac{p\{b_{i+1}(X_{i+1}(s)) - b_i(X_i(s))\}}{X_{i+1,i}(s)^{p+1}} \, ds\Big].$$

Since $b_{i+1} \geq b_i$ and b_i is Lipschitz continuous, we have

$$\sum_{i=1}^{d-1} \mathbb{E}\big[X_{i+1,i}(t \wedge \tau_N)^{-p}\big]$$

$$\leq \sum_{i=1}^{d-1} \mathbb{E}\big[X_{i+1,i}(0)^{-p}\big] - \sum_{i=1}^{d-1} \mathbb{E}\bigg[\int_0^{t \wedge \tau_N} \frac{p\{b_i(X_{i+1}(s)) - b_i(X_i(s))\}}{X_{i+1,i}(s)^{p+1}} \, ds\bigg]$$

$$\leq \sum_{i=1}^{d-1} \mathbb{E}\big[X_{i+1,i}(0)^{-p}\big] + p\|b\|_{\mathrm{Lip}} \int_0^t \sum_{i=1}^{d-1} \mathbb{E}\big[X_{i,i+1}(s \wedge \tau_N)^{-p}\big] \, ds.$$

Using Gronwall's inequality, we get

$$\sum_{i=1}^{d-1} \mathbb{E}\big[X_{i+1,i}(t \wedge \tau_N)^{-p}\big] \leq \bigg(\sum_{i=1}^{d-1} \mathbb{E}\big[X_{i+1,i}(0)^{-p}\big]\bigg) e^{pt\|b\|_{\mathrm{Lip}}}.$$

Letting $N \to \infty$, we conclude the proof of the lemma. $\square$

LEMMA 3.5. *Suppose that Assumption* 3.1 *holds. Assume that* $\frac{3\gamma}{d\sigma_d^2} \geq 2$, $p \in [1, \frac{3\gamma}{d\sigma_d^2} - 1]$, $T > 0$, $\mathbb{E}[|X(0)|^p] < \infty$ *and* $\mathbb{E}[X_{i+1,i}(0)^{-p}] < \infty$ *for each* $i = 1, \ldots, d - 1$. *Then there exists a finite constant* C *such that*

(3.4) $$\sup_{0 \leq t \leq T} \sup_{1 \leq i \leq d} \mathbb{E}\big[|X_i(t \wedge \tau)|^p\big] \leq C.$$

PROOF. Since $|b_i(x)| \leq |b_i(0)| + \|b\|_{\mathrm{Lip}}|x|$ for any $x \in \mathbb{R}$, we have

$$|X_i(t \wedge \tau_N)| \leq |X_i(0)| + |b_i(0)|t + \|b\|_{\mathrm{Lip}} \int_0^{t \wedge \tau_N} |X_i(s)| \, ds$$

$$+ \sum_{j \neq i} \int_0^{t \wedge \tau_N} \frac{\gamma}{|X_{i,j}(s)|} \, ds + \sum_{j=1}^d \bigg|\int_0^{t \wedge \tau_N} \sigma_{i,j}(X(s)) \, dW_j(s)\bigg|.$$

A simple calculation yields

$$\frac{|X_i(t \wedge \tau_N)|^p}{(2d+2)^{p-1}} \leq |X_i(0)|^p + |b_i(0)|^p t^p + \|b\|_{\mathrm{Lip}}^p t^{p-1} \int_0^t |X_i(s \wedge \tau_N)|^p \, ds$$

$$+ t^{p-1} \sum_{j \neq i} \int_0^t \frac{\gamma^p}{|X_{i,j}(s \wedge \tau_N)|^p} \, ds$$

$$+ \sum_{j=1}^d \bigg|\int_0^{t \wedge \tau_N} \sigma_{i,j}(X(s)) \, dW_j(s)\bigg|^p.$$

Denote $C_0 = (\sum_{i=1}^{d-1} \mathbb{E}[X_{i+1,i}(0)^{-p}]) e^{pT\|b\|_{\mathrm{Lip}}}$. From Lemma 3.4, Burkholder–Davis–Gundy's inequality and the boundedness of $\sigma_{i,j}$, by taking expectation,

$$\frac{\mathbb{E}[|X_i(t \wedge \tau_N)|^p]}{(2d+2)^{p-1}} \leq \mathbb{E}\big[|X_i(0)|^p\big] + |b_i(0)|^p t^p$$

$$+ \|b\|_{\mathrm{Lip}}^p t^{p-1} \int_0^t \mathbb{E}\big[|X_i(s \wedge \tau_N)|^p\big] \, ds$$

$$+ (d-1)t^p \gamma^p C_0 + c(p)dt^{p/2}\sigma_d^p.$$

It then follows from Gronwall's inequality that $\mathbb{E}[|X_i(t \wedge \tau_N)|^p]$ is bounded by

$$(2d+2)^{p-1}\big(\mathbb{E}[|X_i(0)|^p] + |b_i(0)|^p t^p + (d-1)t^p \gamma^p C_0 + c(p)dt^{p/2}\sigma_d^p\big)$$
$$\times e^{(2d+2)^{p-1}\|b\|_{\mathrm{Lip}}^p t^{p-1}}.$$

Letting $N \to \infty$, we obtain

$$\mathbb{E}[|X_i(t \wedge \tau)|^p] \le (2d+2)^{p-1}\big(\mathbb{E}[|X_i(0)|^p] + |b_i(0)|^p t^p$$
$$+ (d-1)t^p \gamma^p C_0 + c(p)dt^{p/2}\sigma_d^p\big)e^{(2d+2)^{p-1}\|b\|_{\mathrm{Lip}}^p t^{p-1}}.$$

This implies the assertion of Lemma 3.5. $\square$

The main result of this section reads as follows.

THEOREM 3.6.   *Suppose that Assumption* 3.1 *holds. Assume that* $\frac{3\gamma}{d\sigma_d^2} \ge 2$, $p \in [1, \frac{3\gamma}{d\sigma_d^2} - 1]$, $\mathbb{E}[|X(0)|^p] < \infty$ *and* $\mathbb{E}[X_{i+1,i}(0)^{-p}] < \infty$ *for each* $i = 1, \ldots, d-1$. *Then equation* (3.2) *has a unique strong solution* $X(t)$ *such that* $X(t) \in \Delta_d$ *almost surely for all* $t > 0$. *Moreover, for any* $T > 0$, *there exists a finite constant* C *such that for any* $0 \le s < t \le T$,

$$(3.5) \qquad\qquad \sup_{i=1,\ldots,d} \mathbb{E}[|X_i(t) - X_i(s)|^p] \le C(t-s)^{p/2}.$$

PROOF.   By applying Lemma 3.4 and Lemma 3.5 with $p = 1$, we deduce that $\tau = \infty$ which implies that the equation (3.2) has a unique global strong solution $X(t)$ whose values are in Δ_d for all $t > 0$.

Now we consider the second statement (3.5). For any $0 \le s < t \le T$,

$$\frac{|X_i(t) - X_i(s)|^p}{(2d+1)^{p-1}} \le |b_i(0)|^p (t-s)^p + \|b\|_{\mathrm{Lip}}^p (t-s)^{p-1} \int_s^t |X_i(u)|^p \, du$$
$$+ \sum_{j \ne i}(t-s)^{p-1}\int_s^t \frac{\gamma^p}{|X_{i,j}(u)|^p}\, du$$
$$+ \sum_{j=1}^d \left|\int_s^t \sigma_{i,j}(X(u))\, dW_j(u)\right|^p.$$

It follows from Lemma 3.4, estimate (3.4) and Burkholder–Davis–Gundy's inequality that

$$\mathbb{E}[|X_i(t) - X_i(s)|^p] \le C(t-s)^p + \sum_{j=1}^d c(p)\mathbb{E}\left[\left(\int_s^t \sigma_{i,j}^2(X(u))\, du\right)^{p/2}\right]$$
$$\le C(t-s)^{p/2}.$$

This concludes the second assertion (3.5). $\square$

REMARK 3.7.   Under the assumption of Theorem 3.6, it is straightforward to verify that the Hypothesis 2.7 holds with $\hat{p} = p$.

REMARK 3.8.   The existence and uniqueness of a noncolliding solution in this paper are established under stricter conditions on γ/σ^2 than in [4, 21] and [7]. Note that these papers only considered a particular case of systems (1.1) where each coordinate X_i is driven by its own Brownian motion (which is independent of the other ones) and $\sigma_{i,j}(X(t)) = \delta_{i,j}\sigma_i(X_i(t))$, where $\delta_{i,j}$ is the Dirac delta function. In fact, the later more restrictive condition allows to prove existence and uniqueness for a more general class of equations where the driving Brownian motions can be correlated. More importantly, that condition allows us

to obtain the moment estimate (3.5) which is the key to study the strong rate of convergence for the discrete approximation for equation (1.1).

3.2. *Brownian particles with nearest neighbor repulsion.* In this section, we consider the process $X = (X_1, \ldots, X_d)$ given by the following SDEs:

$$(3.6) \quad \begin{cases} dX_1(t) = \left\{ \dfrac{\gamma}{X_1(t) - X_2(t)} + b_1(X_1(t)) \right\} dt + \displaystyle\sum_{j=1}^{d} \sigma_{1,j}(X(t)) \, dW_j(t), \\[2mm] dX_i(t) = \left\{ \dfrac{\gamma}{X_i(t) - X_{i-1}(t)} + \dfrac{\gamma}{X_i(t) - X_{i+1}(t)} + b_i(X_i(t)) \right\} dt \\[2mm] \qquad\quad + \displaystyle\sum_{j=1}^{d} \sigma_{i,j}(X(t)) \, dW_j(t), \quad i = 2, \ldots, d-1, \\[2mm] dX_d(t) = \left\{ \dfrac{\gamma}{X_d(t) - X_{d-1}(t)} + b_d(X_d(t)) \right\} dt \\[2mm] \qquad\quad + \displaystyle\sum_{j=1}^{d} \sigma_{d,j}(X(t)) \, dW_j(t), \end{cases}$$

with $X(0) \in \Delta_d$. Let the Assumptions (A1)–(A4) hold. Since the coefficients of equation (3.6) are locally Lipschitz continuous in Δ_d, given $X(0) \in \Delta_d$, equation (3.6) has a unique strong local solution up to the stopping time τ defined by (3.1).

REMARK 3.9. These kinds of systems were studied in [7, 15, 22]. In particular, [7] considered the following SDEs:

$$\begin{cases} dX_1(t) = \dfrac{\gamma}{X_1(t) - X_2(t)} \, dt + \sigma_1(X_1(t)) \, dW_1(t), \\[2mm] dX_i(t) = \left\{ \dfrac{\gamma}{X_i(t) - X_{i-1}(t)} + \dfrac{\gamma}{X_i(t) - X_{i+1}(t)} \right\} dt \\[2mm] \qquad\quad + \sigma_i(X_i(t)) \, dW_i(t), \quad i = 2, \ldots, d-1, \\[2mm] dX_d(t) = \dfrac{\gamma}{X_d(t) - X_{d-1}(t)} \, dt + \sigma_d(X_d(t)) \, dW_d(t). \end{cases}$$

It is shown that the system has a unique strong solution with no collisions and no explosions if $d = 3$, $\gamma \geq \frac{3}{4}$ and $|\sigma_i| \leq 1$.

In the following, we apply the method introduced in the previous sections, which is essentially different from the one in [7], to study the existence, uniqueness, noncollision and nonexplosions of the solution to the general equation (3.6).

LEMMA 3.10. *For any $d \geq 3$ and $p \geq 0$, there exists a constant $\overline{\chi}(d, p) < 2$ such that*

$$\sum_{i=1}^{d-2} \left\{ \frac{1}{(x_{i+2} - x_{i+1})(x_{i+1} - x_i)^{p+1}} + \frac{1}{(x_{i+2} - x_{i+1})^{p+1}(x_{i+1} - x_i)} \right\}$$

$$\leq \overline{\chi}(d, p) \sum_{i=1}^{d-1} \frac{1}{(x_{i+1} - x_i)^{p+2}},$$

for any $(x_1, \ldots, x_d) \in \Delta_d$.

PROOF. Denote $L = \sum_{i=1}^{d-1} \frac{1}{(x_{i+1} - x_i)^{p+2}}$ and $\xi_i = \frac{1}{(x_{i+1} - x_i)L^{1/(p+2)}}$. We have $\sum_i \xi_i^{p+2} = 1$. Denote

$$\overline{\chi}(d, p) = \sup\left\{ \sum_{i=1}^{d-2} (\xi_{i+1}\xi_i^p + \xi_{i+1}^p\xi_i) : \sum_{i=1}^{d-1} \xi_i^{p+2} = 1 \right\}.$$

Since the set $S_{p,d-1}^+ := \{\xi = (\xi_1, \ldots, \xi_{d-1}) \in \mathbb{R}_+^{d-1} : \sum_{i=1}^{d-1} \xi_i^{p+2} = 1\}$ is a compact subset of $\mathbb{R}^{d-1}$, the supremum is attainable. On the other hand, for any $\xi \in S_{p,d-1}^+$,

$$2 - \sum_{i=1}^{d-2} (\xi_{i+1}\xi_i^{p+1} + \xi_{i+1}^{p+1}\xi_i) = 2 \sum_{i=1}^{d-1} \xi_i^{p+2} - \sum_{i=1}^{d-2} (\xi_{i+1}\xi_i^{p+1} + \xi_{i+1}^{p+1}\xi_i)$$

$$= \sum_{i=1}^{d-2} (\xi_i - \xi_{i+1})(\xi_i^{p+1} - \xi_{i+1}^{p+1}) + \xi_1^{p+2} + \xi_{d-1}^{p+2}.$$

The last term is strictly positive since for any nonnegative constants a and b the quantity $(a - b)(a^{p+1} - b^{p+1})$ is nonnegative and it equals to zero if and only if $a = b$. This implies the desired result. $\square$

We denote $X_{i+1,i}(t) = X_{i+1}(t) - X_i(t)$. Let τ_N be defined as in (3.3).

LEMMA 3.11.   *Suppose that Assumption* 3.1 *holds. Let p be a positive number satisfying* $\frac{\gamma}{2\sigma_d^2} \geq \frac{p+1}{2 - \overline{\chi}(d,p)}$. *Suppose that* $\mathbb{E}[X_{i+1,i}(0)^{-p}] < \infty$ *for each* $i = 1, \ldots, d - 1$. *Then for any* $T > 0$, *it holds that*

$$\sum_{i=1}^{d-1} \sup_{0 \leq t \leq T} \mathbb{E}[X_{i+1,i}(t \wedge \tau)^{-p}] \leq \left(\sum_{i=1}^{d-1} \mathbb{E}[X_{i+1,i}(0)^{-p}] \right) e^{pT \|b\|_{\text{Lip}}}.$$

In particular,

$$\sup_{i<j} \sup_{0 \leq t \leq T} \mathbb{E}[X_{j,i}(t \wedge \tau)^{-p}] \leq \left(\sum_{i=1}^{d-1} \mathbb{E}[X_{i+1,i}(0)^{-p}] \right) e^{pT \|b\|_{\text{Lip}}}.$$

PROOF.   By using Itô's formula, for each $2 \leq i \leq d - 2$, we have

$$X_{i+1,i}(t \wedge \tau_N)^{-p}$$

$$= X_{i+1,i}(0)^{-p}$$

$$+ \int_0^{t \wedge \tau_N} \frac{p\gamma}{X_{i+1,i}(s)^{p+1}} \left\{ \frac{-2}{X_{i+1,i}(s)} + \frac{1}{X_{i+2,i+1}(s)} + \frac{1}{X_{i,i-1}(s)} \right\} ds$$

$$- \int_0^{t \wedge \tau_N} \frac{p\{b_{i+1}(X_{i+1}(s)) - b_i(X_i(s))\}}{X_{i+1,i}(s)^{p+1}} ds$$

$$+ \sum_{k=1}^d \int_0^{t \wedge \tau_N} \frac{p(p+1)|\sigma_{i+1,k}(X(s)) - \sigma_{i,k}(X(s))|^2}{2 X_{i+1,i}(s)^{p+2}} ds$$

$$- \sum_{k=1}^d \int_0^{t \wedge \tau_N} \frac{p\{\sigma_{i+1,k}(X(s)) - \sigma_{i,k}(X(s))\}}{X_{i+1,i}(s)^{p+1}} dW_k(s).$$

In addition, we have

$$X_{2,1}(t \wedge \tau_N)^{-p}$$

$$= X_{2,1}(0)^{-p} + \int_0^{t \wedge \tau_N} \frac{p\gamma}{X_{2,1}(s)^{p+1}} \left\{ \frac{-2}{X_{2,1}(s)} + \frac{1}{X_{3,2}(s)} \right\} ds$$

$$- \int_0^{t \wedge \tau_N} \frac{p\{b_2(X_2(s)) - b_1(X_1(s))\}}{X_{2,1}(s)^{p+1}} ds$$

$$+ \sum_{k=1}^{d} \int_0^{t \wedge \tau_N} \frac{p(p+1)|\sigma_{2,k}(X(s)) - \sigma_{1,k}(X(s))|^2}{2X_{2,1}(s)^{p+2}} \, ds$$

$$- \sum_{k=1}^{d} \int_0^{t \wedge \tau_N} \frac{p\{\sigma_{2,k}(X(s)) - \sigma_{1,k}(X(s))\}}{X_{2,1}(s)^{p+1}} \, dW_k(s)$$

and

$$X_{d,d-1}(t \wedge \tau_N)^{-p}$$

$$= X_{d,d-1}(0)^{-p} + \int_0^{t \wedge \tau_N} \frac{p\gamma}{X_{d,d-1}(s)^{p+1}} \left\{ \frac{-2}{X_{d,d-1}(s)} + \frac{1}{X_{d-1,d-2}(s)} \right\} ds$$

$$- \int_0^{t \wedge \tau_N} \frac{p\{b_d(X_d(s)) - b_{d-1}(X_{d-1}(s))\}}{X_{d,d-1}(s)^{p+1}} \, ds$$

$$+ \sum_{k=1}^{d} \int_0^{t \wedge \tau_N} \frac{p(p+1)|\sigma_{d,k}(X(s)) - \sigma_{d-1,k}(X(s))|^2}{2X_{d,d-1}(s)^{p+2}} \, ds$$

$$- \sum_{k=1}^{d} \int_0^{t \wedge \tau_N} \frac{p\{\sigma_{d,k}(X(s)) - \sigma_{d-1,k}(X(s))\}}{X_{d,d-1}(s)^{p+1}} \, dW_k(s).$$

Since $\frac{\gamma}{d\sigma_d^2} \geq \frac{p+1}{2 - \overline{\chi}(d,p)}$, by applying Lemma 3.10 we obtain

$$\sum_{i=1}^{d-1} \mathbb{E}[X_{i+1,i}(t \wedge \tau_N)^{-p}]$$

$$\leq \sum_{i=1}^{d-1} \mathbb{E}[X_{i+1,i}(0)^{-p}] - \sum_{i=1}^{d-1} \mathbb{E}\left[\int_0^{t \wedge \tau_N} \frac{p\{b_{i+1}(X_{i+1}(s)) - b_i(X_i(s))\}}{X_{i+1,i}(s)^{p+1}} \, ds \right].$$

The proof is concluded by following the same argument as in the proof of Lemma 3.4. □

By using Lemma 3.11 and adapting the argument of the previous sections, we can show the following result.

THEOREM 3.12. *Suppose that Assumption 3.1 holds. Assume that there exists constant $p \geq 1$ such that $\frac{\gamma}{2\sigma_d^2} \geq \frac{p+1}{2 - \overline{\chi}(d,p)}$, $\mathbb{E}[|X(0)|^p] < \infty$ and $\mathbb{E}[X_{i+1,i}(0)^{-p}] < \infty$ for each $i = 1, \ldots, d - 1$. Then the equation (3.6) has a unique strong solution $X(t)$ such that $X(t) \in \Delta_d$ almost surely for all $t > 0$. Moreover, Hypothesis 2.7 holds for $\hat{p} = p$.*

APPENDIX

A.1. Numerical approximation for system of equation (2.1). In this section, we discuss how to approximate the solution of the system of equations (2.1). Denote $x_i = \xi_{i+1} - \xi_i$ and $a_i = b_{i+1} - b_i$. We can rewrite (2.1) as the following system of equations of variables x_i, $1 \leq i \leq d - 1$:

$$x_i = a_i + \frac{2c_{i,i+1}}{x_i} + \sum_{j<i} \left(\frac{c_{i+1,j}}{x_j + \cdots + x_i} - \frac{c_{i,j}}{x_j + \cdots + x_{i-1}} \right)$$

(A.1)

$$- \sum_{j>i+1} \left(\frac{c_{i+1,j}}{x_{i+1} + \cdots + x_{j-1}} - \frac{c_{i,j}}{x_i + \cdots + x_{j-1}} \right).$$

It is clear that there is a one-to-one correspondence between $(\xi_i)_{1 \le i \le d}$ and $(x_i)_{1 \le i \le d-1}$. Note that since both systems (2.1) and (A.1) are highly nonlinear and very stiff, it is very hard to find an effective numerical approximation scheme for them in the general case. In the following, we will construct an iterative scheme for the system of equations (A.1) and show its convergence in some particular cases.

We first consider the case that $c_{i,j} = 0$ for all i, j satisfying $|i - j| \ge 2$ which corresponds to the system of Brownian particles with nearest neighbor repulsion. We denote $c_{i,i+1} = c_i$ and $k = d - 1$ for the sake of simplicity.

PROPOSITION A.1.   *Let $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$ and $c_i > 0$ for all $i = 1, \ldots, k$. The following system of equations*:

(A.2)
$$\begin{cases} x_1 - \dfrac{2c_1}{x_1} = a_1 - \dfrac{c_2}{x_2}, \\ x_i - \dfrac{2c_i}{x_i} = a_i - \dfrac{c_{i-1}}{x_{i-1}} - \dfrac{c_{i+1}}{x_{i+1}}, \quad i = 2, \ldots, k-1, \\ x_k - \dfrac{2c_k}{x_k} = a_k - \dfrac{c_{k-1}}{x_{k-1}} \end{cases}$$

has a unique solution $(x_1^*, \ldots, x_k^*) \in \mathbb{R}_+^k$. Moreover, if we consider the sequence

$$x_i^{(0)} = \frac{1}{2}\left(a_i + \sqrt{a_i^2 + 8c_i}\right), \quad i = 1, \ldots, k,$$

$$\begin{cases} x_1^{(n+1)} = \dfrac{1}{2}\left(a_1 - \dfrac{c_2}{x_2^{(n)}} + \sqrt{\left(a_1 - \dfrac{c_2}{x_2^{(n)}}\right)^2 + 8c_1}\right), \\ x_i^{(n+1)} = \dfrac{1}{2}\left(a_i - \dfrac{c_{i-1}}{x_{i-1}^{(n)}} - \dfrac{c_{i+1}}{x_{i+1}^{(n)}} + \sqrt{\left(a_i - \dfrac{c_{i-1}}{x_{i-1}^{(n)}} - \dfrac{c_{i+1}}{x_{i+1}^{(n)}}\right)^2 + 8c_i}\right), \\ \quad i = 2, \ldots, k-1, n \ge 0, \\ x_k^{(n+1)} = \dfrac{1}{2}\left(a_k - \dfrac{c_{k-1}}{x_{k-1}^{(n)}} + \sqrt{\left(a_k - \dfrac{c_{k-1}}{x_{k-1}^{(n)}}\right)^2 + 8c_k}\right), \end{cases}$$

then for each $i = 1, \ldots, k$, the sequence $x_i^{(n)}$ decreases to x_i^* as n tends to infinity.

PROOF.   The existence and uniqueness of a solution to (A.2) is a direct consequence of Proposition 2.2. It is clear that $x_i^{(0)} - \frac{2c_i}{x_i^{(0)}} = a_i$, and

$$\begin{cases} x_1^{(n+1)} - \dfrac{2c_1}{x_1^{(n+1)}} = a_1 - \dfrac{c_2}{x_2^{(n)}}, \\ x_i^{(n+1)} - \dfrac{2c_i}{x_i^{(n+1)}} = a_i - \dfrac{c_{i-1}}{x_{i-1}^{(n)}} - \dfrac{c_{i+1}}{x_{i+1}^{(n)}}, \quad i = 2, \ldots, k-1, \\ x_k^{(n+1)} - \dfrac{2c_k}{x_k^{(n+1)}} = a_k - \dfrac{c_{k-1}}{x_{k-1}^{(n)}}. \end{cases}$$

Note that if $c > 0$ then the mapping $x \mapsto x - \frac{2c}{x}$ is strictly increasing on $(0, +\infty)$. Since $x_i^{(n+1)} - \frac{2c_i}{x_i^{(n+1)}} < a_i = x_i^{(0)} - \frac{2c_i}{x_i^{(0)}}$, we have $x_i^{(n+1)} < x_i^{(0)}$ for all $n \ge 0$, $1 \le i \le k$. In particular, we have $x_i^{(1)} < x_i^{(0)}$. Using the induction method, we obtain that for each i, the sequence

$(x_i^{(n)})_{n\geq 0}$ is a decreasing sequence of positive numbers. Indeed, suppose that $x_i^{(n+1)} < x_i^{(n)}$ for all $i = 1, 2, \ldots, k$. Then for any $i = 2, \ldots, k - 1$, it holds

$$\left(x_i^{(n+1)} - \frac{2c_i}{x_i^{(n+1)}}\right) - \left(x_i^{(n+2)} - \frac{2c_i}{x_i^{(n+2)}}\right)$$

$$= c_{i-1}\left(\frac{1}{x_{i-1}^{(n+1)}} - \frac{1}{x_{i-1}^{(n)}}\right) + c_{i+1}\left(\frac{1}{x_{i+1}^{(n+1)}} - \frac{1}{x_{i+1}^{(n)}}\right) > 0,$$

which implies $x_i^{(n+1)} > x_i^{(n+2)}$ for $i = 2, \ldots, k - 1$. A similar argument yields that $x_i^{(n+1)} > x_i^{(n+2)}$ for $i = 1$ and $i = k$ as well. Therefore, for each i, the sequence $(x_i^{(n)})_{n\geq 0}$ converges to the desired limits x_i^*. $\square$

Next, we consider the system (2.1) when $d = 3$.

PROPOSITION A.2. *Let $a, b \in \mathbb{R}$. The following system of equations:*

$$\begin{cases} x - \dfrac{2}{x} = a - \dfrac{1}{y} + \dfrac{1}{x+y}, \\ y - \dfrac{2}{y} = b - \dfrac{1}{x} + \dfrac{1}{x+y}, \end{cases}$$

has a unique solution $(x^*, y^*) \in \mathbb{R}_+^2$. Moreover, if we consider the sequence

$$x_1 = \frac{1}{2}(a + \sqrt{a^2 + 8}), \qquad y_1 = \frac{1}{2}\left(b - \frac{|a| + \sqrt{2}}{2} + \sqrt{\left(b - \frac{|a| + \sqrt{2}}{2}\right)^2 + 6}\right),$$

and

$$\begin{cases} x_{n+1} = \dfrac{1}{2}\left(a - \dfrac{1}{y_n} + \dfrac{1}{x_n + y_n} + \sqrt{\left(a - \dfrac{1}{y_n} + \dfrac{1}{x_n + y_n}\right)^2 + 8}\right), \\ y_{n+1} = \dfrac{1}{2}\left(b - \dfrac{1}{x_n} + \dfrac{1}{x_n + y_n} + \sqrt{\left(b - \dfrac{1}{x_n} + \dfrac{1}{x_n + y_n}\right)^2 + 8}\right). \end{cases}$$

Then

$$\lim x_n = x^* \quad and \quad \lim y_n = y^*.$$

PROOF. *Step 1*: It is clear that

$$\begin{cases} x_{n+1} - \dfrac{2}{x_{n+1}} = a - \dfrac{1}{y_n} + \dfrac{1}{x_n + y_n}, \\ y_{n+1} - \dfrac{2}{y_{n+1}} = b - \dfrac{1}{x_n} + \dfrac{1}{x_n + y_n}. \end{cases}$$

Since $x_{n+1} - \frac{2}{x_{n+1}} < a$ then $x_{n+1} < \frac{1}{2}(\sqrt{a^2 + 8} + a)$ for all $n \geq 1$. Similarly, $y_{n+1} < \frac{1}{2}(\sqrt{b^2 + 8} + b)$ for all $n \geq 1$.

Step 2: We show that $y_1 < y_3$. Indeed, we have

$$y_3 - \frac{2}{y_3} > b - \frac{1}{x_2}$$

$$= b - \frac{1}{4}\left(\sqrt{\left(a - \frac{1}{y_1} + \frac{1}{x_1 + y_1}\right)^2 + 8} - a + \frac{1}{y_1} - \frac{1}{x_1 + y_1}\right).$$

Applying the simple estimate, $\sqrt{a^2 + b^2} \leq |a| + |b|$, we get

$$\sqrt{\left(a - \frac{1}{y_1} + \frac{1}{x_1 + y_1}\right)^2 + 8} \leq |a| + \frac{1}{y_1} + \frac{1}{x_1 + y_1} + \sqrt{8}.$$

Therefore,

$$y_3 - \frac{2}{y_3} > b - \frac{1}{2y_1} - \frac{|a| + \sqrt{2}}{2} = y_1 - \frac{2}{y_1}.$$

This implies $y_3 > y_1$.

Step 3: Since the function $x \mapsto \frac{1}{2}(x + \sqrt{x^2 + 8})$ is strictly increasing on $\mathbb{R}$, for any $k \geq 0$, we have the following relation:

$$x_{k+2} < x_{k+4} \quad \Leftrightarrow \quad \frac{-1}{y_{k+1}} + \frac{1}{x_{k+1} + y_{k+1}} < \frac{-1}{y_{k+3}} + \frac{1}{x_{k+3} + y_{k+3}}$$

(A.3)

$$\Leftrightarrow \quad y_{k+1} + \frac{y_{k+1}^2}{x_{k+1}} < y_{k+3} + \frac{y_{k+3}^2}{x_{k+3}}.$$

Similarly, for any $k \geq 0$, we have

(A.4) $$y_{k+2} > y_{k+4} \quad \Leftrightarrow \quad x_{k+3} + \frac{x_{k+3}^2}{y_{k+3}} < x_{k+1} + \frac{x_{k+1}^2}{y_{k+1}}.$$

Since $x_1 > x_3 > 0$ and $y_3 > y_1 > 0$, it follows from the relations (A.3) and (A.4) with $k = 0$ that $x_2 < x_4$ and $y_2 > y_4$. Using the relations (A.3) and (A.4) again with $k = 1$ yields $x_3 > x_5$ and $y_3 < y_5$. By repeating this argument, we get

(A.5) $$\begin{cases} y_1 < y_3 < y_5 < y_7 < \cdots, \\ y_2 > y_4 > y_6 > y_8 > \cdots, \\ x_1 > x_3 > x_5 > x_7 > \cdots, \\ x_2 < x_4 < x_6 < x_8 < \cdots. \end{cases}$$

Step 4: It follows from Step 1 and (A.5) that the sequences (y_{2k+1}), (y_{2k}), (x_{2k+1}), (x_{2k}) converge to $\hat{y}_1$, $\hat{y}_2$, $\hat{x}_1$, $\hat{x}_2$, respectively. Moreover, $\hat{y}_1$, $\hat{y}_2$, $\hat{x}_1$, $\hat{x}_2$ satisfy

$$\begin{cases} \hat{y}_1 - \dfrac{2}{\hat{y}_1} = b - \dfrac{1}{\hat{x}_2} + \dfrac{1}{\hat{x}_2 + \hat{y}_2}, \\ \hat{y}_2 - \dfrac{2}{\hat{y}_2} = b - \dfrac{1}{\hat{x}_1} + \dfrac{1}{\hat{x}_1 + \hat{y}_1}, \\ \hat{x}_1 - \dfrac{2}{\hat{x}_1} = a - \dfrac{1}{\hat{y}_2} + \dfrac{1}{\hat{x}_2 + \hat{y}_2}, \\ \hat{x}_2 - \dfrac{2}{\hat{x}_2} = a - \dfrac{1}{\hat{y}_1} + \dfrac{1}{\hat{x}_1 + \hat{y}_1}. \end{cases}$$

The first two equations imply

$$(\hat{y}_2 - \hat{y}_1)\left(1 + \frac{2}{\hat{y}_1 \hat{y}_2} - \frac{1}{(\hat{x}_1 + \hat{y}_1)(\hat{x}_2 + \hat{y}_2)}\right)$$

$$= (\hat{x}_2 - \hat{x}_1)\left(\frac{1}{\hat{x}_1 \hat{x}_2} - \frac{1}{(\hat{x}_1 + \hat{y}_1)(\hat{x}_2 + \hat{y}_2)}\right),$$

while the last two equations imply

$$(\hat{x}_2 - \hat{x}_1)\left(1 + \frac{2}{\hat{x}_1\hat{x}_2} - \frac{1}{(\hat{x}_1 + \hat{y}_1)(\hat{x}_2 + \hat{y}_2)}\right)$$

$$= (\hat{y}_2 - \hat{y}_1)\left(\frac{1}{\hat{y}_1\hat{y}_2} - \frac{1}{(\hat{x}_1 + \hat{y}_1)(\hat{x}_2 + \hat{y}_2)}\right).$$

We show that $\hat{x}_1 = \hat{x}_2$ and $\hat{y}_1 = \hat{y}_2$ by contradiction method. Indeed, suppose that $\hat{x}_1 \neq \hat{x}_2$ then $\hat{y}_1 \neq \hat{y}_2$ and

$$\left(1 + \frac{2}{\hat{y}_1\hat{y}_2} - \frac{1}{(\hat{x}_1 + \hat{y}_1)(\hat{x}_2 + \hat{y}_2)}\right)\left(1 + \frac{2}{\hat{x}_1\hat{x}_2} - \frac{1}{(\hat{x}_1 + \hat{y}_1)(\hat{x}_2 + \hat{y}_2)}\right)$$

$$= \left(\frac{1}{\hat{x}_1\hat{x}_2} - \frac{1}{(\hat{x}_1 + \hat{y}_1)(\hat{x}_2 + \hat{y}_2)}\right)\left(\frac{1}{\hat{y}_1\hat{y}_2} - \frac{1}{(\hat{x}_1 + \hat{y}_1)(\hat{x}_2 + \hat{y}_2)}\right).$$

This is not true since the term on the right-hand side is always strictly greater than the one on the left-hand side. It means that $\hat{x}_1 = \hat{x}_2$ and $\hat{y}_1 = \hat{y}_2$. $\quad\square$

REMARK A.3. The iterative method in Proposition A.2 can be generalized for the case that $d \geq 4$, but that result, unfortunately, cannot be.

A.2. Proof of Lemma 3.3. For each $(x_1, \ldots, x_d) \in \Delta_d$, we denote

$$S_1 = \sum_{i=1}^{d-2} \sum_{k=i+2}^{d} \frac{1}{(x_{i+1} - x_i)^p (x_{i+1} - x_k)(x_i - x_k)},$$

$$S_2 = \sum_{i=2}^{d-1} \sum_{k=1}^{i-1} \frac{1}{(x_{i+1} - x_i)^p (x_{i+1} - x_k)(x_i - x_k)}.$$

Using Young's inequality,

$$\frac{p}{p+2}a^{p+2} + \frac{1}{p+2}b^{p+2} + \frac{1}{p+2}c^{p+2} \geq a^p bc, \quad a, b, c > 0,$$

we get

$$S_1 = \sum_{i=1}^{d-2} \sum_{k=i+2}^{d} \frac{1}{(k-i-1)(k-i)} \frac{1}{(x_{i+1} - x_i)^p \frac{x_k - x_{i+1}}{k-i-1} \frac{x_k - x_i}{k-i}}$$

$$\leq \sum_{i=1}^{d-2} \sum_{k=i+2}^{d} \frac{1}{(k-i-1)(k-i)} \left\{ \frac{p}{p+2} \frac{1}{(x_{i+1} - x_i)^{p+2}} \right.$$

$$\left. + \frac{1}{p+2} \frac{1}{(\frac{x_k - x_{i+1}}{k-i-1})^{p+2}} + \frac{1}{p+2} \frac{1}{(\frac{x_k - x_i}{k-i})^{p+2}} \right\}.$$

Next, using the convexity of the function $a \mapsto a^{-(p+2)}$, we have the following estimate:

$$(A.6) \qquad \frac{1}{(\frac{a_1 + \cdots + a_k}{k})^{p+2}} \leq \frac{1}{k}\left(\frac{1}{a_1^{p+2}} + \cdots + \frac{1}{a_k^{p+2}}\right), \quad k \geq 1, a_1, \ldots, a_k > 0.$$

Since $x_k - x_{i+1} = \sum_{j=i+1}^{k-1}(x_{j+1} - x_j)$ and $x_k - x_i = \sum_{j=i}^{k-1}(x_{j+1} - x_j)$, by applying the inequality (A.6), we get

$$S_1 \leq \sum_{i=1}^{d-2} \sum_{k=i+2}^{d} \frac{1}{(k-i-1)(k-i)} \left\{ \frac{p}{p+2} \frac{1}{(x_{i+1} - x_i)^{p+2}} \right.$$

$$+ \frac{1}{p+2} \frac{1}{k-i-1} \sum_{j=i+1}^{k-1} \frac{1}{(x_{j+1}-x_j)^{p+2}}$$

$$+ \frac{1}{p+2} \frac{1}{k-i} \sum_{j=i}^{k-1} \frac{1}{(x_{j+1}-x_j)^{p+2}} \biggr\}$$

$$= S_{11} + S_{12},$$

where

$$S_{11} = \sum_{i=1}^{d-2} \sum_{k=i+2}^{d} \biggl\{ \frac{p}{p+2} \frac{1}{(k-i-1)(k-i)} + \frac{1}{p+2} \frac{1}{(k-i-1)(k-i)^2} \biggr\}$$

$$\times \frac{1}{(x_{i+1}-x_i)^{p+2}},$$

$$S_{12} = \frac{1}{p+2} \sum_{i=1}^{d-2} \sum_{k=i+2}^{d} \frac{1}{(k-i)(k-i-1)} \biggl(\frac{1}{k-i-1} + \frac{1}{k-i} \biggr)$$

$$\times \sum_{j=i+1}^{k-1} \frac{1}{(x_{j+1}-x_j)^{p+2}}.$$

We have

$$S_{11} = \sum_{i=1}^{d-2} \biggl\{ \frac{p}{p+2} \sum_{k=i+2}^{d} \biggl(\frac{1}{k-i-1} - \frac{1}{k-i} \biggr)$$

$$+ \frac{1}{p+2} \sum_{k=i+2}^{d} \frac{1}{(k-i-1)(k-i)^2} \biggr\} \frac{1}{(x_{i+1}-x_i)^{p+2}}$$

$$= \sum_{i=1}^{d-2} \biggl\{ \frac{p}{p+2} \biggl(1 - \frac{1}{d-i} \biggr) + \frac{1}{p+2} \sum_{k=1}^{d-i-1} \frac{1}{k(k+1)^2} \biggr\} \frac{1}{(x_{i+1}-x_i)^{p+2}}.$$

Since $\{(i,j,k) \in \mathbb{N}^3 : 1 \le i \le d-2, i+2 \le k \le d, i+1 \le j \le k-1\} = \{(i,j,k) \in \mathbb{N}^3 : 2 \le j \le d-1, 1 \le i \le j-1, j+1 \le k \le d\}$, we can rewrite S_{12} as

$$S_{12} = \frac{1}{p+2} \sum_{j=2}^{d-1} \sum_{i=1}^{j-1} \sum_{k=j+1}^{d} \frac{1}{(k-i)(k-i-1)} \biggl(\frac{1}{k-i-1} + \frac{1}{k-i} \biggr)$$

$$\times \frac{1}{(x_{j+1}-x_j)^{p+2}}$$

$$= \frac{1}{p+2} \sum_{j=2}^{d-1} \sum_{i=1}^{j-1} \sum_{k=j+1}^{d} \biggl(\frac{1}{(k-i-1)^2} - \frac{1}{(k-i)^2} \biggr) \frac{1}{(x_{j+1}-x_j)^{p+2}}$$

$$= \frac{1}{p+2} \sum_{j=2}^{d-1} \sum_{i=1}^{j-1} \biggl(\frac{1}{(j-i)^2} - \frac{1}{(d-i)^2} \biggr) \frac{1}{(x_{j+1}-x_j)^{p+2}}$$

$$= \frac{1}{p+2} \sum_{j=2}^{d-1} \left(\sum_{k=1}^{j-1} \frac{1}{k^2} - \sum_{k=d-j+1}^{d-1} \frac{1}{k^2} \right) \frac{1}{(x_{j+1} - x_j)^{p+2}}$$

$$= \frac{1}{p+2} \sum_{i=2}^{d-1} \left(\sum_{k=1}^{i-1} \frac{1}{k^2} - \sum_{k=d-i+1}^{d-1} \frac{1}{k^2} \right) \frac{1}{(x_{i+1} - x_i)^{p+2}},$$

where we replace the index j by i at the last equality. Therefore,

$$S_1 \le \sum_{i=1}^{d-1} \left\{ \frac{p}{p+2} \left(1 - \frac{1}{d-i} \right) \right.$$

$$\left. + \frac{1}{p+2} \left(\sum_{k=1}^{d-i-1} \frac{1}{k(k+1)^2} + \sum_{k=1}^{i-1} \frac{1}{k^2} - \sum_{k=d-i+1}^{d-1} \frac{1}{k^2} \right) \right\} \frac{1}{(x_{i+1} - x_i)^{p+2}},$$

where we shall use from now on the convention that $\sum_{i=m}^{n} a_i = 0$ if $m > n$. By following a similar argument, we can bound S_2 as

$$S_2 \le \sum_{i=1}^{d-1} \left\{ \frac{p}{p+2} \left(1 - \frac{1}{i} \right) \right.$$

$$\left. + \frac{1}{p+2} \left(\sum_{k=1}^{i-1} \frac{1}{k(k+1)^2} + \sum_{k=1}^{d-i-1} \frac{1}{k^2} - \sum_{k=i+1}^{d-1} \frac{1}{k^2} \right) \right\} \frac{1}{(x_{i+1} - x_i)^{p+2}}.$$

Therefore,

$$(A.7) \qquad S_1 + S_2 \le \sum_{i=1}^{d-1} \frac{\varphi_i}{(x_{i+1} - x_i)^{p+2}},$$

where φ_i is defined by

$$\varphi_i = \frac{p}{p+2} \left(2 - \frac{1}{i} - \frac{1}{d-i} \right)$$

$$+ \frac{1}{p+2} \left\{ \sum_{k=1}^{i-1} \left(\frac{1}{k^2} + \frac{1}{k(k+1)^2} \right) + \sum_{k=1}^{d-i-1} \left(\frac{1}{k^2} + \frac{1}{k(k+1)^2} \right) \right.$$

$$\left. - \sum_{k=i+1}^{d-1} \frac{1}{k^2} - \sum_{k=d-i+1}^{d-1} \frac{1}{k^2} \right\}.$$

From the fact that $\sum_{k=1}^{n} \{ \frac{1}{k^2} + \frac{1}{k(k+1)^2} \} = 2 - \frac{1}{n+1} - \frac{1}{(n+1)^2}$, we have

$$\varphi_i = \frac{p}{p+2} \left(2 - \frac{1}{i} - \frac{1}{d-i} \right) + \frac{1}{p+2} \left\{ 4 - \frac{1}{i} - \frac{1}{d-i} - \sum_{k=i}^{d-1} \frac{1}{k^2} - \sum_{k=d-i}^{d-1} \frac{1}{k^2} \right\}$$

$$\le \frac{p}{p+2} \left(2 - \frac{1}{i} - \frac{1}{d-i} \right)$$

$$+ \frac{1}{p+2} \left\{ 4 - \frac{1}{i} - \frac{1}{d-i} - \sum_{k=i}^{d-1} \frac{1}{k(k+1)} - \sum_{k=d-i}^{d-1} \frac{1}{k(k+1)} \right\}$$

$$= \frac{p}{p+2} \left(2 - \frac{1}{i} - \frac{1}{d-i} \right) + \frac{1}{p+2} \left\{ 4 - \frac{2}{i} - \frac{2}{d-i} + \frac{2}{d} \right\}.$$

By using the estimate $\frac{1}{i} + \frac{1}{d-i} \geq \frac{4}{d}$, we get

$$\varphi_i \leq 2 - \frac{3}{d}, \quad i = 1, \ldots, d-1.$$

This estimate together with (A.7) implies the desired result.

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