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ANNEALED LIMIT THEOREMS FOR THE ISING MODEL ON RANDOM REGULAR GRAPHS

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In a recent paper, Giardinà et al. [*ALEA Lat. Am. J. Probab. Math. Stat.* **13** (2016) 121–161] have proved a law of large number and a central limit theorem with respect to the annealed measure for the magnetization of the Ising model on some random graphs, including the random 2-regular graph. In this paper, we present a new proof of their results which applies to all random regular graphs. In addition, we prove the existence of annealed pressure in the case of configuration model random graphs.

1. Introduction.

1.1. *Motivation*. The ferromagnetic Ising model is one of the most well-known models in statistical physics describing cooperative behaviors. In this model, each vertex in a graph is assigned by one spin that can be one of two states, $+1$, or -1, while the configuration probability is given by the Boltzmann–Gibbs measure. These spins cooperatively interact with each other toward alignment: spins of vertices connected by edges tend to be at the same state.

The Ising model on regular lattices has been studied carefully by many authors, resulting in numerous beautiful results; see, for example, [18, 19]. Recently, a lot of attention has been draw into investigating this model on class of random graphs [1, 2, 5, 6, 8, 10–13, 16, 17, 20, 21] as a prototype of interactions in real networks. In the new framework, there are two sources of randomness that are the law of spin configurations and the law of random graphs. As other dynamics on random environments, the way to combine these sources of randomness leads to studies in *annealed* and *quenched* settings.

In the quenched setting, a graph sample is fixed then the configuration probability is defined according to this realization of the graph. This setting is particularly relevant to describe interactions in physical environments in which the evolution of environment (random graphs) occurs extremely slow compared with the one of interactions. Since last two decades, the quenched state has been developed in

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great generality. Indeed, most of results in this setting holds for large class of random graphs that are locally tree-like. In particular, the convergence of thermodynamic quantities are proved by Dembo and Montanari in [6], the critical exponents are derived by Dommers et al. in [12]. The weak limit of quenched measure, the metastability of Glauber dynamics, and the scaling limits for magnetization have been shown in [20], [10] and [16], respectively; see also [2, 7] for other results.

On the other hand, in the annealed setting, the configuration probability is defined by taking into account the information of all graph samples. From the point of view of dynamics in random environment, in the annealed state, the environment (random graphs) evolves much faster than dynamics (spins interaction). Hence, the environment seen from spins comprise all possible realizations of the random graph. The annealed state is relevant to describe dynamics on complex networks that rapidly change in time, such as networks of social acquaintances or brain; see [3, 17] for more discussion. In contrast to the complete picture of quenched state, there are only several results on the annealed one. The main reason is that in the annealed setting, it is not enough to control the behavior of model on typical samples of graphs. There are probably some rare samples that give a nontrivial contribution. Studying them often leads to a very challenging topic, the large deviation properties of random graphs. Due to this difficulty, the annealed Ising model has been studied on only certain classes of graphs. In particular, the authors of [11, 17] consider rank-one inhomogeneous random graphs, random 2-regular graphs and configuration model with degrees 1 and 2. In this paper, we will study the case of random regular graphs with a degree larger than 2.

1.2. *Model definitions.* We give in this part, the definitions of random regular graphs, the annealed Ising model and thermodynamics quantities.

1.2.1. *Configuration model random graph.* Let us give a definition following [23] of the configuration model with prescribed degree sequence. For each n, let $V_n = \{v_1, \dots, v_n\}$ be the vertex set of a graph G_n and let $\mathbf{D} = (D_i)_{i \leq n}$ be a sequence of integers. We construct the edge set of G_n as follows. First, we assume that $\ell_n = \sum_1^n D_i$ is even (if not increase one of the D_i's by 1, which makes no difference in what follows). For each vertex v_i, start with D_i half-edges incident to v_i. Then we denote by $\mathcal{H}$ the set of all the half-edges. Select one of them h_1 arbitrarily and then choose a half-edge h_2 uniformly from $\mathcal{H} \setminus \{h_1\}$, and match h_1 and h_2 to form an edge. Next, select arbitrarily another half-edge h_3 from $\mathcal{H} \setminus \{h_1, h_2\}$ and match it to another h_4 uniformly chosen from $\mathcal{H} \setminus \{h_1, h_2, h_3\}$. Then continue this procedure until there are no more half-edges. We finally get a random multigraph that may have self-loops and multiple edges between vertices satisfying the degree of v_i is D_i for all i. We denote the obtained graph by $G_n(\mathbf{D})$.

For $d \geq 1$, if $D_i = d$ for all $i = 1, \dots, n$ we call $G_n(\mathbf{D})$ the *random d-regular graph*, and denote it by $G_{n,d}$.

1.2.2. *Annealed measure*. The annealed Ising model has been defined in [17] as follows. Let G_n be a random multigraph with n vertices $v_1, \dots, v_n$. Let $\Omega_n = \{-1, 1\}^n$ be the space of spin configurations. For any $\sigma = (\sigma_1, \dots, \sigma_n)$ in Ω_n, its energy is given by the Hamiltonian function

$$H(\sigma) = -\beta \sum_{i \le j} k_{i,j} \sigma_i \sigma_j - B \sum_{i=1}^{n} \sigma_i,$$

where $k_{i,j}$ is the number of edges between v_i and v_j, where $\beta \ge 0$ is the inverse temperature and $B \in \mathbb{R}$ is the uniform external magnetic field.

Then the configuration probability is given by what they call the *annealed measure*:

$$\mu_n(\sigma) = \frac{\mathbb{E}(\exp(-H(\sigma)))}{\mathbb{E}(Z_n(\beta, B))},$$

where $\mathbb{E}$ denotes the expectation with respect to the random graph, and $Z_n(\beta, B)$ is the partition function:

$$Z_n(\beta, B) = \sum_{\sigma \in \Omega_n} \exp\bigl(-H(\sigma)\bigr).$$

1.2.3. *Thermodynamic quantities*. In [17], the authors also define the thermodynamic quantities in finite volume of the annealed model:

(i) The annealed pressure is given by

$$\psi_n(\beta, B) = \frac{1}{n} \log \mathbb{E}\bigl(Z_n(\beta, B)\bigr).$$

(ii) The annealed magnetization is given by

$$M_n(\beta, B) = \mathbb{E}_{\mu_n}\left(\frac{S_n}{n}\right),$$

where S_n is the total spin, that is, $S_n = \sigma_1 + \cdots + \sigma_n$, and $\mathbb{E}_{\mu_n}$ is the expectation on Ω_n w.r.t. μ_n. After a simple computation, we get

$$M_n(\beta, B) = \frac{\partial}{\partial B} \psi_n(\beta, B).$$

(iii) The annealed susceptibility is given by

$$\chi_n(\beta, B) = \operatorname{Var}_{\mu_n}\left(\frac{S_n}{\sqrt{n}}\right),$$

with $\operatorname{Var}_{\mu_n}$ the variance on Ω_n w.r.t. μ_n. We also can prove that

$$\chi_n(\beta, B) = \frac{\partial}{\partial B} M_n(\beta, B) = \frac{\partial^2}{\partial B^2} \psi_n(\beta, B).$$

When the sequence $(M_n(\beta, B))_n$ converges to a limit, say $\mathcal{M}(\beta, B)$, we define the spontaneous magnetization as $\mathcal{M}(\beta, 0^+) = \lim_{B\to 0^+} \mathcal{M}(\beta, B)$. Then the critical inverse temperature is defined as

$$\beta_c = \inf\{\beta > 0 : \mathcal{M}(\beta, 0^+) > 0\}.$$

Finally, the uniqueness region of the existence of the limit magnetization is defined as

$$\mathcal{U} = \{(\beta, B) : \beta \geq 0, B \neq 0 \text{ or } 0 < \beta < \beta_c, B = 0\}.$$

1.3. *Main results*. In [17], the authors study this Ising model on the rank-one inhomogeneous random graph, the random 2-regular graph and the configuration model with degrees 1 and 2. After determining limits of thermodynamic quantities and the critical inverse temperature, they prove laws of large numbers (LLN) and central limit theorems (CLT) with respect to the annealed measure for the total spin. Our main contribution in this paper is to generalize their results to the class of all random regular graphs and prove the existence of annealed pressure in the case of the configuration model.

Our first result is on the convergence of thermodynamic quantities when the number of vertices tends to infinity.

THEOREM 1.1 (The thermodynamic limits). *Let us consider the Ising model on the random d-regular graph with $d \geq 2$. Then the following assertions hold*:

(i) *For all $\beta \geq 0$ and $B \in \mathbb{R}$, the annealed pressure converges*

$$\begin{aligned}\lim_{n\to\infty} \psi_n(\beta, B) &= \psi(\beta, B) \\ &= \frac{\beta d}{2} - B + \max_{0\leq t\leq 1}\big[(t-1)\log(1-t) - t\log t + 2Bt + dF(t)\big],\end{aligned}$$

where

$$F(t) = \int_0^{u(t)} \log f(s)\, ds,$$

with $u(t) = \min\{t, 1-t\}$ and

$$f(s) = \frac{e^{-2\beta}(1-2s) + \sqrt{1 + (e^{-4\beta} - 1)(1-2s)^2}}{2(1-s)}.$$

(ii) *For all $(\beta, B) \in \mathcal{U}$, the magnetization converges*

$$\lim_{n\to\infty} M_n(\beta, B) = \mathcal{M}(\beta, B) = \frac{\partial}{\partial B}\psi(\beta, B).$$

Moreover, the critical inverse temperature is

$$\beta_c = \operatorname{atanh}\bigl(1/(d-1)\bigr) = \begin{cases} \frac{1}{2}\log\Bigl(\frac{d}{d-2}\Bigr) & \text{if } d \geq 3, \\ \infty & \text{if } d = 2. \end{cases}$$

(iii) *For all $(\beta, B) \in \mathcal{U}$, the annealed susceptibility converges*

$$\lim_{n\to\infty} \chi_n(\beta, B) = \chi(\beta, B) = \frac{\partial^2}{\partial B^2}\psi(\beta, B).$$

Based on the thermodynamic limits theorem, we obtain a law of large number and a central limit theorem for the total spin.

THEOREM 1.2 (Annealed LLN). *Suppose that $(\beta, B) \in \mathcal{U}$. Then for any $\varepsilon > 0$, there exists a positive constant $L = L(\varepsilon)$, such that for all sufficiently large n*

$$\mu_n\left(\left|\frac{S_n}{n} - \mathcal{M}(\beta, B)\right| > \varepsilon\right) \leq \exp(-nL),$$

where $\mathcal{M}(\beta, B)$ is defined in Theorem 1.1(ii).

THEOREM 1.3 (Annealed CLT). *For all $(\beta, B) \in \mathcal{U}$, the total spin under the annealed measure satisfies a central limit theorem*: *for any $t \in \mathbb{R}$,*

$$\mu_n\left(\frac{S_n - \mathbb{E}_{\mu_n}(S_n)}{\sqrt{n}} > t\right) \longrightarrow \frac{1}{\sqrt{2\pi\chi^2}}\int_t^\infty \exp\left(\frac{-s^2}{2\chi^2}\right) ds,$$

with $\chi = \chi(\beta, B)$ as in Theorem 1.1(iii).

In the low temperature regime ($\beta > \beta_c$) and in the absence of external field ($B = 0$), the magnetization does not converges to a constant. However, similar to the Curie–Weiss model, the law of magnetization converges to a combination of two Dirac measures.

PROPOSITION 1.4. *Suppose that $\beta > \beta_c$ and $B = 0$. Then for $\nu = \lim_{B\to 0^+} \mathcal{M}(\beta, B)$, with $\mathcal{M}(\beta, B)$ as in Theorem 1.1, we have as $n \to \infty$,*

$$\mu_n\left(\left|\frac{S_n}{n} - \nu\right| \leq n^{-1/6}\right) \to 1/2 \quad \text{and} \quad \mu_n\left(\left|\frac{S_n}{n} + \nu\right| \leq n^{-1/6}\right) \to 1/2.$$

In the proof of Proposition 1.4, we can improve the exponent $1/6$ to any positive constant strictly smaller than $1/2$. Furthermore, we believe that using the same analysis for Theorem 1.3, it is possible to prove conditional central limit theorems: for any $t \in \mathbb{R}$,

$$\mu_n\left(\frac{S_n \pm n\nu}{\sqrt{n}} > t \,|\, S_n \lessgtr 0\right) \longrightarrow \frac{1}{\sqrt{2\pi\sigma^2}}\int_t^\infty \exp\left(\frac{-s^2}{2\sigma^2}\right) ds,$$

with $\sigma^2 = \lim_{B\to 0^+} \chi(\beta, B)$. A similar result has been proved for the Curie–Weiss model in [15].

Our result on the existence of annealed pressure in the case of the configuration model with general degree distributions is stated in Section 7, due to its complexity.

1.4. *Discussion.* Let us give here some comments on the approach, consequence and extension of our results.

(i) *On the strategy of proofs*. Structure of the random d-regular graph strongly depends on d. When d increases, the graph becomes more and more complicated. In the case $d = 2$, the annealed Ising model on the graph is well studied in [17]. Their approach is based on the fact that every random 2-regular graph consists of a collection of cycles and the partition function on a cycle can be computed explicitly. However, when $d \geq 3$, this particular fact does not hold anymore. On the other hand, we realize that for any spin configuration, the Hamiltonian can be expressed in terms of β, B and the number of disagreeing edges (the edges whose two extremities have different spins). Moreover, using the exchangeablity of random regular graphs, we show in (2.2) and (3.3) that for any given configuration, the law of disagreeing edges depends only on the number of positive spins in this configuration. Thus the Hamiltonians of configurations with a fixed number of positive spins have the same law. Hence we show that the expectation of the partition function has the form $\sum_{i\leq n} \binom{n}{i}\theta(i, \beta, B)$. Furthermore, as $n \to \infty$,

$$\frac{1}{n}\log\sum_{i=0}^{n}\binom{n}{i}\theta(i, \beta, B) = \max_{0\leq i\leq n}\frac{1}{n}\log\left[\binom{n}{i}\theta(i, \beta, B)\right] + o(1).$$

This explains the form of the annealed pressure $\psi(\beta, B)$ in Theorem 1.1(i), which somehow looks like a large deviation result.

To prove the limit theorems, we use the same general strategy as in [16, 17]. More precisely, we define the sequence of cumulant generating functions as

$$c_n(t) = \frac{1}{n}\log \mathbb{E}_{\mu_n}\big(\exp(tS_n)\big).$$

By definition of the annealed measure, we have

$$c_n(t) = \psi_n(\beta, B + t) - \psi_n(\beta, B).$$

Then by Theorem 1.1 the sequence $(c_n(t))_n$ converges to

$$c(t) = \psi(\beta, B + t) - \psi(\beta, B).$$

In [16], Sections 2.1 and 2.2, the authors show that if the function $c(t)$ is differentiable at 0 then the sequence $(S_n/n)_n$ converges in probability exponentially fast to $c'(0)$ w.r.t. μ_n. That means, for any real number $\varepsilon > 0$, there exists a positive constant $L = L(\varepsilon)$, such that for all n large enough

$$\mu_n\left(\left|\frac{S_n}{n} - c'(0)\right| > \varepsilon\right) \leq \exp(-nL).$$

We will show in Section 4 that the function $\psi(\beta, B)$ is differentiable with respect to B. Thus $c(t)$ is differentiable and the annealed LLN follows.

On the other hand, by using Theorem A.8.7(a) in [14], the central limit theorem in Theorem 1.3 follows from the convergence of generating function of the normalized sum, that is, for any fixed number $t > 0$,

$$\mathbb{E}_{\mu_n}\left(\exp\left(\frac{t(S_n - \mathbb{E}_{\mu_n}(S_n))}{\sqrt{n}}\right)\right) \longrightarrow \exp\left(\frac{\chi(\beta, B)t^2}{2}\right).$$

The authors in [17], Section 3.2, show that this convergence holds if the following condition is satisfied: For any fixed number $t > 0$ and for any sequence (t_n) satisfying $t_n \in [0, t/\sqrt{n}]$, one has

$$c_n''(t_n) \to \chi(\beta, B).$$

We refer to Lemma 5.1 for the proof of this condition.

(ii) *On the case* $d = 2$. We show in Proposition 3.2 that with $d = 2$, the annealed pressure is exactly solved and agrees the result obtained in [17], where the limit theorems have been proved. Hence, in Sections 4, 5, 6 we only study limit theorems for the case $d \geq 3$.

(iii) *On the critical behavior of the annealed model*. In Theorem 1.3 and Proposition 1.4, we study the behavior of total spin S_n as n tends to infinity for the full space of parameters except the critical case $(\beta, B) = (\beta_c, 0)$. In a companion paper [5], we fill this gap by showing a nonstandard scaling limit of S_n. More precisely, we prove that at criticality S_n scaled by $n^{3/4}$ converges in law to a non-Gaussian distribution whose density is proportional to $\exp(-cx^4)$, for some positive constant $c = c(d)$. This phenomena has been also observed for Curie–Weiss model in [14], for the Ising model on inhomogeneous random graphs in [11] and on the square lattice in [4]. We refer the reader to [11] for more discussion on the universality of the critical nature of total spin.

(iv) *On the similarities and differences between annealed and quenched models*. Theorem 1.1(ii) shows that the annealed Ising model undergoes a phase transition at the critical inverse temperature $\beta_c = \operatorname{atanh}(1/(d-1))$, which is equal to the critical value of the quenched Ising model. Moreover, by an analytic proof, we show in Proposition 3.2 that the annealed and quenched pressures are actually the same. We also remark that the existence of annealed pressure have been first proved in [8]. As a consequence, all the thermodynamic limits of the annealed and quenched models are identical, and these two models should behave alike. In fact, limit theorems similar to our results have been proved for quenched model in [16]. This similarity has been conjectured in [17], Section 1.5.1.

However, the nature of the proofs for the existence of the annealed pressure and quenched pressures are different. In [6], by verifying the validity of the Bethe prediction, Dembo and Montanari prove the existence of quenched pressure for generic graph sequences converging locally to trees. In fact, they show that the

Boltmann–Gibbs measure on typical realizations of random graphs converges locally to the one on the limiting trees with homogeneous degree distribution. On the other hand, there is no general approach to prove the existence of annealed measure. Indeed, for the case of inhomogeneous random graphs, the authors of [17] exploit the independent structure of edge distribution, while for the case of random regular graphs (and configuration model also), we combine the exchangeability of random graphs with many combinatorial computations.

(v) *On the generalizations*. In Section 6, we study the Ising model on the configuration model with general degree distributions. Comparing with the case of random regular graphs, we have additionally a source of randomness coming from the sequence of degrees. This randomness makes the problem much more difficult. In particular, the annealed pressure obtained in Proposition 7.3 is so complicated that we can not even prove its differentiability. Without the differentiability, we cannot go further to the other thermodynamic quantities or limit theorems. It is worth noting that in [17], the authors conjecture that when the degrees of vertices fluctuate, the annealed and quenched models behave differently. In particular, they guess that the critical temperatures in these two models are not equal.

Another natural question is to generalize our result to the Potts' model where the spin of vertex may take q values with $q \geq 3$, and the Hamiltonian is proportional to the number of agreeing edges. Our method possibly applies for this model, but it would require much work. The symmetry property that the measures of configurations with a given structure of spins (the profile of the number of spins in each value) are equal will continue to hold for the Potts' model. However, the Hamiltonian of configurations is more complicated than that of the Ising model. Indeed, there are now $q(q-1)/2$ types of disagreeing edges instead of 1 type as in the Ising model. Hence a recursive relation between agreeing and disagreeing edges would be much harder than the one for the Ising model obtained in Section 2.

(vi) *On the organization of the paper*. In Section 2, we prove a key lemma for random 1-regular graph and an auxiliary lemma used in the proof of the existence of annealed pressure. In Section 3, we study the annealed pressure and prove Theorem 1.1(i). In Section 4, we consider the magnetization, prove Theorem 1.1(ii) and Theorem 1.2. In Section 5, we prove Theorem 1.1(iii) and Theorem 1.3. In Section 6, we prove Proposition 1.4. In Section 7, we prove the existence of the annealed pressure in the case of general configuration models. The Appendix is devoted to prove some technical points of our proofs.

2. Preliminaries.

2.1. *A key lemma on random* 1*-regular graph*. The random 1-regular graph will be employed several times in the proofs, so we distinguish its set of vertices from that of the $G_{n,d}$. More precisely, we denote by $\bar{V}_m = \{w_1, \dots, w_m\}$ the set of vertices of $G_{m,1}$.

We now explain the role of $G_{m,1}$ in our arguments. We show in (3.1) that the Hamiltonian of a given configuration can be expressed in terms of the number of disagreeing edges. By the construction of the configuration model, we have a relation between the number of disagreeing edges of $G_n(\mathbf{D})$ and that of $G_{\ell_n,1}$ with $\ell_n = D_1 + \cdots + D_n$. More concretely, for $A \subset V_n$, let us denote

$$e(A, A^c) = \#\{\text{edges between } A \text{ and } A^c \text{ in } G_n(\mathbf{D})\}.$$

For any $1 \le k \le m$, we define $\bar{U}_k = \{w_1, \dots, w_k\}$, $\bar{U}_k^c = \bar{V}_m \setminus \bar{U}_k$, and

$$X(k, m) = \#\{\text{edges between } \bar{U}_k \text{ and } \bar{U}_k^c \text{ in } G_{m,1}\}. \tag{2.1}$$

It directly follows from the construction of the configuration model that

$$e(A, A^c) \overset{(\mathcal{D})}{=} X(\ell_A, \ell_n), \tag{2.2}$$

where

$$\ell_A = \sum_{i=1}^{n} D_i \mathbb{I}(v_i \in A) \quad \text{and} \quad \ell_n = \ell_{V_n},$$

where $\mathbb{I}(B)$ stands for the indicator function of B. The relation (2.2) allows us to reduce problems on disagreeing edges of configuration models (or Hamiltonian of Ising model) to the one of random 1-regular graphs.

We will see in (3.2) that the generating function of the number of disagreeing edges plays a central role in the partition function. Thanks to (2.2), we only need to study the generating functions of the number of disagreeing edges in random 1-regular graphs. For $k \le m$, define

$$g(\beta, k, m) := \mathbb{E}(\exp(-2\beta X(k, m))). \tag{2.3}$$

The asymptotic behavior of $g(\beta, k, m)$ is described in the following lemma.

LEMMA 2.1. *For all $\beta \ge 0$, there exists a positive constant $C = C(\beta)$, such that for all m large enough the following assertions hold*:

(i) *For all* $0 \le k \le \ell \le m$,

$$\left|[\log g(\beta, k, m) - mF(k/m)] - [\log g(\beta, \ell, m) - mF(\ell/m)]\right| \le \frac{C|k - \ell|}{m}.$$

(ii) *We have*

$$\max_{0 \le k \le m} \left| \frac{\log g(\beta, k, m)}{m} - F(k/m) \right| \le \frac{C}{m},$$

with $F(t)$ as in Theorem 1.1.

PROOF. We observe that $g(\beta, 0, m) = 1$ and $F(0) = 0$. Hence, (ii) is a direct consequence of (i). We first claim that to prove (i), it suffices to show

$$\text{(i) holds for all } 0 \le k \le \ell \le [m/2]. \tag{2.4}$$

Indeed, we observe that for all $0 \le k \le m$,

$$X(k, m) \overset{(\mathcal{D})}{=} X(m - k, m).$$

Thus

$$g(\beta, k, m) = g(\beta, m - k, m). \tag{2.5}$$

Moreover, we have $F(t) = F(1 - t)$ for all $t \in [0, 1]$. Hence for all $k \le m$,

$$F\left(\frac{k}{m}\right) = F\left(\frac{m-k}{m}\right). \tag{2.6}$$

Combining (2.5) and (2.6), we get that for $0 \le k \le [m/2] < \ell \le m$,

$$\begin{aligned} &\big|[\log g(\beta, k, m) - mF(k/m)] - [\log g(\beta, \ell, m) - mF(\ell/m)]\big| \\ &\quad = \left|[\log g(\beta, k, m) - mF(k/m)] - \left[\log g(\beta, m - \ell, m) - mF\left(\frac{m-\ell}{m}\right)\right]\right| \\ &\quad \le \frac{C|k - (m - \ell)|}{m} \le \frac{C|k - \ell|}{m}, \end{aligned}$$

by using (2.4) for $0 \le k, m - \ell \le [m/2]$. Hence (i) holds for $0 \le k \le [m/2] < \ell \le m$. Similarly, we can also prove that (i) holds for $[m/2] \le k \le \ell \le m$, and thus (i) follows.

We now prove (2.4). The proof of (2.4) is long and divided into four parts: recursive formula for $g(k, m)$, reduced sequence of $g(k, m)$, approximation of the reduced sequence and conclusion.

I. Recursive formula. We claim that for all $k \le [m/2]$,

$$X(k, m) \overset{(\mathcal{D})}{=} \begin{cases} X(k - 2, m - 2) & \text{with prob. } (k - 1)/(m - 1), \\ 1 + X(k - 1, m - 2) & \text{with prob. } (m - k)/(m - 1). \end{cases} \tag{2.7}$$

Indeed, we remind the construction of the random 1-regular graph: to each vertex in $\bar{V}_m$ we attach an half-edge, then we pair these half-edges uniformly. Let us denote by $\bar{\mathcal{U}}_k$ (resp., $\bar{\mathcal{U}}_k^c$) the set of half-edges that incident to $\bar{U}_k$ (resp., $\bar{U}_k^c$). Suppose that we start the procedure of pairing half-edges with an element in $\bar{\mathcal{U}}_k$, say h_1. Then there are two possibilities. First, with probability $(m - k)/(m - 1)$, the half-edge h_1 is paired with an element in $\bar{\mathcal{U}}_k^c$. This pairing gives an edge between $\bar{U}_k$ and $\bar{U}_k^c$. After this step, there remains $m - 2$ half-edges including $k - 1$, ones belonging to $\bar{\mathcal{U}}_k$. Hence $X(k, m)$ has the same law as $1 + X(k - 1, m - 2)$. Second, with probability $(k - 1)/(m - 1)$, the half-edge h_1 is paired with an element in $\bar{\mathcal{U}}_k$, and that does not give an edge between $\bar{U}_k$ and $\bar{U}_k^c$. Thus after this step, $X(k, m)$ has the same law as $X(k - 2, m - 2)$.

Now applying (2.7) we obtain

$$
\begin{aligned}
g(\beta,k,m) &= \mathbb{E}\big(e^{-2\beta X(k,m)}\big) \\
&= \frac{k-1}{m-1}\mathbb{E}\big(e^{-2\beta X(k-2,m-2)}\big) + \frac{m-k}{m-1}\mathbb{E}\big(e^{-2\beta[1+X(k-1,m-2)]}\big) \qquad (2.8)\\
&= \frac{k-1}{m-1} g(\beta,k-2,m-2) + \frac{(m-k)e^{-2\beta}}{m-1} g(\beta,k-1,m-2).
\end{aligned}
$$

As for (2.7), starting with an half-edge in $\bar{\mathcal{U}}_k^c$, we get

$$
X(k,m) \overset{(\mathcal{D})}{=} \begin{cases} X(k,m-2) & \text{with prob. } (m-k-1)/(m-1), \\ 1 + X(k-1,m-2) & \text{with prob. } k/(m-1). \end{cases}
$$

Hence

$$
g(\beta,k,m) = \frac{(m-k-1)}{m-1} g(\beta,k,m-2) + \frac{ke^{-2\beta}}{m-1} g(\beta,k-1,m-2). \qquad (2.9)
$$

It follows from (2.8) and (2.9) that

$$
\begin{aligned}
g(\beta,k,m-2) = {}& \frac{(m-2k)e^{-2\beta}}{m-k-1} g(\beta,k-1,m-2) \\
&+ \frac{k-1}{m-k-1} g(\beta,k-2,m-2).
\end{aligned} \qquad (2.10)
$$

We replace $m-2$ by m in (2.10) and obtain a recursive formula

$$
\begin{aligned}
g(\beta,k,m) = {}& \frac{(m-2k+2)e^{-2\beta}}{m-k+1} g(\beta,k-1,m) \\
&+ \frac{k-1}{m-k+1} g(\beta,k-2,m).
\end{aligned} \qquad (2.11)
$$

II. Reduced sequence. Define for all $1 \leq i \leq m$

$$
h(i,m) = \frac{g(\beta,i,m)}{g(\beta,i-1,m)}.
$$

Note that here for simplicity we remove the notation β in the sequence h. Then we have

$$
\log g(\beta,k,m) = \sum_{i=1}^{k} \log h(i,m). \qquad (2.12)
$$

Moreover, by (2.11),

$$
h(k,m) = \frac{(m-2k+2)e^{-2\beta}}{m-k+1} + \frac{k-1}{(m-k+1)h(k-1,m)}. \qquad (2.13)
$$

Observe that $g(\beta, 0, m) = 1$ and $g(\beta, 1, m) = e^{-2\beta}$, since $X(0, m) = 0$ and $X(1, m) = 1$. Thus $h(1, m) = c$, with

$$c = e^{-2\beta} \in (0, 1).$$

In addition, by replacing k by $k + 1$ in (2.13), we get

$$h(k+1, m) = \frac{c(m-2k)}{m-k} + \frac{k}{(m-k)h(k, m)}. \tag{2.14}$$

III. Approximation of $h(k, m)$. By numerical analysis, we find that $h(k + 1, m)$ and $h(k, m)$ are very close when m tends to infinity. Hence, from (2.14) it is natural to expect that $h(k, m)$ is approximated by the solution of the fixed-point equation

$$\theta_k = \frac{c(m-2k)}{m-k} + \frac{k}{(m-k)\theta_k}.$$

Going further to approximate the sequence $(h(k, m))$, we consider the following functional equation:

$$\theta = \frac{c(1-2t)}{1-t} + \frac{t}{\theta(1-t)}. \tag{2.15}$$

The positive solution of this equation is

$$\theta = f(t) := \frac{c(1-2t) + \sqrt{1 + (c^2-1)(1-2t)^2}}{2(1-t)}. \tag{2.16}$$

We claim the following estimates on $f(t)$ and $h(k, m)$:

- For all $t \in [0, 1/2]$,

$$c \le f(t) \le 1. \tag{2.17}$$

- There exists a positive constant $A = A(\beta) \ge 1$ such that, for all $t \in (0, 1/2)$,

$$1/A \le f'(t) \le A \quad \text{and} \quad |f''(t)| \le A. \tag{2.18}$$

- There exists a positive constant $\varkappa$ such that, for all m and $0 \le k \le [m/2]$,

$$\left| h(k, m) - f\left(\frac{k-1}{m}\right) \right| \le \frac{\varkappa}{m}. \tag{2.19}$$

Note that the bound for $f''(t)$ in (2.18) is not used in the proof of (2.4), but it is needed for the proof of (2.19). The proof of (2.17), (2.18) and (2.19) is long and complicated, so we put it in the Appendix.

IV. Conclusion. Assuming these claims (2.17), (2.18), (2.19), we now prove (2.4). By (2.17) and (2.19), for all m large enough and $0 \le k \le [m/2]$,

$$c/2 \le \min\{h(k, m), f(k/m)\}. \tag{2.20}$$

Using the mean value theorem, we have, for all $x, y > 0$,

$$|\log x - \log y| \le \frac{|x-y|}{\min\{x, y\}}. \tag{2.21}$$

Using (2.12), (2.19), (2.20) and (2.21), we get that, for all $0 \le k \le \ell \le [m/2]$,

$$\begin{aligned}\left|\log g(k,m) - \log g(\ell,m) + \sum_{i=k+1}^{\ell} \log f\left(\frac{i-1}{m}\right)\right| \\ &= \left|\sum_{i=k+1}^{\ell} \bigl[\log h(i,m) - \log f\bigl((i-1)/m\bigr)\bigr]\right| \\ &\le \sum_{i=k+1}^{\ell} \bigl|\log h(i,m) - \log f\bigl((i-1)/m\bigr)\bigr| \\ &\le \frac{2}{c}\sum_{i=k+1}^{\ell} \bigl|h(i,m) - f\bigl((i-1)/m\bigr)\bigr| \le \frac{2\varkappa(\ell-k)}{cm}.\end{aligned} \tag{2.22}$$

Similarly,

$$\begin{aligned}\left|\frac{\log f(i/m)}{m} - \int_{i/m}^{(i+1)/m} \log f(s)\,ds\right| &\le \int_{i/m}^{(i+1)/m} \bigl|\log f(i/m) - \log f(s)\bigr|\,ds \\ &\le \frac{2}{c}\int_{i/m}^{(i+1)/m} \bigl|f(i/m) - f(s)\bigr|\,ds \\ &\le \frac{2A}{c}\int_{i/m}^{(i+1)/m} \bigl|(i/m) - s\bigr|\,ds \\ &= \frac{A}{m^2 c}.\end{aligned} \tag{2.23}$$

Here, for the third inequality, we have used (2.18) and the mean value theorem. It follows from (2.22) and (2.23) that, for all $0 \le k \le \ell \le [m/2]$,

$$\left|\log g(k,m) - \log g(\ell,m) + m\int_{k/m}^{\ell/m} \log f(s)\,ds\right| \le \left(\frac{2\varkappa + A}{c}\right)\left(\frac{\ell-k}{m}\right),$$

which proves Lemma 2.1(i). □

2.2. *An auxiliary lemma.* The following result will be used in the proof of the existence of the annealed pressure.

LEMMA 2.2. *The following assertions hold*:

(i) *Let* $G(t)$ *be a continuous function on* $[0, 1]$. *Then*

$$\lim_{n\to\infty} \max_{0\le j\le n} G(j/n) = \sup_{0\le t\le 1} G(t).$$

(ii) *Let* $(G_n(t))_n$ *be a sequence of functions on* [0, 1], *which converges pointwise to a function* $G(t)$. *Suppose that there exists a positive constant* C *and a sequence* (ε_n) *tending to* 0, *such that for all* $0 \le s, t \le 1$ *and* $n \ge 1$,

$$\left|G_n(s) - G_n(t)\right| \le C|s-t| + \varepsilon_n.$$

Then $G(t)$ *is a Lipschitz function. Moreover, for any continuous function* $H(t)$ *on* [0, 1], *we have*

$$\lim_{n\to\infty} \max_{0\le j\le n} \left[H(j/n) + G_n(j/n)\right] = \sup_{0\le t\le 1} \left[H(t) + G(t)\right].$$

The results of this lemma are standard in real analysis, so we safely leave the proof to the reader.

2.3. *Notation*. If f and g are two real functions, we write $f = \mathcal{O}(g)$ if there exists a constant $C > 0$, such that $f(x) \le Cg(x)$ for all x; $f \asymp g$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$; $f = o(g)$ if $g(x)/f(x) \to 0$ as $x \to \infty$. Let $(f(j,n))_{1\le j\le n}$ and $(g(j,n))_{1\le j\le n}$ be two sequences of real numbers. The notion $f(j,n) = \mathcal{O}(g(j,n))$ [or $f(j,n) = o(g(j,n))$] is taken uniformly in all $j \le n$. For any real number x, let $[x]$ denote the integer part of x.

3. The annealed pressure.

The first step (which is one of the most important steps) in studying the Ising model is the task of understanding the partition function and the pressure. As mentioned in the Introduction, we will write the Hamiltonian in term of the number of disagreeing edges. Then using the symmetry of random regular graphs, we can investigate the annealed pressure. Let us be more precise now.

We fix an integer $d \ge 2$. Then for any positive integer n, we consider the random d-regular graph whose the vertex set is $V_n = \{v_1, \dots, v_n\}$. For any spin configuration $\sigma \in \Omega_n$, define

$$\sigma_+ = \{v_i : \sigma_i = 1\} \quad \text{and} \quad \sigma_- = \{v_i : \sigma_i = -1\}.$$

Then

$$\sum_{i=1}^n \sigma_i = 2|\sigma_+| - n,$$

$$\sum_{i\le j} k_{i,j}\sigma_i\sigma_j = (dn)/2 - 2e(\sigma_+, \sigma_-),$$

where

$$e(\sigma_+, \sigma_-) = \#\{\text{edges between } \sigma_+ \text{ and } \sigma_-\}.$$

Therefore,

$$H_n(\sigma) = \left(B - \frac{\beta d}{2}\right)n + 2\beta e(\sigma_+, \sigma_-) - 2B|\sigma_+|. \tag{3.1}$$

Thus

$$\mathbb{E}(e^{-H_n(\sigma)}) = e^{(\frac{\beta d}{2}-B)n}\mathbb{E}(e^{-2\beta e(\sigma_+,\sigma_-)})e^{2B|\sigma_+|}. \tag{3.2}$$

By (2.2), if $|\sigma_+| = |\sigma'_+|$ then

$$e(\sigma_+,\sigma_-) \overset{(\mathcal{D})}{=} e(\sigma'_+,\sigma'_-) \overset{(\mathcal{D})}{=} X(d|\sigma_+|, dn)). \tag{3.3}$$

Hence

$$\begin{aligned}
&\sum_{\sigma\in\Omega_n} \mathbb{E}(e^{-2\beta e(\sigma_+,\sigma_-)})e^{2B|\sigma_+|} \\
&\quad = \sum_{j=0}^{n} e^{2Bj} \sum_{\substack{\sigma\in\Omega_n \\ |\sigma_+|=j}} \mathbb{E}(e^{-2\beta e(\sigma_+,\sigma_-)}) \\
&\quad = \sum_{j=0}^{n} \binom{n}{j} e^{2Bj}\mathbb{E}(e^{-2\beta X(dj,dn)}) \\
&\quad = \sum_{j=0}^{n} \binom{n}{j} e^{2Bj} g(\beta, dj, dn),
\end{aligned} \tag{3.4}$$

with $g(\beta,k,m)$ defined as in (2.3) for all $k \le m$. Therefore,

$$\mathbb{E}(Z_n(\beta,B)) = e^{(\frac{\beta d}{2}-B)n} \times \sum_{j=0}^{n} \binom{n}{j} e^{2Bj} g(\beta, dj, dn). \tag{3.5}$$

PROOF OF THEOREM 1.1(i). By (3.5), we have

$$\begin{aligned}
&\frac{1}{n}\log\mathbb{E}(Z_n(\beta,B)) \\
&\quad = \frac{\beta d}{2} - B + \max_{0\le j\le n}\left[\frac{\log\binom{n}{j}}{n} + 2B\frac{j}{n} + \frac{\log g(\beta,dj,dn)}{n}\right] + o(1).
\end{aligned}$$

On the other hand, it follows from Stirling's formula that

$$\frac{\log\binom{n}{j}}{n} = \frac{j}{n}\log\left(\frac{n}{j}\right) + \frac{n-j}{n}\log\left(\frac{n}{n-j}\right) + o(1).$$

Combining the last two equations and Lemma 2.1(ii), we obtain

$$\frac{1}{n}\log\mathbb{E}(Z_n(\beta,B)) = \frac{\beta d}{2} - B + \max_{0\le j\le n} L(j/n) + o(1), \tag{3.6}$$

where $L(t)$ is a continuous function on [0, 1] defined by

$$L(t) = -t\log(t) + (t-1)\log(1-t) + 2Bt + dF(t).$$

Now the result follows from (3.6) and Lemma 2.2(i). □

An explicit formula for the function $F(t)$ is given in the following lemma.

LEMMA 3.1. *For $t \le 1/2$, we have*

$$F(t) = t \log f(t) + \frac{1}{2}\log(1-t) + \frac{1}{2}\log(1+e^{-2\beta}) + \frac{1}{2}\log\left[1 + \frac{e^{-2\beta}(2t-1)}{(1-t)(f(t)+1)}\right].$$

For $t \in (1/2, 1)$, we have $F(t) = F(1-t)$.

As we will see in the Appendix, this representation of $F(t)$ plays an important role in the proof of Proposition 3.2 below, where we establish the equality between quenched and annealed pressures. An explicit formula of the quenched measure $\tilde{\psi}(\beta, B)$ has been derived in [6], Theorem 2.4, and it will be recalled in (A.26) in the Appendix. The following result says that $\tilde{\psi}(\beta, B)$ and $\psi(\beta, B)$ are actually the same.

PROPOSITION 3.2. *For all $\beta > 0$ and $B \in \mathbb{R}$, we have $\psi(\beta, B) = \tilde{\psi}(\beta, B)$. In particular, when $d = 2$,*

$$\psi(\beta, B) = \beta + \log\left(\cosh(B) + \sqrt{\sinh^2(B) + e^{-4\beta}}\right),$$

which agrees with the result obtained in [17].

The proof of Lemma 3.1 and Proposition 3.2 is put in the Appendix.

4. The annealed magnetization and the strong law of large numbers. In this section, we prove the existence of the annealed magnetization and Theorem 1.2 following the strategy mentioned in the Introduction.

PROOF OF THEOREM 1.1(ii). We state the following claims which we prove below:

- Claim 1. For any $\beta \ge 0$, the function $\psi(\beta, \cdot)$ is differentiable at every point $B \neq 0$.
- Claim 2. For any $d \ge 3$,

$$\beta_c = \operatorname{atanh}(1/(d-1)) = \frac{1}{2}\log\left(\frac{d}{d-2}\right).$$

  Moreover, for any $\beta \in (0, \beta_c)$, the function $\psi(\beta, \cdot)$ is differentiable at $B = 0$.

Assuming these claims, Theorem 1.1(ii) follows. Indeed, using similar arguments as in the proof of Theorem 1.1(ii) in [17], we can show that for all $(\beta, B) \in \mathcal{U}$, the annealed magnetization $(M_n(\beta, B))$ converges to

$$\mathcal{M}(\beta, B) := \frac{\partial \psi(\beta, B)}{\partial B}. \tag{4.1}$$

This together with the claims 1 and 2 implies Theorem 1.1(ii). $\square$

PROOF OF CLAIM 1. We consider here the case $B > 0$, the other one can be handled similarly. We first define some functions on [0, 1]:

$$
\begin{aligned}
I(t) &= (t-1)\log(1-t) - t\log t, \\
H(t) &= I(t) + d\int_0^{u(t)} \log f(s)\,ds, \qquad (4.2)\\
L(t) &= H(t) + 2Bt,
\end{aligned}
$$

with the convention $I(0) = I(1) = 0$. By Theorem 1.1(i),

$$\psi(\beta, B) = (\beta d)/2 - B + \max_{0\le t\le 1} L(t). \tag{4.3}$$

Observe that $H(t) = H(1-t)$ for all $t \in [0, 1]$, and $Bt \le B(1-t)$ for all $t \le 1/2$. Hence $L(t) = H(t) + 2Bt$ attains the maximum at a point in $[1/2, 1]$. We consider the derivative of $L(t)$ on $[1/2, 1]$:

$$L'(t) = H'(t) + 2B = \log\left(\frac{1-t}{t}\right) - d\log f(1-t) + 2B.$$

We have $L'(1/2) = 2B > 0$ and $L'(1^-) = -\infty$, so the maximum point of $L(t)$ is a solution of the equation

$$L'(t) = \log\left(\frac{1-t}{t}\right) - d\log f(1-t) + 2B = 0. \tag{4.4}$$

Claim 1*. Equation (4.4) has a unique solution t_* in $(1/2, 1)$, and $L''(t_*) \neq 0$.

Assuming this claim, we can deduce from the implicit function theorem that the function t_* is differentiable with respect to B. Thus the function $\psi(\beta, \cdot)$ is also differentiable and Claim 1 follows. Moreover,

$$\frac{\partial}{\partial B}\psi(\beta, B) = -1 + H'(t_*)\frac{\partial t_*}{\partial B} + 2t_* + 2B\frac{\partial t_*}{\partial B} = -1 + 2t_*. \tag{4.5}$$

Now we prove Claim 1*. Since $L'(1/2) = 2B > 0$ and $L'(1^-) = -\infty$, the function $L'(t)$ has at least one root in $(1/2, 1)$. Suppose that $L'(t)$ has more than one root in $(1/2, 1)$. Then $L''(t)$ has at least two roots in $(1/2, 1)$. We consider the following equation in $(1/2, 1)$:

$$L''(t) = \frac{-1}{t(1-t)} - \frac{dx'(t)}{x(t)} = 0, \tag{4.6}$$

where $x(t) = f(1-t)$. Since $f(t)$ satisfies (2.15), we have

$$x(t) = \frac{c(2t-1)}{t} + \frac{1-t}{tx(t)},$$

with

$$c = e^{-2\beta} \in (0, 1).$$

After some computation, we get

$$\frac{x'(t)}{x(t)} = \frac{cx(t) - 1}{t[c(2t-1)x(t) + 2 - 2t]}. \tag{4.7}$$

Using this and (4.6), we obtain

$$L''(t) = \frac{-d(1-t)(cx(t)-1) - c(2t-1)x(t) + 2 - 2t}{t(1-t)[c(2t-1)x(t) + 2 - 2t]}. \tag{4.8}$$

Hence, by simple computations we have

$$L''(t) = 0 \quad \Leftrightarrow \quad \sqrt{1 + (c^2-1)(2t-1)^2} = \frac{2t(d-2)(1-t)}{c(d(1-t) + 2t - 1)} - c(2t-1),$$

from which it follows that

$$\begin{aligned} 1 + (c^2-1)(2t-1)^2 = {} & \frac{4t^2(d-2)^2(1-t)^2}{c^2(d(1-t)+2t-1)^2} + c^2(2t-1)^2 \\ & - \frac{4t(d-2)(1-t)(2t-1)}{(d(1-t)+2t-1)}, \end{aligned}$$

or equivalently

$$c^2[(d-2)^2(t-t^2) + d - 1] = t(1-t)(d-2)^2. \tag{4.9}$$

Since $d \geq 3$, equation (4.9) is equivalent to

$$t^2 - t + \frac{c^2(d-1)}{(d-2)^2(1-c^2)} = 0. \tag{4.10}$$

Observe that the sum of two solutions of (4.10) is 1. Hence (4.10) has at most one solution in $(1/2, 1)$. Therefore, $L''(t)$ has at most one root in $(1/2, 1)$. Hence the equation $L'(t) = 0$ has a unique solution in $(1/2, 1)$, say t_*. Now we show $L''(t_*) \neq 0$ by contradiction. Suppose that $L''(t_*) = 0$. Then t_* must be a solution of (4.10). Hence

$$\frac{c^2(d-1)}{(d-2)^2(1-c^2)} = t_* - t_*^2 < \frac{1}{4}.$$

Thus

$$c < \frac{d-2}{d}. \tag{4.11}$$

Since $L'(1/2) > 0$ and $L'(t_*) = 0$, there exists $u \in (1/2, t_*)$, such that $L''(u) < 0$. On the other hand, by (4.8) and (4.11),

$$L''(1/2) = -4 - 2d(c-1) > 0.$$

Since $L''(u)L''(1/2) < 0$, the function $L''(t)$ has a root in $(1/2, u)$. Hence $L''(t)$ has at least two roots in $(1/2, 1)$, which leads a contradiction. Therefore, $L''(t_*) \neq 0$ and Claim 1* follows. □

PROOF OF CLAIM 2. Claim 2 is a direct consequence of the following claims.

- Claim 2a. If $\beta > \operatorname{atanh}(1/(d-1))$ then

$$\lim_{B \searrow 0} \mathcal{M}(\beta, B) = -1 + 2t_+ > 0,$$

where t_+ is the unique root in $(1/2, 1)$ of the function $H'(t)$.
- Claim 2b. If $0 < \beta < \operatorname{atanh}(1/(d-1))$, then $H'(t)$ is strictly decreasing on $(0, 1)$ and has a unique root $t_0 = 1/2$. Moreover, the function $\psi(\beta, \cdot)$ is differentiable at $B = 0$ and

$$\lim_{B \searrow 0} \mathcal{M}(\beta, B) = \frac{\partial}{\partial B} \psi(\beta, B)|_{B=0} = 0.$$

We first prove Claim 2a. Observe that $H'(1/2) = 0$ and $H'(1^-) = -\infty$. Moreover, by (4.8)

$$H''(1/2) = -4 - 2d\bigl(e^{-2\beta} - 1\bigr) > 0,$$

since

$$\beta > \operatorname{atanh}\bigl(1/(d-1)\bigr) = \frac{1}{2} \log\left(\frac{d}{d-2}\right).$$

Therefore, $H'(t)$ has at least one root in $(1/2, 1)$. Using the same arguments as in the proof of Claim 1*, $H'(t)$ has at most one root in $(1/2, 1)$. Thus it has a unique root t_+ in $(1/2, 1)$. Moreover, $H'(t) > 0$ for all $t \in (1/2, t_+)$ and $H'(t) < 0$ for all $t \in (t_+, 1)$.

On the other hand, $0 = L'(t_*) = H'(t_*) + 2B$. Hence $H'(t_*) = -2B < 0$ when $B > 0$. Therefore, $t_* > t_+$ for all $B > 0$. In addition, $\lim_{B \searrow 0} t_* = t_+$. Hence by (4.5), we have

$$\lim_{B \searrow 0} \mathcal{M}(\beta, B) = -1 + 2t_+ > 0,$$

which implies Claim 2a.

We now prove Claim 2b. Assume that

$$0 < \beta < \frac{1}{2} \log\left(\frac{d}{d-2}\right). \tag{4.12}$$

We first show that $H''(t) < 0$ for all $t \in (0, 1)$. We consider here the case $t \geq 1/2$, the other one is similar. Using (4.8) and the same calculation as for (4.9), we have

$$\begin{aligned} &H''(t) = L''(t) < 0 \\ &\quad \Leftrightarrow \quad \sqrt{1 + \bigl(e^{-4\beta} - 1\bigr)(2t-1)^2} > \frac{2t(d-2)(1-t)}{e^{-2\beta}(d(1-t) + 2t - 1)} - e^{-2\beta}(2t-1), \end{aligned}$$

from which it follows that

$$e^{-4\beta}\bigl[(d-2)^2\bigl(t - t^2\bigr) + d - 1\bigr] > t(1-t)(d-2)^2. \tag{4.13}$$

Under the condition (4.12), the inequality (4.13) is a consequence of the following:

$$(d-2)^2\big(t-t^2\big)+d-1>t(1-t)d^2$$
$$\Leftrightarrow \quad 1>4t(1-t),$$

which holds for all $t\in(0,1)\setminus\{1/2\}$. For $t=1/2$,

$$H''(1/2)=-4-2d\big(e^{-2\beta}-1\big)<0,$$

by (4.12). In conclusion, $H''(t)<0$ for all $t\in(0,1)$, and thus $H'(t)$ is strictly decreasing and has a unique zero at $t=1/2$. Now applying the implicit function theorem for the function $L'(t)$, we get that t_*, the solution of the equation $L'(t)=0$, is differentiable with respect to B at 0. Thus the function $\psi(\beta,\cdot)$ is also differentiable at $B=0$ and

$$\lim_{B\searrow 0}\mathcal{M}(\beta,B)=\frac{\partial}{\partial B}\psi(\beta,B)\bigg|_{B=0}=\lim_{B\to 0}-1+2t_*=-1+2t_0=0. \tag{4.14}$$

This implies the claim 2b. □

PROOF OF THEOREM 1.2. As mentioned in the Introduction, the exponentially strong law of large numbers for the magnetization follows from the differentiability of the pressure $\psi(\beta,B)$ with respect to B, by using the same arguments in the proof of [17], Theorem 1.2. □

5. The annealed susceptibility and the central limit theorem. We have shown that for all $(\beta,B)\in\mathcal{U}$,

$$\frac{\partial}{\partial B}\psi(\beta,B)=-1+2t_*,$$

where t_* is the solution of the equation

$$L'(t)=H'(t)+2B=0,$$

with $L(t)$ and $H(t)$ as in (4.2). Moreover, we showed that t_* is a differentiable function with respect to B. Hence

$$H''(t_*)\frac{\partial t_*}{\partial B}+2=0,$$

and thus

$$\frac{\partial t_*}{\partial B}=\frac{-2}{H''(t_*)}.$$

Therefore,

$$\chi(\beta,B):=\frac{\partial^2}{\partial B^2}\psi(\beta,B)=2\frac{\partial t_*}{\partial B}=\frac{-4}{H''(t_*)}. \tag{5.1}$$

Let us recall the sequence of cumulant generating functions

$$c_n(t)=\psi_n(\beta,B+t)-\psi_n(\beta,B).$$

LEMMA 5.1. *Suppose that $(\beta, B) \in \mathcal{U}$. Then for any positive constant t and any sequence (t_n) satisfying $t_n \le t/\sqrt{n}$, we have*

$$c_n''(t_n) \to \chi(\beta, B).$$

PROOF OF THEOREM 1.1(iii). The result is a consequence of Lemma 5.1 with $t_n \equiv 0$. □

PROOF OF THEOREM 1.3. As mentioned in the Introduction, the central limit theorem is a consequence of Lemma 5.1 by applying the same arguments as in the proof of [17], Theorem 1.6, and [14], Theorem A.8.7. □

PROOF OF LEMMA 5.1. We consider here the case $B \ge 0$, the other one can be handled similarly. Thanks to (5.1), we only need to show that for any positive constant t and any sequence (t_n) satisfying $t_n \in [0, t/\sqrt{n}]$,

$$c_n''(t_n) \to \frac{-4}{H''(t_*)}. \tag{5.2}$$

It follows from (3.5) that, for all $s > 0$,

$$c_n''(s) = \frac{\partial^2}{\partial B^2} \log Z_n(\beta, B+s) = \frac{4}{n}\left(\frac{T_{2,n}(s)}{T_n(s)} - \left(\frac{T_{1,n}(s)}{T_n(s)}\right)^2\right), \tag{5.3}$$

where

$$T_n(s) = \sum_{j=0}^{n} \binom{n}{j} e^{2(B+s)j} g(\beta, dj, dn),$$

$$T_{1,n}(s) = \sum_{j=0}^{n} \binom{n}{j} e^{2(B+s)j} g(\beta, dj, dn) j,$$

$$T_{2,n}(s) = \sum_{j=0}^{n} \binom{n}{j} e^{2(B+s)j} g(\beta, dj, dn) j^2.$$

Let us define

$$j_* = [nt_*].$$

We will show that the values of $T_n(s)$, $T_{1,n}(s)$, $T_{2,n}(s)$ are concentrated around the j_*th term of each sum if $s = \mathcal{O}(1/\sqrt{n})$. We fix a positive constant t and a sequence (t_n) satisfying $t_n \in [0, t/\sqrt{n}]$. Define

$$\bar{T}_n(t_n) = \sum_{|j-j_*| \ge n^{5/6}} x_j(n) \quad \text{and} \quad \hat{T}_n(t_n) = \sum_{|j-j_*| < n^{5/6}} x_j(n),$$

$$\bar{T}_{1,n}(t_n) = \sum_{|j-j_*| \ge n^{5/6}} j x_j(n) \quad \text{and} \quad \hat{T}_{1,n}(t_n) = \sum_{|j-j_*| < n^{5/6}} j x_j(n),$$

$$\bar{T}_{2,n}(t_n) = \sum_{|j-j_*| \ge n^{5/6}} j^2 x_j(n) \quad \text{and} \quad \hat{T}_{2,n}(t_n) = \sum_{|j-j_*| < n^{5/6}} j^2 x_j(n),$$

where

$$x_j(n) = \binom{n}{j} e^{2(B+t_n)j} g(\beta, dj, dn).$$

To prove (5.2), it suffices to show that

$$\left| \left[\frac{T_{2,n}(t_n)}{T_n(t_n)} - \left(\frac{T_{1,n}(t_n)}{T_n(t_n)} \right)^2 \right] - \left[\frac{\hat{T}_{2,n}(t_n)}{\hat{T}_n(t_n)} - \left(\frac{\hat{T}_{1,n}(t_n)}{\hat{T}_n(t_n)} \right)^2 \right] \right| \le \frac{3}{n^3} \tag{5.4}$$

and

$$\frac{4}{n} \left(\frac{\hat{T}_{2,n}(t_n)}{\hat{T}_n(t_n)} - \left(\frac{\hat{T}_{1,n}(t_n)}{\hat{T}_n(t_n)} \right)^2 \right) \longrightarrow \frac{-4}{H''(t_*)} \qquad \text{as } n \to \infty. \tag{5.5}$$

Before proving (5.4) and (5.5), we give some estimates on the terms $(x_j(n))$. Let us start with binomial coefficients. By Stirling's approximation (see [22]), for all $j \ge 1$,

$$\begin{aligned} \log(\sqrt{2\pi j}) + j \log j - j + \frac{1}{12j+1} &\le \log(j!) \\ &\le \log(\sqrt{2\pi j}) + j \log j - j + \frac{1}{12j}. \end{aligned}$$

Using this approximation, we can easily prove that

$$\binom{n}{j} \le e^{nI(j/n)} \qquad \text{for all } j = 0, \dots, n$$

and

$$\binom{n}{j} = \bigl(1 + \mathcal{O}(n^{-1})\bigr) \sqrt{\frac{n}{2\pi j(n-j)}} \times e^{nI(j/n)}, \qquad \text{for } |j - j_*| < n^{5/6},$$

where the function $I(t)$ is defined in (4.2). Therefore, for all $j \le n$,

$$\begin{aligned} x_j(n) &\le e^{nI(j/n)+2(B+t_n)j} g(\beta, dj, dn) \\ &= \exp\bigl(n\bigl[I(j/n) + 2Bj/n + dF(j/n)\bigr] \\ &\quad + \bigl[\log g(\beta, dj, dn) - ndF(j/n)\bigr] + 2t_n j\bigr) \\ &= \exp\bigl(nL(j/n) + \bigl[\log g(\beta, dj, dn) - ndF(j/n)\bigr] + 2t_n j\bigr), \end{aligned} \tag{5.6}$$

with $L(t)$ as in (4.2). Similarly, for n large enough and $|j - j_*| < n^{5/6}$,

$$\begin{aligned} x_j(n) &= \bigl(1 + \mathcal{O}(n^{-1})\bigr) \sqrt{\frac{n}{2\pi j(n-j)}} \times e^{nI(j/n)+2(B+t_n)j} g(\beta, dj, dn) \\ &= \bigl(1 + \mathcal{O}(n^{-1})\bigr) \sqrt{\frac{n}{2\pi j(n-j)}} \\ &\quad \times \exp\bigl(nL(j/n) + \bigl[\log g(\beta, dj, dn) - ndF(j/n)\bigr] + 2t_n j\bigr). \end{aligned} \tag{5.7}$$

We have some observations on the function $L(t)$ and its derivatives. Since $L(t)$ attains the maximum at a unique point $t_* \in (0, 1)$:

(O1) $L'(t_*) = 0$ and $L''(t_*) < 0$.

(O2) There exists a positive constant ε, such that for all $\kappa \le \varepsilon$,

$$\max_{|t-t_*|\ge\kappa} L(t) = \max\{L(t_* - \kappa), L(t_* + \kappa)\}.$$

(O3) For $\delta = (1 - t_*)/2$, the functions $|L'(t)|$, $|L''(t)|$, $|L'''(t)|$ are uniformly bounded in $(t_* - \delta, t_* + \delta)$.

I. Proof of (5.4). By (5.6) and (5.7), we have for all $|j - j_*| \ge n^{5/6}$,

$$\begin{aligned}\frac{x_j(n)}{x_{j_*}(n)} \le 2\sqrt{\frac{2\pi j_*(n-j_*)}{n}} \exp\bigl(2t_n(j - j_*) \\ + n\bigl[L(j/n) - L(j_*/n)\bigr] \\ + \bigl[\log g(\beta, dj, dn) - ndF(j/n)\bigr] \\ - \bigl[\log g(\beta, dj_*, dn) - ndF(j_*/n)\bigr]\bigr).\end{aligned} \tag{5.8}$$

For n large enough [such that $n^{-1/6} \le \varepsilon$ as in (O2)] and for all $|j - j_*| \ge n^{5/6}$, we have

$$L(j/n) - L(j_*/n) \le \max\{L(j_*/n \pm n^{-1/6}) - L(j_*/n)\}. \tag{5.9}$$

By (O3) and Taylor's expansion, we get

$$\begin{aligned}&L(j_*/n \pm n^{-1/6}) - L(j_*/n) \\ &\quad = \pm n^{-1/6} L'(j_*/n) + n^{-1/3} L''(j_*/n)/2 + \mathcal{O}(n^{-1/2}).\end{aligned}$$

Similarly,

$$L'(j_*/n) = L'(t_*) + \mathcal{O}(|(j_*/n) - t_*|) = \mathcal{O}(1/n), \tag{5.10}$$

since $L'(t_*) = 0$ and $|(j_*/n) - t_*| \le 1/n$. Therefore,

$$\begin{aligned}&n\bigl(L(j_*/n \pm n^{-1/6}) - L(j_*/n)\bigr) \\ &\quad = n^{2/3} L''(j_*/n)/2 + \mathcal{O}(n^{1/2}).\end{aligned} \tag{5.11}$$

On the other hand, since $L''(t_*) < 0$ and the sequence (j_*/n) converges to t_*, for all n large enough

$$L''(j_*/n) \le L''(t_*)/2.$$

Combining this with (5.9), (5.11) gives that for all $|j - j_*| \ge n^{5/6}$,

$$n\bigl(L(j/n) - L(j_*/n)\bigr) \le n^{2/3} L''(t_*)/8. \tag{5.12}$$

We now come back to the formula (5.8). Observe that

$$\sqrt{\frac{j_*(n-j_*)}{n}} \le \frac{\sqrt{n}}{2}. \tag{5.13}$$

On the other hand, by Lemma 2.1(ii), for all $j \le n$,

$$\left|\log g(\beta, dj, dn) - ndF(j/n)\right| = \mathcal{O}(1). \tag{5.14}$$

Since $t_n \le t/\sqrt{n}$, we have

$$t_n(j - j_*) = \mathcal{O}(\sqrt{n}). \tag{5.15}$$

It follows from (5.8), (5.12), (5.13), (5.14), (5.15) that, for n large enough and $|j - j_*| \ge n^{5/6}$,

$$\frac{x_j(n)}{x_{j_*}(n)} \le \sqrt{2\pi n}\exp\bigl(n^{2/3}L''(t_*)/8 + \mathcal{O}(\sqrt{n})\bigr) \le n^{-9}, \tag{5.16}$$

since $L''(t_*) < 0$. Therefore,

$$\bar{T}_n(t_n), \bar{T}_{1,n}(t_n), \bar{T}_{2,n}(t_n) \le \frac{x_{j_*}(n)}{n^6} \le \frac{\hat{T}_n(t_n)}{n^6}.$$

On the other hand,

$$\hat{T}_{1,n}(t_n), \hat{T}_{2,n}(t_n) \le n^2 \hat{T}_n(t_n).$$

Hence

$$\begin{aligned}
&\left|\frac{T_{2,n}(t_n)}{T_n(t_n)} - \frac{\hat{T}_{2,n}(t_n)}{\hat{T}_n(t_n)}\right| \\
&\quad = \left|\frac{\bar{T}_{2,n}(t_n) + \hat{T}_{2,n}(t_n)}{\bar{T}_n(t_n) + \hat{T}_n(t_n)} - \frac{\hat{T}_{2,n}(t_n)}{\hat{T}_n(t_n)}\right| \\
&\quad \le \frac{\hat{T}_n(t_n)\bar{T}_{2,n}(t_n) + \hat{T}_{2,n}(t_n)\bar{T}_n(t_n)}{\hat{T}_n(t_n)^2} \\
&\quad \le \frac{1+n^2}{n^6}.
\end{aligned} \tag{5.17}$$

Similarly, we also have

$$\left|\frac{T_{1,n}(t_n)}{T_n(t_n)} - \frac{\hat{T}_{1,n}(t_n)}{\hat{T}_n(t_n)}\right| \le \frac{1+n^2}{n^6}.$$

Since $T_{1,n}(t_n) \le nT_n(t_n)$ and $\hat{T}_{1,n}(t_n) \le n\hat{T}_n(t_n)$,

$$\frac{T_{1,n}(t_n)}{T_n(t_n)} + \frac{\hat{T}_{1,n}(t_n)}{\hat{T}_n(t_n)} \le 2n.$$

Combining the last two inequalities, we get

$$\left|\left(\frac{T_{1,n}(t_n)}{T_n(t_n)}\right)^2 - \left(\frac{\hat{T}_{1,n}(t_n)}{\hat{T}_n(t_n)}\right)^2\right| \le \frac{2(1+n^2)}{n^5}. \tag{5.18}$$

Using (5.17) and (5.18), we can deduce (5.4).

II. Proof of (5.5).

IIa. Expression of $\hat{T}_n(t_n)$, $\hat{T}_{1,n}(t_n)$ *and* $\hat{T}_{2,n}(t_n)$. Observe that

$$\hat{T}_n(t_n) = \sum_{|j-j_*|<n^{5/6}} x_j(n) = x_{j_*}(n) B_n,$$

where

$$B_n = \sum_{|j-j_*|<n^{5/6}} \frac{x_j(n)}{x_{j_*}(n)}. \tag{5.19}$$

Similarly, we also have

$$\begin{aligned}\hat{T}_{1,n}(t_n) &= \sum_{|j-j_*|<n^{5/6}} j x_j(n)\\ &= x_{j_*}(n)\Bigg[\sum_{|j-j_*|<n^{5/6}} (j-j_*)\frac{x_j(n)}{x_{j_*}(n)} + j_* \sum_{|j-j_*|<n^{5/6}} \frac{x_j(n)}{x_{j_*}(n)}\Bigg]\\ &= x_{j_*}(n)[B_{1,n} + j_* B_n],\end{aligned}$$

where

$$B_{1,n} = \sum_{|j-j_*|<n^{5/6}} (j-j_*)\frac{x_j(n)}{x_{j_*}(n)}, \tag{5.20}$$

and

$$\begin{aligned}\hat{T}_{2,n}(t_n) &= \sum_{|j-j_*|<n^{5/6}} j^2 x_j(n)\\ &= x_{j_*}(n)\Bigg[\sum_{|j-j_*|<n^{5/6}} (j-j_*)^2\frac{x_j(n)}{x_{j_*}(n)}\\ &\quad + 2j_* \sum_{|j-j_*|<n^{5/6}} (j-j_*)\frac{x_j(n)}{x_{j_*}(n)} + j_*^2 \sum_{|j-j_*|<n^{5/6}} \frac{x_j(n)}{x_{j_*}(n)}\Bigg]\\ &= x_{j_*}(n)\big[B_{2,n} + 2j_* B_{1,n} + j_*^2 B_n\big],\end{aligned}$$

where

$$B_{2,n} = \sum_{|j-j_*|<n^{5/6}} (j-j_*)^2\frac{x_j(n)}{x_{j_*}(n)}. \tag{5.21}$$

Therefore,

$$\begin{aligned}
&\frac{\hat{T}_{2,n}(t_n)}{\hat{T}_n(t_n)} - \left(\frac{\hat{T}_{1,n}(t_n)}{\hat{T}_n(t_n)}\right)^2 \\
&\qquad = \frac{B_{2,n} + 2j_* B_{1,n} + j_*^2 B_n}{B_n} - \left(\frac{B_{1,n} + j_* B_n}{B_n}\right)^2 \\
&\qquad = \frac{B_{2,n}}{B_n} - \left(\frac{B_{1,n}}{B_n}\right)^2.
\end{aligned} \tag{5.22}$$

IIb. Estimate of the quotient $x_j(n)/x_{j_*}(n)$. By (5.7), for $|j - j_*| < n^{5/6}$

$$\begin{aligned}
\frac{x_j(n)}{x_{j_*}(n)} = (1 + \mathcal{O}(n^{-1}))\sqrt{\frac{j_*(n-j_*)}{j(n-j)}} \exp\bigl(2t_n(j - j_*) \\
+ n[L(j/n) - L(j_*/n)] \\
+ [\log g(\beta, dj, dn) - ndF(j/n)] \\
- [\log g(\beta, dj_*, dn) - ndF(j_*/n)]\bigr).
\end{aligned} \tag{5.23}$$

Since $j_* = [nt_*]$, with $t_* \in (\frac{1}{2}, 1)$ fixed, we have for n large enough and $|j - j_*| < n^{5/6}$,

$$\sqrt{\frac{j_*(n-j_*)}{j(n-j)}} = 1 + \mathcal{O}(|j - j_*|/n) = 1 + \mathcal{O}(n^{-1/6}). \tag{5.24}$$

It follows from Lemma 2.1(i) that, for all j,

$$\begin{aligned}
&\bigl|[\log g(\beta, dj, dn) - ndF(j/n)] - [\log g(\beta, dj_*, dn) - ndF(j_*/n)]\bigr| \\
&\qquad = \mathcal{O}(|j - j_*|/n).
\end{aligned} \tag{5.25}$$

As for (5.11), by using (O3), (5.10) and Taylor's expansion we have for all $|j - j_*| < n^{5/6}$,

$$\begin{aligned}
&n\bigl(L(j/n) - L(j_*/n)\bigr) \\
&\quad = n\left(L'\left(\frac{j_*}{n}\right)\left(\frac{j - j_*}{n}\right) + L''\left(\frac{j_*}{n}\right)\frac{(j-j_*)^2}{2n^2} + \mathcal{O}\left(\left(\frac{j-j_*}{n}\right)^3\right)\right) \\
&\quad = L''\left(\frac{j_*}{n}\right)\frac{(j-j_*)^2}{2n} + \mathcal{O}\left(\frac{j-j_*}{n}\right) + \mathcal{O}\left(\frac{(j-j_*)^3}{n^2}\right) \\
&\quad = \left[L''\left(\frac{j_*}{n}\right) + \mathcal{O}(n^{-1/6})\right]\frac{(j-j_*)^2}{2n} + \mathcal{O}(n^{-1/6}),
\end{aligned}$$

where in the last line we used that

$$\mathcal{O}\left(\frac{(j-j_*)^3}{n^2}\right) = \frac{(j-j_*)^2}{2n}\mathcal{O}\left(\frac{(j-j_*)}{n}\right) = \frac{(j-j_*)^2}{2n}\mathcal{O}(n^{-1/6}).$$

On the other hand,

$$L''(j_*/n) = H''(j_*/n) = H''(t_*) + \mathcal{O}(1/n).$$

Therefore,

$$n\big(L(j/n) - L(j_*/n)\big) = \left[\frac{H''(t_*)}{2} + \mathcal{O}\big(n^{-1/6}\big)\right]\frac{(j-j_*)^2}{n} + \mathcal{O}\big(n^{-1/6}\big). \tag{5.26}$$

Let us define

$$\alpha_* := H''(t_*)/2 = L''(t_*)/2 < 0.$$

Using (5.23), (5.24), (5.25) and (5.26), we get that for any $\varepsilon \in (0, |\alpha_*|/8)$, for all n large enough and $|j - j_*| < n^{5/6}$,

$$\frac{x_j(n)}{x_{j_*}(n)} \le (1+\varepsilon)\exp\biggl((\alpha_* + \varepsilon)\frac{(j-j_*)^2}{n} + 2t_n(j-j_*)\biggr) \tag{5.27}$$

and

$$\frac{x_j(n)}{x_{j_*}(n)} \ge (1-\varepsilon)\exp\biggl((\alpha_* - \varepsilon)\frac{(j-j_*)^2}{n} + 2t_n(j-j_*)\biggr). \tag{5.28}$$

IIc. Estimate of B_n, $B_{1,n}$ and $B_{2,n}$. Using the integral approximation and the fact that $t_n\sqrt{n}$ is uniformly bounded, we can prove that for all $\alpha < 0$,

$$\begin{aligned}
&\sum_{|j-j_*|<n^{5/6}} \exp\biggl(\frac{\alpha(j-j_*)^2}{n} + 2t_n(j-j_*)\biggr)\\
&\quad = \sum_{|j|<n^{5/6}} \exp\biggl(\alpha\frac{j^2}{n} + 2(t_n\sqrt{n})\frac{j}{\sqrt{n}}\biggr)\\
&\quad = \sum_{j=-\infty}^{\infty} \exp\biggl(\alpha\frac{j^2}{n} + 2(t_n\sqrt{n})\frac{j}{\sqrt{n}}\biggr) + o(1)\\
&\quad = \sqrt{n}\int_{-\infty}^{\infty} e^{\alpha x^2 + 2(t_n\sqrt{n})x}\,dx + \mathcal{O}(1).
\end{aligned} \tag{5.29}$$

Similarly,

$$\begin{aligned}
&\sum_{|j-j_*|<n^{5/6}} (j-j_*)\exp\biggl(\frac{\alpha(j-j_*)^2}{n} + 2t_n(j-j_*)\biggr)\\
&\quad = n\int_{-\infty}^{\infty} x e^{\alpha x^2 + 2(t_n\sqrt{n})x}\,dx + \mathcal{O}(\sqrt{n})
\end{aligned} \tag{5.30}$$

and

$$\sum_{|j-j_*|<n^{5/6}} (j-j_*)^2 \exp\biggl(\frac{\alpha(j-j_*)^2}{n} + 2t_n(j-j_*)\biggr) \tag{5.31}$$
$$= n\sqrt{n}\int_{-\infty}^{\infty} x^2 e^{\alpha x^2 + 2(t_n\sqrt{n})x}\, dx + \mathcal{O}(n).$$

Now we are in the position to estimate B_n, $B_{1,n}$ and $B_{2,n}$. It follows from (5.19), (5.27), (5.28) and (5.29) that, for any $\varepsilon \in (0, |\alpha_*|/8)$, and n large enough

$$(1-2\varepsilon)\sqrt{n}A(\alpha_* - \varepsilon, t_n\sqrt{n}) \le B_n \le (1+2\varepsilon)\sqrt{n}A(\alpha_* + \varepsilon, t_n\sqrt{n}), \tag{5.32}$$

where for $\alpha < 0$ and $\gamma \in \mathbb{R}$,

$$A(\alpha,\gamma) := \int_{-\infty}^{\infty} e^{\alpha x^2 + \gamma x}\, dx.$$

Similarly, using (5.20), (5.27), (5.28), (5.30),

$$(1-2\varepsilon)nA_1(\alpha_* - \varepsilon, t_n\sqrt{n}) \le B_{1,n} \le (1+2\varepsilon)nA_1(\alpha_* + \varepsilon, t_n\sqrt{n}), \tag{5.33}$$

where

$$A_1(\alpha,\gamma) := \int_{-\infty}^{\infty} x e^{\alpha x^2 + \gamma x}\, dx.$$

Combining (5.21), (5.27), (5.28), (5.31), we have

$$(1-2\varepsilon)n\sqrt{n}A_2(\alpha_* - \varepsilon, t_n\sqrt{n}) \tag{5.34}$$
$$\le B_{2,n} \le (1+2\varepsilon)n\sqrt{n}A_2(\alpha_* + \varepsilon, t_n\sqrt{n}),$$

where

$$A_2(\alpha,\gamma) := \int_{-\infty}^{\infty} x^2 e^{\alpha x^2 + \gamma x}\, dx.$$

IId. Conclusion. We observe that the derivatives with respect to α at α_* of the functions $A(\alpha,\gamma)$, $A_1(\alpha,\gamma)$ and $A_2(\alpha,\gamma)$ are bounded. Hence, for any $t > 0$, there exists a positive constant $C = C(t)$, such that for all $|\gamma| \le t$,

$$\bigl|A(\alpha_* \pm \varepsilon, \gamma) - A(\alpha_*, \gamma)\bigr| \le CA(\alpha_*, \gamma)\varepsilon,$$
$$\bigl|A_1(\alpha_* \pm \varepsilon, \gamma) - A_1(\alpha_*, \gamma)\bigr| \le CA_1(\alpha_*, \gamma)\varepsilon,$$
$$\bigl|A_2(\alpha_* \pm \varepsilon, \gamma) - A_2(\alpha_*, \gamma)\bigr| \le CA_2(\alpha_*, \gamma)\varepsilon.$$

Combining these above estimates with (5.32), (5.33), (5.34), we get that for any $\varepsilon \in (0, |\alpha_*|/8)$ and n large enough,

$$\lambda_\varepsilon^- \sqrt{n}A(\alpha_*, 2t_n\sqrt{n}) \le B_n \le \lambda_\varepsilon^+ \sqrt{n}A(\alpha_*, 2t_n\sqrt{n}), \tag{5.35}$$

$$\lambda_\varepsilon^- nA_1(\alpha_*, 2t_n\sqrt{n}) \le B_{1,n} \le \lambda_\varepsilon^+ nA_1(\alpha_*, 2t_n\sqrt{n}), \tag{5.36}$$

$$\lambda_\varepsilon^- n\sqrt{n}A_2(\alpha_*, 2t_n\sqrt{n}) \le B_{2,n} \le \lambda_\varepsilon^+ n\sqrt{n}A_2(\alpha_*, 2t_n\sqrt{n}), \tag{5.37}$$

where

$$\lambda_\varepsilon^- = (1 - C\varepsilon)(1 - 2\varepsilon), \qquad \lambda_\varepsilon^+ = (1 + C\varepsilon)(1 + 2\varepsilon).$$

It follows from (5.22), (5.35), (5.36) and (5.37) that

$$\begin{aligned}
&\frac{4}{n}\left[\frac{\hat{T}_{2,n}(t_n)}{\hat{T}_n(t_n)} - \left(\frac{\hat{T}_{1,n}(t_n)}{\hat{T}_n(t_n)}\right)^2\right] \\
&\quad = \frac{4}{n}\left[\frac{B_{2,n}}{B_n} - \left(\frac{B_{1,n}}{B_n}\right)^2\right] \\
&\quad \le 4\left[\left(\frac{\lambda_\varepsilon^+}{\lambda_\varepsilon^-}\right)\frac{A_2(\alpha_*, 2t_n\sqrt{n})}{A(\alpha_*, 2t_n\sqrt{n})} - \left(\frac{\lambda_\varepsilon^-}{\lambda_\varepsilon^+}\right)^2\left(\frac{A_2(\alpha_*, 2t_n\sqrt{n})}{A(\alpha_*, 2t_n\sqrt{n})}\right)^2\right]
\end{aligned} \tag{5.38}$$

and

$$\begin{aligned}
&\frac{4}{n}\left[\frac{\hat{T}_{2,n}(t_n)}{\hat{T}_n(t_n)} - \left(\frac{\hat{T}_{1,n}(t_n)}{\hat{T}_n(t_n)}\right)^2\right] \\
&\quad = \frac{4}{n}\left[\frac{B_{2,n}}{B_n} - \left(\frac{B_{1,n}}{B_n}\right)^2\right] \\
&\quad \ge 4\left[\left(\frac{\lambda_\varepsilon^-}{\lambda_\varepsilon^+}\right)\frac{A_2(\alpha_*, 2t_n\sqrt{n})}{A(\alpha_*, 2t_n\sqrt{n})} - \left(\frac{\lambda_\varepsilon^+}{\lambda_\varepsilon^-}\right)^2\left(\frac{A_2(\alpha_*, 2t_n\sqrt{n})}{A(\alpha_*, 2t_n\sqrt{n})}\right)^2\right].
\end{aligned} \tag{5.39}$$

Note that $A(\alpha, \gamma)$, $A_1(\alpha, \gamma)$, $A_2(\alpha, \gamma)$ are related to moments of the normal distribution with mean $\gamma/(2\alpha)$ and variance $1/(-2\alpha)$. By some simple calculus, we have

$$\frac{A_2(\alpha, \gamma)}{A(\alpha, \gamma)} - \left(\frac{A_1(\alpha, \gamma)}{A(\alpha, \gamma)}\right)^2 = \frac{-1}{2\alpha}.$$

Thus

$$\frac{A_2(\alpha_*, 2t_n\sqrt{n})}{A(\alpha_*, 2t_n\sqrt{n})} - \left(\frac{A_1(\alpha_*, 2t_n\sqrt{n})}{A(\alpha_*, 2t_n\sqrt{n})}\right)^2 = \frac{-1}{2\alpha_*} = \frac{-1}{H''(t_*)}. \tag{5.40}$$

Using (5.38), (5.39), (5.40), and the fact that $\lambda_\varepsilon^-/\lambda_\varepsilon^+ \to 1$ as $\varepsilon \to 0$, we can deduce that

$$\frac{4}{n}\left[\frac{\hat{T}_{2,n}(t_n)}{\hat{T}_n(t_n)} - \left(\frac{\hat{T}_{1,n}(t_n)}{\hat{T}_n(t_n)}\right)^2\right] \to \frac{-4}{H''(t_*)},$$

as $n \to \infty$. We complete the proof of (5.5). □

6. Proof of Proposition 1.4. In this section we assume that $\beta > \beta_c$ and $B = 0$. Then for all $\sigma \in \Omega_n$,

$$\mu_n(\sigma) = \mu_n(-\sigma). \tag{6.1}$$

By this symmetry of μ_n, we observe that Proposition 1.4 follows if, as $n \to \infty$,

$$\mu_n\left(\left|\frac{S_n}{n} - \nu\right| < n^{-1/6}\right) \longrightarrow \frac{1}{2}, \tag{6.2}$$

with $\nu = \lim_{B\to 0^+} \mathcal{M}(\beta, B)$. We now prove (6.2) using the same strategy as in Section 5. By (3.2) and (3.5), we have

$$\mu_n(\sigma) = \frac{g(\beta, d|\sigma_+|, dn)}{\sum_{j=0}^n \binom{n}{j} g(\beta, dj, dn)}. \tag{6.3}$$

We have proved in Claim 2a in Section 4 that on $(\frac{1}{2}, 1)$, the function $H'(t)$ has a unique zero t_+, which is the maximum point of $H(t)$. In addition, $t_* \to t_+$ as $B \to 0^+$, with t_* the unique zero of $L'(t)$. Let us define

$$\nu = 2t_+ - 1 = \lim_{B\to 0^+} 2t_* - 1 = \lim_{B\to 0^+} \mathcal{M}(\beta, B).$$

Since $S_n = 2|\sigma_+| - n$,

$$\mu_n\left(\left|\frac{S_n}{n} - \nu\right| < n^{-1/6}\right) = \mu_n\left(\left|\frac{|\sigma_+|}{n} - t_+\right| < \frac{1}{2n^{1/6}}\right).$$

Combining this with (6.3), we get

$$\mu_n\left(\left|\frac{S_n}{n} - \nu\right| < \frac{1}{2n^{1/6}}\right) = \frac{1}{\sum_{j=0}^n y_j(n)} \times \sum_{j=0}^n y_j(n)\mathbb{I}\left(\left|\frac{j}{n} - t_+\right| < \frac{1}{2n^{1/6}}\right), \tag{6.4}$$

where

$$y_j(n) = \binom{n}{j} g(\beta, dj, dn).$$

By (2.5), we have $y_j(n) = y_{n-j}(n)$. Hence

$$\sum_{j=0}^n y_j(n) = 2\left(\sum_{j=[n/2]+1}^n y_j(n) + z/2\right) =: 2R_n,$$

where

$$z = \begin{cases} y_{[n/2]}(n) & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Let us define

$$\hat{R}_n = \sum_{j=0}^n y_j(n)\mathbb{I}\left(\left|\frac{j}{n} - t_+\right| < \frac{1}{2n^{1/6}}\right) \quad \text{and} \quad \bar{R}_n = R_n - \hat{R}_n.$$

Note that $t_+ \in (\frac{1}{2}, 1)$, so all the indices in the definition of $\hat{R}_n$ are in the sum R_n. Now we observe that by (6.4), equation (6.2) is equivalent to

$$\frac{\hat{R}_n}{R_n} \to 1. \tag{6.5}$$

Using the same idea of the proof of Lemma 5.1, we define

$$j_+ = [nt_+].$$

By analogous arguments for (5.8) with the notice that if $B = 0$, $H(\cdot) = L(\cdot)$, we can show that for all j

$$\begin{aligned}\frac{y_j(n)}{y_{j_+}(n)} \leq \sqrt{2\pi n}\exp\bigl(&n\bigl[H(j/n) - H(j_+/n)\bigr] \\ &+ \bigl[\log g(\beta, dj, dn) - ndF(j/n)\bigr] \\ &- \bigl[\log g(\beta, dj_+, dn) - ndF(j_+/n)\bigr]\bigr).\end{aligned}$$

Using the same arguments for (5.16), we also have that for all j satisfying $j \geq [n/2]$ and $|(j/n) - t_+| \geq n^{-1/6}$,

$$y_j(n) \leq \frac{y_{j_+}(n)}{n^9}. \tag{6.6}$$

Note that here $H(t)$ and t_+ play the same role of $L(t)$ and t_* as in the proof of (5.16). Using (6.6), we get

$$\bar{R}_n \leq \frac{y_{j_+}(n)}{n^8} \leq \frac{\hat{R}_n}{n^8}.$$

Thus

$$\frac{\hat{R}_n}{R_n} = \frac{\hat{R}_n}{\bar{R}_n + \hat{R}_n} \longrightarrow 1,$$

and (6.5) follows.

7. The annealed pressure of Ising model on the configuration model.

Let G_n be the configuration model whose the vertex set is $V_n = \{v_1, \ldots, v_n\}$ and the degrees of vertices (D_i) are i.i.d. integer-valued random variables with the same distribution as D. Assume that

$$\mathbb{E}(e^{sD}) < \infty \qquad \text{for all } s \in \mathbb{R}. \tag{7.1}$$

Notice that the condition (7.1) is necessary, since without it the partition function has infinite expectation when β is large enough.

Now we study the annealed pressure of the Ising model on G_n. We use the same notation as in the proof of Theorem 1.1(i). Observe that for all $\sigma \in \Omega_n$,

$$\sum_{i=1}^n \sigma_i = 2|\sigma_+| - n,$$

$$\sum_{i \leq j} k_{i,j}\sigma_i\sigma_j = \ell_n/2 - 2e(\sigma_+, \sigma_-),$$

where for all $1 \le j \le n$,

$$\ell_j = D_1 + \cdots + D_j.$$

Using (2.2) and the fact that $(D_i)_{1\le i\le n}$ are i.i.d. random variables, we have, if $|\sigma_+| = |\sigma'_+|$,

$$\ell_n/2 - 2e(\sigma_+, \sigma_-) \overset{(\mathcal{D})}{=} \ell_n/2 - 2e(\sigma'_+, \sigma'_-).$$

Hence, using the same arguments as for Theorem 1.1(i), we obtain

$$\mathbb{E}(Z_n(\beta, B)) = e^{-Bn} \sum_{j=0}^{n} \binom{n}{j} e^{2Bj} b(\beta, j, n), \tag{7.2}$$

where

$$b(\beta, j, n) = \mathbb{E}(\exp[\beta\ell_n/2 - 2\beta e(U_j, U_j^c)]),$$

with

$$U_j = \{v_1, \dots, v_j\}.$$

Using (2.2) once again, we have

$$\mathcal{L}(e(U_j, U_j^c) \mid (D_i)_{1\le i\le n}) = \mathcal{L}(X(\ell_j, \ell_n) \mid (D_i)_{1\le i\le n}),$$

where $X(k, m)$ is defined as in (2.1) for all $k \le m$. Hence

$$\mathbb{E}_{(D_i)}(\exp[-2\beta e(U_j, U_j^c)]) = g(\beta, \ell_j, \ell_n),$$

where $\mathbb{E}_{(D_i)}$ is the expectation w.r.t. configuration model conditioning on the sequence of degrees $(D_i)_{i\le n}$, and $g(\beta, k, m)$ is defined as in (2.3). Thus

$$\mathbb{E}_{(D_i)}(\exp[\beta\ell_n/2 - 2\beta e(U_j, U_j^c)]) = \exp(\beta\ell_n/2) g(\beta, \ell_j, \ell_n).$$

Therefore,

$$\begin{aligned} b(\beta, j, n) &= \bar{\mathbb{E}}(\mathbb{E}_{(D_i)}(\exp[\beta\ell_n/2 - 2\beta e(U_j, U_j^c)])) \\ &= \bar{\mathbb{E}}(\exp(\beta\ell_n/2) g(\beta, \ell_j, \ell_n)), \end{aligned}$$

where $\bar{\mathbb{E}}$ is the expectation w.r.t. the sequence of degrees $(D_i)_{i\le n}$. By Lemma 2.1(ii), there is a positive constant $C = C(\beta)$, such that for all $j \le n$

$$\exp(-C + \ell_n F(\ell_j/\ell_n)) \le g(\beta, \ell_j, \ell_n) \le \exp(C + \ell_n F(\ell_j/\ell_n)),$$

with $F(t)$ as in Theorem 1.1(i). Hence

$$\left| \frac{\log b(\beta, j, n)}{n} - \frac{\log \bar{\mathbb{E}}(\exp[\ell_n(\beta/2 + F(\ell_j/\ell_n))])}{n} \right| \le \frac{C}{n}. \tag{7.3}$$

For each $\beta \geq 0$, we define a sequence of functions on [0, 1] as follows:

$$G_n(\beta, t) = \frac{1}{n} \log \bar{\mathbb{E}}\big(\exp[\ell_n(\beta/2 + F(\ell_{[nt]}/\ell_n))]\big).$$

To study the limit of the sequence of functions $(G_n(\beta, t))_n$, we need a large deviation result for the vector $(\ell_{[nt]}, \ell_n)$. We use the standard notion of large deviation principle (LDP) as in [9]. Let (X_i) be a sequence of i.i.d. random variables. Suppose that for all $s \in \mathbb{R}$

$$\Lambda(s) = \mathbb{E}(e^{sX_1}) < \infty.$$

Let us define for $t \in [0, 1]$

$$Z_n(t) = \frac{1}{n} \sum_{i=1}^{[nt]} X_i.$$

Let ν_n be the law of $Z_n(\cdot)$ in $L_\infty([0, 1])$.

LEMMA 7.1 ([9], Lemma 5.1.8). *Let $\mathcal{Q}$ denote the collection of all ordered finite subsets of* $(0, 1]$. *For any $q = \{0 < t_1 < \cdots < t_{|q|} \leq 1\} \in \mathcal{Q}$ and $f : [0, 1] \to \mathbb{R}$, let $p_q(f)$ denote the vector $(f(t_1), \dots, f(t_{|q|})) \in \mathbb{R}^{|q|}$. Then the sequence of laws $(\nu_n \circ p_q^{-1})_n$ satisfies the LDP in $\mathbb{R}^{|q|}$ with the rate function*

$$R_q(z) = \sum_{i=1}^{|q|} (t_i - t_{i-1}) \Lambda^*\left(\frac{z_i - z_{i-1}}{t_i - t_{i-1}}\right),$$

where $z = (z_1, \dots, z_{|q|})$, $z_0 = t_0 = 0$, and

$$\Lambda^*(x) = \sup_{s \in \mathbb{R}} \{xs - \Lambda(s)\}.$$

Using this result, we can show the convergence of the sequence $(G_n(\beta, t))_n$.

LEMMA 7.2. *For all $\beta \geq 0$, the following assertions hold*:

(i) *There exists a positive constant C, such that for all $0 \leq s, t \leq 1$ and $n \geq 1$,*

$$\big|G_n(\beta, t) - G_n(\beta, s)\big| \leq C\left(|t - s| + \frac{1}{n}\right).$$

(ii) *For all $t \in [0, 1]$, we have*

$$\lim_{n \to \infty} G_n(\beta, t) = G(\beta, t),$$

where

$$G(\beta, t) = \sup_{a,b} \left\{ b\big(\beta/2 + F(a/b)\big) - t\Lambda^*\left(\frac{a}{t}\right) - (1 - t)\Lambda^*\left(\frac{b - a}{1 - t}\right) \right\},$$

with

$$\Lambda^*(x) = \sup_{s \in \mathbb{R}} \{xs - \Lambda(s)\},$$

and

$$\Lambda(s) = \log \bar{\mathbb{E}}(\exp(sD)).$$

Moreover, $G(\beta, t)$ *is a Lipschitz function*.

PROOF. We first prove (i). Observe that

$$0 \le F(t) \le 1 \quad \text{and} \quad \max_{t \in [0,1]} |F'(t)| = \max_{t \in [0,1/2]} |\log f(t)| \le 2\beta,$$

since the function $F(t)$ is symmetric about $1/2$ and $e^{-2\beta} \le f(\beta, t) \le 1$ for all $t \in [0, 1/2]$. Therefore,

$$\beta/2 + \max_{t \in [0,1]} (|F(t)| + |F'(t)|) \le r := 1 + 5\beta/2. \tag{7.4}$$

We claim that

$$\left|\log \bar{\mathbb{E}}(e^{\ell_n[\beta/2+F(\ell_j/\ell_n)]}) - \log \bar{\mathbb{E}}(e^{\ell_n[\beta/2+F(\ell_{j-1}/\ell_n)]})\right| \le C, \tag{7.5}$$

where

$$C = \max\{\log \bar{\mathbb{E}}(e^{3rD}) - \log \bar{\mathbb{E}}(e^{-2rD}), \log \bar{\mathbb{E}}(e^{2rD}) - \log \bar{\mathbb{E}}(e^{-3rD})\}.$$

Assuming (7.5), we can easily prove (i). Indeed, by repeatedly applying (7.5), we have for all $i \le j \le n$,

$$\left|\log \bar{\mathbb{E}}(e^{\ell_n[\beta/2+F(\ell_i/\ell_n)]}) - \log \bar{\mathbb{E}}(e^{\ell_n[\beta/2+F(\ell_j/\ell_n)]})\right| \le C|i - j|. \tag{7.6}$$

Thus

$$\begin{aligned}
&\left|G_n(\beta, t) - G_n(\beta, s)\right| \\
&\quad = \frac{1}{n}\left|\log \bar{\mathbb{E}}(e^{\ell_n[\beta/2+F(\ell_{[nt]}/\ell_n)]}) - \log \bar{\mathbb{E}}(e^{\ell_n[\beta/2+F(\ell_{[ns]}/\ell_n)]})\right| \\
&\quad \le C\left(|t - s| + \frac{1}{n}\right),
\end{aligned}$$

which implies (i).

PROOF OF (7.5). The idea is simple: using the mean value theorem and (7.4), we have for all $1 \le j \le n$,

$$\ell_n\left|F(\ell_j/\ell_n) - F(\ell_{j-1}/\ell_n)\right| \le \max_{t \in [0,1]} |F'(t)| D_j \le r D_j. \tag{7.7}$$

Hence (7.5) would immediately follow if ℓ_n and D_i are independent. Since this fact is not true, we break ℓ_n into two independent parts D_j and $\ell_{n,j}$, with

$$\ell_{n,j} = \ell_n - D_j.$$

We have

$$\begin{aligned}
&\ell_n\big(\beta/2 + F(\ell_{j-1}/\ell_n)\big)\\
&\quad= \ell_{n,j}\big(\beta/2 + F(\ell_{j-1}/\ell_n)\big) + D_j\big(\beta/2 + F(\ell_{j-1}/\ell_n)\big)\\
&\quad= \ell_{n,j}\big(\beta/2 + F(\ell_{j-1}/\ell_{n,j})\big) + \ell_{n,j}\big(F(\ell_{j-1}/\ell_n) - F(\ell_{j-1}/\ell_{n,j})\big)\\
&\qquad + D_j\big(\beta/2 + F(\ell_{j-1}/\ell_n)\big).
\end{aligned}$$

Using the mean value theorem and (7.4), we get

$$\big|\ell_{n,j}\big(F(\ell_{j-1}/\ell_n) - F(\ell_{j-1}/\ell_{n,j})\big) + D_j\big(\beta/2 + F(\ell_{j-1}/\ell_n)\big)\big| \le 2rD_j.$$

Therefore,

$$\big|\ell_n\big(\beta/2 + F(\ell_{j-1}/\ell_n)\big) - \ell_{n,j}\big(\beta/2 + F(\ell_{j-1}/\ell_{n,j})\big)\big| \le 2rD_j. \tag{7.8}$$

It follows from (7.7) and (7.8) that

$$\big|\ell_n\big(\beta/2 + F(\ell_j/\ell_n)\big) - \ell_{n,j}\big(\beta/2 + F(\ell_{j-1}/\ell_{n,j})\big)\big| \le 3rD_j. \tag{7.9}$$

On the other hand,

$$\ell_{n,j}\big(\beta/2 + F(\ell_{j-1}/\ell_{n,j})\big) \qquad \text{is independent of } D_j. \tag{7.10}$$

Using (7.9) and (7.10), we obtain

$$\bar{\mathbb{E}}\big(e^{-3rD_j}\big) \le \frac{\bar{\mathbb{E}}(e^{\ell_n[\beta/2+F(\ell_j/\ell_n)]})}{\bar{\mathbb{E}}(e^{\ell_{n,j}[\beta/2+F(\ell_{j-1}/\ell_{n,j})]})} \le \bar{\mathbb{E}}\big(e^{3rD_j}\big).$$

Similarly, using (7.8) and (7.10), we have

$$\bar{\mathbb{E}}\big(e^{-2rD_j}\big) \le \frac{\bar{\mathbb{E}}(e^{\ell_n[\beta/2+F(\ell_{j-1}/\ell_n)]})}{\bar{\mathbb{E}}(e^{\ell_{n,j}[\beta/2+F(\ell_{j-1}/\ell_{n,j})]})} \le \bar{\mathbb{E}}\big(e^{2rD_j}\big).$$

Combining the last two inequalities gives (7.5).

We now prove (ii). Applying Lemma 7.1 for $q = \{t_1 = t < t_2 = 1\}$, we get that the law of $\frac{1}{n}(\ell_{[nt]}, \ell_n)$ satisfies the LDP in $\mathbb{R}^2$ with the rate function

$$I(a,b) = t\Lambda^*\left(\frac{a}{t}\right) + (1-t)\Lambda^*\left(\frac{b-a}{1-t}\right),$$

where Λ^* is defined as in the statement of (ii). Therefore, using Varadhan's lemma (see, e.g., [9], Theorem 4.3.1), the sequence of functions $(G_n(\beta,\cdot))_n$ converges point-wise to the function $G(\beta,\cdot)$ defined as in the statement of (ii). Moreover, applying Lemma 2.2(ii) and Part (i), we obtain that $G(\beta,t)$ is a Lipschitz function. $\square$

PROPOSITION 7.3. *For all $\beta \geq 0$ and $B \in \mathbb{R}$, the annealed pressure converges to a limit given by*

$$\phi(\beta, B) = -B + \max_{0 \leq t \leq 1} \left[t \log\left(\frac{1}{t}\right) + (1-t)\log\left(\frac{1}{1-t}\right) + 2Bt + G(\beta, t) \right],$$

with $G(\beta, t)$ as in Lemma 7.2.

PROOF. Using (7.2), (7.3) and Stirling's formula, we get

$$\begin{aligned} &\frac{\log \mathbb{E}(Z_n(\beta, B))}{n} \\ &\quad = -B + \max_{0 \leq j \leq n} \left[\frac{\log \binom{n}{j}}{n} + 2B\frac{j}{n} + G_n(\beta, j/n) \right] + o(1) \qquad (7.11) \\ &\quad = -B + \max_{0 \leq j \leq n} \big[S(j/n) + G_n(\beta, j/n) \big] + o(1), \end{aligned}$$

where $S(t)$ is continuous function on [0, 1] defined by

$$S(t) = -t \log t + (t-1)\log(1-t) + 2Bt.$$

Now it follows from (7.11), Lemmas 7.2(ii) and 2.2(ii) that

$$\lim_{n \to \infty} \frac{\log \mathbb{E}(Z_n(\beta, B))}{n} = -B + \max_{0 \leq t \leq 1} \big[S(t) + G(\beta, t) \big],$$

which proves Proposition 7.3. □

REMARK 7.4. We can slightly extend Proposition 7.3 as follows. Let $(X_n)_{n \geq 1}$ and X be integer valued random variables satisfying

$$\sup_{s \in \mathbb{R}} \mathbb{E}\big(e^{sX}\big) < \infty \quad \text{and} \quad \mathbb{E}\big(e^{sX_n}\big) \to \mathbb{E}\big(e^{sX}\big) \qquad \forall s \in \mathbb{R}. \qquad (7.12)$$

For each n, let $(X_{n,i})_{i \leq n}$ be a sequence of i.i.d. random variables with the same distribution as X_n. Let G_n be the configuration model random graph of size n with sequence of degrees given by $(X_{n,i})_{i \leq n}$. Then Proposition 7.3 still holds for the annealed Ising model on G_n. A nice example of degree distribution is $X_n = \mathrm{Bin}(n, \gamma/n)$ and $X = \mathrm{Poi}(\gamma)$ for some $\gamma > 0$. This case is of particular interest due the closeness between the configuration models and Galton–Watson trees.

APPENDIX

A.1. Complement of the proof of Lemma 2.1. We first recall the formula of $f(t)$

$$f(t) = \frac{c(1-2t) + \sqrt{1 + (c^2-1)(1-2t)^2}}{2(1-t)}, \qquad (A.1)$$

which satisfies the fixed-point equation

$$\theta = \frac{c(1-2t)}{1-t} + \frac{t}{\theta(1-t)}, \tag{A.2}$$

with $c = e^{-2\beta}$.

PROOF OF (2.17) AND (2.18). It follows from (A.1) and (A.2) that, for all $0 \le t \le 1/2$,

$$f(t) \le \frac{(1-2t)+1}{2(1-t)} = 1 \tag{A.3}$$

and

$$f'(t) = \frac{-c}{(1-t)^2} + \frac{1}{f(t)(1-t)^2} - \frac{f'(t)t}{f(t)^2(1-t)}.$$

Hence

$$f'(t)\left(1 + \frac{t}{f(t)^2(1-t)}\right) = \frac{(1/f(t)) - c}{(1-t)^2} > 0. \tag{A.4}$$

Thus $f(t)$ is increasing in $(0, 1/2)$. Therefore,

$$c = f(0) \le f(t) \le 1, \tag{A.5}$$

which implies (2.17). It follows from (A.4) and (A.5) that, for all $t \in (0, 1/2)$,

$$\frac{1-c}{1+1/c^2} \le f'(t) \le \frac{4(1-c^2)}{c}. \tag{A.6}$$

Similarly,

$$f''(t) = \frac{-2c}{(1-t)^3} + \frac{2}{f(t)(1-t)^3} - \frac{2f'(t)}{f(t)^2(1-t)^2} - \frac{t}{1-t}\left(\frac{f''(t)f(t)^2 - 2f(t)f'(t)^2}{f(t)^4}\right).$$

Hence

$$f''(t)\left(1 + \frac{t}{(1-t)f(t)^2}\right) = \frac{2(1/f(t) - c)}{(1-t)^3} - \frac{2f'(t)}{(1-t)f(t)^2}\left(\frac{1}{1-t} - \frac{tf'(t)}{f(t)}\right).$$

Using this together with (A.5) and (A.6), we can show that there is a positive constant $A = A(c)$, such that for all $t \in (0, 1/2)$,

$$1/A \le f'(t) \le A \quad \text{and} \quad |f''(t)| \le A. \tag{A.7}$$

Thus (2.18) holds. □

PROOF OF (2.19). Let us recall the sequence $h(k,m)$ defined in Section 2: $h(1,m)=c$ and for $k\le[m/2]$,

$$h(k+1,m)=\frac{c(m-2k)}{m-k}+\frac{k}{(m-k)h(k,m)}. \tag{A.8}$$

We define

$$K=\left[\frac{A^2c^2+2}{c^3}\right],$$

with A as in (A.7). We first claim that for $m\ge 4K$ and $1\le k\le k_*:=[m/2]-K$,

$$f\big((k-1)/m\big)\le h(k,m)\le f(k/m). \tag{A.9}$$

Assuming (A.9), we now prove (2.19). Let us define for $k\le[m/2]$,

$$a_k=\big|h(k,m)-f\big((k-1)/m\big)\big|.$$

By (A.9), for $0\le k\le k_*$ we have

$$a_k\le\big|f(k/m)-f\big((k-1)/m\big)\big|\le A/m, \tag{A.10}$$

by using the mean value theorem and (A.7). To estimate (a_k) with $k\ge k_*$, we need some bounds on $h(k,m)$. By (2.14), we have for all $k_*\le k\le[m/2]=k_*+K$,

$$\frac{1}{2h(k,m)}\le h(k+1,m)\le c+\frac{1}{h(k,m)}.$$

Moreover, $c\le h(k_*,m)\le 1$ by (A.5) and (A.9). Thus there exists a positive constant $\Theta=\Theta(K,c)\ge 1$, such that for all $k_*\le k\le[m/2]$,

$$1/\Theta\le h(k,m)\le\Theta. \tag{A.11}$$

By (A.1), we have for $k\le[m/2]$,

$$f(k/m)=\frac{c(m-2k)}{m-k}+\frac{k}{(m-k)f(k/m)}. \tag{A.12}$$

Then using (A.8) and (A.12), we get that for $k_*\le k\le[m/2]$,

$$\begin{aligned}
a_{k+1}&=\frac{k}{m-k}\left|\frac{1}{h(k,m)}-\frac{1}{f(k/m)}\right|\\
\text{[use (A.5), (A.11)]}\qquad&\le\frac{\Theta|h(k,m)-f(k/m)|}{c}\\
&\le\frac{\Theta|h(k,m)-f((k-1)/m)|}{c}\\
&\quad+\frac{\Theta|f(k,m)-f((k-1)/m)|}{c}\\
\text{[use (A.7)]}\qquad&\le\frac{\Theta a_k}{c}+\frac{\Theta A}{mc}.
\end{aligned} \tag{A.13}$$

Using (A.13), we can prove by induction on t that for all $k_* \le k_* + t \le [m/2]$,

$$a_{k_*+t} \le \left(\frac{\Theta}{c}\right)^t a_{k_*} + \frac{\Theta A}{mc} \sum_{i=0}^{t-1} \left(\frac{\Theta}{c}\right)^i \le \left(\frac{\Theta}{c}\right)^t a_{k_*} + \frac{A}{m}\left(\frac{\Theta}{c}\right)^{t+1}. \tag{A.14}$$

Using (A.10) and (A.14), we obtain that for all $k \le [m/2]$,

$$a_k \le \varkappa/m,$$

with

$$\varkappa = A\left[\left(\frac{\Theta}{c}\right)^K + \left(\frac{\Theta}{c}\right)^{K+1}\right],$$

which implies (2.19).

We now prove (A.9) by induction on k. For $k = 1$, we have

$$c = h(1, m) = f(0/m) \le f(1/m),$$

since $f(t)$ is increasing. Suppose that (A.9) holds for all $k \le k_* - 1 = [m/2] - K - 1$. We now show that it holds for $k + 1$. Using (A.8) and (A.12) and $h(k, m) \le f(k/m)$, we get

$$\begin{aligned} h(k+1, m) &= \frac{c(m-2k)}{m-k} + \frac{k}{(m-k)h(k,m)} \\ &\ge \frac{c(m-2k)}{m-k} + \frac{k}{(m-k)f(k/m)} \\ &= f(k/m). \end{aligned} \tag{A.15}$$

Similarly, using $f((k-1)/m) \le h(k, m)$, we obtain

$$\begin{aligned} h(k+1, m) &\le \frac{c(m-2k)}{m-k} + \frac{k}{(m-k)f((k-1)/m)} \\ &= f(k/m) + \frac{k}{m-k}\left(\frac{1}{f((k-1)/m)} - \frac{1}{f(k/m)}\right) \\ &\le f(k/m) + \frac{k}{m-k}\left(\frac{f(k/m) - f((k-1)/m)}{f((k-1)/m)^2}\right), \end{aligned}$$

since $f(t)$ is increasing in [0, 1/2]. Let us define for $k \le k_*$

$$b_k = f(k/m) - f\big((k-1)/m\big).$$

Then

$$\begin{aligned} &f\big((k+1)/m\big) - h(k+1, m) \\ &\quad \ge b_{k+1} - \frac{k}{m-k}\frac{b_k}{f((k-1)/m)^2} \\ &\quad = b_{k+1} - b_k + b_k\left(1 - \frac{k}{(m-k)f((k-1)/m)^2}\right). \end{aligned} \tag{A.16}$$

Using the mean value theorem, we get

$$b_k = \frac{f'(y_k)}{m} \quad \text{and} \quad b_{k+1} = \frac{f'(y_{k+1})}{m}, \tag{A.17}$$

for some $y_k \in ((k-1)/m, k/m)$ and $y_{k+1} \in (k/m, (k+1)/m)$. Using the mean value theorem, (A.7) and the fact that $|y_k - y_{k+1}| \le 2/m$, we have

$$b_{k+1} - b_k = \frac{f'(y_{k+1}) - f'(y_k)}{m} \ge \frac{-2}{m^2} \max_{y_k \le t \le y_{k+1}} |f''(t)| \ge \frac{-2A}{m^2}. \tag{A.18}$$

Using (A.12), we obtain

$$\begin{aligned} 1 - \frac{(k-1)}{(m-k+1)f((k-1)/m)^2} &= \frac{c(m-2k+2)}{(m-k+1)f((k-1)/m)} \\ &\ge \frac{c(m-2k+2)}{(m-k+1)}, \end{aligned}$$

since $f(t) \le 1$ for all $t \le 1/2$. On the other hand, for $k \le [m/2]$,

$$\begin{aligned} \left| \frac{k}{(m-k)f((k-1)/m)^2} - \frac{(k-1)}{(m-k+1)f((k-1)/m)^2} \right| &\le \frac{4}{mf((k-1)/m)^2} \\ &\le \frac{4}{mc^2}, \end{aligned}$$

since $c \le f(t)$ for all $t \le 1/2$. Combining the last two inequalities yields that

$$\begin{aligned} 1 - \frac{k}{(m-k)f((k-1)/m)^2} &\ge \frac{c(m-2k+2)}{(m-k+1)} - \frac{4}{mc^2} \\ &\ge \frac{c(m-2k+2)}{m} - \frac{4}{mc^2}. \end{aligned} \tag{A.19}$$

It follows from (A.16), (A.17), (A.18), (A.19) that

$$\begin{aligned} f\big((k+1)/m\big) - h(k+1,m) &\ge -\frac{2A}{m^2} + \frac{f'(y_k)}{m}\left(\frac{c(m-2k+2)}{m} - \frac{4}{mc^2}\right) \\ &\ge -\frac{2A}{m^2} + \frac{1}{Am}\left(\frac{c(m-2k+2)}{m} - \frac{4}{mc^2}\right) \\ &\ge 0, \end{aligned} \tag{A.20}$$

by using (A.7) and the fact that

$$k \le [m/2] - \left[\frac{A^2c^2+2}{c^3}\right].$$

It follows from (A.15) and (A.20) that the induction step from k to $k+1$ holds. Thus the proof of (A.9) is complete. □

A.2. Proof of Lemma 3.1. Assume that $t \le 1/2$. Using integration by parts, we have

$$F(t) = \int_0^t \log f(s)\,ds = t \log f(t) - \int_0^t \frac{f'(s)}{f(s)} s\,ds. \tag{A.21}$$

We have $f(s) = A(s)/B(s)$, where

$$A(s) = e^{-2\beta}(1-2s) + \sqrt{1 + \big(e^{-4\beta} - 1\big)(2s-1)^2} \quad \text{and} \quad B(s) = 2(1-s).$$

Moreover,

$$\frac{A'(s)}{A(s)} = \frac{1}{2s(1-s)}\left[1 - 2s - \frac{e^{-2\beta}}{\sqrt{1 + (e^{-4\beta}-1)(2s-1)^2}}\right]$$

and

$$\frac{B'(s)}{B(s)} = \frac{-1}{1-s}.$$

Hence

$$\begin{aligned} \frac{f'(s)s}{f(s)} &= s\left[\frac{A'(s)}{A(s)} - \frac{B'(s)}{B(s)}\right] \\ &= \frac{1}{2(1-s)} - \frac{e^{-2\beta}}{2(1-s)\sqrt{1 + (e^{-4\beta}-1)(2s-1)^2}}. \end{aligned} \tag{A.22}$$

Combining (A.21) and (A.22) gives that

$$\begin{aligned} F(t) &= t \log f(t) + \frac{1}{2}\log(1-t) \\ &\quad + \int_0^t \frac{e^{-2\beta}}{2(1-s)\sqrt{1 + (e^{-4\beta}-1)(2s-1)^2}}\,ds. \end{aligned} \tag{A.23}$$

Let $\alpha = \sqrt{1 - e^{-4\beta}} \in (0, 1)$. Then by computation and changing variables, we have

$$\begin{aligned} J &= \int_0^t \frac{e^{-2\beta}}{(1-s)\sqrt{1 + (e^{-4\beta}-1)(2s-1)^2}}\,ds \\ (u = 1-2s) \qquad &= \int_{1-2t}^1 \frac{\sqrt{1-\alpha^2}}{(1+u)\sqrt{1-\alpha^2 u^2}}\,du \\ \big[v = \arcsin(\alpha u)\big] \qquad &= \int_{\arcsin(\alpha(1-2t))}^{\arcsin(\alpha)} \frac{\sqrt{1-\alpha^2}}{\alpha + \sin v}\,dv \\ \big[w = \tan(v/2)\big] \qquad &= \int_{w_1}^{w_2} \frac{2\sqrt{1-\alpha^2}}{\alpha w^2 + 2w + \alpha}\,dw \\ &= \log\left(\frac{\alpha w + 1 - \sqrt{1-\alpha^2}}{\alpha w + 1 + \sqrt{1-\alpha^2}}\right)\Bigg|_{w_1}^{w_2}. \end{aligned}$$

For $x \in (0,1)$, we have

$$\tan\left(\frac{\arcsin(x)}{2}\right) = \frac{1-\sqrt{1-x^2}}{x}.$$

Thus

$$w_2 = \frac{1-\sqrt{1-\alpha^2}}{\alpha} \quad \text{and} \quad w_1 = \frac{1-\sqrt{1-\alpha^2(1-2t)^2}}{\alpha(1-2t)}.$$

Therefore,

$$\begin{aligned} J &= \log(1-e^{-2\beta}) \\ &\quad - \log\left(\frac{(1-e^{-4\beta})(1-2t)+(1-e^{-2\beta})[1+\sqrt{1+(e^{-4\beta}-1)(1-2t)^2}]}{(1-e^{-4\beta})(1-2t)+(1+e^{-2\beta})[1+\sqrt{1+(e^{-4\beta}-1)(1-2t)^2}]}\right) \\ &= \log(1+e^{-2\beta}) \\ &\quad - \log\left(\frac{(1+e^{-2\beta})(1-2t)+1+\sqrt{1+(e^{-4\beta}-1)(1-2t)^2}}{(1-e^{-2\beta})(1-2t)+1+\sqrt{1+(e^{-4\beta}-1)(1-2t)^2}}\right) \\ &= \log(1+e^{-2\beta}) + \log\left(\frac{1-t+tf(1-t)}{(1-t)(f(t)+1)}\right). \end{aligned}$$

Combining this with (A.23), we obtain

$$\begin{aligned} F(t) &= t\log f(t) + \frac{1}{2}\log(1-t) + \frac{1}{2}\log(1+e^{-2\beta}) \\ &\quad + \frac{1}{2}\log\left[1+\frac{e^{-2\beta}(2t-1)}{(1-t)(f(t)+1)}\right]. \end{aligned} \tag{A.24}$$

Thus Lemma 3.1 follows.

A.3. Proof of Proposition 3.2. We first recall the formula for the quenched pressure determined in [6]. Suppose that $\beta > 0$ and $B > 0$. Let h_* be the positive solution of a fixed-point equation:

$$h = B + (d-1)\operatorname{atanh}(\tanh(\beta)\tanh(h)). \tag{A.25}$$

Then

$$\begin{aligned} \tilde{\psi}(\beta, B) &= \frac{d}{2}\log(\cosh(\beta)) - \frac{d}{2}\log(1+\tanh(\beta)\tanh(h_*)^2) \\ &\quad + \log[e^B(1+\tanh(\beta)\tanh(h_*))^d \\ &\quad + e^{-B}(1-\tanh(\beta)\tanh(h_*))^d]. \end{aligned} \tag{A.26}$$

For $B < 0$, one has $\tilde{\psi}(\beta, B) = \tilde{\psi}(\beta, -B)$, and $\tilde{\psi}(\beta, 0) = \lim_{B\to 0^+}\tilde{\psi}(\beta, B)$. In this subsection, we will show that

$$\tilde{\psi}(\beta, B) = \psi(\beta, B). \tag{A.27}$$

We prove here the case $B > 0$, then the other case follows from the fact that both of the functions $\tilde{\psi}(\beta, \cdot)$ and $\psi(\beta, \cdot)$ are even. We have proved in Sections 3 and 4 that for $B > 0$,

$$\psi(\beta, B) = \frac{\beta d}{2} - B + L(t_*), \tag{A.28}$$

where

$$L(t) = -t\log(t) + (t-1)\log(1-t) + 2Bt + dF(t),$$

and $t_* \in (1/2, 1)$ is the unique solution of the equation

$$L'(t) = \log\left(\frac{1-t}{t}\right) - d\log f(1-t) + 2B = 0. \tag{A.29}$$

We claim a relation between h_* and t_*, which we will prove later:

$$2t_* - 1 = \tanh\bigl(h_* + \operatorname{atanh}\bigl(\tanh(\beta)\tanh(h_*)\bigr)\bigr). \tag{A.30}$$

Assuming (A.30), we now prove (A.27).

Expression of $\tilde{\psi}(\beta, B)$. Let us denote

$$u_* = \tanh(\beta)\tanh(h_*).$$

Applying the function tanh to both sides of equation (A.25), we get

$$\begin{aligned}\tanh(h_*) &= \tanh\bigl(B + (d-1)\operatorname{atanh}(u_*)\bigr)\\ &= \frac{e^{2B}(1+u_*)^{d-1} - (1-u_*)^{d-1}}{e^{2B}(1+u_*)^{d-1} + (1-u_*)^{d-1}}.\end{aligned} \tag{A.31}$$

Thus

$$1 + \tanh(\beta)\tanh(h_*)^2 = 1 + u_*\tanh(h_*) = \frac{e^{2B}(1+u_*)^{d} + (1-u_*)^{d}}{e^{2B}(1+u_*)^{d-1} + (1-u_*)^{d-1}}.$$

Therefore, using (A.26) we have

$$\begin{aligned}\tilde{\psi}(\beta, B) = {}& \frac{d}{2}\log\bigl(\cosh(\beta)\bigr)\\ & - \frac{d}{2}\log\left[\frac{e^{2B}(1+u_*)^{d} + (1-u_*)^{d}}{e^{2B}(1+u_*)^{d-1} + (1-u_*)^{d-1}}\right]\\ & + \log\bigl[e^{B}(1+u_*)^d + e^{-B}(1-u_*)^d\bigr].\end{aligned} \tag{A.32}$$

Expression of $\psi(\beta, B)$. We first display t_* and $e^{-2\beta}$ in term of u_*. Using (A.30), we have

$$2t_* - 1 = \tanh\bigl(B + d\operatorname{atanh}(u_*)\bigr) = \frac{e^{2B}(1+u_*)^{d} - (1-u_*)^{d}}{e^{2B}(1+u_*)^{d} + (1-u_*)^{d}}. \tag{A.33}$$

Thus

$$
\begin{aligned}
t_* &= \frac{e^{2B}(1+u_*)^d}{e^{2B}(1+u_*)^d + (1-u_*)^d}, \\
1 - t_* &= \frac{(1-u_*)^d}{e^{2B}(1+u_*)^d + (1-u_*)^d}.
\end{aligned} \tag{A.34}
$$

On the other hand, using (A.31) we get

$$
\begin{aligned}
e^{-2\beta} &= \frac{1-\tanh(\beta)}{1+\tanh(\beta)} \\
&= \frac{1 - \frac{u_*}{\tanh(h_*)}}{1 + \frac{u_*}{\tanh(h_*)}} \\
&= (1-u_*^2)\frac{e^{2B}(1+u_*)^{d-2} - (1-u_*)^{d-2}}{e^{2B}(1+u_*)^d - (1-u_*)^d}.
\end{aligned} \tag{A.35}
$$

Since $t_* > 1/2$, we have $F(t_*) = F(1-t_*)$. Thus using (A.24),

$$
\begin{aligned}
F(t_*) &= F(1-t_*) \\
&= (1-t_*)\log f(1-t_*) + \frac{1}{2}\log t_* + \frac{1}{2}\log(1+e^{-2\beta}) \\
&\quad + \frac{1}{2}\log\left[1 + \frac{e^{-2\beta}(1-2t_*)}{t_*(f(1-t_*)+1)}\right].
\end{aligned} \tag{A.36}
$$

Since t_* is the solution of (A.29), we have

$$
f(1-t_*) = e^{\frac{2B}{d}}\left(\frac{1-t_*}{t_*}\right)^{\frac{1}{d}}. \tag{A.37}
$$

Using (A.37) and (A.34),

$$
f(1-t_*) = \frac{1-u_*}{1+u_*}. \tag{A.38}
$$

Combining (A.33), (A.34), (A.35) and (A.38) yields that

$$
1 + \frac{e^{-2\beta}(1-2t_*)}{t_*(f(1-t_*)+1)} = \frac{(1+u_*)^2}{2} \times \frac{e^{2B}(1+u_*)^{d-1} + (1-u_*)^{d-1}}{e^{2B}(1+u_*)^d}. \tag{A.39}
$$

Using (A.36), (A.37) and (A.39),

$$
\begin{aligned}
dF(t_*) &= (1-t_*)\left(2B + \log\left(\frac{1-t_*}{t_*}\right)\right) \\
&\quad + \frac{d}{2}\log(1+e^{-2\beta}) + d\log(1+u_*)
\end{aligned}
$$

$$-\frac{d}{2}\log 2+\frac{d}{2}\log\biggl(t_*\frac{e^{2B}(1+u_*)^{d-1}+(1-u_*)^{d-1}}{e^{2B}(1+u_*)^d}\biggr)$$

$$=(2-2t_*)B+(1-t_*)\log\biggl(\frac{1-t_*}{t_*}\biggr)+\frac{d}{2}\log\bigl[\bigl(1+e^{-2\beta}\bigr)/2\bigr]$$

[use (A.34)] $$+d\log(1+u_*)+\frac{d}{2}\log\biggl(\frac{e^{2B}(1+u_*)^{d-1}+(1-u_*)^{d-1}}{e^{2B}(1+u_*)^d+(1-u_*)^d}\biggr).$$

Hence

$$\begin{aligned}\psi(\beta,B)&=\frac{\beta d}{2}-B+(t_*-1)\log(1-t_*)-t_*\log t_*+2Bt_*+dF(t_*)\\&=\frac{d}{2}\bigl(\beta+\log\bigl[\bigl(1+e^{-2\beta}\bigr)/2\bigr]\bigr)+B-\log t_*+d\log(1+u_*)\\&\quad+\frac{d}{2}\log\biggl(\frac{e^{2B}(1+u_*)^{d-1}+(1-u_*)^{d-1}}{e^{2B}(1+u_*)^d+(1-u_*)^d}\biggr)\\&=\frac{d}{2}\log\bigl(\cosh(\beta)\bigr)+\log\bigl(e^Bt_*^{-1}(1+u_*)^d\bigr)\\&\quad+\frac{d}{2}\log\biggl(\frac{e^{2B}(1+u_*)^{d-1}+(1-u_*)^{d-1}}{e^{2B}(1+u_*)^d+(1-u_*)^d}\biggr)\\&=\frac{d}{2}\log\bigl(\cosh(\beta)\bigr)+\log\bigl(e^B(1+u_*)^d+e^{-B}(1-u_*)^d\bigr)\\&\quad+\frac{d}{2}\log\biggl(\frac{e^{2B}(1+u_*)^{d-1}+(1-u_*)^{d-1}}{e^{2B}(1+u_*)^d+(1-u_*)^d}\biggr),\end{aligned}$$

where for the last line, we used (A.34). Using this equation with (A.32), we obtain that

$$\psi(\beta,B)=\tilde{\psi}(\beta,B),$$

which proves (A.27). For $d=2$, it has been shown in [16, 17] that

$$\psi(\beta,B)=\tilde{\psi}(\beta,B)=\beta+\log\bigl(\cosh(B)+\sqrt{\sinh^2(B)+e^{-4\beta}}\bigr).$$

PROOF OF (A.30). Let us denote

$$v_*=\tanh\bigl(h_*+\operatorname{atanh}\bigl(\tanh(\beta)\tanh(h_*)\bigr)\bigr).$$

We claim the following identity (**E**): For all $x>0$ and $y\in\mathbb{R}$, if

(A.40) $$v=\tanh\bigl(y+\operatorname{atanh}\bigl(\tanh(x)\tanh(y)\bigr)\bigr),$$

then

(A.41) $$\frac{e^{-2x}v+\sqrt{1+(e^{-4x}-1)v^2}}{v+1}=\frac{\cosh(x-y)}{\cosh(x+y)}.$$

Assuming (**E**), we can prove (A.30). Indeed, using (A.41) we get

$$\frac{e^{-2\beta}v_* + \sqrt{1+(e^{-4\beta}-1)v_*^2}}{v_*+1} = \frac{\cosh(\beta-h_*)}{\cosh(\beta+h_*)}. \tag{A.42}$$

Since h_* is the solution of (A.25), we have

$$v_* = \tanh\bigl(B + d\,\mathrm{atanh}\bigl(\tanh(\beta)\tanh(h_*)\bigr)\bigr).$$

Applying the function atanh to the both sides of the above equation, we obtain

$$\frac{1}{2}\log\biggl(\frac{1+v_*}{1-v_*}\biggr) = B + \frac{d}{2}\log\biggl(\frac{\cosh(\beta+h_*)}{\cosh(\beta-h_*)}\biggr).$$

Combining this with (A.42), we have

$$\log\biggl(\frac{1-v_*}{1+v_*}\biggr) - d\log\biggl(\frac{e^{-2\beta}v_* + \sqrt{1+(e^{-4\beta}-1)v_*^2}}{v_*+1}\biggr) + 2B = 0, \tag{A.43}$$

or equivalently, by using (A.29),

$$L'\biggl(\frac{v_*+1}{2}\biggr) = 0. \tag{A.44}$$

It follows from (A.29) and (A.44) that t_* and $(v_*+1)/2$ are solutions in $(1/2, 1)$ of the equation $L'(x) = 0$. We have proved in Claim 1^* in Section 4 that this equation has unique solution. Thus $t_* = (v_*+1)/2$, and (A.30) follows.

We now prove the identity (**E**). Applying the function atanh to the both sides of (A.40) gives that

$$\frac{1+v}{1-v} = \frac{e^{2y}\cosh(x+y)}{\cosh(x-y)}. \tag{A.45}$$

Hence, (A.41) is equivalent to

$$(1-v)e^{2y} = e^{-2x}v + \sqrt{1+\bigl(e^{-4x}-1\bigr)v^2},$$

or

$$\begin{cases} (1-v)e^{2y} - e^{-2x}v \ge 0, \\ \bigl((1-v)e^{2y} - e^{-2x}v\bigr)^2 = 1 + \bigl(e^{-4x}-1\bigr)v^2. \end{cases} \tag{A.46}$$

We have

$$(1-v)e^{2y} - e^{-2x}v \ge 0 \quad \Leftrightarrow \quad e^{2x+2y} \ge \frac{v}{1-v}.$$

On the other hand,

$$\frac{v}{1-v} \le \frac{1+v}{1-v} = \frac{e^{2y}\cosh(x+y)}{\cosh(x-y)} = \frac{e^{y-x}+e^{3y+x}}{e^{x-y}+e^{y-x}} \le e^{2x+2y},$$

since $x>0$. Hence the inequality in (A.46) holds. The equation in (A.46) is equivalent to

$$
\begin{aligned}
&(1-v)^2e^{4y}-2v(1-v)e^{2y-2x}=1-v^2\\
&\Leftrightarrow\quad \left(\frac{1-v}{1+v}\right)e^{4y}-\frac{2v}{1+v}e^{2y-2x}=1\\
&\Leftrightarrow\quad \left(\frac{1-v}{1+v}\right)e^{4y}+\left(\frac{1-v}{1+v}-1\right)e^{2y-2x}=1\\
&\Leftrightarrow\quad \left(\frac{1-v}{1+v}\right)(e^{4y}+e^{2y-2x})=1+e^{2y-2x}\\
\text{[using (A.45)]}\quad &\Leftrightarrow\quad e^{-2y}\left(\frac{e^{x-y}+e^{y-x}}{e^{x+y}+e^{-x-y}}\right)(e^{4y}+e^{2y-2x})=1+e^{2y-2x}\\
&\Leftrightarrow\quad e^{3y-x}+e^{y-3x}=e^{3y-x}+e^{y-3x},
\end{aligned}
$$

which holds for all x, y. In conclusion, (A.46) holds, so (A.41) follows. □

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