SUPER-REPLICATION WITH FIXED TRANSACTION COSTS

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We study super-replication of contingent claims in markets with fixed transaction costs. This can be viewed as a stochastic impulse control problem with a terminal state constraint. The first result in this paper reveals that in reasonable continuous time financial market models the super-replication price is prohibitively costly and leads to trivial buy-and-hold strategies. Our second result derives nontrivial scaling limits of super-replication prices for binomial models with small fixed costs.

1. Introduction. This paper deals with super-replication of European options in a market where trading incurs fixed transaction costs. Most papers dealing with fixed transaction costs explore the problem of optimal portfolio choice (see, for instance, [1, 9, 17, 23, 24] and [25]). Much fewer papers (see [15] and [21]) discuss no arbitrage criteria for fixed transaction costs and, to the best of our knowledge, the problem of super-replicating a contingent claim with fixed costs has not been considered in the literature before.

By contrast, for the case of proportional transaction costs, the topic of super-replication is widely studied. In [8], it was conjectured that, in the Black–Scholes model with proportional transaction costs, the cheapest way to super-replicate a call option is to buy one unit of stock right at the start and hold it until maturity. This conjecture was proved by many authors (see, e.g., [5, 13, 14, 22, 28] and for game options in [7]). A natural way to overcome this negative result was proposed by Kusuoka in [18]. He considered scaling limits of the classical Cox–Ross–Rubinstein model of a complete binomial market and showed that, when transaction costs are also rescaled properly, super-replication prices converge to what is now known as a *G*-expectation in the sense of Peng ([26]).

The present paper is a first step in the development of the above theory for the fixed transaction costs case. The setup of fixed transaction costs corresponds to the case where any (nonzero) transaction incurs a fixed cost of $\kappa > 0$, regardless of the trading volume. Clearly, this leads to discontinuous, nonconvex wealth dynamics which induce a stochastic impulse control problem with a novel terminal state

constraint. In particular, convex duality methods, which played a key role in the theory of proportional transaction costs (or their convex generalizations), are not available here.

As a first result, we show in Theorem 3.1 that, in a continuous time financial market with a risky asset exhibiting conditionally full support (see [13]), the cheapest way to super-replicate a convex option is again to apply a trivial buy-and-hold strategy. Hence, Theorem 3.1 can be viewed as an analog for fixed costs of the result in [13] which was obtained for the case of proportional transaction costs. By contrast to the classical duality used in [13], our proof uses the impulse control structure directly.

The second result in the present paper deals with the limiting behavior of super-replication prices in the Cox–Ross–Rubinstein binomial models of [4]. Specifically, we consider a sequence of binomial models with constant volatility and study the asymptotic behavior of the super-replication prices for convex payoffs when the time step goes to zero and the fixed transaction costs are scaled linearly as a function of the time step. In Theorem 4.1, we characterize the scaling limit as a stochastic volatility control problem defined on Wiener space.

Our proof relies heavily on the fact that the payoff of the European option is a convex function of the risky asset. Under this assumption, we derive a nonstandard dual representation for super-replication prices in the binomial models. This representation allow us to obtain the limit behavior of the super-replication prices by modifying ideas from [18]. We emphasize that without the convexity condition on the payoff the analysis is more complicated and remains an open question.

Closely related is the topic of *approximate hedging* which deals with the construction of portfolio strategies with terminal wealth close to the payoff of the derivative security. Approximate hedging in the context of market frictions is going back to the pioneering work of Leland [19] who considers a Black–Scholes model with vanishing proportional transaction costs. This approach is studied rigorously and extended (beyond Black–Scholes and beyond vanishing proportional transaction costs) in [3, 10, 11, 16, 20, 27]. The triviality of super-replication prices established in our Theorem 3.1 can also be viewed as a motivation for the study of approximate hedging in the fixed transaction costs setup.

The paper is organized as follows. In Section 2, we formalize the super-replication problem with fixed costs. Section 3 shows that, in models with conditional full support, trivial buy-and-hold strategies yield optimal super-replications of convex payoffs. In Section 4, we give the scaling limit of super-replication prices with small fixed costs. The proof of this result is prepared by a dual representation for our super-replication prices discussed in Section 4.1 and accomplished in Sections 4.2 and 4.3 by using tools from weak convergence of stochastic processes to analyze the asymptotic behavior of the dual terms.

2. Super-replication with fixed transaction costs. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ be a filtered probability space with a progressively measurable process $S > 0$ which

we take to describe the price evolution of some financial asset with initial price $S_0 = s_0 > 0$. The asset is traded at strictly positive fixed costs $\kappa > 0$ per transaction and so an investor with a bank account (that for simplicity bears no interest) can change her position only finitely often. We take $T = 1$ to be the investor's time horizon and so the times of intervention are given by a family of stopping times $\mathbb{T} = (\tau_i)_{i=1,2,\dots}$ such that

$$0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq T = 1 \qquad \text{with } \tau_i < \tau_{i+1} \text{ on } \{\tau_i < 1\}.$$

Let us denote by $\mathscr{T}$ the class of all such families $\mathbb{T}$ for which the number of interventions by time $T = 1$ is finite almost surely:

$$N(\mathbb{T}) := \sup\{i = 0, 1, \dots : \tau_i < 1\} < \infty, \qquad \mathbb{P}\text{-a.s.}$$

Notice that for simplicity we do not count a possible initial intervention at time $\tau_0 = 0$.

Assume our investor seeks to hedge an option with $\mathscr{F}_1$-measurable payoff $F \geq 0$ at maturity $T = 1$ by an investment strategy $(\mathbb{T}, \mathbb{H})$ where $\mathbb{H} = (h_i)_{i=0,1,\dots}$ describes the $\mathscr{F}_{\tau_i}$-measurable number of assets h_i to be held, respectively, over each period $(\tau_i, \tau_{i+1}]$, $i = 0, 1, \dots$. Keeping in mind the fixed transaction costs $\kappa > 0$ and the free trade at time 0, the investor's gains from trading will by time $t \leq 1$ have accrued to

$$G^\kappa(\mathbb{T}, \mathbb{H})_t := \sum_{i=0,1,\dots} h_i (S_{\tau_{i+1} \wedge t} - S_{\tau_i \wedge t}) - \kappa \sup\{i = 0, 1, \dots : \tau_i < t\}.$$

To rule out the possibility of doubling strategies, the investor can only use admissible strategies from the set

$$\mathscr{A} := \big\{ (\mathbb{T}, \mathbb{H}) : G^\kappa(\mathbb{T}, \mathbb{H}) \text{ bounded from below by a constant } \mathbb{P}\text{-a.s.} \big\}.$$

The option's super-replication price is then given by

$$\mathbb{V}^\kappa(F) := \inf\big\{ x \in \mathbb{R} : x + G^\kappa(\mathbb{T}, \mathbb{H})_1 \geq F \; \mathbb{P}\text{-a.s. for some } (\mathbb{T}, \mathbb{H}) \in \mathscr{A} \big\}.$$

Determining this super-replication price amounts to solving an impulse control problem with terminal state constraint, a task which cannot be carried out explicitly without further assumptions. We will show however that for convex payoffs it can be computed in models with conditional full support (Section 3). At the other end of the modeling spectrum, we consider binomial models converging to a Black–Scholes dynamic, for which we compute the scaling limit for suitably scaled fixed costs (Section 4).

REMARK 2.1. In the frictionless case $\kappa = 0$ with continuous stock prices, the above super-replication price is the classical one even given the constraint to an almost surely finite number of trades. This follows readily from the fact that the wealth process of any continuous-time trading strategy can be approximated uniformly (in time and almost all scenarios) by piecewise constant (admissible) trading strategies (Lemma A.3 in [22]).

3. Buy-and-hold with conditional full support. In this section, we consider a continuous model $S = (S_t)_{t \in [0,1]}$ exhibiting *conditional full support* as discussed by, for example, [13]:

(1) $\qquad \text{supp} \, \mathbb{P}[S|_{[t,1]} \in \cdot | \mathscr{F}_t] = C_{S_t}^+[t, 1], \qquad \mathbb{P}\text{-a.s. for any } t \in [0, 1],$

where, for $y \geq 0$, $C_y^+[t, 1]$ denotes the space of all continuous paths $[t, 1] \to \mathbb{R}_+$ starting in y at time t.

THEOREM 3.1. *For any financial model exhibiting conditional full support in the sense of* (1), *the super-replication price with fixed transaction costs $\kappa > 0$ of any convex payoff $F = f(S_1)$ with $f : [0, \infty) \to \mathbb{R}$ continuous and convex is*

(2) $\qquad \mathbb{V}^\kappa \big(f(S_1) \big) = f(0) + s_0 f'(\infty) \qquad \text{where } f'(\infty) := \sup_{s > 0} f'(s).$

In case $f'(\infty) < \infty$, a super-hedge with initial capital $\mathbb{V}^\kappa(f(S_1))$ is to buy $h_0 := f'(\infty)$ units of the asset at time $\tau_0 = 0$ and hold these until $T = 1$.

PROOF. That the right-hand side of (2) is sufficient for super-replication is trivial if $f'(\infty) = \infty$. If $f'(\infty) < \infty$, we can consider the described buy-and-hold strategy which yields

$$G^\kappa(\mathbb{T}, \mathbb{H})_1 = f'(\infty)(S_1 - S_0) \geq f'_-(S_1)S_1 - f'(\infty)S_0$$
$$\geq f(S_1) - f(0) - f'(\infty)S_0,$$

where both estimates are due to the convexity of f. This shows that $x_0 := f(0) + f'(\infty)S_0$ is enough to super-replicate $F = f(S_1)$.

Now consider $x < x_0$ and take a strategy $(\mathbb{T}, \mathbb{H})$ with gains process $G := G^\kappa(\mathbb{T}, \mathbb{H})$ such that $x + G_1 \geq F = f(S_1)$. We will show that such a strategy cannot be admissible. Specifically, with $\beta < f'(\infty)$ such that $x = f(0) + \beta S_0$, we will argue that

$$A_n := \big\{ \tau_n < 1, S_{\tau_n} < 2/\delta, x + G_{\tau_n} < f(0) + \beta S_{\tau_n} - n\kappa/2 \big\}$$

has positive probability for all $n = 1, 2, \ldots$, where $\delta \in (0, 1/s_0)$ is chosen small enough to ensure

$$f(0) + \beta s - \kappa < f(s) \qquad \text{for all } s < \delta \text{ and all } s > 1/\delta.$$

Such a choice of δ is possible since f is continuous at zero and convex on $[0, \infty)$ with $f'(\infty) > \beta$. Since $\kappa > 0$, it then follows that $x + G_{\tau_n} < f(0) + 2\beta/\delta - n\kappa/2$ on the set A_n with positive probability, $n = 1, 2, \ldots$, and so $G = G^\kappa(\mathbb{T}, \mathbb{H})$ is not bounded from below by a constant and $(\mathbb{T}, \mathbb{H})$ cannot be admissible.

We will prove $\mathbb{P}[A_n] > 0$, $n = 0, 1, \ldots$, by induction. By our choices of $\delta < 1/s_0$ and of β, we even have $\mathbb{P}[A_0] = 1$. Now assume, by way of contradiction,

that $\mathbb{P}[A_n] > 0$, but $\mathbb{P}[A_{n+1}] = 0$ for some n. Observe that on $A_n \cap \{\tau_{n+1} < 1\}$ we can estimate

$$
\begin{aligned}
x + G_{\tau_{n+1}} \\
(3) \qquad = x + G_{\tau_n} &+ \beta(S_{\tau_{n+1}} - S_{\tau_n}) + (h_n - \beta)(S_{\tau_{n+1}} - S_{\tau_n}) - \kappa \\
&< f(0) + \beta S_{\tau_{n+1}} - (n+1)\kappa/2 + (h_n - \beta)(S_{\tau_{n+1}} - S_{\tau_n}) - \kappa/2.
\end{aligned}
$$

Hence, A_{n+1} contains the set $A_n \cap \{\tau_{n+1} < 1\} \cap B_n$ where

$$
B_n := \Big\{ \sup_{\tau_n \le t \le 1} S_t < 2/\delta, \ \sup_{\tau_n \le t \le 1} \big\{ (h_n - \beta)(S_t - S_{\tau_n}) \big\} \le \kappa/2 \Big\}.
$$

Notice that $\tau_{n+1} = 1$ must hold almost surely on $A_n \cap B_n$ since, with $\mathbb{P}[A_{n+1}] = 0$, we also have

$$
0 = \mathbb{P}\big[A_n \cap \{\tau_{n+1} < 1\} \cap B_n\big] = \mathbb{P}\big[A_n \cap B_n\big] - \mathbb{P}\big[A_n \cap \{\tau_{n+1} = 1\} \cap B_n\big].
$$

Now, $A_n \cap B_n$ contains $A_n \cap \{h_n \ge \beta\} \cap C_n$ where

$$
C_n := B_n \cap \{S_1 \ge S_{\tau_n} \vee 1/\delta\}.
$$

On $A_n \cap \{h_n \ge \beta\} \cap C_n$, however, the super-replication property is violated since, on this set, we have $\tau_{n+1} = 1$ almost surely and estimate (3) gives

$$
x + G_1 = x + G_{\tau_{n+1}} < f(0) + \beta S_1 - (n+1)\kappa/2 < f(S_1)
$$

by choice of δ and definition of C_n. Hence, we deduce

$$
0 = \mathbb{P}\big[A_n \cap \{h_n \ge \beta\} \cap C_n\big] = \mathbb{E}\big[1_{A_n \cap \{h_n \ge \beta\}} \mathbb{P}[C_n | \mathscr{F}_{\tau_n}]\big].
$$

As the conditional full support property (1) holds also at stopping times when it holds at deterministic times (see Lemma 2.9 in [13]), we have $\mathbb{P}[C_n | \mathscr{F}_{\tau_n}] > 0$ almost surely on $A_n \cap \{h_n \ge \beta\}$. So the above identity yields that in fact $\mathbb{P}[A_n \cap \{h_n \ge \beta\}] = 0$.

Similarly, we will argue next that $\mathbb{P}[A_n \cap \{h_n < \beta\}] = 0$ so that in conjunction with $\mathbb{P}[A_n \cap \{h_n \ge \beta\}] = 0$ we arrive at the contradiction $\mathbb{P}[A_n] = 0$, completing our proof. Thus, let us first observe that $A_n \cap B_n$ contains $A_n \cap \{h_n < \beta\} \cap \tilde{C}_n$ where

$$
\tilde{C}_n := B_n \cap \{S_1 \le S_{\tau_n} \wedge \delta\}.
$$

Up to a $\mathbb{P}$-null set, however, we still have $A_n \cap \{h_n < \beta\} \cap \tilde{C}_n \subset \{\tau_{n+1} = 1\}$ and the super-replication property is again violated since, on this set, estimate (3) gives

$$
x + G_1 = x + G_{\tau_{n+1}} < f(0) + \beta S_1 - (n+1)\kappa/2 < f(S_1)
$$

by choice of δ and definition of $\tilde{C}_n$. Observing that also $\mathbb{P}[\tilde{C}_n | \mathscr{F}_{\tau_n}] > 0$ almost surely on $A_n \cap \{h_n < \beta\} \in \mathscr{F}_{\tau_n}$ allows us to deduce by the same reasoning as used for C_n that indeed $\mathbb{P}[A_n \cap \{h_n < \beta\}] = 0$. $\quad\square$

REMARK 3.1.   If we restrict ourselves to admissible strategies, the conditional full support property (1) guarantees absence of arbitrage (as it also does for proportional transaction costs; see [12, 13]). Indeed, assume that for a trading strategy $(\mathbb{T}, \mathbb{H})$ we have $G^\kappa(\mathbb{T}, \mathbb{H})_1 \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(G^\kappa(\mathbb{T}, \mathbb{H})_1 > 0) > 0$. Then, similarly to the above proof, we can argue by induction that for any $n = 1, 2, \ldots$, $\mathbb{P}(\tau_n < 1, G^\kappa(\mathbb{T}, \mathbb{H})_{\tau_n} < -n\kappa/2) > 0$, and thus, the gain process $G^\kappa(\mathbb{T}, \mathbb{H})_t, t \geq 0$ is not uniformly bounded from below. So $(\mathbb{T}, \mathbb{H})$ is not admissible. For more refined no arbitrage criteria, we refer to [15, 21].

4. Scaling limit of binomial superreplication prices.   In this section, we consider binomial Cox–Ross–Rubinstein models with fixed transaction costs and describe the scaling limit of super-replication prices for convex claims. To wit, we let $\Omega = \{-1, +1\}^{\bar{\mathbb{N}}_0}$, put $\zeta_i(\omega) = \omega_i$ for $\omega = (\omega_0, \omega_1, \ldots) \in \Omega$ and let $\mathbb{P}$ be the measure under which the ζ_i are i.i.d. with $\mathbb{P}[\zeta_i = 1] = 1/2$. The n-period binomial price process can now be specified as

$$(4) \qquad S_t^{(n)} = s_0 \exp\left(\frac{\sigma}{\sqrt{n}} \sum_{i=1}^{[nt]} \zeta_i\right), \qquad t \in [0, T],$$

and the underlying filtration $(\mathscr{F}_t^{(n)})$ is the one generated by $S^{(n)}$.

Obviously, when considered under their respective equivalent martingale measures $\mathbb{P}^{(n)} \approx \mathbb{P}$, these Cox–Ross–Rubinstein models $S^{(n)}$, $n = 1, 2, \ldots$, converge to a Black–Scholes model with constant volatility $\sigma > 0$. In light of Theorem 3.1, it is clear that in order to get a nontrivial limit for the corresponding super-replication prices with fixed transaction costs, one has to rescale the fixed costs suitably. Our next result shows that the correct scaling is of the order $1/n$ and it identifies the resulting scaling limit as a G-expectation with penalty involving stochastic volatility models. These are specified as martingale exponentials

$$(5) \qquad S_t^{(\nu)} = s_0 \exp\left(\int_0^t \nu_u \, dW_u - \frac{1}{2} \int_0^t \nu_u^2 \, du\right), \qquad t \in [0, 1],$$

where W is a standard Brownian motion on some complete probability space $(\Omega^W, \mathscr{F}^W, \mathbb{P}^W)$ and where ν is taken from the set $\mathscr{A}^W$ of all bounded, real-valued processes $\nu \geq \sigma$ on this space which are progressively measurable with respect to the augmented filtration $(\mathscr{F}_t^W)_{t \in [0,1]}$ generated by W.

THEOREM 4.1.   *For a convex payoff $F = f(S_1)$ with continuous, convex $f :$ $[0, \infty) \to \mathbb{R}$ with polynomial growth, the scaling limit of super-replication prices in the binomial models* (4) *with fixed costs κ/n, $n = 1, 2, \ldots$, is*

$$(6) \qquad \lim_{n \to \infty} \mathbb{V}^{\kappa/n}(f(S_1^{(n)})) = \inf_{\sigma \leq \nu \in \mathscr{A}^W} \mathbb{E}^W\left[f(S_1^{(\nu)}) + \kappa \int_0^1 g(\nu_t^2/\sigma^2) \, dt\right],$$

where $g : [1, \infty) \to (0, 1]$ is the linear interpolation supported by $g(n) = 1/n$, $n = 1, 2, \ldots$ and where the infimum is taken over all the probability spaces and volatility processes $\nu \geq \sigma$ described above.

The proof of Theorem 4.1 is prepared by a duality result for super-replication with fixed costs presented in Section 4.1. Section 4.2 then establishes ">" and Section 4.3 proves "≤" in (6), completing the proof.

Let us explain the intuition behind the above result. As will also be revealed by our proof below, the local volatility pattern v can be viewed as a continuous-time measure of trading activity. For this pattern to attain the infimum in (6), it has to trade off the option price $\mathbb{E}^W[f(S_1^{(v)})]$ against the expected costs $\mathbb{E}^W[\int_0^1 g(v_t^2/\sigma^2)\,dt]$. Indeed, since f is convex, the option price is increasing as a function of the volatility pattern $v \geq \sigma$, and thus would be minimized by $v \equiv \sigma$. This choice, however, incurs the maximum penalty as g is decreasing. This increased reference volatility is reminiscent of Leland's frictional trading recipe which suggests to use a delta hedging strategy with increased local volatility for approximate hedges with vanishing proportional transaction costs.

4.1. *Duality for binomial models with fixed transaction costs.* The starting point for the proof of Theorem 4.1 is a form of dual characterization of super-replication prices with fixed costs in binomial models which works for the special case of convex payoff profiles.

To specify this duality, let us fix $n \in \{1, 2, \ldots\}$ and consider the class $\mathscr{T}^{(n)}$ of systems $\mathbb{T} = \{0 = \tau_0 \leq \cdots \leq \tau_n = 1\} \in \mathscr{T}$ of $(\mathscr{F}_t^{(n)})$-stopping times with values in $\{0/n, 1/n, \ldots, 1\}$ such that if $\tau_{k+1}(\omega) < 1$ then

(7)
$$\xi(\omega) \equiv +1 \qquad \text{for all } i \in \{n\tau_k(\omega) + 1, \ldots, n\tau_{k+1}(\omega)\} \quad \text{or}$$
$$\xi(\omega) \equiv -1 \qquad \text{for all } i \in \{n\tau_k(\omega) + 1, \ldots, n\tau_{k+1}(\omega)\}.$$

In other words, the stopping times $\tau_k \leq \tau_{k+1}$ are such that $\tau_{k+1}(\omega) = 1$ in scenarios ω where $S^{(n)}(\omega)$ is not strictly increasing or strictly decreasing between $\tau_k(\omega)$ and $\tau_{k+1}(\omega)$. Also, already at time τ_k it is known by how many downward steps and how many upward steps the next stop τ_{k+1} will be reached. In other words, for suitable functions $\phi_k^\downarrow, \phi_k^\uparrow : \mathbb{R}_+^{k+1} \to \mathbb{N}$, the number of these steps can be written in the form $\phi_k^\downarrow(S_0^{(n)}, \ldots, S_{\tau_k}^{(n)})$ and $\phi_k^\uparrow(S_0^{(n)}, \ldots, S_{\tau_k}^{(n)})$, respectively, for each $k = 0, 1, \ldots$. Now let $\mathbb{Q}(\mathbb{T}) \ll \mathbb{P}$ be the unique martingale measure for $(S_{\tau_k}^{(n)})_{k=0,1,\ldots}$ with respect to $(\mathscr{F}_{\tau_k})_{k=0,1,\ldots}$ such that (7) holds also for $\mathbb{Q}(\mathbb{T})$-almost every ω with $\tau_{k+1}(\omega) = 1$. Hence, $\mathbb{Q}(\mathbb{T})$ only gives probability to the set of scenarios ω in which the terminal value $S_1^{(n)}(\omega)$ is reached from the latest $S_{\tau_k}^{(n)}(\omega)$ with $\tau_k(\omega) < 1$ in a strictly monotone way.

LEMMA 4.1. *In the n-step binomial model* (4) *with fixed transaction costs* $\kappa > 0$, *the super-replication costs of a payoff* $F = f(S_1^{(n)})$ *with* f *convex on* $(0, \infty)$ *are*

(8)
$$\mathbb{V}^\kappa(f(S_1^{(n)})) = \inf_{\mathbb{T} \in \mathscr{T}^{(n)}} \mathbb{E}_{\mathbb{Q}(\mathbb{T})}[f(S_1^{(n)}) + \kappa N(\mathbb{T})].$$

PROOF.   Let us start by proving "$\geq$" in (8). So take $x \in \mathbb{R}$ and an admissible $(\mathbb{T}, \mathbb{H})$ such that $x + G(\mathbb{T}, \mathbb{H})_1 \geq f(S_1^{(n)})$. By removing stopping points from $\mathbb{T} = \{\tau_k\}_{k=0,1,\ldots}$ if necessary we obtain a possibly coarser stopping system $\tilde{\mathbb{T}} = \{\tilde{\tau}_k\}_{k=0,1,\ldots}$ from our special class $\mathscr{T}^{(n)}$ such that, under the unique martingale measure $\mathbb{Q}(\tilde{\mathbb{T}})$ for $(S_{\tilde{\tau}_k}^{(n)})_{k=0,1,\ldots}$, we have $\tau_k = \tilde{\tau}_k$ almost surely for $k = 0, 1, \ldots$. Therefore, we still have the super-replication property $x + G(\tilde{\mathbb{T}}, \mathbb{H})_1 \geq f(S_1^{(n)})$ $\mathbb{Q}(\tilde{\mathbb{T}})$-a.s. This allows us to conclude

$$x = \mathbb{E}_{\mathbb{Q}(\tilde{\mathbb{T}})}\left[x + \sum_k h_k(S_{\tilde{\tau}_{k+1}}^{(n)} - S_{\tilde{\tau}_k}^{(n)})\right] \geq \mathbb{E}_{\mathbb{Q}(\tilde{\mathbb{T}})}[f(S_1^{(n)}) + \kappa N(\tilde{\mathbb{T}})]$$

as we wanted to show.

We next establish "$\leq$" in (8). To this end, fix $\mathbb{T} \in \mathscr{T}^{(n)}$, put $\mathbb{Q} := \mathbb{Q}(\mathbb{T})$, and denote $x := \mathbb{E}_{\mathbb{Q}}[f(S_1^{(n)}) + \kappa N(\mathbb{T})]$. Observe that, under $\mathbb{Q}$, the (frictionless) financial market with stock price process $(S_{\tau_k}^{(n)})_{k=0,1,\ldots,n}$ is a binomial market, and hence complete. The unique martingale measure is $\mathbb{Q}$. Thus, there exist measurable functions $\psi_k : \mathbb{R}_+^k \to \mathbb{R}$, $k = 0, 1, \ldots, n$ such that

$$(9) \quad x + \sum_{k=0}^{n-1} \psi_k(S_{\tau_1}^{(n)}, \ldots, S_{\tau_k}^{(n)})(S_{\tau_{k+1}}^{(n)} - S_{\tau_k}^{(n)}) = f(S_1^{(n)}) + \kappa N(\mathbb{T}), \qquad \mathbb{Q}\text{-a.s.}$$

Let us now use these maps ψ_k, $k = 0, 1, \ldots, n$, in order to construct a super-replicating strategy for our n-step binomial market with fixed transaction costs κ. For this, it will be convenient to consider the obvious expansion of our binomial model (4) from $[0, T] = [0, 1]$ to all of $[0, \infty)$. Let $\mathbb{P}^{(n)}$ still denote its locally equivalent martingale measure. Use the mappings $\phi^{\downarrow}$ and $\phi^{\uparrow}$ associated with the stopping system $\mathbb{T}$ to define another system of stopping times $\tilde{\mathbb{T}}$ by $\tilde{\tau}_0 := 0$ and, for $k = 0, 1, \ldots,$

$$
\begin{aligned}
(10) \quad \tilde{\tau}_{k+1} &:= \min\left\{t > \tilde{\tau}_k : \ln\left(\frac{S_t^{(n)}}{S_{\tilde{\tau}_k}^{(n)}}\right)/\frac{\sigma}{\sqrt{n}} = \phi_k^{\uparrow}(S_{\tilde{\tau}_1}^{(n)}, \ldots, S_{\tilde{\tau}_k}^{(n)})\right\} \\
&\wedge \min\left\{t > \tilde{\tau}_k : \ln\left(\frac{S_t^{(n)}}{S_{\tilde{\tau}_k}^{(n)}}\right)/\frac{\sigma}{\sqrt{n}} = -\phi_k^{\downarrow}(S_{\tilde{\tau}_1}^{(n)}, \ldots, S_{\tilde{\tau}_k}^{(n)})\right\}.
\end{aligned}
$$

Clearly, these successive two-sided level passage times $\tilde{\tau}_1, \ldots, \tilde{\tau}_n$ are finite $\mathbb{P}^{(n)}$-almost surely, with $\tilde{\tau}_n \geq 1$. To obtain a strategy on $[0, 1]$ we truncate and consider the trading strategy $(\hat{\mathbb{T}}, \mathbb{H})$ intervening at times $\hat{\tau}_k := \tilde{\tau}_k \wedge 1$ according to $\mathbb{H} = (h_k)_{k=0,1,\ldots}$ where

$$h_k := \psi_k(S_{\tilde{\tau}_1}^{(n)}, \ldots, S_{\tilde{\tau}_k}^{(n)}), \qquad k = 0, 1, \ldots.$$

In order to conclude our assertion, it is now sufficient to show that $x + G(\hat{\mathbb{T}}, \mathbb{H})_1 \geq f(S_1^{(n)})$ $\mathbb{P}^{(n)}$-a.s. In fact, we will argue that

(11) $$x + G(\tilde{\mathbb{T}}, \mathbb{H})_{\tilde{\tau}_n} \geq f(S_{\tilde{\tau}_n}^{(n)}) \quad \text{and} \quad \tilde{\tau}_n \geq 1, \qquad \mathbb{P}^{(n)}\text{-a.s.,}$$

which entails our assertion because

$$x + G(\hat{\mathbb{T}}, \mathbb{H})_1 = x + G(\tilde{\mathbb{T}}, \mathbb{H})_1 \geq \mathbb{E}_{\mathbb{P}^{(n)}}\big[x + G(\tilde{\mathbb{T}}, \mathbb{H})_{\tilde{\tau}_n} | \mathscr{F}_1^{(n)}\big]$$

$$\geq \mathbb{E}_{\mathbb{P}^{(n)}}\big[f(S_{\tilde{\tau}_n}^{(n)}) | \mathscr{F}_1^{(n)}\big] \geq f(S_1^{(n)}).$$

Here, the first estimate holds because $G(\tilde{\mathbb{T}}, \mathbb{H})$ is a super-martingale under $\mathbb{P}^{(n)}$, the second estimate is due to (11), and the final one is due to Jensen's inequality for the convex function f and the $\mathbb{P}^{(n)}$-martingale $S^{(n)}$ (which is uniformly bounded up to time $\tilde{\tau}_n$).

It remains to prove (11). For this, observe that by construction both sides of this inequality are functionals of $(S_{\tilde{\tau}_k}^{(n)})_{k=0,1,\dots}$. Moreover, this process is a binomial martingale under $\mathbb{P}^{(n)}$ with exactly the same jump characteristics as $(S_{\tau_k}^{(n)})_{k=0,1,\dots}$ under $\mathbb{Q}$ and, therefore,

$$\text{Law}\big((S_{\tilde{\tau}_k}^{(n)})_{k=0,1,\dots} | \mathbb{P}^{(n)}\big) = \text{Law}\big((S_{\tau_k}^{(n)})_{k=0,1,\dots} | \mathbb{Q}\big).$$

As a consequence, (11) is immediate from (9). $\quad\square$

For later use, let us also note the following lemma which illustrates the trade-off to be struck in our dual description of the super-replication problem: For a convex payoff, $\mathbb{E}_{\mathbb{Q}(\mathbb{T})}[f(S_1^{(n)})]$ may decrease when we add stops to $\mathbb{T}$ while of course any added stop will let the number of interventions $N(\mathbb{T})$ increase.

LEMMA 4.2. *If $\mathbb{T}' \in \mathscr{T}^{(n)}$ is a refinement of $\mathbb{T} \in \mathscr{T}^{(n)}$ in the sense that for any τ_k from $\mathbb{T}$, we have*

$$\tau_k = \max\{\tau'_{k'} \in \mathbb{T}' | \tau'_{k'} \leq \tau_k\},$$

then for any convex payoff profile $f : (0, \infty) \to \mathbb{R}$ we have

$$\mathbb{E}_{\mathbb{Q}(\mathbb{T}')}[f(S_1^{(n)})] \leq \mathbb{E}_{\mathbb{Q}(\mathbb{T})}[f(S_1^{(n)})].$$

PROOF. The measure $\mathbb{Q}(\mathbb{T})$ is a martingale measure for $(S_{\tau_k}^{(n)})_{k=0,1,\dots}$ that is absolutely continuous with respect to $\mathbb{P}$ and which attains the frictionless super-replication price of the convex payoff $f(S_1^{(n)})$ when trading is allowed only at times contained in $\mathbb{T}$. Obviously, refining $\mathbb{T}$ to $\mathbb{T}' \in \mathscr{T}^{(n)}$ offers more flexibility to find super-replication strategies, and thus cannot lead to a higher super-replication price. $\quad\square$

4.2. *Proof of the upper bound for super-replication prices.* In this section, we will prove that "$\geq$" holds in our formula (6) for the scaling limit. More precisely, we will establish

$$(12) \qquad \liminf_n \mathbb{V}^{\kappa/n}\big(f(S_1^{(n)})\big) \geq \inf_{\sigma \leq v \in \mathscr{A}^W} \mathbb{E}^W\left[f(S_1^{(v)}) + \kappa \int_0^1 g(v_t^2/\sigma^2)\,dt\right].$$

Without loss of generality (by passing to a subsequence) we assume that the limit $\lim_n \mathbb{V}^{\kappa/n}(f(S_1^{(n)}))$ exists in $[0, \infty]$.

By Lemma 4.1, we can find, for $n = 1.2, \ldots$, stopping systems $\mathbb{T}_0^{(n)} \in \mathscr{T}^{(n)}$ such that

$$\mathbb{V}^{\kappa/n}\big(f(S_1^{(n)})\big) \geq \mathbb{E}_{\mathbb{Q}(\mathbb{T}_0^{(n)})}\left[f(S_1^{(n)}) + \frac{\kappa}{n}N(\mathbb{T}_0^{(n)})\right] - \frac{1}{n}.$$

Hence, the $\liminf$ in (12) can be estimated if we get an understanding, as $n \uparrow \infty$, of the joint law of $S_1^{(n)}$ and $N(\mathbb{T}_0^{(n)})$ under $\mathbb{Q}(\mathbb{T}_0^{(n)})$. While tightness of this sequence of laws is not obvious, it can be established for a suitable refinement of $\mathbb{T}_0^{(n)}$ using an argument which we adapt from Kusuoka [18]. To this end, fix $m \in \{1, 2, \ldots\}$ and refine $\mathbb{T}_0^{(n)}$ if necessary in such a way that at most m steps are taken between any two stopping times. This gives us a stopping system $\mathbb{T}^{(n)} = \{\tau_k^{(n)}\}_{k=0,1,\ldots} \in \mathscr{T}^{(n)}$ with $N(\mathbb{T}_0^{(n)}) \geq N(\mathbb{T}^{(n)}) - [(n-1)/m]$ and

$$(13) \qquad\qquad \tau_{k+1}^{(n)} - \tau_k^{(n)} \leq \frac{m}{n} \qquad \text{on } \{\tau_{k+1}^{(n)} < 1\}.$$

In light of Lemma 4.2, we can now conclude that $\mathbb{Q}^{(n)} := \mathbb{Q}(\mathbb{T}^{(n)})$ satisfies

$$(14) \qquad \mathbb{V}^{\kappa/n}\big(f(S_1^{(n)})\big) \geq \mathbb{E}_{\mathbb{Q}^{(n)}}\left[f(S_1^{(n)}) + \frac{\kappa}{n}N(\mathbb{T}^{(n)})\right] - \frac{1}{n} - \frac{\kappa}{n}[(n-1)/m].$$

Hence (12) will be established upon letting $m \uparrow \infty$ once we can show that the $\liminf_{n\uparrow\infty}$ of the expectations in (14) is not smaller than the right-hand side of (12) for each $m = 1, 2, \ldots$. This will be accomplished using Kusuoka's tightness argument for which we consider the processes $M^{(n)}, n = 1, 2, \ldots$, given by

$$(15) \quad \begin{aligned} M_1^{(n)} &:= S_1^{(n)}, \\ M_t^{(n)} &:= S_{\tau_{k+1}^{(n)}}^{(n)} \qquad \text{for } t \in \big[\tau_k^{(n)} + 1/n, \tau_{k+1}^{(n)} + 1/n\big) \cap [0, 1), k = 0, 1, \ldots. \end{aligned}$$

Observe that $M^{(n)}$ is a version of the $\mathbb{Q}^{(n)}$-martingale with terminal value S_1^n:

$$M_t^{(n)} = \mathbb{E}_{\mathbb{Q}^{(n)}}\big[S_1^{(n)} | \mathscr{F}_t^{(n)}\big], \qquad t \in [0, 1], \mathbb{Q}^{(n)}\text{-a.s.}$$

LEMMA 4.3. *Suppose $\mathbb{T}^{(n)} \in \mathscr{T}^{(n)}, n = 1, 2, \ldots$, are partitions of $[0, 1]$ such that (13) holds $\mathbb{Q}^{(n)}$-almost surely where $\mathbb{Q}^{(n)} = \mathbb{Q}(\mathbb{T}^{(n)})$. Then the sequence of distributions $(\mathrm{Law}(S^{(n)}|\mathbb{Q}^{(n)}))_{n=1,2,\ldots}$ is tight on the Skorohod space $\mathbb{D}[0, 1]$. Any*

weak accumulation point is the law of a strictly positive continuous martingale M (in its own filtration) under some probability measure $\hat{\mathbb{P}}$ such that

(16) $\mathbb{E}_{\hat{\mathbb{P}}}\Big[\max_{t \in [0,1]} (M_t)^p\Big] \leq \sup_{n=1,2,\dots} \mathbb{E}_{\mathbb{Q}^{(n)}}\Big[\max_{t \in [0,1]} (M_t^{(n)})^p\Big] < \infty$ *for any $p \geq 0$.*

Moreover, the stochastic logarithm L of M/s_0, that is, the continuous local martingale L such that $M = s_0 \mathscr{E}(L)$, has quadratic variation $\langle L \rangle$ absolutely continuous with respect to Lebesgue measure with density $v^2 := d\langle L \rangle/dt \geq \sigma^2$.

In addition, along a suitable subsequence, we have the weak convergence

$$\mathrm{Law}\bigg(S^{(n)}, \int_0^{\cdot} \alpha_s^{(n)}\, ds \,\Big|\, \mathbb{Q}^{(n)}\bigg) \to \mathrm{Law}\bigg(M, \int_0^{\cdot} \frac{1}{2}(v_t^2/\sigma^2 - 1)\, dt \,\Big|\, \hat{\mathbb{P}}\bigg), \qquad n \uparrow \infty,$$

on $\mathbb{D}[0, 1] \times \mathbb{D}[0, 1]$ where

(17) $$\alpha_t^{(n)} := \frac{\sqrt{n}}{\sigma} |M_t^{(n)} - S_t^{(n)}|/S_t^{(n)}, \qquad t \in [0, 1]$$

with $M^{(n)}$ given by (15).

PROOF.   From (13), it follows that $\alpha_t^{(n)}$ is $\mathbb{Q}^{(n)}$ a.s. uniformly bounded (in n and t), and so the tightness of $(\mathrm{Law}(S^{(n)}|\mathbb{Q}^{(n)}))_{n=1,2,\dots}$ and the estimate (16) follow from Propositions 4.8 and 4.27 in [18]. The second part of the lemma follows from Lemma 7.1 in [8].   □

By Skorohod's representation theorem, we can find processes $\hat{S}^{(n)}, \hat{M}^{(n)}, \hat{\alpha}^{(n)}$, $n = 1, 2, \dots$, on a common probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{P}})$ which have for each $n = 1, 2, \dots$ the same joint law as their counterparts $(S^{(n)}, M^{(n)}, \alpha^{(n)})$ under $\mathbb{Q}^{(n)}$ and which are such that $(\hat{S}^{(n)}, \hat{M}^{(n)}, \int_0^{\cdot} \hat{\alpha}_u^{(n)}\, du)$ converges $\hat{\mathbb{P}}$-almost surely uniformly in time to $(\hat{M}, \hat{M}, \int_0^{\cdot} \frac{\hat{v}_t^2 - \sigma^2}{2\sigma}\, dt)$ where $\hat{M} = s_0 \mathscr{E}(\hat{L})$ is a continuous $\hat{\mathbb{P}}$-martingale with finite moments of arbitrary order and $\hat{v}^2$ is the density of the quadratic variation of its stochastic logarithm $\hat{L}$ with respect to Lebesgue measure.

Moreover, for any $n = 1, 2, \dots$, we can define a system $\hat{\mathbb{T}}^{(n)}$ of stopping times $\hat{\tau}_k^{(n)}, k = 0, 1, \dots$, for the filtration generated by $\hat{S}^{(n)}$ such that also the joint $\hat{\mathbb{P}}$-law of these with $(\hat{S}^{(n)}, \hat{M}^{(n)})$ coincides with the joint law under $\mathbb{Q}^{(n)}$ of the stopping times $\tau_k^{(n)}, k = 0, 1, \dots$, with $(S^{(n)}, M^{(n)})$. In particular, we conclude from (13) that

$$\hat{\tau}_{k+1}^{(n)} - \hat{\tau}_k^{(n)} \leq \frac{m}{n}, \qquad \hat{\mathbb{P}}\text{-a.s.}$$

From (4) and (15), we thus get the Taylor expansion

$$\hat{\alpha}_t^{(n)} = n\hat{\tau}_{k+1}^{(n)} - [nt] + O(m^2/\sqrt{n})$$

$$\text{for } t \in \big[\hat{\tau}_k^{(n)} + 1/n, \hat{\tau}_{k+1}^{(n)} + 1/n\big) \cap [0, 1]\ \hat{\mathbb{P}}\text{-a.s.},$$

where the absolute values of the $O(m^2/\sqrt{n})$-terms are uniformly in time and in $\hat{\mathbb{P}}$-a.e. scenario less than or equal to $m^2/\sqrt{n}$.

The last observations allow us to apply Lemma 4.4 below [with $b(t) := \hat{v}_t^2/\sigma^2$] to get the estimate

$$(18) \qquad \liminf_n \frac{N(\hat{\mathbb{T}}^{(n)})}{n} \geq \int_0^1 g(\hat{v}_t^2/\sigma^2)\,dt, \qquad \hat{\mathbb{P}}\text{-a.e.,}$$

where g is the linearly interpolating function defined in Theorem 4.1.

Taking $\liminf_n$ in (14) now gives

$$\liminf_n \mathbb{V}^{\kappa/n}\big(f(S_1^{(n)})\big) \geq \liminf_n \mathbb{E}_{\mathbb{Q}^{(n)}}\big[f(S_1^{(n)}) + \kappa N(\mathbb{T}^{(n)})/n\big] - \frac{\kappa}{m}$$

$$= \liminf_n \mathbb{E}_{\hat{\mathbb{P}}}\big[f(\hat{S}_1^{(n)}) + \kappa N(\hat{\mathbb{T}}^{(n)})/n\big] - \frac{\kappa}{m}$$

$$\geq \mathbb{E}_{\hat{\mathbb{P}}}\left[f(\hat{M}_1) + \kappa \int_0^1 g(\hat{v}_t^2/\sigma^2)\,dt\right] - \frac{\kappa}{m},$$

where the final step is due to Fatou's lemma and (18). Applying a randomization technique similar to Lemma 7.2 in [8] and letting $m \uparrow \infty$ now proves our assertion (12).

LEMMA 4.4. *For $n = 1, 2, \ldots$, let $\mathbb{T}^{(n)} = \{0 = t_0^{(n)} \leq t_1^{(n)} \leq \cdots \leq t_n^{(n)} = 1\}$ be deterministic partitions of $[0, 1]$ such that $nt_k^{(n)} \in \{0, 1, \ldots\}$ and $t_{k+1}^{(n)} - t_k^{(n)} \leq m/n$ for $k = 0, 1, \ldots, n-1$. Suppose the functions*

$$a^{(n)}(t) := nt_{k+1}^{(n)} - [nt], \qquad t_k^{(n)} < t \leq t_{k+1}^{(n)} \qquad \text{for } k = 0, 1, \ldots, n-1,$$

satisfy

$$(19) \qquad \int_0^{\cdot} a^{(n)}(t)\,dt \to \int_0^{\cdot} \frac{1}{2}(b(t) - 1)\,dt \qquad \text{uniformly on } [0, 1]$$

for some $b \in L^1([0, 1], dt)$. Then we have

$$\liminf_n \frac{N(\mathbb{T}^{(n)})}{n} \geq \int_0^1 g(b(t))\,dt,$$

where g is the linearly interpolating function defined in Theorem 4.1.

PROOF. Without loss of generality (by passing to a sub-sequence) we assume that $\lim_{n\to\infty} N(\mathbb{T}^{(n)})/n$ exists. For any n introduce the function $b_n : [0, 1] \to [1, \infty)$ by $b_n(T) = 0$ and

$$b_n(t) = n(t_{k+1}^{(n)} - t_k^{(n)}), \qquad t_k^{(n)} \leq t < t_{k+1}^{(n)}, \qquad k = 0, 1, \ldots, n-1,$$

for which we notice that

$$(20) \qquad \frac{N(\mathbb{T}^{(n)})}{n} = \int_0^1 \frac{1}{b_n(t)}\,dt.$$

Simple calculations yield

$$\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \left[\frac{1}{2}(b_n(t) - 1) - a_n(t) \right] dt = 0.$$

This together with (19) and the fact that $n(t_{k+1}^{(n)} - t_k^{(n)})$ is bounded uniformly in k and n gives

$$(21) \qquad \int_0^t b_n(u)\, du \to \int_0^t b(u)\, du \qquad \text{uniformly in } t \in [0, 1].$$

The Komlos lemma (see Lemma A 1.1 in [6]) implies that there exists a sequence of functions $\tilde{b}_n \in \text{conv}(b_n, b_{n+1}, \ldots)$, $n = 1, 2, \ldots$, such that $\tilde{b}_n$ converges Lebesgue-almost everywhere to a function $\tilde{b}$. In fact, $\tilde{b} = b$ a.e. since by dominated convergence and (21) we get

$$\int_0^t \tilde{b}(u)\, du = \lim_{n \to \infty} \int_0^t \tilde{b}_n(u)\, du$$

$$= \lim_{n \to \infty} \int_0^t b_n(u) = \int_0^t b(u)\, du \qquad \text{for any } t \in [0, 1].$$

Finally, from (20), the fact that the function g is convex and continuous with $g(b_n) = \frac{1}{b_n}$ (as b_n is integer valued) we obtain

$$\lim_n \frac{N(\mathbb{T}^{(n)})}{n} = \lim_n \int_0^1 g(b_n(t))\, dt \geq \lim_n \int_0^1 g(\tilde{b}_n(t)) = \int_0^1 g(b(t))\, dt$$

and the result follows. $\quad \square$

4.3. *Proof of the lower bound for super-replication prices.* In this section, we will establish "$\leq$" for our formula (6) for the scaling limit of super-replication prices. More precisely, we will prove

$$(22) \qquad \limsup_n \mathbb{V}^{\kappa/n}(f(S_1^{(n)})) \leq \mathbb{E}^W \left[f(S_1^{(v)}) + \kappa \int_0^1 g(v_t^2/\sigma^2)\, dt \right]$$

for any volatility process $v \geq \sigma$ in $\mathscr{A}^W$ on some filtered probability space $(\Omega^W, \mathscr{F}^W, (\mathscr{F}_t^W), \mathbb{P}^W)$ supporting a Brownian motion W as considered in Theorem 4.1. In fact, it suffices to show this for piecewise constant v.

LEMMA 4.5. *For any $v \in \mathscr{A}^W$ and any $\varepsilon > 0$, there is $\tilde{v} \in \mathscr{A}^W$ of the simple form*

$$(23) \qquad \tilde{v}_t = \sum_{j=0}^J \sigma \sqrt{\rho_j(S_{t_0}^{(\tilde{v})}, \ldots, S_{t_j}^{(\tilde{v})})} 1_{(t_j, t_{j+1}]}(t)$$

for some times $0 = t_0 < t_1 < \cdots < t_J = 1$ *and continuous bounded functions* $\rho_j :$ $\mathbb{R}^{j+1} \to [1, \infty)$ *such that*

$$\left| \mathbb{E}^W \left[f(S_1^{(\nu)}) + \kappa \int_0^1 g(\nu_t^2 / \sigma^2) \, dt \right] - \mathbb{E}^W \left[f(S_1^{(\tilde{\nu})}) + \kappa \int_0^1 g(\tilde{\nu}_t^2 / \sigma^2) \, dt \right] \right| < \varepsilon.$$

PROOF.   Let $\nu \in \mathscr{A}^W$ and let C be a constant such that $\nu \leq C$ a.s. Using similar density arguments as in Lemma 3.4 in [2] (for $d = 1$) we get that there exists a sequence $\nu^{(n)}, n = 1, 2, \ldots$, such that $\nu^{(n)} \leq C$ is of the simple form given by (23) and $\nu^{(n)} \to \nu$ $\mathbb{P}^W \otimes dt$-a.e. This together with the uniform integrability (due to $\nu^{(n)} \leq C$) of the sequence $f(S_1^{(\nu^{(n)})}) + \kappa \int_0^1 g((\nu_t^{(n)})^2 / \sigma^2) \, dt, n = 1, 2, \ldots$, implies the assertion. $\square$

In the proof of (22), we can assume without loss of generality that $\lim_n \mathbb{V}^{\kappa/n}(f(S_1^{(n)}))$ exists. The duality result in Lemma 4.1 suggests to construct a sequence of stopping systems $\mathbb{T}^{(n)} \in \mathscr{T}^{(n)}$ with respect to $(\mathscr{F}_t^{(n)}), n = 1, 2, \ldots$, such that under the associated measures $\mathbb{Q}^{(n)} := \mathbb{Q}(\mathbb{T}^{(n)})$ the processes $S^{(n)}$ of (4) converge in law to $S^{(\nu)}$. This will be done next.

To fix ideas, let us first focus on the initial period $[t_0, t_1) = [0, t_1)$ where we wish to obtain the constant $\nu_0^2 = \sigma^2 \rho_0 \in [0, \infty)$ as the limiting local variance. Inspection of the argument in the previous section suggests that for $\rho_0 \in \{1, 2, \ldots\}$ this can be accomplished by stopping any ρ_0 consecutive upwards or downwards steps (and not stop before the end in scenarios without this monotonicity property). For ν_0^2 between natural multiples of σ^2, though, we have to mix stopping after $[\rho_0]$ steps and after $[\rho_0] + 1$ steps in just the right proportions. For instance, if we want to obtain asymptotically the local variance $1.5\sigma^2$ (i.e., $\rho_0 = 1.5$), we just alternate between stopping after $[\rho_0] = 1$ steps and after $[\rho_0] + 1 = 2$ steps in the same direction (and again do not stop before the end in all scenarios which are incompatible with this).

In general, the following construction will work: For $j = 0, \ldots, J$, we subdivide the time interval $[[nt_j]/n, [nt_{j+1}]/n)$ into $[nt_{j+1}] - [nt_j] \approx n(t_{j+1} - t_j) = O(n)$ periods of length $1/n$. These $O(n)$ periods can be covered by $\sqrt{n(t_{j+1} - t_j)} = O(\sqrt{n})$ blocks of the same number $\sqrt{n(t_{j+1} - t_j)} = O(\sqrt{n})$ of successive time points. Denote by

$$\rho_j^{(n)} := \rho_j \left(S_{[nt_0]/n}^{(n)}, \ldots, S_{[nt_j]/n}^{(n)} \right)$$

a proxy for the multiple of σ^2 we want to implement asymptotically as local variance over the interval $[t_j, t_{j+1})$. Take $\lambda_j^{(n)}$ to be the unique solution $\lambda \in (0, 1]$ of

$$\rho_j^{(n)} = \lambda [\rho_j^{(n)}] + (1 - \lambda)([\rho_j^{(n)}] + 1).$$

In each of the above $O(\sqrt{n})$ blocks of length $O(\sqrt{n})$, we will first stop every time after $[\rho_j^{(n)}]$ steps have been made by the binomial model consecutively in the same direction (i.e., all upwards or all downwards) and we will not stop at all before reaching the time horizon $T = 1$ in scenarios where different directions are taken in this period. This continues until we have covered a fraction of $\lambda_j^{(n)}$ of the present block's $O(\sqrt{n})$ periods. For the remaining fraction $1 - \lambda_j^{(n)}$ of periods in this block, we will proceed similarly but with a rhythm of stopping every $[\rho_j^{(n)}] + 1$ steps instead of $[\rho_j^{(n)}]$. After that, we repeat this procedure for all of the $O(\sqrt{n})$ blocks we separated the interval $[[nt_j]/n, [nt_{j+1}]/n)$ into in the beginning. Then we proceed similarly with the next interval $[[nt_{j+1}]/n, [nt_{j+2}]/n)$ until all of these intervals are treated.

Let us next analyze the asymptotic transaction costs and variance which this procedure entails. We can do this separately on each of the intervals $[t_j, t_{j+1})$, $j = 0, \ldots, J$. So fix such a j and let n_1 and n_2 denote the number of times where we stop every $[\rho_j^{(n)}]$ and $[\rho_j^{(n)}] + 1$ binomial steps, respectively. Then we have

$$n_1[\rho_j^{(n)}] + n_2([\rho_j^{(n)}] + 1) = \sqrt{n(t_{j+1} - t_j)} + O(1)$$

and, in order to obtain the right asymptotic variance for $M^{(n)}$ constructed from the thus obtained $\tau_j^{(n)}$s as in (15), we want to have at the same time that

$$n_1[\rho_j^{(n)}]^2 + n_2([\rho_j^{(n)}] + 1)^2 = \rho_j^{(n)}\sqrt{n(t_{j+1} - t_j)} + O(1).$$

We conclude

$$\frac{n_1}{\sqrt{n(t_{j+1} - t_j)}} = \frac{1 + [\rho_j^{(n)}] - \rho_j^{(n)}}{[\rho_j^{(n)}]} + O(1/\sqrt{n}),$$

$$\frac{n_2}{\sqrt{n(t_{j+1} - t_j)}} = \frac{\rho_j^{(n)} - [\rho_j^{(n)}]}{1 + [\rho_j^{(n)}]} + O(1/\sqrt{n}),$$

and the fraction of periods covered in $[\rho_j^{(n)}]$ steps, respectively, is the desired

$$\lambda_j^{(n)} = \frac{n_1[\rho_j^{(n)}]}{\sqrt{n(t_{j+1} - t_j)}} + O(1/\sqrt{n}) = 1 + [\rho_j^{(n)}] - \rho_j^{(n)} + O(1/\sqrt{n}).$$

The transaction costs on this block are equal to $(n_1 + n_2)\kappa/n$, and so we conclude that the transaction costs on the whole interval $[[nt_j]/n, [nt_{j+1}]/n)$ amount to

$$(24) \qquad \sqrt{n(t_{j+1} - t_j)}(n_1 + n_2)\kappa/n = \kappa(t_{j+1} - t_j)g(\rho_j^{(n)}) + O(1/\sqrt{n}).$$

Furthermore, we get that for any $t \in (t_j, t_{j+1})$ the process $\alpha^{(n)}$ as in (17) satisfies

$$\int_{t_j}^{t} \alpha_s^{(n)} \, ds = O(1/\sqrt{n}) + \frac{t - t_j}{\sqrt{n(t_{j+1} - t_j)}} (0 + 1 + 2 + \cdots + [\rho_j^{(n)}] - 1) n_1$$

(25)
$$+ \frac{t - t_j}{\sqrt{n(t_{j+1} - t_j)}} (0 + 1 + 2 + \cdots + [\rho_j^{(n)}]) n_2$$

$$= O(1/\sqrt{n}) + \frac{\rho_j^{(n)} - 1}{2} (t - t_j).$$

Having constructed for $n = 1, 2, \ldots$ a system of stopping times $\mathbb{T}^{(n)} = \{\tau_k^{(n)}\} \in \mathscr{T}^{(n)}$, we can let $\mathbb{Q}^{(n)} := \mathbb{Q}(\mathbb{T}^{(n)})$ denote the associated martingale measure for $(S_{\tau_k^{(n)}}^{(n)})_{k=0,1,\ldots}$. Observe that along with the functions ρ_j also the $\rho_j^{(n)}$ are bounded uniformly, say by a constant $m \in \{1, 2, \ldots\}$. As a consequence, the increments between any two successive intervention times are bounded by m/n $\mathbb{Q}^{(n)}$-almost surely as in (13). We can thus invoke Lemma 4.3 to conclude that, possibly along a subsequence again denoted by n, we have the weak convergence

$$\mathrm{Law}\left(S^{(n)}, \int_0^{\cdot} \alpha_s^{(n)} \, ds \,\Big|\, \mathbb{Q}^{(n)} \right) \to \mathrm{Law}\left(M, \int_0^{\cdot} \frac{1}{2}(\hat{v}_t^2/\sigma^2 - 1) \, dt \,\Big|\, \hat{\mathbb{P}} \right), \qquad n \uparrow \infty,$$

on $\mathbb{D}[0, 1] \times \mathbb{D}[0, 1]$ for some $\hat{v} \geq \sigma$ with $\hat{v}^2 = d\langle L \rangle / dt$ for the stochastic logarithm L of $M = s_0 \mathscr{E}(L)$. In fact, M and $\hat{v}$ are just copies, respectively, of our original $S^{(v)}$ and v; see Lemma 4.6 below.

Just as after Lemma 4.3 in the previous section, we now use Skorohod's representation theorem to see that without loss of generality we can assume to have $\hat{S}^{(n)}$, $\hat{M}^{(n)}$ and $\hat{\alpha}^{(n)}$ on $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{P}})$ which have, for each $n = 1, 2, \ldots$, the same joint law as their counterparts $(S^{(n)}, M^{(n)}, \alpha^{(n)})$ under $\mathbb{Q}^{(n)}$ and which are such that, as $n \uparrow \infty$,

(26)
$$\left(\hat{S}^{(n)}, \hat{M}^{(n)}, \int_0^{\cdot} \hat{\alpha}_u^{(n)} \, du \right) \to \left(M, M, \int_0^{\cdot} \frac{1}{2}(\hat{v}_t^2/\sigma^2 - 1) \, dt \right)$$

uniformly in time $\hat{\mathbb{P}}$-almost surely, where $\hat{v}^2 := d\langle L \rangle / dt$ for the stochastic logarithm L of M. For $n = 1, 2, \ldots$, we can also reconstruct from $\hat{S}^{(n)}$ a system of stopping times $\hat{\mathbb{T}}^{(n)}$ for the filtration generated by $\hat{S}^{(n)}$ which corresponds to our $\mathbb{T}^{(n)}$ constructed above. From (25)–(26) and the fact that the functions ρ_j, $j = 1, \ldots, J$ are continuous, it follows that

(27)
$$\hat{v}_t = \sum_{j=0}^{J} \sigma \sqrt{\rho_j(M_{t_0}, \ldots, M_{t_j})} \mathbb{1}_{(t_j, t_{j+1}]}(t), \qquad \mathbb{P} \otimes dt\text{-a.e.}$$

The proof of (22) is now completed by arguing that

$$
\begin{aligned}
\mathbb{V}^{\kappa/n}\big(f\big(S_1^{(n)}\big)\big) &\leq \mathbb{E}_{\mathbb{Q}^{(n)}}\bigg[f\big(S_1^{(n)}\big) + \frac{\kappa}{n} N\big(\mathbb{T}^{(n)}\big)\bigg] \\
&= \mathbb{E}_{\hat{\mathbb{P}}}\big[f\big(\hat{S}_1^{(n)}\big)\big] + \mathbb{E}_{\hat{\mathbb{P}}}\big[\kappa N\big(\hat{\mathbb{T}}^{(n)}\big)/n\big] \\
&\to \mathbb{E}_{\hat{\mathbb{P}}}\big[f(M_1)\big] + \mathbb{E}_{\hat{\mathbb{P}}}\bigg[\kappa \int_0^1 g\big(\hat{v}_t^2/\sigma^2\big)\,dt\bigg] \\
&= \mathbb{E}^W\bigg[f\big(S_1^{(v)}\big) + \kappa \int_0^1 g\big(v_t^2/\sigma^2\big)\,dt\bigg].
\end{aligned}
$$

Here, the estimate in the first line is immediate from Lemma 4.1 and the first identity is due to our Skorohod representation. The convergence $\mathbb{E}_{\hat{\mathbb{P}}}[f(\hat{S}_1^{(n)})] \to \mathbb{E}_{\hat{\mathbb{P}}}[f(M_1)]$ is due to dominated convergence since uniform integrability follows from the polynomial growth of f and (16); the convergence of the other expectations also follows by dominated convergence since $N(\hat{\mathbb{T}}^{(n)})/n \in [0, 1]$, $n = 1, 2, \ldots$, and since (24) in conjunction with (27) yields $\hat{\mathbb{P}}$-a.s. convergence of the costs $\kappa N(\hat{\mathbb{T}}^{(n)})/n$ to $\kappa \int_0^1 g(\hat{v}_t^2/\sigma^2)\,dt$. The final identity is immediate from Lemma 4.6 below.

LEMMA 4.6. *We have*

$$
\text{(28)} \qquad \text{Law}\big(S^{(v)}|\mathbb{P}^W\big) = \text{Law}(M|\hat{\mathbb{P}}).
$$

PROOF. Let us prove by induction that, for any $j = 0, 1, \ldots, J$, the distribution of $M|_{[0,t_j]}$ is equal to the distribution of $S^{(v)}|_{[0,t_j]}$. For $j = 0$, the statement is trivial. Assume that the statement is correct for j. Define the stochastic process

$$
B_t = \frac{1}{\sigma\sqrt{\rho_j(M_{t_0}, \ldots, M_{t_j})}} \int_{t_j}^{t+t_j} \frac{dM_u}{M_u}, \qquad t \in [0, t_{j+1} - t_j].
$$

From the Lévy theorem and (27), it follows that B is a Brownian motion on $[0, t_{j+1} - t_j]$ independent of $M|_{[0,t_j]}$. Clearly, for $t \in [t_j, t_{j+1}]$,

$$
\text{(29)} \quad M_t = M_{t_j}\exp\big(\sigma\sqrt{\rho_j(M_{t_0}, \ldots, M_{t_j})}B_{t-t_j} - \sigma^2\rho_j(M_{t_0}, \ldots, M_{t_j})(t-t_j)/2\big).
$$

On the other hand, for $t \in [t_j, t_{j+1}]$,

$$
\text{(30)} \quad
\begin{aligned}
S_t^{(v)} = S_{t_j}^{(v)} \exp\big(&\sigma\sqrt{\rho_j\big(S_{t_0}^{(v)}, \ldots, S_{t_j}^{(v)}\big)}\hat{B}_{t-t_j} \\
&- \sigma^2\rho_j\big(S_{t_0}^{(v)}, \ldots, S_{t_j}^{(v)}\big)(t-t_j)/2\big),
\end{aligned}
$$

where $\hat{B}_t = W_{t+t_j} - W_{t_j}$, $t \geq 0$ is a Brownian motion independent of $S^{(v)}|_{[0,t_j]}$. From (29)–(30) and the induction assumption, we get that the distribution of $M|_{[0,t_{j+1}]}$ coincides with the distribution of $S^{(v)}|_{[0,t_{j+1}]}$ as required. Hence, the distribution of M is the same as that of $S^{(v)}$. $\square$

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