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APPROXIMATION OF STOCHASTIC PROCESSES BY NONEXPANSIVE FLOWS AND COMING DOWN FROM INFINITY[1]

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This paper deals with the approximation of semimartingales in finite dimension by dynamical systems. We give trajectorial estimates uniform with respect to the initial condition for a well-chosen distance. This relies on a nonexpansivity property of the flow and allows to consider non-Lipschitz vector fields. The fluctuations of the process are controlled using the martingale technics and stochastic calculus.

Our main motivation is the trajectorial description of stochastic processes starting from large initial values. We state general properties on the coming down from infinity of one-dimensional SDEs, with a focus on stochastically monotone processes. In particular, we recover and complement known results on Λ-coalescent and birth and death processes. Moreover, using Poincaré's compactification techniques for flows close to infinity, we develop this approach in two dimensions for competitive stochastic models. We thus classify the coming down from infinity of Lotka–Volterra diffusions and provide uniform estimates for the scaling limits of competitive birth and death processes.

CONTENTS

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1. Introduction. The approximation of stochastic processes has been largely developed and we refer, for example, to [17, 20] for general statements both for deterministic approximation and study of the fluctuations. Particular attention has been paid to random perturbation of dynamical systems [18, 28] and the study of fluid and scaling limits of random models; see [12] for a survey about approximation of Markov chains. In this paper, we are interested in stochastic processes $(X_t : t \geq 0)$ taking values in a Borel subset E of $\mathbb{R}^{\mathrm{d}}$, which can be written as

$$X_t = X_0 + \int_0^t \psi(X_s)\,ds + R_t,$$

where R is a semimartingale. We aim at proving that X remains close to the flow $\phi(x_0, t) = x_t$ given by

$$x_t = x_0 + \int_0^t \psi(x_s)\,ds.$$

The point here is to estimate the probability of this event uniformly with respect to the initial condition $x_0 \in D$, when the drift term ψ may be non-Lipschitz on D. Our main motivation for such estimates is the description of the coming down from infinity, which amounts to let the initial condition x_0 go to infinity, and the uniform scaling limits of stochastic processes describing population models on unbounded domains.

The approach relies on a contraction property of the flow, which provides stability on the dynamics. This notion is used in particular in control theory. More precisely, we say that the vector field ψ is nonexpansive on a domain D when it prevents two trajectories from moving away for the euclidean norm on a subset D of $\mathbb{R}^{\mathrm{d}}$. This amounts to

$$\forall x, y \in D, \qquad \big(\psi(x) - \psi(y)\big)\boldsymbol{.}(x - y) \leq 0,$$

where $\boldsymbol{.}$ is the usual scalar product on $\mathbb{R}^{\mathrm{d}}$. Actually, the distance between two solutions may increase provided that this increase is not too fast. This allows to deal with additional Lipschitz component or bounded perturbation in the flow and it is required for the applications considered here. Thus we are working with (L, α) *nonexpansive vector fields.*

DEFINITION 1.1. The vector field $\psi : D \to \mathbb{R}^{\mathrm{d}}$ is (L, α) nonexpansive on $D \subset \mathbb{R}^{\mathrm{d}}$ if for any $x, y \in D$,

$$\big(\psi(x) - \psi(y)\big).(x - y) \leq L\|x - y\|_2^2 + \alpha\|x - y\|_2.$$

The nonexpansivity property ensures that the drift term cannot make the distance between the stochastic process X and the dynamical system x explode because of small fluctuations due to the perturbation R. To control the size of these fluctuations, we use martingale technics in Section 2: let us mention [12] in the context of scaling limits and [6] for a pioneering work on the speed of coming down from infinity of Λ-coalescents.

These results are developed and specified when X satisfies a Stochastic Differential Equation (SDE), in Section 3, which allows a diffusion component and random jumps given by a Poisson point measure. This covers the range of our applications. We then estimate the probability that the stochastic process remains close to the dynamical system when a transformation of the domain provides (L, α)-nonexpansivity of the associated flow. These estimates hold for any $x_0 \in D$ and a well-chosen distance d, which is bound to capture the fluctuations of X around the flow ϕ. Informally, we obtain that for any $\varepsilon > 0$,

$$\mathbb{P}_{x_0}\Big(\sup_{t \leq T \wedge T_D(x_0)} d\big(X_t, \phi(x_0, t)\big) \geq \varepsilon\Big) \leq C_T \int_0^T \overline{V}_{d,\varepsilon}(x_0, t)\, dt, \tag{1}$$

where $T_D(x_0)$ corresponds to the exit time of the domain D for the flow ϕ started at x_0. The transformation F of the flow is of class $\mathcal{C}^2$, so that we can use the stochastic calculus. The distance d is inherited from this transformation and of the form

$$d(x, y) = \big\|F(x) - F(y)\big\|_2.$$

The perturbation needs to be controlled for this distance d in a tube around the trajectory of the dynamical system and

$$\overline{V}_{d,\varepsilon}(x_0, t) = \sup_{\substack{x \in E \\ d(x, \phi(x_0, t)) \leq \varepsilon}} \big\{\varepsilon^{-2}\big\|V_F(x)\big\|_1 + \varepsilon^{-1}\big\|\widetilde{b}_F(x)\big\|_1\big\},$$

where V_F will be given by the quadratic variation of $F(X)$ and $\widetilde{b}_F$ will be an additional approximation term arising from Itô's formula applied to $F(X)$.

Relevant choices of F will be illustrated through several examples. First, they are linked to the geometry of the flow and allow to change the metric so that (L, α) nonexpansivity property is guaranteed. We refer to the last section for a family of transformations covering the full domain for a two-dimensional competitive model. Second, these transformations F need to reduce enough fluctuations so that these latter can be integrated along the trajectory, see in particular the different functions involved in Section 4.2.

The estimate (1) becomes uniform with respect to $x_0 \in D$ as soon as $\overline{V}_{d,\varepsilon}(x_0, .)$ can be bounded by an integrable function of the time. It allows then to characterize the coming down from infinity for stochastic differentials equations in $\mathbb{R}^{\mathrm{d}}$. Roughly speaking, we consider an unbounded domain D and let T go to 0 to derive from (1) that for any $\varepsilon > 0$,

$$\lim_{T\to 0} \sup_{x_0\in D} \mathbb{P}_{x_0}\Big(\sup_{t\le T} d\big(X_t, \phi(x_0, t)\big) \geq \varepsilon\Big) = 0.$$

Letting then x_0 go to infinity enables to describe the coming down from infinity of processes in several ways. First, the control of the fluctuations of the process X for large initial values by a dynamical system gives a way to prove the tightness of $\mathbb{P}_{x_0}$ for $x_0 \in D$. Moreover, we can link in general the coming down from infinity of the process X to the coming down from infinity of the flow ϕ, in the vein of [5, 6, 25], which focus respectively on Λ coalescence, Ξ coalescent and birth and death processes.

In dimension 1, following [5, 16], we use a monotonicity property to identify the limiting values of $\mathbb{P}_{x_0}$ as $x_0 \to \infty$ and we determine when the process comes down from infinity and how it comes down from infinity (Section 4). In particular, we recover the speed of coming down from infinity of Λ-coalescent [6] with $F = \log$ and in that case V_F is bounded. In [6], the short time behavior of the log of the number of blocks is captured and the nonexpansivity argument for the flow is replaced by a technical result relying on the monotonicity of suitable functions in dimension 1 (Lemma 10 therein). We also recover some results of [5] for birth and death processes and we can provide finer estimates for regularly varying death rates. Here, F is polynomial and V_F is unbounded so this latter has to be controlled along the trajectory of the dynamical system. Finally, we consider the example of transmission control protocol which is nonstochastically monotone and $F(x) = \log(1 + \log(1 + x))$ is required to control its (very) large fluctuations for large values.

In higher dimension, the coming down from infinity of a dynamical system is a more delicate problem in general. Poincaré has initiated a theory to study dynamical systems close to infinity, which is particularly powerful for polynomial vector fields (see, e.g., Chapter 5 in [16]). We develop this approach for competitive Lotka–Volterra models in dimension 2 in Section 5.1, which was a main motivation for this work. We classify the ways the dynamical system can come down from infinity and describe the counterpart for the stochastic process, which differs when the dynamical system is getting close from the boundary of $(0,\infty)^2$.

The uniform estimates (1) can also be used to prove scaling limits of stochastic processes X^K to dynamical systems, which are uniform with respect to the initial condition, without requiring Lipschitz property for the vector field ψ. The results involve a suitable distance d as introduced above to capture the fluctuations of the process

$$\lim_{K\to\infty} \sup_{x_0\in D} \mathbb{P}_{x_0}\Big(\sup_{t\le T} d\big(X_t^K, \phi(x_0, t)\big) \geq \varepsilon\Big) = 0,$$

for some fixed $T, \varepsilon > 0$. It is illustrated in this paper by the convergence of birth and death processes with competition to Lotka–Volterra competitive dynamical system in Section 5.2.

Let us end up with other motivations for this work, some of which being linked to works in progress. First, our original motivation for studying the coming down from infinity is the description of the time for extinction for competitive models in varying environment. Roughly speaking, competitive periods make large sizes of populations quickly decrease, which can be captured by the coming down from infinity. Direction and speed of coming down from infinity are then involved to quantify the time of extinction or determine coexistence of populations. Let us also note that the approach developed here could be extended to the varying environment framework by comparing the stochastic process to a nonautonomous dynamical system. Second, the coming down from infinity is linked to the uniqueness of the quasistationary distribution; see [29] for birth and death processes and [9] for some diffusions. Recently, the coming down from infinity has appeared as a key assumption for the geometric convergence of the conditioned process to the quasistationary distribution, uniformly with respect to the initial distribution. We refer to [11] for details; see, in particular, Assumption (A1) therein.

NOTATION. In the whole paper, $\boldsymbol{.}$ stands for the canonical scalar product on $\mathbb{R}^{\mathrm{d}}$, $\|\boldsymbol{.}\|_2$ the associated Euclidean norm and $\|\boldsymbol{.}\|_1$ the L^1 norm.

For convenience, we write $x = (x^{(i)} : i = 1, \dots, \mathrm{d}) \in \mathbb{R}^{\mathrm{d}}$ a row vector of real numbers. The product xy for $x, y \in \mathbb{R}^{\mathrm{d}}$ is the vector $z \in \mathbb{R}^{\mathrm{d}}$ such that $z_i = x_i y_i$.

We denote by $\overline{B}(x, \varepsilon) = \{y \in \mathbb{R}^{\mathrm{d}} : \|y - x\|_2 \leq \varepsilon\}$ the Euclidean closed ball centered in x with radius ε. More generally, we note $\overline{B}_d(x, \varepsilon) = \{y \in O : d(x, y) \leq \varepsilon\}$ the closed ball centered in $x \in O$ with radius ε associated with the application $d : O \times O \to \mathbb{R}^+$.

When $\chi = (\chi^{(1)}, \dots, \chi^{(d)})$ is differentiable on an open set of $\mathbb{R}^{\mathrm{d}}$ and takes values in $\mathbb{R}^{\mathrm{d}}$, we denote by J_χ its Jacobian matrix and

$$\big(J_\chi(x)\big)_{i,j} = \frac{\partial}{\partial x_j} \chi^{(i)}(x) \qquad (i, j = 1, \dots, \mathrm{d}).$$

We write F^{-1} the reciprocal function of a bijection F and A^{-1} the inverse of an invertible matrix A. Moreover, the transpose of a matrix A is denoted by A^*.

By convention, we assume that $\sup \varnothing = 0$, $\sup[0, \infty) = +\infty$, $\inf \varnothing = \infty$ and if $x, y \in \mathbb{R} \cup \{\infty\}$, we write $x \wedge y$ for the smallest element of $\{x, y\}$.

We write $d(x) \sim_{x \to a} g(x)$ when $d(x)/g(x) \to 1$ as $x \to a$.

We also use notation $\int_{\cdot}^{a} f(x)\,dx < \infty$ (resp., $= \infty$) for $a \in [0, \infty]$ when there exists $a_0 \in (a, \infty)$ such that $\int_{a_0}^{a} f(x)\,dx$ is well defined and finite (resp., infinite).

Finally, we denote by $\langle M \rangle$ the predictable quadratic variation of a continuous local martingale M and by $|A|$ the total variation of a process A and by $\Delta X_s = X_s - X_{s-}$ the jump at time s of a càdlàg process X.

Outline of the paper. In the next section, we provide general results for dynamical systems perturbed by semimartingales using the nonexpansivity of the flow and martingale inequality. In Section 3, we derive approximations results for the Markov process described by SDE. It relies on a transformation F of the process for which we apply the results of Section 2. An extension of the result by adjunction of nonexpansive domains is provided and required for the applications of the last section. We then study the coming down from infinity for one dimensional SDEs in Section 4, with a focus on stochastically monotone processes. Finally, we compare the coming down from infinity of two-dimensional competitive Lotka–Volterra diffusions with the coming down from infinity of Lotka–Volterra dynamical systems and prove uniform approximations of these latter by birth and death processes.

2. Random perturbation of dynamical systems. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)_{t\geq 0}$ a filtration of $\mathcal{F}$, which satisfies the usual conditions. We assume that X is a $\mathcal{F}_t$-adapted càdlàg process on $[0, \infty)$ which takes its values in a Borel subset E of $\mathbb{R}^{\mathrm{d}}$ and satisfies for every $t \geq 0$,

$$X_t = X_0 + \int_0^t \psi(X_s)\, ds + R_t,$$

where $X_0 \in E$ a.s., ψ is a Borel measurable function from $\mathbb{R}^{\mathrm{d}}$ to $\mathbb{R}^{\mathrm{d}}$ locally bounded and $(R_t : t \geq 0)$ is a càdlàg $\mathcal{F}_t$-semimartingale. Moreover, the process R is decomposed as

$$R_t = A_t + M_t, \qquad M_t = M_t^c + M_t^d,$$

with A_t a càdlàg $\mathcal{F}_t$-adapted process with a.s. bounded variations paths, M_t^c a continuous $\mathcal{F}_t$-local martingale, M_t^d a càdlàg $\mathcal{F}_t$-local martingale purely discontinuous and $R_0 = A_0 = M_0 = M_0^c = M_0^d = 0$. Let us observe that such a decomposition may be nonunique.

We assume that ψ is locally Lipchitz on a (nonempty) open set E' of $\mathbb{R}^d$ and consider the solution $x = \phi(x_0, .)$ of

$$x_t = x_0 + \int_0^t \psi(x_s)\, ds$$

for $x_0 \in E'$. This solution exists, belongs to E' and is unique on some time interval $[0, T'(x_0))$, where $T'(x_0) \in (0, \infty]$. Then, to compare the process X to the solution x, we define the maximal gap before t:

$$S_t := \sup_{s\leq t} \|X_s - x_s\|_2$$

for any $t < T'(x_0)$. We also set

$$\begin{aligned} T_{D,\varepsilon}(x_0) = \sup\{t \in [0, T'(x_0)) : \forall s \leq t, x_s \in D \text{ and} \\ \overline{B}(x_s, \varepsilon) \cap E \subset D\} \in [0, \infty] \end{aligned} \tag{2}$$

the last time when x_t and its ε-neighborhood in E belong to a domain D. As mentioned in the Introduction, the key property to control the distance between $(X_t : t \geq 0)$ and $(x_t : t \geq 0)$ before time $T_{D,\varepsilon}(x_0)$ is the (L, α) nonexpansivity property of ψ on D, in the sense of Definition 1.1.

When $\alpha = 0$, we simply say that ψ is L nonexpansive on D. If additionally $L = 0$, we say that ψ is nonexpansive on D. We first note that in dimension 1, the fact that ψ is nonexpansive simply means that ψ is nonincreasing. More generally, when ψ is differentiable on a convex open set O which contains D, ψ is L nonexpansive on D if for any $x \in O$,

$$\mathrm{Sp}(J_\psi + J_\psi^*) \subset (-\infty, 2L],$$

where $\mathrm{Sp}(J_\psi + J_\psi^*)$ is the spectrum of the symmetric linear operator (and hence diagonalizable) $J_\psi + J_\psi^*$; see Table 1 in [1] for details and more general results and the last section for an application. Finally, we observe that

$$\psi = B + \chi = B + f + g$$

is (L, α) nonexpansive on D if B is a vector field whose Euclidean norm is bounded by α on D and χ is L nonexpansive on D. Moreover $\chi = f + g$ is L nonexpansive on D if f is Lipschitz with constant L and g is nonexpansive on D.

For convenience and use of the Gronwall lemma, we also introduce for $L, \alpha \geq 0$ and $\varepsilon > 0$,

$$T_\varepsilon^{L,\alpha} = \sup\{T \geq 0 : 4\alpha T \exp(2LT) \leq \varepsilon\} \in (0, \infty], \tag{3}$$

which is infinite if and only if $\alpha = 0$, that is, as soon as the vector field ψ is L nonexpansive.

2.1. *Trajectorial control for perturbed nonexpansive dynamical systems.* The following lemma gives the trajectorial result which allows to control the gap between the stochastic process $(X_t : t \geq 0)$ and the dynamical system $(x_t : t \geq 0)$ by the size of the fluctuations of the semimartingale $(R_t : t \geq 0)$ and the gap between the initial positions. The control of fluctuations involves the following quantity, which is defined for all $t < T'(x_0)$ and $\varepsilon > 0$:

$$\widetilde{R}_t^\varepsilon = \|X_0 - x_0\|_2^2 + \mathbf{1}_{\{S_{t-} \leq \varepsilon\}}\left[2\int_0^t (X_{s-} - x_s).dR_s + \big\|[M]_t\big\|_1\right],$$

where $\int_0^t (X_{s-} - x_s).dR_s$ is a stochastic integral and $[M] = [X] = [R]$ is the quadratic variation of the semimartingale R. We refer to Chapter I, Theorem 4.31 in [20] for the existence of stochastic integral of càglàd (and thus predictable locally bounded) process with respect to semimartingale. Moreover, the expression of the quadratic variation ensures that

$$\big\|[M]_t\big\|_1 = \big\|[X]_t\big\|_1 = \big\|\langle M^c\rangle_t\big\|_1 + \sum_{s \leq t} \|\Delta X_s\|_2^2; \tag{4}$$

see, for example, Chapter I, Theorem 4.52 in [20]. Unless otherwise specified, the identities hold *almost surely (a.s.)*.

LEMMA 2.1. *Assume that ψ is (L,α) nonexpansive on some domain $D \subset E'$ and let $\varepsilon > 0$.*

Then for any $x_0 \in E'$ and $T < T_{D,\varepsilon}(x_0) \wedge T_\varepsilon^{L,\alpha}$, we have

$$\{S_T \geq \varepsilon\} \subset \Big\{\sup_{t\leq T} \widetilde{R}_t^\varepsilon > \eta^2\Big\},$$

where $\eta = \varepsilon \exp(-LT)/\sqrt{2}$.

PROOF. Let $x_0 \in E'$. First, we consider the quadratic variation of $(X_t - x_t : 0 \leq t < T'(x_0))$:

$$[X - x]_t = [M]_t = (X_t - x_t)^2 - (X_0 - x_0)^2 - 2\int_0^t (X_{s-} - x_s)\, d(X_s - x_s),$$

for $t < T'(x_0)$; see, for example, Chapter I, Definition 4.4.45 in [20] or use Itô's formula. Summing the coordinates of $[M]_t$ and using the definitions of X and x, we get

$$\begin{aligned}\|X_t - x_t\|_2^2 = \|X_0 - x_0\|_2^2 &+ 2\int_0^t (X_{s-} - x_s)\boldsymbol{.}\big(\psi(X_{s-}) - \psi(x_s)\big)\, ds\\ &+ 2\int_0^t (X_{s-} - x_s)\boldsymbol{.}\, dR_s + \big\|[M]_t\big\|_1.\end{aligned}$$

Moreover, for any $s < T_{D,\varepsilon}(x_0)$, $x_s \in D$ and $X_{s-} \in D$ on the event $\{S_{s-} \leq \varepsilon\}$. So using that ψ is (L,α) nonexpansive on D,

$$\begin{aligned}&\mathbf{1}_{\{S_{s-}\leq\varepsilon\}}(X_{s-} - x_s)\boldsymbol{.}\big(\psi(X_{s-}) - \psi(x_s)\big)\\ &\qquad \leq \mathbf{1}_{\{S_{s-}\leq\varepsilon\}}\big(L\|X_{s-} - x_s\|_2^2 + \alpha\|X_{s-} - x_s\|_2\big).\end{aligned}$$

Then for any $t < T_{D,\varepsilon}(x_0)$,

$$\begin{aligned}\mathbf{1}_{\{S_{t-}\leq\varepsilon\}}\|X_t - x_t\|_2^2 \leq \mathbf{1}_{\{S_{t-}\leq\varepsilon\}}\Big[&2L\int_0^t \|X_s - x_s\|_2^2\, ds + 2\alpha\int_0^t \|X_s - x_s\|_2\, ds\\ &+ \|X_0 - x_0\|_2^2 + 2\int_0^t (X_{s-} - x_s)\boldsymbol{.}\, dR_s + \big\|[M]_t\big\|_1\Big]\end{aligned}$$

and by definition of $\widetilde{R}^\varepsilon$,

$$\mathbf{1}_{\{S_{t-}\leq\varepsilon\}} S_t^2 \leq 2L\int_0^t \mathbf{1}_{\{S_{s-}\leq\varepsilon\}} S_s^2\, ds + 2\alpha t\varepsilon + \sup_{s\leq t}\widetilde{R}_s^\varepsilon.$$

By the Gronwall lemma, we obtain for any $T < T_{D,\varepsilon}(x_0)$ and $t \leq T$,

$$\mathbf{1}_{\{S_{t-}\leq\varepsilon\}} S_t^2 \leq \Big(2\alpha T\varepsilon + \sup_{s\leq T}\widetilde{R}_s^\varepsilon\Big)e^{2LT}.$$

Moreover, for $T < T_\varepsilon^{L,\alpha}$, we have $2\alpha T e^{2LT} < \frac{\varepsilon}{2}$ and

$$(2\alpha T\varepsilon + \eta^2)e^{2LT} < \varepsilon^2,$$

recalling that $\eta = \varepsilon/(\sqrt{2}\exp(LT))$. Then

$$\Big\{\sup_{s\leq T} \widetilde{R}_s^\varepsilon \leq \eta^2\Big\} \subset \Big\{\sup_{t\leq T} \mathbf{1}_{\{S_{t-}\leq\varepsilon\}} S_t^2 < \varepsilon^2\Big\}. \tag{5}$$

Denoting

$$T_{\mathrm{exit}} = \inf\{s < T_{D,\varepsilon}(x_0) \wedge T_\varepsilon^{L,\alpha} : S_s \geq \varepsilon\},$$

and recalling that S is càdlàg, we have $S_{T_{\mathrm{exit}}-} \leq \varepsilon$ and $S_{T_{\mathrm{exit}}} \geq \varepsilon$ on the event $\{T_{\mathrm{exit}} \leq T\}$, so using (5) at time $t = T_{\mathrm{exit}}$ ensures that

$$\{T_{\mathrm{exit}} \leq T\} \subset \Big\{\sup_{s\leq T} \widetilde{R}_s^\varepsilon > \eta^2\Big\},$$

which completes the proof. □

2.2. *Nonexpansivity and perturbation by martingales.* We use now martingale maximal inequality to estimate the probability that the distance between the process $(X_t : t \geq 0)$ and the dynamical system $(x_t : t \geq 0)$ goes beyond some level $\varepsilon > 0$. Such arguments are classical and have been used in several contexts; see, in particular, [12] for a survey and applications in scaling limits and [6] for the coming down from infinity of Λ-coalescent, which have both inspired the results below.

PROPOSITION 2.2. *Assume that ψ is (L, α) nonexpansive on some domain $D \subset E'$ and let $\varepsilon > 0$.*

Then for any $x_0 \in E'$ and $T < T_{D,\varepsilon}(x_0) \wedge T_\varepsilon^{L,\alpha}$, for any $p \geq 1/2$ and $q \geq 0$,

$$\begin{aligned}
&\mathbb{P}(S_T \geq \varepsilon) \\
&\quad\leq \mathbb{P}\Big(\|X_0 - x_0\|_2 \geq \varepsilon\frac{e^{-LT}}{2\sqrt{2}}\Big) + C_q\frac{e^{2qLT}}{\varepsilon^q}\mathbb{E}\Big(\Big(\int_0^T \mathbf{1}_{\{S_{s-}\leq\varepsilon\}}\, d\big\||A|_s\big\|_1\Big)^q\Big) \\
&\qquad + C_{p,\mathrm{d}}\frac{e^{4pLT}}{\varepsilon^{2p}}\Big[\mathbb{E}\Big(\Big(\int_0^T \mathbf{1}_{\{S_{t-}\leq\varepsilon\}}\, d\big\|\langle M^c\rangle_t\big\|_1\Big)^p\Big) \\
&\qquad + \mathbb{E}\Big(\Big(\sum_{t\leq T}\mathbf{1}_{\{S_{t-}\leq\varepsilon\}}\|\Delta X_t\|_2^2\Big)^p\Big)\Big],
\end{aligned}$$

for some positive constants C_q (resp. $C_{p,\mathrm{d}}$) which depend only on q (resp., p, d).

PROOF. By definition of $\widetilde{R}^\varepsilon$,

$$\Big\{\sup_{t\leq T} \widetilde{R}_t^\varepsilon \geq \eta^2\Big\} \subset \Big\{\|X_0 - x_0\|_2^2 \geq \frac{\eta^2}{4}\Big\} \cup B_\eta,$$

where $B_\eta = \{\sup_{t\le T} \mathbf{1}_{\{S_{t-}\le\varepsilon\}} \int_0^t (X_{s-} - x_s)\boldsymbol{.}dR_s \ge \eta^2/8\} \cup \{\sup_{t\le T} \mathbf{1}_{\{S_{t-}\le\varepsilon\}} \times \|[M]_t\|_1 \ge \eta^2/4\}$. Recalling that $R_t = A_t + M_t$ and (4),

$$\begin{aligned} B_\eta \subset & \left\{\sup_{t\le T}\int_0^t \mathbf{1}_{\{S_{s-}\le\varepsilon\}}(X_{s-}-x_s)\boldsymbol{.}dA_s \ge \frac{\eta^2}{16}\right\} \\ & \cup \left\{\sup_{t\le T}\int_0^t \mathbf{1}_{\{S_{s-}\le\varepsilon\}}(X_{s-}-x_s)\boldsymbol{.}\,dM_s \ge \frac{\eta^2}{16}\right\} \\ & \cup \left\{\int_0^T \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\|\langle M^c\rangle_t\|_1 \ge \frac{\eta^2}{8}\right\} \cup \left\{\sum_{t\le T}\mathbf{1}_{\{S_{t-}\le\varepsilon\}}\|\Delta X_t\|_2^2 \ge \frac{\eta^2}{8}\right\}. \end{aligned}$$

We also know from Lemma 2.1 that

$$\{S_T \ge \varepsilon\} \subset \left\{\sup_{s\le T} \widetilde{R}_t \ge \eta^2\right\}$$

and using the Markov inequality yields

$$\begin{aligned} &\mathbb{P}(S_T \ge \varepsilon) \\ &\quad \le \mathbb{P}\left(\|X_0 - x_0\|_2^2 \ge \frac{\eta^2}{4}\right) + \mathbb{P}(B_\eta) \\ &\quad \le \mathbb{P}\left(\|X_0 - x_0\|_2^2 \ge \frac{\eta^2}{4}\right) + \left(\frac{16}{\eta^2}\right)^q \mathbb{E}\left(\sup_{t\le T}\left|\int_0^t \mathbf{1}_{\{S_{s-}\le\varepsilon\}}(X_{s-}-x_s)\boldsymbol{.}dA_s\right|^q\right) \\ &\qquad + \left(\frac{16}{\eta^2}\right)^{2p} \mathbb{E}\left(\sup_{t\le T}\left|\int_0^t \mathbf{1}_{\{S_{s-}\le\varepsilon\}}(X_{s-}-x_s)\boldsymbol{.}dM_s\right|^{2p}\right) \\ &\qquad + \left(\frac{8}{\eta^2}\right)^{p} \mathbb{E}\left(\left(\int_0^T \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\|\langle M^c\rangle_t\|_1\right)^p\right) \\ &\qquad + \left(\frac{8}{\eta^2}\right)^{p} \mathbb{E}\left(\left[\sum_{t\le T}\mathbf{1}_{\{S_{t-}\le\varepsilon\}}\|\Delta X_t\|_2^2\right]^p\right). \end{aligned} \tag{6}$$

First, using that $|f_s\boldsymbol{.}dg_s| \le \|f_s\|_2 d\||g|_s\|_1$ since $|f_s^{(i)}| \le \|f_s\|_2$, we have for $t \le T$,

$$\begin{aligned} \left|\int_0^t \mathbf{1}_{\{S_{s-}\le\varepsilon\}}(X_{s-}-x_s)\boldsymbol{.}dA_s\right| &\le \int_0^t \mathbf{1}_{\{S_{s-}\le\varepsilon\}}\|X_{s-}-x_s\|_2\, dA_s^1 \\ &\le \varepsilon\int_0^T \mathbf{1}_{\{S_{s-}\le\varepsilon\}}\, dA_s^1, \end{aligned} \tag{7}$$

where $A_s^1 := \||A|_s\|_1$ is the sum of the coordinates of the total variations of the process A.

Second, the Burkholder–Davis–Gundy inequality (see [14], 93, Chapter VII, page 287) for the local martingale

$$N_t = \int_0^t \mathbf{1}_{\{S_{s-}\le\varepsilon\}}(X_{s-}-x_s)\boldsymbol{.}dM_s$$

ensures that there exists $C_p > 0$ such that

$$\mathbb{E}\Big(\sup_{t\le T}|N_t|^{2p}\Big) \le C_p\mathbb{E}\big([N]_T^p\big).$$

Writing the coordinates of X, M and x, respectively, $(X^{(i)} : i = 1, \dots, \mathrm{d})$, $(M^{(i)} : i = 1, \dots, \mathrm{d})$ and $(x^{(i)} : i = 1, \dots, \mathrm{d})$ and adding that

$$\begin{aligned}[N]_T &= \int_0^T \sum_{i,j=1}^{\mathrm{d}} \mathbf{1}_{\{S_{s-}\le\varepsilon\}}\big(X_{s-}^{(i)} - x_s^{(i)}\big)\big(X_{s-}^{(j)} - x_s^{(j)}\big)d\big|\big[M^{(i)}, M^{(j)}\big]\big|_s \\ &\le \varepsilon^2 \int_0^T \sum_{i,j=1}^{\mathrm{d}} \mathbf{1}_{\{S_{s-}\le\varepsilon\}} d\big[M^{(i)}, M^{(j)}\big]_s\end{aligned}$$

and that $d|[M^{(i)}, M^{(j)}]|_s \le d[M^{(i)}]_s + d[M^{(j)}]_s$, we obtain

$$\begin{aligned}&\mathbb{E}\bigg(\sup_{t\le T}\bigg|\int_0^t \mathbf{1}_{\{S_{s-}\le\varepsilon\}}(X_{s-} - x_s).dM_s\bigg|^{2p}\bigg) \\ &\quad\le C_{p,\mathrm{d}}\varepsilon^{2p}\mathbb{E}\bigg(\bigg(\int_0^T \sum_{i=1}^{\mathrm{d}} \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\big[M^{(i)}\big]_t\bigg)^p\bigg) \\ &\quad\le C'_{p,\mathrm{d}}\varepsilon^{2p}\bigg[\mathbb{E}\bigg(\bigg(\int_0^T \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\big\|\langle M^c\rangle_t\big\|_1\bigg)^p\bigg) \\ &\qquad + \mathbb{E}\bigg(\bigg(\sum_{t\le T} \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\|\Delta X_t\|_2^2\bigg)^p\bigg)\bigg],\end{aligned}\tag{8}$$

for some positive constants $C_{p,\mathrm{d}}$ and $C'_{p,\mathrm{d}}$, where we recall that $[M^{(i)}]_t = \langle M^{c,(i)}\rangle_t + \sum_{s\le t}(\Delta X_s^{(i)})^2$. Plugging (7) and (8) in (6), we get

$$\begin{aligned}\mathbb{P}(S_T \ge \varepsilon) &\le \mathbb{P}\bigg(\|X_0 - x_0\|_2^2 \ge \frac{\eta^2}{4}\bigg) + \bigg(\frac{16\varepsilon}{\eta^2}\bigg)^q \mathbb{E}\bigg(\bigg(\int_0^T \mathbf{1}_{\{S_{s-}\le\varepsilon\}}\, dA_s^1\bigg)^q\bigg) \\ &\quad + C''_{p,\mathrm{d}}\bigg(\frac{\varepsilon^{2p}}{\eta^{4p}} + \frac{1}{\eta^{2p}}\bigg)\bigg[\mathbb{E}\bigg(\bigg(\int_0^T \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\big\|\langle M^c\rangle_t\big\|_1\bigg)^p\bigg) \\ &\quad + \mathbb{E}\bigg(\bigg(\sum_{t\le T} \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\|\Delta X_t\|_2^2\bigg)^p\bigg)\bigg]\end{aligned}$$

for some $C''_{p,\mathrm{d}}$ positive. Recalling that $\eta = \varepsilon/(\sqrt{2}\exp(LT))$ completes the proof. $\square$

3. Uniform estimates for stochastic differential equations. In this section, we assume that $X = (X^{(i)} : i = 1, \dots, \mathrm{d})$ is a càdlàg Markov process which takes values in $E \subset \mathbb{R}^d$ and is the unique strong solution of the following SDE on $[0, \infty)$:

$$X_t = x_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s + \int_0^t \int_{\mathcal{X}} H(X_{s-}, z) N(ds, dz) + \int_0^t \int_{\mathcal{X}} G(X_{s-}, z) \widetilde{N}(ds, dz),$$

a.s. for any $x_0 \in E$, where $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ is a measurable space,

- $B = (B^{(i)} : i = 1, \dots, \mathrm{d})$ is a d-dimensional Brownian motion;
- N is a Poisson Point Measure (PPM) on $\mathbb{R}^+ \times \mathcal{X}$ with intensity $ds q(dz)$, where q is a σ-finite measure on $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$; and $\widetilde{N}$ is the compensated measure of N.
- N and B are independent;
- $b = (b^{(i)} : i = 1, \dots, \mathrm{d})$, $\sigma = (\sigma_j^{(i)} : i, j = 1, \dots, \mathrm{d})$, H and G are Borel measurable functions locally bounded, which take values respectively in $\mathbb{R}^{\mathrm{d}}$, $\mathbb{R}^{2\mathrm{d}}$, $\mathbb{R}^{\mathrm{d}}$ and $\mathbb{R}^{\mathrm{d}}$.

Moreover, we follow the classical convention (see Chapter II in [19]) and we assume that $HG = 0$, G is bounded and for any $t \geq 0$,

$$\int_0^t \int_{\mathcal{X}} \big|H(X_{s-}, z)\big| N(ds, dz) < \infty \qquad \text{a.s.},$$

$$\mathbb{E}\left(\int_0^t \int_{\mathcal{X}} \|G(X_{s-\wedge\sigma_n}, z)\|_2^2 \, ds\, q(dz)\right) < \infty,$$

for some sequence of stopping time $\sigma_n \uparrow \infty$. We dot not discuss here the conditions which ensure the strong existence and uniqueness of this SDE for any initial condition. This will be standard results for the examples considered in this paper and we refer to [13] for some general statement relevant in our context.

3.1. *Main result.* We need a transformation F to construct a suitable distance and evaluate the gap between the process X and the associated dynamical system on a domain D.

ASSUMPTION 3.1. (i) The domain D is an open subset of $\mathbb{R}^{\mathrm{d}}$ and the function F is defined on an open set O which contains $\overline{D \cup E}$.

(ii) $F \in \mathcal{C}^2(O, \mathbb{R}^{\mathrm{d}})$ and F is a bijection from D into $F(D)$ and its Jacobian J_F is invertible on D.

(iii) For any $x \in E$,

$$\int_{\mathcal{X}} \big|F\big(x + H(x, z)\big) - F(x)\big| q(dz) < \infty$$

and the function $x \in E \to h_F(x) = \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz)$ can be extended to the domain $\overline{D \cup E}$. This extension h_F is locally bounded on $\overline{D \cup E}$ and locally Lipschitz on D.

(iv) The function b is locally Lipschitz on D.

Under this assumption, F is a $\mathcal{C}^2$ diffeomorphism from D into $F(D)$ and $F(D)$ is an open subset of $\mathbb{R}^d$. We require in (iii) that the large jumps of $F(X)$ can be compensated. This assumption could be relaxed by letting the large jumps which could not be compensated in an additional term with finite variations, that is, using the term A_t of the semimartingale R_t in the previous section. But that will not be useful for the applications given here. Under Assumption 3.1, we set $b_F = b + J_F^{-1} h_F$, which is well defined and locally Lipschitz on D. We note that for any $x \in E \cap D$,

$$b_F(x) = b(x) + J_F(x)^{-1}\left(\int_{\mathcal{X}} \bigl[F\bigl(x + H(x,z)\bigr) - F(x)\bigr] q(dz)\right).$$

We introduce the flow ϕ_F associated to b_F and defined for $x_0 \in D$ as the unique solution of

$$\phi_F(x_0, 0) = x_0, \qquad \frac{\partial}{\partial t}\phi_F(x_0,t) = b_F\bigl(\phi_F(x_0,t)\bigr),$$

for $t \in [0, T_D(x_0))$, where $T_D(x_0) \in (0, \infty]$ is the maximal time until which the solution exists and belongs to D. We observe that when $H = 0$, then $b_F = b$ and $\phi_F = \phi$ do not depend on the transformation F.

We introduce now the vector field ψ_F defined by

$$\psi_F = (J_F b_F) \circ F^{-1} = (J_F b + h_F) \circ F^{-1}$$

on the open set $F(D)$. We also set, for any $x \in E$,

$$\begin{aligned}\widetilde{b}_F(x) = {}& \frac{1}{2} \sum_{i,j=1}^{\mathrm{d}} \frac{\partial^2 F}{\partial x_i \partial x_j}(x) \sum_{k=1}^{d} \sigma_k^{(i)}(x) \sigma_k^{(j)}(x) \\ & + \int_{\mathcal{X}} \bigl[F\bigl(x + G(x,z)\bigr) - F(x) - J_F(x) G(x,z)\bigr] q(dz).\end{aligned} \tag{9}$$

Let us note that the generator of X is given by $\mathcal{L}F = \psi_F \circ F + \widetilde{b}_F$. The term $\widetilde{b}_F$ is not contributing significantly to the coming down from infinity in the examples we consider here, and thus considered as an approximation term. On the contrary, we need to introduce

$$\begin{aligned}V_F(x) = {}& \sum_{i,j,k=1}^{\mathrm{d}} \frac{\partial F}{\partial x_i}(x) \frac{\partial F}{\partial x_j}(x) \sigma_k^{(i)}(x) \sigma_k^{(j)}(x) \\ & + \int_{\mathcal{X}} \bigl[F\bigl(x + H(x,z) + G(x,z)\bigr) - F(x)\bigr]^2 q(dz)\end{aligned} \tag{10}$$

for $x \in E$, to quantify the fluctuations of the process due to the martingale parts. Finally, we use the following application defined on O (and thus on $D \cup E$) to compare the process X and the flow ϕ_F:

$$d_F(x,y) = \bigl\| F(x) - F(y) \bigr\|_2.$$

We observe that d is (indeed) a distance (at least) on D and in the examples below it is actually a distance on $D \cup E$. We recall notation (3) and the counterpart of (2) is defined by

$$T_{D,\varepsilon,F}(x_0) = \sup\{t \in [0, T_D(x_0)) : \forall s \le t, \overline{B}_{d_F}(\phi_F(x_0, s), \varepsilon) \cap E \subset D\}. \tag{11}$$

THEOREM 3.2. *Under Assumption 3.1, we assume that ψ_F is (L, α) nonexpansive on $F(D)$.*

Then for any $\varepsilon > 0$ and $x_0 \in E \cap D$ and $T < T_{D,\varepsilon,F}(x_0) \wedge T_\varepsilon^{L,\alpha}$, we have

$$\mathbb{P}_{x_0}\Big(\sup_{t \le T} d_F(X_t, \phi_F(x_0, t)) \ge \varepsilon\Big) \le C_{\mathrm{d}} e^{4LT} \int_0^T \overline{V}_{F,\varepsilon}(x_0, s)\, ds,$$

where C_{d} is a positive constant depending only on the dimension d *and*

$$\overline{V}_{F,\varepsilon}(x_0, s) = \sup_{\substack{x \in E \\ d_F(x, \phi_F(x_0,s)) \le \varepsilon}} \{\varepsilon^{-2} \|V_F(x)\|_1 + \varepsilon^{-1} \|\widetilde{b}_F(x)\|_1\}. \tag{12}$$

We refer to the two next sections for examples and applications, which involve different choices for F and (L, α) nonexpansivity with potentially α or L equal to 0. The key assumption concerns the nonexpansivity of ψ_F, which need to be combined with control of the fluctuations V_F. Before the proof of Theorem 3.2, let us illustrate the condition of L nonexpansivity of ψ_F by considering the diffusion case ($q = 0$ and X continuous). This will be useful in Section 5.

EXAMPLE. We recall from the first section (or Table 1 in [1]) that when $F(D)$ is convex and ψ_F is differentiable on $F(D)$, ψ_F is L nonexpansive on $F(D)$ iff $\mathrm{Sp}(J_{\psi_F}(y) + J^*_{\psi_F}(y)) \subset (-\infty, 2L]$ for any $y \in F(D)$. In the case $q = 0$, choosing

$$F(x) = (f_i(x_i) : i = 1, \dots, \mathrm{d})$$

and setting $A(x) = J_{\psi_F}(F(x))$, we have for any $i, j = 1, \dots, d$ such that $i \ne j$,

$$A_{ij}(x) = \frac{f_i'(x_i)}{f_j'(x_j)} \frac{\partial}{\partial x_j} b^{(i)}(x), \qquad A_{ii}(x) = \frac{\partial}{\partial x_i} b^{(i)}(x) + \frac{f_i''(x_i)}{f_i'(x_i)} b^{(i)}(x). \tag{13}$$

Then ψ_F is L nonexpansive on $F(D)$ iff the largest eigenvalue of $A(x) + A^*(x)$ is less than $2L$ for any $x \in D$.

PROOF OF THEOREM 3.2. Under Assumption 3.1, we can further assume that $F \in \mathcal{C}^2(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$. Indeed, we can consider φF where $\varphi \in \mathcal{C}^\infty(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$ is equal to 0 on the complementary set of O and to 1 on $\overline{D \cup E}$, since these two sets are disjoint closed sets, using, for example, the smooth Urysohn lemma. This allows to extend F from $\overline{D \cup E}$ to $\mathbb{R}^d$ in such a way that $F \in \mathcal{C}^2(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$.

Applying now Itô's formula to $F(X_t)$ (see Chapter 2, Theorem 5.1 in [19]), we have

$$\begin{aligned} F(X_t) = F(x_0) &+ \int_0^t J_F(X_s)b(X_s)\,ds \\ &+ \int_0^t \int_E [F(X_{s-} + H(X_{s-}, z)) - F(X_{s-})] N(ds, dz) \\ &+ \int_0^t \sum_{i,j=1}^{\mathrm{d}} \frac{\partial F}{\partial x_i}(X_s)\sigma_j^{(i)}(X_s)\,dB_s^{(j)} \\ &+ \int_0^t \int_E [F(X_{s-} + G(X_{s-}, z)) - F(X_{s-})] \widetilde{N}(ds, dz) \\ &+ \int_0^t \widetilde{b}_F(X_s)\,ds \end{aligned}$$

for $t \geq 0$. Then the $\mathcal{F}_t$-semimartingale $Y_t = F(X_t)$ takes values in $F(E)$ and can be written as

$$Y_t = F(x_0) + \int_0^t \psi(Y_s)\,ds + A_t + M_t^c + M_t^d, \tag{14}$$

where ψ, A, M^c and M^d are defined as follows. First, we consider the Borel locally bounded function $\psi(y) = 1_{\{y \in F(D)\}}\psi_F(y)$ for $y \in \mathbb{R}^{\mathrm{d}}$, so writing $\hat{b}_F(x) = J_F(x)b(x) + h_F(x)$ for $x \in E$, we have $\psi(Y_s) = 1_{\{Y_s \in F(D)\}}\hat{b}_F(X_s)$. Moreover,

$$A_t = \int_0^t (\widetilde{b}_F(X_s) + \mathbf{1}_{\{Y_s \notin F(D)\}}\hat{b}_F(X_s))\,ds$$

is a continuous $\mathcal{F}_t$-adapted process with a.s. bounded variations paths and

$$M_t^c = \int_0^t \sum_{i,j=1}^{\mathrm{d}} \frac{\partial F}{\partial x_i}(X_s)\sigma_j^{(i)}(X_s)\,dB_s^{(j)}$$

is a continuous $\mathcal{F}_t$-local martingale and writing $K = G + H$ and using Assumption 3.1(iii),

$$M_t^d = \int_0^t \int_{\mathcal{X}} [F(X_{s-} + K(X_{s-}, z)) - F(X_{s-})] \widetilde{N}(ds, dz)$$

is a càdlàg $\mathcal{F}_t$-local martingale purely discontinuous.

We observe that the dynamical system $y_t = F(\phi_F(x_0, t))$ satisfies for $t < T(x_0)$,

$$y_0 = F(x_0), \qquad y_t' = J_F(\phi_F(x_0, t))b_F(\phi_F(x_0, t)) = \psi_F(y_t) = \psi(y_t),$$

since $\psi_F = \psi$ on $F(D)$. This flow is thus associated with the vector field ψ and ψ is locally Lipschitz on $F(D)$. Moreover, recalling the definition (2) and setting $E' = F(D)$, $T'(y_0) = T_D(x_0)$, the first time $T_{F(D),\varepsilon}(y_0)$ when $(y_t)_{t \geq 0}$ starting from y_0 is at distance ε from the boundary of $F(D)$ for the Euclidean distance is larger than $T_{D,\varepsilon,F}(x_0)$ defined by (11):

$$\begin{aligned}T_{F(D),\varepsilon}(y_0) &= \sup\{t \in [0, T'(y_0)) : \forall s \le t, \overline{B}(y_s, \varepsilon) \cap F(E) \subset F(D)\}\\ &\ge T_{D,\varepsilon,F}(x_0).\end{aligned}$$

Adding that ψ is (L, α) nonexpansive on $F(D)$, we apply now Proposition 2.2 to Y with $p = q = 1$ and $Y_0 = y_0 = F(x_0)$. Then, for any $T < T_{D,\varepsilon,F}(x_0) \wedge T_\varepsilon^{L,\alpha}$, we get

$$\begin{aligned}\mathbb{P}(S_T \ge \varepsilon) \le C_{\mathrm{d}} e^{4LT}\bigg[&\varepsilon^{-1}\mathbb{E}\bigg(\int_0^T \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\big\||A|_t\big\|_1\bigg)\\ &+ \varepsilon^{-2}\mathbb{E}\bigg(\int_0^T \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\big\|\langle M^c\rangle_t\big\|_1\bigg)\\ &+ \varepsilon^{-2}\mathbb{E}\bigg(\sum_{t\le T} \mathbf{1}_{\{S_{t-}\le\varepsilon\}} \|\Delta Y_t\|_2^2\bigg)\bigg]\end{aligned} \tag{15}$$

for some constant C_{d} positive, where $S_t = \sup_{s\le t} \|Y_s - y_s\|_2$. Using now

$$\langle M^c\rangle_t = \int_0^t \sum_{i,j,k=1}^{\mathrm{d}} \frac{\partial F}{\partial x_i}(X_s) \frac{\partial F}{\partial x_j}(X_s) \sigma_k^{(i)}(X_s) \sigma_k^{(j)}(X_s)\, ds,$$

we get

$$\begin{aligned}&\int_0^T \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\big\|\langle M^c\rangle_t\big\|_1\\ &\quad\le \int_0^T \sup_{\substack{x\in E\\ d_F(x,\phi_F(x_0,t))\le\varepsilon}} \left\{\sum_{i,j,k,l=1}^{\mathrm{d}} \frac{\partial F^{(l)}}{\partial x_i}(x) \frac{\partial F^{(l)}}{\partial x_j}(x) \sigma_k^{(i)}(x) \sigma_k^{(j)}(x)\right\} dt,\end{aligned}$$

since $S_t = \sup_{s\le t} \|Y_s - y_s\|_2 = \sup_{s\le t} d_F(X_s, \phi_F(x_0, s))$. Similarly,

$$\begin{aligned}&\mathbb{E}\bigg(\sum_{t\le T} \mathbf{1}_{\{S_{t-}\le\varepsilon\}} \|\Delta Y_t\|_2^2\bigg)\\ &\quad= \mathbb{E}\bigg(\int_0^T \int_{\mathcal{X}} \mathbf{1}_{\{S_{t-}\le\varepsilon\}} \big\|F\big(X_{t-} + K(X_{t-}, z)\big) - F(X_{t-})\big\|_2^2\, dt\, q(dz)\bigg)\\ &\quad\le \int_0^T \sup_{\substack{x\in E\\ d_F(x,\phi_F(x_0,t))\le\varepsilon}} \left\{\int_{\mathcal{X}} \big\|F\big(x + K(x, z)\big) - F(x)\big\|_2^2 q(dz)\right\} dt\end{aligned}$$

and combining the two last inequalities we get

$$\mathbb{E}\left(\int_0^T \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\big\|\langle M^c\rangle_t\big\|_1\right) + \mathbb{E}\left(\sum_{t\le T} \mathbf{1}_{\{S_{t-}\le\varepsilon\}} \|\Delta Y_t\|_2^2\right)$$

$$\le \int_0^T \sup_{\substack{x\in E\\ d_F(x,\phi_F(x_0,t))\le\varepsilon}} \big\|V_F(x)\big\|_1\, dt. \tag{16}$$

Finally, on the event $\{S_{t-}\le\varepsilon\}$, $Y_{t-}=F(X_{t-})\in F(D)$ for any $t\le T$ since $T<T_{D,\varepsilon,F}(x_0)$, so

$$\mathbb{E}\left(\int_0^T \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\big\||A|_t\big\|_1\right) \le \int_0^T \mathbf{1}_{\{S_{t-}\le\varepsilon\}} \big\|\widetilde{b}_F(X_{t-})\big\|_1\, dt$$

$$\le \int_0^T \sup_{\substack{x\in E\\ d_F(x,\phi_F(x_0,t))\le\varepsilon}} \big\|\widetilde{b}_F(x)\big\|_1\, dt \tag{17}$$

and the conclusion comes by plugging the two last inequalities in (15). $\square$

3.2. *Adjunction of nonexpansive domains.* We relax here the assumptions required for Theorem 3.2. Indeed finding a transformation which guarantees nonexpansivity of the flow is delicate in general. Adjunction of simple transformations is relevant for covering the whole state space and performing computations. It will be useful for the study of two-dimensional competitive processes in Section 5. Let us note that the trajectorial estimates obtained previously are well adapted to gluing domains, while this is a delicate problem for controls of stochastic processes relying for instance on Lyapounov functions. Thus, we decompose the domain D as follows.

ASSUMPTION 3.3. (i) The domains D and $(D_i : i=1,\dots,N)$ are open subsets of $\mathbb{R}^{\mathrm{d}}$ and F_i are $\mathbb{R}^{\mathrm{d}}$ valued functions from an open set O_i which contains $\overline{D_i}$ and

$$D\subset \bigcup_{i=1}^N D_i, \qquad F_i\in\mathcal{C}^2\big(O_i,\mathbb{R}^{\mathrm{d}}\big).$$

Moreover, F_i is a bijection from D_i into $F(D_i)$ whose Jacobian matrix is invertible on D_i.

(ii) There exist a distance d on $\bigcup_{i=1}^N D_i\cup E$ and $c_1,c_2>0$ such that for any $i\in\{1,\dots,N\}$, $x,y\in D_i$,

$$c_1 d(x,y)\le \big\|F_i(x)-F_i(y)\big\|_2\le c_2 d(x,y).$$

(iii) For each $i\in\{1,\dots,N\}$, for any $x\in E\cap D_i$,

$$\int_{\mathcal{X}} \big|F_i\big(x+H(x,z)\big)-F_i(x)\big|\, q(dz)<\infty$$

and the function $x \in E \cap D_i \to h_{F_i}(x) = \int_{\mathcal{X}}[F_i(x + H(x,z)) - F_i(x)]q(dz)$ can be extended to $\overline{D_i}$.

Moreover, this extension is locally bounded on $\overline{D_i}$ and locally Lipschitz on D_i.
(iv) The function b is locally Lipschitz on $\bigcup_{i=1}^N D_i$.

Second, we consider the flow associated to the vector field b_{F_i}, where b_{F_i} is defined as previously by $b_{F_i}(x) = b(x) + J_{F_i}(x)^{-1}h_{F_i}(x)$ and is locally Lipschitz on the domain D_i. But now ϕ may go from one domain to an other. To glue the estimates obtained in the previous part by adjunction of domains, we need to bound the number of times κ ϕ changes domain. More precisely, we consider a nonautonomous flow $\phi(.,.)$ such that $\phi(x_0, 0) = x_0$ for $x_0 \in D$ and let $\varepsilon_0 \in (0,1)$, $\kappa \geq 1$ and $(t_k(.) : k \leq \kappa)$ be a sequence of elements of $[0,\infty]$ such that $0 = t_0(x_0) \leq t_1(x_0) \leq \cdots \leq t_\kappa(x_0)$ for $x_0 \in D$, which meet the following assumption.

ASSUMPTION 3.4. For any $x_0 \in D$, $\phi(x_0, .)$ is continuous on $[0, t_\kappa(x_0))$ and for any $k \leq \kappa - 1$, there exists $n_k(x_0) \in \{1,\dots,N\}$ such that for any $t \in (t_k(x_0), t_{k+1}(x_0))$,

$$\overline{B}_d(\phi(x_0,t),\varepsilon_0) \subset D_{n_k(x_0)} \quad \text{and} \quad \frac{\partial}{\partial t}\phi(x_0,t) = b_{F_{n_k(x_0)}}(\phi(x_0,t)).$$

This nonautonomous flow ϕ will be used in the continuous case in Section 5. Then we recall that $b_F = b$ does not depend on the transformation F and the flow ϕ will be simply given by $\phi(x_0, 0) = x_0$, $\frac{\partial}{\partial t}\phi(x_0,t) = b(\phi(x_0,t))$ as expected.

Recalling notation $\psi_F = (J_F b_F) \circ F^{-1}$ and the expressions of $T_\varepsilon^{L,\alpha}$ and $\widetilde{b}_F$ and V_F given respectively in (3), (9) and (10), the result can be stated as follows.

THEOREM 3.5. *Under Assumptions 3.3 and 3.4, we assume that for each $i \in \{1,\dots,N\}$, ψ_{F_i} is (L_i,α_i) nonexpansive on $F_i(D_i)$ and let $T_0 \in (0,\infty)$.*

Then for any $\varepsilon \in (0, \underline{\varepsilon}]$ and $T < \min\{T_\varepsilon^{L_i,\alpha_i} : i = 1,\dots,N\} \wedge t_\kappa(x_0) \wedge T_0$ and $x_0 \in E \cap D$,

$$\mathbb{P}_{x_0}\Big(\sup_{t \leq T} d(X_t, \phi(x_0,t)) \geq \varepsilon\Big) \leq C \sum_{k=0}^{\kappa-1} \int_{t_k(x_0)\wedge T}^{t_{k+1}(x_0)\wedge T} \overline{V}_{d,\varepsilon}(F_{n_k(x_0)}, x_0, t)\, dt,$$

with $\underline{\varepsilon}$ and C positive constants which depend (only) on d, c_1, c_2, $(L_i)_{i=1,\dots,N}$, κ, *ε_0 and T_0; and*

$$\overline{V}_{d,\varepsilon}(F, x_0, s) = \sup_{\substack{x \in E \\ d(x,\phi(x_0,s)) \leq \varepsilon}} \{\varepsilon^{-2}\|V_F(x)\|_1 + \varepsilon^{-1}\|\widetilde{b}_F(x)\|_1\}.$$

The proof relies also on Proposition 2.2 but it is technically more involved than the proof of Theorem 3.2. We observe that T_0 could be chosen equal to ∞ in this statement in the case where $L_i = 0$ for any $i \in \{1, \dots, \kappa\}$. We need now the following constants:

$$b_k(x_0, T) = 2\sqrt{2}\exp(L_{n_k(x_0)}T), \qquad a_k(x_0, T) = \frac{c_2}{c_1} b_k(x_0, T),$$
$$\varepsilon_k(x_0, T) = \frac{c_1 \varepsilon_0}{c_2 b_k(x_0, T)} = \frac{\varepsilon_0}{a_k(x_0, T)},$$

for $k = 0, \dots, \kappa - 1$ and observe that $a_k(x_0, T) \geq 1$.

LEMMA 3.6. *Under the assumptions of Theorem 3.5, for any $x_0 \in E \cap D$, $k \in \{0, \dots, \kappa - 1\}$, $T < T_\varepsilon^{L_{n_k(x_0)}, \alpha_{n_k(x_0)}} \wedge t_\kappa(x_0)$ and $\varepsilon \in (0, \varepsilon_k(x_0, T)]$, we have*

$$\begin{aligned}
\mathbb{P}_{x_0}\Big(\sup_{t_k(x_0)\wedge T \leq t \leq t_{k+1}(x_0)\wedge T} d\big(X_t, \phi(x_0, t)\big) \geq \varepsilon a_k(x_0, T)\Big) \\
\leq \mathbb{P}(d\big(X_{t_k(x_0)}, \phi\big(x_0, t_k(x_0)\big)\big) \geq \varepsilon) \\
+ C \int_{t_k(x_0)\wedge T}^{t_{k+1}(x_0)\wedge T} \overline{V}_{d, \varepsilon a_k(x_0, T)}(F_{n_k(x_0)}, x_0, s)\, ds,
\end{aligned}$$

where C is a positive constant which depends only on d *and c_1 and $L_{n_k(x_0)}$.*

PROOF. Let us fix $k \in \{0, \dots, \kappa - 1\}$ and $x_0 \in E \cap D$. We write $L = L_{n_k(x_0)}$, $\alpha = \alpha_{n_k(x_0)}$, $F = F_{n_k(x_0)}$ and $D = D_{n_k(x_0)}$ for simplicity and consider $T < T_\varepsilon^{L,\alpha} \wedge t_\kappa(x_0)$. As at the beginning of the previous proof, we can assume that $F \in \mathcal{C}^2(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$ and recall that F is bijection from D into $F(D)$. We note that $z_0 = \phi(x_0, t_k(x_0)) \in D$ by Assumption 3.4 and the solution z of $z_t' = b_F(z_t)$ is well defined on a nonempty (maximal) time interval since b_F is locally Lipschitz on D using Assumption 3.3. By uniqueness in the Cauchy–Lipschitz theorem, $z_t = \phi(x_0, t_k(x_0) + t)$ for $t \in [t_k(x_0), t_{k+1}(x_0))$. We write now $\widetilde{X}_t = X_{t_k(x_0)+t}$ and the counterpart of (14) for $Y_t = F(\widetilde{X}_t)$ is

$$Y_t = Y_0 + \int_0^t \psi(Y_s)\, ds + A_t + M_t^c + M_t^d, \tag{18}$$

for $t \geq 0$, where $\psi(y) = 1_{\{y \in F(D)\}} \psi_F(y)$,

$$M_t^c = \int_0^t \sum_{i,j=1}^{\mathrm{d}} \frac{\partial F}{\partial x_i}(\widetilde{X}_s) \sigma_j^{(i)}(\widetilde{X}_s)\, dB_s^{(j)}$$

and we make here the following decomposition for A and M^d. Using Assumption 3.3(iii) for the compensation of jumps when $\widetilde{X}_{s-} \in D$, we set

$$\begin{aligned}
A_t = \int_0^t \big(\widetilde{b}_F(\widetilde{X}_s) + \mathbf{1}_{\{F(\widetilde{X}_s) \notin F(D)\}} J_F(\widetilde{X}_s) b(\widetilde{X}_s) \\
- \mathbf{1}_{\{\widetilde{X}_s \notin D, F(\widetilde{X}_s) \in F(D)\}} h_F \circ F^{-1}(Y_s)\big)\, ds \\
+ \int_0^t \int_{\mathcal{X}} \mathbf{1}_{\{\widetilde{X}_{s-} \notin D\}} \big[F\big(\widetilde{X}_{s-} + H(\widetilde{X}_{s-}, z)\big) - F(\widetilde{X}_{s-})\big] N(ds, dz),
\end{aligned}$$

which is a process with a.s. finite variations paths; and

$$M_t^d = \int_0^t \int_{\mathcal{X}} [F(\widetilde{X}_{s-} + G(\widetilde{X}_{s-}, z)) - F(\widetilde{X}_{s-})]\widetilde{N}(ds, dz)$$
$$+ \int_0^t \int_{\mathcal{X}} \mathbf{1}_{\{\widetilde{X}_{s-} \in D\}} [F(\widetilde{X}_{s-} + H(\widetilde{X}_{s-}, z)) - F(\widetilde{X}_{s-})]\widetilde{N}(ds, dz)$$

is a càdlàg $\mathcal{F}_t$-local martingale purely discontinuous.

Moreover, by Assumptions 3.4 and 3.3(ii), for any $t < t_{k+1}(x_0) - t_k(x_0)$, $x_t \in D$, $y_t = F(x_t) \in F(D)$ and satisfies $y_t' = \psi(y_t)$ and for any $\varepsilon \in (0, c_1\varepsilon_0]$,

$$\overline{B}(y_t, \varepsilon) \cap F(E) \subset F(\overline{B}_{d_F}(z_t, \varepsilon)) \subset F(\overline{B}_d(z_t, \varepsilon/c_1)) \subset F(D).$$

Adding that $\psi = \psi_F$ is (α, L) nonexpansive on $F(D)$, we can apply Proposition 2.2 to the process Y on $F(D)$ for $p = q = 1$ and $E' = F(D)$ and get for any $\varepsilon \in (0, c_1\varepsilon_0]$,

$$\mathbb{P}_{x_0}\Big(\sup_{t\le T_1} \|Y_t - y_t\|_2 \ge \varepsilon\Big)$$
$$\le \mathbb{P}(\|Y_0 - y_0\|_2 \ge \varepsilon/b_k(x_0, T_0)) + C\varepsilon^{-1}\mathbb{E}\Big(\int_0^{T_1} \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\big\||A|_t\big\|_1\Big)$$
$$+ C\varepsilon^{-2}\Big[\mathbb{E}\Big(\int_0^{T_1} \mathbf{1}_{\{S_{t-}\le\varepsilon\}}\, d\big\|\langle M^c\rangle_t\big\|_1\Big) + \mathbb{E}\Big(\sum_{t\le T_1} \mathbf{1}_{\{S_{t-}\le\varepsilon\}} \|\Delta Y_t\|_2^2\Big)\Big]$$

for any $T_1 < T_\varepsilon^{L,\alpha} \wedge (t_{k+1}(x_0) - t_k(x_0))$, where C is positive constant depending on $L_{n_k(x_0)}$ and d. Following (16) and (17) in the proof of Theorem 3.2, we obtain

$$\mathbb{P}_{x_0}\Big(\sup_{[t_k(x_0)\wedge T, t_{k+1}(x_0)\wedge T)} d_F(X_t, \phi(x_0, t)) \ge \varepsilon\Big)$$
$$\le \mathbb{P}(d_F(X_{t_k(x_0)}, x_{t_k(x_0)})) \ge \varepsilon/b_k(x_0, T))$$
$$+ C' \int_{t_k(x_0)\wedge T}^{t_{k+1}(x_0)\wedge T} \overline{V}_{F,\varepsilon}(x_0, s)\, ds \tag{19}$$

for some constant C' depending also only of L and d, where $\overline{V}_{F,\varepsilon}$ has been defined in (12). Using again Assumption 3.3(ii) to replace d_F by d above, we have

$$\{d(X_{t_k(x_0)}, \phi(x_0, t_k(x_0))) < \varepsilon/(c_2 b_k(x_0, T))\}$$
$$\subset \{d_F(X_{t_k(x_0)}, \phi(x_0, t_k(x_0))) < \varepsilon/b_k(x_0, T)\}$$

and

$$\overline{V}_{F,\varepsilon}(x_0, s) \le (c_1^{-1} + c_1^{-2})\overline{V}_{d,\varepsilon/c_1}(F, x_0, s)$$

and we obtain

$$\begin{aligned}
&\mathbb{P}_{x_0}\Big(\sup_{[t_k(x_0)\wedge T, t_{k+1}(x_0)\wedge T)} d\big(X_t, \phi(x_0,t)\big) \geq \varepsilon/c_1\Big)\\
&\quad \leq \mathbb{P}\bigg(d\big(X_{t_k(x_0)}, \phi(x_0, t_k(x_0))\big) \geq \frac{\varepsilon}{c_2 b_k(x_0,T)}\bigg)\\
&\qquad + C'\big(c_1^{-1}+c_1^{-2}\big)\int_{t_k(x_0)\wedge T}^{t_{k+1}(x_0)\wedge T} \overline{V}_{d,\varepsilon/c_1}(F, x_0, s)\,ds.
\end{aligned}$$

Using the quasi-left continuity of X, this inequality can be extended to the closed interval $[t_k(x_0)\wedge T, t_{k+1}(x_0)\wedge T]$ for $k < \kappa - 1$. This completes the proof by replacing ε by $\varepsilon c_2 b_k(x_0, T)$. $\square$

PROOF OF THEOREM 3.5. We write $T_m = T_0 \wedge \min\{T_\varepsilon^{L_i,\alpha_i} : i = 1,\dots,N\} \wedge t_\kappa(x_0) \in (0,\infty)$ and set

$$\underline{\varepsilon} = \inf\big\{\varepsilon_k(x_0,T) : k = 1,\dots,N; x_0 \in E\cap D; T < T_0\big\} \in (0,\infty).$$

Lemma 3.6 and the Markov property at time $t_k(x_0)\wedge T$ ensure that for any $\varepsilon \in (0,\underline{\varepsilon}]$, $x_0 \in E\cap D$, $T \in (0,T_m)$,

$$\begin{aligned}
&\mathbb{P}_{x_0}\Big(\sup_{[t_k(x_0), t_{k+1}(x_0)\wedge T]} d\big(X_t,\phi(x_0,t)\big) \geq \varepsilon a_k(x_0,T),\\
&\qquad \sup_{[0, t_k(x_0)\wedge T]} d\big(X_t,\phi(x_0,t)\big) < \varepsilon\Big)\\
&\quad \leq C\int_{t_k(x_0)\wedge T}^{t_{k+1}(x_0)\wedge T} \overline{V}_{d,\varepsilon a_k(x_0,T)}(F_{n_k(x_0)}, x_0, s)\,ds
\end{aligned}$$

for each $k = 0,\dots,\kappa-1$, by setting $C = \max\{C_{\mathrm{d},c_1,L_i} : i = 1,\dots,N\}$.

Denoting $A_k(x_0,T) = \Pi_{i\leq k} a_i(x_0,T)$ and recalling that $a_i(x_0,T) \geq 1$, by iteration we obtain for $\varepsilon \leq \underline{\varepsilon}/A_\kappa(x_0,t)$ and $T < T_m$ that

$$\begin{aligned}
&\mathbb{P}_{x_0}\bigg(\bigcup_{k=0}^{\kappa-1}\Big\{\sup_{[t_k(x_0), t_{k+1}(x_0)\wedge T]} d\big(X_t,\phi(x_0,t)\big) \geq \varepsilon A_k(x_0,T)\Big\}\bigg)\\
&\quad \leq C\sum_{k=0}^{\kappa-1}\int_{t_k(x_0)\wedge T}^{t_{k+1}(x_0)\wedge T} \overline{V}_{d,\varepsilon A_k(x_0,T)}(F_{n_k(x_0)}, x_0, s)\,ds,
\end{aligned}$$

since $X_0 = x_0 = \phi(x_0, 0)$. This ensures that for any $T < T_m$,

$$\begin{aligned}
&\mathbb{P}_{x_0}\Big(\sup_{0\leq t\leq T} d\big(X_t,\phi(x_0,t)\big) \geq \varepsilon A_\kappa(x_0,T)\Big)\\
&\qquad \leq C A_\kappa(x_0,T)^2 \sum_{k=0}^{\kappa-1}\int_{t_k(x_0)\wedge T}^{t_{k+1}(x_0)\wedge T} \overline{V}_{d,\varepsilon A_\kappa(x_0,T)}(F_{n_k(x_0)}, x_0, s)\,ds.
\end{aligned}$$

Recalling that $(n_k(x_0) : k = 0, \dots, \kappa)$ takes value in a finite set, $A_\kappa(x_0, T)$ is bounded for $x_0 \in E \cap D$ and $T \in [0, T_0)$ by a constant depending only on κ, $(L_i : i = 1, \dots, N)$, c_1 and c_2. This yields the result. $\square$

4. Coming down from infinity for one-dimensional stochastic differential equations. In this section, we assume that $E \subset \mathbb{R}$ and $+\infty$ is a limiting value of E and $D = (a, \infty)$ for some $a \in (0, \infty)$. Following the beginning of the previous section, we consider a càdlàg Markov process X which takes values in E and assume that it is the unique strong solution of the following SDE on $[0, \infty)$:

$$X_t = x_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s + \int_0^t \int_{\mathcal{X}} H(X_{s-}, z) N(ds, dz) + \int_0^t \int_{\mathcal{X}} G(X_{s-}, z) \widetilde{N}(ds, dz),$$

for any $x_0 \in E$, where we recall that $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ is a measurable space; B is a Brownian motion; N is a Poisson point measure on $\mathbb{R}^+ \times \mathcal{X}$ with intensity $ds q(dz)$; N and B are independent and $HG = 0$. We make the following assumption, which is a slightly stronger counterpart of Assumption 3.1 and is convenient for the study of the coming down infinity in dimension 1.

ASSUMPTION 4.1. Let $F \in \mathcal{C}^2((a', \infty), \mathbb{R})$, for some $a' \in [-\infty, a)$ such that $\overline{E} \subset (a', \infty)$:

(i) For any $x > a$, $F'(x) > 0$ and $F(x) \to \infty$ as $x \to \infty$.
(ii) For any $x \in E$, $\int_{\mathcal{X}} |F(x + H(x, z)) - F(x)| q(dz) < \infty$.

The function $x \in E \to h_F(x) = \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz)$ can be extended to $\overline{E} \cup [a, \infty)$.

This extension is locally bounded on $\overline{E} \cup [a, \infty)$ and locally Lipschitz on (a, ∞).

(iii) b is locally Lipschitz on (a, ∞).
(iv) The function $b_F = b + h_F / F'$ is negative on (a, ∞).

Following the previous sections, we consider now the flow ϕ_F given for $x_0 \in (a, \infty)$ by

$$\phi_F(x_0, 0) = x_0, \qquad \frac{\partial}{\partial t} \phi_F(x_0, t) = b_F\big(\phi_F(x_0, t)\big),$$

which is well and uniquely defined and belongs to (a, ∞) on a maximal time interval denoted by $[0, T(x_0))$, where $T(x_0) \in (0, \infty]$. We first observe that $x_0 \to \phi_F(x_0, t)$ is increasing where it is well defined. This can be seen by recalling that the local Lipschitz property ensures the uniqueness of solutions, and thus prevents the trajectories from intersecting. Then $T(x_0)$ is increasing and its limit when

$x_0 \uparrow \infty$ is denoted by $T(\infty)$ and belong to $(0, \infty]$. Moreover, the flow starting from infinity is well defined by a monotone limit:

$$\phi_F(\infty, t) = \lim_{x_0 \to \infty} \phi_F(x_0, t)$$

for any $t \in [0, T(\infty))$. Finally, under Assumption 4.1, for $x_0 \in (a, \infty)$, $b_F(x_0) < 0$ and for any $t < T(x_0)$, $\int_{x_0}^{\phi_F(x_0,t)} 1/b_F(x)\, dx = t$. This yields the following classification.

Either

$$\int_{.}^{\infty} \frac{1}{-b_F(x)} < +\infty,$$

and then

$$\phi_F(\infty, t) = \inf\left\{u \geq 0 : \int_{.}^{\infty} \frac{1}{-b_F(x)}\, dx < t\right\} < \infty$$

for any $t \in (0, T(\infty))$. We say that the dynamical system *instantaneously comes down from infinity*. Moreover, the application $t \in [0, T(\infty)) \to \phi(\infty, t) \in \overline{\mathbb{R}}$ is continuous, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is endowed with the distance

$$\overline{d}(x, y) = \left|e^{-x} - e^{-y}\right|. \tag{20}$$

Otherwise, $T(\infty) = \infty$ and $\phi(\infty, t) = \infty$ for any $t \in [0, \infty)$.

Our aim now is to derive an analogous classification for stochastic differential equations using the results of the previous section. Letting the process start from infinity requires additional work. We give first a condition useful for the identification of the limiting values of $(\mathbb{P}_x : x \in E)$ when $x \to \infty$.

DEFINITION 4.2. The process X is stochastically monotone if for all $x_0, x_1 \in E$ such that $x_0 \leq x_1$, for all $t > 0$ and $x \in \mathbb{R}$, we have

$$\mathbb{P}_{x_0}(X_t \geq x) \leq \mathbb{P}_{x_1}(X_t \geq x).$$

The Λ-coalescent, the birth and death process, continuous diffusions with strong pathwise uniqueness and several of their extensions satisfy this property, while, for example, the transmission control protocol does not. We refer to the examples of forthcoming Section 4.2 for details.

4.1. *Weak convergence and coming down from infinity.* We recall that $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ endowed with $\overline{d}$ defined by (20) is polish and the notation of the previous section become $\psi_F = (F' b_F) \circ F^{-1}$, $\widetilde{b}_F(x) = F''(x)\sigma(x)^2 + \int_{\mathcal{X}} [F(x + G(x,z)) - F(x) - F'(x)G(x,z)] q(dz)$ and $V_F(x) = (F'(x)\sigma(x))^2 + \int_{\mathcal{X}} [F(x + H(x,z) + G(x,z)) - F(x)]^2 q(dz)$.

In this section, we introduce

$$\hat{V}_{F,\varepsilon}(a, t) = \sup_{x \in E \cap \mathfrak{D}_{F,\varepsilon}(a,t)} \{\varepsilon^{-2} V_F(x) + \varepsilon^{-1} \widetilde{b}_F(x)\},$$

where for convenience we use the extension $F(\infty)=\infty$ and we set

$$\mathfrak{D}_{F,\varepsilon}(a,t)=\{x\in(a,\infty):F(x)\le F(\phi(\infty,t))+\varepsilon\}.$$

Finally, we make the following key assumption to use the results of the previous section.

ASSUMPTION 4.3. The vector field ψ_F is (L,α) nonexpansive on $(F(a),\infty)$ and for any $\varepsilon>0$,

$$\int_0^{\cdot}\hat{V}_{F,\varepsilon}(a,t)\,dt<\infty. \tag{21}$$

Let us remark that ψ_F is (L,α) nonexpansive on $(F(a),\infty)$ iff for all $y_1>y_2>F(a)$, $\psi_F(y_1)\le\psi_F(y_2)+L(y_1-y_2)+\alpha$. This means that for all $x_1>x_2>a$, $F'(x_1)b(x_1)+h_F(x_1)\le F'(x_2)b(x_2)+h_F(x_2)+L(F(x_1)-F(x_2))+\alpha$.

Let us now give sufficient conditions for the convergence of $(\mathbb{P}_x)_{x\in E}$ as $x\to\infty$. For that purpose, we introduce the modulus

$$w'(f,\delta,[A,B])=\inf_{\mathbf{b}}\max_{\ell=0,\dots,L-1}\sup_{b_\ell\le s,t<b_{\ell+1}}\overline{d}(f_s,f_t), \tag{22}$$

where the infimum extends over all subdivisions $\mathbf{b}=(b_\ell,\ell=0,\dots,L)$ of $[A,B]$ which are δ-sparse. We refer to Chapter 3 in [8] for details on the Skorokhod topology.

PROPOSITION 4.4. *We assume that X is stochastically monotone*:

(i) *If $E=\{0,1,2,\dots\}$, then $(\mathbb{P}_x)_{x\in E}$ converges weakly as $x\to\infty$ in the space of probability measures on $\mathbb{D}([0,T],\overline{\mathbb{R}})$.*

(ii) *If Assumptions 4.1 and 4.3 hold and $\int_{\cdot}^{\infty}\frac{1}{-b_F(x)}\,dx<+\infty$ and for any $K>0$ and $\varepsilon>0$,*

$$\lim_{\delta\to 0}\sup_{x\in E,x\le K}\mathbb{P}_x(w'(X,\delta,[0,T])\ge\varepsilon)=0, \tag{23}$$

then $(\mathbb{P}_x)_{x\in E}$ converges weakly as $x\to\infty$ in the space of probability measures on $\mathbb{D}([0,T],\overline{\mathbb{R}})$.

The convergence result (i) concerns the discrete case $\sigma=0$. It has been obtained in [15] when the limiting probability $\mathbb{P}_\infty$ is known a priori and the process comes down from infinity. The proof of the tightness for (i) follows [15] and relies on the monotonicity and the fact that the states are noninstantaneous, which is here due to our càdlàg assumption for any initial state space. The identification of the limit is derived directly from the monotonicity and the proof of (i) is actually a direct extension of Lemma 2.1 in [5]. This proof is omitted.

The tightness argument for (ii) is different and can be applied to processes with a continuous part and extended to larger dimensions. The control of the fluctuations of the process for large values relies on the approximation by the continuous dynamical system ϕ_F using Assumption 4.3 and the previous section. Then the tightness on compacts sets is guaranteed by (23). The proof is given below.

In the next result, we assume that $(\mathbb{P}_x)_{x\in E}$ converges weakly and $\mathbb{P}_\infty$ is then well defined as the limiting probability. We determine under our assumptions when (and how) the process comes down from infinity. More precisely, we link the coming down from infinity of the process X to that of the flow ϕ_F, in the vein of [5, 6, 25] who considered some classes of discrete processes; see below for details.

THEOREM 4.5. *We assume that Assumptions 4.1 and 4.3 hold and that $(\mathbb{P}_x : x \in E)$ converges weakly as $x \to \infty$ in the space of probability measures on $\mathbb{D}([0, T], \overline{\mathbb{R}})$ to $\mathbb{P}_\infty$:*

(i) *If*

$$\int_{.}^{\infty} \frac{1}{-b_F(x)}\,dx < +\infty,$$

then

$$\mathbb{P}_\infty(\forall t > 0 : X_t < +\infty) = 1 \quad \textit{and} \quad \mathbb{P}_\infty\Big(\lim_{t\downarrow 0+} F(X_t) - F\big(\phi_F(\infty, t)\big) = 0\Big) = 1.$$

(ii) *Otherwise*, $\mathbb{P}_\infty(\forall t \ge 0 : X_t = +\infty) = 1$.

After the proof given below, we consider examples with different size of fluctuations at infinity. For Λ-coalescent, we recover the speed of coming down from infinity of [6] using $F = \log$ and in that case V_F is bounded. For birth and death processes with polynomial death rates, fluctuations are smaller and we use $F(x) = x^\beta$ ($\beta < 1$) and get a finer approximation of the process coming down from infinity by a dynamical system. But V_F is no longer bounded and has to be controlled along the dynamical system coming down from infinity. When proving that some birth and death processes or transmission control protocol do not come down from infinity, $\mathfrak{D}_{F,\varepsilon}(a, t)$ is nonbounded and we are looking for F increasing slowly enough so that V_F is bounded to check (21); see the next section for details.

The proofs of the two last results need the following lemma. We recall notation $D = (a, \infty)$, $d_F(x, y) = |F(x) - F(y)|$ and $T_{D,\varepsilon,F}(x_0)$, respectively, $T_\varepsilon^{L,\alpha}$ given in (11), respectively, (3).

LEMMA 4.6. *Under Assumptions 4.1 and 4.3, for any $\varepsilon > 0$, $x_0 \in E \cap D$ and $T < T_{D,\varepsilon,F}(x_0) \wedge T_\varepsilon^{L,\alpha}$, we have*

$$\mathbb{P}_{x_0}\Big(\sup_{t\le T} d_F\big(X_t, \phi_F(x_0, t)\big) \ge \varepsilon\Big) \le C(\varepsilon, T),$$

where

$$C(\varepsilon, T) = C \exp(4LT) \int_0^T \hat{V}_{F,\varepsilon}(a,t)\,dt$$

goes to 0 *when* $T \to 0$ *and* C *is a positive constant.*

PROOF. Assumption 3.1 and the (L, α) nonexpansivity of ψ_F are guaranteed respectively by Assumptions 4.1 and 4.3, with here $O = (a', \infty)$ and $D = (a, \infty)$. Thus, we can apply Theorem 3.2 on the domain D and for any $x_0 \in D \cap E$ and $\varepsilon > 0$ and $T < T_{D,\varepsilon,F}(x_0) \wedge T_\varepsilon^{L,\alpha}$, we have

$$\mathbb{P}_{x_0}\Bigl(\sup_{t<T} d_F\bigl(X_t, \phi_F(x_0,t)\bigr) \geq \varepsilon\Bigr) \leq C \exp(4LT) \int_0^T \overline{V}_{F,\varepsilon}(x_0, s)\,ds.$$

Now let $t < T_{D,\varepsilon,F}(x_0)$ and $x \in E$ such that $d_F(x, \phi_F(x_0,t)) \leq \varepsilon$. Then $x > a$ and $F(a) < F(x) \leq F(\phi_F(x_0,t)) + \varepsilon$ and combining the monotonicities of the flow ϕ_F and the function F,

$$F(a) < F(x) \leq F\bigl(\phi_F(\infty, t)\bigr) + \varepsilon,$$

since $\phi(x_0, t) > a$. Thus $x \in \mathfrak{D}_{F,\varepsilon}(a,t)$ and

$$\overline{V}_{F,\varepsilon}(x_0,t) \leq \hat{V}_{F,\varepsilon}(a,t),$$

which completes the proof, since the behavior of $C(\varepsilon, T)$ when $T \to 0$ comes from (21). $\square$

PROOF OF THE PROPOSITION 4.4(ii). The fact that X is a stochastically monotone Markov process ensures that for all $x_0, x_1 \in E$, $x_0 \leq x_1$, $k \geq 0$, $0 \leq t_1 \leq \cdots \leq t_k$, $a_1, \dots, a_k \in \mathbb{R}$,

$$\mathbb{P}_{x_0}(X_{t_1} \geq a_1, \dots, X_{t_k} \geq a_k) \leq \mathbb{P}_{x_1}(X_{t_1} \geq a_1, \dots, X_{t_k} \geq a_k).$$

It can be shown by induction for $k \geq 1$ by using the Markov property at time t_1 and writing $X_{t_1}^{x_1} = X_{t_1}^{x_0} + B$, where X^x is the process X starting at x and B is a nonnegative random variable $\mathcal{F}_{t_1}$)-measurable. Then

$$\mathbb{P}_{x_0}(X_{t_1} \geq a_1, \dots, X_{t_k} \geq a_k)$$

converges as $x_0 \to \infty$ ($x_0 \in E$) by monotonicity, which identifies the finite dimensional limiting distributions of $(\mathbb{P}_x : x \in E)$ when $x \to \infty$.

Let us turn to the proof of the tightness in the Skorokhod space $\mathbb{D}([0,T], \overline{\mathbb{R}})$ and fix $\eta > 0$. The flow ϕ_F comes down instantaneously from infinity since $\int_\infty^{\cdot} 1/b_F(x) < \infty$. Thus, we can choose $T_0 \in (0, T(\infty))$ such that $\phi_F(\infty, T_0) \in D$. Using also that F tends to ∞, let us now fix $K_1 \in [\phi_F(\infty, T_0), \infty)$ and $\varepsilon \in (0, \eta]$ such that $\overline{d}(K_1, \infty) \leq \eta$ and for any $x \geq K_1$ and $y \in \mathbb{R}$ such that $d_F(x,y) < \varepsilon$, we have $\overline{B}_{d_F}(x, \varepsilon) \subset D$ and $\overline{d}(x,y) < \eta$. By continuity and monotonicity of $t \to$

$\phi_F(\infty, t)$, there exists $T_1 \in (0, T_0]$ such that $\phi_F(\infty, T_1) = K_1 + 1$. Adding that $T(x_0) \uparrow T(\infty)$ and $\phi_F(x_0, T_1) \uparrow \phi_F(\infty, T_1)$ as $x_0 \uparrow \infty$, we have $\phi_F(x_0, T_1) \geq K_1$ for any x_0 large enough and then $T_{D,\varepsilon,F}(x_0) \geq T_1$. Thus, Lemma 4.6 ensures that for any x_0 large enough and $T < T_1 \wedge T_\varepsilon^{L,\alpha}$,

$$\limsup_{x_0 \to \infty, x_0 \in E} \mathbb{P}_{x_0}\Big(\sup_{t \leq T} d_F\big(X_t, \phi_F(x_0, t)\big) \geq \varepsilon\Big) \leq C(\varepsilon, T), \tag{24}$$

where $C(\varepsilon, T) \to 0$ as $T \to 0$. Let now $T_2 \in (0, T_1 \wedge T_\varepsilon^{L,\alpha})$ such that $C(\varepsilon, T_2) \leq \eta$. Using that for any $t \in [0, T_2]$, $\phi_F(x_0, t) \geq K_1$ and $\overline{d}(\phi_F(x_0, t), \infty) \leq \eta$ for x_0 large enough,

$$\begin{aligned}\Big\{\sup_{t \leq T_2} \overline{d}(X_t, \infty) \geq 2\eta\Big\} &\subset \Big\{\sup_{t \leq T_2} \overline{d}\big(\phi_F(x_0, t), X_t\big) \geq \eta\Big\} \\ &\subset \Big\{\sup_{t \leq T_2} d_F\big(X_t, \phi_F(x_0, t)\big) \geq \varepsilon\Big\}.\end{aligned}$$

Writing $K = F^{-1}(F(\phi(\infty, T_2)) + \eta)$ and using that $\phi_F(x_0, T_2) \uparrow \phi_F(\infty, T_2) \in D$, we have also

$$\{X_{T_2} \geq K\} \subset \big\{F(X_{T_2}) \geq F\big(\phi_F(\infty, T_2)\big) + \eta\big\} \subset \big\{d_F\big(X_{T_2}, \phi_F(x_0, T_2)\big) \geq \varepsilon\big\},$$

since F' is positive on D and $\eta \geq \varepsilon$. Then (24) and the two last inclusions ensure that

$$\mathbb{P}_{x_0}\Big(\Big\{\sup_{t \leq T_2} \overline{d}(X_t, \infty) \geq 2\eta\Big\} \cup \{X_{T_2} \geq K\}\Big) \leq \eta$$

for x_0 large enough. Moreover, by (23), for any $T \geq T_2$, for δ small enough,

$$\sup_{x \in E; x \leq K} \mathbb{P}_x\big(w'\big(X, \delta, [0, T - T_2]\big) \geq 2\eta\big) \leq \eta.$$

Combining these two last bounds at time T_2 by the Markov property, we get that for x_0 large enough and δ small enough, $\mathbb{P}_{x_0}(w'(X, \delta, [0, T]) \geq 2\eta) \leq 2\eta$. The tightness is proved. $\square$

PROOF OF THEOREM 4.5. We fix $\varepsilon > 0$ and let $T_0 \in (0, T(\infty) \wedge T_\varepsilon^{L,\alpha})$ such that $\overline{B}_{d_F}(\phi_F(\infty, T_0), 2\varepsilon) \subset D$. We observe that $T_{D,\varepsilon,F}(x_0) \geq T_0$ for x_0 large enough since $\phi_F(x_0, T_0) \uparrow \phi_F(\infty, T_0)$ as $x_0 \uparrow \infty$ and $t \in [0, T(x_0)) \to \phi_F(x_0, t)$ decreases. We apply Lemma 4.6 and get for any $T < T_0$,

$$\limsup_{x_0 \to \infty, x_0 \in E} \mathbb{P}_{x_0}\Big(\sup_{t \leq T} d_F\big(X_t, \phi_F(x_0, t)\big) \geq \varepsilon\Big) \leq C(\varepsilon, T), \tag{25}$$

where $C(\varepsilon, T) \to 0$ as $T \to 0$.

We first consider the case (i) and fix now also $t_0 \in (0, T_0)$. The flow ϕ_F comes down from infinity instantaneously, so $\phi_F(\infty, t) < \infty$ on $[t_0, T]$. By Dini's theorem, $\phi_F(x_0, .)$ converges to $\phi_F(\infty, .)$ uniformly on $[t_0, T]$, using the monotonicity of the convergence and the continuity of the limit. We obtain from (25) that for any $T < T_0$,

$$\limsup_{x_0\to\infty, x_0\in E} \mathbb{P}_{x_0}\Big(\sup_{t_0\le t\le T} d_F\big(X_t, \phi_F(\infty, t)\big) \ge 2\varepsilon\Big) \le C(\varepsilon, T),$$

and the weak convergence of $(\mathbb{P}_x : x \in E)$ to $\mathbb{P}_\infty$ yields

$$\mathbb{P}_\infty\Big(\sup_{t_0\le t\le T} d_F\big(X_t, \phi_F(\infty, t)\big) > 2\varepsilon\Big) \le C(\varepsilon, T).$$

Letting $t_0 \downarrow 0$ and then $T \downarrow 0$ ensures that

$$\lim_{T\to 0} \mathbb{P}_\infty\Big(\sup_{0<t\le T} d_F\big(X_t, \phi_F(\infty, t)\big) > 2\varepsilon\Big) = 0.$$

Then $\mathbb{P}_\infty(\lim_{t\downarrow 0+} F(X_t) - F(\phi_F(\infty, t)) = 0) = 1$ and $\mathbb{P}_\infty(\forall t > 0 : X_t < \infty) = 1$, which proves (i).

For the case (ii), that is, $\int_\infty^{\cdot} 1/b_F(x) = \infty$, we recall that $T(\infty) = \infty$, so (25) yields

$$\mathbb{P}_\infty\Big(F(X_T) < \limsup_{x_0\to\infty} F\big(\phi(x_0, T)\big) - A\Big) \le C(A, T)$$

for any $T \in (0, T_\varepsilon^{L,\alpha})$. Adding that $F(\phi(x_0, T)) \uparrow F(\phi(\infty, T)) = F(\infty) = \infty$ as $x_0 \uparrow \infty$,

$$\mathbb{P}_\infty(X_T < \infty) \le C(A, T).$$

Since $\phi(\infty, t) = \infty$ for any $t \ge 0$, $\mathfrak{D}_{F,A}(a, t) = (a, \infty)$ for any $A > 0$. Then $C(A, T) \le \frac{1}{A} C(1, T)$ for $A \ge 1$ and $C(A, T) \to 0$ as $A \to \infty$, since $C(1, T) < \infty$ by (21). We get $\mathbb{P}_\infty(X_T = \infty) = 1$ for any $T > 0$, which completes the proof recalling that X is a càdlàg Markov process under $\mathbb{P}_\infty$. $\square$

4.2. *Examples and applications.* We consider here examples of processes in one dimension and recover some known results. We also get new estimates and we illustrate the assumptions required and the choice of F. Thus, we recover classical results on the coming down from infinity for Λ-coalescent and refine some of them for birth and death processes. Here, $b, \sigma = 0$ and the condition allowing the compensation of jumps (Assumption 4.1(ii)) will be obvious. We also provide a criterion for the coming down from infinity of the transmission control protocol, which is a piecewise deterministic Markov process with $b \ne 0, \sigma = 0$. Several extensions of these results could be achieved, such as mixing branching coalescing processes or additional catastrophes. They are left for future works, while the next section considers diffusions in higher dimension.

4.2.1. Λ-*coalescent* [6, 26, 27]. Pitman [26] has given a Poissonian representation of Λ-coalescent. We recall that Λ is a finite measure on $[0,1]$ and we set $\nu(dy)=y^{-2}\Lambda(dy)$. Without loss of generality, we assume that $\Lambda[0,1]=1$ and for simplicity, we focus on coalescent without Kingman part and assume $\Lambda(\{0\})=0$. We consider a Poisson point process on $(\mathbb{R}^+)^2$ with intensity $dt\,\nu(dy)$: each atom (t,y) yields a coalescence event where each block is picked independently with probability y and all the blocks picked merge into a single bock. Then the numbers of blocks jump from n to $B_{n,y}+1_{B_{n,y}<n}$, where $B_{n,y}$ follows a binomial distribution with parameter $(n,1-y)$. Thus, the number of blocks X_t at time t is the solution of the SDE

$$X_t=X_0-\int_0^t\int_0^1\int_{[0,1]^{\mathbb{N}}}\left(-1+\sum_{1\le i\le X_{s-}}1_{u_i\le y}\right)^+N(ds,dy,du),$$

where N is a PPM with intensity on $\mathbb{R}^+\times[0,1]\times[0,1]^{\mathbb{N}}$ with intensity $dt\nu(dy)\,du$. Thus here $E=\{1,2,\dots\}$, $\mathcal{X}=[0,1]\times[0,1]^{\mathbb{N}}$ is endowed with the cylinder σ-algebra of Borelian sets of $[0,1]$, $q(dy\,du)=\nu(dy)\,du$ where du is the uniform measure on $[0,1]^{\mathbb{N}}$, $b=\sigma=G=0$ and

$$H(x,z)=H\bigl(x,(y,u)\bigr)=-\left(-1+\sum_{1\le i\le x}1_{u_i\le y}\right)^+.$$

We follow [6] and we denote for $x\in(1,\infty)$,

$$F(x)=\log(x),\qquad \psi(x)=\int_{[0,1]}\bigl(e^{-xy}-1+xy\bigr)\nu(dy).$$

In particular, F meets the Assumption 4.1(i) with $a>0$ and $a'=0$. Moreover, for every $x\in\mathbb{N}$,

$$\begin{aligned}h_F(x)&=\int_{\mathcal{X}}\bigl[F\bigl(x+H(x,z)\bigr)-F(x)\bigr]q(dz)\\&=\int_{\mathcal{X}}\log\left(\frac{x+H(x,z)}{x}\right)q(dz)\\&=\int_{[0,1]}\nu(dy)\mathbb{E}\left(\log\left(\frac{B_{x,y}+1_{B_{x,y}<x}}{x}\right)\right)=-\frac{\psi(x)}{x}+h(x),\end{aligned}$$

where h is bounded thanks to Proposition 7 in [6]. Thus h can be extended to a bounded $\mathcal{C}^1$ function on $(0,\infty)$ and Assumption 4.1(ii) is satisfied. Moreover,

$$\psi_F(x)=h_F\bigl(F^{-1}(x)\bigr)=-\frac{\psi(\exp(x))}{\exp(x)}+h\bigl(\exp(x)\bigr)$$

and Lemma 9 in [6] ensures that $x\in(1,\infty)\to\psi(x)/x$ is increasing. Then ψ_F is $(0,2\|h\|_\infty)$ nonexpansive on $(0,\infty)$. Besides

$$b_F(x)=F'(x)^{-1}h_F(x)=-\psi(x)+xh(x).$$

Adding that $\psi(x)/x \to \infty$ as $x \to \infty$, we get $b_F(x) < 0$ for x large enough and Assumption 4.1(iv) is checked. Finally, $\widetilde{b}_F = 0$ since $\sigma = 0$ and $G = 0$ and the second part of Proposition 7 in [6] ensures that

$$V_F(x) = \int_{\mathcal{X}} \bigl[F\bigl(x + H(x,z)\bigr) - F(x)\bigr]^2 q(dz)$$
$$= \int_{[0,1]} \nu(dy)\mathbb{E}\biggl(\biggl(\log\biggl(\frac{B_{x,y} + 1_{B_{x,y}<x}}{x}\biggr)\biggr)^2\biggr)$$

is bounded for $x \in \mathbb{N}$ and so is $\widehat{V}_{F,\varepsilon}$. Then Assumptions 4.1 and 4.3 are satisfied with $F = \log$, $a' = 0$ and a large enough. Moreover, $(\mathbb{P}_x : x \in \mathbb{N})$ converges weakly to $\mathbb{P}_\infty$, which can be seen here from Proposition 4.4(i) since X is stochastically monotone. Thus Theorem 4.5 can be applied and writing $w_t = \phi_F(\infty, t)$, we have:

(i) *If* $\int_{\cdot}^{\infty} \frac{1}{-b_F(x)} < +\infty$, *then* $w_t \in \mathcal{C}^1((0,\infty),(0,\infty))$, $w_t' = -\psi(w_t) + w_t h(w_t)$ *for* $t > 0$ *and*

$$\mathbb{P}_\infty(\forall t > 0 : X_t < \infty) = 1 \quad \text{and} \quad \mathbb{P}_\infty\Bigl(\lim_{t\downarrow 0+} \frac{X_t}{w_t} = 0\Bigr) = 1.$$

(ii) *Otherwise* $\mathbb{P}_\infty(\forall t \geq 0 : X_t = +\infty) = 1$.

To compare with known results, let us note that $b_F(x) \sim -\psi(x)$ as $x \to \infty$, so that we recover here the criterion of coming down from infinity obtained in [7]. This latter is equivalent to the criterion initially proved in [27]. Finally,

$$w_t \sim_{t\downarrow 0+} v_t \qquad \text{where } v_t = \inf\biggl\{s > 0 : \int_s^\infty \frac{1}{\psi(x)}\,dx < t\biggr\}$$

satisfies $v_t' = \psi(v_t)$ for $t > 0$. We recover the speed of coming down from infinity of [6].

4.2.2. *Birth and death processes* [5, 29]. We consider a birth and death process X and we denote by λ_k (resp., μ_k) the birth rate (resp., the death rate) when the population size is equal to $k \in E = \{0, 1, 2, \dots\}$. We assume that $\mu_0 = \lambda_0 = 0$ and $\mu_k > 0$ for $k \geq 1$ and we denote

$$\pi_1 = \frac{1}{\mu_1}, \qquad \pi_k = \frac{\lambda_1 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} \qquad (k \geq 2).$$

We also assume that

$$\sum_{k\geq 1} \frac{1}{\lambda_k \pi_k} = \infty. \tag{26}$$

Then the process X is well defined on E and eventually becomes extinct a.s. [21, 22], that is, $T_0 = \inf\{t > 0 : X_t = 0\} < \infty$ a.s. It is the unique strong solution on E

of the following SDE:

$$X_t = X_0 + \int_0^t \int_0^\infty (1_{z \le \lambda_{X_{s-}}} - 1_{\lambda_{X_{s-}} < z \le \lambda_{X_{s-}} + \mu_{X_{s-}}}) N(ds, dz),$$

where N is a Poisson Point Measure with intensity $ds\,dz$ on $[0, \infty)^2$. Lemma 2.1 in [5] ensures that $(\mathbb{P}_x)_{x \in E}$ converges weakly to $\mathbb{P}_\infty$. It can also be derived from Proposition 4.4(i) since X is stochastically monotone. Under the extinction assumption (26), the following criterion for the coming down from infinity is well known [2]:

$$S = \lim_{n \to \infty} \mathbb{E}_n(T_0) = \sum_{i \ge 1} \pi_i + \sum_{n \ge 1} \frac{1}{\lambda_n \pi_n} \sum_{i \ge n+1} \pi_i < +\infty. \tag{27}$$

The speed of coming down from infinity of birth and death processes has been obtained in [5] for regularly varying death rate (with index $\varrho > 1$) and a birth rate negligible compared to the death rate. Let us here get a finer result for a relevant subclass which allows rather simple computations and describes competitive model in population dynamics. It contains in particular the logistic birth and death process.

PROPOSITION 4.7. *We assume that there exist $b \ge 0$, $c > 0$ and $\varrho > 1$ such that*

$$\lambda_k = bk, \qquad \mu_k = ck^\varrho \qquad (k \ge 0).$$

Then for any $\alpha \in (0, 1/2)$,

$$\mathbb{P}_\infty\Big(\lim_{t \downarrow 0+} t^{\alpha/(1-\varrho)}(X_t/w_t - 1) = 0\Big) = 1,$$

where

$$w_t \sim_{t \downarrow 0+} \big[ct/(\varrho - 1)\big]^{1/(1-\varrho)}.$$

This complements the results obtained in [5], where it was shown that $X_t/w_t \to 1$ as $t \downarrow 0$. The proof used the decomposition of the trajectory in terms of the first hitting time of integers, which works well (in one dimension) when simultaneous deaths cannot occur. The fact that X satisfies a central limit theorem when $t \to 0$ under $\mathbb{P}_\infty$ (see Theorem 5.1 in [5]) ensures that the previous result is sharp in the sense that it does not hold for $\alpha \ge 1/2$.

Before the proof, we consider the critical case where the death rate is slightly larger than the birth rate. We recover here the criterion for the coming down from infinity using Theorem 4.5. We complement this result by providing estimates both when the process comes and does not come from infinity. The function f_β defined by

$$f_\beta(x) = \int_2^{2+x} 1/\sqrt{y \log(y)^\beta}\, dy$$

provides the best distance (i.e., the fastest increasing function going to infinity) allowing to compare the process and the flow by bounding the quadratic variation. It allows in particular to capture the fluctuations when they do not come down from infinity, see (ii) below.

PROPOSITION 4.8. *We assume that there exist $b \geq 0$, $c > 0$ and $\beta > 0$ such that*

$$\lambda_k = bk, \qquad \mu_k = ck \log(k+1)^\beta \qquad (k \geq 0).$$

(i) *If $\beta > 1$, then $\mathbb{P}_\infty(\forall t > 0 : X_t < +\infty) = 1$ and*

$$\mathbb{P}_\infty\Big(\lim_{t \downarrow 0+} f_\beta(X_t) - f_\beta(w_t) = 0\Big) = 1,$$

where $w_t = \phi_{f_\beta}(\infty, t) \in \mathcal{C}^1((0,\infty),(0,\infty))$.

(ii) *If $\beta \leq 1$, $\mathbb{P}_\infty(\forall t \geq 0 : X_t = +\infty) = 1$ and for any $\varepsilon > 0$,*

$$\lim_{T \to 0} \limsup_{x_0 \to \infty, x_0 \in \mathbb{N}} \mathbb{P}_{x_0}\Big(\sup_{t \leq T} \big| f_\beta(X_t) - f_\beta\big(\phi_{f_\beta}(x_0,t)\big)\big| \geq \varepsilon\Big) = 0.$$

We do not provide more explicit estimates for the flow ϕ_{f_β} or for w_t in short time for that case and we turn to the proof of the two previous propositions. Let us specify notation for the birth and death process. Here, $\chi = [0,\infty)$, $q(dz) = dz$ and

$$H(x,z) = 1_{z \leq \lambda_x} - 1_{\lambda_x < z \leq \lambda_x + \mu_x}.$$

Letting $F \in \mathcal{C}^1((-1,\infty),\mathbb{R})$, we have $\int_{\mathcal{X}} |F(x+H(x,z)) - F(x)| q(dz) < \infty$ and

$$h_F(x) = \big(F(x+1) - F(x)\big)\lambda_x + \big(F(x-1) - F(x)\big)\mu_x$$

for $x \in \{0,1,\dots\}$. For the classes of birth and death rates λ, μ considered in the two previous propositions, h_F is well defined on $(-1,\infty)$ by the identity above and $h_F \in \mathcal{C}^1((-1,\infty),\mathbb{R})$. Assumption 4.1(ii) will be checked with $a' = -1$. Finally,

$$V_F(x) = \big(F(x+1) - F(x)\big)^2\lambda_x + \big(F(x) - F(x-1)\big)^2\mu_x.$$

PROOF OF PROPOSITION 4.7. We consider $\alpha \in (0, 1/2)$ and

$$F(x) = (1+x)^\alpha \qquad \big(\alpha \in (0,1/2)\big).$$

Then $F'(x) > 0$ for $x > -1$ and $h_F(x) = ((x+2)^\alpha - (x+1)^\alpha)bx + (x^\alpha - (x+1)^\alpha)cx^\varrho$ and there exists $a > 0$ such that $h'_F(x) < 0$ for $x \geq a$. This ensures that Assumption 4.1 is checked with $a' = -1$ and a. Moreover, $\psi_F = h_F \circ F^{-1}$ is nonincreasing and thus nonexpansive on $(F(a),\infty)$.

Adding that here

$$h_F(x) \sim_{x\to\infty} -c\alpha x^{\varrho+\alpha-1}$$

we get

$$b_F(x) = F'(x)^{-1} h_F(x) = -c(1+x)^{\varrho} + \mathcal{O}\big(x^{\max(\varrho-1,1)}\big). \qquad (x\to\infty) \tag{28}$$

By integrating the inverse of this identity, we obtain

$$\phi_F(\infty,t) \sim_{t\downarrow 0+} \big[ct/(\varrho-1)\big]^{1/(1-\varrho)}. \tag{29}$$

Finally,

$$\begin{aligned} V_F(x) &= \big((x+2)^\alpha - (x+1)^\alpha\big)^2 bx + \big((x+1)^\alpha - x^\alpha\big)^2 cx^{\varrho} \\ &\sim \alpha^2 c x^{\varrho+2\alpha-2} \qquad (x\to\infty). \end{aligned}$$

Adding that for any $T>0$, there exists $c_0>0$ such that $\phi(\infty,t)\le c_0 t^{1/(1-\varrho)}$ for $t\in[0,T]$ and that $F^{-1}(y) = y^{1/\alpha}-1$, then for any $\varepsilon>0$, there exists $c_0'>0$ such that for any $t\le T$,

$$\begin{aligned} \hat{V}_{F,\varepsilon}(a,t) &\le \varepsilon^{-2} \sup\big\{V_F(x) : 0\le x\le \big((\phi_F(\infty,t)+1)^\alpha + \varepsilon\big)^{1/\alpha} - 1\big\} \\ &\le c_0'\big(t^{1/(1-\varrho)}\big)^{\varrho+2\alpha-2}. \end{aligned}$$

Using that $(\varrho+2\alpha-2)/(1-\varrho) = -1 + (2\alpha-1)/(1-\varrho) > -1$ since $\alpha<1/2$, we obtain

$$\int_0^{\cdot} \hat{V}_{F,\varepsilon}(a,t)\,dt < \infty.$$

Thus Assumptions 4.1 and 4.3 are satisfied and Theorem 4.5(i) can be applied, since $\int_{\cdot}^{\infty} -1/b_F(x)\,dx < \infty$. Defining $w_t = \phi_F(\infty,t)$, we get $\mathbb{P}_\infty(\lim_{t\downarrow 0+} X_t^\alpha - w_t^\alpha = 0) = 1$ for any $\alpha\in(0,1/2)$. Writing

$$X_t^\alpha - w_t^\alpha = w_t^\alpha\big((1+(X_t/w_t - 1))^\alpha - 1\big)$$

and using a Taylor expansion completes the proof recalling (29). $\square$

PROOF OF PROPOSITION 4.8. The criterion $\beta>1$ for the coming down from infinity can be derived easily from the criterion $S<\infty$ recalled in (27). It is also a consequence of Theorem 4.5 using $F(x) = (1+x)^\alpha$ as in the previous proof and the integrability criterion for $\int_{\cdot}^{\infty} 1/b_F(x)$, using that $b_F(x) = h_F(x)/F'(x) \sim -cx\log(x+1)^\beta$ as $x\to\infty$.

Let us turn to the proof of the estimates (i)–(ii) and take $F = f_\beta$. Then $F(x)\to\infty$ as $x\to\infty$,

$$h_F(x) = bx\int_{2+x}^{3+x} \frac{1}{\sqrt{y\log(y)^\beta}}\,dy - cx\log(x)^\beta \int_{1+x}^{2+x} \frac{1}{\sqrt{y\log(y)^\beta}}\,dy$$

and its derivative is negative for x large enough. Then Assumption 4.1 is satisfied with again $a' = 1$ and a large enough. So $\psi_F(x) = h_F(F^{-1}(x))$ is decreasing, and thus nonexpansive for x large enough. Moreover, there exists $C > 0$ such that

$$V_F(x) \le Cx \log(x)^\beta \left(\int_{1+x}^{2+x} \frac{1}{\sqrt{y \log(y)^\beta}} \, dy \right)^2.$$

So V_F is bounded and Assumption 4.3 is satisfied. Then (i) comes from Theorem 4.5(i) and (ii) comes from Lemma 4.6 observing that $T_{D,\varepsilon,F}(x_0) \to \infty$ as $x_0 \to \infty$. $\square$

4.2.3. *Transmission control protocol.* The Transmission control protocol [3] is a model for transmission of data, mixing a continuous (positive) drift which describes the growth of the data transmitted and jumps due to congestions, where the size of the data are divided by two. Then the size X_t of data at time t is given by the unique strong solution on $[0, \infty)$ of

$$X_t = x_0 + bt - \int_0^t \mathbf{1}_{\{u \le r(X_{s-})\}} \frac{X_{s-}}{2} N(ds, du),$$

where $x_0 \ge 0$, $b > 0$, $r(x)$ is a continuous positive nondecreasing function and N is PPM on $[0, \infty)^2$ with intensity $ds\,du$. This is a classical example of the piecewise deterministic Markov process. Usually, $r(x) = cx^\beta$, with $\beta \ge 0$, $c > 0$. The choice of F is a bit more delicate here owing to the size and intensity of the fluctuations. Consider F such that $F'(x) > 0$ for $x > 0$. Now $E = [0, \infty)$, $h_F(x) = r(x)(F(x/2) - F(x))$,

$$b_F = b + h_F/F', \qquad \psi_F = (bF' + h_F) \circ F^{-1}.$$

Finally,

$$V_F(x) = r(x)\big(F(x/2) - F(x)\big)^2$$

and we cannot use $F(x) = (1 + x)^\gamma$ or $F(x) = \log(1 + x)^\gamma$ since then the second part of Assumption 4.3 does not hold. We need to reduce the size of the jumps even more and take $F(x) = \log(1 + \log(1 + x))$. The model is not stochastically monotone but Lemma 4.6 can be used to get the following result, which yields a criterion for the coming down from infinity.

PROPOSITION 4.9. (i) *If there exists $c > 0$ and $\beta > 1$ such that $r(x) \ge c \log(1 + x)^\beta$ for any $x \ge 1$, then for any $T > 0$, $\eta > 0$, there exists K such that*

$$\inf_{x_0 \ge 0} \mathbb{P}_{x_0}(\exists t \le T : X_t \le K) \ge 1 - \eta.$$

(ii) *If there exists $c > 0$ and $\beta \le 1$ such that $r(x) \le c \log(1 + x)^\beta$ for any $x \ge 0$, then for any $T, K > 0$,*

$$\lim_{x_0 \to \infty} \mathbb{P}_{x_0}(\exists t \le T : X_t \le K) = 0.$$

Thus, in the first regime, the process comes down instantaneously and a.s. from infinity, while in the second regime it stays at infinity, even if $\mathbb{P}_\infty$ has not been constructed here. In particular, if $r(x) = cx^\beta$ and $\beta, c > 0$, the process comes down instantaneously from infinity. If $\beta = 0$, it does not, which can actually be seen easily since in the case $r(.) = c$, $X_t \geq (x_0 + bt)/2^{N_t}$, where N_t is a Poisson process with rate c and the right-hand side goes to ∞ as $x_0 \to \infty$ for any $t \geq 0$.

PROOF OF PROPOSITION 4.9. Here, $E = [0, \infty)$ and we consider

$$F(x) = \log\bigl(1 + \log(1 + x)\bigr)$$

on (a', ∞) where $a' \in (-1, 0)$ is chosen such that $\log(1 + a') > -1$. Then

$$F'(x) = \frac{1}{(1 + x)(1 + \log(1 + x))} > 0.$$

Moreover,

$$F(x/2) - F(x) = \log\bigl(1 - \varepsilon(x)\bigr),$$

where

$$\varepsilon(x) = 1 - \frac{1 + \log(1 + x/2)}{1 + \log(1 + x)} = \frac{\log(2) + \mathcal{O}(1/(1 + x))}{1 + \log(1 + x)}.$$

We consider now

$$r(x) = c \log(1 + x)^\beta$$

with $c > 0$ and $\beta \in [0, 2]$. We get

$$\begin{aligned} b_F(x) &= b + c \log(1 + x)^\beta (1 + x)\bigl(1 + \log(1 + x)\bigr) \log\bigl(1 - \varepsilon(x)\bigr) \\ &\sim -c \log(2) x \log(x)^\beta \end{aligned}$$

as $x \to \infty$. Thus, Assumptions 4.1 is satisfied for a' and a large enough. Moreover,

$$\int_\infty^{\cdot} \frac{1}{b_F(x)}\, dx < +\infty \qquad \text{if and only if } \beta > 1.$$

We observe that when $\beta \leq 1$, $bF' + h_F$ is bounded. Adding that $h_F'(x) = c\beta(x + 1)^{-1} \log(1 + x)^{\beta - 1}(F(x/2) - F(x)) + c \log(1 + x)^\beta (F'(x/2)/2 - F'(x))$, we get $(bF' + h_F)'(x) < 0$ for x large enough when $\beta > 1$. Thus for any $\beta \geq 0$, $\psi_F = (bF' + h_F) \circ F^{-1}$ is $(0, \alpha)$ nonexpansive on $(F(a), \infty)$, for some $\alpha > 0$ and a large enough. Finally,

$$V_F(x) = c \log(1 + x)^\beta \log\bigl(1 - \varepsilon(x)\bigr)^2 \sim c \log(x)^{\beta - 2}$$

as $x \to \infty$ and V_F is bounded for $\beta \leq 2$. So Assumptions 4.1 and 4.3 are satisfied for a' and a large enough and we can apply Lemma 4.6. We get for any $x_0 \geq 0$ and $T > 0$,

$$\mathbb{P}_{x_0}\Bigl(\sup_{t \leq T} \bigl|F(X_t) - F\bigl(\phi_F(x_0, t)\bigr)\bigr| \geq A\Bigr) \leq C(A, T), \tag{30}$$

for A large enough, where $C(A,T) \to 0$ as $T \to 0$ and $C(A,T) \le C.T. \times \sup_{x\ge 0} V_F(x)/A^2$.

We can now prove (i) and let $\beta > 1$. There exists $\widetilde{c} > 0$ such that

$$\widetilde{r}(x) = \widetilde{c}\log(1+x)^{\beta\wedge 2}$$

satisfies for any $x \ge 1$ and $y \in [x, 2x]$, $r(x) \ge \widetilde{r}(y)$. By a coupling argument, we can construct a TCP associated with the rate of jumps $\widetilde{r}$ such that $X_t \le \widetilde{X}_t$ a.s. for $t \in [0, \inf\{s \ge 0 : X_s \le 1\})$. Then $\phi_F(x_0, t) \le \phi_F(\infty, t) < \infty$ since $\beta > 1$ ensures that the dynamical system comes down from infinity. Letting $T \to 0$ in (30) yields (i).

To prove (ii), we use similarly the coupling $X_t \ge \widetilde{X}_t$ with $\widetilde{r}(x) = \widetilde{c}\log(1+x)^{\beta}$ and $\beta \le 1$ and let now $A \to \infty$ in (30). This completes the proof since V_F bounded ensures that $C(A,T) \to 0$. □

4.2.4. *Logistic Feller diffusions* [9] *and perspectives*. The coming down from infinity of diffusions of the form

$$dZ_t = \sqrt{\gamma Z_t}\,dB_t + h(Z_t)\,dt$$

has been studied in [9] and is linked to the uniqueness of the quasi-stationary distribution (Theorem 7.3). Writing $X_t = 2\sqrt{Z_t/\gamma}$, it becomes

$$dX_t = dB_t - q(X_t)\,dt,$$

where $q(x) = x^{-1}(1/2 - 2h(\gamma x^2/4)/\gamma)$. Under some assumptions (see Remark 7.4 in [9]), the coming down from infinity is indeed equivalent to

$$\int^{\infty} \frac{1}{q(x)}\,dx < \infty,$$

which can be compared to our criterion in Theorem 4.5. Several extensions and new results could be obtained using the results of this section. In particular, one may be interested to mix a diffusion part for competition, negative jumps due to coalescence and branching events. In that vein, let us mention [24]. This is one motivation to take into account the compensated Poisson measure in the definition of the process X, so that Lévy processes and CSBP may be considered in general. It is left for future stimulating works. Let us here simply mention that a class of particular interest is given by the logistic Feller diffusion:

$$dZ_t = \sqrt{\gamma Z_t}\,dB_t + \left(\tau Z_t - aZ_t^2\right)dt.$$

The next part is determining the speed of coming down from infinity of this diffusion. This part actually deals more generally with the two dimensional version of this diffusion, where nonexpansivity and the behavior of the dynamical system are more involved.

5. Uniform estimates for two-dimensional competitive Lotka–Volterra processes. We consider the historical Lotka–Volterra competitive model for two species. It is given by the unique solution $x_t = (x_t^{(1)}, x_t^{(2)})$ of the following ODE on $[0, \infty)$:

$$
\begin{aligned}
(x_t^{(1)})' &= x_t^{(1)}(\tau_1 - a x_t^{(1)} - c x_t^{(2)}),\\
(x_t^{(2)})' &= x_t^{(2)}(\tau_2 - b x_t^{(2)} - d x_t^{(1)}),
\end{aligned}
\tag{31}
$$

starting from $x_0 = (x_0^{(1)}, x_0^{(2)})$, where $a, b, c, d \geq 0$. The associated flow is denoted by ϕ:

$$\phi : [0, \infty)^2 \times [0, \infty) \to [0, \infty)^2, \qquad \phi(x_0, t) = x_t = (x_t^{(1)}, x_t^{(2)}).$$

The coefficients a and b are the intraspecific competition rates and c, d are the interspecific competition rates. We assume that

$$a, b, c, d > 0$$

or $a, b > 0$ and $c = d = 0$, so that our results cover the (simpler) case of one-single competitive (logistic) model. It is well known [4, 23] that this deterministic model is the large population approximation of individual-based model, namely birth and death processes with logistic competition; see also Section 5.2. Moreover and more generally, when births and deaths are accelerated, these individual-based models converge weakly to the unique strong solution of the following SDE on $[0, \infty)$,

$$
\begin{aligned}
X_t^{(1)} &= x_0^{(1)} + \int_0^t X_t^{(1)}(\tau_1 - a X_s^{(1)} - c X_s^{(2)})\, ds + \int_0^t \sigma_1 \sqrt{X_s^{(1)}}\, dB_s^{(1)}\\
X_t^{(2)} &= x_0^{(2)} + \int_0^t X_t^{(2)}(\tau_2 - b X_s^{(2)} - d X_s^{(1)})\, ds + \int_0^t \sigma_2 \sqrt{X_s^{(2)}}\, dB_s^{(2)},
\end{aligned}
\tag{32}
$$

where B is a two-dimensional Brownian motion. This is the classical Lotka–Volterra diffusion for two competitive species; see, for example, [10] for related issues on quasi-stationary distributions.

In this section, we compare the stochastic Lotka–Volterra competitive processes to the deterministic flow ϕ for two new regimes allowing to capture the behavior of the process for large values. These results rely on the statements of Section 3 which are applied to a well-chosen finite family of transformations among

$$F_{\beta,\gamma}(x) = \begin{pmatrix} x_1^{\beta} \\ \gamma x_2^{\beta} \end{pmatrix}, \qquad x \in (0, \infty)^2, \beta \in (0, 1], \gamma > 0, \tag{33}$$

using the adjunction procedure. Moreover, Poincaré's compactification technics for flows is used to describe and control the coming down from infinity.

First, in Section 5.1, we study the small time behavior of the diffusion $X = (X^{(1)}, X^{(2)})$ starting from large values. We compare the diffusion X to the flow

$\phi(x_0, t)$ for a suitable distance which captures the fluctuations of the diffusion at infinity. We then derive the way the process X comes down from infinity, that is, its direction and its speed. Second, in Section 5.2, we prove that usual scaling limits of competitive birth and death processes (see (38) for a definition) hold uniformly with respect to the initial values, for a suitable distance and relevant set of parameters.

These results give answers to two issues which have motivated this work: first, how classical competitive stochastic models regulate large populations (see, in particular, forthcoming Corollary 5.2); second, can we extend individual based-models approximations of the Lotka–Volterra dynamical system to arbitrarily large initial values and if yes, when and for which distance. These results will be useful for forthcoming works on coexistence of competitive species in varying environment. We believe that the technics developed here allow to study similarly the coming down from infinity of these competitive birth and death processes and other multidimensional stochastic processes.

5.1. *Uniform short time estimates for competitive Lotka–Volterra diffusions.* We consider the domain

$$\mathcal{D}_\alpha = (\alpha, \infty)^2$$

and the distance d_β on $[0, \infty)^2$ defined for $\beta > 0$ by

$$d_\beta(x, y) = \sqrt{|x_1^\beta - y_1^\beta|^2 + |x_2^\beta - y_2^\beta|^2} = \|F_{\beta,1}(x) - F_{\beta,1}(y)\|_2. \tag{34}$$

We recall that $a, b, c, d > 0$ or ($a = b > 0$ and $c = d = 0$) and we define

$$T_D(x_0) = \inf\{t \geq 0 : \phi(x_0, t) \notin D\} \tag{35}$$

the first time when the flow ϕ starting from x_0 exits D.

THEOREM 5.1. *For any $\beta \in (0, 1)$, $\alpha > 0$ and $\varepsilon > 0$,*

$$\lim_{T \to 0} \sup_{x_0 \in \mathcal{D}_\alpha} \mathbb{P}_{x_0}\Big(\sup_{t \leq T \wedge T_{\mathcal{D}_\alpha}(x_0)} d_\beta\big(X_t, \phi(x_0, t)\big) \geq \varepsilon\Big) = 0.$$

This yields a control of the stochastic process X defined in (32) by the dynamical system for large initial values and times small enough. We are not expecting that this control hold outside $\mathcal{D}_\alpha$. Indeed, the next result shows that the process and the dynamical system coming from infinity have a different behavior when they come close to the boundary of $(0, \infty)^2$. It is naturally due to the diffusion component and the absorption at the boundary.

The proof cannot be achieved for $\beta = 1$ since then the associated quadratic variations are not integrable at time 0. Heuristically, $\sqrt{Z_t}\, dB_t$ is of order $\sqrt{1/t}\, dB_t$ for small times. This latter does not become small when $t \to 0$ and the fluctuations do not vanish for d_1 in short time.

We denote $\widehat{(x, y)} \in (-\pi, \pi]$ the oriented angle in the trigonometric sense between two nonzero vectors of $\mathbb{R}^2$ and if $ab \neq cd$, we write

$$x_\infty = \frac{1}{ab - cd}(b - c, a - d). \tag{36}$$

The following classification yields the way the diffusion comes down from infinity.

COROLLARY 5.2. *We assume that $\sigma_1 > 0$, $\sigma_2 > 0$ and let $x_0 \in (0, \infty)^2$:*

(i) *If $a > d$ and $b > c$, then for any $\eta \in (0, 1)$ and $\varepsilon > 0$,*

$$\lim_{T\to 0} \limsup_{r\to\infty} \mathbb{P}_{rx_0}\Big(\sup_{\eta T \le t \le T} \|tX_t - x_\infty\|_2 \ge \varepsilon\Big) = 0.$$

If furthermore x_0 is collinear to x_∞, the previous limit holds also for $\eta = 0$.

(ii) *If $a < d$ and $b < c$ and $\widehat{(x_\infty, x_0)} \neq 0$, then for any $T > 0$,*

$$\lim_{r\to\infty} \mathbb{P}_{rx_0}\big(\inf\{t \ge 0 : X_t^{(i)} = 0\} \le T\big) = 1,$$

where $i = 1$ when $\widehat{(x_\infty, x_0)} \in (0, \pi/2]$ and $i = 2$ when $\widehat{(x_\infty, x_0)} \in [-\pi/2, 0)$.

(iii) *If $(a \le d$ and $b > c)$ or if $(a < d$ and $b \ge c)$, then for any $T > 0$,*

$$\lim_{r\to\infty} \mathbb{P}_{rx_0}\big(\inf\{t \ge 0 : X_t^{(2)} = 0\} \le T\big) = 1.$$

(iv) *If $a = d$ and $b = c$, then for any $\varepsilon > 0$,*

$$\lim_{T\to 0} \limsup_{r\to\infty} \mathbb{P}_{rx_0}\Big(\sup_{t\le T} \big\|tX_t - \big(ax_0^{(1)} + bx_0^{(2)}\big)^{-1} x_0\big\|_2 \ge \varepsilon\Big) = 0.$$

In the first case (i), the diffusion X and the dynamical system x come down from infinity in a single direction x_∞, with speed proportional to $1/t$. They only need a short time at the beginning to find this direction. This short time quantified by η here could be made arbitrarily small when x_0 becomes large. Let us also observe that the one-dimensional logistic Feller diffusion X_t is given by $X_t^{(1)}$ for $c = d = 0$. Thus, taking x_0 collinear to x_∞, (i) yields the speed of coming down from infinity of one-dimensional logistic Feller diffusions:

$$\lim_{T\to 0} \lim_{r\to\infty} \mathbb{P}_r\Big(\sup_{t\le T} |atX_t - 1| \ge \varepsilon\Big) = 0. \tag{37}$$

In the second case (ii), the direction taken by the dynamical system and the process depends on the initial direction. The dynamical system then goes to the boundary of $(0, \infty)^2$ without reaching it. But the fluctuations of the process make it reach the boundary and one species becomes extinct. When the process starts in the direction of x_∞, additional work would be required to describe its behavior, linked to the behavior of the dynamical system around the associated unstable variety coming from infinity.

In the third case (iii), the dynamical system ϕ goes to the boundary $(0, \infty) \times \{0\}$ when coming down from infinity, wherever it comes from. Then, as above, the diffusion $X^{(2)}$ hits 0. Let us note that, even in that case, the dynamical system may then go to a coexistence fixed point or to a fixed point where only the species 2 survives. This latter event occurs when $\tau_2/b > \tau_1/c$, $\tau_2/d > \tau_1/a$ and is illustrated in the third simulation below. Obviously, the symmetric situation happens when ($b \le c$ and $d < a$) or ($b < c$ and $d \le a$). Moreover, in cases (ii)–(iii), the proof tells us that when X hits the axis, it is not close from $(0, 0)$. Then it becomes a one-dimensional Feller logistic diffusion whose coming down infinity has been given above; see (37).

In the case (iv), the process comes down in the direction of its initial value, at speed a/t.

See Figure 1 below for simulations illustrating these four regimes.

Finally, let us note that this raises several questions on the characterization of a process starting from infinity in dimension 2. In particular, informally, the process coming down from infinity in a direction x_0 which is not x_∞ has a discontinuity at time 0 in the cases (i)–(ii)–(iii).

5.2. *Uniform scaling limits of competitive birth and death processes.* Let us deal finally with competitive birth and death processes and consider their scaling limits to the Lotka–Volterra dynamical system ϕ given by (31). These scaling limits are usual approximations in large populations of dynamical system by individual-based model; see, for example, [4, 23]. We provide here estimates which are uniform with respect to the initial values in a cone in the interior of $(0, \infty)^2$, for a distance capturing the large fluctuations of the process at infinity. The birth and death rates of the two species are given for population sizes $n_1, n_2 \ge 0$ and $K \ge 1$ by

$$\lambda_1^K(n_1, n_2) = \lambda_1 n_1, \qquad \mu_1^K(n_1, n_2) = \mu_1 n_1 + a n_1.\frac{n_1}{K} + c n_1.\frac{n_2}{K}$$

for the first species and by

$$\lambda_2^K(n_1, n_2) = \lambda_2 n_2, \qquad \mu_2^K(n_1, n_2) = \mu_2 n_2 + b n_2.\frac{n_2}{K} + d n_2.\frac{n_1}{K}$$

for the second species. We assume that

$$\lambda_1 - \mu_1 = \tau_1, \qquad \lambda_2 - \mu_2 = \tau_2.$$

Dividing the number of individuals by K, the normalized population size X^K satisfies

$$X_t^K = x_0 + \int_0^t \int_{[0,\infty)} H^K(X_{s-}, z) N(ds, dz), \tag{38}$$

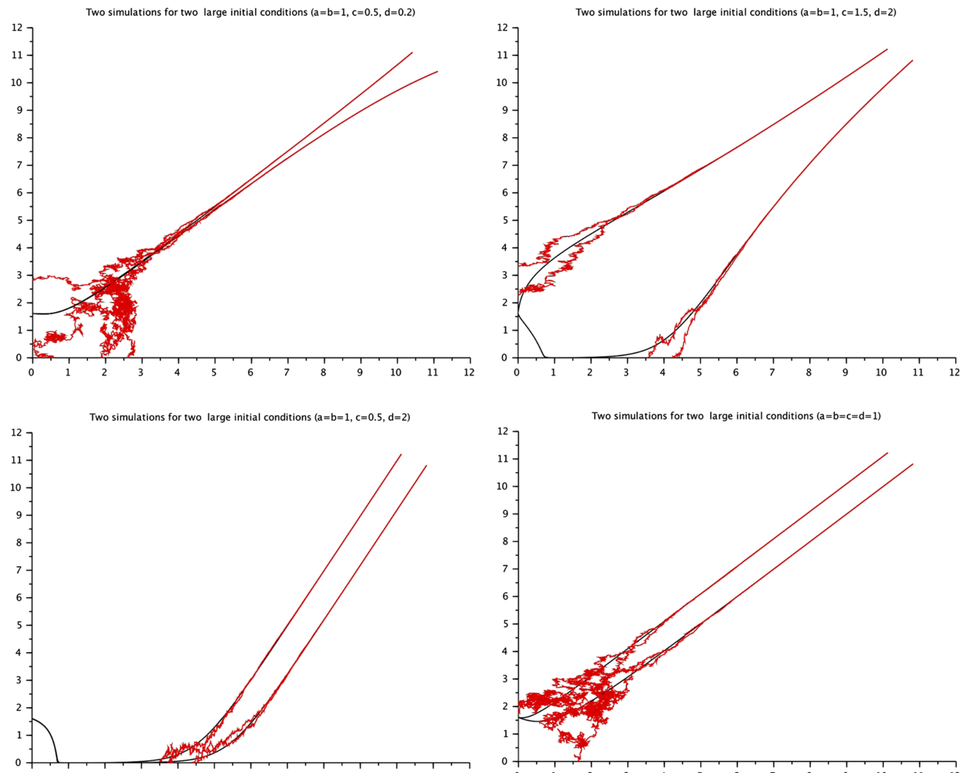

FIG. 1. *Simulations. We consider two large initial values* x_0 *such that* $\|x_0\|_1 = 10^5$. *We plot the dynamical system* (*in black line*) *and two realizations of the diffusion* (*in red line*) *starting from these two initial values. In each simulation,* $\tau_1 = 1$, $\tau_2 = 4$ *and the solutions of the dynamical system converge to the fixed point where only the second species survives. The coefficient diffusion terms are* $\sigma_1 = \sigma_2 = 10$. *We plot here* $G(x_t)$ *and* $G(X_t)$, *where* $G(x, y) = (X, Y) = (\log(1+x), \log(1+y))$, *to zoom on the behavior of the process when coming close to the axes. The four regimes* (*i*)–(*ii*)–(*iii*)–(*iv*) *of the corollary above, which describe the coming down from infinity, are successively illustrated. One can also compare with the pictures of Section* 5.3 *describing the flow.*

where writing $\tau_1^K = \lambda_1^K + \mu_1^K$ for convenience,

$$H^K(x, z) = \frac{1}{K}\begin{pmatrix} \mathbf{1}_{\{z \le \lambda_1^K(Kx)\}} - \mathbf{1}_{\{\lambda_1^K(Kx) \le z \le \tau_1^K(Kx)\}} \\ \mathbf{1}_{\{0 \le z - \tau_1^K(Kx) \le \lambda_2^K(Kx)\}} - \mathbf{1}_{\{\lambda_2^K(Kx) \le z - \tau_1^K(Kx) \le \lambda_2^K(Kx) + \mu_2^K(Kx)\}} \end{pmatrix} \tag{39}$$

and N is a PPM on $[0, \infty) \times [0, \infty)$ with intensity $ds\, dz$. We set

$$\mathfrak{D}_\alpha = \{(x_1, x_2) \in (\alpha, \infty)^2 : x_1 \ge \alpha x_2, x_2 \ge \alpha x_1\},$$

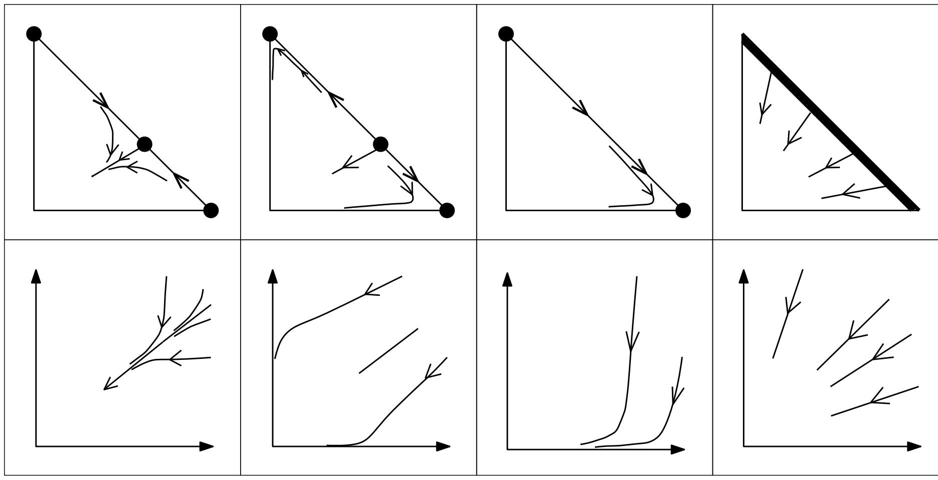

FIG. 2. *Flow close to infinity. We draw the four regimes of the compactified flow* Φ *starting close or on the boundary* $\partial\mathcal{S}$ *and below the associated behavior of the original flow* ϕ *on* $[0,\infty)^2$. *The fixed points of the boundary are fat.*

which is required both for the control of the flow and of the fluctuations. We only consider here the case

$$(40)\qquad \begin{aligned}&(b>c>0 \text{ and } a>d>0) \quad \text{or} \quad (a,b>0 \text{ and } c=d=0) \quad \text{or}\\ &(a=d>0 \text{ and } b=c>0)\end{aligned}$$

since we know from the previous corollary that it gives the cases when the flow does not go instantaneously to the boundary of $(0,\infty)^2$ in short time when coming from infinity. Thus the flow does not exit from $\mathfrak{D}_\alpha$ instantaneously, which would prevent the uniformity in the convergence below. This corresponds to the cases $\ell = x_\infty$ and $\ell = \widehat{x_0}$ in the forthcoming Lemma 5.7(ii) and Figure 2.

THEOREM 5.3. *For any* $T>0$, $\beta\in(0,1/2)$ *and* $\alpha,\varepsilon>0$, *there exists* $C>0$ *such that for any* $K\geq 0$,

$$\sup_{x_0\in\mathfrak{D}_\alpha\cap\mathbb{N}^2/K} \mathbb{P}_{x_0}\Big(\sup_{t\leq T} d_\beta\big(X_t^K,\phi(x_0,t)\big)\geq\varepsilon\Big)\leq\frac{C}{K}.$$

The proof, which is given below, relies on (L,α_K) nonexpansivity of the flow associated with X^K, with $\alpha_K\to 0$. Additional work should allow to make T go to infinity when K goes to infinity. The critical power $\beta=1/2$ is reminiscent from results obtained for one-dimensional logistic birth and death process in Proposition 4.7 in Section 4.2.2.

5.3. *Nonexpansivity of the flow and Poincaré's compactification.* The proofs of the three previous statements of this section rely on the following lemmas.

The first one provides the domains where the transformation $F_{\beta,\gamma}$ yields a nonexpansive vector field. It is achieved by determining the spectrum of the symmetrized operator of the Jacobian matrix of $\psi_{F_{\beta,\gamma}}$ and provide a covering of the state space. This is the key ingredient to use the results of Section 3 for the study of the coming down from infinity of Lotka–Volterra diffusions (Theorem 5.1) and the proof of the scaling limits of birth and death processes (Theorem 5.3).

We also need to control the flow ϕ when it comes down from infinity. The lemmas of Section 5.3.2 describe the dynamics of the flow and provide some additional results useful for the proofs. These proofs rely on the extension of the flow on the boundary at infinity, using Poincaré's technics. Finally, we combine these results in Sections 5.3.3 and 5.3.4 and decompose the whole trajectory of the flow in a finite number of time intervals during which it belongs to a domain where nonexpansivity holds for one of the transformation $F_{\beta,\gamma}$.

As one can see on spectral computations below, nonexpansivity holds in a cone. We recall that a cone is a subset $\mathcal{C}$ of $\mathbb{R}^2$ such that for all $x \in \mathcal{C}$ and $\lambda > 0$, $\lambda x \in \mathcal{C}$. We use the convex components of open cones, which are open convex cones. For S a subset of $\mathbb{R}^2$, we denote by $\overline{S}$ the closure of S. Recalling notation of Section 3, we have here $E = [0,\infty)^2$, $\mathrm{d} = 2$ and

$$\psi_F = (J_F b) \circ F^{-1}$$

$$\text{where } b(x) = b(x_1, x_2) = \begin{pmatrix} \tau_1 x_1 - a x_1^2 - c x_1 x_2 \\ \tau_2 x_2 - b x_2^2 - d x_1 x_2 \end{pmatrix}. \tag{41}$$

5.3.1. *Nonexpansitivity in cones.* Let us write $\overline{\tau} = \max(\tau_1, \tau_2)$ and

$$q_\beta = 4ab(1+\beta)^2 + 4(\beta^2 - 1)cd$$

for convenience and consider the open cones of $(0,\infty)^2$ defined by

$$\begin{aligned} D_{\beta,\gamma} = \{ x \in (0,\infty)^2 : {} & 4\beta(1+\beta)(a\,dx_1^2 + bcx_2^2) + q_\beta x_1 x_2 \\ & - (c\gamma^{-1} x_1^\beta x_2^{1-\beta} - d\gamma x_1^{1-\beta} x_2^\beta)^2 > 0 \}. \end{aligned} \tag{42}$$

LEMMA 5.4. *Let $\beta \in (0,1]$ and $\gamma > 0$.*

The vector field $\psi_{F_{\beta,\gamma}}$ is $\overline{\tau}$ nonexpansive on each convex component of the open cone $F_{\beta,\gamma}(D_{\beta,\gamma})$.

In the particular case $a, b > 0$ and $c = d = 0$, for any $\beta \in (0,1]$ and $\gamma > 0$, $D_{\beta,\gamma} = (0,\infty)^2$. But this fact does hold in general. We need the transformations $F_{\beta,\gamma}$ for well-chosen values of γ to get the nonexpansivity property of the flow on unbounded domains. Let us also note that $(0,\infty)^2$ is not coverable by a single domain of the form $D_{\beta,\gamma}$ in general and the adjunction procedure of Section 3.2 will be needed.

PROOF OF LEMMA 5.4. We write for $y=(y_1,y_2)\in[0,\infty)^2$,

$$\psi_{F_{\beta,\gamma}}(y)=\psi_1(y)+\psi_{2,\beta,\gamma}(y), \tag{43}$$

where

$$\psi_1(y)=\begin{pmatrix}\beta\tau_1 y_1\\ \beta\tau_2 y_2\end{pmatrix},\qquad \psi_{2,\beta,\gamma}(y)=-\begin{pmatrix}\beta y_1(a y_1^{1/\beta}+c\gamma^{-1/\beta}y_2^{1/\beta})\\ \beta y_2(b\gamma^{-1/\beta}y_2^{1/\beta}+d y_1^{1/\beta})\end{pmatrix}.$$

First, ψ_1 is Lipschitz on $[0,\infty)^2$ with constant $\overline{\tau}$ since $\beta\in(0,1]$. Moreover, writing $A_{\beta,\gamma}(x)=J_{\psi_{2,\beta,\gamma}}(F_{\beta,\gamma}(x))$, we have for any $x\in[0,\infty)^2$,

$$\begin{aligned}&A_{\beta,\gamma}(x)+A^*_{\beta,\gamma}(x)\\ &\quad=-\begin{pmatrix}2a(1+\beta)x_1+2c\beta x_2 & c\gamma^{-1}x_1^{\beta}x_2^{1-\beta}+d\gamma x_2^{\beta}x_1^{1-\beta}\\ c\gamma^{-1}x_1^{\beta}x_2^{1-\beta}+d\gamma x_2^{\beta}x_1^{1-\beta} & 2b(1+\beta)x_2+2d\beta x_1\end{pmatrix}.\end{aligned}$$

This can be seen using (13) or by a direct computation. We consider now the trace and the determinant of this matrix:

$$T(x)=\mathrm{Tr}\big(A_{\beta,\gamma}(x)+A^*_{\beta,\gamma}(x)\big),\qquad \Delta(x)=\det\big(A_{\beta,\gamma}(x)+A^*_{\beta,\gamma}(x)\big). \tag{44}$$

As $\beta>0$ and $x\in(0,\infty)^2$, $T(x)<0$, while

$$\begin{aligned}\Delta(x)&=\big(2a(1+\beta)x_1+2c\beta x_2\big)\big(2b(1+\beta)x_2+2d\beta x_1\big)\\ &\quad-\big(c\gamma^{-1}x_1^{\beta}x_2^{1-\beta}+d\gamma x_2^{\beta}x_1^{1-\beta}\big)^2.\end{aligned} \tag{45}$$

It is positive when $x=(x_1,x_2)\in D_{\beta,\gamma}$ and then the spectrum of $A_{\beta,\gamma}(x)+A^*_{\beta,\gamma}(x)$ is included in $(-\infty,0]$. Recalling Table 1 in [1] or the beginning of Section 2, this ensures that $\psi_{2,\beta,\gamma}$ is nonexpansive on the open convex components of $F_{\beta,\gamma}(D_{\beta,\gamma})$. Then $\psi_{F_{\beta,\gamma}}$ is $\overline{\tau}$ nonexpansive on the open convex components of $F_{\beta,\gamma}(D_{\beta,\gamma})$. Let us finally observe that $D_{\beta,\gamma}$, and thus $F_{\beta,\gamma}(D_{\beta,\gamma})$ are open cones, which completes the proof of the lemma. $\square$

We define now

$$\mathcal{C}_{\eta,\beta,\gamma}=\big\{x\in(0,\infty)^2: x_1/x_2\in(0,\eta)\cup(x_{\beta,\gamma}-\eta,x_{\beta,\gamma}+\eta)\cup(1/\eta,\infty)\big\},$$

writing $x_{\beta,\gamma}=(d\gamma^2/c)^{1/(2\beta-1)}$ when it is well defined. The next result ensures that these domains provide a covering by cones for which nonexpansivity hold. The case $c=d=0$ is obvious and we focus on the general case.

LEMMA 5.5. *Assume that* $a,b,c,d>0$. *Let* $\gamma>0$, $\beta\in(0,1)-\{1/2\}$ *such that* $q_\beta>0$.

There exists $\eta>0$ *and* $A>0$ *and* $\mu>0$ *such that*:

(i) $\mathcal{C}_{\eta,\beta,\gamma}\subset D_{\beta,\gamma}$.

(ii) *for any* y, y' *which belong both to a same convex component of the cone* $F_{\beta,\gamma}(\mathcal{C}_{\eta,\beta,\gamma})$ *and to the complementary set of* $B(0,A)$, *then*

$$(\psi_{F_{\beta,\gamma}}(y)-\psi_{F_{\beta,\gamma}}(y')).(y-y')\le -\mu.(\|y\|_2\wedge\|y'\|_2).\|y-y'\|_2^2. \tag{46}$$

PROOF. (i) The inclusion $\{x\in(0,\infty)^2 : x_1=x_\gamma x_2\}\subset D_{\beta,\gamma}$ comes from the fact that

$$x_1=(d\gamma^2/c)^{1/(2\beta-1)}x_2 \qquad \text{implies that } (c\gamma^{-1}x_1^{\beta}x_2^{1-\beta}-d\gamma x_1^{1-\beta}x_2^{\beta})^2=0$$

and the fact that $q_\beta>0$. The inclusion $\{x\in(0,\infty)^2 : x_1/x_2\in(0,\eta)\cup(1/\eta,\infty)\}\subset D_{\beta,\gamma}$ is obtained by bounding

$$(c\gamma^{-1}x_1^{\beta}x_2^{1-\beta}-d\gamma x_1^{1-\beta}x_2^{\beta})^2\le(c\gamma^{-1}\eta^{1-\beta}+d\gamma\eta^{\beta})^2x_1^2$$

when $x_2\le\eta x_1$. Indeed, $a,d>0$ and letting η be small enough such that $4\beta(1+\beta)ad>(c\gamma^{-1}\eta^{1-\beta}+d\gamma\eta^{\beta})^2$ yields the result since $\beta\in(0,1)$.

(ii) Recalling notation (44), for any $x\in[0,\infty)^2-\{(0,0)\}$, $T(x)<0$ and the value of $\Delta(x)$ is given by (45). Let $x_0\neq 0$ such that $\Delta(x_0)>0$, then there exist $v_1,v_2>0$ and some open ball $\mathcal{V}(x_0)$ centered in x_0, such that for any $x\in\mathcal{V}(x_0)$, we have $-v_1\le T(x)<0$ and $\Delta(x)\ge v_2$. Writing $\{-\mu_1(x),-\mu_2(x)\}$ with $0\le\mu_1\le\mu_2$ the two negative eigenvalues of the symmetric matrix $A_{\beta,\gamma}(x)+A^*_{\beta,\gamma}(x)$, we have for for any $\lambda>0$ and $x\in\mathcal{V}(x_0)$

$$\mu_1(\lambda x)=\frac{\mu_1(\lambda x)\mu_2(\lambda x)}{\mu_2(\lambda x)}\ge\frac{\mu_1(\lambda x)\mu_2(\lambda x)}{\mu_1(\lambda x)+\mu_2(\lambda x)}=\frac{\Delta(\lambda x)}{-T(\lambda x)}=\lambda\frac{\Delta(x)}{-T(x)}\ge\lambda\frac{v_2}{v_1},$$

since Δ (resp., T) gives the product (resp., the sum) of the two eigenvalues. We obtain that there exists $\mu>0$ such that for any x in the convex cone $\mathcal{C}(x_0)$ generated by $\mathcal{V}(x_0)$, the spectrum of $A_{\beta,\gamma}(x)+A^*_{\beta,\gamma}(x)$ is included in $(-\infty,-2\mu\|x\|_2]$. Recalling that $A_{\beta,\gamma}=J_{\psi_{2,\beta,\gamma}}\circ F_{\beta,\gamma}$ and $\beta\le 1$, there exists $\widetilde{\mu}$ such that the spectrum of $J_{\psi_{2,\beta,\gamma}}(y)+J^*_{\psi_{2,\beta,\gamma}}(y)$ is included in $(-\infty,-2\widetilde{\mu}\|y\|_2]$ for any $y\in F_{\beta,\gamma}(\mathcal{C}(x_0))$ such that $\|y\|_2\ge 1$. Then

$$(\psi_{2,\beta,\gamma}(y)-\psi_{2,\beta,\gamma}(y')).(y-y')\le-\widetilde{\mu}.(\|y\|_2\wedge\|y'\|_2).\|y-y'\|_2^2,$$

for any y, y' in a convex set containing $F_{\beta,\gamma}(\mathcal{C}(x_0))\cap B(0,1)^c$; see again Table 1 in [1] for details. Recalling now (43) and that ψ_1 is Lipschitz with constant $\overline{\tau}$, there exists $A>0$ such that

$$(\psi_{F_{\beta,\gamma}}(y)-\psi_{F_{\beta,\gamma}}(y')).(y-y')\le-\frac{1}{2}\widetilde{\mu}.(\|y\|_2\wedge\|y'\|_2).\|y-y'\|_2^2$$

for any $y,y'\in B(0,A)^c$ which belong to convex component of $F_{\beta,\gamma}(\mathcal{C}(x_0))$. We conclude by choosing $\eta>0$ such that $\mathcal{C}_{\eta,\beta,\gamma}\subset\bigcup_{x_0\in\{x_\gamma,(0,1),(1,0)\}}\mathcal{C}(x_0)$. □

5.3.2. *Poincaré's compactification and coming down from infinity of the flow.* To describe the coming down from infinity of the flow ϕ, we use the following compactification $\mathcal{K}$ of $[0,\infty)^2$:

$$\mathcal{K}(x)=\mathcal{K}(x_1,x_2)=\left(\frac{x_1}{1+x_1+x_2},\frac{x_2}{1+x_1+x_2},\frac{1}{1+x_1+x_2}\right)=(y_1,y_2,y_3).$$

The application $\mathcal{K}$ is a bijection from $[0,\infty)^2$ into the simplex $\mathcal{S}$ defined by

$$\mathcal{S}=\{y\in[0,1]^2\times(0,1]:y_1+y_2+y_3=1\}\subset\overline{\mathcal{S}}=\{y\in[0,1]^3:y_1+y_2+y_3=1\}.$$

We note ∂S the outer boundary of $\mathcal{S}$:

$$\begin{aligned}\partial S&=\overline{\mathcal{S}}-\mathcal{S}\\&=\{(y_1,1-y_1,0):y_1\in[0,1]\}\\&=\Big\{\lim_{r\to\infty}\mathcal{K}(rx):x\in[0,\infty)^2-\{(0,0)\}\Big\}.\end{aligned}$$

The key point to describe the direction of the dynamical system ϕ coming from infinity is the following change of time. It allows to extend the flow on the boundary and is an example of Poincaré's compactification technics, which is particularly powerful for polynomial vector field [16]. More precisely, we consider the flow Φ of the dynamical system on $\overline{\mathcal{S}}$ given for $z_0\in\overline{\mathcal{S}}$ and $t\geq 0$ by

$$\Phi(z_0,0)=z_0,\qquad \frac{\partial}{\partial t}\Phi(z_0,t)=H\big(\Phi(z_0,t)\big), \tag{47}$$

where H is the Lipschitz function on $\overline{\mathcal{S}}$ defined by

$$\begin{aligned}H^{(1)}(y_1,y_2,y_3)&=y_1y_2\big[(b-c)y_2+(d-a)y_1\big]\\&\quad+y_1y_3\big[(\tau_1-\tau_2-c)y_2-ay_1+y_3\tau_1\big],\\H^{(2)}(y_1,y_2,y_3)&=y_1y_2\big[(a-d)y_1+(c-b)y_2\big]\\&\quad+y_2y_3\big[(\tau_2-\tau_1-d)y_1-by_2+y_3\tau_2\big],\\H^{(3)}(y_1,y_2,y_3)&=y_3\big(ay_1^2+by_2^2+(c+d)y_1y_2-\tau_1y_1y_3-\tau_2y_2y_3\big).\end{aligned} \tag{48}$$

The study of Φ close to $\partial\mathcal{S}$ is giving us the behavior of ϕ close to infinity using the change of time $\varphi\in\mathcal{C}^1([0,\infty)^2\times[0,\infty),[0,\infty))$ defined by

$$\varphi(x_0,0)=x_0,\qquad \frac{\partial}{\partial t}\varphi(x_0,t)=1+\|\phi(x_0,t)\|_1.$$

LEMMA 5.6. *For any $x_0\in[0,\infty)^2$ and $t\geq 0$,*

$$\mathcal{K}\big(\phi(x_0,t)\big)=\Phi\big(\mathcal{K}(x_0),\varphi(x_0,t)\big).$$

PROOF. We denote by $(y_t : t \geq 0)$ the image of the dynamical system $(x_t : t \geq 0)$ through $\mathcal{K}$: $y_t = \mathcal{K}(x_t) = \mathcal{K}(\phi(x_0, t))$. Then

$$y_t' = G(x_t) = G \circ \mathcal{K}^{-1}(y_t)$$

where $G = (G^{(1)}, G^{(2)}, G^{(3)})$ is given by

$$G^{(1)}(x_1, x_2) = \frac{(d-a)x_1^2 x_2 + (b-c)x_1 x_2^2 + (\tau_1 - \tau_2 - c)x_1 x_2 - a x_1^2 + \tau_1 x_1}{(1 + x_1 + x_2)^2},$$

$$G^{(2)}(x_1, x_2) = \frac{(c-b)x_2^2 x_1 + (a-d)x_2 x_1^2 + (\tau_2 - \tau_1 - d)x_2 x_1 - b x_2^2 + \tau_2 x_2}{(1 + x_1 + x_2)^2},$$

$$G^{(3)}(x_1, x_2) = \frac{a x_1^2 + b x_2^2 + (c+d)x_1 x_2 - \tau_1 x_1 - \tau_2 x_2}{(1 + x_1 + x_2)^2}.$$

Using that $x_1 = y_1/y_3$ and $x_2 = y_2/y_3$ and recalling the definition (48) of H, we have

$$G \circ \mathcal{K}^{-1}(y) = \frac{1}{y_3} H(y) \tag{49}$$

for $y = (y_1, y_2, y_3) \in \mathcal{S}$. The key point in the theory of Poincaré is that H is continuous on $\overline{\mathcal{S}}$ and that the trajectories of the dynamical system $(z_t : t \geq 0)$ associated to the vector field H:

$$z_t' = H(z_t)$$

are the same than the trajectories of $(y_t : t \geq 0)$ whose vector field is $G \circ \mathcal{K}^{-1}$. Indeed the positive real number $1/y_3$ only changes the norm of the vector field, and thus the speed at which the same trajectory is covered. The associated change of time $v_t = \varphi(x_0, t)$ such that

$$z_{v_t} = y_t = \mathcal{K}(x_t)$$

can now be simply computed. Indeed $(z_{v_t})' = H(y_t) v_t'$ coincides with $y_t' = G \circ \mathcal{K}^{-1}(y_t)$ as soon as

$$v_t' = \frac{1}{y_t^{(3)}} = \frac{1}{\mathcal{K}^{(3)}(\phi(x_0, t))} = 1 + \|\phi(x_0, t)\|_1,$$

using (49). This completes the proof. $\square$

To describe the direction from which the flow ϕ comes down from infinity, we introduce the hitting times of cones centered in x:

$$\begin{aligned} t_-(x_0, x, \varepsilon) &= \inf_{s \geq 0} \{\widehat{(x_s, x)} \in [-\varepsilon, +\varepsilon]\}, \\ t_+(x_0, x, \varepsilon) &= \inf_{s \geq t_-(x_0, x, \varepsilon)} \{\widehat{(x_s, x)} \notin [-2\varepsilon, +2\varepsilon]\}, \end{aligned} \tag{50}$$

where we recall that $x_s = \phi(x_0, s)$ and $\inf \varnothing = \infty$. The directions ℓ of the coming down from infinity are defined by:

- $\ell = x_\infty$ if $b > c$ and $a > d$, where x_∞ has been defined in (36).
- $\ell = (1/a, 0)$ if $b > c$ and $a \le d$; or if $b \ge c$ and $a < d$; or if $c > b$ and $d > a$ and $\widehat{(x_0, x_\infty)} > 0$.
- $\ell = (0, 1/b)$ if $a > d$ and $b \le c$; or if $a \ge d$ and $b < c$; or if $c > b$ and $d > a$ and $\widehat{(x_0, x_\infty)} < 0$.
- $\ell = \widehat{x_0}$ if $a = d$ and $b = c$, where $\widehat{x_0} = x_0/(ax_0^{(1)} + bx_0^{(2)})$ for any $x_0 \in (0, \infty)^2$.

The proof of the direction relies on the previous compactification result. We can then specify the speed of coming down from infinity of the flow ϕ since the problem is reduced to one single dimension where computations can be easily achieved.

LEMMA 5.7. (i) *For any* $T > 0$, *there exists* $c_T > 0$ *such that* $\|\phi(x_0, t)\|_1 \le c_T/t$ *for all* $x_0 \in [0, \infty)^2$ *and* $t \in (0, T]$.

(ii) *For all* $x_0 \in (0, \infty)^2$ *and* $\varepsilon > 0$,

$$\lim_{r\to\infty} t_-(rx_0, \ell, \varepsilon) = 0, \qquad \lim_{r\to\infty} t_+(rx_0, \ell, \varepsilon) > 0.$$

(iii) *Moreover*,

$$\lim_{t\to 0} \limsup_{r\to\infty} \big| \|t\phi(rx_0, t)\|_1 - \|\ell\|_1 \big| = 0.$$

PROOF. (i) Using $a > 0$, we first observe that $(x_t^{(1)})' \le -a(x_t^{(1)})^2/2$ in the time intervals when $x_t^{(1)} \ge 2\tau_1/a$. Solving $(x_t^{(1)})' = -(x_t^{(1)})^2 a/2$ proves (i).

(ii) We use the notation (47) and (48) above and the dynamics of $z_t = \Phi(z_0, t)$ on the invariant set $\partial\mathcal{S}$ is simply given by the vector field $H(y_1, y_2, 0)$ for $y_1 \in [0, 1]$, $y_1 + y_2 = 1$:

$$H^{(1)}(y_1, y_2, 0) = -H^{(2)}(y_1, y_2, 0) = y_1 y_2\big[(b - c)y_2 + (d - a)y_1\big].$$

The two points $(1, 0, 0)$ and $(0, 1, 0)$ on $\partial\mathcal{S}$ are invariant for the dynamical system $(z_t : t \ge 0)$.

Let us first consider the case when $a \ne d$ or $b \ne c$. There is an additional invariant point in ∂S if and only if

$$(b - c)(a - d) > 0.$$

Thus, if $(b - c)(a - d) \le 0$, $H^{-1}((0, 0, 0)) \cap \partial\mathcal{S} = \{(1, 0, 0), (0, 1, 0)\}$ and z_t starting from the boundary $\partial\mathcal{S}$ goes either to $(1, 0, 0)$ whatever its initial value z_0 in the interior of the boundary; or to $(0, 1, 0)$ whatever its initial value z_0 in the interior of the boundary. These cases are inherited from the sign of $b - c$, which provides the stability of the fixed points $(1, 0, 0)$ and $(0, 1, 0)$. Then by Lemma 5.6 the dynamical system $z_{\varphi(x_0,t)} = \mathcal{K}(x_t)$ starting close to the boundary $\partial\mathcal{S}$ goes:

- either to $(1, 0, 0)$; and then $\widehat{(x_t, \ell)}$ becomes small, where $\ell = (1/a, 0)$.
- or to $(0, 1, 0)$; and then $\widehat{(x_t, \ell)}$ becomes small, where $\ell = (0, 1/b)$.

More precisely, z issued from $\mathcal{K}(\phi(rx_0,t))$ reaches any neighborhood of $(1,0,0)$ or $(0,1,0)$ in a time which is bounded for r large enough. Adding that $\partial\varphi(rx_0,t)/\partial t = 1 + \|\phi(rx_0,t)\|_1$ is large before $z_{\varphi(rx_0,.)}$ has reached this neighborhood ensures that this reaching time is arbitrarily small for $\mathcal{K}^{-1}(\phi(rx_0,.))$ when r is large. This proves that $t_-(rx_0,\ell,\varepsilon)\to 0$ as $r\to\infty$. Moreover, $t_+(rx_0,\ell,\varepsilon)$ is not becoming close to 0 as $r\to\infty$ since the speed of the dynamical system $\phi(rx_0,.)$ is bounded on the compacts sets of $[0,\infty)^2$.

Otherwise, $(b-c)(a-d)>0$ and

$$H^{-1}\big((0,0,0)\big)\cap\partial\mathcal{S}=\big\{(1,0,0),(0,1,0),z_\infty\big\},$$

where z_∞ is the unique invariant point in the interior of the boundary

$$z_\infty=\frac{1}{b-c+a-d}(b-c,a-d,0).$$

Then we need to see if z_∞ is repulsive or attractive on the invariant set $\partial\mathcal{S}$. In the case $c>b$ and $d>a$, this point is attractive and z_∞ is a a saddle and

$$z_\infty=\lim_{r\to\infty}\mathcal{K}(rx_\infty).$$

So Lemma 5.6 now ensures that the dynamical system x_t takes the direction $\ell=x_\infty$ when starting from a large initial value. As in the previous case, $t_-(rx_0,\ell,\varepsilon)\to 0$ and $t_+(rx_0,\ell,\varepsilon)$ does not.

In the case $b<c$ and $a<d$, y_∞ is a source and the dynamical system z_t either goes to $(1,0,0)$ (and then $\ell=(1/a,0)$) or to $(0,1,0)$ (and then $\ell=(0,1/b)$). This depends on the position of the initial value with respect to the second unstable variety, and thus on the sign of $(\widehat{x_0,x_\infty})$.

Finally, the case $a=d$, $b=c$ is handled similarly noting that the whole set $\partial\mathcal{S}$ is invariant.

(iii) We know from (ii) that the direction of the dynamical system coming from infinity is ℓ and we reduce now its dynamics close to infinity to a one-dimensional and solvable problem. Indeed, let us write

$$x_t(r)=\phi(rx_0,t)$$

and focus on the case $\ell^{(1)}\neq 0$. First, we observe that for any $T>0$, there exists $M_T>0$ such that for any $t\in[0,T]$ and $r\geq 1$,

$$x_t^{(2)}(r)\leq M_T x_t^{(1)}(r). \tag{51}$$

Indeed $\mathcal{K}(x_t)=z_{v_t}$ does not come close to the boundary $\{(0,u,1-u):u\in[0,1]\}$ on compact time intervals when $\ell^{(1)}\neq 0$. Plugging (51) in (31) provides a lower bound for $x_t^{(1)}(r)$ and we obtain for any $\varepsilon>0$,

$$t_1(\varepsilon)=\liminf_{r\to\infty}\inf\big\{t\geq 0: x_t^{(1)}(r)<\big(|\tau_1|+1\big)/\varepsilon\big\}\in(0,\infty].$$

Moreover, by definition (50), for any $\varepsilon > 0$ and $r > 0$ and $t \in [t_-(rx_0, \ell, \varepsilon), t_+(rx_0, \ell, \varepsilon)]$, we have $\widehat{(x_t(r), \ell)} \le 2\varepsilon$ and

$$\left| \frac{x_t^{(2)}(r)}{x_t^{(1)}(r)} - \frac{\ell^{(2)}}{\ell^{(1)}} \right| \le u(\varepsilon), \tag{52}$$

where $u(\varepsilon) \in [0, \infty]$ and $u(\varepsilon) \to 0$ as $\varepsilon \to 0$. We write

$$\theta_\ell = \frac{\ell^{(2)}}{\ell^{(1)}}, \qquad t_-(r) = t_-(rx_0, \ell, \varepsilon), \qquad t_+(r) = t_+(rx_0, \ell, \varepsilon) \wedge t_1\big(u(\varepsilon)\big)$$

for convenience. Plugging (52) in the first equation of (31) yields for any $t \in [t_-(r), t_+(r)]$ and $r \ge 1$,

$$-\big(a + c\theta_\ell + (1+c)u(\varepsilon)\big) \le \frac{(x_t^{(1)}(r))'}{(x_t^{(1)}(r))^2} \le -\big(a + c\theta_\ell - (1+c)u(\varepsilon)\big).$$

We get by integration, for any ε small enough,

$$\begin{aligned} &\frac{1}{(a + c\theta_\ell + (1+c)u(\varepsilon))(t - t_-(r)) + 1/x_{t_-(r)}^{(1)}(r)} \\ &\qquad \le x_t^{(1)}(r) \le \frac{1}{(a + c\theta_\ell - (1+c)u(\varepsilon))(t - t_-(r)) + 1/x_{t_-(r)}^{(1)}(r)}. \end{aligned}$$

Using (ii), $t_-(r) \to 0$ and $t_+ = \liminf t_+(r) > 0$ as $r \to \infty$. Moreover, $\ell^{(1)} \neq 0$ ensures that $x_{t_-(r)}^{(1)}(r) \to \infty$ as $r \to \infty$. Then for any ε positive small enough and $t \le t_+$,

$$\begin{aligned} \frac{1}{a + c\theta_\ell + (1+c)u(\varepsilon)} &\le \liminf_{r\to\infty} t x_t^{(1)}(r) \\ &\le \limsup_{r\to\infty} t x_t^{(1)}(r) \le \frac{1}{a + c\theta_\ell - (1+c)u(\varepsilon)}. \end{aligned}$$

Letting finally $\varepsilon \to 0$, $u(\varepsilon) \to 0$ and we obtain

$$\lim_{t\to 0} \limsup_{r\to\infty} \big| t x_t^{(1)}(r) - 1/(a + c\theta_\ell) \big| = 0.$$

Using again (52) provides the counterpart for $t x_t^{(2)}$ and completes the proof in the case $\ell^{(1)} \neq 0$. The case $\ell^{(2)} \neq 0$ is treated similarly. $\square$

5.3.3. *Approximation of the flow of scaled birth and death processes.* We use notation of Section 3 for

$$X^K = \begin{pmatrix} X^{K,(1)} \\ X^{K,(2)} \end{pmatrix}$$

with here $E = \{0, 1, 2, \dots\}^2$, $\chi = [0, \infty)$, $q(dz) = dz$ and

$$h_F^K(x) = \int_0^\infty \big[F\big(x + H^K(x, z)\big) - F(x)\big]\, dz,$$

where H^K is defined in (39). Recalling the definition of $F_{\beta,\gamma}$ from (33), we get

$$h^K_{F_{\beta,\gamma}}(x)$$

$$= \begin{pmatrix} \lambda_1 K x_1\big((x_1 + 1/K)^\beta - x_1^\beta\big) + K x_1(\mu_1 + a x_1 + c x_2)\big((x_1 - 1/K)^\beta - x_1^\beta\big) \\ \gamma\lambda_2 K x_2\big((x_2 + 1/K)^\beta - x_2^\beta\big) + \gamma K x_2(\mu_2 + b x_2 + d x_1)\big((x_2 - 1/K)^\beta - x_2^\beta\big) \end{pmatrix}. \tag{53}$$

We consider

$$b^K_{F_{\beta,\gamma}} = J^{-1}_{F_{\beta,\gamma}} h^K_{F_{\beta,\gamma}}, \qquad \psi^K_{F_{\beta,\gamma}} = h^K_{F_{\beta,\gamma}} \circ F^{-1}_{\beta,\gamma}$$

and we recall that $\mathfrak{D}_\alpha = \{(x_1, x_2) \in (\alpha, \infty)^2 : x_1 \geq \alpha x_2, x_2 \geq \alpha x_1\}$ and

$$b(x) = \begin{pmatrix} \tau_1 x_1 - a x_1^2 - c x_1 x_2 \\ \tau_2 x_2 - b x_2^2 - d x_1 x_2 \end{pmatrix}, \qquad \psi_{F_{\beta,\gamma}} = (J_{F_{\beta,\gamma}} b) \circ F^{-1}_{\beta,\gamma}.$$

To compare these quantities and approximate the flow associated with b^K, we introduce

$$\Delta^K_{\beta,\gamma}(x) = \frac{\beta(\beta - 1)}{2K} \begin{pmatrix} (a x_1 + c x_2) x_1^{\beta - 1} \\ \gamma (b x_2 + d x_1) x_2^{\beta - 1} \end{pmatrix}.$$

LEMMA 5.8. *For any $\alpha > 0$ and $\beta \in (0, 1]$ and $\gamma > 0$, there exists $C > 0$ such that for any $x \in \mathfrak{D}_\alpha$ and $y \in F_{\beta,\gamma}(\mathfrak{D}_\alpha)$ and $K \geq 2/\alpha$:*

(i)

$$\big\| h^K_{F_{\beta,\gamma}}(x) - J_{F_{\beta,\gamma}}(x) b(x) - \Delta^K_{\beta,\gamma}(x) \big\|_2 \leq \frac{C}{K} \|x\|_2^{\beta - 1}.$$

(ii)

$$\big\| b^K_{F_{\beta,\gamma}}(x) - b(x) \big\|_2 \leq \frac{C}{K} \|x\|_2.$$

(iii)

$$\psi^K_{F_{\beta,\gamma}}(y) = \psi_{F_{\beta,\gamma}}(y) + \Delta^K_{\beta,\gamma}\big(F^{-1}_{\beta,\gamma}(y)\big) + R^K_{\beta,\gamma}\big(F^{-1}_{\beta,\gamma}(y)\big),$$

where $\|R^K_{\beta,\gamma}(x)\|_2 \leq C/K$.

(iv) *Moreover, $\psi^K_{F_{\beta,\gamma}}$ is $(C, C/K)$ nonexpansive on each convex component of $F_{\beta,\gamma}(D_{\beta,\gamma} \cap \mathfrak{D}_\alpha)$, where we recall that $D_{\beta,\gamma}$ is defined in (42).*

(v) *Finally,*

$$\big\| \psi^K_{F_{\beta,\gamma}}(y) - \psi_{F_{\beta,\gamma}}(y) \big\|_2 \leq C \frac{1 + \|y\|}{K}.$$

PROOF. First, by the Taylor–Lagrange formula applied to $(1+h)^\beta$, there exists $c_0>0$ such that

$$\left|\left(z+\frac{\delta}{K}\right)^\beta - z^\beta - \frac{\delta}{K}\beta z^{\beta-1} - \frac{\delta^2}{2K^2}\beta(\beta-1)z^{\beta-2}\right| \le \frac{c_0}{K^2}z^{\beta-3}$$

for any $z>\alpha$ and $K\ge 2/\alpha$ and $\delta\in\{-1,1\}$, since $h=\delta/(Kz)\in(-1/2,1/2)$. Using then (53) and

$$J_{F_{\beta,\gamma}}(x) = \begin{pmatrix} \beta x_1^{\beta-1} & 0 \\ 0 & \gamma\beta x_2^{\beta-1}\end{pmatrix},$$

$$J_{F_{\beta,\gamma}}(x)b(x) = \begin{pmatrix} \beta x_1^{\beta-1}x_1(\tau_1-ax_1-cx_2) \\ \gamma\beta x_2^{\beta-1}x_2(\tau_2-bx_2-dx_1)\end{pmatrix}$$

yields (i), since $\|x\|_2$, x_1 and x_2 are equivalent up to a positive constant when $x\in\mathfrak{D}_\alpha$. We immediately get (iii) since $\|x\|_2^{\beta-1}$ is bounded on $[\alpha,\infty)^2$ when $\beta\le 1$.

Then (i) and the fact that there exists $c_0>0$ such that for any $x\in\mathfrak{D}_\alpha$ and $u\in[0,\infty)^2$,

$$\|J_{F_{\beta,\gamma}}(x)^{-1}\Delta^K_{\beta,\gamma}(x)\|_2 \le c_0\frac{\|x\|_2}{K}, \qquad \|J_{F_{\beta,\gamma}}(x)^{-1}u\|_2\le c_0\|x\|_2^{1-\beta}\|u\|_2$$

proves (ii).

We observe that $\Delta^K_{\beta,\gamma}\circ F^{-1}_{\beta,\gamma}$ is uniformly Lipschitz on $F_{\beta,\gamma}(\mathfrak{D}_\alpha)$ with constant L since its partial derivative are bounded on this domain. Recalling then from Lemma 5.4(i) that $\psi_{F_{\beta,\gamma}}$ is $\bar\tau$ nonexpansive on $F_{\beta,\gamma}(D_{\beta,\gamma})$, the decomposition (iii) ensures that $\psi^K_{F_{\beta,\gamma}}$ is $(\bar\tau+L, C/K)$ nonexpansive on $F_{\beta,\gamma}(D_{\beta,\gamma}\cap\mathfrak{D}_\alpha)$. So (iv) holds.

Finally, using (iii) and adding that

$$\sup_{y\in F_{\beta,\gamma}(\mathfrak{D}_\alpha),K\ge1} K\frac{\|\Delta^K_{\beta,\gamma}(F^{-1}_{\beta,\gamma}(y))\|_2}{\|y\|_2} = \sup_{x\in\mathfrak{D}_\alpha,K\ge1} K\frac{\|\Delta^K_{\beta,\gamma}(x)\|_2}{\|F_{\beta,\gamma}(x)\|_2}<\infty$$

proves (v) and completes the proof. □

5.3.4. *Adjunction of open convex cones*. We decompose the trajectory of the flow in $\mathcal{D}_\alpha=(\alpha,\infty)^2$ into time intervals where a nonexpansive transformation can be found. This relies on the next lemma and the results of Section 5.3.1. Recall from (35) that $T_D(x_0)$ is the exit time of D for the flow started from x_0. Moreover $d_\beta(x,y)=\|F_{\beta,1}(x)-F_{\beta,1}(y)\|_2$ from (34), while the definition of ℓ is given in previous Section 5.3.2.

LEMMA 5.9. (i) *Let* $\alpha>0$, $\beta\in(0,1]$, $N\in\mathbb{N}$ *and* $(C_i)_{i=1,\dots,N}$ *be a family of open convex cones of* $(0,\infty)^2$ *such that*

$$(0,\infty)^2=\bigcup_{i=1}^N C_i.$$

Then there exists $\kappa \in \mathbb{N}$ *and* $\varepsilon_0 > 0$ *and* $(t_k(x_0) : k = 0, \dots, \kappa)$ *and* $(n_k(x_0) : k = 1, \dots, \kappa - 1)$ *such that for any* $x_0 \in \mathcal{D}_\alpha$,

$$0 = t_0(x_0) \le t_1(x_0) \le \cdots \le t_\kappa(x_0) = T_{\mathcal{D}_\alpha}(x_0), \qquad n_k(x_0) \in \{1, \dots, N\}$$

and for any $k \le \kappa - 1$ *and* $t \in [t_k(x_0), t_{k+1}(x_0))$, *we have*

$$\overline{B}_{d_\beta}\big(\phi(x_0, t), \varepsilon_0\big) \subset C_{n_k(x_0)}.$$

(ii) *In the case* $\ell = x_\infty \in (0, \infty)^2$, *for any* $x_0 \in (0, \infty)^2$ *and* $\varepsilon > 0$,

$$\liminf_{r \to \infty} T_{\mathcal{D}_\varepsilon}(r x_0) > 0.$$

(iii) *In the case* $\ell = (1/a, 0)$, *for any* $x_0 \in (0, \infty)^2$ *and* $\varepsilon > 0$ *and* $T > 0$, *for* r *large enough*,

$$T_{\mathcal{D}_\varepsilon}(r x_0) = \inf\big\{t \ge 0 : \phi(r x_0, t) \in [0, \infty) \times [0, \varepsilon]\big\} \le T.$$

(iv) *Under Assumption* (40), *for any* $\alpha_0 > 0$,

$$\inf_{x_0 \in \mathfrak{D}_{\alpha_0}} T_{\mathfrak{D}_\alpha}(x_0) \stackrel{\alpha \to 0}{\longrightarrow} +\infty.$$

PROOF. (i) We define

$$C_i^\varepsilon = \big\{x \in \mathcal{D}_\alpha \cap C_i : \overline{B}_{d_\beta}(x, \varepsilon) \subset C_i\big\}$$

and we first observe that for ε small enough,

$$\bigcup_{i=1}^{N} C_i^{2\varepsilon} = \mathcal{D}_\alpha,$$

since $\beta > 0$ and the open convex cones C_i are domains between two half-lines of $(0, \infty)^2$ and their collection for $i = 1, \dots, N$ covers $(0, \infty)^2$. We define $t_0(x_0) = 0$ and

$$n_0(x_0) = \min\big\{i = 1, \dots, N : x_0 \in C_i^{2\varepsilon}\big\},$$
$$t_1(x_0) = \inf\big\{t \ge 0 : \phi(x_0, t) \notin C_{n_0(x_0)}^\varepsilon\big\}$$

and by recurrence for $k \ge 1$,

$$n_k(x_0) = \min\big\{i = 1, \dots, N : \phi\big(x_0, t_k(x_0)\big) \in C_i^{2\varepsilon}\big\},$$
$$t_{k+1}^i(x_0) = \inf\big\{t \ge t_k(x_0) : \phi(x_0, t) \notin C_{n_k(x_0)}^\varepsilon\big\}.$$

Let us now prove that

$$M(x_0) = \max\big\{k : t_k(x_0) < \infty\big\}$$

is bounded by $2N$. Indeed, the direction of variation of the angle

$$\Theta_t = \widehat{(0, x_t)} = \mathrm{Arctan}\big(x_t^{(2)}/x_t^{(1)}\big)$$

is given by the sign of

$$\frac{\partial}{\partial t}\log\big(x_t^{(2)})/x_t^{(1)}\big) = \tau_2 - \tau_1 + (a-d)x_t^{(1)} + (c-b)x_t^{(2)}.$$

If the flow crosses the line $\{(x,y)\in\mathbb{R}^2 : \tau_1 - \tau_2 + (d-a)x + (b-c)y = 0\}$ at some time, then it enters a stable subdomain, since the vector field ψ is entrant at the boundary. This means that it stays inside this domain and the angle is monotone from this time. As a consequence, the sense of variations of Θ can change at most once, when it crosses the line, if it does. This ensures that the dynamical system may at most visit each cone $(C_i : i = 1,\dots,N)$ twice, in a monotone way, so $\sup M \le 2N$. This provides the expected construction.

(ii) comes simply from Lemma 5.6 which ensures that in the case $\ell = x_\infty$, the dynamical system comes down from infinity in the interior of $(0,\infty)^2$, see also the first picture in Figure 2 above.

(iii) We use again the dynamical system $(z_t : t \ge 0)$ given by Φ and defined in (47). More precisely, the property here comes from the continuity of the associated flow with respect to the initial condition. Indeed, in the case $\ell = (1/a, 0)$, the trajectories of $(z_t : t \ge 0)$ starting from r large go to $(1,0,0)$ along the boundary $\partial\mathcal{S}$ and then remain close to boundary $\{(u,0,1-u) : u \in [u_0,1]\}$ for some fixed $u_0 < 1$. This ensures that $(x_t : t \ge 0)$ exits from $\mathcal{D}_\varepsilon$ through $(0,\infty)\times\{\varepsilon\}$ and in finite time for r large enough. The fact that this exit time $T_{\mathcal{D}_\varepsilon}(rx_0)$ goes to zero as $r\to\infty$ is due to the fact that the dynamics of $(x_t : t \ge 0)$ is an acceleration of that of $(z_t : t \ge 0)$ when starting close to infinity, with time change $1 + \|\phi(x_0,t)\|_1$.

Finally, (iv) is a consequence of Lemma 5.7. Indeed, noticing that Assumption (40) ensures that $\ell \in \{x_\infty, x_0\}$ and using that the speed of the dynamical system is bounded on compact sets, the dynamical system does not come arbitrarily close to the boundary $(0,\infty)^2$ in a given time interval and more precisely the time to reach the boundary of $\mathfrak{D}_\alpha$ goes to infinity as α goes to 0. $\square$

LEMMA 5.10. *Let $\beta \in (0,1) - \{1/2\}$ such that $q_\beta = 4ab(1+\beta)^2 + 4cd(\beta^2 - 1) > 0$ and $\alpha > 0$.*

There exists $N \ge 1$, $(\gamma_i : i = 1,\dots,N) \in (0,\infty)^N$, convex cones $(C_i : i = 1,\dots,N)$, $\kappa \in \mathbb{N}$, $\varepsilon_0 > 0$, $0 = t_0(x_0) \le t_1(x_0) \le \cdots \le t_\kappa(x_0) = T_{\mathcal{D}_\alpha}(x_0)$ and $n_k(x_0) \in \{1,\dots,N\}$ such that:

(i) *For each $i = 1,\dots,N$, $\psi_{F_{\beta,\gamma_i}}$ is $\overline{\tau}$ nonexpansive on $F_{\beta,\gamma_i}(C_i)$ and $\bigcup_{i=1}^N C_i = (0,\infty)^2$.*

(ii) *For any $x_0 \in \mathcal{D}_\alpha$, $k = 0,\dots,\kappa-1$, $t \in (t_k(x_0), t_{k+1}(x_0))$,*

$$\overline{B}_{d_\beta}\big(\phi(x_0,t),\varepsilon_0\big) \subset C_{n_k(x_0)} \cap \mathcal{D}_{\alpha/2}.$$

(iii) *Finally, for K large enough, there exists a continuous flow ϕ^K such that for any $x_0 \in \mathfrak{D}_\alpha$, $\phi^K(x_0,0) = x_0$ and for any $k = 0,\dots,\kappa-1$ and $t \in$*

$(t_k(x_0), t_{k+1}(x_0) \wedge T_{\mathfrak{D}_\alpha}(x_0))$,

$$\overline{B}_{d_\beta}(\phi^K(x_0,t), \varepsilon_0/2) \subset C_{n_k(x_0)} \cap \mathfrak{D}_{\alpha/2} \quad \textit{and} \quad \frac{\partial}{\partial t}\phi^K(x_0,t) = b^K_{F_{n_k(x_0)}}(\phi^K(x_0,t))$$

and for any $T > 0$,

$$\sup_{\substack{x_0 \in \mathfrak{D}_\alpha, \\ t < T_{\mathfrak{D}_\alpha}(x_0) \wedge T}} d_\beta(\phi^K(x_0,t), \phi(x_0,t)) \stackrel{K\to\infty}{\longrightarrow} 0. \tag{54}$$

PROOF. We only deal with the case $c \neq 0$ (and then $d \neq 0$). Indeed, we recall from Lemma 5.4 that the proofs of (i)–(ii) in the case $c = d = 0$ is obvious, since one can take $N = 1$ and $C_1 = (0,\infty)^2$. Moreover, the proof of (iii) is simplified in that case.

By Lemma 5.5, for any $\gamma > 0$, there exists $\eta(\beta,\gamma) > 0$ such that $\mathcal{C}_{\eta(\beta,\gamma),\beta,\gamma} \subset D_{\beta,\gamma}$ and (46) holds for some $A_{\beta,\gamma}, \mu_{\beta,\gamma} \geq 0$. The collection of the convex components of $(\mathcal{C}_{\eta(\beta,\gamma),\beta,\gamma} : \gamma > 0)$ covers $(0,\infty)^2$, since it contains the half lines $\{(x_1,x_2) \in (0,\infty) : x_1 = x_2 x_\gamma\}$ and $\{x_\gamma : \gamma > 0\} = (0,\infty)$. We underline that this collection also contains the cones $\{(x_1,x_2) \in (0,\infty)^2 : x_1 < \eta(\beta,\gamma)x_2\}$ and $\{(x_1,x_2) \in (0,\infty)^2 : x_2 < \eta(\beta,\gamma)x_1\}$. Then, by a compactness argument, we can extract a finite covering of $(0,\infty)^2$ from this collection of open convex cones. This means that there exists $N \geq 1$ and $(\gamma_i : i = 1,\dots,N) \in (0,\infty)^N$ and convex cones $(C_i : i = 1,\dots,N)$ such that $\bigcup_{i=1}^N C_i = (0,\infty)^2$ and $C_i \subset \mathcal{C}_{\eta(\beta,\gamma_i),\beta,\gamma_i}$. By Lemma 5.4, $\psi_{F_{\beta,\gamma_i}}$ is $\overline{\tau}$ is nonexpansive on $F_{\beta,\gamma_i}(C_i)$ for each $i = 1,\dots,N$, which proves (i).

We let now $\alpha > 0$. The point (ii) is a direct consequence of Lemma 5.9(i) applied to the covering $(C_i : i = 1,\dots,N)$ of $(0,\infty)^2$. Indeed, one just need to choose ε_0 small enough so that $\overline{B}_{d_\beta}(x,\varepsilon_0) \subset \mathcal{D}_{\alpha/2}$ for any $x \in \mathcal{D}_\alpha$.

Let us now deal with (iii). First, from the proof of (i) and writing $F_i = F_{\beta,\gamma_i}$, $A_i = A_{\beta,\gamma_i}$ and $\mu_i = \mu_{\beta,\gamma_i}$, (46) becomes

$$\begin{aligned}&(\psi_{F_i}(y) - \psi_{F_i}(y')) . (y - y') \\ &\quad \leq -\mu_i(\|y\|_2 \wedge \|y'\|_2)\|y - y'\|_2^2,\end{aligned} \tag{55}$$

for any $i = 1,\dots,N$ and $y, y' \in F_i(C_i) \cap B(0,A_i)^c$, since $F_i(C_i)$ is convex by construction and included in $F_i(C_{\eta(\beta,\gamma_i),\beta,\gamma_i})$.

We define the flow ϕ_i^K associated to $b^K_{F_i}$ on C_i:

$$\phi_i^K(x_0,0) = x_0, \qquad \frac{\partial}{\partial t}\phi_i^K(x_0,t) = b^K_{F_i}(\phi_i^K(x_0,t))$$

for $x_0 \in C_i$ and $t < T_i^K(x_0)$, where $T_i^K(x_0)$ is the maximal time when this flow is well defined and belongs to C_i. We consider the image $\widetilde{\phi}_i^K(y_0,t) = F_i(\phi_i^K(F_i^{-1}(y_0),t))$ of this flow. It satisfies

$$\widetilde{\phi}_i^K(y_0,t) = y_0, \qquad \frac{\partial}{\partial t}\widetilde{\phi}_i^K(y_0,t) = \psi^K_{F_i}(\widetilde{\phi}_i^K(y_0,t))$$

for any $y_0 \in F_i(C_i)$ and $t < T_i^K(F_i^{-1}(y_0))$. Similarly, writing $\widetilde{\phi}_i(y_0,t) = F_i(\phi(F_i^{-1}(y_0),t))$, we have

$$\widetilde{\phi}_i(y_0,t) = y_0, \qquad \frac{\partial}{\partial t}\widetilde{\phi}_i(y_0,t) = \psi_{F_i}(\widetilde{\phi}_i(y_0,t))$$

for any $y_0 \in F_i(C_i)$ and $t < T_{C_i}(F_i^{-1}(y_0))$.

Combining (55) with Lemma 5.8 (v) and observing that $\|y\|_2 \wedge \|y'\|_2 \geq \|y\|_2(1-\varepsilon_0/A)$ when $y' \in B(y,\varepsilon_0)$ and $\|y\| \geq A$, the assumptions of Lemma 6.1 in the Appendix are met for ψ_{F_i} and $\psi^K_{F_i}$ on the domain $F_i(C_i \cap \mathfrak{D}_{\alpha/2})$. We apply this lemma with $\eta = Kr_K$. It ensures that for any $T>0$ and any sequence $r_K \to 0$,

$$\sup_{\substack{y_0\in F_i(C_i\cap\mathfrak{D}_\alpha), y_1\in\overline{B}(y_0,r_K)\\ t<\widetilde{T}_{i,\varepsilon_0}(y_0)\wedge T}} \|\widetilde{\phi}_i^K(y_1,t) - \widetilde{\phi}_i(y_0,t)\|_2 \stackrel{K\to\infty}{\longrightarrow} 0,$$

where $\widetilde{T}_{i,\varepsilon}(y_0) = \sup\{t \in (0, T_{C_i}(F_i^{-1}(y_0))) : \forall s \leq t, \overline{B}(\widetilde{\phi}_i(y_0,s),\varepsilon) \subset F_i(C_i \cap \mathfrak{D}_{\alpha/2})\}$. Then

$$\sup_{\substack{x_0\in C_i\cap\mathfrak{D}_\alpha, x_1\in\overline{B}_{d_\beta}(x_0,r_K)\\ t<T_{i,\varepsilon_0}(x_0)\wedge T}} d_\beta(\phi_i^K(x_1,t),\phi(x_0,t)) \stackrel{K\to\infty}{\longrightarrow} 0, \tag{56}$$

where $T_{i,\varepsilon}(x_0) = \sup\{t \in (0, T_{C_i}(x_0)) : \forall s \leq t, \overline{B}_{d_\beta}(\phi(x_0,t),\varepsilon) \subset C_i \cap \mathfrak{D}_{\alpha/2}\}$. From (ii), we also know that $\overline{B}_{d_\beta}(\phi(x_0,t),\varepsilon_0) \subset C_{n_k(x_0)} \cap \mathfrak{D}_{\alpha/2}$ for $t \in [t_k(x_0), t_{k+1}(x_0) \wedge T_{\mathfrak{D}_\alpha}(x_0))$, so

$$\sup_{\substack{x_0\in\mathfrak{D}_\alpha\\ x_1\in\overline{B}_{d_\beta}(\phi(x_0,t_k(x_0)),r_K)\\ t\in[t_k(x_0),t_{k+1}(x_0)\wedge T_{\mathfrak{D}_\alpha}(x_0)\wedge T)}} d_\beta(\phi^K_{n_k(x_0)}(x_1,t-t_k(x_0)),\phi(x_0,t)) \stackrel{K\to\infty}{\longrightarrow} 0.$$

Then for K large enough, we construct the continuous flow ϕ^K inductively for $k=0,\dots,\kappa-1$ such that for any $x_0 \in \mathfrak{D}_\alpha$,

$$\phi^K(x_0,0) = x_0, \qquad \phi^K(x_0,t) = \phi^K_{n_k(x_0)}(\phi^K(x_0,t_k(x_0)), t-t_k(x_0))$$

for any $t \in [t_k(x_0), t_{k+1}(x_0) \wedge T_{\mathfrak{D}_\alpha}(x_0))$. This construction satisfies

$$\sup_{\substack{x_0\in\mathfrak{D}_\alpha,\\ t\in[t_k(x_0),t_{k+1}(x_0)\wedge T_{\mathfrak{D}_\alpha}(x_0)\wedge T)}} d_\beta(\phi^K(x_0,t),\phi(x_0,t)) \stackrel{K\to\infty}{\longrightarrow} 0$$

and for K large enough, for any $t \in [t_k(x_0), t_{k+1}(x_0) \wedge T_{\mathfrak{D}_\alpha}(x_0))$,

$$\overline{B}_{d_\beta}(\phi^K(x_0,t),\varepsilon_0/2) \subset C_{n_k(x_0)} \cap \mathfrak{D}_{\alpha/2}.$$

Adding that ϕ_i^K is the flow associated with the vector field $b^K_{F_i}$ completes the proof. $\square$

5.4. *Proofs of Theorem* 5.1, *Corollary* 5.2 *and Theorem* 5.3. We can now prove Theorem 5.1 for the diffusion X defined by (32) using the results of Section 3. Here, $E=[0,\infty)^2$, $d=2$, $q=0$ ($H=G=0$), $\sigma_j^{(i)}=0$ if $j\neq i$ and

$$\sigma_1^{(1)}(x)=\sigma_1\sqrt{x_1},\qquad \sigma_2^{(2)}(x)=\sigma_2\sqrt{x_2}.$$

Moreover, $b_{F_{\beta,\gamma}}=b$ is given by (41), $\psi_{F_{\beta,\gamma}}=(J_{F_{\beta,\gamma}}b_{F_{\beta,\gamma}})\circ F_{\beta,\gamma}^{-1}$ and

$$\widetilde{b}_{F_{\beta,\gamma}}(x)=\frac{1}{2}\sum_{i=1}^{2}\frac{\partial^2 F_{\beta,\gamma}}{\partial^2 x_i}(x)\sigma_i^{(i)}(x)^2=\frac{1}{2}\beta(\beta-1)\begin{pmatrix}\sigma_1^2x_1^{\beta-1}\\ \gamma\sigma_2^2x_2^{\beta-1}\end{pmatrix}\tag{57}$$

and

$$V_{F_{\beta,\gamma}}(x)=\sum_{i=1}^{2}\left(\frac{\partial F_{\beta,\gamma}}{\partial x_i}(x)\sigma_i^{(i)}(x)\right)^2=\beta^2\begin{pmatrix}\sigma_1^2x_1^{2\beta-1}\\ (\gamma\sigma_2)^2x_2^{2\beta-1}\end{pmatrix}.\tag{58}$$

PROOF OF THEOREM 5.1. Let $\beta\in(1/2,1)$ close enough to 1 so that $q_\beta=4ab(1+\beta)^2+4cd(\beta^2-1)>0$. Using Lemma 5.10(ii), we can check Assumptions 3.3 and 3.4 of Section 3 with $D=\mathcal{D}_\alpha$, $D_i=C_i\cap\mathcal{D}_{\alpha/2}$, $O_i=\mathcal{D}_{\alpha/4}$ ($i=1,\dots,N$), $d=d_\beta$ and ϕ defined by (31). Moreover, writing $F_i=F_{\beta,\gamma_i}$ for convenience, Lemma 5.10(i) ensures that ψ_{F_i} is $\overline{\tau}$ nonexpansive on $F_i(D_i)$. We recall also that $T_\varepsilon^{\bar{\tau},0}=\infty$ and apply then Theorem 3.5 to the diffusion X and get for any ε small enough, for any $T<1$ and $x_0\in\mathcal{D}_\alpha$,

$$\begin{aligned}\mathbb{P}_{x_0}\Big(&\sup_{t\le T\wedge T_{\mathcal{D}_\alpha}(x_0)}d_\beta\big(X_t,\phi(x_0,t)\big)\ge\varepsilon\Big)\\&\le C\sum_{k=0}^{\kappa-1}\int_{t_k(x_0)\wedge T}^{t_{k+1}(x_0)\wedge T}\overline{V}_{d_\beta,\varepsilon}(F_{n_k(x_0)},x_0,t)\,dt\end{aligned}$$

for some positive constant C, by a.s. continuity of $d_\beta(X_t,\phi(x_0,t))$ at time $T\wedge T_{\mathcal{D}_\alpha}(x_0)$. We need now to control $\overline{V}$. First, we recall from Lemma 5.10(ii) that $\overline{B}_{d_\beta}(\phi(x_0,t),\varepsilon_0)\subset\mathcal{D}_{\alpha/2}$ for $x_0\in\mathcal{D}_\alpha$ and $t<T_{\mathcal{D}_\alpha}(x_0)$. Then we use (57) to see that $\widetilde{b}_{F_i}$ is bounded on $\mathcal{D}_{\alpha/2}$, so

$$c_i'(\varepsilon):=\sup_{\substack{x_0\in\mathcal{D}_\alpha,t<T_{\mathcal{D}_\alpha}(x_0)\\ d_\beta(x,\phi(x_0,t))\le\varepsilon}}\|\widetilde{b}_{F_i}(x)\|_1<\infty$$

for $\varepsilon\le\varepsilon_0$. Moreover, plugging Lemma 5.7(i) into (58) to control V_{F_i}, there exists $c_i''(\varepsilon)>0$ such that for any $x_0\in\mathcal{D}_\alpha$ and $t<T_{\mathcal{D}_\alpha}(x_0)$,

$$\begin{aligned}\overline{V}_{d_\beta,\varepsilon}(F_i,x_0,t)&=\sup_{\substack{x\in[0,\infty)^2\\ d_\beta(x,\phi(x_0,t))\le\varepsilon}}\{\varepsilon^{-2}\|V_{F_i}(x)\|_1+\varepsilon^{-1}\|\widetilde{b}_{F_i}(x)\|_1\}\\&\le\varepsilon^{-2}\frac{c_i''(\varepsilon)}{t^{2\beta-1}}+\varepsilon^{-1}c_i'(\varepsilon).\end{aligned}$$

Adding that $\int_0^{\cdot}(\varepsilon^{-2}\frac{c_i''(\varepsilon)}{t^{2\beta-1}} + \varepsilon^{-1}c_i'(\varepsilon))\,dt < \infty$ for $\beta < 1$, we get

$$\lim_{T\downarrow 0}\sup_{x_0\in\mathcal{D}_\alpha}\mathbb{P}_{x_0}\Bigl(\sup_{t\le T\wedge T_{\mathcal{D}_\alpha}(x_0)} d_\beta\bigl(X_t,\phi(x_0,t)\bigr)\ge\varepsilon\Bigr)=0$$

for ε small enough. This completes the proof for $\beta < 1$ close enough to 1, which is enough to conclude, since $d_{\beta'}$ is dominated by d_β on $\mathcal{D}_\alpha$ if $\beta' \le \beta$. $\square$

We can now describe the coming down from infinity of the competitive diffusion X.

PROOF OF COROLLARY 5.2. Let us deal with (i), so $\ell = x_\infty \in (0,\infty)^2$ and we fix $x_0 \in (0,\infty)^2$ and $\eta \in (0,1)$. First, plugging Lemma 5.7(ii) and (iii) in the inequality

$$\bigl\|tx_t(r)-x_\infty\bigr\|_2 \le \bigl|\bigl\|tx_t(r)\bigr\|_1 - \|x_\infty\|_1\bigr| + \min\bigl(\bigl\|tx_t(r)\bigr\|_2, \|x_\infty\|_2\bigr)\bigl|\sin(\widehat{x_t,x_\infty})\bigr|$$

ensures that

$$\lim_{T\to 0}\limsup_{r\to\infty}\sup_{\eta T\le t\le T}\bigl\|tx_t(r)-x_\infty\bigr\|_2 = 0. \tag{59}$$

Moreover, for any $\varepsilon > 0$, Lemma 5.9(ii) ensures that

$$\liminf_{r\to\infty} T_{\mathcal{D}_\varepsilon}(rx_0) > 0,$$

where we recall definition (35) for the exit time $T_{\mathcal{D}_\varepsilon}(.)$. Writing again $x_t(r) = \phi(rx_0,t)$ for convenience, Theorem 5.1 ensures that for any $\beta \in (0,1)$,

$$\lim_{T\to 0}\limsup_{r\to\infty}\mathbb{P}_{rx_0}\Bigl(\sup_{t\le T} d_\beta\bigl(X_t,x_t(r)\bigr)\ge\varepsilon\Bigr)=0.$$

Then, using that $d_\beta(tx,ty) = t^\beta d_\beta(x,y)$ and $\|tx_t(r)\|_1$ is bounded for $t \le 1$ and $r > 0$ by Lemma 5.7(i), the last limit yields

$$\lim_{T\to 0}\limsup_{r\to\infty}\mathbb{P}_{rx_0}\Bigl(\sup_{t\le T}\bigl\|tX_t - tx_t(r)\bigr\|_2\ge\varepsilon\Bigr)=0, \tag{60}$$

for any $\varepsilon > 0$, since the Euclidean distance is uniformly continuous from the bounded sets of $[0,\infty)^2$ endowed with d_β to $\mathbb{R}^+$ endowed with the absolute value.

Combining (59) and (60) ensures that for any $\varepsilon > 0$,

$$\lim_{T\to 0}\limsup_{r\to\infty}\mathbb{P}_{rx_0}\Bigl(\sup_{\eta T\le t\le T}\|tX_t - x_\infty\|_2\ge\varepsilon\Bigr)=0.$$

This proves the first part of (i). The second part of (i) (resp., the proof of (iv)) is obtained similarly just by noting that $t_-(rx_0,x_\infty,\varepsilon) = 0$ (resp., $t_-(rx_0,\widehat{x_0},\varepsilon) = 0$) if x_0 is collinear to x_∞.

For the cases (ii)–(iii), we know from Lemma 5.7 that the dynamical system is going to the boundary of $(0,\infty)^2$ in short time. Let us deal with the case

$$\ell = (1/a, 0)$$

and the case $\ell = (0, 1/b)$ would be handled similarly. We fix $x_0 \in (0,\infty)^2$, $T_0 > 0$, $\varepsilon \in (0,1]$, $\eta > 0$ and $\beta \in (0,1)$. By Theorem 5.1, there exists $T \le T_0$ such that for r large enough

$$\mathbb{P}_{rx_0}\Big(\sup_{t \le T \wedge T_{\mathcal{D}_\varepsilon}(rx_0)} d_\beta\big(X_t, x_t(r)\big) \ge \varepsilon\Big) \le \eta.$$

By Lemma 5.9(iii), for r large enough, we have $T_{\mathcal{D}_\varepsilon}(rx_0) = \inf\{t \ge 0 : x_t^{(2)}(r) \le \varepsilon\} \le T$. Thus,

$$\mathbb{P}_{rx_0}\big(d_\beta\big(X_{T_{\mathcal{D}_\varepsilon}(rx_0)}, x_{T_{\mathcal{D}_\varepsilon}(rx_0)}(r)\big) \ge \varepsilon\big) \le \eta \quad \text{and} \quad x^{(2)}_{T_{\mathcal{D}_\varepsilon}(rx_0)}(r) = \varepsilon.$$

Fix now $c \ge 1$ such that $c^\beta \ge 2$. We get

$$\begin{aligned}\mathbb{P}_{rx_0}\big(X^{(2)}_{T_{\mathcal{D}_\varepsilon}(rx_0)} \ge c\varepsilon\big) &= \mathbb{P}_{rx_0}\big(\big(X^{(2)}_{T_{\mathcal{D}_\varepsilon}(rx_0)}\big)^\beta - \varepsilon^\beta \ge \big(c^\beta - 1\big)\varepsilon^\beta\big) \\ &\le \mathbb{P}_{rx_0}\big(d_\beta\big(X_{T_{\mathcal{D}_\varepsilon}(rx_0)}, x_{T_{\mathcal{D}_\varepsilon}(rx_0)}(r)\big) \ge \varepsilon\big) \le \eta,\end{aligned}$$

since $\varepsilon^\beta \ge \varepsilon$. By the Markov property and the fact that the boundaries of $[0,\infty)^2$ are absorbing, we obtain for r large enough

$$\begin{aligned}\mathbb{P}_{rx_0}\big(X^{(2)}_{2T_0} = 0\big) &\ge \mathbb{P}\big(X^{(2)}_{T_{\mathcal{D}_\varepsilon}(rx_0)} \le c\varepsilon, \exists t \in \big[T_{\mathcal{D}_\varepsilon}(rx_0), T_{\mathcal{D}_\varepsilon}(rx_0) + T_0\big] : X^{(2)}_t = 0\big) \\ &\ge (1-\eta)p(c\varepsilon),\end{aligned}$$

where, using that $X^{(2)}$ is stochastically smaller than a one-dimensional Feller diffusion Z with diffusion coefficient $\sigma_2 \neq 0$,

$$p(z) = \mathbb{P}_z(Z_{T_0} = 0).$$

Thus, $\lim_{z \downarrow 0+} p(z) = 1$ and letting $\varepsilon \to 0$ in the previous inequality yields

$$\liminf_{r \to \infty} \mathbb{P}_{rx_0}\big(X^{(2)}_{2T_0} = 0\big) \ge 1 - \eta.$$

Letting $\eta \to 0$ completes the proof of (ii)–(iii). $\square$

Recalling the notation of Section 5.3.3, we finally prove Theorem 5.3.

PROOF OF THEOREM 5.3. Let $T_0 > 0$ and $\beta \in (0, 1/2)$ and $\alpha_0 > \alpha > 0$. We first observe that assumption (40) ensures that $q_\beta = 4ab(1+\beta)^2 + 4cd(\beta^2 - 1) > 0$. Using Lemma 5.10(iii), Assumptions 3.3 and 3.4 are satisfied for the process X^K, with the domains $D = \mathfrak{D}_\alpha$ and $D_i = C_i \cap \mathfrak{D}_{\alpha/2}$, the continuous flow ϕ^K, the transformations $F_i = F_{\beta,\gamma_i}$, the times $t_k(.) \wedge T_{\mathfrak{D}_\alpha}(.)$ and the integers $n_k(.)$. Recalling that C_i is convex and $C_i \subset C_{\eta(\beta,\gamma_i),\beta,\gamma_i} \subset D_{\beta,\gamma_i}$, we know from

Lemma 5.8(iv) that $\psi^K_{F_i}$ is $(c_i, c_i/K)$ nonexpansive on $F_i(D_i)$ for some constant $c_i \geq 0$. Thus, we apply Theorem 3.5 and there exists $\underline{\varepsilon} = \underline{\varepsilon}^K$ which does not depend on K so that for any $K \geq 1$, $\varepsilon \in (0, \underline{\varepsilon}]$, $T < \min(T_\varepsilon^{c_i, c_i/K} : i = 1, \dots, N) \wedge (T_0 + 1)$ and $x_0 \in \mathfrak{D}_\alpha$,

$$\begin{aligned}\mathbb{P}_{x_0}\Bigl(&\sup_{t < T \wedge T_{\mathfrak{D}_\alpha}(x_0)} d_\beta\bigl(X_t^K, \phi^K(x_0, t)\bigr) \geq \varepsilon\Bigr)\\ &\leq C \sum_{k=0}^{\kappa-1} \int_{t_k(x_0)\wedge T}^{t_{k+1}(x_0)\wedge T} \overline{V}^K_{d_\beta,\varepsilon}(F_{n_k(x_0)}, x_0, t)\, dt,\end{aligned}$$

where C is positive constant which does not depend on K, x_0 and

$$\overline{V}^K_{d_\beta,\varepsilon}(F_i, x_0, t) = \sup\bigl\{\varepsilon^{-2}\bigl\|V^K_{F_i}(x)\bigr\|_1 : x \in [0,\infty)^2, d_\beta\bigl(x, \phi^K(x_0,t)\bigr) \leq \varepsilon\bigr\}.$$

Moreover, for K large enough, we have $4c_i T_0 \exp(2L_i T_0) < K\varepsilon$, so that $T_0 < T_\varepsilon^{c_i, c_i/K}$ for $i = 1, \dots, N$ and

$$\begin{aligned}\mathbb{P}_{x_0}\Bigl(&\sup_{t < T_0 \wedge T_{\mathfrak{D}_\alpha}(x_0)} d_\beta\bigl(X_t^K, \phi^K(x_0, t)\bigr) \geq \varepsilon\Bigr)\\ &\leq C \sum_{k=0}^{\kappa-1} \int_{t_k(x_0)\wedge T_0}^{t_{k+1}(x_0)\wedge T_0} \overline{V}^K_{d_\beta,\varepsilon}(F_{n_k(x_0)}, x_0, t)\, dt.\end{aligned}\tag{61}$$

Adding that

$$\begin{aligned}V^K_{F_{\beta,\gamma}}(x) &= \begin{pmatrix} V^{K,(1)}_{F_{\beta,\gamma}}(x) \\ V^{K,(2)}_{F_{\beta,\gamma}}(x)) \end{pmatrix}\\ &= \int_0^\infty \bigl(F_{\beta,\gamma}\bigl(x + H^K(x,z)\bigr) - F_{\beta,\gamma}\bigl(x + H^K(x,z)\bigr)\bigr)^2\, dz\end{aligned}$$

and recalling (39) and writing $\gamma_1 = 1$, $\gamma_2 = \gamma$, we have for $i \in \{1,2\}$ and $x \in \mathcal{D}_\alpha$,

$$\begin{aligned}V^{K,(i)}_{F_{\beta,\gamma}}(x) &= \gamma_i\bigl[\lambda_i^K(Kx)\bigl((x_i + 1/K)^\beta - x_i^\beta\bigr)^2\\ &\quad + \mu_i^K(Kx)\bigl((x_i - 1/K)^\beta - x_i^\beta\bigr)^2\bigr]\\ &\leq \frac{\text{cst}}{K} x_i^{2\beta-2} x_i(1 + x_1 + x_2)\end{aligned}$$

for some cst > 0, which depends on β, γ, α and can now change from line to line.

Then for $x \in \mathfrak{D}_\alpha$,

$$\bigl\|V^K_{F_{\beta,\gamma}}(x)\bigr\|_1 \leq \frac{\text{cst}}{K}\bigl(x_1^{2\beta}(1 + x_2/x_1) + x_2^{2\beta}(1 + x_1/x_2)\bigr) \leq \frac{\text{cst}}{K}\bigl(x_1^{2\beta} + x_2^{2\beta}\bigr).$$

Moreover from Lemma 5.10(iii) that for K large enough, $\overline{B}_{d_\beta}(\phi^K(x_0,t), \varepsilon_0/2) \subset \mathfrak{D}_{\alpha/2}$ for any $x_0 \in \mathfrak{D}_\alpha$ and $t < t_\kappa(x_0)$. Combining the last part of Lemma 5.10(iii) and Lemma 5.7(i), $\|\phi^K(x_0,t)\|_1 \le c_T/t$ for $t \in [0,T]$. We obtain that for any $x_0 \in \mathfrak{D}_\alpha$ and $\varepsilon \le \varepsilon_0/2$,

$$\int_{t_k(x_0)\wedge T_0}^{t_{k+1}(x_0)\wedge T_0} \overline{V}^K_{d_\beta,\varepsilon}(F_{n_k(x_0)}, x_0, t)\,dt \le \varepsilon^{-2}\frac{\text{cst}}{K}\int_{t_k(x_0)\wedge T_0}^{t_{k+1}(x_0)\wedge T_0} t^{-2\beta}\,dt$$

for $k \in \{0,\dots,\kappa-1\}$. Using the fact that $\int_0^{\cdot} t^{-2\beta}\,dt < \infty$ for $\beta < 1/2$, we get

$$\sum_{k=0}^{\kappa-1}\int_{t_k(x_0)\wedge T_0}^{t_{k+1}(x_0)\wedge T_0} \overline{V}_{d,\varepsilon}(F_{n_k(x_0)}, x_0, t)\,dt \le \varepsilon^{-2}\frac{\text{cst}}{K}.$$

Recall now from Lemma 5.9(iv) that under Assumption (40), we can choose $\alpha \in (0,\alpha_0)$ small enough so that $T_{\mathfrak{D}_\alpha}(x_0) \ge T_0$ for any $x_0 \in \mathfrak{D}_{\alpha_0}$. Using also (54), (61) becomes

$$\sup_{x_0\in\mathfrak{D}_{\alpha_0}} \mathbb{P}_{x_0}\Big(\sup_{t<T_0} d_\beta\big(X^K_t, \phi(x_0,t)\big) \ge \varepsilon\Big) \le \varepsilon^{-2}\frac{C}{K},$$

for $\varepsilon \le \underline{\varepsilon} \wedge \varepsilon_0/2$ and K large enough, where C is a positive constant which does not depend on K. $\square$

APPENDIX

We need the following estimates. We assume that ψ and ψ^K are locally Lipschitz vectors fields on the closure $\overline{D}$ of an open domain $D \subset \mathbb{R}^{\mathrm{d}}$ and their respective flows on D are ϕ and ϕ^K. We assume that there are well defined and belongs to D, respectively, until a maximal time T_D and T_D^K. We write again $T_{D,\varepsilon}(x_0) = \sup\{t \ge 0 : \forall s < T(x_0), \overline{B}(\phi(x_0,s),\varepsilon) \subset D\}$.

LEMMA 6.1. *We assume that there exist $A \ge 1$, $c, \mu > 0$ and $\varepsilon \in (0,1]$ such that*

$$\big(\psi(x)-\psi(y)\big).(x-y) \le -\mu\|x\|_2\|x-y\|_2^2 \tag{62}$$

for any $x \in D \cap B(0,A)^c$ and $y \in \overline{B}(x,\varepsilon)$ and

$$\big\|\psi(x)-\psi^K(x)\big\|_2 \le c\frac{1+\|x\|_2}{K} \tag{63}$$

for any $x \in D$ and $K \ge 1$. Then, writing $M = 3c/\mu$, there exists $L \ge 0$ such that for all $T \ge 0$, $\eta > 0$, $K \ge 2\max(M,\eta)\exp((L+1/M)T)/\varepsilon$, $x_0 \in D$ and $x_1 \in \overline{B}(x_0,\eta/K)$, we have $T_D^K(x_1) \ge T_{D,\varepsilon}(x_0)$ and

$$\sup_{t<T_{D,\varepsilon}(x_0)\wedge T} \big\|\phi(x_0,t)-\phi^K(x_1,t)\big\|_2 \le \frac{\max(M,\eta)\exp((L+M)T)}{K}.$$

PROOF. Let $T > 0$ and $K \geq 2\max(M,\eta)\exp((L+1/M)T)/\varepsilon$, so that

$$\max(M,\eta)/K \leq \max(M,\eta)e^{(L+1/M)T}/K \leq \varepsilon/2.$$

Write

$$x_t = \phi(x_0,t), \qquad x_t^K = \phi^K(x_1,t), \qquad T^K = T_D(x_0) \wedge T_D^K(x_1)$$

for convenience and consider the time

$$t_1^K = \inf\{t \in [0,T^K) : \|x_t - x_t^K\|_2^2 \geq M/K\} \in (0,\infty].$$

Let us assume that $t_1^K < T_{D,\varepsilon}(x_0) \wedge T \wedge T^K$ and set

$$t_2 = \inf\{t \in (t_1^K, T^K) : \|x_t - x_t^K\|_2^2 \geq \varepsilon \text{ or } \|x_t - x_t^K\|_2^2 < M/K\}.$$

We show now that for any $t \in [t_1^K, t_2^K \wedge T^K)$, we have

$$\begin{aligned}\frac{d}{dt}\|x_t - x_t^K\|_2^2 &= 2(\psi(x_t) - \psi^K(x_t^K)).(x_t - x_t^K)\\ &\leq 2(L+1/M)\|x_t - x_t^K\|_2^2 \qquad (64)\end{aligned}$$

to get from the Gronwall inequality and $\|x_{t_1^K} - x_{t_1^K}^K\|_2 \leq \max(M,\eta)/K$ that

$$\|x_t - x_t^K\|_2 \leq \max(M,\eta)\exp((L+1/M)T)/K.$$

This will be enough to prove the lemma since the right-hand side is smaller than $\varepsilon/2$.

First, using that on the closure of $D \cap B(0, A+1)$, ψ is Lipschitz and that $K\|\psi^K(.) - \psi(.)\|_2$ is bounded on $D \cap B(0, A+1)$ by (63), there exists $L > 0$ such that

$$\|\psi(x) - \psi^K(y)\|_2 \leq L(\|x-y\|_2 + 1/K),$$

for any $x, y \in D \cap B(0, A+1)$. Then, using the Cauchy–Schwarz inequality, for any $t \in [t_1^K, t_2^K \wedge T^K)$ such that $x_t \in B(0, A)$,

$$\begin{aligned}\frac{d}{dt}\|x_t - x_t^K\|_2^2 &\leq 2\|x_t - x_t^K\|_2\|\psi(x_t) - \psi^K(x_t^K)\|_2\\ &\leq 2L\|x_t - x_t^K\|_2^2 + \frac{2}{K}\|x_t - x_t^K\|_2\\ &\leq 2(L+1/M)\|x_t - x_t^K\|_2^2\end{aligned}$$

since $\|x_t - x_t^K\|_2 \geq M/K$ for $t \leq t_2^K$. This proves (64) when $x_t \in B(0, A)$.

To conclude, we consider $t \in [t_1^K, t_2^K \wedge T^K]$ such that $x_t \in B(0, A)^c$. Then (62) and (63) and the Cauchy–Schwarz inequality give

$$\begin{aligned}\frac{d}{dt}\|x_t - x_t^K\|_2^2 &= 2(\psi(x_t) - \psi(x_t^K)).(x_t - x_t^K) \\ &\quad + 2(\psi(x_t^K) - \psi^K(x_t^K)).(x_t - x_t^K) \\ &\le 2\left(-\mu\|x_t\|_2\|x_t - x_t^K\|_2 + c\frac{1 + \|x_t^K\|_2}{K}\right)\|x_t - x_t^K\|_2.\end{aligned}$$

Moreover, $\|x_t\|_2 \ge A \ge 1$ and $x_t^K \in \overline{B}(x_t, \varepsilon)$, so

$$1 + \|x_t^K\|_2 \le 1 + \|x_t\|_2 + \|x_t^K - x_t\|_2 \le 3\|x_t\|_2,$$

and adding that $\|x_t - x_t^K\|_2 \ge M/K = 3c/(K\mu)$ since $t \le t_2^K$, we get

$$\frac{d}{dt}\|x_t - x_t^K\|_2^2 \le 0.$$

This completes the proof of (64), and thus of the lemma. $\square$

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