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CONTINUOUS-TIME DUALITY FOR SUPERREPLICATION WITH TRANSIENT PRICE IMPACT

BY PETER BANK* AND YAN DOLINSKY†,‡,[1]

*Technische Universität Berlin**, Hebrew University*† *and Monash University*‡

We establish a superreplication duality in a continuous-time financial model as in (Bank and Voß (2018)) where an investor's trades adversely affect bid- and ask-prices for a risky asset and where market resilience drives the resulting spread back towards zero at an exponential rate. Similar to the literature on models with a constant spread (cf., e.g., *Math. Finance* **6** (1996) 133–165; *Ann. Appl. Probab.* **20** (2010) 1341–1358; *Ann. Appl. Probab.* **27** (2017) 1414–1451), our dual description of superreplication prices involves the construction of suitable absolutely continuous measures with martingales close to the unaffected reference price. A novel feature in our duality is a liquidity weighted L^2-norm that enters as a measurement of this closeness and that accounts for strategy dependent spreads. As applications, we establish optimality of buy-and-hold strategies for the superreplication of call options and we prove a verification theorem for utility maximizing investment strategies.

1. Introduction. Financial models with transaction costs have been a great source of intriguing challenges for stochastic analysis and control theory. Starting with [17, 20, 43] strong emphasis has been put on the singular control problems that emerge in models with a constant spread. The duality theory for these models is now developed in great detail (see [14, 17, 28, 32–34]). This has been used to study utility maximization via its relation to shadow prices [9, 18, 19, 24, 35] and has also been instrumental in the development of asymptotic approaches for small transaction costs ([42] and the references therein).

While convenient mathematically, the assumption of a constant spread is justified only for very liquid assets. Less liquid assets will have a spread that widens when a large transaction is being executed and, upon completion of the transaction, the spread will decrease again due to market resilience. This is well known in the order execution literature [23, 39] where one derives optimal schedules for unwinding large positions that account for such (at least partially) transient price impact.

Following the approach proposed in [8], we introduce a model with transient price impact that allows for impact from both buying and selling a risky asset. We

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even allow for stochastic market depth and resilience that merely have to satisfy a certain monotonicity assumption required to obtain convex wealth dynamics. Instead of the utility maximization problem of interest in [8], we focus here on the fundamental problem of superreplicating an arbitrary contingent claim in a cost optimal way. For models with a constant spread, the duality theory of this problem is well understood in terms of consistent price systems that are based on the construction of measures with martingales that do not deviate from the asset price by more than the exogenously given spread; see [41] and the reference therein. This structure is recovered here but, due to the endogenous nature of our spreads, we also have to optimally determine these. Our main result, Theorem 3.2, shows how to suitably penalize possible choices by a liquidity-dependent L^2-norm, characterizing the superreplication costs in the form of a convex risk measure. An interesting point to observe is that, contrary to the models with exogenous spread, our model does not require any notion of admissibility for our trading strategies. As already observed in a model with purely temporary price impact in [26], this is due to the impossibility to scale strategies at will since such scaling incurs superlinearly growing costs.

The proof of this result rests on a particularly convenient expression for the terminal wealth resulting from a strategy that also reveals the convexity of this functional. As usual, a lower bound on superreplication costs is comparably easy to obtain given the consistent price system structure imposed by our dual variables. The proof of absence of a duality gap, that is, establishing an upper bound is more involved. The first step is a rather standard separation argument (Lemma 4.1) which gives us a suitable pricing measure. As a second step, we introduce the martingale of the consistent price system as a Lagrange multiplier enforcing the terminal liquidation constraint (Lemma 4.2). The crucial third step is the construction of a suitable spread and the identification of its liquidity dependent L^2-norm as the correct penalty term for our duality (Lemma 4.3). This is made possible by applying a stochastic representation theorem from [4] which so far was used only in connection with one-sided singular control problems [6, 7, 15, 22] and here finds its first application in a two-sided control problem with bounded variation rather than increasing controls.

As an application, we show that also in our transient price impact model the best way to superreplicate a call option is, under natural conditions, to buy and hold the asset until maturity. This is in line with results on models with exogenous spread; cf. [13, 27, 30, 37, 38, 44]. We also provide a verification result for identifying utility maximizing strategies by the construction of suitable shadow prices similar to results with fixed spread [18, 35] and to a result with purely temporary price impact [26].

2. Trading in a transient price impact model. We consider a financial model with transient price impact similar to [8]. Specifically, we consider a "large" investor who can invest in a riskless savings account bearing zero interest (for simplicity) and whose trades into and out of a risky asset move bid- and ask prices that, in addition, are also driven by some exogenous noise. This noise will be specified by a continuous, adapted process $P = (P_t)_{t\geq 0}$ on a filtered probability space $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ where $\mathscr{F}_0$ is generated by the $\mathbb{P}$-null sets. We will assume that the filtration is continuous.

ASSUMPTION 2.1. All $(\mathscr{F}_t)_{t\geq 0}$-martingales have a continuous version.

REMARK 2.2. This assumption is satisfied, for example, if $(\mathscr{F}_t)_{t\geq 0}$ is generated by a Brownian motion. It rules out complete surprises as generated, for instance, by the jumps of Poisson processes. The assumption ensures that there will not be any common jumps of trading policies and our martingale prices to be introduced later and it will allow us to apply a stochastic representation theorem from [4] which is key for our analysis. From the duality theory of proportional transaction costs (see in particular [18]), it is known that exogenous jumps lead to the need for làdlàg strategies and a considerably more delicate analysis which, in our context for strategy-dependent spreads, we have to leave for future research. Moreover, jumps, also by P or the market depth process δ to be introduced shortly, would pose the challenge to specify what information on the jump is available when to the investor and how he can act on it. While certainly relevant from a financial-economic point of view, these questions are also beyond the scope of the present paper.

The large investor's trading strategy is described by his given initial holdings $x_0 \in \mathbb{R}$ and a right-continuous, predictable process $X = (X_t)_{t\geq 0}$ of bounded variation specifying the number of risky assets held at any time. We denote by $X^{\uparrow}$ and $X^{\downarrow}$ the right-continuous predictable increasing and decreasing part resulting from the Hahn-decomposition of

$$X_t = x_0 + X_t^{\uparrow} - X_t^{\downarrow}, \quad t \geq 0, \qquad X_{0-} \triangleq x_0, \qquad X_{0-}^{\uparrow} \triangleq X_{0-}^{\downarrow} \triangleq 0.$$

The set of all such strategies will be denoted by $\mathscr{X}$.

Trades will permanently affect the mid-price P^X which, in line with [29], we let take the linear form

$$P_t^X \triangleq P_t + \iota X_t, \quad t \geq 0, \qquad P_{0-}^X \triangleq P_0 + \iota x_0,$$

for some impact parameter $\iota \geq 0$. Trades will in addition drive bid- and ask-prices away from the mid-price. Without further interventions, market resilience lets bid- and ask-prices then gradually revert toward the mid-price. We model this by letting the half-spread follow the dynamics

$$d\zeta_t^X = \frac{1}{\delta_t}\left(dX_t^{\uparrow} + dX_t^{\downarrow}\right) - r_t \zeta_t^X\, dt, \quad \zeta_{0-}^X \triangleq \zeta_0, \tag{2.1}$$

for a given initial value $\zeta_0 \geq 0$ and a given market depth process δ and resilience rate r.

REMARK 2.3. One way to interpret these spread dynamics is to think of trades eating into their respective side of the limit order book, widening the spread to an extent which depends on the current order book height δ_t while the market's resilience ensures that, without further trades, the spread will diminish at the exponential rate r_t. For simplicity, we assume the order book height at any time to be constant across ticks and identical for the ask- and the bid-side. More flexible nonlinear spread dynamics as in [1, 40] are conceivable but beyond the scope of the present paper. Note also that the mesoscopic time-scale underlying our model does not allow us to accommodate all the market microstructure effects so crucial for high-frequency trading, but instead suggests to view our model's market depth and resilience processes also as mesoscopic specifications of these market characteristics that would in practice need to be calibrated, for example, to moving averages of order book heights and order flow dynamics accounting for both limit and market orders; see [16] for an empirical study in this vein that also supports linear price impact specifications as in our stylized model.

We will require the following regularity of market depth δ and resilience rate r.

ASSUMPTION 2.4. The market depth $\delta = (\delta_t)_{t\geq 0} > 0$ is continuous and adapted. The resilience rate $r = (r_t)_{t\geq 0} \geq 0$ is predictable and such that δ and ρ are bounded away from zero and infinity where

$$\rho_t \triangleq \exp\left(\int_0^t r_s\,ds\right), \quad t \geq 0.$$

Moreover, the resilience rate dominates the changes in market depth in the sense that

$$\kappa_t \triangleq \delta_t/\rho_t^2 \text{ is strictly decreasing in } t \geq 0. \tag{2.2}$$

REMARK 2.5. As will become apparent in Lemma 4.1, Condition (2.2) is needed to ensure that the wealth dynamics are convex. When δ is absolutely continuous it amounts to the requirement

$$\frac{1}{2}\frac{d}{dt}\log\delta_t < r_t, \quad t \geq 0,$$

that is, relative changes in the market's depth have to be dominated by the market's resilience. In particular, Assumption 2.4 holds when δ and r are strictly positive constants and we confine ourselves to a finite trading period $[0, T]$ as is natural for the superreplication duality discussed shortly. The question whether one can develop a duality theory without this assumption we have to leave for future research; see, however, [5] for considerations in this direction in a deterministic order execution problem.

By time $T \in (0, \infty)$, the induced investor's cash position will have evolved from its given initial value $\xi_0 \in \mathbb{R}$ to ξ_T^X as determined by the profits and losses made from trading in and out of the risky asset. These trades are executed half the spread away from the mid-price P^X and so the terminal cash position is

$$\xi_T^X \triangleq \xi_0 - \int_{[0,T]} P_t^X \circ dX_t - \int_{[0,T]} \zeta_t^X \circ d\big(X_t^{\uparrow} + X_t^{\downarrow}\big). \tag{2.3}$$

REMARK 2.6. The above $\circ$-integrals are understood in the sense that for two RCLL processes X, Y with X of bounded variation, we let

$$\int_{[0,T]} Y_t \circ dX_t \triangleq \int_{[0,T]} \frac{1}{2}(Y_{t-} + Y_{t+})\, dX_t,$$

where on the right-hand side we have a standard Lebesgue-integral with respect to the signed measure dX. In (2.3), this way of integrating accounts for the fact that, when buying assets in a bulk $\Delta X_t^{\uparrow} > 0$, both the mid-pricee P^X and the half-spread ζ^X will increase only gradually during the order execution, letting the investor effectively trade at the average between pre- and post-transaction mid-price and the average between pre- and post-transaction half spread. We refer to [39] for similar considerations in an order execution framework. Alternatively, it is possible to consider $\int Y \circ dX$ as a Marcus integral for our controlled system. For our linear impact specification, this amounts to the Stratonovich-like integral (2.3); cf. [11] and the references therein.

3. Duality for superreplication of contingent claims. Let us now consider the classical superreplication problem for a cash-settled European contingent claim with $\mathscr{F}_T$-measurable payoff $H \geq 0$ at time $T \geq 0$ that is *not* affected by the large investor's trades, but exogenously given, for instance, as a functional of the given unaffected price process P. We will give a dual description of such an exogenous payoff's superreplication costs

$$\pi(H) \triangleq \inf\big\{\xi_0 \in \mathbb{R} : \xi_T^X \geq H \text{ for some } X \in \mathscr{X} \text{ with } X_T = 0\big\}. \tag{3.1}$$

REMARK 3.1. We have to confine ourselves to claims whose payoff are not affected by the large investor because we have to preserve the convexity of the superreplication problem. Of course, pricing and hedging claims with payoffs that can be affected (or even manipulated) by the large investor is a practically (and in the aftermath possibly judicially) most relevant problem. But this would require a rather product-specific analysis and is thus beyond the scope of this duality paper. See, however, the PDE approaches in, for example, [10, 12] as well as [3, 31] for some results in this direction. Note also that for a vanilla option, whose payoff only depends on the terminal mid-price at time T, the liquidation constraint $X_T = 0$ ensures that $P_T^X = P_T$ and thus prevents any manipulation possibilities, making our duality result below applicable to these products.

On the dual side of our description of superreplication costs, the market frictions will be captured by the optional random measure μ that, under Assumption 2.4, is induced by the continuous increasing process $-\kappa = -\delta/\rho^2$ on $(0, T)$ with point mass $\kappa_T = \delta_T/\rho_T^2$ in T:

$$\mu(dt) \triangleq 1_{(0,T)}(t)|d\kappa_t| + \kappa_T \mathrm{Dirac}_T(dt). \tag{3.2}$$

With this notation, we can now formulate our main result.

THEOREM 3.2. *Under Assumptions 2.1 and 2.4, the superreplication costs (3.1) of a contingent claim $H \geq 0$ have the dual description*

$$\pi(H) = \sup_{(\mathbb{Q},M,\alpha)} \left\{ \mathbb{E}_{\mathbb{Q}}[H] - \frac{1}{2}\|\alpha - \zeta_0\|^2_{L^2(\mathbb{Q}\otimes\mu)} - M_0 x_0 - \frac{1}{2}\iota x_0^2 \right\} > -\infty, \tag{3.3}$$

where the supremum is taken over all triples $(\mathbb{Q}, M, \alpha)$ of probability measures $\mathbb{Q} \ll \mathbb{P}$ on $\mathscr{F}_T$, martingales $M \in \mathscr{M}^2(\mathbb{Q})$ and all optional $\alpha \in L^2(\mathbb{Q} \otimes \mu)$ which control the fluctuations of P in the sense that

$$|P_t - M_t| \leq \frac{\rho_t}{\delta_t} \mathbb{E}_{\mathbb{Q}}\left[\int_{[t,T]} \alpha_u \mu(du) \Big| \mathscr{F}_t \right], \quad 0 \leq t \leq T. \tag{3.4}$$

3.1. *Comparison with other superreplication duality formulae.* Let us discuss the above duality formula for superreplication prices by comparing it with other such dualities obtained in different financial models.

First, the supremum on the right-hand side of (3.3) includes all measures $\mathbb{Q} \ll \mathbb{P}$ for which P is a square-integrable martingale (if there are any). For these, one can choose $M = P$ and $\alpha = \zeta_0$ to satisfy the constraint (3.4) and obtain that $\pi(H) \geq \mathbb{E}_{\mathbb{Q}}[H] - x_0 P_0$ when ignoring permanent impact ($\iota = 0$). This inequality is clearly in line with the classical frictionless super-replication duality. (Notice that the value of the initial position $x_0 P_0$ is subtracted here because $\pi(H)$ in (3.2) describes the superreplication costs in cash required when starting with a position of x_0 in the risky asset.)

Let us next turn to models with transaction costs arising from a fixed spread. Adjusting the multiplicative settings considered in [14, 17, 18] to an additive one as considered here leads to consistent price systems given by $\mathbb{P}$-martingales (Z^0, Z^1) with $Z^0_0 = 1$, $Z^0_T > 0$ such that the distance of $M \triangleq Z^1/Z^0$ to P is dominated by the (constant for simplicity) half-spread λ which one has to pay on top of P when buying and which is subtracted from the proceeds when selling a unit of the risky asset. One can then define $\mathbb{Q}$ by $d\mathbb{Q}/d\mathbb{P} \triangleq Z^0_T$ and put $\alpha \triangleq \lambda$ to obtain a triple $(\mathbb{Q}, M, \alpha)$ as required by our duality formula, for example, in any model with zero resilience ($r = 0$, $\rho = 1$) and initial spread $\zeta_0 = \lambda$. Indeed, (3.4) does hold for any market depth $\delta > 0$ (which has to be decreasing to meet Assumption 2.4) since then

$$\frac{\rho_t}{\delta_t} \mathbb{E}_{\mathbb{Q}}\left[\int_{[t,T]} \alpha_u \mu(du) \Big| \mathscr{F}_t \right] = \frac{\rho_t}{\delta_t} \lambda \mathbb{E}_{\mathbb{Q}}\left[\mu\big([t,T]\big) | \mathscr{F}_t\right] = \frac{\rho_t}{\delta_t} \lambda \kappa_t = \lambda.$$

As a result, ignoring possible permanent impact ($\iota = 0$), $\pi(H) \geq \mathbb{E}_{\mathbb{Q}}[H] - Z_0^1 x_0$, in line with the superreplication results for models with fixed spread.

Observe that, contrary to these models, our setting with spread impact does not require any notion of admissibility for trading strategies. Also, in our model we have, regardless of the initial position x_0, that $\pi(0) > -\infty$ for *any* choice of (continuous) price process P. Hence, even for specifications allowing for the most egregious arbitrage in a fixed-spread model (let alone in a frictionless one), there is no way to reach zero terminal wealth from arbitrarily low initial cash positions. This is due to the fact that scaling favorable strategies ultimately turns these unfavorable as transaction costs effectively grow quadratically when scaling a strategy, not just linearly as in any setting with a fixed spread. This effect has been observed in an Almgren–Chriss [2]-style model with temporary rather than transient market impact in [26]. Like our superreplication cost formula, theirs takes the form of a convex risk measure rather than a coherent one as found for the fixed spread models. This is again due to the nonlinear scaling of transaction costs.

3.2. *Applications*. To illustrate the usefulness of the above duality result, let us derive in this section the superreplication costs of a call option and show how to verify optimality of a proposed investment strategy.

3.2.1. *Superreplicating call options*. As a first application of our superreplication duality, let us verify that also in our model with strategy-dependent spread, buy-and-hold is the best way to superreplicate a call option

$$H = (P_T - k)^+ \quad \text{with } k \geq 0,$$

at least if liquidity coefficients are deterministic and if the unaffected price P satisfies the conditional full-support property (see [27])

$$\operatorname{supp} \mathbb{P}\big[(P_u)_{t \leq u \leq T} \in \cdot \,|\, \mathscr{F}_t\big] = C_{P_t}\big([t, T], \mathbb{R}_+\big), \quad 0 \leq t \leq T, \tag{3.5}$$

where, for $p \geq 0$, $C_p([t, T], \mathbb{R}_+)$ denotes the class of continuous, nonnegative functions f on $[t, T]$ with $f(t) = p$.

COROLLARY 3.3. *Let Assumption 2.1 hold true and let market depth and resilience be deterministic and satisfy Assumption 2.4. In addition, suppose P is strictly positive with the conditional full support property (3.5). Then, for an investor with initial position $x_0 \leq 1$, the superreplication cost of a cash-settled call option is*

$$\begin{aligned} \pi\big((P_T - k)^+\big) = P_0(1 - x_0) - \frac{1}{2}\iota x_0^2 + \zeta_0(1 - x_0) + \frac{(1 - x_0)^2}{2\delta_0} \\ + \frac{\zeta_0 + (1 - x_0)/\delta_0}{\rho_T} + \frac{1}{2\delta_T} \end{aligned} \tag{3.6}$$

and it is attained by holding one unit of the risky asset over $[0, T)$ to be sold at time T.

PROOF. Let us consider the strategy that immediately takes its position in the risky asset to one unit and keeps it there until unwinding it in the end:

$$\widehat{X}^{\uparrow} \triangleq (1-x_0)1_{[0,T]}, \qquad \widehat{X}^{\downarrow} \triangleq 1_{\{T\}}, \qquad \widehat{X} = 1_{[0,T)}.$$

When starting with the cash position ξ_0 given by the right-hand side of (3.6) this leads by (2.3) to the terminal wealth

$$\xi_T^{\widehat{X}} = P_T \geq (P_T - k)^+ = H.$$

Here, the estimate holds true as P is nonnegative. So the right-hand side of (3.6) is sufficient initial cash to superreplicate the call.

We will use our duality formula from Theorem 3.2 to show that $\varepsilon > 0$ less than this amount is not sufficient. To this end, we choose

$$\alpha_t \triangleq \zeta_0 + \frac{1-x_0}{\delta_0} + \frac{\rho_T}{\delta_T} 1_{\{T\}}(t), \quad 0 \leq t \leq T.$$

Clearly, there exists a Lipschitz continuous, nonincreasing deterministic function $g : [0,T] \to \mathbb{R}$ with

$$\begin{aligned}
g_0 &= \frac{\int_{[0,T]} \alpha_u \mu(du)}{\delta_0}, \\
g_T &= -\frac{\alpha_T}{\rho_T}, \\
|g_t| &\leq \frac{\rho_t}{\delta_t} \int_{[t,T]} \alpha_u \mu(du), \quad 0 \leq t \leq T.
\end{aligned}$$

Lemma 3.4 below yields a probability measure $\mathbb{Q} \ll \mathbb{P}$ with $\mathbb{Q}(P_T > \varepsilon) < \varepsilon$ and a square integrable $\mathbb{Q}$-martingale M such that

$$|g_t + P_t - M_t| < \varepsilon \inf_{0 \leq t \leq T} \frac{\rho_t}{\delta_t} \mu\bigl([t,T]\bigr), \quad 0 \leq t \leq T.$$

Hence, the triple $(\mathbb{Q}, M, \alpha + \varepsilon)$ is as requested by our duality Theorem 3.2. Using the simple inequality $P_T \leq (P_T - k)^+ + \varepsilon + k 1_{\{P_T > \varepsilon\}}$, we thus obtain

$$\begin{aligned}
\pi(H) &\geq \mathbb{E}_{\mathbb{Q}}[P_T - M_T] + M_0 - \mathbb{E}_{\mathbb{Q}}\bigl[P_T - (P_T - k)^+\bigr] \\
&\quad - \frac{1}{2}\|\alpha - \zeta_0\|^2_{L^2(\mathbb{Q}\otimes\mu)} - M_0 x_0 - \frac{1}{2}\iota x_0^2 \\
&\geq \frac{\alpha_T}{\rho_T} + (1-x_0)P_0 + (1-x_0)\frac{\int_{[0,T]} \alpha_u \mu(du)}{\delta_0} \\
&\quad - \frac{1}{2}\int_{[0,T]} |\alpha_u - \zeta_0|^2 \, d\mu(u) - \frac{1}{2}\iota x_0^2 - O(\varepsilon).
\end{aligned}$$

The result follows by using $\mu([0,T]) = \delta_0$ and taking $\varepsilon \downarrow 0$. $\square$

LEMMA 3.4. *Suppose $P > 0$ exhibits the conditional full support property (3.5) and let $g : [0, T] \to \mathbb{R}$ be Lipschitz-continuous and nonincreasing. Then, for any $\varepsilon > 0$, there is a probability measure $\mathbb{Q} \ll \mathbb{P}$ and a square-integrable $\mathbb{Q}$-martingale M such that $\mathbb{Q}$-almost surely*

$$|g_t + P_t - M_t| < \varepsilon, \quad 0 \le t \le T \tag{3.7}$$

and

$$\mathbb{Q}(P_T > \varepsilon) < \varepsilon.$$

PROOF. Without loss of generality, we can assume that $T = 1$ and fix $0 < \varepsilon < 1$. It will be convenient to denote increments of a given process (X_t) by $\Delta_n^N X \triangleq X_{\frac{n}{N}} - X_{\frac{n-1}{N}}$ where $N \in \mathbb{N}$ and $n = 1, \dots, N$. For such n, N and for $\sigma > 0$, consider the disjoint events $A_{+,n}^{N,\sigma}$ and $A_{-,n}^{N,\sigma}$ given by

$$A_{\pm,n}^{N,\sigma} \triangleq \left\{\Delta_n^N (P + g) = \pm\big(P_{\frac{n-1}{N}} \wedge N^{1/4}\big)\frac{\sigma}{\sqrt{N}} + o \text{ for some } o \in \big[0, 1/N^2\big]\right\}$$
$$\cap \left\{\max_{\frac{n-1}{N} \le t \le \frac{n}{N}} |P_t - P_{\frac{n-1}{N}}| \le \varepsilon/3\right\}.$$

For $N > (6\sigma/\varepsilon)^4$ (as assumed henceforth), the path properties described for P in the definition of both $A_{+,n}^{N,\sigma}$ and $A_{-,n}^{N,\sigma}$ are met for any given $P_{\frac{n-1}{N}} > 0$ by nonempty open sets of continuous paths taking values in $(0, \infty)$. Indeed, the latter nonnegativity requirement is allowed here because g is assumed to be nonincreasing (where $\Delta_n^N g \le 0$). Therefore, the conditional full support property (3.5) ensures that

$$\mathbb{P}\big[A_{+,n}^{N,\sigma} | \mathscr{F}_{\frac{n-1}{N}}\big] > 0 \quad \text{and} \quad \mathbb{P}\big[A_{-,n,}^{N,\sigma} | \mathscr{F}_{\frac{n-1}{N}}\big] > 0, \quad n = 1, \dots, N.$$

So there is $\mathbb{Q}^{N,\sigma} \ll \mathbb{P}$ for which

$$\mathbb{Q}^{N,\sigma}\big[A_{+,n}^{N,\sigma} | \mathscr{F}_{\frac{n-1}{N}}\big] = \mathbb{Q}^{N,\sigma}\big[A_{-,n}^{N,\sigma} | \mathscr{F}_{\frac{n-1}{N}}\big] = \frac{1}{2}, \quad n = 1, \dots, N;$$

for instance $\mathbb{Q}^{N,\sigma}$ with density

$$\frac{d\mathbb{Q}^{N,\sigma}}{d\mathbb{P}} \triangleq \prod_{n=1,\dots,N} \frac{1}{2}\left(\frac{1_{A_{+,n}^{N,\sigma}}}{\mathbb{P}[A_{+,n}^{N,\sigma} | \mathscr{F}_{\frac{n-1}{N}}]} + \frac{1_{A_{-,n}^{N,\sigma}}}{\mathbb{P}[A_{-,n}^{N,\sigma} | \mathscr{F}_{\frac{n-1}{N}}]}\right)$$

will do. In conjunction with the definition of $A_{\pm,n}^{N,\sigma}$, this ensures that $\mathbb{Q}^{N,\sigma}$-a.s.

$$\big|\mathbb{E}_{\mathbb{Q}^{N,\sigma}}\big[\Delta_n^N (P + g) | \mathscr{F}_{\frac{n-1}{N}}\big]\big| \le \frac{1}{N^2}, \quad n = 1, \dots, N,$$

and thus, the auxiliary discrete-time martingale

$$\tilde{M}_n \triangleq P_0 + g_0 + \sum_{m=1}^{n} \big(P_{\frac{m}{N}} - \mathbb{E}_{\mathbb{Q}^{N,\sigma}}[P_{\frac{m}{N}} | \mathscr{F}_{\frac{m-1}{N}}]\big), \quad n = 0, \dots, N,$$

satisfies $\mathbb{Q}^{N,\sigma}$-a.s.

$$|P_{\frac{n}{N}} + g_{\frac{n}{N}} - \tilde{M}_n| \le \frac{1}{N}, \quad n = 0, \dots, N. \tag{3.8}$$

Combining this with the Lipschitz-continuity of g and the $\varepsilon/3$-bound on the fluctuations of P over any time interval of length $\frac{1}{N}$ from the definition of $A^{N,\sigma}_{\pm,n}$ yields

$$|\tilde{M}_n - \tilde{M}_{n-1}| \le \frac{\varepsilon}{2}, \quad n = 1, \dots, N, \mathbb{Q}^{N,\sigma}\text{-a.s.}$$

for $N > N_0$, where $N_0(\sigma)$ depends only on σ, ε, and the Lipschitz constant L of g.

We conclude that the bounded $\mathbb{Q}^{N,\sigma}$-martingale given by

$$M^{N,\sigma}_t \triangleq \mathbb{E}_{\mathbb{Q}^{N,\sigma}}[\tilde{M}_N | \mathscr{F}_t], \quad 0 \le t \le T,$$

satisfies $M^{N,\sigma}_{\frac{n}{N}} = \tilde{M}_n$, $n = 0, \dots, N$, and

$$\max_{n=1,\dots,N} \max_{\frac{n-1}{N} \le t \le \frac{n}{N}} \left| M^{N,\sigma}_t - M^{N,\sigma}_{\frac{n-1}{N}} \right| \le \frac{\varepsilon}{2}, \quad \mathbb{Q}^{N,\sigma}\text{-a.s.}$$

This together with (3.8) and the $\varepsilon/3$-bound on the fluctuations of P from the definition of $A^{N,\sigma}_{\pm,n}$ gives that $g + P - M^{N,\sigma}$ satisfies the required bound (3.7) $\mathbb{Q}^{N,\sigma}$-a.s. for $N > N_0(\sigma)$.

It remains to argue that σ and then $N > N_0(\sigma)$ can be chosen such that $\mathbb{Q} \triangleq \mathbb{Q}^{N,\sigma}$ from the above construction also satisfies the second requirement $\mathbb{Q}(P_1 > \varepsilon) < \varepsilon$. To this end, note that the difference equation

$$\begin{aligned} Z^{N,\sigma}_0 &\triangleq P_0, \\ \Delta^N_n Z^{N,\sigma} &\triangleq (Z^{N,\sigma}_{\frac{n-1}{N}} \wedge N^{1/4}) \frac{\sigma}{\sqrt{N}} (1_{A^{N,\sigma}_{+,n}} - 1_{A^{N,\sigma}_{-,n}}) + \frac{L+1}{N}, \quad n = 1, \dots, N, \end{aligned} \tag{3.9}$$

yields a process $Z^{N,\sigma}$ dominating P in the sense that $Z^{N,\sigma}_{\frac{n}{N}} \ge P_{\frac{n}{N}}$, $n = 0, \dots, N$, $\mathbb{Q}^{N,\sigma}$-a.s., as follows readily by induction using the definition of $A^{N,\sigma}_{\pm,n}$ and the Lipschitz continuity of g. Theorem 4.4 in [21] in conjunction with (3.8) yields that, as $N \uparrow \infty$, the distribution of $Z^{N,\sigma}_1$ under $\mathbb{Q}^{N,\sigma}$ converges to the distribution of $Z^{(\sigma)}_1$ where $Z^{(\sigma)}$ is the (unique) solution of the linear SDE

$$Z^{(\sigma)}_0 = P_0, \qquad dZ^{(\sigma)}_t = Z^{(\sigma)}_t \sigma \, dW_t + (L+1) \, dt$$

for some standard Brownian motion W. In view of (3.9), we can thus choose σ and $N > N_0(\sigma)$ to fulfill the requirement $\mathbb{Q}^{N,\sigma}(P_1 > \varepsilon) < \varepsilon$ provided that $Z^{(\sigma)}_1$ converges to 0 in probability as $\sigma \uparrow \infty$. For this, observe that

$$Z^{(\sigma)}_1 = P_0 e^{\sigma W_1 - \sigma^2/2} \left(1 + \int_0^1 (L+1) e^{-\sigma W_t + \sigma^2 t/2} \, dt \right)$$

$$\leq P_0 e^{\sigma W_1 - \sigma^2/2} + P_0(L+1)e^{-\sigma^2/(2\ln\sigma)} \int_0^{1-1/\ln\sigma} e^{\sigma(W_1 - W_t)}\,dt$$
$$+ P_0(L+1)\int_{1-1/\ln\sigma}^{1} e^{\sigma(W_1-W_t)-\sigma^2(1-t)/2}\,dt.$$

Clearly, the first two summands in the last expression vanish almost surely while, due to Fubini's theorem, the expectation of the last one is

$$\mathbb{E}\left(\int_{1-1/\ln\sigma}^{1} e^{\sigma(W_1-W_t)-\sigma^2(1-t)/2}\,dt\right) = \frac{1}{\ln\sigma} \to 0$$

for $\sigma \uparrow \infty$. This shows that indeed $\lim_{\sigma\uparrow\infty} Z_1^{(\sigma)} = 0$ in probability and the proof is completed. □

3.2.2. *Utility maximization by duality.* Superreplication duality is often used to study utility maximization problems which, in turn, allow for less conservative and practically more useful contingent claim valuation paradigms such as indifference pricing. While this paper has to leave indifference valuation for future research, let us note here a verification theorem to illustrate the suitability of our duality concepts for this theory.

COROLLARY 3.5. *Let Assumptions 2.1 and 2.4 hold and consider a strictly concave, increasing and differentiable utility function u for which*

$$\sup_{X\in\mathscr{X} \text{ with } X_T=0} \mathbb{E}[u(\xi_T^X)\vee 0] < \infty.$$

Suppose $\widehat{X} \in \mathscr{X}$ with $\widehat{X}_T = 0$ yields via

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \triangleq \frac{u'(\xi_T^{\widehat{X}})}{\mathbb{E}[u'(\xi_T^{\widehat{X}})]}$$

a probability measure $\widehat{\mathbb{Q}} \ll \mathbb{P}$ which allows for a shadow price *$\widehat{M}$ for spread dynamics*

$$\widehat{\lambda}_t \triangleq \frac{\rho_t}{\delta_t}\mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[t,T]} \widehat{\alpha}_u \mu(du)\middle|\mathscr{F}_t\right], \quad 0 \leq t \leq T,$$

with $\widehat{\alpha} \triangleq \rho\zeta^{\widehat{X}} \in L^2(\widehat{\mathbb{Q}}\otimes\mu)$, that is, for a $\widehat{\mathbb{Q}}$-square integrable martingale $\widehat{M}$ such that

$$P_t - \widehat{\lambda}_t \leq \widehat{M}_t \leq P + \widehat{\lambda}_t, \quad 0 \leq t \leq T, \tag{3.10}$$

with equality almost surely holding true in the first and second estimate on the support of $d\widehat{X}^{\downarrow}$ and $d\widehat{X}^{\uparrow}$, respectively.

Then $\widehat{X}$ yields the highest expected utility $\mathbb{E}[u(\xi_T^X)]$ among all strategies $X \in \mathscr{X}$ with $X_T = 0$.

The proof of this corollary will follow readily from considerations required for the proof of Theorem 3.2. We thus postpone it to the end of Section 4.2. We adopted the notion of shadow prices from the theory of optimal investment with proportional transaction costs (see, e.g., [17, 19, 35]) where the martingales $\widehat{M}$ with the stated flat-off conditions are constructed explicitly or emerge from duality of utility maximization. In our setting, the construction of shadow prices is more challenging as the spread $\widehat{\lambda}$ is not given exogenously. It is thus not obvious how to construct optimal investment policies $\widehat{X}$ from the above verification result. See, however, [8] for a convex analytic approach to exponential utility maximization when P is a Brownian motion with drift and δ and r are constant.

4. Proof of the duality theorem.

4.1. *Preliminaries*. Let us prepare the proof of Theorem 3.2 by rewriting the profits and losses from trading in our price impact model.

LEMMA 4.1. *Suppose Assumption 2.4 holds true. Then, for any strategy $X \in \mathscr{X}$ with $X_T = 0$, we have*

$$\xi_T^X = v_0 - \Lambda_T^X, \tag{4.1}$$

where

$$v_0 \triangleq \xi_0 + \frac{1}{2}\left(\iota x_0^2 + \delta_0 \zeta_0^2\right) \tag{4.2}$$

and

$$\Lambda_T^X \triangleq \int_{[0,T]} P_t \, dX_t + \frac{1}{2} \int_{[0,T]} \left(\eta_t^X\right)^2 \mu(dt) \tag{4.3}$$

with

$$\eta_t^X \triangleq \rho_t \zeta_t^X = \zeta_0 + \int_{[0,t]} \frac{\rho_s}{\delta_s} \, d\left(X_s^{\uparrow} + X_s^{\downarrow}\right), \quad 0 \le t \le T. \tag{4.4}$$

Moreover, there is a constant $C > 0$, depending only on the bounds on δ/ρ from Assumption 2.4, such that, for any $X \in \mathscr{X}$, we have

$$X_T^{\uparrow} + X_T^{\downarrow} \le C\Big(l + \sup_{0 \le t \le T} |P_t|\Big) \quad \text{on } \left\{\Lambda_T^X \le l^2\right\}. \tag{4.5}$$

Finally, the mapping $X \mapsto \Lambda_T^X$ is convex and lower semicontinuous. More precisely, if $X^n \in \mathscr{X}$ converges weakly to $X \in \mathscr{X}$ in the sense that almost surely $X^{n,\uparrow}$ and $X^{n,\downarrow}$ converge weakly as Borel-measures on $[0, T]$ to, respectively, some adapted, right continuous, increasing A and B with $X = x_0 + A - B$, $A_{0-} = B_{0-} = 0$, then almost surely

$$\liminf_n \Lambda_T^{X^n} \ge \Lambda_T^X. \tag{4.6}$$

PROOF. 1. Let us first prove our formula (4.1) for ξ_T^X. For the integral of the mid-price, we get by continuity of P that

$$\begin{aligned}\int_{[0,T]} P_t^X \circ dX_t &= \int_{[0,T]} P_t\, dX_t + \iota \int_{[0,T]} X_t \circ dX_t \\ &= \int_{[0,T]} P_t\, dX_t + \iota \frac{1}{2}(X_T^2 - x_0^2),\end{aligned} \tag{4.7}$$

where the last identity is due to the chain rule for Stratonovich integrals. Similarly, using $\zeta^X = \eta^X/\rho$ and $d(X_t^\uparrow + X_t^\downarrow) = \frac{\delta_t}{\rho_t}\, d\eta_t^X$, we get

$$\begin{aligned}\int_{[0,T]} \zeta_t^X \circ d(X_t^\uparrow + X_t^\downarrow) &= \int_{[0,T]} \frac{\delta_t}{\rho_t^2} \eta_t^X \circ d\eta_t^X = \int_{[0,T]} \kappa_t \circ d\left(\frac{1}{2}(\eta_t^X)^2\right) \\ &= \kappa_T \frac{1}{2}(\eta_T^X)^2 - \delta_0 \frac{1}{2}\zeta_0^2 - \int_{(0,T)} \frac{1}{2}(\eta_t^X)^2\, d\kappa_t \\ &= \frac{1}{2}\int_{[0,T]} (\eta_t^X)^2 \mu(dt) - \frac{1}{2}\delta_0\zeta_0^2.\end{aligned} \tag{4.8}$$

Combining (4.7) with (4.8), we obtain (4.1) when $X_T = 0$.

2. For $X \in \mathscr{X}$, it follows from the definition (4.3) of Λ_T^X that on $\{\Lambda_T^X \le l^2\}$ we have

$$\begin{aligned} l^2 + \sup_{t\in[0,T]} |P_t|(X_T^\uparrow + X_T^\downarrow) &\ge l^2 - \int_{[0,T]} P_t\, dX_t \\ &\ge \frac{1}{2}\int_0^T (\eta_t^X)^2 \mu(dt) \ge (X_T^\uparrow + X_T^\downarrow)^2/C\end{aligned}$$

for some constant $C > 0$ only depending on the bounds on δ and ρ from Asssumption 2.4. Hence, $x \triangleq X_T^\uparrow + X_T^\downarrow$ is such that $x^2 \le C(px + l^2)$ for $p \triangleq \sup_{t\in[0,T]} |P_t|$. This implies (4.5).

3. Let $X_0, X_1 \in \mathscr{X}$ and observe that then $\frac{1}{2}(X_0^\uparrow + X_1^\uparrow) - \frac{1}{2}(X_0^\downarrow + X_1^\downarrow)$ is a decomposition of $X \triangleq \frac{1}{2}(X_0 + X_1)$ into the difference of two right-continuous increasing processes. It follows that $\frac{1}{2}(X_0^\uparrow + X_1^\uparrow) - X^\uparrow$ and $\frac{1}{2}(X_0^\downarrow + X_1^\downarrow) - X^\downarrow$ are increasing and so $0 \le \eta^X \le \frac{1}{2}(\eta^{X_0} + \eta^{X_1})$. In light of (4.3), this yields the convexity of Λ^X.

Similarly, for X^n converging to $X = x_0 + A - B$ as described in the lemma, $A - X^\uparrow$ and $B - X^\downarrow$ are increasing. Hence, we have $\eta_t^{X^n} \to \eta_t^{x_0+A+B} \ge \eta_t^X$ in $t = T$ and in every point of continuity t for $A + B$. By continuity of P, we also have

$$\lim_n \int_{[0,T]} P\, dX^n = \int_{[0,T]} P\, dX.$$

So lower semicontinuity of $X \mapsto \Lambda_T^X$ is a consequence of (4.3) and Fatou's lemma. $\square$

4.2. *Proof of the lower bound.* Observe first that the supremum in (3.3) is greater than $-\infty$. Indeed we can take any $\mathbb{Q}^0 \ll \mathbb{P}$ for which $\alpha_t^0 \triangleq \sup_{0\le s\le t} |P_s \rho_s|$, $0 \le t \le T$, is in $L^2(\mathbb{Q}^0 \otimes \mu)$ and let $M^0 \triangleq 0$ to obtain a triple $(\mathbb{Q}^0, M^0, \alpha^0)$ satisfying the constraint (3.4). Indeed, we then have

$$\begin{aligned}\frac{\rho_t}{\delta_t}\mathbb{E}_{\mathbb{Q}^0}\left[\int_{[t,T]} \alpha_u^0 \mu(du)\middle|\mathscr{F}_t\right] &\ge \frac{\rho_t}{\delta_t}\alpha_t^0 \mathbb{E}_{\mathbb{Q}^0}\big[\mu\big([t,T]\big)|\mathscr{F}_t\big] \\ &= \frac{\alpha_t^0}{\rho_t} \ge |P_t| = \big|P_t - M_t^0\big|.\end{aligned}$$

Hence, the supremum in (3.3) cannot be $-\infty$.

To prove that it gives a lower bound, consider $\xi_0 \in \mathbb{R}$ and $X \in \mathscr{X}$ with $X_T = 0$ such that $\xi_T^X \ge H \ge 0$ and let $(\mathbb{Q}, M, S)$ be a triple as in Theorem 3.2.

LEMMA 4.2. *We have*

$$X_T^{\uparrow} + X_T^{\downarrow}, \qquad \sup_{0\le t\le T} |P_t| \in L^2(\mathbb{Q}). \tag{4.9}$$

PROOF. By Doob's maximal inequality, $\sup_{t\in[0,T]} |M_t| \in L^2(\mathbb{Q})$. Similarly, $\alpha \in L^2(\mathbb{Q} \otimes \mu)$ yields that also the supremum over $[0, T]$ of the right-hand side of (3.4) is in $L^2(\mathbb{Q})$. Together with our previous observation, this implies that also $\sup_{0\le t\le T} |P_t| \in L^2(\mathbb{Q})$. Square-integrability of $X_T^{\uparrow} + X_T^{\downarrow}$ is now immediate from (4.5) with $l^2 \triangleq v_0 = \xi_T^X + \Lambda_T^X \ge \Lambda_T^X$ because $\xi_T^X \ge H \ge 0$ almost surely. $\square$

By Lemma 4.1, the superreplication property of X is tantamount to

$$v_0 \ge H + \int_{[0,T]} P_t\, dX_t + \frac{1}{2}\int_{[0,T]} \big(\eta_t^X\big)^2 \mu(dt). \tag{4.10}$$

Observe that by (3.4) we can estimate

$$\begin{aligned}\int_{[0,T]} P_t\, dX_t &= \int_{[0,T]} (P_t - M_t)\, dX_t + \int_{[0,T]} M_t\, dX_t \\ &\ge -\int_{[0,T]} |P_t - M_t|\big(dX_t^{\uparrow} + dX_t^{\downarrow}\big) - M_0 x_0 - \int_0^T X_t\, dM_t \qquad (4.11)\\ &= -\int_{[0,T]} |P_t - M_t|\frac{\delta_t}{\rho_t}\, d\eta_t^X - M_0 x_0 - \int_0^T X_t\, dM_t,\end{aligned}$$

where we first used integration by parts and $X_T = 0$ and then that (2.1) gives $d\eta_t^X = \rho_t/\delta_t(dX_t^{\uparrow} + dX_t^{\downarrow})$. Square-integrability of M and (4.9) yield $\mathbb{E}_{\mathbb{Q}}[\int_0^T X_t^2\, d[M]_t^{1/2}] < \infty$, ensuring that $\int_0^{\cdot} X_t\, dM_t$ is a true martingale. Hence, taking expectation in (4.11) we find

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\Big[\int_{[0,T]} P_t\,dX_t\Big] \\
&\geq -\mathbb{E}_{\mathbb{Q}}\Big[\int_{[0,T]} |P_t - M_t|\frac{\delta_t}{\rho_t}\,d\eta_t^X + M_0x_0\Big] \qquad (4.12)\\
&\geq -\mathbb{E}_{\mathbb{Q}}\Big[\int_{[0,T]} \mathbb{E}_{\mathbb{Q}}\Big[\int_{[t,T]} \alpha_u\mu(du)\Big|\mathscr{F}_t\Big]\,d\eta_t^X + M_0x_0\Big] \qquad (4.13)\\
&= -\mathbb{E}_{\mathbb{Q}}\Big[\int_{[0,T]}\int_{[0,u]} d\eta_t^X\alpha_u\mu(du) + M_0x_0\Big]\\
&= -\mathbb{E}_{\mathbb{Q}}\Big[\int_{[0,T]} (\eta_u^X - \zeta_0)\alpha_u\mu(du) + M_0x_0\Big],
\end{aligned}
$$

where in the second estimate we used (3.4) and the first identity follows from Fubini's theorem in conjunction with the observation that the conditional expectation in (4.13) can be dropped as it gives the optional projection of $(\int_{[t,T]}\alpha_u\mu(du))_{0\leq t\leq T}$.

Now we take expectation in (4.10) and use the preceding estimate to obtain

$$
\begin{aligned}
v_0 &\geq \mathbb{E}_{\mathbb{Q}}\Big[H + \int_{[0,T]}\Big\{\frac{1}{2}(\eta_t^X)^2 - (\eta_t^X - \zeta_0)\alpha_t\Big\}\mu(dt) - M_0x_0\Big]\\
&= \mathbb{E}_{\mathbb{Q}}\Big[H + \int_{[0,T]}\Big\{\frac{1}{2}(\eta_t^X - \alpha_t)^2 - \frac{1}{2}(\alpha_t - \zeta_0)^2 + \frac{1}{2}\zeta_0^2\Big\}\mu(dt) - M_0x_0\Big]\\
&\geq \mathbb{E}_{\mathbb{Q}}\Big[H + \int_{[0,T]}\Big\{-\frac{1}{2}(\alpha_t - \zeta_0)^2 + \frac{1}{2}\zeta_0^2\Big\}\mu(dt) - M_0x_0\Big]\\
&= \mathbb{E}_{\mathbb{Q}}[H] - \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\Big[\int_{[0,T]}(\alpha_t - \zeta_0)^2\mu(dt)\Big] + \frac{1}{2}\zeta_0^2\delta_0 - M_0x_0,
\end{aligned}
$$

where in the last step we used that $\mu([0,T]) = \kappa_0 = \delta_0$. Recalling the definition (4.2) of v_0, this gives

$$\xi_0 \geq \mathbb{E}_{\mathbb{Q}}[H] - \frac{1}{2}\|\alpha - \zeta_0\|^2_{L^2(\mathbb{Q}\otimes\mu)} - M_0x_0 - \frac{1}{2}\iota x_0^2,$$

which yields the claimed lower bound.

It is at this point easy to also give the *proof of the verification result stated in Corollary* 3.5. For this, take any $X \in \mathscr{X}$ and note that, by concavity of u,

$$u(\xi_T^X) - u(\xi_T^{\widehat{X}}) \leq u'(\xi_T^{\widehat{X}})(\xi_T^X - \xi_T^{\widehat{X}}).$$

Taking expectations under $\mathbb{P}$ and recalling the definition of $\widehat{\mathbb{Q}}$, it thus suffices to argue

$$\mathbb{E}_{\widehat{\mathbb{Q}}}[\xi_T^X] \leq \mathbb{E}_{\widehat{\mathbb{Q}}}[\xi_T^{\widehat{X}}]. \qquad (4.14)$$

For this, note that from (4.1), we have

$$\mathbb{E}_{\widehat{\mathbb{Q}}}[\xi_T^X] = v_0 - \mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[0,T]} P_t\, dX_t\right] - \frac{1}{2}\mathbb{E}_{\widehat{\mathbb{Q}}}\left[\frac{1}{2}\int_{[0,T]} (\eta_t^X)^2 \mu(dt)\right].$$

Proceeding as for (4.12), (4.13), we estimate

$$\mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[0,T]} P_t\, dX_t\right] \geq -\mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[0,T]} (\eta_u^X - \zeta_0)\widehat{\alpha}_u \mu(du) + \widehat{M}_0 x_0\right]$$

and observe that for $X = \widehat{X}$ we actually get an equality here due to the support assumption (3.10). Therefore,

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{Q}}}[\xi_T^X] &\leq v_0 + \mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[0,T]} (\eta_t^X - \zeta_0)\widehat{\alpha}_t - \frac{1}{2}(\eta_t^X)^2 \mu(dt) + \widehat{M}_0 x_0\right] \\ &= v_0 + \widehat{M}_0 x_0 + \mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[0,T]} \left\{-\frac{1}{2}(\eta_t^X - \widehat{\alpha}_t)^2 + \frac{1}{2}(\widehat{\alpha}_t - \zeta_0)^2 - \frac{1}{2}\zeta_0^2\right\}\mu(dt)\right] \\ &\leq v_0 + \widehat{M}_0 x_0 + \mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[0,T]} \left\{\frac{1}{2}(\widehat{\alpha}_t - \zeta_0)^2 - \frac{1}{2}\zeta_0^2\right\}\mu(dt)\right],\end{aligned}$$

where, again, we have equality everywhere for $X = \widehat{X}$ by choice of $\widehat{\alpha} = \eta^{\widehat{X}}$. It follows that (4.14) does hold true as remained to be shown.

4.3. *Proof of the upper bound.* In order to prove "$\leq$" in our dual description (3.3), we have to construct for any $\widehat{\xi}_0 < \pi(H)$ a triple $(\widehat{\mathbb{Q}}, \widehat{M}, \widehat{\alpha})$ as considered in Theorem 3.2 such that

$$\widehat{\xi}_0 < \mathbb{E}_{\widehat{\mathbb{Q}}}[H] - \frac{1}{2}\|\widehat{\alpha} - \zeta_0\|^2_{L^2(\widehat{\mathbb{Q}}\otimes\mu)} - \widehat{M}_0 x_0 - \frac{1}{2}\iota x_0^2. \tag{4.15}$$

Observe that, by changing to an equivalent measure if necessary, we can assume without loss of generality that

$$H \in L^1(\mathbb{P}), \qquad \sup_{0\leq t\leq T} |P_t| \in L^6(\mathbb{P}). \tag{4.16}$$

For notational convenience, let us introduce the class

$$\mathscr{X}^2 \triangleq \{X \in \mathscr{X} : X_T^\uparrow + X_T^\downarrow \in L^2(\mathbb{P})\}$$

and let us denote by

$$\widehat{v}_0 \triangleq \widehat{\xi}_0 + \frac{1}{2}(\iota x_0^2 + \delta_0 \zeta_0^2) \tag{4.17}$$

the constant from (4.2) corresponding to $\xi_0 = \widehat{\xi}_0$.

We start with the construction of $\widehat{\mathbb{Q}}$ which emerges from a standard separation argument.

LEMMA 4.3. *There is a probability measure $\widehat{\mathbb{Q}}$ with bounded density with respect to $\mathbb{P}$ such that*

$$\widehat{v}_0 < \mathbb{E}_{\widehat{\mathbb{Q}}}[H] + \inf_{X \in \mathscr{X}^2 \text{ with } X_T = 0} \mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X]. \tag{4.18}$$

PROOF. In light of our expression (4.1) for the investor's terminal cash position, the condition $\widehat{\xi}_0 < \pi(H)$ translates into

$$H - \widehat{v}_0 \notin \mathscr{C} \triangleq \{-\Lambda_T^X - A : X \in \mathscr{X} \text{ with} X_T = 0, A \in L_+^0(\mathscr{F}_T)\}. \tag{4.19}$$

We will argue below that $\mathscr{C}$ is a convex and closed subset of $L^0(\mathscr{F}_T)$. It follows then that $\mathscr{C} \cap L^1(\mathbb{P})$ is a convex and closed subset of $L^1(\mathbb{P})$ that, by (4.19), does not contain $H - \widehat{v}_0 \in L^1(\mathbb{P})$. By the Hahn–Banach separation theorem we can thus find $Z \in L^\infty(\mathscr{F}_T) - \{0\}$ such that

$$\mathbb{E}[Z(H - \widehat{v}_0)] > \sup_{C \in \mathscr{C} \cap L^1(\mathbb{P})} \mathbb{E}[ZC]. \tag{4.20}$$

Since $L_-^1(\mathbb{P}) - \Lambda_T^0 \subset \mathscr{C}$, we must have $Z \geq 0$ almost surely. We can therefore define a probability measure $\widehat{\mathbb{Q}} \ll \mathbb{P}$ via

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \triangleq \frac{Z}{\mathbb{E}[Z_T]}.$$

Then (4.20) readily yields (4.18) upon observing that for $X \in \mathscr{X}^2$ we have $\Lambda_T^X \in L^1(\mathbb{P})$ due to Assumption 2.4 and (4.16).

It remains to prove that $\mathscr{C}$ is indeed a convex, closed subset of $L^0(\mathscr{F}_T)$. Convexity is immediate from the convexity of $X \mapsto \Lambda_T^X$ established in Lemma 4.1. For closedness, take $X^n \in \mathscr{X}$ with $X_T^n = 0$ and $A^n \in L_+^0(\mathscr{F}_T)$, $n = 1, 2, \dots$, such that $\Lambda_T^{X^n} + A^n$ converges in $L^0(\mathbb{P})$ or, without loss of generality, even almost surely to some finite limit L. We have to show that $-L \in \mathscr{C}$, that is,

$$L \geq \Lambda_T^X \quad \text{for some } X \in \mathscr{X}.$$

By the given convergence, $\sup_n \Lambda_T^{X^n}$ is finite almost surely. Hence, by our estimate (4.5) also $\sup_n (X_T^{n,\uparrow} + X_T^{n,\downarrow})$ is finite almost surely. In particular, $\operatorname{conv}(X_T^{n,\uparrow} + X_T^{n,\downarrow}, n = 1, 2, \dots)$ is bounded almost surely, and thus in probability. So, by a Komlos lemma as Lemma 3.4 of [25] or Lemma 3.1 in [8], there is a cofinal sequence of convex combinations $\tilde{X}^n$ of $X^n, X^{n+1}, \dots$, such that almost surely $\tilde{X}^{n,\uparrow}$ and $\tilde{X}^{n,\downarrow}$ converge weakly as Borel-measures on $[0, T]$ to, respectively, A and B, two adapted, right continuous, and increasing processes with $A_{0-} = B_{0-} = 0$. By lower semicontinuity and convexity of $X \mapsto \Lambda_T^X$ (see (4.6) in Lemma 4.1), it follows that for $X \triangleq x_0 + A - B \in \mathscr{X}$ we indeed have

$$\Lambda_T^X \leq \liminf_n \Lambda_T^{\tilde{X}^n} \leq \liminf_n \Lambda_T^{X^n} \leq L$$

as desired. $\square$

The martingale $\widehat{M}$ is constructed as a Lagrange multiplier for the constraint $X_T = 0$ in the infimum of (4.18).

LEMMA 4.4. *We have*

$$\inf_{X\in\mathscr{X}^2 \text{ with } X_T=0} \mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X] = \sup_{M\in\mathscr{M}^2(\widehat{\mathbb{Q}})} \inf_{X\in\mathscr{X}^2} \mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X - M_T X_T]. \tag{4.21}$$

In conjunction with (4.18), this lemma shows in particular that there is $\widehat{M} \in \mathscr{M}^2(\widehat{\mathbb{Q}})$ with

$$\widehat{v}_0 < \mathbb{E}_{\widehat{\mathbb{Q}}}[H] + \inf_{X\in\mathscr{X}^2} \mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X - \widehat{M}_T X_T]. \tag{4.22}$$

PROOF. We start by observing that

$$\begin{aligned}\inf_{X\in\mathscr{X}^2 \text{ with } X_T=0} \mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X] &= \lim_n \inf_{X\in\mathscr{X}^2}\{\mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X] + n\|X_T\|_{L^2(\widehat{\mathbb{Q}})}\} \\ &= \lim_n \inf_{X\in\mathscr{X}^2} \sup_{\|M_T\|_{L^2(\widehat{\mathbb{Q}})}\le n} \mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X - M_T X_T].\end{aligned} \tag{4.23}$$

Indeed, the second identity is immediate as is "$\ge$" in the first line. For "$\le$" there, take $X^n \in \mathscr{X}^2$ such that $\mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^{X^n}] + n\|X_T^n\|_{L^2(\widehat{\mathbb{Q}})}$ approaches the limit in the first line. Then $\sup_n \mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^{X^n}] < \infty$ and, by convexity of $X \mapsto \Lambda_T^X$, we even have $\sup_{X\in\mathrm{conv}(X^n, n=1,2,\dots)} \mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X] < \infty$. It thus follows from (4.5) that $\mathrm{conv}(X_T^{n,\uparrow} + X_T^{n,\downarrow}, n = 1, 2, \dots)$ is bounded in $L^2(\widehat{\mathbb{Q}})$. In particular, it is bounded in L^0 and we can thus apply a Komlos result such as Lemma 3.1 in [8] to obtain $\tilde{X}^n \in \mathrm{conv}(X^n, X^{n+1}, \dots)$, $n = 1, 2, \dots$ that converge to some $\tilde{X} \in \mathscr{X}$ in the way required for the lower semicontinuity statement (4.6) in Lemma 4.1. We claim that

$$\tilde{X}_T = 0 \quad \text{with } \mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^{\tilde{X}}] \le \lim_n\{\mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^{\tilde{X}^n}] + n\|\tilde{X}_T^n\|_{L^2(\widehat{\mathbb{Q}})}\}. \tag{4.24}$$

Then, since by construction of the $(\tilde{X}^n)_{n=1,2,\dots}$ this limit coincides with the one in (4.23), we obtain that "$\le$" must hold there. For the proof of (4.24), note that $(\Lambda_T^{\tilde{X}^n})$ is bounded in $L^1(\widehat{\mathbb{Q}})$ because $\mathrm{conv}(X_T^{n,\uparrow} + X_T^{n,\downarrow}, n = 1, 2, \dots)$ is bounded in $L^2(\widehat{\mathbb{Q}})$. With the limit in (4.24) finite, this implies $\|\tilde{X}_T^n\|_{L^2(\widehat{\mathbb{Q}})} \to 0$ and so indeed $\tilde{X}_T = 0$. For the estimate in (4.24), observe that by Fatou's lemma and the lower semicontinuity of $X \mapsto \Lambda^X$ it suffices to show that $(\Lambda_T^{\tilde{X}^n} \wedge 0)_{n=1,2,\dots}$ is uniformly $\widehat{\mathbb{Q}}$-integrable. This, in turn, follows by observing that due to Hölder's inequality (with $p = 4$, $q = 4/3$),

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{Q}}}[|\Lambda_T^{\tilde{X}^n} \wedge 0|^{3/2}] &\le \mathbb{E}_{\widehat{\mathbb{Q}}}\left[\left|\int_{[0,T]} P_t\, d\tilde{X}_t^n \wedge 0\right|^{3/2}\right] \\ &\le \mathbb{E}_{\widehat{\mathbb{Q}}}\Big[\sup_{0\le t\le T} |P_t|^{3/2}(\tilde{X}_T^{n,\uparrow} + \tilde{X}_T^{n,\downarrow})^{3/2}\Big]\end{aligned}$$

$$\le \mathbb{E}_{\widehat{\mathbb{Q}}}\Big[\sup_{0\le t\le T}|P_t|^6\Big]^{1/4}\mathbb{E}_{\widehat{\mathbb{Q}}}\big[\big(\tilde{X}_T^{n,\uparrow}+\tilde{X}_T^{n,\downarrow}\big)^2\big]^{3/4}$$

is bounded because of (4.16) and because of the already established $L^2(\widehat{\mathbb{Q}})$-boundedness of $\operatorname{conv}(X_T^{n,\uparrow}+X_T^{n,\downarrow}, n=1,2,\dots)$.

With (4.23) established, we obtain our assertion (4.21) from the minimax relation

$$\begin{aligned}\inf_{X\in\mathscr{X}^2}\sup_{\|M_T\|_{L^2(\widehat{\mathbb{Q}})}\le n}&\mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X - M_T X_T]\\ &=\sup_{\|M_T\|_{L^2(\widehat{\mathbb{Q}})}\le n}\inf_{X\in\mathscr{X}^2}\mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X - M_T X_T].\end{aligned}\tag{4.25}$$

For this, we endow $\mathscr{X}^2$ with the $L^2(\mathbb{P})$-norm of the ω-wise total variation of its elements, $\|X\| \triangleq \mathbb{E}_{\mathbb{P}}[(X_T^{\uparrow}+X_T^{\downarrow})^2]^{1/2}$, and the $L^2(\widehat{\mathbb{Q}})$-ball with the weak topology. Then both of these sets are convex subsets of topological vector spaces and the latter set is even compact. Moreover, $(X, M_T)\mapsto \mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X - M_T X_T]$ is continuous and convex in X and continuous and concave (even affine) in M_T. We can thus apply Sion's minimax theorem [36] to obtain (4.25). $\square$

Our final lemma constructs $\widehat{\alpha}$.

LEMMA 4.5. *There is an optional $\widehat{\alpha}\in L^2(\widehat{\mathbb{Q}}\otimes\mu)$ such that*

$$|P_t-\widehat{M}_t|\le\frac{\rho_t}{\delta_t}\mathbb{E}_{\widehat{\mathbb{Q}}}\Big[\int_{[t,T]}\widehat{\alpha}_u\mu(du)\Big|\mathscr{F}_t\Big],\quad 0\le t\le T,$$

and

$$\inf_{X\in\mathscr{X}^2}\mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X-\widehat{M}_T X_T] = -\frac{1}{2}\|\widehat{\alpha}-\zeta_0\|^2_{L^2(\widehat{\mathbb{Q}}\otimes\mu)}-\widehat{M}_0x_0+\frac{1}{2}\zeta_0^2\delta_0.\tag{4.26}$$

PROOF. We first use integration by parts along with the observation that $\mathbb{E}_{\widehat{\mathbb{Q}}}[\int_0^T X_t^2\,d[\widehat{M}]_t^{1/2}]<\infty$ for $X\in\mathscr{X}^2$ to obtain that for such X we can write

$$\begin{aligned}&\mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X-\widehat{M}_T X_T]\\ &\quad=\mathbb{E}_{\widehat{\mathbb{Q}}}\Big[\int_{[0,T]}(P_t-\widehat{M}_t)\,dX_t+\frac{1}{2}\int_{[0,T]}(\eta_t^X)^2\mu(dt)-\widehat{M}_0x_0\Big]\\ &\quad=\mathbb{E}_{\widehat{\mathbb{Q}}}\Big[-\int_{[0,T]}|P_t-\widehat{M}_t|\,d\tilde{X}_t+\frac{1}{2}\int_{[0,T]}(\eta_t^{\tilde{X}})^2\mu(dt)\Big]-\widehat{M}_0x_0,\end{aligned}$$

where $\tilde{X}_t\triangleq x_0-\int_{[0,t]}\operatorname{sign}(P_s-M_s)\,dX_s$, $0\le t\le T$, satisfies $\eta^X=\eta^{\tilde{X}}$. So the infimum in (4.26) coincides with the infimum of this last expectation over all $\tilde{X}\in$

$\mathscr{X}^2$. In fact, it coincides with its infimum over all increasing and bounded $\tilde{X} \in \mathscr{X}$:

$$\inf_{X\in\mathscr{X}^2} \mathbb{E}_{\widehat{\mathbb{Q}}}[\Lambda_T^X - \widehat{M}_T X_T]$$
$$= \inf_{\tilde{X}\in\mathscr{X} \text{ incr., bdd.}} \mathbb{E}_{\widehat{\mathbb{Q}}}\left[-\int_{[0,T]} |P_t - \widehat{M}_t|\, d\tilde{X}_t + \frac{1}{2}\int_{[0,T]} (\eta_t^{\tilde{X}})^2 \mu(dt)\right] - \widehat{M}_0 x_0.$$

It thus remains to show that this last infimum is

$$\begin{aligned}&\inf_{\tilde{X}\in\mathscr{X} \text{ incr., bdd.}} \mathbb{E}_{\widehat{\mathbb{Q}}}\left[-\int_{[0,T]} |P_t - \widehat{M}_t|\, d\tilde{X}_t + \frac{1}{2}\int_{[0,T]} (\eta_t^{\tilde{X}})^2 \mu(dt)\right]\\ &\quad= -\frac{1}{2}\|\widehat{\alpha} - \zeta_0\|^2_{L^2(\widehat{\mathbb{Q}}\otimes\mu)} + \frac{1}{2}\zeta_0^2\delta_0\end{aligned} \tag{4.27}$$

for some $\widehat{\alpha} \in L^2(\widehat{\mathbb{Q}} \otimes \mu)$.

We will argue below that there is a progressively measurable process a with upper-right continuous paths such that $\sup_{\tau\le v\le \cdot} a_v \in L^1(\widehat{\mathbb{Q}} \otimes \mu)$ with

$$|P_\tau - \widehat{M}_\tau| \frac{\delta_\tau}{\rho_\tau} = \mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[\tau,T]} \sup_{\tau\le v\le u} a_v \mu(du) \middle| \mathscr{F}_\tau\right] \tag{4.28}$$

for any stopping time $\tau \le T$, that is, such that the left-hand side in (4.28) is the $\widehat{\mathbb{Q}}$-optional projection of the μ-integral on the right-hand side. Therefore, we get for any increasing and bounded $\tilde{X} \in \mathscr{X}$ that

$$\begin{aligned}&\mathbb{E}_{\widehat{\mathbb{Q}}}\left[-\int_{[0,T]} |P_t - \widehat{M}_t|\, d\tilde{X}_t + \frac{1}{2}\int_{[0,T]} (\eta_t^{\tilde{X}})^2 \mu(dt)\right]\\ &\quad= \mathbb{E}_{\widehat{\mathbb{Q}}}\left[-\int_{[0,T]}\int_{[t,T]} \sup_{t\le v\le u} a_v \mu(du) \frac{\rho_t}{\delta_t}\, d\tilde{X}_t + \frac{1}{2}\int_{[0,T]} (\eta_u^{\tilde{X}})^2 \mu(du)\right]\\ &\quad= \mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[0,T]}\left\{\frac{1}{2}(\eta_u^{\tilde{X}})^2 - \int_{[0,u]} \sup_{t\le v\le u} a_v\, d\eta_t^{\tilde{X}}\right\}\mu(du)\right],\end{aligned} \tag{4.29}$$

where for the second equality we applied Fubini's theorem and used that by monotonicity of $\tilde{X}$ and (4.4) we have $\frac{\rho_t}{\delta_t}\, d\tilde{X}_t = d\eta_t^{\tilde{X}}$. Introducing

$$\widehat{\alpha}_u \triangleq \sup_{0\le v\le u} a_v \vee \zeta_0, \quad 0 \le u \le T,$$

we can estimate the expression in $\{\dots\}$ in (4.29) by

$$\frac{1}{2}(\eta_u^{\tilde{X}})^2 - \int_{[0,u]} \sup_{t\le v\le u} a_v\, d\eta_t^{\tilde{X}}$$
$$\ge \frac{1}{2}(\eta_u^{\tilde{X}})^2 - \int_{[0,u]} \widehat{\alpha}_u\, d\eta_t^{\tilde{X}} = \frac{1}{2}(\eta_u^{\tilde{X}})^2 - \widehat{\alpha}_u(\eta_u^{\tilde{X}} - \zeta_0) \tag{4.30}$$

$$= \frac{1}{2}(\eta_u^{\tilde{X}} - \widehat{\alpha}_u)^2 - \frac{1}{2}(\widehat{\alpha}_u - \zeta_0)^2 + \frac{1}{2}\zeta_0^2 \ge -\frac{1}{2}(\widehat{\alpha}_u - \zeta_0)^2 + \frac{1}{2}\zeta_0^2, \tag{4.31}$$

which does not depend on the choice of increasing, bounded $\tilde{X} \in \mathscr{X}$. Combining (4.29) with this estimate thus gives

$$
\begin{aligned}
&\inf_{\tilde{X}\in\mathscr{X} \text{ incr., bdd.}} \mathbb{E}_{\widehat{\mathbb{Q}}}\Big[-\int_{[0,T]} |P_t - \widehat{M}_t|\, d\tilde{X}_t + \frac{1}{2}\int_{[0,T]} (\eta_t^{\tilde{X}})^2 \mu(dt)\Big] \\
&\geq \mathbb{E}_{\widehat{\mathbb{Q}}}\Big[\int_{[0,T]} \Big\{-\frac{1}{2}(\widehat{\alpha}_u - \zeta_0)^2 + \frac{1}{2}\zeta_0^2\Big\}\mu(du)\Big] \\
&= -\frac{1}{2}\|\widehat{\alpha} - \zeta_0\|^2_{L^2(\widehat{\mathbb{Q}}\otimes\mu)} + \frac{1}{2}\zeta_0^2\delta_0,
\end{aligned}
$$

which proves "$\geq$" in our assertion (4.27).

It remains to argue that, in fact, equality holds true, which in particular includes showing $\widehat{\alpha} \in L^2(\widehat{\mathbb{Q}} \otimes \mu)$. We start by observing that $\widehat{\alpha}$ is at least in $L^1(\widehat{\mathbb{Q}} \otimes \mu)$ because $\sup_{0\leq v\leq .} a \in L^1(\widehat{\mathbb{Q}} \otimes \mu)$. Moreover, $\widehat{\alpha}$ is increasing from ζ_0 and it is right-continuous and adapted by the upper-right continuity and progressive measurability of a. We can thus consider the increasing $\widehat{X} \in \mathscr{X}$ with $\eta^{\widehat{X}} = \widehat{\alpha}$. For $\tilde{X} = \widehat{X}$, we clearly have equality in (4.31), and, in fact, also in (4.30). Indeed, by construction, $\widehat{X}$, and thus $\eta^{\widehat{X}}$ increase only at times t when our process a reaches a new maximum beyond ζ_0 so that $\sup_{t\leq v\leq u} a_v = \sup_{0\leq v\leq u} a_v = \widehat{\alpha}_u$ for any $u \geq t$ at these times. Now, with $\tilde{X} = \widehat{X} \wedge n$ in (4.29) we get from these considerations that

$$
\begin{aligned}
&\mathbb{E}_{\widehat{\mathbb{Q}}}\Big[-\int_{[0,T]} |P_t - \widehat{M}_t|\, d(\widehat{X}\wedge n)_t + \frac{1}{2}\int_{[0,T]} (\eta_t^{\widehat{X}\wedge n})^2 \mu(dt)\Big] \qquad (4.32)\\
&\quad = \mathbb{E}_{\widehat{\mathbb{Q}}}\Big[\int_{[0,T]} \Big\{\frac{1}{2}(\eta_u^{\widehat{X}\wedge n})^2 - \int_{[0,u]} \sup_{t\leq v\leq u} a_v\, d\eta_t^{\widehat{X}\wedge n}\Big\}\mu(du)\Big] \\
&\quad = \int_{\{\widehat{X}\leq n\}} \Big(-\frac{1}{2}(\widehat{\alpha} - \zeta_0)^2 + \frac{1}{2}\zeta_0^2\Big)\, d(\widehat{\mathbb{Q}} \otimes \mu) \\
&\qquad + \int_{\{\widehat{X}> n\}} \Big(\frac{1}{2}(\eta^{\widehat{X}\wedge n})^2 - \widehat{\alpha}\eta^{\widehat{X}\wedge n} + \widehat{\alpha}\zeta_0\Big)\, d(\widehat{\mathbb{Q}} \otimes \mu). \qquad (4.33)
\end{aligned}
$$

Once we know that $\widehat{\alpha} = \eta^{\widehat{X}} \geq \eta^{\widehat{X}\wedge n}$ is in $L^2(\widehat{\mathbb{Q}} \otimes \mu)$, we can use, respectively, monotone and dominated convergence to let $n \uparrow \infty$ in the preceding expression and conclude that

$$
\begin{aligned}
&\inf_{\tilde{X}\in\mathscr{X} \text{ incr., bdd.}} \mathbb{E}_{\widehat{\mathbb{Q}}}\Big[-\int_{[0,T]} |P_t - \widehat{M}_t|\, d\tilde{X}_t + \frac{1}{2}\int_{[0,T]} (\eta_t^{\tilde{X}})^2 \mu(dt)\Big] \\
&\leq \int_{\Omega\times[0,T]} \Big(-\frac{1}{2}(\widehat{\alpha} - \zeta_0)^2 + \frac{1}{2}\zeta_0^2\Big)\, d(\widehat{\mathbb{Q}} \otimes \mu) + 0 \\
&= -\frac{1}{2}\|\widehat{\alpha} - \zeta_0\|^2_{L^2(\widehat{\mathbb{Q}}\otimes\mu)} + \frac{1}{2}\zeta_0^2\delta_0
\end{aligned}
$$

as remained to be shown for our claim (4.27). Now, use the estimate

$$\mathbb{E}_{\widehat{\mathbb{Q}}}\left[-\int_{[0,T]}|P_t-\widehat{M}_t|\,d(\widehat{X}\wedge n)_t+\frac{1}{2}\int_{[0,T]}(\eta_t^{\widehat{X}\wedge n})^2\mu(dt)\right]$$
$$\geq -\left\|\sup_{0\le t\le T}\left(|P_t-M_t|\frac{\delta_t}{\rho_t}\right)\right\|_{L^2(\widehat{\mathbb{Q}})}\|\eta_T^{\widehat{X}\wedge n}-\zeta_0\|_{L^2(\widehat{\mathbb{Q}})}+\frac{1}{2}\|\eta^{\widehat{X}\wedge n}\|^2_{L^2(\widehat{\mathbb{Q}}\otimes\mu)}$$

to see that if $\widehat{\alpha}=\eta^{\widehat{X}}$ was not in $L^2(\widehat{\mathbb{Q}}\otimes\mu)$ then the expectation in (4.32) would tend to $+\infty$ by monotone convergence as $n\uparrow\infty$. At the same time, though, the first integral in (4.33) would converge to $-\infty$. Moreover, $\widehat{\alpha}\in L^1(\widehat{\mathbb{Q}}\otimes\mu)$ ensures that the contribution of $\widehat{\alpha}\zeta_0$ to the second $\widehat{\mathbb{Q}}\otimes\mu$-integral there vanishes for $n\uparrow\infty$. By choice of $\widehat{X}$, we have $\widehat{\alpha}=\eta^{\widehat{X}}\geq\eta^{\widehat{X}\wedge n}$, so that the remaining contribution from this integral is less than or equal to 0. Hence, the assumption $\widehat{\alpha}\notin L^2(\widehat{\mathbb{Q}}\otimes\mu)$ leads us to the contradiction that the identical quantities in (4.32) and (4.33) would converge to $+\infty$ and $-\infty$ at the same time when $n\uparrow\infty$.

For the completion of our proof, we still need to construct the process a from (4.28). It will be obtained by the representation theorem from [4]. For this we note that, while having full support on $[0,T]$ by Assumption 2.4, our measure μ is not directly applicable for this representation theorem since it has an atom at time T. We thus replace it with the atomless optional random measure $\tilde{\mu}(dt)=1_{[0,T)}(t)\mu(dt)+\lambda e^{-\lambda(t-T)}1_{[T,\infty)}(t)\,dt$ on $[0,\infty)$ where $\lambda\triangleq\mu(\{T\})$. We also extend $Y_t\triangleq|P_t-\widehat{M}_t|\frac{\delta_t}{\rho_t}$, $0\le t\le T$, to a process on $[0,\infty)$ by letting $Y_t\triangleq Y_Te^{-\lambda(t-T)}$ for $t\geq T$ and we let $\mathscr{F}_t\triangleq\mathscr{F}_T$ for $t\geq T$. Then, by Assumption 2.1, the process Y is adapted, continuous with limit $\lim_{t\uparrow\infty}Y_t=0$ and it is of class (D) since it has an integrable upper bound because of $M\in\mathscr{M}^2(\widehat{\mathbb{Q}})$ and (4.16). We thus can apply Theorem 3 of [4] in connection with their Remark 2.1 to obtain an upper-right continuous, progressively measurable a such that for any stopping time τ we have $\sup_{\tau\le v\le .}a_v\in L^1(\widehat{\mathbb{Q}}\otimes\tilde{\mu})$ with

$$Y_\tau=\mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[\tau,\infty)}\sup_{\tau\le v\le u}a_v\tilde{\mu}(du)\middle|\mathscr{F}_\tau\right].$$

In fact, for $t\geq T$, one readily checks that $a_t=a_T=Y_T$ will do. Therefore, we get by uniqueness of a that for any stopping time $\tau\le T$ the above representation amounts to

$$|P_\tau-\widehat{M}_\tau|\frac{\delta_\tau}{\rho_\tau}=Y_\tau=\mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[\tau,T)}\sup_{\tau\le v\le u}a_v\tilde{\mu}(du)+\int_{[T,\infty)}\sup_{\tau\le v\le T}a_v\tilde{\mu}(dt)\middle|\mathscr{F}_\tau\right]$$
$$=\mathbb{E}_{\widehat{\mathbb{Q}}}\left[\int_{[\tau,T)}\sup_{\tau\le v\le u}a_v\mu(du)+\sup_{\tau\le v\le T}a_v\mu(\{T\})\middle|\mathscr{F}_\tau\right]$$

as requested. □

The proof of the upper bound in our duality (3.3) of Theorem 3.2 is now easy to complete. Indeed, the constructed triple $(\widehat{\mathbb{Q}},\widehat{M},\widehat{\alpha})$ is as requested by our theorem.

Moreover, recalling the definition (4.17) of $\widehat{v}_0$ and combining (4.22) with (4.26) gives the desired upper bound (4.15).

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INSTITUT FÜR MATHEMATIK
TECHNISCHE UNIVERSITÄT BERLIN
STRASSE DES 17. JUNI 136
10623 BERLIN
GERMANY
E-MAIL: bank@math.tu-berlin.de

DEPARTMENT OF STATISTICS
HEBREW UNIVERSITY OF JERUSALEM
MOUNT SCOPUS, JERUSALEM 91905
ISRAEL
AND
SCHOOL OF MATHEMATICS
MONASH UNIVERSITY
WELLINGTON RD, CLAYTON VIC 3800
AUSTRALIA
E-MAIL: yan.dolinsky@mail.huji.ac.il