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AFFINE PROCESSES BEYOND STOCHASTIC CONTINUITY

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In this paper, we study time-inhomogeneous affine processes beyond the common assumption of stochastic continuity. In this setting, times of jumps can be both inaccessible and predictable. To this end, we develop a general theory of finite dimensional affine semimartingales under very weak assumptions. We show that the corresponding semimartingale characteristics have affine form and that the conditional characteristic function can be represented with solutions to measure differential equations of Riccati type. We prove existence of affine Markov processes and affine semimartingales under mild conditions and elaborate on examples and applications including affine processes in discrete time.

CONTENTS

1. Introduction. The importance of jumps at predictable or predetermined times is widely acknowledged in the financial literature; see, for example, [1, 11, 13, 18, 26, 29, 31–33]. This is due to the fact that a surprisingly large amount of jumps or, more generally, rapid changes in stock prices or other financial time series occur in correspondence with announcements released at scheduled, and

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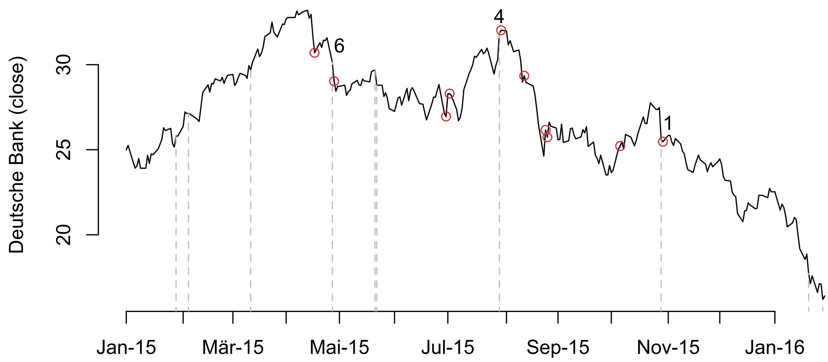

FIG. 1. *Chart of the stock price of Deutsche Bank. The vertical lines represent dates which have been announced in the previous annual reports of* 2013 *and* 2014, *for example, annual and quarterly reports and shareholder meetings. We marked the* 10 *largest one-day movements by circles; three (the largest, and the* 4*th and* 6*th largest) of them occurred at pre-announced dates.*

hence predictable times (see, e.g., [27]). A prominent example is the jump of the EUR/GBP exchange rate on the 23rd of June in 2016 when it became clear that the British referendum on membership in the EU will come out in favor of Brexit. In addition, large jumps in stock prices frequently coincide with the release of quarterly reports or earnings announcements. (See Figure 1 for an example and [12] for further empirical support). Econometric models incorporating such jumps at predetermined times were studied and tested on market data in [32]; see also [34] and [16, 17].

While affine processes are a prominent model class for interest rates or stochastic volatility, they have only been considered under the assumption of stochastic continuity, which precludes jumps at predictable times. This assumption is dropped in this paper, and we study affine processes only under very mild assumptions, which allow for jumps to occur at both predictable and totally inaccessible times.

The defining property of affine processes is the exponential affine form of the conditional characteristic function which allows for rich structural properties while retaining tractability due to the representation of the conditional characteristic function in terms of ordinary differential equations, the so-called 'generalized Riccati equations'. In subsequent research, further applications have been explored (e.g., [9, 24, 25]) as well as extensions of the state space (e.g., [5, 6]) and most notably an extension to time-inhomogeneous affine processes in [14].

In Remark 2.11 of [14], the author conjectures that his results can also be obtained on the level of semimartingales omitting the assumption of stochastic continuity. Here we confirm this conjecture by generalizing the result in [14] to affine semimartingales with singular continuous and discontinuous characteristics and only locally integrable parameters. This result is complemented by existence results for affine Markov processes and affine semimartingales under certain mild assumptions. Furthermore, we provide a variety of examples and applications. In particular, we propose an affine term-structure framework that allows for discontinuities at previously fixed time points.

The paper at hand is structured as follows. The next section revisits some facts about semimartingales before stating the definition of *affine semimartingales* and introducing certain technical assumption. After proving first results, we define the concept of a *good parameter set* in Section 3 which is a key ingredient of our first main result, the characterization Theorem 3.2. Section 4 discusses the relation between affine Markov processes and affine semimartingales as well as the important case of infinitely divisible processes. Section 5 is devoted to the existence of affine Markov processes and affine semimartingales under certain conditions on their good parameter set. Examples and applications are explained in Section 6 which concludes the paper with the introduction of a new *affine term-structure framework*. Details about measure differential equations that appear in the characterization and existence results instead of the ODEs appearing in [10] and [14], are postponed to the Appendix.

2. Preliminaries.

2.1. *Affine semimartingales.* Consider a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, P)$ with filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions. A stochastic process X taking values in $\mathbb{R}^d$ is called *càdlàg* if all its paths are right continuous with left limits. For a càdlàg process X, we define X_- and ΔX by

$$\begin{cases} X_{0-} = X_0, \qquad X_{t-} = \lim_{s\uparrow t} X_s \quad \text{for } t > 0, \\ \Delta X_t = X_t - X_{t-}. \end{cases}$$

In particular, note that $\Delta X_0 = 0$ and that X can be recovered from X_- by taking right limits.

A *semimartingale* is a process X with decomposition $X = X_0 + N + M$ where X_0 is $\mathscr{F}_0$-measurable, N is càdlàg, adapted, has paths of finite variation over each finite interval with $N_0 = 0$ and M is a local martingale starting in 0. We will always consider a càdlàg version of the semimartingale X.

To the jumps of X, we associate an integer-valued random measure μ^X by

$$\mu^X(dt, dx) = \sum_{s\geq 0} \mathbb{1}_{\{\Delta X_s \neq 0\}} \delta_{(s,\Delta X_s)}(dt, dx); \tag{1}$$

here δ_a is the Dirac measure at point a. We denote the compensator, or the dual predictable projection, of the random measure μ^X by ν. This is the unique predictable random measure which renders stochastic integrals with respect to $\mu^X - \nu$ local martingales.

We briefly recall the well-known concept of *characteristics* of a semimartingale, cf. [21], Chapter II: a semimartingale X with decomposition $X = X_0 + N + M$ is called *special* if N is predictable. In this case, the decomposition is unique, and we call it the *canonical decomposition*. The local martingale part M can be decomposed in a continuous local martingale part, which we denote by X^c, and a purely discontinuous local martingale part, $X - X^c$. We fix a truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$ which is a bounded function satisfying $h(x) = x$ in a neighborhood of 0. Then $\check{X}(h) = \sum_{s \le \cdot}(\Delta X_s - h(\Delta X_s))$ and $X(h) = X - \check{X}(h)$ both define d-dimensional stochastic processes. Note that $\Delta X(h) = h(\Delta X)$, such that $X(h)$ has bounded jumps. The resulting process is a special semimartingale and we denote its canonical decomposition by

$$X(h) = X_0 + B(h) + M(h),$$

with a predictable process of finite variation $B(h)$ and a local martingale $M(h)$. The *characteristics* of the semimartingale X is the triplet (B, C, ν) where $B = B(h)$, $C = (C^{ij})$ with $C^{ij} = \langle X^{i,c}, X^{j,c}\rangle$ and $\nu = \nu^X$ is the compensator of μ^X defined in equation (1). For additional facts on semimartingales and stochastic analysis, we refer to [21].

Let $D \subset \mathbb{R}^d$ be a closed convex cone of full dimension, that is, a convex set, closed under multiplication with positive scalars, and with linear hull equal to $\mathbb{R}^d$. An important example is the set $\mathbb{R}^m_{\ge 0} \times \mathbb{R}^n$ with $m + n = d$, which was used as the 'canonical state-space' for affine processes in [10, 14]. For u, w in $\mathbb{C}^d$, we set $\langle u, w\rangle = \sum_{i=1}^d u_i w_i$ and denote the real part of u by $\operatorname{Re} u$. Moreover, we define the *complex dual cone* of the state space D by

$$\mathcal{U} := \{u \in \mathbb{C}^d : \langle \operatorname{Re} u, x\rangle \le 0 \text{ for all } x \in D\}. \tag{2}$$

For the canonical state space, $\mathcal{U}$ equals $\mathbb{C}^m_{\le 0} \times i\mathbb{R}^n$, where $\mathbb{C}_{\le 0} = \{u \in \mathbb{C} : \operatorname{Re} u \le 0\}$, which coincides with the definition used in [10].[1] We are now prepared to state the central definition of this paper.

DEFINITION 2.1. Let X be a càdlàg semimartingale, taking values in D. The process X is called an *affine semimartingale*, if there exist $\mathbb{C}$ and $\mathbb{C}^d$-valued deterministic functions $\phi_s(t, u)$ and $\psi_s(t, u)$, continuous in $u \in \mathcal{U}$ and with $\phi_s(t, 0) = 0$ and $\psi_s(t, 0) = 0$, such that

$$E[e^{\langle u, X_t\rangle} | \mathscr{F}_s] = \exp(\phi_s(t, u) + \langle \psi_s(t, u), X_s\rangle) \tag{3}$$

for all $0 \le s \le t$ and $u \in \mathcal{U}$. Moreover, X is called *time homogeneous*, if $\phi_s(t, u) = \phi_0(t - s, u)$ and $\psi_s(t, u) = \psi_0(t - s, u)$, again for all $0 \le s \le t$ and $u \in \mathcal{U}$.

Note that the left-hand side of (3) is always well defined and bounded in absolute value by 1, due to the definition of $\mathcal{U}$.

[1]We use this notation in analogous fashion for $>$, $<$ or $\ge$ instead of $\le$ and with $\mathbb{R}$ instead of $\mathbb{C}$.

REMARK 2.2. Comparing Definition 2.1 with the definition of an *affine process* in [10] (which treats the time-homogeneous case) and [14] (which treats the time-inhomogeneous case), we have replaced the Markov assumption of [10, 14] with a semimartingale assumption. In view of [10], Theorem 2.12, this seems to slightly restrict the scope of the definition, since it excludes nonconservative processes. On the other hand, and this is the central point of our paper, we do not impose a stochastic continuity assumption on X, as has been done in [10, 14]. It turns out that omitting this assumption leads to a significantly larger class of stochastic processes and to a substantial extension of the results in [10, 14]. Section 4 contains further results on the relation between affine semimartingales and affine Markov processes; in particular, we show in Lemma 4.3 that affine semimartingales are Markovian under mild conditions.

To continue, we introduce an important condition on the support of the process X. Recall that the support of a generic random variable X is the smallest closed set C such that $P(X \in C) = 1$; we denote this set by $\mathrm{supp}(X)$. For a set A, we write $\mathrm{conv}(A)$ for its convex hull, that is, the smallest convex set containing A.

CONDITION 2.3. We say that an affine semimartingale X has *support of full convex span*, if $\mathrm{conv}(\mathrm{supp}(X_t)) = D$ for all $t > 0$.

Under Condition 2.3, ϕ and ψ are uniquely specified.

LEMMA 2.4. *Let X be an affine semimartingale satisfying the support Condition 2.3. Then $\phi_s(t, u)$ and $\psi_s(t, u)$ are uniquely specified by (3) for all $0 < s \le t$ and $u \in \mathcal{U}$.*

PROOF. Fix $0 < s \le t$ and suppose that $\widetilde{\phi}_s(t, u)$ and $\widetilde{\psi}_s(t, u)$ are also continuous in $u \in \mathcal{U}$ and satisfy (3). Write $p_s(t, u) := \widetilde{\phi}_s(t, u) - \phi_s(t, u)$ and $q_s(t, u) := \widetilde{\phi}_s(t, u) - \phi_s(t, u)$. Due to (3), it must hold that

$$p_s(t, u) + \langle q_s(t, u), X_s \rangle \quad \text{takes values in } \{2\pi i k : k \in \mathbb{N}\} \text{ a.s. } \forall u \in \mathcal{U}.$$

However, the set $\mathcal{U}$ is simply connected, and hence its image under a continuous function must also be simply connected. It follows that $u \mapsto p_s(t, u) + \langle q_s(t, u), X_s \rangle$ is constant on $\mathcal{U}$ and, therefore, equal to $p_s(t, 0) + \langle q_s(t, 0), X_s \rangle = 0$. Hence,

$$p_s(t, u) + \langle q_s(t, u), x \rangle = 0,$$

for all $x \in \mathrm{supp}(X_s)$ and $u \in \mathcal{U}$. Taking convex combinations, the equality can be extended to $x \in D$. Since D has full linear span, we conclude that $p_s(t, u) = 0$ and $q_s(t, u) = 0$ for all $u \in \mathcal{U}$, completing the proof. □

DEFINITION 2.5. An affine semimartingale is called *quasiregular*, if the following hold:

(i) The functions ϕ and ψ are of finite variation in s and càdlàg in both s and t. More precisely, we assume that for all $(t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$,

$$s \mapsto \phi_s(t, u) \quad \text{and} \quad s \mapsto \psi_s(t, u)$$

are càdlàg functions of finite variation on $[0, t]$, and for all $(s, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$

$$t \mapsto \phi_s(t, u) \quad \text{and} \quad t \mapsto \psi_s(t, u)$$

are càdlàg functions on $[s, \infty)$.

(ii) For all $0 < s \leq t$, the functions

$$u \mapsto \phi_{s-}(t, u) \quad \text{and} \quad u \mapsto \psi_{s-}(t, u)$$

are continuous on $\mathcal{U}$.

REMARK 2.6. Definition 2.5 should be compared to the assumptions imposed in [10] and [14]. In both papers, technical 'regularity conditions' are defined. In [10, 14], ϕ and ψ are automatically continuous in their first argument, due to the stochastic continuity of X. In addition, they are assumed continuously differentiable from the right, with a derivative that is continuous in u. Thus, (i) and (ii) are clearly milder than the regularity assumptions in [10] or [14].

2.2. *First results on ϕ and ψ.* We proceed to show first analytic results on the functions ϕ and ψ from (3).

LEMMA 2.7. *Let X be an affine semimartingale satisfying the support Condition 2.3. Then*:

(i) *the function $u \mapsto \phi_s(t, u)$ maps $\mathcal{U}$ to $\mathbb{C}_{\leq 0}$ and $u \mapsto \psi(t, u)$ maps $\mathcal{U}$ to $\mathcal{U}$, for all $0 < s \leq t$,*

(ii) *ϕ and ψ satisfy the* semi-flow property, *that is, for all $0 < s \leq r \leq t$ and $u \in \mathcal{U}$,*

$$\begin{aligned} &\phi_s(t, u) = \phi_r(t, u) + \phi_s\big(r, \psi_r(t, u)\big), \qquad \phi_t(t, u) = 0, \\ &\psi_s(t, u) = \psi_s\big(r, \psi_r(t, u)\big), \qquad \psi_t(t, u) = u. \end{aligned} \tag{4}$$

PROOF. To show the first property, recall that by equation (3) we have

$$E\big[e^{\langle u, X_t\rangle} | \mathscr{F}_s\big] = \exp\big(\phi_s(t, u) + \langle\psi_s(t, u), X_s\rangle\big) \tag{5}$$

for all $u \in \mathcal{U}$ and $0 \leq s \leq t$. Since $\langle \operatorname{Re} u, X_t\rangle \leq 0$, a.s., the left-hand side is bounded by one in absolute value. Thus, also

$$\operatorname{Re}\phi_s(t, u) + \langle \operatorname{Re}\psi_s(t, u), X_s\rangle \leq 0, \quad \text{a.s.}$$

and consequently

$$\operatorname{Re}\phi_s(t,u) + \langle \operatorname{Re}\psi_s(t,u), x\rangle \le 0, \quad \text{for all } x \in \operatorname{supp}(X_s).$$

Taking arbitrary convex combinations of these inequalities and using that, by Condition 2.3, $\operatorname{conv}(\operatorname{supp}(X_s)) = D$ we obtain that the inequality must in fact hold for all $x \in D$. Since D is a cone this implies that $\operatorname{Re}\phi_s(t,u) \le 0$ and $\psi_s(t,u) \in \mathcal{U}$, proving (i).

To show the semi-flow equations we apply iterated expectations to the left-hand side of (5), yielding

$$\begin{aligned} E\big[E\big[e^{\langle u, X_t\rangle}|\mathscr{F}_r\big]|\mathscr{F}_s\big] &= E\big[\exp\big(\phi_r(t,u) + \langle \psi_r(t,u), X_r\rangle\big)|\mathscr{F}_s\big] \\ &= \exp\big(\phi_s(r,u) + \phi_s\big(r, \psi_r(t,u)\big) + \langle \psi_s\big(r, \psi_r(t,u)\big), X_s\rangle\big). \end{aligned}$$

Note that the exponent on the right-hand side is continuous in u and that the same holds true for (5). By the same argument as in the proof of Lemma 2.4, we conclude that

$$\phi_s(t,u) + \langle \psi_s(t,u), x\rangle = \phi_s(r,u) + \phi_s\big(r, \psi_r(t,u)\big) + \langle \psi_s\big(r, \psi_r(t,u)\big), x\rangle,$$

for all $x \in D$. Since the linear hull of D is $\mathbb{R}^d$ the semi-flow equations (4) follow. Note that the terminal conditions $\psi_t(t,u) = u$ and $\phi_t(t,u) = 0$ are a simple consequence of $E[\exp(\langle u, X_t\rangle)|\mathscr{F}_t] = \exp(\langle u, X_t\rangle)$ and the uniqueness property from Lemma 2.4. $\square$

REMARK 2.8. Note that $s = 0$ is excluded from the semi-flow equations, since Condition 2.3 does not apply to the initial value X_0 of X. However, as soon as quasiregularity is imposed, the càdlàg property of ϕ and ψ immediately allows to extend the semi-flow equations also to $s = 0$.

REMARK 2.9. To express the semi-flow equations in a more succinct matter, it is sometimes convenient to introduce the following 'big-flow' notation. Define the set $\widehat{\mathcal{U}} := \mathbb{C}_{\le 0} \times \mathcal{U}$ and denote its elements by $\widehat{u} = (u_0, u)$. Define

$$\Psi_s(t, \widehat{u}) := \begin{pmatrix} \phi_s(t,u) + u_0 \\ \psi_s(t,u) \end{pmatrix}.$$

Part (i) of Lemma 2.7 is equivalent to the claim that $u \mapsto \Psi_s(t,u)$ maps $\widehat{\mathcal{U}}$ to $\widehat{\mathcal{U}}$ and part (ii) is equivalent to

$$\Psi_s(t, \widehat{u}) = \Psi_s\big(r, \Psi_r(t, \widehat{u})\big), \qquad \Psi_t(t, \widehat{u}) = \widehat{u},$$

for all $0 < s \le r \le t$ and $\widehat{u} \in \widehat{\mathcal{U}}$.

LEMMA 2.10. *Let X be a quasiregular affine semimartingale. Then, for all $u \in \mathcal{U}$,*

$$E\left[e^{\langle u, X_{t-}\rangle} | \mathscr{F}_s\right] = \exp\left(\phi_s(t-, u) + \langle \psi_s(t-, u), X_s\rangle\right), \quad \forall 0 \le s < t, \tag{6}$$

$$E\left[e^{\langle u, X_t\rangle} | \mathscr{F}_{s-}\right] = \exp\left(\phi_{s-}(t, u) + \langle \psi_{s-}(t, u), X_{s-}\rangle\right), \quad \forall 0 < s \le t. \tag{7}$$

If in addition X satisfies the support Condition 2.3, it also holds that

$$E\left[e^{\langle u, \Delta X_t\rangle} | \mathscr{F}_{t-}\right] = \exp\left(-\Delta\phi_t(t, u) - \langle \Delta\psi_t(t, u), X_{t-}\rangle\right), \tag{8}$$

for all $(t, u) \in \mathbb{R}_{\ge 0} \times \mathcal{U}$.

PROOF. The first expression, (6), follows by taking left limits in t on both sides of (3). On the right-hand side, the limit is well defined by the càdlàg property of ϕ and ψ in t. On the left-hand side, dominated convergence and the càdlàg property of X yield (6). Equation (7) follows from a similar argument, now taking left limits in s. Indeed, note that for any integrable random variable Y, martingale convergence yields that that $\lim_{\epsilon\downarrow 0} E[Y|\mathscr{F}_{s-\epsilon}] = E[Y|\mathscr{F}_{s-}]$. Equation (8) follows by evaluating (7) at $s = t$ and noting that $\Delta\phi_t(t, u) = \phi_t(t, u) - \phi_{t-}(t, u) = -\phi_{t-}(t, u)$, and $\Delta\psi_t(t, u) = \psi_t(t, u) - \psi_{t-}(t, u) = u - \psi_{t-}(t, u)$, due to Lemma 2.7. □

LEMMA 2.11. *Let X be a quasiregular affine semimartingale satisfying the support Condition 2.3. Then:*

(i) *for all $(s, u) \in \mathbb{R}_{\ge 0} \times \mathcal{U}$ the functions*

$$t \mapsto \phi_{s-}(t, u), \qquad t \mapsto \psi_{s-}(t, u)$$

are càdlàg on $[s, \infty)$.

(ii) *The 'double limits' $\phi_{s-}(t-, u)$ and $\psi_{s-}(t-, u)$ are well defined and independent of the order of limits, that is,*

$$\lim_{\epsilon\downarrow 0} \psi_{s-}(t - \epsilon, u) = \lim_{\delta\downarrow 0} \psi_{s-\delta}(t-, u),$$

and similarly for ϕ.

(iii) *The semi-flow equations (4) still hold when s is replaced by $s-$ or t is replaced by $t-$ (or both).*

(iv) *It holds that*

$$E\left[e^{\langle u, X_{t-}\rangle} | \mathscr{F}_{s-}\right] = \exp\left(\phi_{s-}(t-, u) + \langle \psi_{s-}(t-, u), X_{s-}\rangle\right),$$

for all $0 < s \le t$ and $u \in \mathcal{U}$.

(v) *For all $u \in \mathcal{U}$ and $0 \le s < t$, it holds that*

$$\begin{aligned} \Delta\phi_s(t, u) &= \Delta\phi_s\left(s, \psi_s(t, u)\right), \\ \Delta\psi_s(t, u) &= \Delta\psi_s\left(s, \psi_s(t, u)\right). \end{aligned} \tag{9}$$

PROOF. We show claims (i), (ii) and (iii) for ψ only. The proof can easily be extended to ϕ, for example, by using the 'Big flow' argument of Remark 2.9. To show right continuity in (i), we write

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\psi_{s-}(t+\epsilon,u) &= \lim_{\epsilon\downarrow 0}\psi_{s-}\big(t,\psi_t(t+\epsilon,u)\big)=\psi_{s-}\Big(t,\lim_{\epsilon\downarrow 0}\psi_t(t+\epsilon,u)\Big)\\ &= \psi_{s-}\big(t,\psi_t(t,u)\big)=\psi_{s-}(t,u).\end{aligned}$$

Here we have used the flow property, the continuity of $\psi_{s-}(t,u)$ in u and finally the right continuity of $\psi_s(t,u)$ in t. As for the left limit, the equality

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\psi_{s-}(t-\epsilon,u) &= \lim_{\epsilon\downarrow 0}\psi_{s-}\big(s,\psi_s(t-\epsilon,u)\big)=\psi_{s-}\Big(s,\lim_{\epsilon\downarrow 0}\psi_t(t-\epsilon,u)\Big)\\ &= \psi_{s-}\big(s,\psi_s(t-,u)\big)\end{aligned}$$

shows that the left limit exists. Moreover,

$$\psi_{s-}\big(s,\psi_s(t-,u)\big)=\lim_{\delta\downarrow 0}\psi_{s-\delta}\big(s,\psi_s(t-,u)\big)=\lim_{\delta\downarrow 0}\psi_{s-\delta}(t-,u))$$

shows exchangeability of the limits in (ii). Claim (iii) follows from the semi-flow equations (4) by taking left limits in s, left limits in t, or both. Similarly, claim (iv) follows from (6) by taking left limits in s, or from (7) by taking left limits in t.

For (v), we apply the semi-flow property (4) for $r=s$ and obtain that

$$\Delta\phi_s(t,u)=\phi_s(t,u)-\phi_{s-}(t,u)=\phi_s\big(s,\psi_s(t,u)\big)-\phi_{s-}\big(s,\psi_s(t,u)\big)$$

and the first part of (9) follows. The second part follows analogously. $\square$

3. The characterization of affine semimartingales. In this section, we derive the representation of affine semimartingales via their semimartingale characteristics as well as generalized measure Riccati equations for the coefficients ϕ and ψ. It turns out that the class of affine semimartingales substantially generalizes the class of stochastically continuous affine processes: first, jumps at fixed time points are allowed and second, the jump height may depend on the state of the process.

Throughout, we will use the shorthand notation $\alpha=(\alpha_0,\bar\alpha)$ for a generic $(d+1)$-dimensional vector $\alpha=(\alpha_0,\dots,\alpha_d)$. Moreover, we denote by $\mathcal{S}_+^d$ the convex cone of symmetric positive semidefinite $d\times d$ matrices. Given characteristics (B,C,ν) of a semimartingale X, recall from [21], equation (II.1.23), Proposition II.2.6, that C is always continuous and B can be decomposed as $B=B^c+\sum\Delta B$. Furthermore, also a 'continuous part' ν^c of ν can be defined by

$$\begin{aligned}\mathcal{J}&:=\big\{(\omega,t):\nu(\omega,\{t\},D)>0\big\},\\ \nu^c(\omega,dt,dx)&:=\nu(\omega,dt,dx)\mathbb{1}_{\mathcal{J}^\complement}(\omega,t).\end{aligned}\tag{10}$$

Finally, if one chooses a 'good version' (as we always do) of the characteristics, then

$$\Delta B_t = \int_D h(x)\nu(\{t\}, dx), \tag{11}$$

where h is the truncation function for the jumps; cf. [21], Proposition II.2.9. We introduce the following definition, which will be needed to formulate our main results.

DEFINITION 3.1. Let A be a nondecreasing càdlàg function with continuous part A^c and jump points $J^A := \{t \geq 0 | \Delta A_t > 0\}$. Let $(\gamma, \beta, \alpha, \mu) = (\gamma_i, \beta_i, \alpha_i, \mu_i)_{i \in \{0,\dots,d\}}$ be functions such that $\gamma_0 : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$, $\bar{\gamma} : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$, $\beta_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$, $\alpha_i : \mathbb{R}_{\geq 0} \to \mathcal{S}^d$ and $(\mu_i(t, \cdot))_{t \geq 0}$ are families of (possibly signed) Borel measures on $D \setminus \{0\}$. We call $(A, \gamma, \beta, \alpha, \mu)$ a *good parameter set* if for all $i \in \{0, \dots, d\}$:

(i) α_i and β_i are locally integrable w.r.t. A^c,
(ii) for all compact sets $K \subset D \setminus \{0\}$, $\mu(\cdot, K)$ is locally A^c-integrable,
(iii) $\gamma(t, u) = 0$ for all $(t, u) \in (\mathbb{R}_{\geq 0} \setminus J^A) \times \mathcal{U}$.

THEOREM 3.2. *Let X be a quasiregular affine semimartingale satisfying the support Condition 2.3. Then there exists a good parameter set $(A, \gamma, \beta, \alpha, \mu)$ such that the semimartingale characteristics (B, C, ν) of X w.r.t. the truncation function h satisfy, $\mathbb{P}$-a.s. for any $t > 0$,*

$$B_t^c(\omega) = \int_0^t \Big(\beta_0(s) + \sum_{i=1}^d X_{s-}^i(\omega)\beta_i(s)\Big)\, dA_s^c, \tag{12a}$$

$$C_t(\omega) = \int_0^t \Big(\alpha_0(s) + \sum_{i=1}^d X_{s-}^i(\omega)\alpha_i(s)\Big)\, dA_s^c, \tag{12b}$$

$$\nu^c(\omega, ds, dx) = \Big(\mu_0(s, dx) + \sum_{i=1}^d X_{s-}^i(\omega)\mu_i(s, dx)\Big)\, dA_s^c, \tag{12c}$$

$$\int_D (e^{\langle u, \xi\rangle} - 1)\nu(\omega, \{t\}, d\xi) = \Big(\exp\Big(\gamma_0(t, u) + \big\langle X_{t-}(\omega), \bar{\gamma}(t, u)\big\rangle\Big) - 1\Big). \tag{12d}$$

Moreover, for all $(T, u) \in (0, \infty) \times \mathcal{U}$, the functions ϕ and ψ are absolutely continuous w.r.t. A and solve the following generalized measure Riccati equations: their continuous parts satisfy

$$\frac{d\phi_t^c(T, u)}{dA_t^c} = -F\big(t, \psi_t(T, u)\big), \tag{13}$$

$$\frac{d\psi_t^c(T, u)}{dA_t^c} = -R\big(t, \psi_t(T, u)\big), \tag{14}$$

dA^c-a.e., where

$$
\begin{aligned}
F(s,u) &= \langle \beta_0(s), u\rangle + \frac{1}{2}\langle u, \alpha_0(s)u\rangle + \int_D \big(e^{\langle x,u\rangle} - 1 - \langle h(x), u\rangle\big)\mu_0(s,dx), \\
R_i(s,u) &= \langle \beta_i(s), u\rangle + \frac{1}{2}\langle u, \alpha_i(s)u\rangle + \int_D \big(e^{\langle x,u\rangle} - 1 - \langle h(x), u\rangle\big)\mu_i(s,dx),
\end{aligned} \tag{15}
$$

while their jumps are given by

$$
\begin{aligned}
\Delta\phi_t(T,u) &= -\gamma_0\big(t, \psi_t(T,u)\big), \\
\Delta\psi_t(T,u) &= -\bar{\gamma}\big(t, \psi_t(T,u)\big),
\end{aligned} \tag{16}
$$

and their terminal conditions are

$$
\phi_T(T,u) = 0 \quad \textit{and} \quad \psi_T(T,u) = u. \tag{17}
$$

REMARK 3.3. Note that the parameter set $(A, \gamma, \beta, \alpha, \mu)$ is not uniquely determined: indeed, consider some increasing function A' such that $A \ll A'$ and write $g = \frac{dA}{dA'}$ for the Radon–Nikodym density of A with respect to A'. It is easy to see that all statements of the theorem remain true for the alternative parameter set $(A', \gamma, g\beta, g\alpha, g\mu)$.

REMARK 3.4. We expect that Theorem 3.2 can be extended to affine semimartingales with explosion or killing, by adding a 'fourth characteristic' (cf. [38] and also [3]), which possesses an affine decomposition similar to (12). The rigorous formulation of the corresponding results will not be pursued here, and is left for future research.

The distribution of the jumps of the affine semimartingale occurring at fixed times t can directly be characterized as follows.

LEMMA 3.5. *Let X be a quasiregular affine semimartingale satisfying the support Condition 2.3 and with characteristics (B, C, ν).*

(i) *For any $(t,u) \in (0,\infty) \times \mathcal{U}$,*

$$
\int_D \big(e^{\langle u,\xi\rangle} - 1\big)\nu\big(\omega; \{t\}, d\xi\big) = \exp\big(-\Delta\phi_t(t,u) - \langle \Delta\psi_t(t,u), X_{t-}\rangle\big) - 1. \tag{18}
$$

(ii) *Set*

$$
\begin{aligned}
J^\nu &:= \big\{t > 0 : \mathbb{P}\big(\nu(\omega, \{t\}, D) > 0\big) > 0\big\}, \\
J^{\phi,\psi} &:= \big\{t > 0 : \exists u \in \mathcal{U} \text{ such that } \Delta\phi_t(t,u) \neq 0 \text{ or } \Delta\psi_t(t,u) \neq 0\big\}.
\end{aligned} \tag{19}
$$

Then $J^\nu = J^{\phi,\psi}$.

(iii) *Set $\gamma_0(t,u) = -\Delta\phi_t(t,u)$ and $\bar{\gamma}(t,u) = -\Delta\psi_t(t,u)$. Then (12d) and (16) hold true and $\gamma = (\gamma_0, \bar{\gamma})$ is a good parameter in the sense of Definition 3.1 whenever $J^\nu \subset J^A$.*

PROOF. By definition, $\nu(\{t\}, d\xi)$ is the dual predictable projection of $\delta_{\Delta X_t}(d\xi)$ such that (by Proposition II 1.17 in [21])

$$\int_D (e^{\langle u,\xi\rangle} - 1)\nu(\omega; \{t\}, d\xi) = E\big[(e^{\langle u,\Delta X_t\rangle} - 1)|\mathscr{F}_{t-}\big].$$

Combining with (8), claim (i) follows. For (ii), let $t \in J^\nu$. Then there exists an $u \in \mathcal{U}$, such that the left-hand side of (18) is nonzero. Thus also the right-hand side is nonzero and we conclude that either $\Delta\phi_t(t,u) \neq 0$ or $\Delta\psi_t(t,u) \neq 0$. It follows that $t \in J^{\phi,\psi}$, and hence that $J^\nu \subseteq J^{\phi,\psi}$. For the other direction let $t \in J^{\phi,\psi}$ and choose an $u \in \mathcal{U}$ such that $\Delta\phi_t(t,u) \neq 0$ or $\Delta\psi_t(t,u) \neq 0$. Together with Condition 2.3 on X, we conclude that the right-hand side of (18) is nonzero with strictly positive probability. The same must hold for the left-hand side and we conclude that $t \in J^\nu$, and hence that $J^\nu = J^{\phi,\psi}$. For (iii), note that γ has been defined in such a way that (18) becomes (12d). The jump equations (16) are a direct consequence of (9). If $J^\nu \subset J^A$, then $\gamma(t,u) = 0$ whenever $t \notin J^A$ and it follows that γ is a good parameter. $\square$

We now focus on the continuous parts of the semimartingale characteristics, and make the following definition: For any affine semimartingale X with characteristics (B, C, ν) and for $(T,u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$, we define a complex-valued random measure on $[0, T]$ by

$$\begin{aligned} G(dt,\omega,T,u) :=& \langle \psi_t, dB_t^c(\omega)\rangle + \frac{1}{2}\langle \psi_t, dC_t(\omega)\psi_t\rangle \\ &+ \int_D \big(e^{\langle\psi_t,\xi\rangle} - 1 - \langle\psi_t, h(\xi)\rangle\big)\nu^c(\omega, dt, d\xi), \end{aligned} \tag{20}$$

where we write $\psi_t := \psi_t(T,u)$ for short.

LEMMA 3.6. *Let X be a quasiregular affine semimartingale with a good version of its characteristics (B, C, ν), let $(T,u) \in (0,\infty) \times \mathcal{U}$ and let $G(dt,\omega,T,u)$ be the complex-valued random measure defined in* (20). *It holds that*

$$G(dt;\omega,T,u) + d\phi_t^c(T,u) + \langle X_t(\omega), d\psi_t^c(T,u)\rangle = 0, \quad \mathbb{P}\text{-a.s.}, \tag{21}$$

as identity between measures on $[0, T]$.

PROOF. For $(T,u) \in (0,\infty) \times \mathcal{U}$ consider the process

$$M_t^{u,T} := \mathbb{E}\big[e^{\langle u,X_T\rangle}|\mathscr{F}_t\big] = \exp\big(\phi_t(T,u) + \langle\psi_t(T,u), X_t\rangle\big), \quad t \in [0,T),$$

which is a càdlàg martingale with the terminal value $M_T^{u,T} = \exp(\langle u, X_T\rangle)$. To alleviate notation, we consider (T,u) fixed and write

$$M_t = M_t^{u,T} = \exp\big(\phi_t + \langle\psi_t, X_t\rangle\big),$$

with $\phi_t := \phi_t(T,u)$ and $\psi(t) := \psi_t(T,u)$. Applying the Itô formula for semimartingales (cf. [21], Proposition II.2.42) to M, we obtain a decomposition

$$M_t = L_t + F_t,$$

where L is a local martingale and F is the predictable finite variation process

$$\begin{aligned} F_t := \int_0^t M_{s-}\Big\{ & d\phi_s^c + \langle X_{s-}, d\psi_s^c\rangle + \langle \psi_{s-}, dB_s\rangle + \frac{1}{2}\langle \psi_{s-}, dC_s\psi_{s-}\rangle \\ & + \int_D \big(e^{\Delta\phi_s + \langle \psi_s, X_{s-}+\xi\rangle - \langle \psi_{s-}, X_{s-}\rangle} - 1 - \langle \psi_{s-}, h(\xi)\rangle\big)\nu(\omega, ds, d\xi)\Big\}. \end{aligned} \tag{22}$$

The jump part ΔF vanishes due to Lemma 3.5 and (11), and we are left with the continuous part

$$\begin{aligned} F_t = F_t^c = \int_0^t M_{s-}\Big\{ & d\phi_s^c + \langle X_{s-}, d\psi_s^c\rangle + \langle \psi_{s-}, dB_s^c\rangle + \frac{1}{2}\langle \psi_{s-}, dC_s\psi_{s-}\rangle \\ & + \int_D \big(e^{\langle \psi_{s-}, \xi\rangle} - 1 - \langle \psi_{s-}, h(\xi)\rangle\big)\nu^c(\omega, ds, d\xi)\Big\}. \end{aligned}$$

Recall that M is a martingale, and hence $M \equiv L$ and $F \equiv 0$ on $[0,T]$, $\mathbb{P}$-a.s. With (20), F can be rewritten as

$$F_t = \int_0^t M_{s-}\{d\phi_s^c + \langle X_{s-}, d\psi_s^c\rangle + G(ds;\omega,T,u)\}.$$

Since none of the measures appearing above charges points, the left limits X_{s-}, ψ_{s-} can be substituted by right limits X_s, ψ_s. Moreover, M_{s-} is nonzero everywhere and (21) follows. $\square$

In order to make efficient use of the support Condition 2.3, we introduce the following convention: Given an affine semimartingale X, a tuple $\mathbf{X} = (X^0,\dots,X^d)$ represents $d+1$ *stochastically independent* copies of X. Formally, the tuple $\mathbf{X}$ can be realized on the product space $(\Omega^{(d+1)}, \mathscr{F}^{\otimes(d+1)}, (\mathscr{F}_t^{\otimes(d+1)})_{t\geq 0})$ equipped with the associated product measure. Moreover, for any points $\xi_0,\dots,\xi_d$ in $\mathbb{R}^d$, we define the $(d+1)\times(d+1)$-matrix

$$H(\xi_0,\dots,\xi_n) := \begin{pmatrix} 1 & \xi_0^\top \\ \vdots & \vdots \\ 1 & \xi_n^\top \end{pmatrix}. \tag{23}$$

The matrix-valued process Θ_t is formed by inserting $\mathbf{X} = (X^0,\dots,X^d)$ into H, that is, we set

$$\Theta_t(\omega) = H(X^0,\dots,X^d) = \begin{pmatrix} 1 & X_t^0(\omega)^\top \\ \vdots & \vdots \\ 1 & X_t^d(\omega)^\top \end{pmatrix}. \tag{24}$$

LEMMA 3.7. *Let $s > 0$ and let X be an affine semimartingale satisfying the support Condition 2.3. Then there exists $\epsilon > 0$ and a set $E \in \mathscr{F}_s$ with $\mathbb{P}(E) > 0$, such that the matrices $\Theta_t(\omega)$ and $\Theta_{t-}(\omega)$; are regular for all $(t, \omega) \in (s, s+\epsilon) \times E$.*

PROOF. Define the first hitting time

$$\tau := \inf\{t > s : \Theta_t \text{ singular, or } \Theta_{t-} \text{ singular}\}.$$

Since the set of singular matrices is a closed subset of the vector space of $\mathbb{R}^{(d+1)\times(d+1)}$-matrices, τ is a stopping time; cf. [35], Theorem 1.4. Moreover, by monotone convergence, we have

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\big(\Theta_t \text{ and } \Theta_{t-} \text{ regular for all } t \in (s, s+1/n)\big) &= \lim_{n\to\infty} \mathbb{P}(\tau \geq s + 1/n) \\ &= \mathbb{P}(\tau > s).\end{aligned}$$

If we can show that $\mathbb{P}(\tau > s) > 0$, then the claim follows by choosing N large enough and setting $\epsilon = 1/N$ and $E = \{\tau \geq s + 1/N\}$. But by right continuity of X, the set $\{\omega : \tau(\omega) > s\}$ is equal to $\{\omega : \Theta_s(\omega) \text{ is regular}\}$ and it remains to show that Θ_s is regular with strictly positive probability. By Condition 2.3, it holds that $\operatorname{conv}(\operatorname{supp}(X_s)) = D$ and we can find $d+1$ convex independent points[2] $\xi^0, \dots, \xi^d$ in $\operatorname{supp}(X_s)$. Recalling the definition of H in (23), it follows that $H(\xi^0, \dots, \xi^d)$ is regular. Since the set of regular matrices is open, we find $\delta > 0$ such that even $H(y_0, \dots, y_d)$ is regular for all $y_i \in U_\delta(\xi_i)$, $i \in \{0, \dots, d\}$, where $U_\delta(\xi_i)$ is the open ball of radius δ centered at ξ_i. Now, by independence of $X^0, \dots, X^d$, it follows that

$$\begin{aligned}\mathbb{P}(\Theta_s \text{ is regular}) &\geq \mathbb{P}\big(X_s^i \in U_\delta(\xi_i)\ \forall i \in \{0, \dots, d\}\big) \\ &= \prod_{i=0}^{d} \mathbb{P}\big(X_s \in U_\delta(\xi_i)\big).\end{aligned}$$

Since for each $i \in \{0, \dots, d\}$ the intersection of $U_\delta(\xi_i)$ with the support of X_s is nonempty, all probabilities are strictly positive, and the proof is complete. $\square$

Similar to the $\mathbb{R}^{(d+1)\times(d+1)}$-valued process process $(\Theta_t)_{t\geq 0}$ defined in (24), we define $d+1$ independent copies of the complex-valued random measure $G(dt, \omega, T, u)$ from equation (20) and denote them by $G_0, \dots, G_d$, respectively. With this notation and for any $(T, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$, the $d+1$ corresponding equations

[2] A set of points is called *convex independent* if none of them can be expressed as a convex combination of the remaining points.

(21) can be written in matrix-vector form as

$$\Theta_t(\omega) \cdot \begin{pmatrix} d\phi_t^c(T,u) \\ d\psi_t^{c,1}(T,u) \\ \vdots \\ d\psi_t^{c,d}(T,u) \end{pmatrix} = - \begin{pmatrix} G_0(dt;\omega,T,u) \\ \vdots \\ G_d(dt;\omega,T,u) \end{pmatrix} \tag{25}$$

which holds $\mathbb{P}$-a.s. as an identity between complex-valued measures on $[0,T]$. The next lemma gives a 'local' version of the continuous part of Theorem 3.2.

LEMMA 3.8. *Let X be a quasiregular affine semimartingale satisfying the support Condition 2.3 and let $\tau \in (0,\infty)$ be a deterministic time point. Then there exists an interval $I_\tau := (\tau, \tau+\epsilon)$, where $\epsilon = \epsilon(\tau) > 0$, and good parameters $(A^c, \beta, \alpha, \mu)$ on I_τ. With respect to these parameters, and with F and R as in (15), the measure Riccati equations (13) and (14) hold true for each $(T,u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$ and $t \in I_\tau \cap [0,T]$.*

REMARK 3.9. We emphasize that in this lemma the parameters $(A^c, \beta, \alpha, \mu)$ as well as the functions F and R may depend on τ.

For a semimartingale X, there exists a càdlàg, increasing, predictable, $\mathbb{R}_{\geq 0}$-valued process $\mathcal{A}$ starting in 0 and with continuous part $\mathcal{A}^c$, such that the semimartingale characteristics of X can be 'disintegrated' with respect to $\mathcal{A}$. For the continuous parts (B^c, C, ν^c) of the characteristics, this implies the representation

$$\begin{aligned} B_t^c &= \int_0^t b_s \, d\mathcal{A}_s^c, \\ C_t &= \int_0^t c_s \, d\mathcal{A}_s^c, \\ \nu^c(\omega, dt, dx) &= K_{\omega,t}(dx) \, d\mathcal{A}_t^c(\omega), \end{aligned} \tag{26}$$

where b and c are predictable processes and $K_{\omega,t}(dx)$ a transition kernel from $\Omega \times \mathbb{R}_{\geq 0}$, endowed with the predictable σ-algebra, to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$; see [21], Proposition II.2.9, for further details.

PROOF. Let $X^0, \dots, X^d$ be $d+1$ stochastically independent copies of X. Denote the semimartingale characteristics of X^i by (B^i, C^i, ν^i) and define $G_i(\omega; t, T, u)$ as in (20), $i = 0, \dots, d$. The semimartingale characteristics (B^i, C^i, ν^i) can be disintegrated as in (26). Since we consider only a finite collection of semimartingales, we may assume that the process $\mathcal{A}_s^c(\omega)$ is the same for each X^i.

By Lemma 3.7, there exists an interval $I_\tau = (\tau, \tau+\epsilon)$, $\epsilon > 0$, and a set $E \in \mathscr{F}$ with $\mathbb{P}(E) > 0$ and such that $\Theta_t(\omega)$ is invertible for all $(t,\omega) \in I_\tau \times E$. Multiplying

(25) from the left with the inverse of this matrix yields

$$
(27)\qquad \begin{pmatrix} d\phi_t^c(T,u) \\ d\psi_t^{c,1}(T,u) \\ \vdots \\ d\psi_t^{c,d}(T,u) \end{pmatrix} = -\Theta_t(\omega)^{-1} \cdot \begin{pmatrix} G_0(dt;\omega,T,u) \\ \vdots \\ G_d(dt;\omega,T,u) \end{pmatrix},
$$

as an identity between complex-valued measures on I_τ for all $\omega \in E$. Since $\mathbb{P}(E) > 0$, we can choose some particular $\omega_* \in E$ where (27) holds. Setting

$$
A_t^c := \mathcal{A}_t^c(\omega_*), \quad t \in I_\tau
$$

we observe that $G_i(dt;\omega_*,T,u) \ll dA_t^c$ for each $i \in \{0,\dots,d\}$ and conclude that also the left-hand side of (27) is absolutely continuous with respect to A^c on I_τ. Denote by (b^i, c^i, K^i) the disintegrated semimartingale characteristics of X^i, as in (26). Note that the random measures $G_i(dt;\omega,T,u)$ depend linearly on (b^i, c^i, K^i), which in light of (27) suggests to apply the linear transformation $\Theta_t(\omega)^{-1}$ directly to the disintegrated semimartingale characteristics. Evaluating at ω_*, we hence define the *deterministic* functions $(\beta^i, \alpha^i, \mu^i)_{i\in\{0,\dots,d\}}$ on I_τ by setting

$$
\begin{aligned}
(\beta^0,\beta^1,\dots,\beta^d)_t^\top &:= \Theta_{t-}(\omega_*)^{-1} \cdot (b^0,b^1,\dots,b^d)_t^\top(\omega_*),\\
(\alpha_{kl}^0,\alpha_{kl}^1,\dots,\alpha_{kl}^d)_t^\top &:= \Theta_{t-}(\omega_*)^{-1} \cdot (c_{kl}^0,c_{kl}^1,\dots,c_{kl}^d)_t^\top(\omega_*), \quad k,l \in \{1,\dots,d\},\\
(\mu^0,\mu^1,\dots,\mu^d)_t^\top &:= \Theta_{t-}(\omega_*)^{-1} \cdot (K^0,K^1,\dots,K^d)_t^\top(\omega_*).
\end{aligned}
$$

Using these parameters, the functions F, R can be defined on I_τ as in (15). In combination with (27), it follows that

$$
(28)\qquad
\begin{aligned}
\begin{pmatrix} d\phi_t^c(T,u) \\ d\psi_t^{c,1}(T,u) \\ \vdots \\ d\psi_t^{c,d}(T,u) \end{pmatrix} &= -\Theta_t(\omega_*)^{-1} \cdot \begin{pmatrix} G_0(dt;\omega_*,T,u) \\ \vdots \\ G_d(dt;\omega_*,T,u) \end{pmatrix}\\
&= -\begin{pmatrix} F(t,\psi_t(T,u)) \\ R^1(t,\psi_t(T,u)) \\ \vdots \\ R^d(t,\psi_t(T,u)) \end{pmatrix} dA_t^c
\end{aligned}
$$

for $t \in I_\tau \cap [0,T]$, which yields validity of the Riccati equations (13) and (14) on I_τ. $\square$

PROOF OF THEOREM 3.2. We consider first the continuous parts of the Riccati equations, and thereafter treat their jumps. Applying Lemma 3.8 to each

$\tau \in (0, \infty)$, we obtain a family of intervals I_τ, each with nonempty interior I_τ°, such that $(I_\tau^\circ)_{\tau \in (0,\infty)}$ is an open cover of the positive half-line $(0, \infty)$. Since $\mathbb{R}_{\geq 0}$ can be exhausted by compact sets, such a cover has a countable subcover $\mathcal{S}$. To each interval $I \in \mathcal{S}$, Lemma 3.8 associates good parameters $(A^{c,I}, \beta^I, \alpha^I, \nu^I)$. By countability of $\mathcal{S}$, there exists a continuous common dominating function $A^c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $A^{c,I} \ll A^c$ for all $I \in \mathcal{S}$. As discussed in Remark 3.3, passing from $A^{c,I}$ to A^c has merely the effect of multiplying all parameters with the Radon–Nikodym derivative $\frac{dA^{c,I}}{dA^c}$. Hence, we may assume without loss of generality that $A^{c,I} = A^c$ for each $I \in \mathcal{S}$.

Let now I and $\tilde{I}$ be two intervals with a nonempty intersection, taken from the countable subcover $\mathcal{S}$. Denote by $(A^c, \beta, \alpha, \mu)$ and $(A^c, \tilde{\beta}, \tilde{\alpha}, \tilde{\mu})$ the respective parameter sets obtained for these intervals by application of Lemma 3.8 and by (F, R) and $(\tilde{F}, \tilde{R})$ the corresponding functions defined by (15). We say that these two parameter sets are *compatible* if they agree (up to a dA_t^c-null set) on the intersection $I \cap \tilde{I}$. Once we have shown compatibility for arbitrary intervals I and $\tilde{I}$, it is clear that we can find a single good parameter set (A, β, α, μ), defined on the whole real half-line $\mathbb{R}_{\geq 0}$, such that the Riccati equations (13) and (14) hold true. To condense notation, we introduce the vectors

$$d\Psi_t^c(T, u) := \begin{pmatrix} d\phi_t^c(T, u) \\ d\psi_t^{c,1}(T, u) \\ \vdots \\ d\psi_t^{c,d}(T, u) \end{pmatrix}, \qquad \mathcal{R}(t, u) := \begin{pmatrix} F(t, u) \\ R^1(t, u) \\ \vdots \\ R^d(t, u) \end{pmatrix},$$

$$\tilde{\mathcal{R}}(t, u) := \begin{pmatrix} \tilde{F}(t, u) \\ \tilde{R}^1(t, u) \\ \vdots \\ \tilde{R}^d(t, u) \end{pmatrix}.$$

Applying equation (28) once on the interval I and once on $\tilde{I}$ yields

$$\begin{aligned} \mathcal{R}\big(t, \psi_t(T, u)\big)\, dA_t^c &= d\Psi_t^c(T, u) \\ &= \tilde{\mathcal{R}}\big(t, \psi_t(T, u)\big)\, dA_t^c, \quad t \in I \cap \tilde{I} \cap [0, T]. \end{aligned} \tag{29}$$

Let now $\mathcal{T} \times \mathcal{E}$ be a countable dense subset of $\mathbb{R}_{\geq 0} \times \mathcal{U}$. Taking the union over the countable set $\mathcal{T} \times \mathcal{E}$, we obtain from (29) that

$$\begin{aligned} &\mathcal{R}\big(t, \psi_t(T, u)\big) = \tilde{\mathcal{R}}\big(t, \psi_t(T, u)\big) \\ &\quad \text{for all } (T, u) \in \mathcal{T} \times \mathcal{E} \text{ and } t \in \big(I \cap \tilde{I} \cap [0, T]\big) \setminus N, \end{aligned} \tag{30}$$

where N is a dA_t^c-null set, independent of (T, u).

The next step is to 'evaluate' (30) at $T = t$ and to use that $\psi_t(t, u) = u$ by taking limits in the countable set $\mathcal{T}$. Observe that as functions of Lévy–Khintchine form (cf. (15)) both F and R are continuous in u. By denseness of $\mathcal{T}$ in $\mathbb{R}_{\geq 0}$, we can find a sequence $(T_n) \subseteq \mathcal{T}$ such that $T_n \downarrow t$ as $n \to \infty$.

Together with the right continuity of $\psi_t(T, u)$ in T this yields

$$\mathcal{R}(t,u) = \lim_{n\to\infty} \mathcal{R}\big(t, \psi_t(T_n,u)\big) = \lim_{n\to\infty} \tilde{\mathcal{R}}\big(t, \psi_t(T_n,u)\big) = \tilde{\mathcal{R}}(t,u), \tag{31}$$

for all $u \in \mathcal{E}$. Using continuity of F and R in u, equation (31) can be extended from the dense subset $\mathcal{E}$ to all of $\mathcal{U}$. It is well known that a function of Lévy–Khintchine form determines its parameter triplet uniquely; cf. [37], Theorem 8.1. Hence, we may conclude that

$$\beta_t^i = \tilde{\beta}_t^i, \qquad \alpha_t^i = \tilde{\alpha}_t^i, \qquad \mu_t^i = \tilde{\mu}_t^i,$$

for each $i \in \{0, \dots, d\}$ and $t \in I \cap \tilde{I}$ with exception of the dA_t^c-null set N. This is the desired compatibility property and shows the existence of good parameters $(A^c, \beta, \alpha, \nu)$.

We now turn to the continuous parts of the semimartingale characteristics (B, C, ν) and show (12a), (12b) and (12c). To this end, fix $(T, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$ and let (b, c, K) be the continuous semimartingale characteristics of X, disintegrated with respect to the increasing predictable process $\mathcal{A}_t^c(\omega)$, as in (26). For each $\omega \in \Omega$, write

$$\mathcal{A}_t^c(\omega) = \int_0^t a_s(\omega)\, dA_t^c + \mathcal{S}_t(\omega)$$

for the Lebesgue decomposition of $\mathcal{A}_t^c(\omega)$ with respect to A_t^c.[3] Furthermore, define

$$\begin{aligned} g(\omega, t, T, u) :=& \langle \psi_t, b_t(\omega)\rangle + \frac{1}{2}\langle \psi_t, c_t(\omega)\psi_t\rangle \\ &+ \int_D \big(e^{\langle \psi_t, \xi\rangle} - 1 - \langle \psi_t, h(\xi)\rangle\big) K_t(\omega, d\xi), \end{aligned} \tag{32}$$

which can be considered as the disintegrated analogue of (20). Combining (25) with the Riccati equations, we obtain that

$$\begin{aligned} \Theta_t(\omega; x) \cdot \mathcal{R}\big(t, \psi_t(T,u)\big)\, dA_t^c =& g(\omega, t, u, T) a_t(\omega)\, dA_t^c \\ &+ g(\omega, t, u, t)\, d\mathcal{S}_t(\omega) \end{aligned} \tag{33}$$

for all $(T, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$ and $t \in [0, T]$. By the uniqueness of the Lebesgue decomposition, we conclude that

$$\begin{cases} a_t(\omega) g(\omega, t, T, u) = \Theta_t(\omega) \cdot \mathcal{R}\big(t, \psi_t(T,u)\big), & dA_t^c\text{-a.e.,} \\ g(\omega, t, T, u) = 0, & d\mathcal{S}_t(\omega)\text{-a.e.} \end{cases} \tag{34}$$

---

[3] Note that our argument does not require measurability of $\omega \mapsto a_s(\omega)$ or $\omega \mapsto \mathcal{S}_t(\omega)$.

As in the first part of the proof, we consider a countable dense subset $\mathcal{T} \times \mathcal{E}$ of $\mathbb{R}_{\geq 0} \times \mathcal{U}$. Taking the union over all (T, u) in $\mathcal{T} \times \mathcal{E}$ and repeating the density arguments of (31), we find an dA_t^c-null set N_1 and a $d\mathcal{S}_t(\omega)$-null set N_2, such that

$$\begin{cases} a_t(\omega)g(\omega, t, t, u) = \Theta_t(\omega) \cdot \mathcal{R}(t, u), & \text{for all } t \in \mathbb{R}_{\geq 0} \setminus N_1, u \in \mathcal{E}, \\ g(\omega, t, t, u) = 0, & \text{for all } t \in \mathbb{R}_{\geq 0} \setminus N_2, u \in \mathcal{E}. \end{cases} \tag{35}$$

As functions of u, both sides are of Lévy–Khintchine form. In addition, $\mathcal{E}$ is dense in $\mathcal{U}$, which allows us to conclude from the first equation that

$$a_t(\omega)b_t(\omega) = \Theta_t(\omega) \cdot (\beta_t^0, \dots, \beta_t^d),$$

$$a_t(\omega)c_t(\omega) = \Theta_t(\omega) \cdot (\alpha_t^0, \dots, \alpha_t^d),$$

$$a_t(\omega)K_t(\omega) = \Theta_t(\omega) \cdot (\mu_t^{c,0}, \dots, \mu_t^{c,d})$$

for all $t \in \mathbb{R}_{\geq 0} \setminus N_1$ and from the second equation that

$$b_t(\omega) = 0, \qquad c_t(\omega) = 0, \qquad K_t(\omega) = 0, \qquad d\mathcal{S}_t(\omega)\text{-a.e.}$$

Integrating with respect to $\mathcal{A}_t^c(\omega)$ and adding up yields (12).

To conclude the proof, we finally turn to the discontinuous part. Note that Lemma 3.5 already provides us with parameters γ, a set J^ν and the validity of (12c) and (16). Taking the continuous increasing function A^c from the first part of the proof and inserting jumps of strictly positive hight at each time $t \in J^\nu$, we obtain an increasing function A with continuous part A^c and jump set $J^A = J^\nu$. Note that the heights of the jumps are arbitrary; for example, the values of the summable series $(2^{-n})_{n\in\mathbb{N}}$ can be taken. Together, $(A, \gamma, \alpha, \beta, \mu)$ is now a good parameter set in the sense of Definition 3.1 and all parts of Theorem 3.2 have been shown. □

4. Affine Markov processes and infinite divisibility.

Let X be a Markov process in D (possibly nonconservative) with transition kernels $p_{s,t}(x, B)$, defined for all $0 \leq s \leq t$, $x \in D$ and $B \in \mathcal{B}(D)$. The following definition is analogous to [10], Definition 2.1.

DEFINITION 4.1. A Markov process X in D is called *affine Markov process*, if there exist $\mathbb{C}$- and $\mathbb{C}^d$-valued functions ϕ, ψ, such that the transition kernels of X satisfy

$$\int_D e^{\langle u, \xi \rangle} p_{s,t}(x, d\xi) = e^{\phi_s(t,u) + \langle \psi_s(t,u), x \rangle}, \tag{36}$$

for all $0 \leq s \leq t$, $(x, u) \in D \times \mathcal{U}$.

An affine Markov process need not be a semimartingale, as we show in Example 6.4. However, under mild conditions, affine semimartingales are affine Markov processes. First, note that to every affine semimartingale we can associate transition kernels $p_{s,t}(x, B)$, defined for all $0 \le s \le t$, $B \in \mathcal{B}(D)$ and $x \in \operatorname{supp}(X_s)$, by considering the regular conditional distributions

$$\mathbb{P}(X_t \in B | X_s) = p_{s,t}(X_s, B). \tag{37}$$

By (3), the kernels will satisfy (36) for all $x \in \operatorname{supp}(X_s)$ and the semi-flow equations (4) provide the Chapman–Kolmogorov equations for the kernels $p_{s,t}(x, \cdot)$. It remains to show that the family of transition kernels and the validity of (36) can be extended from $\operatorname{supp}(X_s)$ to D. Apart from the trivial condition $\operatorname{supp}(X_s) = D$ for all $s > 0$, we can give the following sufficient condition:

DEFINITION 4.2. An affine semimartingale X is called infinitely divisible, if the regular conditional distributions $p_{s,t}(X_s, \cdot)$ are infinitely divisible probability measures on D, $\mathbb{P}$-a.s. for any $0 \le s \le t$.

LEMMA 4.3. *Let X be a quasiregular affine semimartingale satisfying the support Condition 2.3. Suppose that*:

(i) $\operatorname{supp}(X_t) = D$ *for all* $t > 0$, *or*
(ii) *X is infinitely divisible.*

Then X can be realized as a conservative affine Markov process with state space D.

PROOF. It suffices to show that the right-hand side of (36) is the Fourier transform of a probability measure on D for all $x \in D$ and $0 < s \le t$. Indeed, if the family $(p_{s,t})_{0 \le s \le t}$ satisfies (36), the semi-flow equations (4) ensure that it satisfies the Chapman–Kolmogorov equations. By the Kolmogorov existence theorem (see, e.g., [23], Theorem 8.4), this guarantees the existence of a unique Markov process with transition kernels $(p_{s,t})_{0 \le s \le t}$. Let $p_{s,t}(x, \cdot)$ be the transition kernels of the semimartingale X, defined by (37). Note that by the affine property (3), these kernels satisfy (36) for all $x \in \operatorname{supp}(X_s)$, and it remains to extend the identity to all $x \in D$. In case (i), this is trivial for $s > 0$, since $\operatorname{supp}(X_s) = D$. In case (ii), by infinite divisibility, there exists, for any $\lambda \in (0, 1)$, a probability kernel $p_{s,t}^{(\lambda)}(x, \cdot)$, such that

$$\int_D e^{\langle u, \xi \rangle} p_{s,t}^{(\lambda)}(x, d\xi) = e^{\lambda \phi_s(t,u) + \langle \psi_s(t,u), \lambda x \rangle}. \tag{38}$$

Fix $x, y \in \operatorname{supp}(X_s)$, $\lambda \in (0, 1)$ and let $z = \lambda x + (1 - \lambda) y$ be a convex midpoint of x and y. At z we define $p_{s,t}(z, \cdot) := p_{s,t}^{\lambda}(x, \cdot) \star p_{s,t}^{(1-\lambda)}(y, \cdot)$, where $\star$ denotes convolution of measures, and obtain

$$\int_D e^{\langle u, \xi \rangle} p_{s,t}(z, d\xi) = e^{\phi_s(t,u) + \langle \psi_s(t,u), \lambda x + (1-\lambda) y \rangle} = e^{\phi_s(t,u) + \langle \psi_s(t,u), z \rangle} \tag{39}$$

that is, (36) has been extended to the convex midpoint $z = \lambda x + (1 - \lambda) y$ of x and y. By Condition 2.3, we have $\operatorname{conv}(\operatorname{supp}(X_s)) = D$ for all $s > 0$, which shows

(36), except at the time point $s = 0$. In both cases (i) and (ii), we can finally use the quasiregularity property of ϕ, ψ to immediately extend (36) to $s = 0$ by taking limits from the right. □

It turns out that infinite divisibility has even stronger implications on the structure of affine semimartingales, in particular, at the deterministic jump times J^A.

LEMMA 4.4. *Let X be an infinitely divisible, quasiregular affine semimartingale satisfying the support Condition 2.3. Then the conditional distribution of ΔX_t given X_{t-} is $\mathbb{P}$-a.s infinitely divisible, for any $t \geq 0$. Moreover, the parameters $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_d)$ in Theorem 3.2 are of the following form: For any $t \in J^A$ and $i \in \{0, \dots, d\}$, there exist $\tilde{\beta}_i(t) \in \mathbb{R}^d$, $\tilde{\alpha}_i(t) \in \mathcal{S}^d$ and a (possibly signed) Borel measure $\tilde{\mu}_i(t, \cdot)$ on $D \setminus \{0\}$, such that*

$$\gamma_i(t,u) = \langle \tilde{\beta}_i(t), u\rangle + \frac{1}{2}\langle u, \tilde{\alpha}_i(t)u\rangle + \int_D \big(e^{\langle x,u\rangle} - 1 - \langle h(x), u\rangle\big)\tilde{\mu}_i(t, dx), \tag{40}$$

for all $u \in \mathcal{U}$.

PROOF. Using Lemma 3.5 and the quasiregularity property from Definition 2.5, we can write

$$\begin{aligned} E\big[e^{\langle u, X_t\rangle} | \mathscr{F}_{t-}\big] &= \exp\big(-\Delta\phi_t(t,u) - \langle \Delta\psi_t(t,u), X_{t-}\rangle\big) \\ &= \lim_{s\uparrow t} \exp\big(\phi_s(t,u) + \langle \psi_s(t,u), X_s\rangle\big) \\ &= \lim_{s\uparrow t} \int_D e^{\langle u,\xi\rangle} p_{s,t}(X_s, d\xi). \end{aligned}$$

Note that the right-hand side is the limit of Fourier–Laplace transforms of infinitely divisible measures on D. The left-hand side is the Fourier–Laplace transform of the distribution of X_t, conditionally on $\mathscr{F}_{t-}$, and we conclude that also this distribution must be infinitely divisible. By Lemma 3.5, $\gamma_0(t,u) = -\Delta\phi_t(t,u)$ and $\gamma_i(t,u) = -\Delta\psi_t^i(t,u)$ for all $i \in \{1, \dots, d\}$. The decomposition (40) then follows from the Lévy–Khintchine formula for infinitely divisible distributions. □

Recall the definition of a *good parameter set* $(A, \gamma, \beta, \alpha, \mu)$ from Definition 3.1, and note that the functions $\beta(t)$, $\alpha(t)$ and $\mu(t, \cdot)$ are only defined up to A^c-null sets. In particular, we can modify β, α, μ at any jump point $t \in J^A$ without affecting the validity of Theorem 3.2. In light of the decomposition (40) of γ, this suggests the following definition.

DEFINITION 4.5. Let $(A, \gamma, \beta, \alpha, \mu)$ be the good parameter set of a quasiregular infinitely divisible affine semimartingale X satisfying the support Condition 2.3. We *enhance* the functions β, α, μ by setting

$$\begin{aligned}\alpha_i(t) &= \frac{1}{\Delta A_t}\tilde{\alpha}_i(t), \qquad \beta_i(t) = \frac{1}{\Delta A_t}\tilde{\beta}_i(t),\\ \mu_i(t, d\xi) &= \frac{1}{\Delta A_t}\tilde{\mu}_i(t, d\xi), \quad \text{for all } t \in J^A,\ i \in \{0, \dots, d\},\end{aligned} \tag{41}$$

with $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\mu}$ as in Lemma 4.4 and refer to (A, β, α, μ) as *enhanced parameter set* of X.

Note that γ does no longer appear in the enhanced parameter set, since it was absorbed into the values of α, β, μ at the time points $t \in J^A$. The enhanced parameters also allow us to combine F with γ and R with $\bar{\gamma}$ by setting

$$\begin{aligned}\mathfrak{F}(t, u) &:= F(t, u)\mathbb{1}_{\{t \notin J^A\}} + \frac{1}{\Delta A_t}\gamma_0(t, u)\mathbb{1}_{\{t \in J^A\}},\\ \mathfrak{R}(t, u) &:= R(t, u)\mathbb{1}_{\{t \notin J^A\}} + \frac{1}{\Delta A_t}\bar{\gamma}(t, u)\mathbb{1}_{\{t \in J^A\}}.\end{aligned}$$

Both $\mathfrak{F}$ and $\mathfrak{R}$ are of Lévy–Khintchine form and the continuous part (13)–(14) and discontinuous part (16) of the measure Riccati equations can be unified into the measure differential equations

$$\begin{aligned}\frac{d\phi_t(T, u)}{dA_t} &= -\mathfrak{F}\big(t, \psi_t(T, u)\big),\\ \frac{d\psi_t(T, u)}{dA_t} &= -\mathfrak{R}\big(t, \psi_t(T, u)\big),\end{aligned}$$

which, together with the terminal conditions (17), are equivalent to the integral equations

$$\phi_t(T, u) = \int_{(t,T]} \mathfrak{F}\big(s, \psi_s(T, u)\big)\, dA_s, \tag{42a}$$

$$\psi_t(T, u) = u + \int_{(t,T]} \mathfrak{R}\big(s, \psi_s(T, u)\big)\, dA_s. \tag{42b}$$

5. Existence of affine Markov processes and affine semimartingales.

In this section, we show, under mild assumptions, the existence of affine semimartingales, using affine Markov processes as an intermediate step. While we have made no restriction on the state space D before, we consider throughout this section only the ‘canonical state space’ (cf. [10, 14])

$$D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n, \qquad m + n = d.$$

Note that for this state space, $\mathcal{U}$ takes the form $\mathcal{U} = \mathbb{C}^m_{\leq 0} \times i\mathbb{R}^n$. In addition, we have

$$\partial\mathcal{U} = i\mathbb{R}^d, \qquad \mathcal{U}^o = \mathbb{C}^m_{<0} \times i\mathbb{R}^n,$$

as in [10]. For notational simplicity, we denote $\mathcal{I} = \{1, \dots, m\}$, $\mathcal{J} = \{m + 1, \dots, d\}$, and $\mathcal{I} \setminus i := \mathcal{I} \setminus \{i\}$, $\mathcal{J} \cup i := \mathcal{J} \cup \{i\}$ for any i. Finally, we introduce the following shorthand notation:

- For two subsets $I, J \subset \{1, \dots, d\}$, we denote by a_{IJ} the submatrix of a with indices in $I \times J$, that is, $a_{IJ} := (a_{ij})_{i \in I, j \in I}$.
- β denotes the matrix with columns $\beta_0, \beta_1, \dots, \beta_d$. We write $\bar{\beta}$ for β with the first column dropped.
- For any $i, k \in \{1, \dots, d\}$, we set $\bar{H}_{ik}(t) := \int_{D \setminus \{0\}} h_i(\xi) \mu_k(t, d\xi)$ whenever the integral is finite. The other values can be chosen arbitrarily, and the resulting matrix is denoted by $\bar{H}(t) = (\bar{H}_{ik}(t))$. Moreover, we define the column vector $H_0(t) := \int_{D \setminus \{0\}} h(\xi) \mu_0(t, d\xi)$.

Recall from Theorem 3.2 that an affine semimartingale X has a good parameter set $(A, \gamma, \alpha, \beta, \mu)$. To show existence of an affine semimartingale given a good parameter set, we also need to take into account the geometry of our state space. In [10], this was done by introducing admissibility conditions on the parameters. In the following definition, we extend this notion to our setting.

DEFINITION 5.1. A good parameter set $(A, \gamma, \alpha, \beta, \mu)$ is called *admissible*, if:

(i) for A^c-almost all $t \in \mathbb{R}_{\geq 0}$,

- $\alpha_i(t) \in \mathcal{S}_+^d$ for all $i \in \{0, \dots, d\}$, $\alpha_{0;\mathcal{I}\mathcal{I}}(t) = 0$, $\alpha_{i;\mathcal{I}\setminus i, \mathcal{I}\setminus i}(t) = 0$ for $i \in \mathcal{I}$, and $\alpha_j(t) = 0$ for $j \in J$,
- $\beta(t) \in \mathbb{R}^{d \times (d+1)}$ such that $\beta_0(t) - H_0(t) \in D$, $\bar{\beta}_{\mathcal{I}\mathcal{J}}(t) = 0$ and $\bar{\beta}_{i(\mathcal{I}\setminus i)}(t) - \bar{H}_{i(\mathcal{I}\setminus i)}(t) \in \mathbb{R}_{\geq 0}^{m-1}$ for all $i \in \mathcal{I}$,
- $\mu(t)$ is a vector of Lévy measures with support on D such that $\mu_j(t) = 0$ for $j \in \mathcal{J}$ and $\mathcal{M}_i(t) < \infty$ for $i \in \mathcal{I} \cup 0$, where

$$\mathcal{M}_i(t) := \int_{D \setminus \{0\}} (\langle h_{\mathcal{I}\setminus i}(\xi), 1 \rangle + \|h_{\mathcal{J}\cup i}(\xi)\|^2) \mu_i(t, d\xi); \tag{43}$$

(ii) for all $t \in J^A$ and all $x \in D$, the function $u \mapsto \exp(\gamma_0(t, u) + \langle \bar{\gamma}(t, u) + u, x \rangle)$ is the Fourier–Laplace transform of a D-valued random variable.

If X is infinitely divisible and (A, α, β, μ) its enhanced parameter set (see Definition 4.5), then (ii) can be replaced by:

(ii′) for all $t \in J^A$ and $i \in \{0, \dots, d\}$,

- $\alpha_i(t) \in \mathcal{S}_+^d$, $\alpha_{i;\mathcal{I}\mathcal{I}}(t) = 0$ for $i \in \mathcal{I} \cup 0$ and $\alpha_j(t) = 0$ for $j \in \mathcal{J}$,
- $\beta_0(t) - H_0(t) \in D$, $\bar{\beta}_{\mathcal{I}\mathcal{J}}(t) = 0$ and $\bar{\beta}_{\mathcal{I}\mathcal{I}}(t) - \bar{H}_{\mathcal{I}\mathcal{I}}(t) + \mathrm{id}_m \in \mathbb{R}_{\geq 0}^m$,
- $\mu_i(t)$ is a Lévy measure on D with $\int_{D \setminus 0} (\langle h_{\mathcal{I}}(\xi), 1 \rangle + \|h_{\mathcal{J}}(\xi)\|^2) \times \mu_i(t, d\xi) < \infty$ for $i \in \mathcal{I} \cup 0$ and $\mu_j = 0$ for $j \in J$.

Note that a zero element on the diagonal of a positive semidefinite matrix implies that the whole corresponding row and column is zero; therefore further restrictions on the elements of α_i can be derived from the above conditions. In addition, we remark that the parameters $\beta(t)$ are unique only up to the choice of the truncation functions h_i, which in turn determine the compensator matrix H; see also [10], Remark 2.13.

PROPOSITION 5.2. *Let X be a quasiregular affine semimartingale satisfying the support Condition 2.3 with good parameter set $(A, \gamma, \alpha, \beta, \mu)$. Suppose that*:

(i) $\operatorname{supp}(X_t) = D$ *for all* $t > 0$, *or*
(ii) X *is infinitely divisible.*

Then the parameters $(A, \gamma, \alpha, \beta, \mu)$ are admissible.

PROOF. By Lemma 4.3, X can be realized as a (time-inhomogeneous) Markov process with transition kernels $p_{s,t}(x, d\xi)$, defined for all $0 \le s \le t$ and $x \in D$. Set $f_u(x) = e^{\langle u,x\rangle}$ for $u \in \mathcal{U}$. Similar to the proofs of admissibility in [10], we consider the following limit:

$$\begin{aligned} G_t f_u(x) &:= \lim_{h\downarrow 0} \frac{\mathbb{E}[f_u(X_t)|X_{t-h} = x] - e^{\langle u,x\rangle}}{A_t - A_{t-h}} \\ &= \lim_{h\downarrow 0} \frac{\exp(\phi_{t-h}(t,u) + \langle \psi_{t-h}(t,u), x\rangle) - e^{\langle u,x\rangle}}{A_t - A_{t-h}}. \end{aligned} \tag{44}$$

For A^c-almost all $t \in \mathbb{R}_{\ge 0}$, there exists a sequence $(h_n)_{n\in\mathbb{N}}$, decreasing to 0, along which the limit exists (cf. the main Theorem in [7] or [2], Theorem 5.8.8). From (7), together with (13) and (14), we can identify the limit to be

$$G_t f_u(x) = \big(F(t,u) + \langle R(t,u), x\rangle\big) f_u(x). \tag{45}$$

For $t \in J^A$, we obtain instead from (8) that

$$G_s f_u(x) = \big(e^{-\Delta\phi_s(s,u) - \langle \Delta\psi_s(s,u), x\rangle} - 1\big) \cdot \frac{1}{\Delta A_s} f_u(x). \tag{46}$$

On the other hand, we can write the limit in terms of the transition kernels of X as

$$\frac{G_t f_u(x)}{f_u(x)} = \lim_{n\to\infty} \frac{1}{A_t - A_{t-h_n}} \bigg(\int_D \big(f_u(\xi - x) - 1\big) p_{t-h_n,t}(x, d\xi) \bigg).$$

By (45) and (46), the above limit exists and is continuous at $u = 0$. If t is a continuity point of A, we interpret the integral term in the last line as the log-characteristic function of a compound Poisson distribution with intensity $1/(A_t - A_{t-h_n})$, which is infinitely divisible. This implies that also their weak limit is infinitely divisible. We conclude that the right-hand side of (45) is the log-characteristic functions of an infinitely divisible distributions and, therefore, of Lévy–Khintchine form. From here, the admissibility of (α, β, μ) at points of continuity of A follows on the same lines as in [10]. For discontinuity points $t \in J^A$ of A, we obtain from (46) that

$$\exp\big(\gamma_0(t,u) + \langle \bar{\gamma}(t,u) + u, x\rangle\big) = \int_D f_u(\xi) p_{t-,t}(x, d\xi),$$

where we have written $p_{t-,t}(x,\cdot)$ for the weak limit of $p_{t-h,t}(x,\cdot)$ as $h \downarrow 0$. Part (ii) of the admissibility conditions follows form the fact that $p_{t-,t}(x,\cdot)$ must be supported on D for all $x \in D$ and $0 \le s \le t$. If X is infinitely divisible, then the decomposition of γ as (40), together with standard support theorems for infinitely divisibly distributions (cf. [37], Theorem 24.10) yield (ii′). $\square$

REMARK 5.3. We illustrate the geometric intuition behind the admissibility conditions (ii′) at deterministic jump times: Consider the state space $D = [0, \infty)$, a deterministic jump time $t \in J^A$ and the simplifying conditions $\alpha_0(t) = \alpha_1(t) = 0$, $\mu_0(t) = \mu_1(t)$, $\beta_0(t) = 0$ and $\beta_1(t) = -1$. In this case, the jump at t is given by

$$\Delta X_t = \beta_1(t) X_{t-} = -X_{t-},$$

such that $X_t = X_{t-} + \Delta X_t = 0$. Thus, the process X takes a jump of *negative height* from X_{t-} to 0; a phenomenon that can not take place for stochastically continuous affine processes on the state space D. Moreover, the jump is guaranteed to land at 0, that is, at the boundary of D. Values of $\beta_1(t) \in (-1, 0)$ also lead to jumps of negative height, but landing within the interior $(0, \infty)$ of D. Values of $\beta_1(t) < -1$ would lead to jumps landing outside of D and are therefore not admissible. The condition on β in (ii′) can be compared to the *tenability conditions* of Pólya urns; cf. [22], equation (1.2).

In the remaining part of the section, we show the following: Given an admissible enhanced parameter set, we can construct a Markov process that is an infinitely divisible affine semimartingale for every starting point in D. In this regard, we require a further integrability assumption.

ASSUMPTION 5.4. Given an enhanced parameter set (A, β, α, μ), assume that α, β and $\mathcal{M}$ defined by (43) are locally integrable with respect to A.

PROPOSITION 5.5. *Let (A, α, β, μ) be an admissible enhanced parameter set satisfying Assumption 5.4. Then, for all $(T, u) \in (0, \infty) \times \mathcal{U}^\circ$ there exists a unique solution $(\phi_\cdot(T,u), \psi_\cdot(T,u))$ on $[0, T]$ to the generalized measure Riccati equations* (12)–(17) (*or equivalently to* (42)).

In the following, let $u = (v, w) \in \mathcal{U}$ with $v \in \mathbb{C}^m_{\le 0}$ and $w \in i\mathbb{R}^n$. We will also use the convention $\int_{(a,b]} = \int_a^b$ to shorten notation in some places.

PROOF. Since an enhanced parameter set is given, the generalized measure Riccati equations (12)–(17) can be combined into (42). It suffices to show existence of a unique global solution to equation (42b), since existence and uniqueness for (42a) then follows by simple integration (note that ϕ does not appear on the right-hand side of (42a)). Due to the admissibility conditions, the equation for ψ can be split into an equation for the components $\psi^{\mathcal{I}} = (\psi^i)$, $i \in \mathcal{I}$ and a decoupled linear equation for the components with $j \in \mathcal{J}$ (see also [10], Section 6), which can be written as

$$\psi_t^{\mathcal{J}}(T, u) = w + \int_t^T \bar{\beta}_{JJ}(s)\psi_s^{\mathcal{J}}(T, u)\, dA_s.$$

This linear equation can be solved according to Example A.4 in the Appendix which yields a function with linear dependency on the starting value w, that is,

$$\psi_t^{\mathcal{J}}(T, u) = w\psi_t^{\mathcal{J}}(T), \qquad \psi^{\mathcal{J}}(T) : [0, T] \to \mathbb{R}^{n\times n}. \tag{47}$$

The existence and uniqueness of a *local* solution to the generalized measure Riccati equation (42b) is a consequence of Theorem A.3 in the Appendix. Indeed, $\mathfrak{R}(t, (v, w))$ is of Lévy–Khintchine form, hence analytical in v by Lemma 5.3(i) in [10], and thus locally Lipschitz continuous in u with a Lipschitz constant that can be chosen A-integrable, due to the integrability of the enhanced parameters (α, β, μ).

To extend the local solution to the entire time horizon, we adopt the proof in [10] to our setting. Let $g(\cdot, T, u)$ be a local solution to the Riccati equations with terminal condition $u \in \mathcal{U}^\circ$ at time T. We have to show that g extends—backwards in time—to a global solution on $[0, T]$. Consider the lifetime of g in $\mathcal{U}^\circ$

$$\tau_{T,u} := \limsup_{n\to\infty}\{t \in \mathbb{R}_+ \mid \|g(t, T, u)\| \geq n \text{ or } g(t, T, u) \in (\mathcal{U}^\circ)^{\complement}\}.$$

For the existence on the entire time horizon $\tau_{T,u}$ has to be zero, for all $u \in \mathcal{U}^\circ$. Similar to [10], equation (6.8), we obtain from the Lévy–Khintchine form of $\mathfrak{R}$ for dA-almost-all t that

$$\operatorname{Re} \mathfrak{R}_i(t, u) \leq C(t)\big((\operatorname{Re} u_i)^2 - \operatorname{Re} u_i\big), \tag{48}$$

where $C(t)$ is a constant independent of u, for all t. The integrability of the parameters of $\mathfrak{R}$ allows to choose C as also being A-integrable. Hence the local solution g satisfies the following integral inequality:

$$\operatorname{Re} g_t^i(T, u) \leq v + \int_{(t,T]} C(s)\big(\big(\operatorname{Re} g_s^i(T, u)\big)^2 - \operatorname{Re} g_s^i(T, u)\big)\, dA_s.$$

By the comparison result Proposition A.5 for measure differential equations, stated in the Appendix, we get

$$\operatorname{Re} g_t^i(T, u) \leq f_t(T, u),$$

where f satisfies

$$f_t(T,u)=\operatorname{Re} v+\int_{(t,T]} C(s)\big(f_s(T,u)^2-f_s(T,u)\big)\,dA_s.$$

Note that for all $K>0$ there exists $c>0$ such that $(x^2-x)<-cx$ as long as $x\in(-K,0)$. Hence, $f_\cdot(T,u)<0$ for all $u\in\mathcal{U}^\circ$.

For the upper bound, we consider the squared norm of $\psi^{\mathcal{I}}$. With the chain rule formula for functions of bounded variation in [4], Theorem 4.1, we can write

$$\begin{aligned}
\|\psi_t^{\mathcal{I}}(T,u)\|^2 &= \|v\|^2+\int_{(t,T]} 2\operatorname{Re}\langle\overline{\psi_s^{\mathcal{I}}(T,u)},\mathfrak{R}^{\mathcal{I}}\big(s,\psi_s^{\mathcal{I}}(T,u),\psi_s^{\mathcal{J}}(T,u)\big)\rangle\,dA_s^c\\
&\quad+\sum_{s\in(t,T]}\|\psi_s^{\mathcal{I}}(T,u)\|^2-\|\psi_{s-}^{\mathcal{I}}(T,u)\|^2\\
&= \|v\|^2+\int_{(t,T]} 2\operatorname{Re}\langle\overline{\psi_s^{\mathcal{I}}(T,u)},\mathfrak{R}^{\mathcal{I}}\big(s,\psi_s^{\mathcal{I}}(T,u),\psi_s^{\mathcal{J}}(T,u)\big)\rangle\,dA_s\\
&\quad-\sum_{s\in(t,T]}\langle\overline{\Delta\psi_s^{\mathcal{I}}(T,u)},\Delta\psi_s^{\mathcal{I}}(T,u)\rangle\\
&\le \|v\|^2+\int_{(t,T]} 2\operatorname{Re}\langle\overline{\psi_s^{\mathcal{I}}(T,u)},\mathfrak{R}^{\mathcal{I}}\big(s,\psi_s^{\mathcal{I}}(T,u),\psi_s^{\mathcal{J}}(T,u)\big)\rangle\,dA_s,
\end{aligned}\tag{49}$$

where we have used $\psi_{s-}^{\mathcal{I}}(T,u)=\psi_s^{\mathcal{I}}(T,u)-\Delta\psi_s^{\mathcal{I}}(T,u)$ in the second line. With

$$K(t,u):=\operatorname{Re} v_i\langle\alpha^i_{\mathcal{J}\mathcal{J}}(t)w,w\rangle+\operatorname{Re}\bar{v}_i\langle\beta_i(t)-H_i(t),u\rangle,$$

we can write

$$\begin{aligned}
\operatorname{Re}\big(\bar{v}_i\mathfrak{R}_i(t,u)\big)&=\alpha^i_{ii}(t)|v_i|^2\operatorname{Re} v_i+K(t,u)\\
&\quad+\operatorname{Re}\bigg(\bar{v}_i\int_{D\setminus\{0\}}\big(e^{\langle u,\xi\rangle}-1-\langle u_{\mathcal{J}\cup i},h_{\mathcal{J}\cup i}(\xi)\rangle\big)\mu_i(t,d\xi)\bigg).
\end{aligned}$$

Using the same calculations as Proposition 6.1 in [10], we obtain the following estimate:

$$\operatorname{Re}\big(\bar{v}_i\mathfrak{R}_i(t,u)\big)\le C_t\big(1+\|w\|^2\big)\big(1+\|v\|^2\big),\quad \forall u=(v,w)\in\mathcal{U}.$$

From the A-integrability of $\mathcal{M}$, it follows that C, which is independent of u, can be chosen A-integrable. Inserting the above equation into (49), we obtain

$$\|\psi_t^{\mathcal{I}}(T,u)\|^2\le\|v\|^2+\int_{(t,T]} C_s\big(1+\|\psi_s^{\mathcal{J}}(T,u)\|^2\big)\big(1+\|\psi_s^{\mathcal{I}}(T,u)\|^2\big)\,dA_s.$$

Gronwalls inequality for measure differential equations (cf. [19], Corollary 19.3.3) yields

$$\|\psi_t^{\mathcal{I}}(T,u)\|^2\le\|v\|^2\exp\bigg(\int_{(t,T]} C_s\big(1+\|\psi_s^{\mathcal{J}}(T,u)\|^2\big)\,dA_s\bigg).\tag{50}$$

With (47), this shows that the solution can not explode, and thus $\tau_{T,u}=0$, that is, we have a solution on $[0,T]$. □

PROPOSITION 5.6. *Let (ϕ, ψ) be a solution to the generalized measure Riccati equations* (13)–(17). *Then it holds that*:

(i) *for each $u \in \mathcal{U}$ and $s < t$ the left limits*

$$\phi_s(t-, u) = \lim_{\varepsilon \downarrow 0} \phi_s(t - \varepsilon), \quad \text{and} \quad \psi_s(t-, u) = \lim_{\varepsilon \downarrow 0} \psi_s(t - \varepsilon, u)$$

exist.

(ii) *For all $u = (v, w) \in \mathcal{U}$ and $s \le t$, $\psi_s^{\mathcal{J}}(t, (v, 0)) = 0$.*

(iii) (ϕ, ψ) *satisfy the semi-flow property*, *that is*, *let $r \le s \le t$ then for all $u \in \mathcal{U}^\circ$*

$$\phi_r(t, u) = \phi_s(t, u) + \phi_r\big(s, \psi_s(t, u)\big) \quad \text{and} \quad \phi_t(t, u) = 0,$$
$$\psi_r(t, u) = \psi_r\big(s, \psi_s(t, u)\big) \quad \text{and} \quad \psi_t(t, u) = u.$$

(iv) *For all $t \in [0, T]$ and $K \subset \mathcal{U}$ compact*,

$$\sup_{u \in K, s \le t} \|\psi_s(t, u)\| < \infty.$$

PROOF. The first assertion follows from the integral representation of ϕ and ψ. The second assertion can be derived directly from the admissibility conditions. Regarding (iii), let $s \le t$, $u \in \mathcal{U}^\circ$ and define

$$f(r) := \psi_r\big(s, \psi_s(t, u)\big), \quad \text{for } 0 \le r \le s.$$

Plugging equation (42b) into the above definition, we see that—on $[0, s]$—f satisfies the same measure Riccati equation as $\psi_r(t, u)$:

$$f(r) = \psi_s(t, u) + \int_{(r,s]} \mathfrak{R}\big(w, f(w)\big)\, dA_s.$$

By uniqueness of the Riccati equation, we infer $f(r) = \psi(r, t, u)$. A simple calculation exploiting the above and equation (42b) shows the equation for ϕ. Assertion (iv) follows readily from equations (47) and (50). $\square$

We are now prepared to state our main result on existence of affine Markov processes and affine semimartingales.

THEOREM 5.7. *Let (A, α, β, μ) be an admissible enhanced parameter set satisfying Assumption* 5.4. *Then there exists an infinitely divisible affine Markov process X* (*cf. Definition* 4.1) *with ϕ, ψ solutions of the associated measure Riccati equations*. *If X is conservative*, *then it is an affine semimartingale with characteristics given by* (12), *for any initial point $X_0 = x \in D$.*

The next result provides a sufficient condition for the conservativeness of X; further conditions can be developed along the lines of [10], Lemma 9.2.

COROLLARY 5.8. *Let X be an affine Markov process as in Theorem 5.7. If, for any $T > 0$, $g \equiv 0$ is the only $\mathbb{R}^m_{\leq 0}$-valued solution to*

$$\frac{dg_t}{dA_t} = -\operatorname{Re}\mathfrak{R}^{\mathcal{I}}(t, g_t), \quad g_T = 0, \tag{51}$$

then X is conservative.

Theorem 5.7 follows almost entirely from the next two propositions.

PROPOSITION 5.9. *Let the assumptions of Theorem 5.7 hold true and let (ϕ, ψ) be solutions to (13)–(17) with admissible parameters. Then there exists an affine Markov process X, unique in law, with state space D and whose transition kernels satisfy the affine property (36) with exponents ϕ and ψ.*

For the proof of this proposition, we introduce the following notation (see [10], Section 7). Let $\mathcal{C}$ denote the convex cone of functions $\phi \colon \mathcal{U} \to \mathbb{C}_{\leq 0}$ of the form

$$\phi(u) = \langle Aw, w\rangle + \langle B, u\rangle - C + \int_{D\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \langle w, h_{\mathcal{J}}(\xi)\rangle\big) M(d\xi) \tag{52}$$

for $u := (v, w) \in \mathcal{U}$, where $A \in \mathcal{S}^d_+$, $B \in D$, $C \in \mathbb{R}_{\geq 0}$ and $M(d\xi)$ is a nonnegative Borel measure on $D \setminus \{0\}$ integrating $\langle \mathbf{1}, h_{\mathcal{I}}(\xi)\rangle + \|h_{\mathcal{J}}(\xi)\|^2$. We denote by $\mathcal{C}^m$ the m-fold Cartesian product of $\mathcal{C}$. Recall from [10], Lemma 7.1, that $\phi \in \mathcal{C}$ if and only if there exists a sub-stochastic measure η on D such that

$$\int_D e^{\langle \xi,u\rangle} \eta(d\xi) = e^{\phi(u)}, \quad \forall u \in \mathcal{U}. \tag{53}$$

PROOF. The proof splits into four steps. First, we show, under some restrictions on the form of $\mathfrak{F}$, and $\mathfrak{R}$, that the solutions (ϕ, ψ) of the generalized measure Riccati equations are in $\mathcal{C} \times \mathcal{C}^d$, which follows similar to Proposition 7.4(ii) in [10]. In concrete terms, suppose that, for all $i \in \mathcal{I}$,

$$\begin{gathered}\int_{D\setminus\{0\}} h_i(\xi)\mu_i(d\xi) < \infty,\\ \alpha_{i,ik} = \alpha_{i,ki} = 0, \text{ for all } k \in \mathcal{J}.\end{gathered} \tag{54}$$

In this case, $\mathfrak{R}^{\mathcal{I}}$ can be written in the form

$$\mathfrak{R}^{\mathcal{I}}_i(t, u) = \tilde{\mathfrak{R}}^{\mathcal{I}}_i(t, u) - c_i(t)v_i, \quad i \in \mathcal{I}$$

with $\tilde{\mathfrak{R}}_i \in \mathcal{C}$, $c_i \geq 0$ dA-a.e. and $c_i(t)\Delta A_t \leq 1$. Therefore, the generalized measure Riccati equation (42b) is equivalent to the following equation:

$$\psi^i_s(t, u) = v_i \mathcal{E}^t_s(-c_i\, dA) + \int_s^t \mathcal{E}^s_r(-c_i\, dA)\tilde{\mathfrak{R}}\big(r, \psi_r(t, u)\big)\, dA_r, \quad i \in \mathcal{I},$$

where

$$\mathcal{E}_s^t(-c_i\,dA) = \exp\left(-\int_s^t c_i(r)\,dA_r^c\right) \prod_{r\in(s,t]} \left(1 - c_i(r)\Delta A_r\right)$$

is the solution to the linear measure differential equation $\frac{dg_t}{dA_t} = c_i(t)g_t$; see Example A.4. Define the iterative sequence

$$\begin{aligned} {}^{(0)}\psi_s^i(t,u) &= v_i, \\ {}^{(k+1)}\psi_s^i(t,u) &= v_i \mathcal{E}_s^t(-c_i\,dA) \\ &\qquad + \int_s^t \mathcal{E}_r^s(-c_i\,dA)\tilde{\mathfrak{R}}_i\left(r, {}^{(k)}\psi_r^{\mathcal{I}}(t,u), \psi_s^{\mathcal{J}}(t,u)\right) dA_r. \end{aligned}$$

By Banachs fixed-point theorem and Helly's selection principle, there is a subsequence of $({}^k\psi^{\mathcal{I}})_{k\in\mathbb{N}}$ that converges pointwise to the solution $\psi^{\mathcal{I}}$ of (42b). By Proposition 7.2 in [10], $\mathcal{C}^m$ is stable under composition and pointwise limits and we conclude that $\psi_s^{\mathcal{I}}(t,\cdot) \in \mathcal{C}^m$. The assertion $\psi_s^{\mathcal{J}}(t,\cdot) \in \mathcal{C}^n$ follows directly from (47). Since $\mathfrak{F}$ is in $\mathcal{C}$ also, $\phi_s(t,\cdot)$ is in $\mathcal{C}$; cf. [10], Proposition 7.2.

Second, we prepare for the approximation argument of part three and establish continuous dependence of a solution to the generalized measure Riccati equations on the right-hand side, that is, convergence in $L^1(dA) \times$ (uoc. on $\mathcal{U}$) of the right-hand side implies convergence of the solution in (dA − a.e.) × (uoc. on $\mathcal{U}$). Here and in the following, 'uoc. on $\mathcal{U}$' means uniformly on compact subsets of $\mathcal{U}$. Indeed, let $K \subseteq \mathcal{U}$ compact and $\mathfrak{R}$, $\tilde{\mathfrak{R}}$ with good, admissible and A-integrable parameters, such that

$$\left\| \sup_{u\in K}\left(\mathfrak{R}(\cdot,u) - \tilde{\mathfrak{R}}(\cdot,u)\right)\right\|_{L^1(dA)} \le \delta. \tag{55}$$

Denote the solution corresponding to $\tilde{\mathfrak{R}}$ by $\tilde{\psi}$ and examine the difference with ψ:

$$\begin{aligned} \left|\psi_t(T,u) - \tilde{\psi}_t(T,u)\right| &\le \int_t^T \left|\mathfrak{R}\left(s,\psi_s(T,u)\right) - \tilde{\mathfrak{R}}\left(s,\tilde{\psi}_s(T,u)\right)\right| dA_s \\ &\le \int_t^T \left|\mathfrak{R}\left(s,\psi_s(T,u)\right) - \mathfrak{R}\left(s,\tilde{\psi}_s(T,u)\right)\right| dA_s \\ &\qquad + \int_t^T \left|\mathfrak{R}\left(s,\tilde{\psi}_s(T,u)\right) - \tilde{\mathfrak{R}}\left(s,\tilde{\psi}_s(T,u)\right)\right| dA_s. \end{aligned}$$

If $\tilde{\psi}$ stays in K, we can estimate the second summand by δ and obtain with Proposition 5.6(iv) in conjunction with the local Lipschitz-continuity of $\mathfrak{R}$ (with A-integrable Lipschitz constant) that

$$\left|\psi_t(T,u) - \tilde{\psi}_t(T,u)\right| \le \delta + \int_t^T L_s\left|\psi_s(T,u) - \tilde{\psi}_s(T,u)\right| dA_s. \tag{56}$$

By Gronwalls lemma for Stieltjes differential equation (cf. [19], Corollary 19.3.3), the difference satisfies

$$\left|\psi_t(T,u) - \tilde{\psi}_t(T,u)\right| \le \delta \exp\left(\int_t^T L_s \, dA_s\right). \tag{57}$$

Now suppose

$$\tau = \sup\left\{t \in [0,T]: \left|\psi_t(T,u) - \tilde{\psi}_t(T,u)\right| > \alpha\right\} > 0. \tag{58}$$

This implies that the difference of ψ and $\tilde{\psi}$ is less than α for all $t \in [\tau, T]$ due to the common terminal value of ψ and $\tilde{\psi}$ and the continuity from the right. By (57), we can choose δ small enough, such that $|\psi_t(T,u) - \tilde{\psi}_t(T,u)| \le \frac{\alpha}{L_\tau \Delta A_\tau + 1} \le \alpha$. Therefore, $\tilde{\psi}$ cannot leave the α-neighborhood continuously, but only by a jump. However, ψ satisfies

$$\Delta\psi_t(T,u) = \mathfrak{R}\big(t, \psi_t(T,u)\big)\Delta A_t$$

at points of discontinuity (similarly for $\tilde{\psi}$) from which it follows that

$$\left|\psi_{\tau-}(T,u) - \tilde{\psi}_{\tau-}(T,u)\right| < \alpha$$

—a contradiction. This proves the continuous dependence on the right-hand side.

Third, we show an analogue of [14], Lemma 5.7, that is, there exists a sequence $(\mathfrak{R}_k)_{k\in\mathbb{N}}$ of functions of Lévy–Khintchine form with admissible parameters satisfying Assumption 5.4 and conditions (54), converging to $\mathfrak{R}$ in $(L^1(dA)) \times$ (uoc. on $\mathcal{U}$).

The construction of the sequence $(\mathfrak{R}_k)_{k\in\mathbb{N}}$ of functions satisfying (54) is the same as in the proof of [14], Lemma 5.7, or [10], page 33. Only the mode of convergence has been strengthened to convergence in $L^1(dA) \times$ (uoc. on $\mathcal{U}$). From [10], page 33, we obtain, for any $i \in \mathcal{I}$, the identity

$$\tilde{\mathfrak{R}}_k^i(t,u) - \mathfrak{R}^i(t,u) = \frac{2}{p_i^*(t)}\left(h_u\left(\frac{\xi^*(t)}{k}\right) - \frac{1}{2}\langle Q(t)u_{\mathcal{J}\cup i}, u_{\mathcal{J}\cup i}\rangle\right), \tag{59}$$

where

$$p^*(t) = \frac{\alpha_{ii}^i(t)}{\|\alpha_{i\mathcal{J}\cup i}^i(t)\|^2}, \qquad \xi^*(t)_{\mathcal{I}\setminus i} = 0,$$

$$\xi(t)_{\mathcal{J}\cup i}^* = \frac{\alpha_{i\mathcal{J}\cup i}^i(t)}{\|\alpha_{i\mathcal{J}\cup i}^i(t)\|}, \qquad Q(t)_{kl} := p^* \frac{\alpha_{ki}^i \alpha_{il}^i}{\alpha_{ii}^i},$$

$$h_u(\xi) = \big(e^{\langle u,\xi\rangle} - 1 - \langle u_{\mathcal{J}\cup i}, h_{\mathcal{J}\cup i}(\xi)\rangle\big)/\big(\langle \mathbf{1}, h_{\mathcal{I}\setminus i}(\xi)\rangle + \|h_{\mathcal{J}\cup i}(\xi)\|\big).$$

We can simplify the expressions in (59) to

$$\frac{1}{2}\langle Q(t)u_{\mathcal{J}\cup i}, u_{\mathcal{J}\cup i}\rangle = 2\sum_{l,m\in\mathcal{J}\cup i} u_l \frac{\alpha_{li}^i(t)\alpha_{im}^i(t)}{\alpha_{ii}^i(t)} u_m.$$

Using the properties of the truncation functions and $\|\xi^*\| = 1$, we obtain for large enough k that

$$\frac{2}{p^*(t)} h_u\left(\frac{\xi^*(t)}{k}\right) \le C \frac{1}{p^*(t)} \left(1 + \|u_{\mathcal{J}\cup i}\|^2 \|\xi^*\|^2\right)$$
$$\le C\left(1 + \|u_{\mathcal{J}\cup i}\|^2\right) \frac{\|\alpha^i_{i\,\mathcal{J}\cup i}(t)\|^2}{\alpha^i_{ii}(t)},$$

where C does not depend on u or ξ^*. Integrability of the above quantities w.r.t. A follows from the positive semidefiniteness of $\alpha(t)$ and the Cauchy–Schwarz inequality. This implies convergence of $\tilde{\mathfrak{R}}_k$ to $\mathfrak{R}$ in $(L^1(dA)) \times (\text{uoc.on}\,\mathcal{U})$ due to the construction of $\tilde{\mathfrak{R}}$.

Finally, we come to the last step. From (53), it now follows that for every $(t, x) \in [0, T] \times D$ and $s \in [0, t]$, there exists a unique, sub-stochastic measure $p_{s,t}(s, \cdot)$ on D with

$$\int_D e^{\langle u,\xi\rangle} p_{s,t}(x, d\xi) = e^{\phi(s,t,u)+\langle\psi(s,t,u),x\rangle}, \quad \forall u \in \mathcal{U}. \tag{60}$$

The semi-flow property of (ϕ, ψ) ensures that the family of measures $(p_{s,t})_{s\le t\in[0,T]}$ satisfies the Chapman–Kolmogorov equations. By the Kolmogorov existence theorem (see [23], Theorem 8.4), there exists a D-valued Markov process X on $[0, T]$, unique in law, with transition kernels $(p_{s,t})_{s\le t\in[0,T]}$. By definition, X satisfies the affine property (36) for all $u \in \mathcal{U}$. $\square$

PROPOSITION 5.10. *Let X be the affine Markov process from Proposition 5.9 started at some $X_0 = x \in D$. If X is conservative, then there is a modification of X which is a càdlàg affine semimartingale.*

PROOF. Let X be the affine Markov process and $(\mathscr{F}_t)_{t\ge0}$ its natural filtration. From (36), we have that

$$M_t^{T,u} := \mathbb{E}\left[e^{\langle u,X_T\rangle} | \mathscr{F}_t\right] = e^{\phi_t(T,u)+\langle\psi_t(T,u),X_t\rangle}, \tag{61}$$

which must be a martingale for all $u \in \mathcal{U}$. Since ϕ and ψ are right continuous in T and càdlàg in t, applying this identity with $t = 0$ shows that X (and, therefore, also every $M^{T,u}$) is right continuous in probability. It follows that the martingale $M^{T,u}$ has a càdlàg modification. Let $u = (v, w)$. By equation (47) $\psi_t^{\mathcal{J}}(T, (v, 0)) = 0$ for all $t < T$, and hence $\langle\psi_t^{\mathcal{I}}(T, (v, 0)), X_t^{\mathcal{I}}\rangle$ are càdlàg semimartingales for $v \in \mathbb{R}^m_-$ on $[0, T]$. For some linearly independent vectors $e_1, \dots, e_m$ in $R^m_{\le0}$, we can find $s \le T$ such that $\psi_t^{\mathcal{I}}(T, e_1), \dots, \psi_t^{\mathcal{I}}(T, e_m)$ are linearly independent for all $t \in (s, T]$. Thus $X^{\mathcal{I}}$ is a semimartingale on $(s, T]$. This can be done for arbitrary T which allows to infer with a covering argument (and right continuity at $t = 0$) that $X^{\mathcal{I}}$ is a semimartingale on $\mathbb{R}_{\ge0}$.

For the real-valued part $X^{\mathcal{J}}$ of the process, we use that, for all $u=(v,w)\in\mathcal{U}^\circ$, the equation for $\psi^{\mathcal{J}}$ reduces to a linear equation with solution $\psi_t^{\mathcal{J}}(T,u)=w\psi_t^{\mathcal{J}}(T)$ (see equation (47)). By the same argument as in [10], Proof of Theorem 2.12, it follows that also $X^{\mathcal{J}}$, is a càdlàg semimartingale. □

We complete the proof of Theorem 5.7 and Corollary 5.8.

PROOF. In light of Propositions 5.9 and 5.10, it only remains to show that the semimartingale triplet of X is given by (12) with the same parameters that were used for the construction of X. To this end, we apply Lemma 3.6 to X, and get similar to equation (25),

$$\Theta_t(\omega)\cdot\begin{pmatrix}F\big(t,\psi_t(T,u)\big)\\ R^1\big(t,\psi_t(T,u)\big)\\ \vdots\\ R^d\big(t,\psi_t(T,u)\big)\end{pmatrix}dA_t^c=\begin{pmatrix}G_0(dt;\omega,T,u)\\ \vdots\\ G_d(dt;\omega,T,u),\end{pmatrix},$$

where F, R on the left-hand side contain the parameters (A,β,α,μ) and G the semimartingale characteristics of X (cf. (20)). We proceed as in the proof of Theorem 3.2 by taking the union over a countable, dense subset $\mathcal{T}\times\mathcal{E}$ of $\mathbb{R}_{\geq 0}\times\mathcal{U}$ and considering the right limits $T\downarrow t$ in the countable set $\mathcal{T}$. Using $\psi_t(t,u)=u$ and the fact that functions of Lévy–Khintchine form determine their parameter triplets uniquely, we derive the continuous part of (12). The equation for ν at jump points follows from Lemmata 3.5 and 4.4, completing the proof of Theorem 5.7.

For the proof of Corollary 5.8, evaluating (36) at $u=0$ yields

$$p_{t,T}(x,D)=\exp\big(\phi_t(T,0)+\langle\psi_t(T,0),x\rangle\big) \tag{62}$$

for all $0\leq t\leq T$ and $x\in D$. Taking into account that $p_{t,T}(x,D)\leq 1$ and that $D=\mathbb{R}^m_{\geq 0}\times\mathbb{R}^n$, we see that $\phi_t(T,0)\leq 0$, $\psi_t^{\mathcal{I}}(T,0)\leq 0$ and $\psi_t^{\mathcal{J}}(T,0)=0$. Writing $g(t):=\psi_t^{\mathcal{I}}(T,0)$ the measure Riccati equation (42b) becomes (51). This equation has the constant solution $g\equiv 0$; if it is the only solution, then $\psi_t^{\mathcal{I}}(T,0)=0$ for all $0\leq t\leq T$. Inserting into (42a), also $\phi_t(T,0)=0$. Together with (62), this shows that $p_{t,T}(x,D)=1$, that is, X is conservative. □

REMARK 5.11. The proof of Theorem 5.7 can easily be adapted to the case where γ_0 is not of the Lévy–Khintchine form (40) at $t\in J^A$, but a general log-characteristic function of a D-valued random variable. This is due to the fact that γ_0 enters only into part (42a), but not into part (42b) of the measure Riccati equation.

6. Examples and applications. We begin this section with some examples that illustrate several aspects of stochastic discontinuities within affine semimartingales. After that, we study affine semimartingales in discrete time in Section 6.1. In Section 6.2, we take a look at the application of affine semimartingales to stock prices with dividends and in Section 6.3 we consider a new class of affine term structure models allowing for stochastic discontinuities.

EXAMPLE 6.1. Consider the following discrete-time variant of the (time-inhomogeneous) Poisson process: let $X_0 = x \in \mathbb{N}$. Furthermore, assume that X is constant except for $t \in \{1, 2, \dots\}$ and assume that $\Delta X_n \in \{0, 1\}$, $n \in \{1, 2, \dots\}$ are independent with $P(\Delta X_n = 1) = p_n \in (0, 1)$. Then X is an affine semimartingale because for $0 \le s \le t$,

$$E\left[e^{uX_t}|\mathscr{F}_s\right] = \exp\left(uX_s + \sum_{s<n\le t, n\in\mathbb{N}} \phi_n(u)\right),$$

where

$$\begin{aligned}\phi_n(u) &= E\left[e^{u\Delta X_n}\right] = e^u\left(p_n + e^{-u}(1-p_n)\right)\\ &= \exp\left(u + \log\left(p_n + e^{-u}(1-p_n)\right)\right).\end{aligned}$$

Clearly, it may happen that $\Delta X_n = 0$ while $\phi(u, n, t) - \phi(u, n-, t) = \phi_n(u) \neq 0$. Stochastic discontinuity is reflected by having jumps at $t \in \{1, 2, \dots\}$ with positive probability. The considered process falls in the class of point processes whose associated jump measure is an *extended* Poisson measure; see II.1c in [21]. In contrast to Poisson processes, X is not quasi-left-continuous. In summary, X is a process with independent increments, but not a time-inhomogeneous Lévy process.

The following example illustrates how one can construct stochastically discontinuous affine semimartingales from stochastically continuous ones, even from affine semimartingales without jumps, through a suitable (discontinuous) time change.

EXAMPLE 6.2. This example is inspired by [17]: consider an affine semimartingale X which is stochastically continuous (as treated in [10] and [14]). We assume that D denotes the state space of the affine semimartingale and that ϕ and ψ are the characteristics of X as in (3).

Let $\{t_1 < \cdots < t_N\} \subset \mathbb{R}_{\ge 0}$ be some time points and $a_i \in \mathbb{R}^d$, $b_i \in \mathbb{R}^{d\times d}$ such that $a_i + b_i \cdot x \in D$ for all $x \in D$, $i = 1, \dots, N$. Then

$$\tilde{X}_t := \sum_{i=1}^{N} \mathbb{1}_{\{t\ge t_i\}}(a_i + b_i \cdot X_t), \quad t \ge 0 \tag{63}$$

is an affine semimartingale in the sense of Definition 2.1. Note that $\tilde{X}$ is in general not stochastically continuous, as it jumps with positive probability at the time points t_i, $i = 1, \dots, N$.

Indeed, by the affine property of X and using iterated conditional expectations, we obtain for $t_k \le t < t_{k+1}$,

$$
\begin{aligned}
E[e^{\langle u, \tilde{X}_t\rangle} | \mathscr{F}_{t_k}] &= E\left[\exp\left(\left\langle u, \sum_{i=1}^{k}(a_i + b_i \cdot X_t)\right\rangle\right) \Big| \mathscr{F}_{t_k}\right] \\
&= e^{\sum_{i=1}^{k}\langle u, a_i\rangle} E\left[\exp\left(\left\langle \sum_{i=1}^{k} u b_i^{\top}, X_t\right\rangle\right) \Big| \mathscr{F}_{t_k}\right] \qquad (64)\\
&= \exp\left(\sum_{i=1}^{k}\langle u, a_i\rangle + \phi_{t_k}(t, u') + \langle \psi_{t_k}(t, u'), X_{t_k}\rangle\right),
\end{aligned}
$$

since X is affine; here we set $u' := \sum_{i=1}^{k} u b_i^{\top}$. The affine characteristics of $\tilde{X}$ are directly obtained from equation (64).

The above example suggests that even more complex variants of the transformation considered in (63) stay in the affine class. The following example shows that this need not always be the case.

EXAMPLE 6.3. Consider an affine process X and let

$$
Y_t = X_t + \mathbb{1}_{\{t \ge 1\}} X_1, \quad t \ge 0.
$$

Then Y is in general not affine because for $1 \le s < t$,

$$
E[e^{uY_t} | \mathscr{F}_s] = e^{uX_1} \cdot e^{\phi_s(t,u) + \psi_s(t,u) X_s} \ne e^{\tilde{\phi}_s(t,u) + \tilde{\psi}_s(t,u) X_s}
$$

as in general $\psi_s(t, u) \ne u$. However, $(X, Y)^{\top}$ is affine, a property prominently used in bond option pricing.

The next example illustrates the possibility of Markov processes with affine Fourier transform, which are not semimartingales.

EXAMPLE 6.4. Given a starting point $x \in \mathbb{R}$ and a function $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$ with $f(0) = 0$, consider the deterministic process $X_t(\omega) = f(t) + x$, $t \ge 0$. Then X is affine in the sense that its Fourier transform has exponential affine form, as

$$
E[e^{uX_t} | \mathscr{F}_s] = e^{u(f(t) - f(s) + X_s)}.
$$

Hence X satisfies equation (2.1) with $\phi_s(t, u) = u(f(t) - f(s))$ and $\psi_s(t, u) = u$. Moreover, X is a Markov process with transition kernel $p_{s,t}(x, d\xi) = \delta_{x+f(t)-f(s)}(d\xi)$. Choosing f of finite variation, the process X is an affine semimartingale and falls into the scope of Definition 2.1—although it does not satisfy the support condition 2.3. However, when f is of infinite variation (e.g., a path of Brownian motion), X is no longer a semimartingale, ϕ does not satisfy the quasiregularity condition 2.5, but X is still an affine Markov process in the sense of Definition 4.1.

In the case of processes with independent increments, the gap to those processes which are also semimartingales can be completely classified; see Section II.4.c in [21]. A complete study of the gap between affine semimartingales and affine Markov processes is beyond the scope of this article.

Other than affine transitions at the discontinuity points $t_1, \dots, t_N$ are also possible, as the following example illustrates.

EXAMPLE 6.5. Let N be a Poisson process with intensity λ. This is also an affine process with affine characteristics $\psi_s(t,u) = u$ and $\phi_s(t,u) = \lambda(t-s)(e^u - 1)$. Let α be a Bernoulli distributed random V variable with $\mathbb{P}(\alpha = -1) = \frac{1}{2}$ and β a standard normal random variable. Further let α, β and N be mutually independent. Consider a (deterministic) time $\tau > 0$ and the process given by

$$X_t = N_t + \mathbb{1}_{\{t \geq \tau\}}(\alpha + \beta\sqrt{N_\tau}), \quad t \geq 0$$

together with the (augmented) filtration generated by $\sigma(N_s, \alpha\mathbb{1}_{\{\tau \leq s\}}, \beta\mathbb{1}_{\{\tau \leq s\}} : s \leq t)$. We compute the conditional characteristic function of X. At first, let $s < \tau \leq t$:

$$\begin{aligned} E\big[e^{\langle u, X_t\rangle} | \mathscr{F}_s\big] &= E\big[E\big[e^{\langle u, N_t + \mathbb{1}_{\{t\geq\tau\}}(\alpha+\beta\sqrt{N_\tau})\rangle} | \mathscr{F}_\tau\big] | \mathscr{F}_s\big] \\ &= e^{\phi_\tau(t,u)} E\big[e^{u\alpha}\big] \cdot E\big[e^{\psi_\tau(t,u)N_\tau + u\beta\sqrt{N_\tau}} | \mathscr{F}_s\big] \\ &= e^{\phi_\tau(t,u)} \frac{1}{2}(e^u + e^{-u}) E\big[e^{(\psi_\tau(t,u)+\frac{1}{2}u^2)N_\tau} | \mathscr{F}_s\big] \\ &= e^{\phi_\tau(t,u)} \frac{1}{2}(e^u + e^{-u}) e^{\psi_s(\tau, \psi_\tau(t,u)+\frac{1}{2}u^2)N_s}. \end{aligned}$$

In the second case where $\tau \leq s \leq t$, we have

$$\begin{aligned} E\big[e^{uX_t} | \mathcal{F}_s\big] &= \exp\big(\phi(s,t,u) + \psi(s,t,u)N_s + u(\alpha + \beta\sqrt{N_\tau})\big) \\ &= \exp\big(\phi(s,t,u) + uX_s\big). \end{aligned}$$

Hence X is an affine process with affine characteristics $\tilde{\phi}$ and $\tilde{\psi}$ given by

$$\begin{aligned} \tilde{\phi}_s(t,u) &= \phi_s(t,u) + \mathbb{1}_{\{s<\tau\leq t\}}\big(\log(\cosh u)\big), \\ \tilde{\psi}_s(t,u) &= \psi_s\left(\tau, \psi_\tau(t,u) + \mathbb{1}_{\{s<\tau\leq t\}}\frac{1}{2}u^2\right) = u + \mathbb{1}_{\{s<\tau\leq t\}}\frac{1}{2}u^2. \end{aligned}$$

Note that the process X does not satisfy the support Condition 2.3, since it is supported on the positive real whole numbers before the jump but might take negative values after τ.

6.1. *Affine processes in discrete time.* In the considered semimartingale approach, affine processes in discrete time can also be embedded into continuous time. This allows us to obtain a full treatment of affine processes in discrete time as special case of our general results. Note that any discrete time process is of finite variation, and hence a semimartingale such that as a matter of fact, Definition 2.1 covers all discrete-time affine processes in finite dimension.

We use the time series notation for a process in discrete time and consider without loss of generality the time points $0, 1, 2, \dots$. Consider a complete probability space $(\Omega, \mathscr{F}, P)$ and a filtration in discrete time $\hat{\mathbb{F}} = (\hat{\mathscr{F}}_n)_{n \geq 0}$.

DEFINITION 6.6. The time series $(\hat{X}_n)_{n \geq 0}$ is called *affine* if it is $\hat{\mathbb{F}}$-adapted and there exist $\mathbb{C}$ and $\mathbb{C}^d$-valued càdlàg functions $\phi_n(m, u)$ and $\psi_n(m, u)$, respectively, such that

$$E\big[e^{\langle u, \hat{X}_m\rangle} | \hat{\mathscr{F}}_n\big] = \exp\big(\phi_n(m, u) + \langle \psi_n(m, u), \hat{X}_n\rangle\big) \tag{65}$$

holds for all $u \in i\mathbb{R}^d$ and $0 \leq n \leq m$, $n, m \in \mathbb{N}_0$. It is called *time homogeneous*, if $\phi_n(m, u) = \phi_0(n - m, u) =: \phi_{m-n}(u)$ and $\psi_n(m, u) = \psi_0(m - n, u) =: \psi_{m-n}(u)$, again for all $u \in i\mathbb{R}^d$ and $0 \leq s \leq t$.

To emphasize the filtration, we are working with, we will sometimes call $\hat{X}$ $\hat{\mathbb{F}}$-affine. We associate to the time series $(\hat{X}_n)_{n \geq 0}$ the piecewise-constant embedding into continuous time

$$X_t = \hat{X}_{[t]}, \quad t \geq 0 \tag{66}$$

with $[t] = n$ if $n \leq t < n + 1$. Then $\hat{X}$ is càdlàg, of finite variation, and hence a semimartingale. In a similar way, we let $\mathscr{F}_t = \hat{\mathscr{F}}_{[t]}$ and obtain the associated filtration in continuous time. Usual conditions are not needed here.

Note that even if the affine time series is time homogeneous, the associated continuous-time affine process X will not be time homogeneous in general: for $0 < \epsilon < 1$,

$$\begin{aligned} E\big[e^{\langle u, X_{m+\epsilon}\rangle} | \mathscr{F}_n\big] &= \exp\big(\phi_n(m + \epsilon, u) + \langle \psi_n(m + \epsilon, u), X_n\rangle\big) \\ &= \exp\big(\phi_n(m, u) + \langle \psi_n(m, u), X_n\rangle\big) \end{aligned}$$

which would give $\phi_{m+\epsilon-n}(u) = \phi_{m-n}(u)$, while on the other hand,

$$\begin{aligned} E\big[e^{\langle u, X_{m+\epsilon/2}\rangle} | \mathscr{F}_{n-\epsilon/2}\big] &= \exp\left(\phi_{n-\frac{\epsilon}{2}}\left(m + \frac{\epsilon}{2}, u\right) + \left\langle \psi_{n-\frac{\epsilon}{2}}\left(m + \frac{\epsilon}{2}, u\right), X_{n-\epsilon/2}\right\rangle\right) \\ &= \exp\big(\phi_{n-1}(m, u) + \langle \psi_{n-1}(m, u), X_{n-1}\rangle\big) \end{aligned}$$

which would give $\phi_{m-n}(u) = \phi_{m-(n-1)}(u)$ thus rendering X to be constant. Time inhomogeneity in discrete time is therefore a strictly weaker concept than in continuous time. However, in the reverse direction we have a positive result.

REMARK 6.7. If X is a time-homogeneous continuous-time $\mathbb{F}$-affine process, it follows immediately that the time-series $\hat{X}$ is $\hat{\mathbb{F}}$-affine and $\hat{X}$ is time homogeneous.

PROPOSITION 6.8. *Let $(\hat{X})$ be an affine time series satisfying the support Condition 2.3. Then ϕ and ψ satisfy the semi-flow property*

$$\begin{aligned} \phi_n(m,u) &= \phi_n\big(n', \psi_{n'}(m,u)\big) + \phi_{n'}(m,u), \\ \psi_n(m,u) &= \psi_n\big(n', \psi_{n'}(m,u)\big) \end{aligned} \tag{67}$$

for all $0 \le n < n' \le m$, $u \in i\mathbb{R}^d$.

PROOF. We apply Theorem 3.2. First, note that

$$z_n(u) = \int_D e^{\langle u,x\rangle} \nu\big(\{n\}, dx\big) = E\big[\mathbb{1}_{\{\Delta X_n \neq 0\}} e^{\langle u, \Delta X_n\rangle} | \mathscr{F}_{n-1}\big].$$

Hence,

$$E\big[e^{\langle u,\Delta X_n\rangle} | \mathscr{F}_{n-1}\big] = z_n(u) + P(\Delta X_n = 0 | \mathscr{F}_{n-1}) = z_n(u) + 1 - z_n(0).$$

This yields by definition that

$$\begin{aligned} E\big[e^{\langle u,\Delta X_n\rangle} | \mathscr{F}_{n-1}\big] &= E\big[e^{\langle u,X_n\rangle} | \mathscr{F}_{n-1}\big] e^{-\langle u, X_{n-1}\rangle} \\ &= e^{\phi_{n-1}(n,u) + \langle \psi_{n-1}(n,u) - u, X_{n-1}\rangle} \end{aligned} \tag{68}$$

and from equation (16) we recover that $\gamma_0(n,u) = -\phi_{n-1}(n,u)$ and $\gamma_i(n,u) = -\psi_{n-1}(n,u) + u$. First, Theorem 3.2 yields that

$$\Delta\phi_{n+1}(m,u) = -\phi_n\big(n+1, \psi_n(m,u)\big),$$

that is,

$$\phi_n(m,u) = \phi_n\big(n+1, \psi_{n+1}(m,u)\big) + \phi_{n+1}(m,u) \tag{69}$$

for $0 \le n < m$ and all $u \in i\mathbb{R}^d$. By induction, we obtain that ϕ satisfies the semi-flow property

$$\phi_n(m,u) = \phi_n\big(n', \psi_{n'}(m,u)\big) + \phi_{n'}(m,u)$$

for all $0 \le n < n' < m$ and $u \in i\mathbb{R}^d$. In similar spirit, Theorem 3.2 yields that

$$\Delta\psi_{n+1}(m,u) = -\psi_n\big(n+1, \psi_{n+1}(m,u)\big) + \psi_{n+1}(m,u)$$

which is equivalent to

$$\psi_n(m,u) = \psi_n\big(n+1, \psi_{n+1}(m,u)\big), \tag{70}$$

and hence the semi-flow property

$$\psi_n(m,u) = \psi_n\big(n', \psi_{n'}(m,u)\big)$$

for all $0 \le n < n' < m$ and $u \in i\mathbb{R}^d$ and the claim follows. $\square$

REMARK 6.9. Despite the semi-flow property one obtains directly from (69) and (70) that ϕ and ψ are unique solutions of the following difference equations:

$$\begin{aligned}\phi_n(n+1) &= F(n,u),\\ \psi_n(n+1,u)-u &= R(n,u),\\ \phi_n(m+1,u) &= F(n,u)+\phi_n\big(m,u+R(m,u)\big),\\ \psi_n(m+1,u) &= \psi_n\big(m,u+R(m,u)\big),\end{aligned}$$

where the functions F and R are defined by the first two equations. With the notation of Theorem 3.2, $F=-\gamma_0$ and $R_i=-\gamma_i$. These equations and the above proposition are the content of Proposition 4.4 in [36]. The authors obtain the result directly from iterated conditional expectations.

EXAMPLE 6.10 (AR(1)). A (time-inhomogeneous) autoregressive time series of order (1) is given by

$$\hat{X}_n=\alpha(n)\hat{X}_{n-1}+\epsilon_n,$$

where we assume that (ϵ_n) are independent (not necessarily identically nor normally distributed). Then $\hat{X}$ is affine as

$$E\big[e^{uX_n}|\hat{\mathscr{F}}_{n-1}\big]=E\big[e^{u\epsilon_n}\big]e^{\alpha(n)X_{n-1}}$$

with $\hat{\mathscr{F}}_{n-1}=\sigma(\hat{X}_0,\dots,X_{n-1})$. The generalization to higher order requires an extension of the state space. So an AR(p) series gives an affine process $(\hat{X}_n,\dots,\hat{X}_{n-p})_{n\geq p}$.

6.2. *Asset prices with dividends*. Dividends and the relationship of a firm's asset prices have been discussed and analyzed a long time, early contributions being, for example, [30, 31] or the approach proposed in [28], for which we propose a dynamic generalization. Most notably, typical continuous-time models incorporate dividends via a dividend yield. While this approach does ease mathematical modelling, it certainly does not reflect empirical facts. In this section, we show how a time-inhomogeneous affine process can be used to model stock price with dividends in an efficient way.

From a general viewpoint, the following example shows how to mix two different time scales (continuous time and discrete time) in a time-inhomogeneous affine model. Moreover, as the discrete-time scale has a certain lag, we also show how past-dependence can be incorporated in the same way (by extension of the state space, of course).

Consider a $d\geq 3$-dimensional affine process X. Let $D:=X^1$ denote the cumulated dividends process where we assume that dividends are paid at the time points $t=1,2,\dots$, that is, D is nondecreasing and constant on each interval $[n,n+1)$, $n\geq 1$. Let X^2 denote the stock price process, that is, the jump of X^2 at dividend payment dates includes subtraction of the dividend payment, ΔX_n^2, plus possibly an additional jump due to new information, for example, by the height of the dividend. We will follow the approach in [28] and assume that the size of the dividend depends linearly on the current year's profit after taxes. In this regard, let X^3 denote the accumulated profits of the current year after taxes, that is, $X_n^3 = 0$ and X_{n-}^3 denotes the accumulated profits of the ith year.

In Lintner's model (see [28]), the current dividend D_n is given by

$$D_n = a + bX_{n-}^3 + cD_{n-} + \epsilon_n,$$

where ϵ_n are mean-zero stochastic error terms. According to Theorem 3.2, X may be chosen affine only if the conditional distribution of the ϵ_n satisfies

$$P(\epsilon_n \in dx | X_{n-}) = \kappa_{0,3}(dx) + \sum_{i=1}^{d} X_{n-}^i \kappa_{i,3}(dx),$$

where for $y \in \mathbb{R}^d$, $\kappa_{i,j}(dx) = \int_{\mathbb{R}^{d-1}} \kappa(dy_1, \dots, dy_{j-1}, dx, dy_{j+1}, dy_d)$. Clearly, this includes for example independent error terms (not necessarily normally distributed). The remaining components of X may be used for modelling stochastic volatility or further covariates.

6.3. *Affine term-structure models.* In this section, we study a new class of term-structure models driven by affine processes. Motivated by our findings in Section 3, where it turned out that the semimartingale characteristics of an affine process X are dominated by an increasing, càdlàg function A, we study the following extension of the seminal Heath–Jarrow–Morton [20] framework: Consider a family of bond prices, given by

$$P(t, T) = \exp\biggl(-\int_{(t,T]} f(t, u)\, dA_u\biggr), \quad 0 \le t \le T \le T^*, \tag{71}$$

with some final time horizon $T^* > 0$. The rate $f(t, T)$ is called *instantaneous forward rate* representing the interest rate contractible at time $t \le T$ for the infinitesimal future time interval $(T, T + dA_T]$; see [15] for details and related literature. The numéraire in this market is assumed to be of the from $\exp(\int_0^t r(s)\, dA_s)$.

The term-structure model proposed here is specified by assuming the following structure of the forward rates:

$$f(t, T) = f(0, T) + \int_0^t a(s, T)\, dX_s, \quad 0 \le t \le T \le T^*, \tag{72}$$

where a is a suitable, deterministic function. The first step will be the derivation of a condition on a which renders discounted bond prices local martingales, thus leading to a bond market satisfying a suitable no-arbitrage property like, for example, NAFL.

Consider a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions and consider for the beginning a d-dimensional, special semimartingale X

with semimartingale characteristics (B, C, ν). As we aim at considering an affine process X, with a view on Theorem 3.2 we additionally assume that X has the canonical representation

$$X = X_0 + B_t + X^c + x * (\mu - \nu), \tag{73}$$

where $dB_t = b_t\, dA_t$, $dC_t = c_t\, dA_t$ and $\nu(dt, dx) = K_t(dx)\, dA_t$ and A is deterministic, càdlàg, increasing with $A_0 = 0$. We define the left-continuous processes $A(\cdot, T)$, $0 < T \le T^*$, by

$$A(t, T) := \int_{[t,T]} a(s, u)\, dA_u, \quad 0 \le t \le T,$$

and require the following technical assumption.

(A1) Assume that $a : [0, T^*]^2 \to \mathbb{R}^d$ is measurable and satisfies

$$\left(\int_{\cdot}^{T^*} |a_i(\cdot, u)|^2\, dA_u \right)^{\frac{1}{2}} \in L(X^i), \quad i = 1, \dots, d,$$

$$\int_0^{T^*} \int_0^{T^*} |a(t, u)||\, dB_t|\, dA_u < \infty, \quad 0 \le t \le T^*,$$

where $L(X^i)$ denotes the set of processes which are integrable in the semimartingale integration sense with respect to the ith coordinate X^i of X, $i = 1, \dots, d$.

PROPOSITION 6.11. *Under* (A1), *discounted bond prices are local martingales if, and only if*:

(i) $r_t = f(t, t)\; dA \otimes d\mathbb{P}$*-almost surely for* $0 \le t \le T^*$, *and*
(ii) *the following condition holds*:

$$\begin{aligned} A(t, T)b_t = {}& \frac{1}{2} A(t, T) c_t A(t, T)^\top \\ & + \int_{\mathbb{R}^d} \big(e^{A(t,T)x} - 1 - A(t, T)x\big) K_t(dx), \end{aligned} \tag{74}$$

$dA \otimes d\mathbb{P}$*-almost surely for* $0 \le t \le T \le T^*$.

PROOF. The proof follows the classical steps in [20], relying on a stochastic Fubini theorem. First note, that discounted bond prices take the form

$$\begin{aligned} \tilde{P}(t, T) = {}& e^{-\int_{(t,T]} f(0,u)\, dA_u} \\ & \times \exp\left(-\int_{(t,T]} \int_0^t a(s, u)\, dX_s\, dA_u - \int_{(0,t]} r_s\, dA_s \right) \\ =: {}& P(0, T) \exp\big(I(t, T)\big). \end{aligned} \tag{75}$$

The dynamics of I can be obtained from the dynamics of the forward rates as

$$\begin{aligned}
\int_{(t,T]} f(t,u)\,dA_u &= \int_{(t,T]} f(0,u)\,dA_u + \int_{(t,T]}\int_0^t a(s,u)\,dX_s\,dA_u \\
&= \int_{(t,T]} f(0,u)\,dA_u + \int_0^t \int_{(t,T]} a(s,u)\,dA_u\,dX_s \\
&= \int_{(t,T]} f(0,u)\,dA_u + \int_0^t \int_{[s,T]} a(s,u)\,dA_u\,dX_s \\
&\quad - \int_0^t \int_{[s,t]} a(s,u)\,dA_u\,dX_s \\
&= \int_{(t,T]} f(0,u)\,dA_u - \int_0^t \int_0^u a(s,u)\,dX_s\,dA_u \\
&\quad + \int_0^t A(s,T)\,dX_s \\
&= \int_0^T f(0,u)\,dA_u - \int_0^t f(u,u)\,dA_u + \int_0^t A(s,T)\,dX_s;
\end{aligned}$$

interchange of the integrals is justified under (A1) by the Fubini theorem, for example, along the lines of [35, 40]. The next step is to represent $\exp(I(\cdot,T)) = \mathcal{E}(\tilde{I}(\cdot,T))$ as a stochastic exponential $\mathcal{E}$ on the modified process $\tilde{I}$ relying on Theorem II.8.10 in [21]. This theorem yields that

$$\tilde{I}(t,T) = \tilde{I}(0,T) + I(t,T) + \frac{1}{2}\langle I^c(\cdot,T)\rangle_t + \left(e^x - 1 - x\right) * \mu^{I(\cdot,T)},$$

where $\mu^{I(\cdot,T)}$ denotes the random measure associated to the jumps of I; see (1). Calculating the above terms under our assumptions together with representation (73) yields that

$$\begin{aligned}
d\tilde{I}(t,T) = \biggl(&-A(t,T)b_t + \frac{1}{2}A(t,T)c_t A(t,T)^\top \\
&+ \int_{\mathbb{R}^d}\left(e^{-A(t,T)x} - 1 + A(t,T)x\right)K(t,dx) \\
&+ \left(f(t,t) - r_t\right)\biggr)\,dA_t + dM_t, \quad 0 \le t \le T
\end{aligned}$$

with a local martingale M. The claim follows by first considering $T = t$, thus yielding (i) and thereafter (ii). For the reverse, observe that (i) and (ii) imply that $\tilde{I}(\cdot,T)$ is a local martingale, and the claim follows. □

Recall the notion of a good parameter set of the affine semimartingale X from Definition 3.1. The following corollary gives a specification of an affine term-structure model in the more classical case, that is, when $\gamma = 0$.

COROLLARY 6.12. *If* (A1) *holds and* X *is a quasiregular affine semimartingale satisfying the support Condition* 2.3 *and with parameter set* $(A, 0, \beta, \alpha, \mu)$, *and if*

$$A(t,T)\beta_i(t) = \frac{1}{2}A(t,T)\alpha_i(t)A(t,T)^\top + \int_{\mathbb{R}^d}\left(e^{A(t,T)x} - 1 - A(t,T)x\right)\mu_i(t,dx), \tag{76}$$

holds for $i = 0, \dots, d$, *then the drift condition* (74) *holds.*

PROOF. The application of Theorem 3.2 yields that $b = \beta_0 + \sum_{i=1}^d X_-^i \beta_i$, with similar expression for a and K. Using linearity and (76), we immediately obtain (74). □

A reverse version of this result is easily obtained requiring additionally linear independence of certain coefficients; see, for example, Section 9.3 in [15].

In the following, we study a variety of extensions of the Vasiček model for incorporating jumps at predictable times. Of course, in a similar manner an extension of the Cox–Ingersoll–Ross model is possible, or one may even extend general stochastically continuous Markov processes in a similar way.

EXAMPLE 6.13 (The Vasiček model). We begin by casting the famous Vasiček model in the above framework. The Vasiček model is a one-factor Gaussian affine model, where the short rate is the strong solution of the stochastic differential equation

$$dr_t = (\alpha + \beta r_t)\,dt + \sigma\,dW_t \tag{77}$$

with a one-dimensional standard Brownian motion W and $\beta \neq 0$, $\sigma > 0$. The bond prices are given in exponential form, such that $P(t,T) = \exp(-\phi(t,T) - \psi(t,T)r_t)$ with ϕ and ψ solving certain Riccati differential equation; see [15], Section 5.4.1, for details. If we embed this approach in our structure given in (71), we may choose $A_t = t$. The dynamics of $f(t,T)$ in this case will depend also on $R_t := \int_0^t r_s\,ds$, such that we utilize the affine process

$$X_t = (t, R_t, r_t)^\top, \quad t \geq 0$$

in (72). We obtain that $b_t = b_t^0 + b_t^1 X_t$ with $b_t^0 = (1, 0, \alpha)^\top$ and $b_t^1 = (0, 1, \beta)$ as well as $c_t = c^0$ where the matrix c^0 has vanishing entries except for $c^0_{3,3} = \sigma^2$. The drift condition (76) now directly implies that for $A(t,T) = (A_1(t,T), A_2(t,T), A_3(t,T))$

$$\begin{aligned} A_2(t,T) &= -\beta A_3(t,T), \\ A_1(t,T) &= \left(A_3(t,T)\right)^2\frac{\sigma^2}{2} - \alpha A_3(t,T). \end{aligned} \tag{78}$$

We have the freedom to choose on component of $A(t,T)$ which we do to match the volatility structure of the Vasiček model, by setting the third component of $A(t,T)$ equal to

$$A_3(t,T) = \beta^{-1}\big(e^{\beta(T-t)} - 1\big).$$

In particular, this choice gives us

$$\begin{aligned}
a_1(t,T) &= \frac{\sigma^2}{\beta}\big(e^{\beta(T-t)} - 1\big) - \alpha e^{\beta(T-t)},\\
a_2(t,T) &= -\beta e^{\beta(T-t)},\\
a_3(t,T) &= e^{\beta(T-t)}.
\end{aligned}$$

It is a straightforward exercise that this specification indeed coincides with the Vasiček model given the explicit expressions for ϕ and ψ in Section 5.4.1 in [15]. In a similar manner, all affine term-structure models can be cast in the framework considered in this section.

EXAMPLE 6.14 (A simple Gaussian term structure model). A review of the above specification points towards the simpler Gaussian model where X is the three-dimensional affine process as above, driven by the Vasiček spot rate, but now we choose

$$A_3(t,T) = (T-t),$$

such that the parameter $a_3(t,T) = 1$ is constant. The drift condition now implies

$$\begin{aligned}
a_2 &= -\beta,\\
A_1(t,T) &= (T-t)^2\frac{\sigma^2}{2} - \alpha(T-t),
\end{aligned}$$

and we obtain a linear term $a_1(t,T) = \sigma^2(T-t) - \alpha$. This Gaussian model is considerably simpler than the Vasiček model, and still has a mean-reversion property (as X has the mean reversion property), but the volatility of the forward rate does not have the dampening factor $e^{\beta(T-t)}$ in the volatility.

Finally, we provide two examples of stochastic discontinuous specifications.

EXAMPLE 6.15 (Example 6.14 with discontinuity). Now we incorporate a stochastic discontinuity at $t = 1$ in the above example and let $A(t) = t + \mathbb{1}_{\{t\geq 1\}}$. The idea is to introduce a single jump at $t = 1$ in the third component and compensate this by a predictable jump in the first coordinate. We begin by describing precisely the model: first,

$$dr_t = (\alpha + \beta r_t)\,dt + \sigma\,dW_t + dJ_t,$$

where $J_t = \mathbb{1}_{\{t \geq 1\}} \xi$ with $\xi \sim \mathcal{N}(0, \gamma^2)$, $\gamma > 0$, being independent of W. Consider

$$X_t = (A_t, R_t, r_t)^\top, \quad t \geq 0,$$

with $R = \int_0^\cdot r_s\, ds$, as above. This construction of X implies that for $t \neq 1$, $b_t^0 = (1, 0, \alpha)^\top$ and $b_t^1 = (0, 1, \beta)^\top$ while for $t = 1$, $b_1^0 = (1, 0, 0)^\top$ and $b_1^1 = 0$. Moreover, for $t \neq 1$, $c_t^0 = c_0$ as in the example above, $c_t^1 = 0$ and, for $t = 1$, we obtain $c_1 = 0$. The kernel K vanishes except for $t = 1$ and is given by $K_1(dx) = \delta_1(dx_1)\phi(\frac{x_3}{\gamma})\, dx_3$ where δ_1 is the Dirac measure at point 1 and ϕ is the standard normal density. It does not depend on ω.

As in Example 6.14, we specify $a_3 = 1$, such that $A_3(t, T) = (T - t) + \mathbb{1}_{\{1 \in [t, T]\}}$. For $t > 1$, the process $A(t, T)$ is exactly as in the previous Example 6.14. For the remaining times, we again use Corollary 6.12; on the one hand, for $i = 1$, the drift condition (76) implies that $A_2(t, T) = -\beta A_3(t, T)$ for all $0 \leq t \leq T$. On the other hand, for $i = 0$, the drift condition can be *separated*. Indeed, as $dA_t = dt + \delta_1(dt)$, we obtain, using $\Delta C \equiv 0$, that (for $t = 1$)

$$A(1, T) b_0^1 = \int_{\mathbb{R}^d} \big(e^{-A(1,T)x} - 1 + A(1, T)x\big) K_{0,1}(dx), \tag{79}$$

and, for $t \neq 1$,

$$A(t, T) b_t^0 = \frac{1}{2} A(t, T) c_t^0 A(t, T)^\top. \tag{80}$$

Now equation (79) gives

$$A_1(1, T) = e^{-A_1(1,T) + \frac{(A_3(1,T)\gamma)^2}{2}} - 1 + A_1(1, T)$$

$$\Leftrightarrow \quad A_1(1, T) = \frac{(A_3(1, T)\gamma)^2}{2}, \tag{81}$$

such that A is specified for $t \in [1, T]$. Finally, for $0 \leq t < 1$, equation (80) implies

$$A_1(t, T) = -\alpha A_3(t, T) + \frac{(A_3(t, T)\sigma)^2}{2}$$

and we conclude our example.

EXAMPLE 6.16 (A discontinuous Vasiček model). We extend the previous example to the Vasiček model in a more general manner. Consider time points $t_1, \ldots, t_n$ which correspond to stochastic discontinuities. Moreover, assume that

$$dr_t = (\alpha + \beta r_t)\, dt + \sigma\, dW_t + dJ_t,$$

where

$$J_t = \sum_{i=1}^n \mathbb{1}_{\{t_i \leq t\}} \xi_i, \quad t \geq 0,$$

with ξ_i being i.i.d. $\sim \mathcal{N}(0,\gamma^2)$, being independent of W. Let $A_t = t + \sum_{i=1}^n \mathbb{1}_{\{t_i \le t\}}$ and consider as above $X = (A, R, r)$. Again, for $t \notin \{t_1, \dots, t_n\}$, $b_t^0 = (1, 0, \alpha)^\top$, $b_t^1 = (0, 1, \beta)^\top$ and $c_t^0 = c_0$, while for $t = t_i$, $b_{t_i}^0 = (1, 0, 0)^\top$, $b_{t_i}^1 = 0$ and $c_{t_i} = 0$. Moreover,

$$K_t(dx) = \mathbb{1}_{\{t \in \{t_1,\dots,t_n\}\}} \delta_1(dx_1) \phi\left(\frac{x_3}{\gamma}\right) dx_3.$$

We begin by specifying $a_3(t,T) = e^{\beta(T-t)}$ as in Example 6.13, such that

$$A_3(t,T) = \beta^{-1}\big(e^{\beta(T-t)} - 1\big) + \sum_{i=1}^n \mathbb{1}_{\{t_i \in [t,T]\}}.$$

Again, we separate the drift condition in continuous and discontinuous part with the aid of Corollary 6.12 yielding directly $A_2(t,T) = -\beta A_3(t,T)$ and $A_1(t,T) = (A_3(t,T))^2 \frac{\sigma^2}{2} - \alpha A_3(t,T)$, for $t \in [0,T] \backslash \{t_1, \dots, t_n\}$, compare equation (78). It remains to compute $A(t_i, T)$ for $t_i \le T$. In this regard, we obtain as in (81) that

$$A(t_i, T) = \frac{(A_3(t_i,T)\gamma)^2}{2}, \quad i = 1, \dots, n, \tag{82}$$

such that the discontinuous Vasiček model is fully specified.

APPENDIX: MEASURE DIFFERENTIAL EQUATIONS

This section recalls and extends some notions and statements about measure differential equations (sometimes also referred to as Stieltjes differential equations) for the special cases needed in this article.

Let A be an increasing function on $\mathbb{R}_{\ge 0}$ with left limits and $F : \mathbb{R}_{\ge 0} \times \mathcal{U} \to \mathcal{U}$, where the space $\mathcal{U}$ is defined in equation (2). Assume $F(\cdot, g(\cdot))$ is A-integrable on some interval $I \subset \mathbb{R}_{\ge 0}$ for all functions $g : \mathbb{R}_{\ge 0} \to \mathcal{U}$ of bounded variation. We consider the equation

$$\frac{dg(t)}{dA_t} = -F\big(t, g(t)\big), \quad g(T) = u, \tag{83}$$

dg/dA denotes the Radon–Nikodym derivative of the measure induced by g with respect to the measure induced by A. We now recall the definition of a solution to a measure differential equation from [8] that we adopt in this article.

DEFINITION A.1. Let S be an open connected set in $\mathcal{U}$ and $T \in I$. A function $g(\cdot) = g(\cdot, T, u)$ will be called a solution of (83) through (T, u) on the interval I if g is right continuous, of bounded variation, $g(T) = u$ and the distributional derivative of g satisfies (83) on (τ, T) for any $\tau < T$ in I.

REMARK A.2. Assume $F(t, g(t))$ is integrable with respect to the Lebesgue–Stieltjes measure dA for each function g of bounded variation. Equivalently to the above definition, g is a solution of (83) through (T, u) on I if and only if it satisfies the integral equation

$$g(t) = u + \int_{(t,T]} F\bigl(s, g(s)\bigr)\, dA_s; \tag{84}$$

see [8] for more details.

We are now going to state and prove a modification of the existence and uniqueness result for measure differential equations in [39]. Define

$$\Omega_b = \bigl\{u \in \mathcal{U} \,|\, |u| < b\bigr\}.$$

THEOREM A.3. *Suppose the following conditions hold*:

(i) *there exists an A-integrable function w such that*

$$\bigl|F(t, u)\bigr| \le w(t) \tag{85}$$

uniformly in $u \in \Omega_b$;

(ii) *F satisfies a Lipschitz condition in u, that is, there exists an A-integrable Lipschitz constant L such that*

$$\bigl|F(t, u_1) - F(t, u_2)\bigr| \le L(t)|u_1 - u_2|$$

for all $u \in \Omega_b$.

Then there exists a unique solution g of (83) *on some interval $(T - a, T]$, $a > 0$, satisfying the terminal condition $g(T) = u$.*

PROOF. First, note that we have the following equation for the jumps of a solution g to (83), for all $t \in \{t \in \mathbb{R}_+ | \Delta A_t \neq 0\}$:

$$\Delta g(t) = -F\bigl(t, g(t)\bigr)\Delta A_t. \tag{86}$$

With $\Delta g(t) = g(t) - g(t-)$, this is an explicit equation for the left limit of g, hence we can assume that A has no jump at the terminal time T, as we can simply compute $g(T-)$ from the terminal value and start from there instead. Even with a time-varying Lipschitz constant, the proof of Theorem 1 in [39] is valid with small adjustments: A is increasing and cádlág. Therefore, there exists $r \in [0, T]$ such that

$$\int_{(r,T]} L(s)\, dA_s < 1$$

and

$$k := |u| + \int_{(r,T]} w(s)\, dA_s < b. \tag{87}$$

Denote the space of cádlág functions f on $(r, T]$ with terminal value $f(T) = u$ and total variation $\|f\| \le k$ by Λ and consider the mapping

$$Kf(t) = u - \int_{(t,T]} F\big(s, f(s)\big)\, dA_s, \quad t \in (r, T].$$

It follows from condition (i) and equation (87) that K maps Λ into itself. From the Lipschitz condition on F, we obtain

$$\|Kf_1 - Kf_2\| \le \|f_1 - f_2\| \int_{(r,T]} L(s)\, dA_s.$$

Hence, K is a contraction on Λ, a closed subspace of the space of càdlàg functions with bounded variation.This implies the existence of a unique fixed point of K, which is the desired local solution of (85). $\square$

EXAMPLE A.4 (The linear equation). Let A as above and $L \in L_1(dA)$ with $L(t)\Delta A_t \ge -1$ for all $t \ge 0$. Consider the linear measure equation

$$\frac{d}{dA_t}\phi(t) = -L(t)\phi(t), \quad \phi(T) = \phi_T \tag{88}$$

on $[0, T]$. The process $\tilde{A}_t := \int_{[0,t]} L(s)\, dA_s$ has finite variation, and thus we can apply [21], Theorem I.4.61, and especially equation (I.4.63) to obtain that the unique, càdlàg solution to the linear equation (88) is given by $\phi(t) = \phi_T \mathcal{E}_t^T(L\, dA)$ where

$$\begin{aligned} \mathcal{E}_t^T(L\, dA) &:= e^{\int_t^T L(s)\, dA_s} \prod_{s\in(t,T]} \big(1 + L(s)\Delta A_s\big) e^{-L(s)\Delta A_s} \\ &= e^{\int_t^T L(s)\, dA_s^c} \prod_{s\in(t,T]} \big(1 + L(s)\Delta A_s\big). \end{aligned}$$

PROPOSITION A.5. *Let f, g be right continuous and absolutely continuous w.r.t. A. If the following conditions hold*:

(i) $f(T) \le g(T)$,

(ii) $\frac{d}{dA_t} f(t) = -F(t, f(t))$ *and* $\frac{d}{dA_t} g(t) = -G(t, g(t))$ *on* $I = [0, T]$, *where* F, G *are locally Lipschitz continuous in the second variable with A-integrable Lipschitz constants*, *and*

(iii) $F(t, u) \le G(t, u)$ *for all* $t \in I$,

then $f(t) \le g(t)$ *for all* $t \in I$.

PROOF. Suppose the conclusion of the proposition does not hold. Let $w = f - g$. Then exists an interval $I' = [t_0, t_1)$ such that w is positive and continuous on I' and $w(t_1) \le 0$. Two cases can occur: $\Delta A_{t_1} = 0$ or $\Delta A_{t_1} \ne 0$.

Consider first the case when there is no jump at t_1. From condition (ii) and (iii), we obtain on $(t_0, t_1]$ that

$$\frac{dw(t)}{dA_t} = G\big(t, g(t)\big) - F\big(t, f(t)\big) \geq G\big(t, g(t)\big) - G\big(t, f(t)\big) \geq -L_t w(t),$$

where L_t is the Lipschitz constant of $G(t, \cdot)$ on the relevant domain. Consider the function $W(t) = w(t)\exp(-\int_t^{t_1} L_s\, dA_s)$ on $(t_0, t_1]$. W is absolutely continuous w.r.t. A and continuous. Furthermore,

$$\frac{dW(t)}{dA_t} = \left(\frac{dw(t)}{dA_t} + L_t w(t)\right) e^{-\int_t^{t_1} L_s\, dA_s} \geq 0, \quad t \in (t_0, t_1].$$

Together with $w(t_1) \leq 0$, it follows that $w(t) \leq 0$ for all $t \in (t_0, t_1]$ contradicting the assumption. Second, if we have a jump at t_1, that is, $\Delta w(t_1) \neq 0$, we immediately get $\Delta w(t_1) < 0$ and, therefore,

$$\begin{aligned} 0 > \Delta w(t_1) &= -\big(F\big(t_1, f(t_1)\big) - G\big(t, g(t_1)\big)\big)\Delta A_{t_1} \\ &\geq -L_{t_1} w(t_1) \Delta A_{t_1}. \end{aligned}$$

Hence, $w(t_1) > 0$; a contradiction. □

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