The Annals of Applied Probability
2019, Vol. 29, No. 1, 89–129
https://doi.org/10.1214/18-AAP1403

PARTICLE SYSTEMS WITH SINGULAR INTERACTION THROUGH HITTING TIMES: APPLICATION IN SYSTEMIC RISK MODELING

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We propose an interacting particle system to model the evolution of a system of banks with mutual exposures. In this model, a bank defaults when its normalized asset value hits a lower threshold, and its default causes instantaneous losses to other banks, possibly triggering a cascade of defaults. The strength of this interaction is determined by the level of the so-called *noncore exposure*. We show that, when the size of the system becomes large, the cumulative loss process of a bank resulting from the defaults of other banks exhibits discontinuities. These discontinuities are naturally interpreted as *systemic events*, and we characterize them explicitly in terms of the level of noncore exposure and the fraction of banks that are "about to default." The main mathematical challenges of our work stem from the very singular nature of the interaction between the particles, which is inherited by the limiting system. A similar particle system is analyzed in [*Ann. Appl. Probab.* **25** (2015) 2096–2133] and [*Stochastic Process. Appl.* **125** (2015) 2451–2492], and we build on and extend their results. In particular, we characterize the large-population limit of the system and analyze the jump times, the regularity between jumps, and the local uniqueness of the limiting process.

1. Introduction. Consider an interconnected system whose components might fail, such as a banking system in which banks may default. The existing approaches to quantitative modeling of such systems, roughly, fall into the following two categories: (1) network models, and (2) particle systems with mean-field interaction. Models of the first category are considered, for example, in [14, 24, 25]. These models are able to capture the current characteristics of the system with high precision and to predict the effects of immediate external shocks. However, to obtain analytical results on the risks associated with a given network (e.g., on the probability of a default cascade of a certain size due to a specific external shock) a limit, as the size of the system goes to infinity, needs to be taken. The results are then expressed in terms of the average values of the members' characteristics. In

Received May 2017; revised November 2017.
[1]Supported in part by NSF Grant DMS-1411824.
[2]Supported in part by NSF Grant DMS-1506290.
MSC2010 subject classifications. 35B65, 35K20, 82C22, 91G80.
Key words and phrases. Banking systems, blow-ups in parabolic partial differential equations, default cascades, interacting particle systems, large system limits, loss of continuity, mean-field models, noncore exposures, nonlinear Cauchy–Dirichlet problems, regularity estimates, self-excitation, singular interaction, systemic crises, systemic risk.

the models of the second category, it is assumed from the very beginning that there are only a few characteristics by which any two particles (e.g., banks) can differ and that the interaction between them is either of a mean-field type, or is given implicitly through a correlation with a common factor. In contrast to the network models, the particle systems are dynamic and allow to investigate how the system evolves over time. Under suitable assumptions, it is possible to derive analytic formulas for the risk of future failures in such systems, in addition to describing the immediate (i.e., static) risk embedded in the system at a given point in time. Analysis of this kind is often carried out in the context of losses due to defaults in a large portfolio; see, for example, [3, 6, 8, 11, 15, 17] and the references therein.

In the present paper, we follow the mean-field approach in modeling the dynamics of an interconnected system of banks. At the same time, we use an explicit (i.e., structural) mechanism of default contagion (i.e., interaction between the particles), which, in particular, differentiates our setting from the models based on interacting default intensities (cf., [8, 15]). The goal of our paper is to provide a method for estimating the proximity of a *systemic failure* (i.e., for quantifying the *systemic risk*), which would allow a regulator to intervene ahead of time. We understand the systemic failure as the occurrence of a "significantly large" default cascade. It has been documented in various studies (cf., [8, 14, 24]) that an interconnected banking system transitions between two regimes: the well-behaved regime, in which the system spends most of its time, and during which the default cascades are very small or do not appear at all, and the systemic crisis regime, which occurs rarely, and which is characterized by large groups of banks defaulting in a short period of time. Even though the presence of such *phase transitions* is well known, it is often difficult to define precisely what constitutes a significantly large cascade. In our setting, the times of such cascades are captured by the discontinuity points of the cumulative loss process in a limiting system, thus providing a natural endogenous definition of a systemic failure.

In addition, we provide an explicit connection between the occurrence of systemic events and the internal characteristics of the banking system, which in principle, can be observed and controlled by a regulator. More specifically, we describe the time of systemic failure in terms of the level of *mutual exposure* of the banks and the fraction of banks in the immediate danger of default. The level of mutual exposure measures the interconnectedness of the system, allowing the firms to lend to and borrow from each other. Such lending and borrowing may decrease the individual risk of a bank, but it also provides channels for spreading losses from individual defaults across the rest of the system (i.e., for creating default contagion). This dual role of mutual exposure has been analyzed, for example, in [2, 14, 16, 24]. The mutual exposure of the banks is also known as the *noncore exposure* (in contrast to the core exposure, which measures how much the banks lend to the real economy), and its effects on the occurrence of systemic crises is analyzed, for example, in [13, 26]. As mentioned above, the level of noncore exposure can be controlled by a financial regulator, and such a control was implemented by the government of South Korea in the aftermath of the financial crisis of 2008 (cf. [1, 26]). The joint effect of the interconnectedness of the system (measured by the level of noncore exposure, in the present case) and the fraction of members in the immediate danger of failure on the occurrence of a large failure cascade has been also investigated in [2, 24, 30].

The mathematical model considered in this paper is based on a system of Brownian particles with singular interaction through hitting times. Systems of this type are considered, for example, in [9] and [10] in connection to a problem from neuroscience, and we use and extend some of the ideas developed therein to establish our results (for alternative models, with smooth interaction, we refer to [18] and [15]). In particular, the convergence of the N-particle system as $N \to \infty$ (Theorem 2.4) is based on appropriate adaptations of the methods presented in [10]. The main original contribution of our work is in the analysis of the limiting process. Namely, Theorem 2.6 establishes regularity of the limiting process, while Theorem 2.7 proves its local uniqueness. Theorem 2.7 amounts to proving the uniqueness of the solution to a nonlocal nonlinear Cauchy–Dirichlet problem. It is worth mentioning that the limiting process of a similar particle system has been analyzed in [9] (see also [4, 5, 7], where the focus is on analyzing the possibility of a jump in the limiting process and on describing stationary solutions when no jumps occur). The paper [9] establishes regularity of the limiting process and proves its uniqueness using a similar Cauchy–Dirichlet problem. However, the main results of [9] require additional assumptions on the strength of interaction in the system, which, in particular, rule out the possibility of a jump in the limiting process. As such jumps have a natural practical interpretation (e.g., as systemic crises, in the application proposed herein), we, specifically, focus on the cases where such jumps may occur. As a consequence, the limiting process and the solution to the associated Cauchy–Dirichlet problem, herein, do not possess as much regularity as in [9], which, naturally, complicates the analysis. Concurrently with our work, a one-dimensional discrete particle system with absorption has been considered in [12], motivated by the recent results [20, 28] on the one-dimensional multiparticle diffusion limited aggregation problem. In [12], the absorption boundary is moved up instead of the remaining particles being shifted down, but the two points of view are easily seen to be equivalent. For the motion of the boundary corresponding to a discrete version of the mechanism in [9, 10] and certain i.i.d. initial conditions, the main result of [12] identifies the scaling limit of the boundary explicitly.

2. Main results. Consider N banks and write $X_t^1, X_t^2, \dots, X_t^N$ for their total asset values at a time $t \geq 0$, discounted according to the (possibly stochastic) growth rate of the overall banking system. A bank i defaults when its total asset value X^i drops below a barrier $\underline{x}^i > 0$. Since each asset value process can be normalized by the corresponding barrier, we may assume without loss of generality that $\underline{x}^1 = \underline{x}^2 = \cdots = \underline{x}^N = 1$. In the *absence of defaults*, we let the asset value processes $X_t^1, X_t^2, \dots, X_t^N$ follow the stochastic differential equations (SDEs):

$$\mathrm{d}X_t^i = X_t^i(\alpha + \sigma^2/2)\,\mathrm{d}t + X_t^i \sigma\,\mathrm{d}B_t^i, \qquad i = 1, 2, \dots, N, \tag{2.1}$$

where $\alpha \in \mathbb{R}$ and $\sigma > 0$ are constants, and $B^1, B^2, \dots, B^N$ are independent standard Brownian motions. Here, α stands for the return associated with the *traditional investments* of a bank (i.e., investments in companies outside of the banking system), and σ is the volatility coefficient of such investments.

Now, suppose that the asset value process of a bank i hits the default barrier $\underline{x}^i = 1$ at a time t, leading to the default of bank i. As a result of the default, the asset values of other banks drop and may immediately cause further defaults, and so on. When the default of a bank causes immediate defaults of other banks, we speak of a *default cascade*. For the sake of tractability, we assume that, if k banks default at time t, then the value of each remaining bank is reduced by the factor

$$\left(1 - \frac{k}{S_{t-}}\right)^{-C},$$

where S_t is the number of banks that have survived up to and including time t, and $C \in [0, 1)$ is a fixed constant. The value of C represents the level of noncore exposure in the banking system. Notice that $(1 - k/S_{t-})^C \approx 1 - Ck/S_{t-}$, if k/S_{t-} is small. In such a case, the proposed loss function represents the losses from default contagion in a banking system in which every bank, in total, borrows from all other banks the fraction C of the average bank's value, with the sizes of individual loans distributed proportionally to the other banks' values. After the default event, the asset value processes of the surviving banks continue to follow the dynamics of (2.1) until one of them hits 1, and so on.

The informal description of the processes $X^1, X^2, \dots, X^N$ in the previous two paragraphs can be formalized as follows. We fix a time horizon $T > 0$, let

$$Y^i := \log X^i, \qquad \tau^i := \inf\{t \in [0, T] : Y_t^i \le 0\}, i = 1, 2, \dots, N \tag{2.2}$$

be the *logarithmic asset values* and the *default times* of the banks, respectively, and denote by

$$S_t := \sum_{i=1}^N \mathbf{1}_{\{\tau^i > t\}}, \qquad t \in [0, T] \tag{2.3}$$

the *size of the banking system*. In addition, for any fixed $t \in [0, T]$ we consider the order statistics

$$Y_{t-}^{(1)} \le Y_{t-}^{(2)} \le \dots \le Y_{t-}^{(S_{t-})} \tag{2.4}$$

of the vector $(Y_{t-}^i : \tau^i \ge t)$. Then the number of defaults at time $t \in [0, T]$ is defined by

$$K_t^d := \left(\inf\left\{k = 1, 2, \dots, S_{t-} : Y_{t-}^{(k)} + C\log\left(1 - \frac{k-1}{S_{t-}}\right) > 0\right\} - 1\right) \wedge S_{t-}, \tag{2.5}$$

with the convention $\inf\varnothing = \infty$. Finally, each of the processes $Y^1, Y^2, \dots, Y^N$ satisfies

$$Y_t^i = \widetilde{Y}_t^i \mathbf{1}_{\{\widetilde{Y}_s^i > 0, s\in[0,t)\}} + \widetilde{Y}_{\tau^i}^i \mathbf{1}_{\{t>\tau^i\}}, \tag{2.6}$$

where

$$\widetilde{Y}_t^i = Y_0^i + \alpha t + \sigma B_t^i + \bigl(1 \wedge (\widetilde{Y}_t^i + 1)^+\bigr) \sum_{u\le t: K_u^d > 0} C \log\biggl(1 - \frac{K_u^d}{S_{u-}}\biggr) \tag{2.7}$$

for $t \in [0, \tau^0) \cap [0, T]$,

$$\widetilde{Y}_t^i = (-1) \wedge \widetilde{Y}_{\tau^0 -}^i + \alpha\bigl(t - \tau^0\bigr) + \sigma\bigl(B_t^i - B_{\tau^0}^i\bigr) \tag{2.8}$$

for $t \in [\tau^0, \infty) \cap [0, T]$, and $\tau^0 := \max_{1\le j\le N} \tau^j$.

The paths of the processes $\widetilde{Y}^1, \widetilde{Y}^2, \dots, \widetilde{Y}^N$ and $Y^1, Y^2, \dots, Y^N$ can be constructed sequentially on the time intervals from 0 to the first default time, from the first default time to the next default time, and so on. It is easy to see that the algebraic equations defining the values of these processes between jumps, of the form

$$\tilde{y} = y_0 + \bigl(1 \wedge (1 + \tilde{y})^+\bigr)c,$$

with constants $y_0 \in \mathbb{R}$ and $c < 0$, are uniquely solvable. The truncation factors $1 \wedge (\widetilde{Y}_t^i + 1)^+$, $i = 1, 2, \dots, N$, included for a purely technical reason, ensure that each process $\widetilde{Y}^i$ does not jump below -1. They are only needed in the proof of convergence of this particle system (Theorem 2.4), and their exact form does not affect the limit. In addition, these factors are constantly equal to 1 on $[0, \tau^i)$, for $i = 1, 2, \dots, N$, respectively, and, therefore, have no effect on the pre-default paths Y_t^i, $t \in [0, \tau^i)$, $i = 1, 2, \dots, N$. We also extend the path of each $\widetilde{Y}^i$ to the interval $[0, T + 1]$ continuously, as follows:

$$\widetilde{Y}_t^i = \widetilde{Y}_T^i + \alpha(t - T) + \sigma\bigl(B_t^i - B_T^i\bigr), \qquad t \in (T, T + 1].$$

The reason we continue the path of $\widetilde{Y}^i$ beyond $\tau^i \wedge T$, rather than stop the process at this time, is that the paths need to be sufficiently "noisy" in order to establish the desired convergence result.

We sometimes refer to the vector of processes $(\widetilde{Y}^1, \widetilde{Y}^2, \dots, \widetilde{Y}^N)$ as the *finite-particle system*. It is worth noting that

$$\sum_{u\le t: K_u^d > 0} C \log\biggl(1 - \frac{K_u^d}{S_{u-}}\biggr) = \sum_{u\le t: K_u^d > 0} C(\log S_u - \log S_{u-}) = C \log \frac{S_t}{N}, \tag{2.9}$$
$$t \in \bigl[0, \tau^0 \wedge T\bigr),$$

hence,

$$\widetilde{Y}^i_t = Y^i_0 + \alpha t + \sigma B^i_t + (1 \wedge (\widetilde{Y}^i_t + 1)^+) C \log\left(\frac{1}{N}\sum_{j=1}^N \mathbf{1}_{\{\tau^j > t\}}\right), \quad t \in [0, \tau^0 \wedge T). \tag{2.10}$$

However, the strong solution of (2.10) is not unique, because the default cascades are not uniquely determined by (2.10) alone.

Being interested in the emergence of large ("systemic") losses due to default cascades, we study the large N asymptotics of the banking system by means of the empirical measures

$$\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{Y^i} \quad \text{and} \quad \widetilde{\mu}^N := \frac{1}{N}\sum_{i=1}^N \delta_{\widetilde{Y}^i}. \tag{2.11}$$

We view μ^N and $\widetilde{\mu}^N$ as random probability measures on the spaces $D([0,T])$ and $D([0,T+1])$, respectively, consisting of real-valued càdlàg paths. The latter are endowed with the Skorokhod M1 topology (see, e.g., [10, 19, 27, 31], for a detailed discussion of the M1 topology). The limiting object associated with the sequence $\widetilde{\mu}^N$, $N \in \mathbb{N}$ turns out to satisfy (see Theorem 2.4 below)

$$\begin{aligned} &\overline{Y}_t = \overline{Y}_0 + \alpha t + \sigma \overline{B}_t + (1 \wedge (\overline{Y}_t + 1)^+)\Lambda_t, \qquad t \in [0, \overline{\tau}^0) \cap [0,T], \\ &\Lambda_t = C \log \mathbb{P}(\overline{\tau} > t), \qquad t \in [0, \overline{\tau}^0) \cap [0,T], \\ &\overline{\tau} = \inf\{t \in [0,T] : \overline{Y}_t \le 0\}, \end{aligned} \tag{2.12}$$

where $\overline{B}$ is a standard Brownian motion independent of $\overline{Y}_0 > 0$, and

$$\overline{\tau}^0 := \inf\Big\{t \in [0,T] : \mathbb{P}\Big(\inf_{s\in[0,t]} \overline{Y}_s \le 0\Big) = 1\Big\}. \tag{2.13}$$

The following definition identifies the relevant class of solutions to the above equation.

DEFINITION 2.1. A real-valued càdlàg process $\overline{Y}_t$, $t \in [0,T]$ is called a *physical solution* to (2.12),[3] if it satisfies (2.12) and, whenever $\Lambda_t \ne \Lambda_{t-}$, for some $t \in [0, \overline{\tau}^0) \cap [0,T]$, we have $\Lambda_{t-} - \Lambda_t \in [\overline{D}^0_t, \overline{D}_t]$, where

$$\overline{D}^0_t := \inf\Big\{y > 0 : y - C\frac{\mathbb{P}(\overline{\tau} \ge t, \overline{Y}_{t-} \in (0,y))}{\mathbb{P}(\overline{\tau} \ge t)} > 0\Big\} < \infty, \tag{2.14}$$

$$\overline{D}_t := \inf\{y > 0 : y - F_t(y) > 0\} < \infty, \tag{2.15}$$

$$F_t(y) := -C \log\Big(1 - \frac{\mathbb{P}(\overline{\tau} \ge t, \overline{Y}_{t-} \in (0,y))}{\mathbb{P}(\overline{\tau} \ge t)}\Big). \tag{2.16}$$

[3] For simplicity, we refer to it as "physical solution."

REMARK 2.2. It is easy to see that, for any $t \in [0, \overline{\tau}^0) \cap [0, T]$, it holds that $\mathbb{P}(\overline{\tau} > t) > 0$, and, hence, all quantities in (2.12), (2.16) are well defined.

REMARK 2.3. It is easy to notice that $\overline{D}_t = F(\overline{D}_t)$, as F_t can only jump upward.

The path $\overline{Y}_t$, $t \in [0, \overline{\tau}) \cap [0, T]$, for a physical solution $\overline{Y}$, should be thought of as the logarithmic asset value process of a *typical* bank in a *large* banking system, which is made precise by the following theorem.

THEOREM 2.4. *Suppose that, for all $N \in \mathbb{N}$, the initial values $Y_0^1, Y_0^2, \dots, Y_0^N$ are i.i.d. according to a probability measure ν on $[0, \infty)$, with a bounded density f_ν vanishing in a neighborhood of 0. Then the sequence of random measures $\widetilde{\mu}^N$, $N \in \mathbb{N}$ is tight with respect to the topology of weak convergence, and every limit point of this sequence belongs with probability one to the space of distributions of physical solutions $\overline{Y}$ for which $\overline{Y}_0 \stackrel{d}{=} \nu$.*

Note that Theorem 2.4, in particular, proves the weak existence of a physical solution. The function Λ in Definition 2.1 represents the aggregate losses (on the logarithmic scale) of a typical bank in a large banking system resulting from the defaults of other banks. When default cascades lead to a jump in Λ, we speak of a *systemic event*. At the *random* time $\overline{\tau}$, the bank in consideration defaults; the *deterministic* quantity $\overline{\tau}^0$ represents the time when the last bank defaults; $\overline{D}_t^0$ and $\overline{D}_t$, respectively, provide the lower and upper bounds on the maximum logarithmic value among the banks defaulting at time t. The term "physical solution" is borrowed from [10].

One of the most interesting characteristics of the proposed model is the time of the first *systemic event*

$$t_{\text{sys}} := \inf\{t \in [0, \overline{\tau}^0) \cap [0, T] : \Lambda_t \neq \Lambda_{t-}\}, \tag{2.17}$$

when a nonnegligible fraction of banks defaults in a short period of time. The time t_{sys} can be viewed as the time of the first *phase transition*, with the banking system passing abruptly from the well-behaved regime to the systemic crisis regime. Note that Definition 2.1 provides a partial characterization of this time in terms of the (observable) distribution of the values of particles that have survived thus far:

$$\begin{aligned} &\inf\{t \in [0, \overline{\tau}^0) \cap [0, T] : \overline{D}_t > 0\} \\ &\qquad \le t_{\text{sys}} \le \inf\{t \in [0, \overline{\tau}^0) \cap [0, T] : \overline{D}_t^0 > 0\}. \end{aligned} \tag{2.18}$$

The heuristic interpretation of the above inequality is that the normalized density of survived particles, $p(t, y)/\mathbb{P}(\overline{\tau} \ge t)$, where $p(t, \cdot)$ is the density of the distribution of $\overline{Y}_{t-}\mathbf{1}_{\{\overline{\tau} \ge t\}}$ restricted to $(0, \infty)$ [see Lemma 5.1 for the existence of $p(t, \cdot)$], near

$y = 0$, can be used to *measure the proximity* to the time t_{sys} of the first systemic event. Simply put, a systemic event occurs when the normalized density at zero exceeds the level $1/C$. This observation yields a natural connection between a systemic event and the two relevant *observable* quantities: the fraction of banks at immediate risk and the level of *noncore exposure*.

It may seem artificial that we require the jumps of Λ to occur at the times when $\overline{D}^0 > 0$, in the definition of physical solution. The following proposition shows that the latter causes no loss of generality (and it is also used in the proof of Theorem 2.4).

PROPOSITION 2.5. *Consider any càdlàg process $\overline{Y}$ satisfying* (2.12), *with the associated Λ and $\overline{\tau}^0$. Then, for any $t \in [0, \overline{\tau}^0)$, if there exists $\eta \geq 0$, s.t.*

$$\frac{\mathbb{P}(\overline{\tau} \geq t, \overline{Y}_{t-} \in (0, y))}{\mathbb{P}(\overline{\tau} \geq t)} \geq \frac{y}{C} \qquad \forall y \in [0, \eta],$$

then, $\Lambda_{t-} - \Lambda_t \geq \eta$.

Indeed, the above proposition shows that $\overline{D}_t^0 > 0$ is a sufficient condition for a jump at time t, for any càdlàg process $\overline{Y}$ satisfying (2.12).

The above discussion motivates the analysis of physical solution in relation to the value of its normalized density at zero. Our next result is concerned with the *regularity* of a physical solution. Lemma 5.2 in [9] establishes the $1/2$-Hölder continuity of the cumulative loss process at any time at which it does not jump. The subsequent results in [9] show that the normalized density at zero vanishes and the cumulative loss process becomes continuously differentiable at all times, if the strength of the interaction C is sufficiently small (the model analyzed in [9] is not exactly the same as the present one, but the arguments used therein can be adapted to the present case). Theorem 2.6, below, fills the gap between the two results: it shows that the cumulative loss process possesses higher Hölder regularity (even though it may not be continuously differentiable) if the normalized density at zero vanishes, without the assumption that C is sufficiently small. In order to state this result, we introduce

$$r_t^* := \lim_{\eta \downarrow 0} \sup_{s \in (0,t]} \operatorname*{ess\,sup}_{y \in (0,\eta)} \frac{p(s, y)}{\mathbb{P}(\overline{\tau} \geq s)}, \tag{2.19}$$

where $p(s, \cdot)$ is the density of the distribution of $\overline{Y}_{s-}\mathbf{1}_{\{\overline{\tau} \geq s\}}$ restricted to $(0, \infty)$.

THEOREM 2.6. *Let $\overline{Y}$ be a physical solution, with the associated Λ, $\overline{\tau}^0$, r^*. Suppose that $\overline{Y}_0$ has a bounded density vanishing in a neighborhood of 0. Consider any $t_0 \in (0, \overline{\tau}^0)$ for which $r_{t_0}^* = 0$. Then, for any $t_0' \in [0, t_0)$ there exist $\widetilde{C} < \infty$ and $\gamma \in (0, 1]$ such that*

$$|\Lambda_t - \Lambda_s| \leq \widetilde{C}|t - s|^{(1+\gamma)/2}, \qquad s, t \in [0, t_0']. \tag{2.20}$$

Next, we turn to the uniqueness of a physical solution. Note that establishing uniqueness is not only interesting in its own right, but it would also strengthen the convergence result significantly. Indeed, once the uniqueness is established, Theorem 2.4 would imply that $\widetilde{\mu}^N$, $N \in \mathbb{N}$ converge to a deterministic limit, which is the law of the unique physical solution. To date, the uniqueness of a general physical solution given by Definition 2.1, or its analogue in [9, 10], remains an open problem. Nevertheless, the uniqueness can be established in a class of sufficiently regular solutions, which can be described via an associated Cauchy–Dirichlet system. Such a uniqueness result is established in [9], under the additional assumption that C is sufficiently small, which ensures that the cumulative loss process is continuously differentiable and, in particular, rules out the possibility of a jump. A local uniqueness result is also established in [9], and it does not require C to be sufficiently small. Nevertheless, the latter result only holds on a time interval on which the cumulative loss process is continuously differentiable. Herein, we do not make an assumption that C is small, as we would like to analyze systems in which the cumulative loss process can jump. In addition, we establish uniqueness on a time interval on which the loss process neither jumps nor possesses a continuous derivative. More specifically, we establish uniqueness up to the time

$$t_{\mathrm{reg}} = \bigl(\sup\bigl\{t \in \bigl(0, \overline{\tau}^0\bigr) : \|\lambda\|_{L^2([0,t])} < \infty\bigr\}\bigr) \wedge T, \tag{2.21}$$

where λ is the weak derivative of Λ, and we use the conventions: $\sup \varnothing = 0$ and $\|\lambda\|_{L^2([0,t])} = \infty$ if Λ is not absolutely continuous on $[0, t]$. The following theorem proves the uniqueness of the *stopped* physical solution $\overline{Y}_{t\wedge\overline{\tau}}$, $t \in [0, t_{\mathrm{reg}})$, in the class of solutions with $t_{\mathrm{reg}} > 0$ and such that $\|\lambda\|_{L^2([0,\cdot])}$ "does not jump to infinity." Moreover, it provides a precise connection between the cumulative loss process and the normalized density on $[0, t_{\mathrm{reg}})$. Therein and throughout the paper, we write $W_2^1([0,\infty))$ ($W_2^{1,2}([0,t] \times [0,\infty))$, resp.) for the Sobolev space of $L^2([0,\infty))$ ($L^2([0,t] \times [0,\infty))$, resp.) functions whose first weak derivatives (first weak derivatives in time and the first two weak derivatives in space, resp.) belong to $L^2([0,\infty))$ ($L^2([0,t] \times [0,\infty))$, resp.), equipped with the associated Sobolev norm.

THEOREM 2.7. *Let ν be a probability measure on* $[0,\infty)$ *admitting a density f_ν in the Sobolev space $W_2^1([0,\infty))$ with $f_\nu(0) = 0$. Then*:

(a) *there exists a physical solution $\overline{Y}$, such that $\overline{Y}_0 \overset{d}{=} \nu$ and the associated t_{reg} and λ satisfy*: $t_{\mathrm{reg}} > 0$ *and*, *if $t_{\mathrm{reg}} < T$*,

$$\lim_{t \uparrow t_{\mathrm{reg}}} \|\lambda\|_{L^2([0,t])} = \infty;$$

(b) *the value of t_{reg} is the same for all physical solutions satisfying the conditions of part* (a), *and the corresponding stopped physical solutions $\overline{Y}_{t\wedge\overline{\tau}}$, $t \in$* $[0, t_{\mathrm{reg}})$ *are indistinguishable*;

(c) $p(\cdot,\cdot)$ *is continuous on* $[0, t_{\mathrm{reg}}) \times [0, \infty)$, *with* $p(\cdot, 0) \equiv 0$; *moreover, the weak derivative* $\partial_y p$ *satisfies* $(\partial_y p)(\cdot, 0) \in L^2_{\mathrm{loc}}([0, t_{\mathrm{reg}}))$ *and*

$$\lambda_t = -C \frac{\sigma^2}{2} \frac{(\partial_y p)(t, 0)}{\int_0^\infty p(t, y)\, dy} \qquad \textit{for almost every } t \in [0, t_{\mathrm{reg}}). \tag{2.22}$$

Parts (a) and (b) of Theorem 2.7 show that the logarithmic asset value of a typical bank in a large banking system behaves according to the unique stopped physical solution until the time $t_{\mathrm{reg}} > 0$, given by (2.21). Theorem 2.7(c) expresses the value of λ_t through the slope of the normalized logarithmic asset value profile of banks that are close to failure at time t.

REMARK 2.8. If $t_{\mathrm{reg}} = T$, for every $T > 0$, then there exists a unique physical solution with locally square integrable λ on the entire time interval $[0, \infty)$. The results of [9] can be used to show that such an extension is possible (even with continuous λ) under additional assumptions on the initial condition and, most importantly, for sufficiently small C (depending on the initial condition).

The rest of the paper is structured as follows. In Section 3, we analyze the Cauchy–Dirichlet problem associated with a stopped physical solution before the explosion of λ in the L^2 norm, which is used in the proof of Theorem 2.7. Section 4 studies the fixed-point problem satisfied by λ until it explodes in the L^2 norm. We use Sobolev norm estimates for solutions to linear parabolic PDEs in [22] (see [23], Chapter III, Section 6) and parabolic Sobolev inequalities (see, e.g., [23], Chapter II, Lemmas 3.3, 3.4) to show that the Banach fixed-point theorem is applicable to a suitable "truncated" fixed-point problem. This yields the existence and uniqueness of the solution to the original fixed-point problem. The latter is used to construct the unique stopped physical solution until the explosion of λ in the L^2 norm, proving Theorem 2.7. It is worth noting that in the proofs of Section 4 we do not rely on Definition 2.1 of a physical solution, but rather work with (2.12) only. Section 5 establishes a priori regularity properties of physical solutions and connects the behavior of the normalized density $p(t, y)/\mathbb{P}(\overline{\tau} \geq t)$ near $y = 0$ with the Hölder continuity of Λ, proving Theorem 2.6. Section 6 provides the proof of Proposition 2.5, which is an adaptation of the arguments used in [10]. Section 7 is devoted to the proof of Theorem 2.4, and it also follows the ideas of [10].

3. Cauchy–Dirichlet problem.

For ν as in Theorem 2.7, $T_1 \in (0, \infty)$, and $\lambda \in L^2([0, T_1])$ consider the Cauchy-Dirichlet problem

$$\partial_t p = -(\alpha + \lambda_t)\partial_y p + \frac{\sigma^2}{2}\partial_y^2 p, \qquad p(0, \cdot) = f_\nu,\ p(\cdot, 0) = 0. \tag{3.1}$$

The next two lemmas investigate its solution p.

LEMMA 3.1. *Let ν be as in Theorem 2.7. Then, for $T_1 \in (0,\infty)$ and $\lambda \in L^2([0,T_1])$, there exists a unique generalized solution p of (3.1) in the space $W_2^{1,2}([0,T_1]\times[0,\infty))$. Moreover, p is nonnegative and satisfies the integrability estimates*

$$
\begin{aligned}
&e^{-\int_0^t (\frac{\alpha+\lambda_s}{\sigma})^2 \,\mathrm{d}s} \int_0^\infty \left(2\Phi\left(\frac{y}{\sigma\sqrt{t}}\right)-1\right)^2 f_\nu(y)\,\mathrm{d}y \\
&\quad \le \int_0^\infty p(t,y)\,\mathrm{d}y \le e^{\frac{1}{2}\int_0^t (\frac{\alpha+\lambda_s}{\sigma})^2 \,\mathrm{d}s} \int_0^\infty \left(2\Phi\left(\frac{y}{\sigma\sqrt{t}}\right)-1\right)^{1/2} f_\nu(y)\,\mathrm{d}y
\end{aligned}
\tag{3.2}
$$

for all $t \in [0,T_1]$, where Φ is the standard Gaussian cumulative distribution function.

LEMMA 3.2. *Let ν be as in Theorem 2.7. Then, for $T_1 \in (0,\infty)$ and $\lambda \in L^2([0,T_1])$, the unique generalized solution $p \in W_2^{1,2}([0,T_1]\times[0,\infty))$ of (3.1) fulfills*

$$
\int_0^\infty p(t,y)\,\mathrm{d}y = \int_0^\infty f_\nu(y)\,\mathrm{d}y - \frac{\sigma^2}{2}\int_0^t (\partial_y p)(s,0)\,\mathrm{d}s, \qquad t \in [0,T_1]. \tag{3.3}
$$

PROOF OF LEMMA 3.1. *Step 1.* The existence and uniqueness of the generalized solution $p \in W_2^{1,2}([0,T_1]\times[0,\infty))$ of (3.1) follow from the results of [23], Chapter III, Section 6 (see [23], Chapter III, Remark 6.3, and note that $-(\alpha+\lambda)$ fulfills the condition (6.26) there). We also refer to the original reference [22].

Now, consider the process $\overline{Z}_t^\lambda$, $t \in [0,T_1]$ defined by

$$
\begin{aligned}
&\overline{Z}_0^\lambda \stackrel{d}{=} \nu, \qquad \overline{Z}_t^\lambda = \overline{Z}_0^\lambda + \alpha t + \int_0^t \lambda_s\,\mathrm{d}s + \sigma \overline{B}_t, \qquad t \in [0, T_1 \wedge \overline{\tau}], \\
&\overline{\tau} = \inf\{t \in [0,T_1] : \overline{Z}_t^\lambda = 0\}, \qquad \overline{Z}_t^\lambda = 0, \qquad \overline{\tau} \le t \le T_1,
\end{aligned}
\tag{3.4}
$$

where $\overline{B}$ is a standard Brownian motion independent of $\overline{Z}_0^\lambda \stackrel{d}{=} \nu$. The Radon–Nikodym and the Girsanov theorems show that the law of $\overline{Z}_t^\lambda$ has a density with respect to that of $(\overline{Z}_0^\lambda + \sigma\overline{B})_{t\wedge\overline{\tau}}$ for all $t \in [0,T_1]$. In particular, the restriction of the law of $\overline{Z}_t^\lambda$ to $(0,\infty)$ possesses a density $\widetilde{p}(t,\cdot)$ with respect to the Lebesgue measure for all $t \in [0,T_1]$. We claim next that the $W_2^{1,2}([0,T_1]\times[0,\infty))$-solution p of (3.1) equals $\widetilde{p}$.

Step 2. We fix a $t \in [0,T_1]$, pick a function $h \in W_2^1([0,\infty))$ with $h(0)=0$, and consider the auxiliary problem

$$
\begin{aligned}
&\partial_s \zeta + (\alpha+\lambda_s)\partial_y \zeta + \frac{\sigma^2}{2}\partial_y^2 \zeta = 0, \\
&\zeta(t,\cdot) = h, \qquad \zeta(\cdot,0) = 0, \qquad \zeta \in W_2^{1,2}\big([0,t]\times[0,\infty)\big).
\end{aligned}
\tag{3.5}
$$

As with the problem (3.1) there exists a unique generalized solution ζ of (3.5). Moreover, for any fixed $K \in (0,\infty)$ and with

$$\overline{\tau}_K := \inf\{s \in [0,T_1] : \overline{Z}_0^\lambda + \sigma \overline{B}_s = K\} \tag{3.6}$$

the PDE in (3.5) and the Itô formula in [21], Section 2.10, Theorem 1, yield

$$\begin{aligned} &\zeta\big(t \wedge \overline{\tau} \wedge \overline{\tau}_K, \big(\overline{Z}_0^\lambda + \sigma \overline{B}\big)_{t\wedge\overline{\tau}\wedge\overline{\tau}_K}\big) \\ &\quad = \zeta\big(0, \overline{Z}_0^\lambda\big) + \int_0^{t\wedge\overline{\tau}\wedge\overline{\tau}_K} (\partial_y \zeta)\big(s, \overline{Z}_0^\lambda + \sigma \overline{B}_s\big)\sigma \,\mathrm{d}\overline{B}_s \\ &\qquad - \int_0^{t\wedge\overline{\tau}\wedge\overline{\tau}_K} (\partial_y \zeta)\big(s, \overline{Z}_0^\lambda + \sigma \overline{B}_s\big)(\alpha + \lambda_s)\,\mathrm{d}s \end{aligned} \tag{3.7}$$

(note that ζ is a continuous bounded function and $(\partial_y\zeta) \in L^4([0,t]\times[0,\infty))$ by the parabolic Sobolev inequality in the form of [23], Chapter II, Lemma 3.3, justifying the applicability of the Itô formula cited). In view of the Girsanov theorem, (3.7) implies

$$\zeta\big(t\wedge\overline{\tau}\wedge\overline{\tau}_K, \overline{Z}^\lambda_{t\wedge\overline{\tau}\wedge\overline{\tau}_K}\big) = \zeta\big(0,\overline{Z}_0^\lambda\big) + \int_0^{t\wedge\overline{\tau}\wedge\overline{\tau}_K} (\partial_y\zeta)\big(s,\overline{Z}_s^\lambda\big)\sigma\,\mathrm{d}\overline{B}_s. \tag{3.8}$$

Next, we combine the Girsanov theorem with Hölder's and Jensen's inequalities to obtain the chain of estimates

$$\begin{aligned} &\mathbb{E}\bigg[\int_0^{t\wedge\overline{\tau}\wedge\overline{\tau}_K} \big((\partial_y\zeta)\big(s,\overline{Z}_s^\lambda\big)\big)^2\,\mathrm{d}s\bigg] \\ &\quad = \mathbb{E}\bigg[e^{-\int_0^t \frac{\alpha+\lambda_s}{\sigma}\,\mathrm{d}\overline{B}_s - \frac{1}{2}\int_0^t (\frac{\alpha+\lambda_s}{\sigma})^2\,\mathrm{d}s} \int_0^{t\wedge\overline{\tau}\wedge\overline{\tau}_K} \big((\partial_y\zeta)\big(s,\overline{Z}_0^\lambda+\sigma\overline{B}_s\big)\big)^2\,\mathrm{d}s\bigg] \\ &\quad \le \mathbb{E}\big[e^{-3\int_0^t \frac{\alpha+\lambda_s}{\sigma}\,\mathrm{d}\overline{B}_s - \frac{3}{2}\int_0^t (\frac{\alpha+\lambda_s}{\sigma})^2\,\mathrm{d}s}\big]^{1/3} \\ &\qquad \times \mathbb{E}\bigg[\bigg(\int_0^{t\wedge\overline{\tau}\wedge\overline{\tau}_K} \big((\partial_y\zeta)\big(s,\overline{Z}_0^\lambda+\sigma\overline{B}_s\big)\big)^2\,\mathrm{d}s\bigg)^{3/2}\bigg]^{2/3} \\ &\quad \le e^{\int_0^t (\frac{\alpha+\lambda_s}{\sigma})^2\,\mathrm{d}s} t^{1/3} \mathbb{E}\bigg[\int_0^{t\wedge\overline{\tau}\wedge\overline{\tau}_K} \big|(\partial_y\zeta)\big(s,\overline{Z}_0^\lambda+\sigma\overline{B}_s\big)\big|^3\,\mathrm{d}s\bigg]^{2/3}. \end{aligned}$$

The latter expression is finite thanks to [21], Section 2.2, Theorem 4, and $(\partial_y\zeta) \in L^6([0,t]\times[0,\infty))$ (a consequence of the parabolic Sobolev inequality in the form of [23], Chapter II, Lemma 3.3). Consequently, taking the expectation on both sides of (3.8) and passing to the limit $K \to \infty$ we get

$$\mathbb{E}\big[\zeta\big(t\wedge\overline{\tau}, \overline{Z}^\lambda_{t\wedge\overline{\tau}}\big)\big] = \mathbb{E}\big[\zeta\big(0,\overline{Z}_0^\lambda\big)\big], \tag{3.9}$$

which can be rewritten as

$$\int_0^\infty h(y)\widetilde{p}(t,y)\,\mathrm{d}y = \int_0^\infty \zeta(0,y) f_\nu(y)\,\mathrm{d}y. \tag{3.10}$$

On the other hand, $p \in W_2^{1,2}([0,T_1] \times [0,\infty))$ implies that the norms $\|p(s,\cdot)\|_{L^2([0,\infty))}$, $s \in [0,t]$ are uniformly bounded due to the continuity of the evaluation maps $p \mapsto p(s,\cdot)$ (see, e.g., [23], Chapter II, Lemma 3.4). This and a density argument invoking the continuity of the evaluation maps one more time show that the weak formulation of the problem (3.1) applies to test functions in $W_2^{1,2}([0,t] \times [0,\infty))$. For the solution ζ of (3.5), it gives

$$\int_0^\infty h(y)p(t,y)\,\mathrm{d}y = \int_0^\infty \zeta(0,y) f_\nu(y)\,\mathrm{d}y, \tag{3.11}$$

which together with (3.10) and the arbitrariness of h, t implies $\widetilde{p} = p$ on $[0,T_1] \times [0,\infty)$.

Step 3. The nonnegativity of p is now an immediate consequence of the nonnegativity of $\widetilde{p}$. In addition, $\int_0^\infty p(t,y)\,\mathrm{d}y$ can be rewritten as

$$\mathbb{E}[\mathbf{1}_{\{\overline{Z}_t^\lambda > 0\}}] = \int_0^\infty \mathbb{E}\big[e^{-\int_0^t \frac{\alpha+\lambda_s}{\sigma}\,\mathrm{d}\overline{B}_s - \frac{1}{2}\int_0^t (\frac{\alpha+\lambda_s}{\sigma})^2\,\mathrm{d}s}\mathbf{1}_{\{y+\sigma\overline{B}_s>0, 0\le s\le t\}}\big] f_\nu(y)\,\mathrm{d}y.$$

At this point, the estimates of (3.2) follow from the Cauchy–Schwarz inequality in the forms

$$\begin{aligned}
&\mathbb{E}\big[e^{\int_0^t \frac{\alpha+\lambda_s}{\sigma}\,\mathrm{d}\overline{B}_s + \frac{1}{2}\int_0^t (\frac{\alpha+\lambda_s}{\sigma})^2\,\mathrm{d}s}\big]^{-1}\mathbb{P}\bigg(\overline{B}_s > -\frac{y}{\sigma}, 0 \le s \le t\bigg)^2 \\
&\quad \le \mathbb{E}\big[e^{-\int_0^t \frac{\alpha+\lambda_s}{\sigma}\,\mathrm{d}\overline{B}_s - \frac{1}{2}\int_0^t (\frac{\alpha+\lambda_s}{\sigma})^2\,\mathrm{d}s}\mathbf{1}_{\{y+\sigma\overline{B}_s>0, 0\le s\le t\}}\big] \\
&\quad \le \mathbb{E}\big[e^{-2\int_0^t \frac{\alpha+\lambda_s}{\sigma}\,\mathrm{d}\overline{B}_s - \int_0^t (\frac{\alpha+\lambda_s}{\sigma})^2\,\mathrm{d}s}\big]^{1/2}\mathbb{P}\bigg(\overline{B}_s > -\frac{y}{\sigma}, 0 \le s \le t\bigg)^{1/2}
\end{aligned}$$

and the reflection principle for Brownian motion. □

We proceed to the proof of Lemma 3.2.

PROOF OF LEMMA 3.2. We pick a sequence h_n, $n \in \mathbb{N}$ of infinitely differentiable functions on $[0,\infty)$ such that:

(i) $h_n(y) = 1$ if $n^{-1} \le y \le n$ and $h_n(y) = 0$ if $y \le (n+1)^{-1}$ or $y \ge n+1$,
(ii) h_n is nondecreasing on $[(n+1)^{-1}, n^{-1}]$ and nonincreasing on $[n, n+1]$,
(iii) $\sup_{n\in\mathbb{N}} \sup_{[n,n+1]} |h_n'| < \infty$ and $\sup_{n\in\mathbb{N}} \sup_{[n,n+1]} |h_n''| < \infty$.

The weak formulation of (3.1) for each such function reads

$$\begin{aligned}
&\int_0^\infty h_n(y)p(t,y)\,\mathrm{d}y - \int_0^\infty h_n(y) f_\nu(y)\,\mathrm{d}y \\
&\quad = \int_0^t (\alpha+\lambda_s)\int_0^\infty h_n'(y)p(s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\qquad + \frac{\sigma^2}{2}\int_0^t \int_0^\infty h_n''(y)p(s,y)\,\mathrm{d}y\,\mathrm{d}s, \qquad t \in [0,T_1].
\end{aligned} \tag{3.12}$$

The monotone convergence theorem implies that the first line in (3.12) tends to

$$\int_0^\infty p(t,y)\,\mathrm{d}y - \int_0^\infty f_\nu(y)\,\mathrm{d}y$$

in the limit $n \to \infty$. Moreover, the first summand on the second line in (3.12) can be rewritten as

$$\begin{aligned}\int_0^t (\alpha+\lambda_s) \int_{(n+1)^{-1}}^{n^{-1}} h_n'(y)\,p(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ + \int_0^t (\alpha+\lambda_s) \int_n^{n+1} h_n'(y)\,p(s,y)\,\mathrm{d}y\,\mathrm{d}s.\end{aligned} \tag{3.13}$$

Combining $p(s,0)=0$, $s\in[0,T]$, the uniform continuity of p on $[0,T_1]\times[0,1]$ (due to the parabolic Sobolev inequality in [23], Chapter II, Lemma 3.3), and property (ii) above we see that the first summand in (3.13) converges to 0 as $n\to\infty$. The same is true for the second summand in (3.13) thanks to property (iii) above, the upper bound of Lemma 3.1, and the dominated convergence theorem.

The second summand on the second line in (3.12) can be recast as

$$\frac{\sigma^2}{2}\int_0^t \int_n^{n+1} h_n''(y)\,p(s,y)\,\mathrm{d}y\,\mathrm{d}s + \frac{\sigma^2}{2}\int_0^t \int_{(n+1)^{-1}}^{n^{-1}} h_n''(y)\,p(s,y)\,\mathrm{d}y\,\mathrm{d}s. \tag{3.14}$$

As $n\to\infty$, the first summand in (3.14) converges to 0 by the same argument as used to analyze the second summand in (3.13). Next, we employ integration by parts to transform the second summand in (3.14) to

$$-\frac{\sigma^2}{2}\int_0^t \int_{(n+1)^{-1}}^{n^{-1}} h_n'(y)(\partial_y p)(s,y)\,\mathrm{d}y\,\mathrm{d}s \tag{3.15}$$

(recall that $p(s,\cdot)\in W_2^1([0,\infty))$, $s\in[0,T_1]$ thanks to the well-definedness of the evaluation maps $p\mapsto p(s,\cdot)$, see [23], Chapter II, Lemma 3.4). The quantity in (3.15) converges to $-\frac{\sigma^2}{2}\int_0^t(\partial_y p)(s,0)\,\mathrm{d}s$ as $n\to\infty$, since

$$\begin{aligned}&\limsup_{n\to\infty}\left|\int_0^t \int_{(n+1)^{-1}}^{n^{-1}} h_n'(y)(\partial_y p)(s,y)\,\mathrm{d}y\,\mathrm{d}s - \int_0^t (\partial_y p)(s,0)\,\mathrm{d}s\right| \\ &\quad = \limsup_{n\to\infty}\left|\int_0^t \int_{(n+1)^{-1}}^{n^{-1}} h_n'(y)\int_0^y \big(\partial_y^2 p\big)(s,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s\right| \\ &\quad \le \limsup_{n\to\infty}\int_0^t \int_0^{n^{-1}} \big|\partial_y^2 p\big|(s,z)\,\mathrm{d}z\,\mathrm{d}s = 0,\end{aligned}$$

where we have relied on properties (i), (ii) above and $p\in W_2^{1,2}([0,T_1]\times[0,\infty))$. All in all, we end up with (3.3) when we take the $n\to\infty$ limit in (3.12). □

4. Regular interval of the physical solution. The next proposition is the key to the proof of Theorem 2.7 and establishes the existence and uniqueness of the solution to the fixed-point problem associated with the function λ in (2.21).

PROPOSITION 4.1. *Let ν be as in Theorem 2.7. Then*:

(a) *there exist a time $t_{\mathrm{reg}} \in (0, T]$ and a function $\lambda \in L^2_{\mathrm{loc}}([0, t_{\mathrm{reg}}))$ such that for all $T_1 \in (0, t_{\mathrm{reg}})$ the unique generalized solution of*

$$\partial_t p = -(\alpha + \lambda_t)\partial_y p + \frac{\sigma^2}{2}\partial_y^2 p, \qquad p(0, \cdot) = f_\nu, \qquad p(\cdot, 0) = 0 \tag{4.1}$$

in $W_2^{1,2}([0, T_1] \times [0, \infty))$ satisfies

$$-C\frac{\sigma^2}{2}\frac{(\partial_y p)(t, 0)}{\int_0^\infty p(t, y)\,\mathrm{d}y} = \lambda_t \qquad \text{for almost every } t \in [0, T_1] \tag{4.2}$$

and $\lim_{T_1 \uparrow t_{\mathrm{reg}}} \|\lambda\|_{L^2([0,T_1])} = \infty$ *if* $t_{\mathrm{reg}} < T$;

(b) *for any $(t_{\mathrm{reg}}, \lambda)$, $(\tilde{t}_{\mathrm{reg}}, \tilde{\lambda})$ such that*:

(i) $t_{\mathrm{reg}}, \tilde{t}_{\mathrm{reg}} \in (0, T]$,

(ii) $\lambda \in L^2_{\mathrm{loc}}([0, t_{\mathrm{reg}}))$, $\tilde{\lambda} \in L^2_{\mathrm{loc}}([0, \tilde{t}_{\mathrm{reg}}))$ *satisfy the fixed-point problem* (4.1), (4.2) *for all* $T_1 \in (0, t_{\mathrm{reg}})$, $T_1 \in (0, \tilde{t}_{\mathrm{reg}})$, *respectively*,

(iii) $\lim_{T_1 \uparrow t_{\mathrm{reg}}} \|\lambda\|_{L^2([0,T_1])} = \infty$ *if* $t_{\mathrm{reg}} < T$, $\lim_{T_1 \uparrow \tilde{t}_{\mathrm{reg}}} \|\tilde{\lambda}\|_{L^2([0,T_1])} = \infty$ *if* $\tilde{t}_{\mathrm{reg}} < T$

it holds $t_{\mathrm{reg}} = \tilde{t}_{\mathrm{reg}}$ and $\lambda = \tilde{\lambda}$ almost everywhere.

PROOF. *Step 1.* Our first aim is to show for all $M \in (0, \infty)$ and all small enough $T_1 = T_1(M) \in (0, T)$ the existence and uniqueness in $L^2([0, T_1])$ for the "truncated" fixed-point problem

$$\begin{aligned} &\partial_t p = -\big(\alpha + \lambda_t^{M,T_1}\big)\partial_y p + \frac{\sigma^2}{2}\partial_y^2 p, \\ &p(0, \cdot) = f_\nu, \qquad p(\cdot, 0) = 0, \qquad p \in W_2^{1,2}\big([0, T_1] \times [0, \infty)\big), \end{aligned} \tag{4.3}$$

$$-C\frac{\sigma^2}{2}\frac{(\partial_y p)(t, 0)}{\int_0^\infty p(t, y)\,\mathrm{d}y} = \lambda_t \qquad \text{for almost every } t \in [0, T_1], \tag{4.4}$$

where

$$\lambda^{M,T_1} := \lambda \mathbf{1}_{\{\|\lambda\|_{L^2([0,T_1])} \le M\}} + \lambda \frac{M}{\|\lambda\|_{L^2([0,T_1])}} \mathbf{1}_{\{\|\lambda\|_{L^2([0,T_1])} > M\}}. \tag{4.5}$$

To this end, it suffices to verify that the mapping taking $L^2([0, T_1])$ functions λ to the left-hand side of (4.4) [with p being the unique generalized solution of (4.3)] is a contraction on $L^2([0, T_1])$, since then the Banach fixed-point theorem can be applied. We observe that the described mapping is well defined with its range contained in $L^2([0,T_1])$ by the assumptions $f_\nu \in W_2^1([0,\infty))$, $f_\nu(0)=0$, the existence and uniqueness result of [23], Chapter III, Remark 6.3, the well-definedness of the evaluation map $p \mapsto (\partial_y p)(\cdot,0)$ of [23], Chapter II, Lemma 3.4, and the lower bound in (3.2). The following two steps are devoted to the proof of the contraction property.

Step 2. Given two $L^2([0,T_1])$ functions λ, $\widetilde{\lambda}$, let p, $\widetilde{p}$ be the corresponding solutions of (4.3) and note that $\Delta := p - \widetilde{p} \in W_2^{1,2}([0,T_1]\times[0,\infty))$ satisfies

$$\partial_t \Delta = -\big(\alpha + \widetilde{\lambda}_t^{M,T_1}\big)\partial_y \Delta + \frac{\sigma^2}{2}\partial_y^2 \Delta + \big(\widetilde{\lambda}_t^{M,T_1} - \lambda_t^{M,T_1}\big)\partial_y p, \qquad \Delta(0,\cdot)=0, \qquad \Delta(\cdot,0)=0. \tag{4.6}$$

The source term in (4.6) admits the norm bound

$$\begin{aligned} &\big\|\big(\widetilde{\lambda}_t^{M,T_1} - \lambda_t^{M,T_1}\big)\partial_y p\big\|_{L^2([0,T_1]\times[0,\infty))} \\ &\quad \le \big\|\widetilde{\lambda}_t^{M,T_1} - \lambda_t^{M,T_1}\big\|_{L^2([0,T_1])} \operatorname*{ess\,sup}_{t\in[0,T_1]} \big\|(\partial_y p)(t,\cdot)\big\|_{L^2([0,\infty))} \\ &\quad \le 2\|\widetilde{\lambda}_t - \lambda_t\|_{L^2([0,T_1])} \operatorname*{ess\,sup}_{t\in[0,T_1]} \big\|(\partial_y p)(t,\cdot)\big\|_{L^2([0,\infty))}. \end{aligned} \tag{4.7}$$

Moreover, the boundedness of the evaluation maps $p \mapsto p(t,\cdot)$ (see [23], Chapter II, Lemma 3.4) and the results of [23], Chapter III, Section 6, used for the solution p of (4.3) give the respective estimates

$$\operatorname*{ess\,sup}_{t\in[0,T_1]} \big\|(\partial_y p)(t,\cdot)\big\|_{L^2([0,\infty))} \le C_1 \|p\|_{W_2^{1,2}([0,T_1]\times[0,\infty))} \le C_2, \tag{4.8}$$

with constants $C_1 = C_1(T) < \infty$ and $C_2 = C_2(\alpha, M, \sigma, \|f_\nu\|_{W_2^1([0,\infty))}, T) < \infty$. In view of (4.7), (4.8), we can now apply [23], Chapter II, Lemma 3.4, and the results of [23], Chapter III, Section 6, to the solution Δ of (4.6) to find

$$\begin{aligned} \operatorname*{ess\,sup}_{t\in[0,T_1]} \big\|(\partial_y \Delta)(t,\cdot)\big\|_{L^2([0,\infty))} &\le C_1 \|\Delta\|_{W_2^{1,2}([0,T_1]\times[0,\infty))} \\ &\le C_3 C_2 \|\widetilde{\lambda}_t - \lambda_t\|_{L^2([0,T_1])}, \end{aligned} \tag{4.9}$$

where the constant $C_3 < \infty$ can be chosen in terms of α, M, σ and T only.

Next, we regard the PDE in (4.6) as a heat equation with Dirichlet boundary conditions and the $L^2([0,T_1]\times[0,\infty))$ source

$$g := -\big(\alpha + \widetilde{\lambda}_t^{M,T_1}\big)\partial_y \Delta + \big(\widetilde{\lambda}_t^{M,T_1} - \lambda_t^{M,T_1}\big)\partial_y p. \tag{4.10}$$

In particular, we can write

$$\Delta(t,y) = \int_0^t \int_0^\infty g(s,z)\psi_\sigma(t-s,z,y)\,\mathrm{d}z\,\mathrm{d}s, \qquad (t,y)\in[0,T_1]\times[0,\infty), \tag{4.11}$$

$$(\partial_y \Delta)(t,y) = \int_0^t \int_0^\infty g(s,z)(\partial_y \psi_\sigma)(t-s,z,y)\,\mathrm{d}z\,\mathrm{d}s, \qquad (t,y) \in [0,T_1] \times [0,\infty), \tag{4.12}$$

where

$$\psi_\sigma(t-s,y,z) := \bigl(2\pi\sigma^2(t-s)\bigr)^{-1/2} \times \biggl(\exp\biggl(-\frac{(y-z)^2}{2\sigma^2(t-s)}\biggr) - \exp\biggl(-\frac{(y+z)^2}{2\sigma^2(t-s)}\biggr)\biggr) \tag{4.13}$$

is the Dirichlet heat kernel on $[0,\infty)$ with the diffusion coefficient σ. It now follows from Fubini's theorem and the triangle inequality (first inequality), Young's inequality for convolution (second inequality), Cauchy–Schwarz inequality (third inequality) and (4.7), (4.8) and (4.9) (fourth inequality) that

$$\begin{aligned}
&\|(\partial_y \Delta)(\cdot,0)\|_{L^2([0,T_1])} \\
&\quad = \biggl\| \int_0^\infty \int_0^t g(s,z)(\partial_y \psi_\sigma)(t-s,z,0)\,\mathrm{d}s\,\mathrm{d}z \biggr\|_{L^2([0,T_1])} \\
&\quad \le \int_0^\infty \biggl\| \int_0^t g(s,z)(\partial_y \psi_\sigma)(t-s,z,0)\,\mathrm{d}s \biggr\|_{L^2([0,T_1])} \mathrm{d}z \\
&\quad \le \int_0^\infty \|g(\cdot,z)\|_{L^2([0,T_1])} \|(\partial_y \psi_\sigma)(\cdot,z,0)\|_{L^1([0,T_1])}\,\mathrm{d}z \\
&\quad \le \|g\|_{L^2([0,T_1]\times[0,\infty))} \biggl(\int_0^\infty \|(\partial_y \psi_\sigma)(\cdot,z,0)\|^2_{L^1([0,T_1])}\,\mathrm{d}z \biggr)^{1/2} \\
&\quad \le C_4 \|\widetilde{\lambda}_t - \lambda_t\|_{L^2([0,T_1])} \biggl(\int_0^\infty \|(\partial_y \psi_\sigma)(\cdot,z,0)\|^2_{L^1([0,T_1])}\,\mathrm{d}z \biggr)^{1/2},
\end{aligned} \tag{4.14}$$

with a constant $C_4 = C_4(\alpha, M, \sigma, \|f_\nu\|_{W_2^1([0,\infty))}, T) < \infty$.

Step 3. Next, we subtract (3.3) for $\widetilde{p}$ from (3.3) for p, apply the triangle and the Cauchy–Schwarz inequalities, and use (4.14) to find

$$\begin{aligned}
\sup_{t\in[0,T_1]} \biggl| \int_0^\infty \Delta(t,y)\,\mathrm{d}y \biggr| &= \frac{\sigma^2}{2} \sup_{t\in[0,T_1]} \biggl| \int_0^t (\partial_y \Delta)(s,0)\,\mathrm{d}s \biggr| \\
&\le \frac{\sigma^2}{2} T_1^{1/2} \|(\partial_y \Delta)(\cdot,0)\|_{L^2([0,T_1])} \\
&\le C_5 T_1^{1/2} \|\widetilde{\lambda}_t - \lambda_t\|_{L^2([0,T_1])},
\end{aligned} \tag{4.15}$$

where the constant $C_5 < \infty$ depends on α, M, σ, f_ν and T only.

In addition, the triangle inequality and the lower bound in (3.2) imply

$$\begin{aligned}
&\left\| \frac{(\partial_y p)(\cdot,0)}{\int_0^\infty p(\cdot,y)\,\mathrm{d}y} - \frac{(\partial_y \widetilde{p})(\cdot,0)}{\int_0^\infty \widetilde{p}(\cdot,y)\,\mathrm{d}y} \right\|_{L^2([0,T_1])} \\
&\quad \le \left\| \frac{1}{\int_0^\infty p(\cdot,y)\,\mathrm{d}y} (\partial_y \Delta)(\cdot,0) \right\|_{L^2([0,T_1])} \\
&\qquad + \left\| (\partial_y \widetilde{p})(\cdot,0) \left(\frac{1}{\int_0^\infty p(\cdot,y)\,\mathrm{d}y} - \frac{1}{\int_0^\infty \widetilde{p}(\cdot,y)\,\mathrm{d}y} \right) \right\|_{L^2([0,T_1])} \\
&\quad \le C_6 \left(\|(\partial_y \Delta)(\cdot,0)\|_{L^2([0,T_1])} + \left\| (\partial_y \widetilde{p})(\cdot,0) \int_0^\infty \Delta(\cdot,y)\,\mathrm{d}y \right\|_{L^2([0,T_1])} \right),
\end{aligned}$$

with a constant $C_6 = C_6(\alpha, M, \sigma, f_\nu, T) < \infty$. In view of (4.14), (4.15) and the boundedness of the evaluation map $\widetilde{p} \mapsto (\partial_y \widetilde{p})(\cdot,0)$ (see [23], Chapter II, Lemma 3.4), the latter upper bound is at most

$$C_7 \left(\left(\int_0^\infty \|(\partial_y \psi_\sigma)(\cdot,z,0)\|^2_{L^1([0,T_1])}\,\mathrm{d}z \right)^{1/2} + T_1^{1/2} \right) \|\widetilde{\lambda}_t - \lambda_t\|_{L^2([0,T_1])}, \tag{4.16}$$

where $C_7 < \infty$ can be chosen in terms of α, M, σ, f_ν and T only. The desired contraction property for small enough $T_1 = T_1(M) \in (0,T)$ readily follows.

Step 4. Now, we let

$$\begin{aligned} t_{\mathrm{reg}} := \sup\{ T_1 \in (0,T) : &\text{ the problem (4.1), (4.2) has a solution} \\ &\lambda \in L^2([0,T_1]) \} \end{aligned} \tag{4.17}$$

and claim that the supremum is taken over a nonempty set. Indeed, for fixed $M \in (0,\infty)$ and a small enough $T_1 = T_1(M) \in (0,T)$ consider the unique solution $\lambda \in L^2([0,T_1])$ of the truncated fixed-point problem (4.3), (4.4). The corresponding solution p of (4.3) satisfies

$$\|-(\alpha + \lambda_t^{M,T_1})\partial_y p\|_{L^2([0,T_1]\times[0,\infty))} \le (\alpha + M) C_2, \tag{4.18}$$

where C_2 is as in (4.8). Repeating the estimates from (4.14) we get therefore

$$\begin{aligned}
&\|(\partial_y p)(\cdot,0)\|_{L^2([0,T_1])} \\
&\quad \le (\alpha + M) C_2 \left(\int_0^\infty \|(\partial_y \psi_\sigma)(\cdot,z,0)\|^2_{L^1([0,T_1])}\,\mathrm{d}z \right)^{1/2} \\
&\qquad + \left\| \int_0^\infty f_\nu(z) (\partial_y \psi_\sigma)(t,z,0)\,\mathrm{d}z \right\|_{L^2([0,T_1])}.
\end{aligned} \tag{4.19}$$

The fixed-point constraint (4.4), the lower bound in (3.2), and the latter inequality give $\|\lambda\|_{L^2([0,T_1])} \le M$ upon decreasing the value of $T_1 = T_1(M) \in (0,T)$ if necessary. Such a T_1 belongs to the set on the right-hand side of (4.17), since $\lambda^{M,T_1} = \lambda$ and consequently λ is a solution of the fixed-point problem (4.1), (4.2).

We show next that for every element T_1 of the set on the right-hand side of (4.17) the corresponding solution of the fixed-point problem (4.1), (4.2) is unique. To this end, for any two solutions $\lambda, \widetilde{\lambda} \in L^2([0,T_1])$ we let $M = 1 + \|\lambda\|_{L^2([0,T_1])} \vee \|\widetilde{\lambda}\|_{L^2([0,T_1])}$. Then, for any $\varepsilon \in (0,T_1]$ the restrictions of both λ and $\widetilde{\lambda}$ to $[0,\varepsilon]$ solve the truncated fixed-point problem (4.3), (4.4) on $[0,\varepsilon]$. Combining this observation with the contraction property established in Steps 1–3 we find an $\varepsilon \in (0,T_1]$ such that $\lambda_t = \widetilde{\lambda}_t$ for almost every $t \in [0,\varepsilon]$.

With this ε and the solution $p \in W_2^{1,2}([0,\varepsilon]\times[0,\infty))$ of the Cauchy–Dirichlet problem in (4.3) we consider the mapping which takes $L^2([\varepsilon,(2\varepsilon)\wedge T_1])$ functions ρ to

$$-C\frac{\sigma^2}{2}\frac{(\partial_y u)(\cdot,0)}{\int_0^\infty u(\cdot,y)\,\mathrm{d}y},$$

where u is the unique solution of

$$\begin{aligned}
&\partial_t u = -\big(\alpha + \rho_t^{M,\varepsilon,(2\varepsilon)\wedge T_1}\big)\partial_y u + \frac{\sigma^2}{2}\partial_y^2 u,\\
&u(\varepsilon,\cdot) = p(\varepsilon,\cdot), \qquad u(\cdot,0) = 0, \\
&u \in W_2^{1,2}\big(\big[\varepsilon,(2\varepsilon)\wedge T_1\big]\times[0,\infty)\big)
\end{aligned}\tag{4.20}$$

and

$$\begin{aligned}
&\rho^{M,\varepsilon,(2\varepsilon)\wedge T_1}\\
&\quad := \rho\mathbf{1}_{\{\|\rho\|_{L^2([\varepsilon,(2\varepsilon)\wedge T_1])}\le (M^2-\|\lambda\|^2_{L^2([0,\varepsilon])})^{1/2}\}}\\
&\qquad + \rho\frac{(M^2-\|\lambda\|^2_{L^2([0,\varepsilon])})^{1/2}}{\|\rho\|_{L^2([\varepsilon,(2\varepsilon)\wedge T_1])}}\mathbf{1}_{\{\|\rho\|_{L^2([\varepsilon,(2\varepsilon)\wedge T_1])}> (M^2-\|\lambda\|^2_{L^2([0,\varepsilon])})^{1/2}\}}.
\end{aligned}\tag{4.21}$$

This mapping is well defined with range contained in $L^2([\varepsilon,(2\varepsilon)\wedge T_1])$, since one can regard u as the restriction of the unique solution of

$$\begin{aligned}
&\partial_t u = -\big(\alpha + \xi_t^{M,(2\varepsilon)\wedge T_1}\big)\partial_y u + \frac{\sigma^2}{2}\partial_y^2 u,\\
&u(0,\cdot) = f_\nu, \qquad u(\cdot,0) = 0, \\
&u \in W_2^{1,2}\big(\big[0,(2\varepsilon)\wedge T_1\big]\times[0,\infty)\big)
\end{aligned}\tag{4.22}$$

to $[\varepsilon,(2\varepsilon)\wedge T_1]\times[0,\infty)$, where

$$\xi_t^{M,(2\varepsilon)\wedge T_1} := \begin{cases}\lambda_t & \text{if } t\in[0,\varepsilon),\\ \rho_t^{M,\varepsilon,(2\varepsilon)\wedge T_1} & \text{if } t \in \big[\varepsilon,(2\varepsilon)\wedge T_1\big],\end{cases}\tag{4.23}$$

and use the assumptions $f_\nu \in W_2^1([0,\infty))$, $f_\nu(0) = 0$, the existence and uniqueness result of [23], Chapter III, Remark 6.3, for (4.22), the second assertion in [23], Chapter II, Lemma 3.4, and the lower bound in (3.2).

Moreover, the described mapping is a contraction on $L^2([\varepsilon, (2\varepsilon) \wedge T_1])$. Indeed, repeating the analysis of Steps 1–3, replacing every occurrence of the interval $[0, \varepsilon]$ by $[\varepsilon, (2\varepsilon) \wedge T_1]$ and estimating

$$\operatorname*{ess\,sup}_{t \in [\varepsilon, (2\varepsilon) \wedge T_1]} \|(\partial_y u)(t, \cdot)\|_{L^2([0,\infty))}, \qquad \|(\partial_y u)(\cdot, 0)\|_{L^2([\varepsilon,(2\varepsilon)\wedge T_1])}$$

via [23], Chapter II, Lemma 3.4, and the results of [23], Chapter III, Section 6, for the problem (4.22) we conclude that the Lipschitz constant of the mapping does not exceed

$$C\frac{\sigma^2}{2}C_7\biggl(\biggl(\int_0^\infty \|(\partial_y \psi_\sigma)(\cdot, z, 0)\|^2_{L^1([0,\varepsilon])}\, \mathrm{d}z\biggr)^{1/2} + \varepsilon^{1/2}\biggr),$$

where one can use the same constant C_7 as in (4.16) [because the initial condition in the problem (4.22) is the same as in the problem (4.3)]. It follows that $\lambda_t = \widetilde{\lambda}_t$ for almost every $t \in [\varepsilon, (2\varepsilon) \wedge T_1]$, as the restrictions of λ and $\widetilde{\lambda}$ to $[\varepsilon, (2\varepsilon) \wedge T_1]$ are both fixed points of the mapping in consideration. A sequential repetition of the same argument on the time intervals

$$\bigl[(2\varepsilon) \wedge T_1, (3\varepsilon) \wedge T_1\bigr], \bigl[(3\varepsilon) \wedge T_1, (4\varepsilon) \wedge T_1\bigr], \ldots$$

yields $\lambda_t = \widetilde{\lambda}_t$ for almost every $t \in [0, T_1]$.

Part (b) of the proposition is an immediate consequence of the just established uniqueness assertion. In addition, the latter allows to combine the solutions of the fixed-point problem (4.1), (4.2) for different elements T_1 of the set on the right-hand side of (4.17) to a function $\lambda \in L^2_{\mathrm{loc}}([0, t_{\mathrm{reg}}))$, with t_{reg} defined via (4.17). To obtain part (a) of the proposition it remains to check $\lim_{T_1 \uparrow t_{\mathrm{reg}}} \|\lambda\|_{L^2([0,T_1])} = \infty$ if $t_{\mathrm{reg}} < T$. If $t_{\mathrm{reg}} < T$ and $\lim_{T_1 \uparrow t_{\mathrm{reg}}} \|\lambda\|_{L^2([0,T_1])} < \infty$ were to hold, then $\lambda \in L^2([0, t_{\mathrm{reg}}])$ would be a solution of the fixed-point problem (4.1), (4.2) on $[0, t_{\mathrm{reg}}]$. In addition, with $p \in W_2^{1,2}([0, t_{\mathrm{reg}}] \times [0, \infty))$ being the corresponding solution of the Cauchy–Dirichlet problem (4.1) the same arguments as in Steps 1–3 and the first paragraph of Step 4 would give the existence of a solution $\rho \in L^2([t_{\mathrm{reg}}, \widehat{T}])$ to the fixed-point problem

$$\begin{aligned} &\partial_t u = -(\alpha + \rho_t)\partial_y u + \frac{\sigma^2}{2}\partial_y^2 u, \\ &u(t_{\mathrm{reg}}, \cdot) = p(t_{\mathrm{reg}}, \cdot), \qquad u(\cdot, 0) = 0, \end{aligned} \tag{4.24}$$

$$-C\frac{\sigma^2}{2}\frac{(\partial_y u)(t, 0)}{\int_0^\infty u(t, y)\,\mathrm{d}y} = \rho_t \qquad \text{for almost every } t \in [t_{\mathrm{reg}}, \widehat{T}] \tag{4.25}$$

for $(\widehat{T} - t_{\mathrm{reg}}) \in (0, T - t_{\mathrm{reg}})$ small enough (note that $p(t_{\mathrm{reg}}, \cdot) \in W_2^1([0, \infty))$ with $p(t_{\mathrm{reg}}, 0) = 0$ thanks to the well-definedness of the evaluation map $p \mapsto p(t_{\mathrm{reg}}, \cdot)$; see [23], Chapter II, Lemma 3.4). The concatenation of λ and ρ would then be a solution of the fixed-point problem (4.1), (4.2) on $[0, \widehat{T}]$, a contradiction to the definition of t_{reg} in (4.17). $\square$

Given an initial condition $\overline{Z}_0 \overset{d}{=} \nu$ as in Theorem 2.7, we define

$$(4.26)\qquad \begin{aligned} &\overline{Z}_t = \overline{Z}_0 + \alpha t + \int_0^t \lambda_s \,\mathrm{d}s + \sigma \overline{B}_t, \qquad t \in [0, t_{\mathrm{reg}} \wedge T \wedge \overline{\tau}), \\ &\overline{\tau} = \inf\{t \in [0,T] : \overline{Z}_t = 0\}, \qquad \overline{Z}_t = 0, \qquad t \in [\overline{\tau}, t_{\mathrm{reg}} \wedge T), \end{aligned}$$

with the pair $(t_{\mathrm{reg}}, \lambda)$ of Proposition 4.1(a). The next proposition establishes that, until the explosion of the weak derivative of the cumulative loss process in the L^2 norm, any physical solution satisfying the conditions in Theorem 2.7(a) and stopped upon hitting 0 must be given by $\overline{Z}$.

PROPOSITION 4.2. *Let ν be as in Theorem 2.7. Then*:

(a) *the process $\overline{Z}$ defined by* (4.26) *satisfies the fixed-point constraint*

$$(4.27)\qquad \lambda_t = C\,\partial_t \log \mathbb{P}(\overline{\tau} > t) \qquad \textit{for almost every } t \in [0, t_{\mathrm{reg}} \wedge T);$$

(b) *for any physical solution $\overline{Y}$ satisfying the conditions in Theorem* 2.7(a), *the corresponding time $t_{\mathrm{reg}} > 0$ and the stopped process $\overline{Y}_{t\wedge\overline{\tau}}$, $t \in [0, t_{\mathrm{reg}} \wedge T)$ coincide with t_{reg} and $\overline{Z}$ in* (4.26).

PROOF. For any $T_1 \in (0, t_{\mathrm{reg}} \wedge T)$, the argument employed in the proof of Lemma 3.1 shows that the densities $p(t,\cdot)$, $t \in [0, T_1]$ of the restrictions of the laws of $\overline{Z}_t$, $t \in [0, T_1]$ to $(0,\infty)$, respectively, form a $W_2^{1,2}([0,T_1]\times[0,\infty))$-solution of (3.1). Consequently, the identity (3.3) and the lower bound in (3.2) reveal the function $t \mapsto \log \int_0^\infty p(t,y)\,\mathrm{d}y$ as absolutely continuous on $[0, T_1]$ with

$$(4.28)\qquad \begin{aligned} &\partial_t \log \int_0^\infty p(t,y)\,\mathrm{d}y \\ &\quad = -\frac{\sigma^2}{2} \frac{(\partial_y p)(t,0)}{\int_0^\infty p(t,y)\,\mathrm{d}y} \qquad \text{for almost every } t \in [0, T_1]. \end{aligned}$$

By combining (4.2) with (4.28), we arrive at (4.27), that is, part (a) of the proposition.

Next, we let λ be the weak derivative of the loss function of a physical solution $\overline{Y}$ as in part (b) of the proposition and $t_{\mathrm{reg}} > 0$ be the explosion time of λ in the L^2 norm. We also fix a $T_1 \in (0, t_{\mathrm{reg}} \wedge T)$ and denote by $p(t,\cdot)$, $t \in [0,T_1]$ the densities of the restrictions of the laws of $\overline{Y}_{t\wedge\overline{\tau}}$, $t \in [0, T_1]$ to $(0,\infty)$, respectively. Then, both (4.27) and (4.28) hold. Moreover, substituting the right-hand side of (4.28) for $\partial_t \log \mathbb{P}(\overline{\tau} > t)$ in (4.27) we get (4.2). Now, it follows from Proposition 4.1(b) that the pair $(t_{\mathrm{reg}}, \lambda)$ is the one of Proposition 4.1(a). Part (b) of the proposition at hand readily follows. □

We conclude the section with the proof of Theorem 2.7.

PROOF OF THEOREM 2.7. Parts (a) and (b) of the theorem follow directly from parts (a) and (b) of Proposition 4.2, respectively. Moreover, for any $T_1 \in (0, t_{\mathrm{reg}})$ the argument used in the proof of Lemma 3.1 identifies the densities $p(t,\cdot)$, $t \in [0, T_1]$ of the restrictions of the laws of $\overline{Y}_{t\wedge\overline{\tau}}$, $t \in [0, T_1]$ to $(0,\infty)$ with the $W_2^{1,2}([0, T_1] \times [0,\infty))$-solution of (3.1). Thus, the restriction $(\partial_y p)(\cdot, 0) \in L^2_{\mathrm{loc}}([0, t_{\mathrm{reg}}))$ of the weak derivative $\partial_y p$ is well defined due to the second assertion in [23], Chapter II, Lemma 3.4, and the characterization (2.22) follows from (4.2). □

5. A priori regularity of physical solutions. We begin this section by stating some elementary properties of physical solutions.

LEMMA 5.1. *Let $\overline{Y}$ be a càdlàg process satisfying* (2.12), *with the associated Λ, $\overline{\tau}^0$ and $\overline{\tau}$. Then, for $t \in (0, \overline{\tau}^0)$*:

(a) *the associated loss function Λ is nonincreasing*;

(b) *the laws of $\overline{Y}_t \mathbf{1}_{\{\overline{\tau}\geq t\}}$ and $\overline{Y}_{t-}\mathbf{1}_{\{\overline{\tau}\geq t\}}$, restricted to $(0,\infty)$, possess densities*; *the latter are bounded by a constant independent of t if the law of $\overline{Y}_0$ possesses a bounded density*;

(c) $\Lambda_{t-} = C \log \mathbb{P}(\overline{\tau} \geq t)$;

(d) $\mathbb{P}(\overline{Y}_{t-} \leq 0) > 0$.

PROOF. Property (a) is immediate from the definition of Λ. To deduce property (b) we notice that, for all $0 < a < b < \infty$,

$$\mathbb{P}\big(\overline{Y}_t \mathbf{1}_{\{\overline{\tau}\geq t\}} \in (a,b)\big) \leq \mathbb{P}\big(\overline{Y}_0 + \sigma \overline{B}_t \in (a - \alpha t - \Lambda_t, b - \alpha t - \Lambda_t)\big),$$
$$\mathbb{P}\big(\overline{Y}_{t-} \mathbf{1}_{\{\overline{\tau}\geq t\}} \in (a,b)\big) \leq \mathbb{P}\big(\overline{Y}_0 + \sigma \overline{B}_t \in (a - \alpha t - \Lambda_{t-}, b - \alpha t - \Lambda_{t-})\big),$$

and the two right-hand sides are bounded above by a constant times $(b - a)$. This constant can be chosen to be the same for all values of t in an interval bounded away from zero, and it is uniform for all $t \geq 0$ if $\overline{Y}_0$ possesses a bounded density. To obtain property (c), we let $s \in [0, t)$, $\varepsilon \in (0, 1)$ and employ the chain of estimates

$$\begin{aligned}
&\mathbb{P}(\overline{\tau} > s) - \mathbb{P}(\overline{\tau} \geq t)\\
&\qquad \leq \mathbb{P}\Big(\overline{Y}_s > 0, \inf_{r\in[s,t)} \overline{Y}_r \leq 0\Big)\\
&\qquad \leq \mathbb{P}\big(\overline{Y}_s \in (0,\varepsilon)\big) + \mathbb{P}\Big(\inf_{r\in[s,t)} \big(\alpha(r-s) + \sigma(\overline{B}_r - \overline{B}_s) + \Lambda_r - \Lambda_s\big) \leq -\varepsilon\Big).
\end{aligned}$$

In view of the existence of Λ_{t-} and property (b), the limit $\varepsilon \downarrow 0$ of the limit superior $s \uparrow t$ of the latter upper bound is 0, and property (c) readily follows. Finally, property (d) is a consequence of

$$\overline{Y}_{t-} \leq \overline{Y}_0 + \alpha t + \sigma \overline{B}_t, \qquad t \in \big(0, \overline{\tau}^0\big),$$

which is in turn due to property (a). □

Let us fix an arbitrary càdlàg process $\overline{Y}$ satisfying (2.12), with the associated Λ and $\overline{\tau}^0$. Recall that $p(t,\cdot)$ denotes the density of the distribution of $\overline{Y}_{t-}\mathbf{1}_{\{\overline{\tau}\geq t\}}$ restricted to $(0,\infty)$, and note that it may no longer be described by the PDE analyzed in the previous sections. In the rest of the section, we establish certain regularity properties of $\overline{Y}$, which, ultimately, allow us to conclude that Λ is Hölder continuous, with a Hölder exponent strictly greater than $1/2$, on any interval on which the density at zero vanishes.

We assume that the law of $\overline{Y}_0$ admits a bounded density and begin with an auxiliary construction. For fixed $t\in[0,\overline{\tau}^0)$ and $\varepsilon\in(0,\infty)$, we consider the sequence of processes $\overline{Y}^n$, $n\in\mathbb{N}$ defined recursively as follows:

$$\overline{Y}^1_s=\overline{Y}_{t-}+(\alpha s+\sigma\widetilde{B}_s)\mathbf{1}_{\{\overline{\tau}\geq t\}},\qquad s\in[0,\varepsilon], \tag{5.1}$$

$$\overline{Y}^n_s=\overline{Y}_{t-}+\big(\alpha s+\sigma\widetilde{B}_s+L^{n-1}\big)\mathbf{1}_{\{\overline{\tau}\geq t\}},\qquad s\in[0,\varepsilon],n\geq 2, \tag{5.2}$$

$$L^n=C\log\mathbb{P}\Big(\overline{\tau}\geq t,\inf_{s\in[0,\varepsilon]}\overline{Y}^n_s>0\Big)-\Lambda_{t-},\qquad n\geq 1, \tag{5.3}$$

where $\widetilde{B}_s:=\overline{B}_{t+s}-\overline{B}_t$, $s\in[0,\varepsilon]$ and $\Lambda_{0-}:=0$. The latter logarithm is well defined, since $t<\overline{\tau}^0$ and $\widetilde{B}$ is independent of $\overline{Y}_s$, $s\in[0,t]$. By Lemma 5.1(c), $\overline{Y}^2_s\leq\overline{Y}^1_s$ for all $s\in[0,\varepsilon]$ with probability one. Then, by induction, we conclude that the sequences $\overline{Y}^n_s$, $n\in\mathbb{N}$ are nonincreasing for all $s\in[0,\varepsilon]$ with probability one.

LEMMA 5.2. *Suppose that the law of $\overline{Y}_0$ possesses a bounded density. Then the following hold for any $t\in[0,\overline{\tau}^0)$:*

(a) *If $p(t,\cdot)$ satisfies*

$$\lim_{\eta\downarrow 0}\operatorname*{ess\,sup}_{y\in(0,\eta)}p(t,y)=0, \tag{5.4}$$

then there is a constant $C_L<\infty$ depending only on $C,\sigma,\Lambda_{t-},\|p(t,\cdot)\|_{L^\infty([0,\infty))}$ such that

$$|L^n|\leq C_L\varepsilon^{1/2} \tag{5.5}$$

for all $n\in\mathbb{N}$ sufficiently large and all $\varepsilon\in(0,\infty)$ sufficiently small, where L^n is defined by (5.3). Moreover, C_L can be chosen to be arbitrarily small, provided $\varepsilon>0$ is small enough.

(b) *If $p(t,\cdot)$ satisfies*

$$p(t,y)\leq\widehat{C}y^\gamma,\qquad y\in(0,\eta) \tag{5.6}$$

with some constants $\widehat{C}<\infty$, $\gamma\in(0,1]$, and $\eta>0$, then there is a constant $C_L<\infty$ depending only on $C,\sigma,\widehat{C},\gamma,\eta,\Lambda_{t-},\|p(t,\cdot)\|_{L^\infty([0,\infty))}$ such that

$$|L^n|\leq C_L\varepsilon^{(1+\gamma)/2} \tag{5.7}$$

for all $n\in\mathbb{N}$ sufficiently large and all $\varepsilon\in(0,\infty)$ sufficiently small.

PROOF. Let $\kappa(y) := \operatorname{ess\,sup}_{z\in(0,y)} p(t,z)$, $y \in (0,\infty)$ and note that in the setting of part (b) it holds $\kappa(y) \le \widehat{C} y^\gamma$, $y \in (0,\infty)$, where we have increased the value of $\widehat{C}$ if necessary [recall Lemma 5.1(b)]. We have the estimates

$$
\begin{aligned}
0 \ge\ & e^{\Lambda_{t-}/C}\big(e^{L^1/C}-1\big) = \mathbb{P}\Big(\overline{\tau}\ge t,\ \inf_{s\in[0,\varepsilon]} \overline{Y}^1_s > 0\Big) - \mathbb{P}(\overline{\tau}\ge t) \\
=\ & \int_0^\infty \Big(\mathbb{P}\Big(\inf_{s\in[0,\varepsilon]} (\alpha s + \sigma \widetilde{B}_s) > -y\Big) - 1\Big) p(t,y)\,\mathrm{d}y \\
\ge\ & -2\int_0^\infty \Phi\Big(\frac{|\alpha|\varepsilon - y}{\sigma\sqrt{\varepsilon}}\Big) p(t,y)\,\mathrm{d}y \ge -2\sqrt{\varepsilon}\int_0^\infty \Phi(1-y/\sigma)\,p(t,y\sqrt{\varepsilon})\,\mathrm{d}y \\
=\ & -2\sqrt{\varepsilon}\int_0^{\iota/\sqrt{\varepsilon}} \Phi(1-y/\sigma)\,p(t,y\sqrt{\varepsilon})\,\mathrm{d}y \\
& \quad - 2\sqrt{\varepsilon}\int_{\iota/\sqrt{\varepsilon}}^\infty \Phi(1-y/\sigma)\,p(t,y\sqrt{\varepsilon})\,\mathrm{d}y \\
\ge\ & -2\sqrt{\varepsilon}\Big(\int_0^\infty \Phi(1-y/\sigma)\kappa(y\sqrt{\varepsilon})\,\mathrm{d}y + \|p(t,\cdot)\|_{L^\infty([0,\infty))} e^{-\varepsilon^{-1/4}}\Big) \\
=:\ & -\sqrt{\varepsilon}C_0(\varepsilon)
\end{aligned}
$$

for all $\iota \in (0,1)$ and sufficiently small $\varepsilon \in (0,\iota^3)$. Here, as before, Φ stands for the standard Gaussian cumulative distribution function. It is clear from (5.4) that $C_0(\varepsilon)\to 0$ as $\varepsilon \downarrow 0$, and we conclude

$$
0 \ge L^1 \ge C\log\big(1 - e^{-\Lambda_{t-}/C}\sqrt{\varepsilon}C_0(\varepsilon)\big) \ge -2Ce^{-\Lambda_{t-}/C}\sqrt{\varepsilon}C_0(\varepsilon) \tag{5.8}
$$

for all sufficiently small $\varepsilon > 0$. In the setting of part (b), we have the additional upper bound

$$
C_0(\varepsilon) \le 2\varepsilon^{\gamma/2}\Big(\widehat{C}\int_0^\infty \Phi(1-y/\sigma) y^\gamma\,\mathrm{d}y + \|p(t,\cdot)\|_{L^\infty([0,\infty))}\Big) =: C_1\varepsilon^{\gamma/2}. \tag{5.9}
$$

For $n \ge 2$, we find

$$
\begin{aligned}
& e^{\Lambda_{t-}/C}\big(e^{L^n/C}-1\big) \\
&\quad = \mathbb{P}\Big(\overline{\tau}\ge t,\ \inf_{s\in[0,\varepsilon]} \overline{Y}^n_s > 0\Big) - \mathbb{P}(\overline{\tau}\ge t) \\
&\quad = \int_0^\infty \Big(\mathbb{P}\Big(\inf_{s\in[0,\varepsilon]} (\alpha s + \sigma \widetilde{B}_s) + L^{n-1} > -y\Big) - 1\Big) p(t,y)\,\mathrm{d}y \\
&\quad \ge -\int_0^{-L^{n-1}} p(t,y)\,\mathrm{d}y - 2\int_{-L^{n-1}}^\infty \Phi\Big(\frac{|\alpha|\varepsilon - y - L^{n-1}}{\sigma\sqrt{\varepsilon}}\Big) p(t,y)\,\mathrm{d}y \\
&\quad \ge -\int_0^{-L^{n-1}} p(t,y)\,\mathrm{d}y - 2\sqrt{\varepsilon}\int_0^\infty \Phi(1-y/\sigma)\,p\big(t, y\sqrt{\varepsilon} - L^{n-1}\big)\,\mathrm{d}y.
\end{aligned} \tag{5.10}
$$

Next, we choose an $\iota \in (0,1)$ small enough, so that for all sufficiently small $\varepsilon \in (0, \iota^3)$ it holds

$$2Ce^{-\Lambda_{t-}/C}\sqrt{\varepsilon}C_0(\varepsilon) \le \frac{2C_6}{1-C_5}\sqrt{\varepsilon} < \iota \tag{5.11}$$

where

$$\begin{aligned}
&C_6 := \frac{CC_3}{1 - C_2\iota\kappa(\iota) - C_3\sqrt{\varepsilon}}, \qquad C_5 := C_4\kappa(\iota) < 1,\\
&C_4 := \frac{CC_2}{1 - C_2\iota\kappa(\iota) - C_3\sqrt{\varepsilon}},\\
&C_3 := 2e^{-\Lambda_{t-}/C}\|p(t,\cdot)\|_{L^\infty([0,\infty))}\int_0^\infty \Phi(1 - y/\sigma)\,\mathrm{d}y,\\
&C_2 := e^{-\Lambda_{t-}/C}.
\end{aligned} \tag{5.12}$$

In particular, (5.11) implies

$$L^1 \ge -2Ce^{-\Lambda_{t-}/C}\sqrt{\varepsilon}C_0(\varepsilon) \ge -\frac{2C_6}{1-C_5}\sqrt{\varepsilon} > -\iota. \tag{5.13}$$

Assuming that

$$L^{n-1} \ge -\frac{2C_6}{1-C_5}\sqrt{\varepsilon} > -\iota \tag{5.14}$$

for some $n \ge 2$, the overall estimate in (5.10) yields

$$\begin{aligned}
e^{\Lambda_{t-}/C}\big(e^{L^n/C} - 1\big) \ge {}& -L^{n-1}\kappa\big(-L^{n-1}\big)\\
& - 2\sqrt{\varepsilon}\|p(t,\cdot)\|_{L^\infty([0,\infty))}\int_0^\infty \Phi(1 - y/\sigma)\,\mathrm{d}y,
\end{aligned} \tag{5.15}$$

so that

$$\begin{aligned}
L^n &\ge C\log\big(1 - C_2\big(-L^{n-1}\big)\kappa\big(-L^{n-1}\big) - C_3\sqrt{\varepsilon}\big) \ge C_5L^{n-1} - C_6\sqrt{\varepsilon}\\
&\ge -\frac{2C_5C_6}{1-C_5}\sqrt{\varepsilon} - C_6\sqrt{\varepsilon} = -\frac{(1+C_5)C_6}{1-C_5}\sqrt{\varepsilon} \ge -\frac{2C_6}{1-C_5}\sqrt{\varepsilon}.
\end{aligned} \tag{5.16}$$

Thus, by induction,

$$L^n \ge -\frac{2C_6}{1-C_5}\sqrt{\varepsilon} =: -C_7\sqrt{\varepsilon} > -\iota, \qquad n \ge 1. \tag{5.17}$$

Finally, we apply (5.10) again to obtain for all sufficiently small $\varepsilon > 0$

$$
\begin{aligned}
e^{\Lambda_{t-}/C}\big(e^{L^n/C}-1\big) &\geq L^{n-1}\kappa\big(-L^{n-1}\big) \\
&\qquad - 2\sqrt{\varepsilon}\int_0^{\iota/\sqrt{\varepsilon}-C_7}\Phi(1-y/\sigma)\kappa\big((y+C_7)\sqrt{\varepsilon}\big)\,\mathrm{d}y \\
&\qquad - 2\sqrt{\varepsilon}\|p(t,\cdot)\|_{L^\infty([0,\infty))}\int_{\iota/\sqrt{\varepsilon}-C_7}^{\infty}\Phi(1-y/\sigma)\,\mathrm{d}y \\
&\geq L^{n-1}\kappa\big(-L^{n-1}\big) - C_8\sqrt{\varepsilon}C_0(\varepsilon)
\end{aligned}
\tag{5.18}
$$

and, hence, $L^n \geq -C_5(-L^{n-1}) - C_9\sqrt{\varepsilon}C_0(\varepsilon)$, $n \geq 2$, for suitable constants $C_8, C_9 < \infty$. Iterating the latter inequality we end up with

$$
0 \geq L^n \geq -\frac{2C_9}{1-C_5}\sqrt{\varepsilon}C_0(\varepsilon) \tag{5.19}
$$

for all $n \in \mathbb{N}$ sufficiently large. Both parts of the lemma readily follow. □

Next, we use the sequence L^n, $n \in \mathbb{N}$ to construct an auxiliary process $\widetilde{Y}$ admitting a comparison to the physical solution $\overline{Y}$.

LEMMA 5.3. *Suppose that the law of $\overline{Y}_0$ possesses a bounded density and the assumptions of part* (a) *or part* (b) *of Lemma 5.2 hold. Then, for all $\varepsilon \in (0,\infty)$ sufficiently small, there is a continuous process $\widetilde{Y}$ satisfying*

$$
\widetilde{Y}_s = \overline{Y}_{t-} + (\alpha s + \sigma\widetilde{B}_s + \widetilde{L})\mathbf{1}_{\{\overline{\tau}\geq t\}}, \qquad s \in [0,\varepsilon], \tag{5.20}
$$

with

$$
\widetilde{L} = C\log\mathbb{P}\Big(\overline{\tau}\geq t,\ \inf_{s\in[0,\varepsilon]}\widetilde{Y}_u > 0\Big) - C\log\mathbb{P}(\overline{\tau}\geq t). \tag{5.21}
$$

Moreover,

$$
\widetilde{L} \geq -C_L\varepsilon^{(1+\gamma)/2}, \tag{5.22}
$$

for all $\varepsilon \in (0,\infty)$ sufficiently small, where C_L is as in the corresponding part of Lemma 5.2 and γ should be set to 0 *in the case of the setting of part* (a) *of Lemma 5.2.*

PROOF. By Lemma 5.2, for $\varepsilon \in (0,\infty)$ sufficiently small, the sequence L^n, $n \in \mathbb{N}$ has a limit $\widetilde{L}$. Hence, the processes $\overline{Y}^n$, $n \in \mathbb{N}$ converge uniformly on $[0,\varepsilon]$ to the process $\widetilde{Y}$ defined by (5.20) with probability one, so that $\inf_{s\in[0,\varepsilon]}\overline{Y}^n_s$, $n \in \mathbb{N}$ tend almost surely to $\inf_{s\in[0,\varepsilon]}\widetilde{Y}_s$. Clearly, the conditional distribution of the latter random variable given $\{\overline{\tau}\geq t\}$ has no atoms, and hence,

$$
\lim_{n\to\infty}\mathbb{P}\Big(\overline{\tau}\geq t,\ \inf_{s\in[0,\varepsilon]}Y^n_s > 0\Big) = \mathbb{P}\Big(\overline{\tau}\geq t,\ \inf_{s\in[0,\varepsilon]}\widetilde{Y}_s > 0\Big), \tag{5.23}
$$

which yields (5.21). The estimate (5.22) follows directly from Lemma 5.2. □

Recall that $\overline{Y}$ is a càdlàg process satisfying (2.12), with the associated Λ, $\overline{\tau}^0$ and that $t \in (0, \overline{\tau}^0)$.

LEMMA 5.4. *Suppose that the law of $\overline{Y}_0$ possesses a bounded density and that Λ is continuous at t and on $[t, t+\varepsilon]$ for some $\varepsilon \in (0,\infty)$ as in Lemma 5.3. Then, with any solution $\widetilde{Y}$ of (5.20), (5.21) for that ε, it holds*

$$\Lambda_{t+s} - \Lambda_t \geq \widetilde{L}, \qquad s \in [0,\varepsilon] \cap [0, \overline{\tau}^0 - t). \tag{5.24}$$

PROOF. Suppose that there exists an $s \in [0,\varepsilon] \cap [0, \overline{\tau}^0 - t)$ such that $\Lambda_{t+s} - \Lambda_t < \widetilde{L}$. Since $\widetilde{L} < 0$, we must have $s > 0$. Due to the continuity of Λ, we can further find an $s' \in (0,\varepsilon)$ such that $\Lambda_{t+s'} - \Lambda_t = \widetilde{L}$ and $\Lambda_{t+s''} - \Lambda_t > \widetilde{L}$ for all $s'' \in [0, s')$. Therefore, for any $s'' \in [0, s']$, the definitions of $\overline{Y}$, $\widetilde{Y}$ and the properties of Brownian motion give

$$\mathbf{1}_{\{\overline{\tau} > t+s''\}} - \mathbf{1}_{\{\overline{\tau} \geq t, \inf_{r \in [0,\varepsilon]} \widetilde{Y}_r > 0\}} \geq 0,$$

$$\mathbb{P}(\mathbf{1}_{\{\overline{\tau} > t+s''\}} - \mathbf{1}_{\{\overline{\tau} \geq t, \inf_{r \in [0,\varepsilon]} \widetilde{Y}_r > 0\}} > 0) > 0.$$

Taking $s'' = s'$, we end up with $\Lambda_{t+s'} - \Lambda_t > \widetilde{L}$, which is the desired contradiction. □

The following proposition shows that the conditions of Lemma 5.2 imply the Hölder continuity of the cumulative loss process.

PROPOSITION 5.5. *Suppose that the law of $\overline{Y}_0$ possesses a bounded density and that for some $t_0 \in (0, \overline{\tau}^0)$, Λ is continuous on $[0, t_0)$ and the assumption of part* (a) *or part* (b) *of Lemma 5.2 applies for all $t \in [0, t_0)$. Then there exist $\widetilde{C} < \infty$ and $\gamma \in [0,1]$ such that*

$$|\Lambda_t - \Lambda_s| \leq \widetilde{C} |t-s|^{(1+\gamma)/2}, \qquad s, t \in [0, t_0), \tag{5.25}$$

where γ can be chosen strictly positive in the case of part (*b*) *of Lemma 5.2.*

PROOF. Combining Lemmas 5.2, 5.3, 5.4 we conclude that for any $t \in [0, t_0)$ there is a constant $C_L < \infty$ such that

$$0 \geq \Lambda_s - \Lambda_t \geq -C_L (s-t)^{(1+\gamma)/2} \tag{5.26}$$

holds for all s in a right neighborhood of t. The proposition now follows by noting that the size of such neighborhoods can be chosen uniformly in t. □

Finally, we recall the definition of r_t^* from (2.19) and connect this quantity to the assumption in part (b) of Lemma 5.2.

PROPOSITION 5.6. *Let* $\overline{Y}$ *be a physical solution,*[4] *with the associated* $\overline{\tau}^0$, r^*, p. *Suppose that* $\overline{Y}_0$ *has a bounded density vanishing in a neighborhood of* 0. *Consider any* $t_0 \in (0, \overline{\tau}^0)$ *for which* $r^*_{t_0} = 0$. *Then, there exist* $\widehat{C}, \eta \in (0, \infty)$ *and* $\gamma \in (0, 1]$ *such that*

$$p(t, y) \le \widehat{C} y^\gamma, \qquad y \in (0, \eta), t \in (0, t_0). \tag{5.27}$$

PROOF. The assumption $r^*_{t_0} = 0$ and Definition 2.1 of a physical solution imply that Λ is continuous on $[0, t_0)$. Moreover, we note that the conditions in Lemma 5.2(a) are satisfied for all $t \in [0, t_0)$. These observations allow us to apply Proposition 5.5, which will be used further in the proof. Next, we fix $s \in (0, t_0)$ and $\chi, y_0 \in (0, \infty)$, pick a function $\phi^\chi \in C^\infty([0, \infty))$ with values in $[0, 1]$ and support contained in $(y_0, y_0 + \chi)$, and define

$$g(t, y) = \mathbb{E}\big[\phi^\chi(\overline{Y}_{t \wedge \overline{\tau}} - y)\big], \qquad (t, y) \in [0, s] \times \mathbb{R}. \tag{5.28}$$

In addition, we let

$$Z_t := -\alpha t + \sigma(\overline{B}_{s-t} - \overline{B}_s) + \Lambda_{s-t} - \Lambda_s, \qquad t \in [0, s], \tag{5.29}$$

and write $\mathbb{F}^Z = (\mathcal{F}^Z_t)_{t \in [0,s]}$ for the filtration generated by Z. We also consider the stopping time with respect to $\mathbb{F}^Z$:

$$\theta := \big(\inf\{t \in [0, s] : Z_t \notin (-y_0, \chi_1 \eta)\}\big) \wedge \big(\chi_2 \eta^2\big) \wedge s, \tag{5.30}$$

where $\chi_1, \chi_2, \eta \in (0, \infty)$ are constants to be specified below. Note that, whenever $\overline{Y}_{s-t} \ge 0$, we have

$$Z_t \ge \overline{Y}_{s-t} - \overline{Y}_s, \tag{5.31}$$

and whenever $\overline{Y}_{s-t}, \overline{Y}_s \ge 0$, we have

$$Z_t = \overline{Y}_{s-t} - \overline{Y}_s. \tag{5.32}$$

The latter always holds on $\{t \le \theta\} \cap \{\overline{Y}_s \ge y_0\}$.

We claim that

$$g(s - t \wedge \theta, Z_{t \wedge \theta}), \qquad t \in [0, s] \tag{5.33}$$

is a martingale with respect to $\mathbb{F}^Z$. To this end, we use that, for any $t \in [0, s]$, $\overline{Y}_{s-t}$ is independent of $\mathcal{F}^Z_t$ and that

$$\mathbf{1}_{\{\overline{Y}_s \ge y_0\}} \mathbf{1}_{\{\inf_{r \in [0,s-t]} \overline{Y}_r > 0\}} \mathbf{1}_{\{t \le \theta\}} = \mathbf{1}_{\{\overline{Y}_s \ge y_0\}} \mathbf{1}_{\{\inf_{r \in [0,s]} \overline{Y}_r > 0\}} \mathbf{1}_{\{t \le \theta\}}, \tag{5.34}$$

[4] Note that, herein, we require that $\overline{Y}$ is a physical solution, as opposed to merely being a solution to (2.12). This assumption is used in the first sentences in the proof.

which yields

$$
\begin{aligned}
&\mathbb{E}\big[\phi^\chi(\overline{Y}_s)\mathbf{1}_{\{\overline{Y}_s\geq y_0\}}\mathbf{1}_{\{\inf_{r\in[0,s]}\overline{Y}_r>0\}} \mid \mathcal{F}_t^Z\big]\mathbf{1}_{\{t\leq\theta\}}\\
&\quad=\mathbb{E}\big[\phi^\chi(\overline{Y}_{s-t}-Z_t)\mathbf{1}_{\{\inf_{r\in[0,s-t]}\overline{Y}_r>0\}} \mid \mathcal{F}_t^Z\big]\mathbf{1}_{\{t\leq\theta\}}\\
&\quad=\mathbb{E}\big[\phi^\chi(\overline{Y}_{s-t}-y)\mathbf{1}_{\{\inf_{r\in[0,s-t]}\overline{Y}_r>0\}}\big]\big|_{y=Z_t}\mathbf{1}_{\{t\leq\theta\}}\\
&\quad=g(s-t,Z_t)\mathbf{1}_{\{t\leq\theta\}},
\end{aligned}
\tag{5.35}
$$

where we have relied on $Z_t\geq -y_0$, $t\leq\theta$. The first expression in (5.35) is a martingale multiplied by $\mathbf{1}_{\{t\leq\theta\}}$, so that the last expression in (5.35) stopped at θ is a martingale.

Applying the optional sampling theorem, we obtain

$$
\begin{aligned}
g(s,0)&=\mathbb{E}\big[g(s-\theta,Z_\theta)\big]\\
&\leq \mathbb{P}(Z_\theta\neq -y_0,\theta<s)\sup_{(t,z)\in[0,s]\times[-y_0,\chi_1\eta]} g(t,z)\\
&\quad+\mathbb{E}\big[g(s-\theta,-y_0)\mathbf{1}_{\{Z_\theta=-y_0\}}\big]+\mathbb{E}\big[g(0,Z_s)\mathbf{1}_{\{Z_\theta\neq -y_0,\theta=s\}}\big]\\
&\leq \mathbb{P}(Z_\theta\neq -y_0,\theta<s)\sup_{(t,z)\in[0,s]\times[-y_0,\chi_1\eta]} g(t,z)\\
&\quad+\sup_{t\in[0,s]} g(t,-y_0)+\sup_{z\in[-y_0,\chi_1\eta]} g(0,z),
\end{aligned}
\tag{5.36}
$$

which implies further

$$
\begin{aligned}
&\Big(\int_{y_0}^{y_0+\chi}\phi^\chi(y)p(s,y)\,\mathrm{d}y\Big)\\
&\quad\leq \mathbb{P}(Z_\theta\neq -y_0,\theta<s)\Big(\sup_{t\in[0,s]}\operatorname*{ess\,sup}_{z\in[0,\chi_1\eta+y_0+\chi]} p(t,z)\Big)\|\phi^\chi\|_{L^1([0,\infty))}\\
&\qquad+\Big(\sup_{t\in[0,s]}\operatorname*{ess\,sup}_{z\in[0,\chi]} p(t,z)\Big)\|\phi^\chi\|_{L^1([0,\infty))}\\
&\qquad+\Big(\operatorname*{ess\,sup}_{z\in[0,\chi_1\eta+y_0+\chi]} p(0,z)\Big)\|\phi^\chi\|_{L^1([0,\infty))}.
\end{aligned}
\tag{5.37}
$$

Letting $y_0\in(0,\eta)$ with $\eta\in(0,\infty)$ sufficiently small we get in the case $s\geq \chi_2\eta^2$

$$
\begin{aligned}
&\mathbb{P}(Z_\theta\neq -y_0,\theta<s)\\
&\quad\leq \mathbb{P}\Big(\inf_{t\in[0,\chi_2\eta^2]}\big(\Lambda_{s-t}-\Lambda_s-\alpha t+\sigma(\overline{B}_{s-t}-\overline{B}_s)\big)>-\eta\Big)\\
&\qquad+\mathbb{P}\Big(\sup_{t\in[0,\chi_2\eta^2]}\big(\Lambda_{s-t}-\Lambda_s-\alpha t+\sigma(\overline{B}_{s-t}-\overline{B}_s)\big)>\chi_1\eta\Big)\\
&\quad\leq\frac{1}{2},
\end{aligned}
\tag{5.38}
$$

once we make $\chi_1, \chi_2 \in (0, \infty)$ sufficiently large, uniformly in η and $s \in (0, t_0)$. Hereby, the second inequality in (5.38) relies on the 1/2-Hölder continuity of Λ, which is in turn due to Proposition 5.5. In the case $s < \chi_2 \eta^2$, we obtain similarly

$$\begin{aligned} &\mathbb{P}(Z_\theta \neq -y_0, \theta < s) \\ &\quad \leq \mathbb{P}\Big(\sup_{t \in [0, \chi_2 \eta^2]} \big(\Lambda_{s-t} - \Lambda_s - \alpha t + \sigma(\overline{B}_{s-t} - \overline{B}_s)\big) > \chi_1 \eta \Big) \leq \frac{1}{2}. \end{aligned} \tag{5.39}$$

Combining (5.37), (5.38), (5.39) and $y_0 \in (0, \eta)$ we end up with

$$\begin{aligned} &\|\phi^\chi\|^{-1}_{L^1([0,\infty))} \int_{y_0}^{y_0+\chi} \phi^\chi(y) p(s, y)\, \mathrm{d}y \\ &\quad \leq \frac{1}{2}\Big(\sup_{t \in [0,s]} \operatorname*{ess\,sup}_{z \in [0, (\chi_1+1)\eta + \chi]} p(t, z)\Big) \\ &\qquad + \Big(\sup_{t \in [0,s]} \operatorname*{ess\,sup}_{z \in [0, \chi]} p(t, z)\Big) + \Big(\operatorname*{ess\,sup}_{z \in [0, (\chi_1+1)\eta + \chi]} p(0, z)\Big). \end{aligned} \tag{5.40}$$

To complete the proof, we consider $\chi \in (0, \eta)$ in (5.40), and choose a sequence of test functions, ϕ^χ, that approximates the indicator function of the set on which $p(s, \cdot)$ is close to its maximum, to obtain

$$\begin{aligned} \operatorname*{ess\,sup}_{z \in [0,\eta]} p(s, z) &\leq \frac{1}{2}\Big(\sup_{t \in [0,s]} \operatorname*{ess\,sup}_{z \in [0, (\chi_1+2)\eta]} p(t, z)\Big) \\ &\quad + \Big(\sup_{t \in [0,s]} \operatorname*{ess\,sup}_{z \in [0, \chi]} p(t, z)\Big) + \Big(\operatorname*{ess\,sup}_{z \in [0, (\chi_1+2)\eta]} p(0, z)\Big). \end{aligned} \tag{5.41}$$

Next, we use $r_s^* = 0$ and take a limit as $\chi \to 0$ in the above, to conclude

$$\begin{aligned} \operatorname*{ess\,sup}_{z \in [0,\eta]} p(s, z) &\leq \frac{1}{2}\Big(\sup_{t \in [0,s]} \operatorname*{ess\,sup}_{z \in [0, (\chi_1+2)\eta]} p(t, z)\Big) \\ &\quad + \Big(\operatorname*{ess\,sup}_{z \in [0, (\chi_1+2)\eta]} p(0, z)\Big). \end{aligned} \tag{5.42}$$

Replacing s by $t \in [0, s]$ and taking the supremum over $t \in [0, s]$ we find therefore

$$\begin{aligned} \sup_{t \in [0,s]} \operatorname*{ess\,sup}_{z \in [0,\eta]} p(t, z) &\leq \frac{1}{2}\Big(\sup_{t \in [0,s]} \operatorname*{ess\,sup}_{z \in [0, (\chi_1+2)\eta]} p(t, z)\Big) \\ &\quad + \Big(\operatorname*{ess\,sup}_{z \in [0, (\chi_1+2)\eta]} p(0, z)\Big). \end{aligned} \tag{5.43}$$

An iteration of this inequality yields

$$\begin{aligned} \sup_{t \in [0,s]} \operatorname*{ess\,sup}_{z \in [0,\eta]} p(t, z) &\leq \frac{1}{2^n}\Big(\sup_{t \in [0,s]} \operatorname*{ess\,sup}_{z \in [0, (\chi_1+2)^n\eta]} p(t, z)\Big) \\ &\quad + 2\Big(\operatorname*{ess\,sup}_{z \in [0, (\chi_1+2)^n\eta]} p(0, z)\Big) \end{aligned} \tag{5.44}$$

for all $n \in \mathbb{N}$. It remains to choose $\widetilde{\eta} \in (0, \infty)$ such that $p(0, \cdot)$ vanishes on $[0, \widetilde{\eta}]$, let $\eta \in (0, \widetilde{\eta}/(\chi_1 + 2))$ and select n as the integer part of $\log(\widetilde{\eta}/\eta)/\log(\chi_1 + 2)$ to deduce

$$\begin{aligned}
&\sup_{t\in[0,s]} \operatorname*{ess\,sup}_{z\in[0,\eta]} p(t, z) \\
&\quad \le 2^{-\log(\widetilde{\eta}/\eta)/\log(\chi_1+2)+1}\Big(\sup_{t\in[0,s]} \operatorname*{ess\,sup}_{z\in[0,\widetilde{\eta}]} p(t, z)\Big) \\
&\quad = 2^{-\log\widetilde{\eta}/\log(\chi_1+2)+1}\Big(\sup_{t\in[0,s]} \operatorname*{ess\,sup}_{z\in[0,\widetilde{\eta}]} p(t, z)\Big)\eta^{\log 2/\log\lambda}.
\end{aligned} \tag{5.45}$$

The proposition follows by noting that the factor in front of $\eta^{\log 2/\log\lambda}$ can be bounded by a constant $\widehat{C} \in (0, \infty)$ independent of $s \in (0, t_0)$. □

Combining Proposition 5.6 with Proposition 5.5, we get Theorem 2.6.

6. Jumps of the cumulative loss process. In this section, we prove Proposition 2.5. The proof is an adaptation of [10], proof of Proposition 2.7. We fix $t \in [0, \overline{\tau}^0)$ and $\eta \ge 0$ satisfying the condition in the proposition, but suppose $\tilde{\eta} := \eta - (\Lambda_{t-} - \Lambda_t) > 0$. Then we obtain by means of the elementary estimate

$$\begin{aligned}
\frac{\Lambda_t - \Lambda_{t-}}{C} &= \log \mathbb{P}(\overline{\tau} > t) - \log \mathbb{P}(\overline{\tau} \ge t) \\
&\le \frac{\mathbb{P}(\overline{\tau} > t) - \mathbb{P}(\overline{\tau} \ge t)}{\mathbb{P}(\overline{\tau} \ge t)} = -\frac{\mathbb{P}(\overline{\tau} = t)}{\mathbb{P}(\overline{\tau} \ge t)}
\end{aligned} \tag{6.1}$$

that, for all $y \in [0, \tilde{\eta}]$,

$$\begin{aligned}
&\frac{\mathbb{P}(\overline{\tau} > t, \overline{Y}_t \in (0, y))}{\mathbb{P}(\overline{\tau} > t)} \\
&\quad \ge \frac{\mathbb{P}(\overline{\tau} > t, \overline{Y}_{t-} \in (0, y + \Lambda_{t-} - \Lambda_t))}{\mathbb{P}(\overline{\tau} \ge t)} \\
&\quad = \frac{\mathbb{P}(\overline{\tau} \ge t, \overline{Y}_{t-} \in (0, y + \Lambda_{t-} - \Lambda_t))}{\mathbb{P}(\overline{\tau} \ge t)} \\
&\qquad - \frac{\mathbb{P}(\overline{\tau} = t, \overline{Y}_{t-} \in (0, y + \Lambda_{t-} - \Lambda_t))}{\mathbb{P}(\overline{\tau} \ge t)} \\
&\quad \ge \frac{y + \Lambda_{t-} - \Lambda_t}{C} + \frac{\Lambda_t - \Lambda_{t-}}{C} = \frac{y}{C}.
\end{aligned} \tag{6.2}$$

Next, we combine the conclusion of (6.2) and Fubini's theorem to get, for $s > t$,

$$\begin{aligned}
\Lambda_s - \Lambda_t &= C \log \mathbb{P}(\overline{\tau} > s) - C \log \mathbb{P}(\overline{\tau} > t) \le C \frac{\mathbb{P}(\overline{\tau} > s) - \mathbb{P}(\overline{\tau} > t)}{\mathbb{P}(\overline{\tau} > t)} \\
&\le - \frac{C}{\mathbb{P}(\overline{\tau} > t)} \\
&\quad \times \mathbb{P}\Big(\overline{Y}_t > 0, \overline{Y}_t - \sup_{r \in [t,s]} \big(\sigma(\overline{B}_t - \overline{B}_r) + \Lambda_t - \Lambda_r\big) \le -|\alpha|(s-t)\Big) \qquad (6.3) \\
&\le - \int_0^{\tilde{\eta}} \mathbb{P}\Big(y - \sup_{r \in [t,s]} \big(\sigma(\overline{B}_t - \overline{B}_r) + \Lambda_t - \Lambda_r\big) \le -|\alpha|(s-t)\Big) \, \mathrm{d}y \\
&\le -\mathbb{E}\Big[\Big(\sup_{r \in [t,s]} \big(\sigma(\overline{B}_t - \overline{B}_r) + \Lambda_t - \Lambda_r\big) - |\alpha|(s-t)\Big) \wedge \tilde{\eta}\Big].
\end{aligned}$$

Moreover, by the right continuity of Λ we have, for $s - t > 0$ small enough, that $\sup_{r \in [t,s]}(\Lambda_t - \Lambda_r) - |\alpha|(s-t) \le \frac{\tilde{\eta}}{2}$. For such $s > t$, one can use the explicit distribution of the Brownian supremum $\sup_{r \in [t,s]}(\sigma(\overline{B}_t - \overline{B}_r))$ to deduce that the last expression of (6.3) does not exceed

$$\begin{aligned}
&- \mathbb{E}\Big[\sup_{r \in [t,s]} \big(\sigma(\overline{B}_t - \overline{B}_r) + \Lambda_t - \Lambda_r\big) - |\alpha|(s-t)\Big] + k(s-t) \\
&\quad = -\mathbb{E}\Big[\sup_{r \in [t,s]} \big(\sigma(\overline{B}_t - \overline{B}_r) + \Lambda_t - \Lambda_r\big)\Big] + \big(|\alpha| + k\big)(s-t), \qquad (6.4)
\end{aligned}$$

where $k \in (0, \infty)$ is a constant depending only on σ and $\tilde{\eta}$.

Finally, assuming an a priori estimate of the form

$$\Lambda_s - \Lambda_t \le -c\sqrt{s-t}, \qquad s \in [t, t+h] \qquad (6.5)$$

for some $h \in (0, 1)$ small enough in the sense above and $c \ge 0$ [note that (6.5) holds at least with $c = 0$], (6.3), (6.4) and the Brownian scaling property lead to the improvement

$$\begin{aligned}
&\Lambda_s - \Lambda_t \le -\Big(\mathbb{E}\Big[\sup_{r \in [0,1]} (\sigma \overline{B}_r + c\sqrt{r})\Big] - \big(|\alpha| + k\big)\sqrt{h}\Big)\sqrt{s-t}, \qquad (6.6) \\
&s \in [t, t+h].
\end{aligned}$$

Repeating the arguments following [10], page 2458, second to last display, word-by-word, we conclude that the possibility of such an improvement contradicts the right continuity of Λ. This completes the proof.

7. Convergence of the finite-particle systems. This last section is devoted to the proof of Theorem 2.4, and we work under the assumptions of that theorem throughout the rest of the paper. Recall the equations (2.7), (2.8), (2.10), satisfied by the finite-particle system $(\widetilde{Y}^1, \widetilde{Y}^2, \dots, \widetilde{Y}^N)$ and the definition (2.5) of the cascade sizes, needed to uniquely determine the finite-particle system. We note that the equations (2.7), (2.8) guarantee that the processes $\widetilde{Y}^1, \widetilde{Y}^2, \dots, \widetilde{Y}^N$ never jump across -1. Our proof follows the line of reasoning in [10], and we often refer to the results established therein.

Our first aim is to establish the tightness of the sequence of empirical measures $\widetilde{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\widetilde{Y}^i}$, $N \in \mathbb{N}$. To this end, we start with the following lemma, which is the analogue of [10], Lemma 5.2.

LEMMA 7.1. *For any* $\chi > 0$, *there exists some* $\upsilon = \upsilon(\chi) \in (0, 1)$ (*independent of* N) *such that*

$$\mathbb{P}\big(\exists t \in [0, \upsilon] : 1 - S_t/N \geq \upsilon^{-1} t^{1/4}\big) \leq \chi. \tag{7.1}$$

PROOF. We introduce the auxiliary particle system $(\widehat{Y}^1, \widehat{Y}^2, \dots, \widehat{Y}^N)$ defined analogously to $(\widetilde{Y}^1, \widetilde{Y}^2, \dots, \widetilde{Y}^N)$, but with the equation (2.10) replaced by

$$\widehat{Y}^i_t = \widetilde{Y}^i_0 + \alpha t + \sigma B^i_t - \frac{1+C}{2}(1 - \widehat{S}_t/N), \qquad t \in [0, T]. \tag{7.2}$$

More specifically, we substitute (2.5) by

$$\widehat{K}^d_t = \left(\inf\left\{k = 1, 2, \dots, \widehat{S}_{t-} : \widehat{Y}^{(k)}_{t-} - \frac{1+C}{2}\frac{k-1}{N} > 0\right\} - 1\right) \wedge \widehat{S}_{t-}, \tag{7.3}$$

and rewrite (2.7) for $(\widehat{Y}^1, \widehat{Y}^2, \dots, \widehat{Y}^N)$ accordingly. Fix an arbitrary $\chi_1 \in (0, 1)$. By repeating the proof of [10], Lemma 5.2, we conclude that for any $\chi > 0$ there exists some $\upsilon = \upsilon(\chi) \in (0, 1)$ (independent of N) such that

$$\mathbb{P}\big(\exists t \in \big[0, \chi_1^4 \upsilon^4\big] : 1 - \widehat{S}_t/N \geq \upsilon^{-1} t^{1/4}\big) \leq \chi. \tag{7.4}$$

In fact, in [10] each particle shifts the locations of the other particles every time it hits a new integer, which makes $\widehat{S}$ even smaller. This observation allows to simplify some parts of the proof of [10], Lemma 5.2, when deriving the estimate (7.4).

If $\chi_1 \in (0, 1)$ is chosen for the following to hold:

$$\begin{aligned} &-\frac{1+C}{2} y_1 \leq C \log(1 - y_1), \\ &-\frac{1+C}{2} y_1 \leq C \log\left(1 - \frac{y_1}{1 - y_2}\right), \qquad y_1, y_2 \in [0, \chi_1), \end{aligned} \tag{7.5}$$

then on the complement of the event in (7.4) we have for all $t \in [0, \chi_1^4 \upsilon^4]$ and all $k = 1, 2, \dots, \widehat{K}^d_t + 1$:

$$\begin{aligned} &-\frac{1+C}{2}(1 - \widehat{S}_t/N) \leq C \log(\widehat{S}_t/N), \\ &-\frac{1+C}{2}\frac{k-1}{N} \leq C \log\left(1 - \frac{k-1}{\widehat{S}_{t-}}\right). \end{aligned} \tag{7.6}$$

These inequalities and induction along the hitting times of zero for the auxiliary particles yield, on the complement of the event in (7.4), for all $t \in [0, \chi_1^4 \upsilon^4]$ and all $i = 1, 2, \dots, N$,

$$\widetilde{Y}^i_{t\wedge\tau^i} \geq \widehat{Y}^i_{t\wedge\widehat{\tau}^i}, \qquad K^d_t \leq \widehat{K}^d_t, \qquad S_t \geq \widehat{S}_t. \tag{7.7}$$

The lemma follows upon decreasing the value of $\upsilon \in (0, 1)$ if necessary. $\square$

The next lemma is needed to prove the upper bound on jump sizes in the definition of a physical solution. It is the analogue of [10], Lemma 5.3, and its proof is a simplified version of the proof of the latter. The present setting allows for a simplification, because each particle can only contribute to the cumulative loss process once, that is, it can only "spike" once, in the terminology of [10].

LEMMA 7.2. *There exist some $C_0 < \infty$, $\varepsilon > 0$ such that for all $r \in (0, 1)$, $t \in [0, T)$ and $s \in (0, (T - t) \wedge \varepsilon)$ one can find an $N_0 = N_0(r, s) \in \mathbb{N}$ with*

$$\begin{aligned}&\mathbb{P}\biggl(\frac{S_{t-}}{N} \geq r, \forall \iota \leq \biggl(\frac{S_{t-} - S_{t+s}}{S_{t-}} - 2s^{1/4}\biggr)^+ :\\ &\qquad \frac{1}{S_{t-}}\bigl|\bigl\{\tau^i \geq t, \widetilde{Y}^i_{t-} + C\log(1-\iota) \leq 2s^{1/4}\bigr\}\bigr| \geq \frac{\iota}{1+s^{1/4}}\biggr)\\ &\qquad \geq \mathbb{P}\biggl(\frac{S_{t-}}{N} \geq r\biggr) - C_0 s\end{aligned} \tag{7.8}$$

for all $N \geq N_0$, where $|\cdot|$ denotes the number of elements of a set.

PROOF. For an $\varepsilon > 0$, we consider arbitrary $r \in (0, 1)$, $t \in [0, T)$, $s \in (0, (T - t) \wedge \varepsilon)$ and work throughout on the event $\{S_{t-}/N \geq r\}$ (in particular, all events are intersected with $\{S_{t-}/N \geq r\}$, and all complements are taken with respect to $\{S_{t-}/N \geq r\}$). Then, for any $k \in \{0, 1, \dots, S_{t-} - S_{t+s}\}$ we have $\sum_{i=1}^N \mathbf{1}_{A^{i,1}(k)} \geq k$, where

$$\begin{aligned}A^{i,1}(k) := \biggl\{&\tau^i \geq t,\\ &\widetilde{Y}^i_{t-} + C\log\biggl(1 - \frac{k}{S_{t-}}\biggr) - \alpha^- s + \sigma \inf_{s'\in[0,s]}\bigl(B^i_{t+s'} - B^i_t\bigr) \leq 0\biggr\}.\end{aligned} \tag{7.9}$$

In addition, we define the events

$$\begin{aligned}&A = \biggl\{\frac{1}{S_{t-}}\sum_{i=1}^N \mathbf{1}_{A^{i,2}} \leq s\biggr\},\\ &A^{i,2} = \Bigl\{\tau^i \geq t, -\alpha^- s + \sigma \inf_{s'\in[0,s]}\bigl(B^i_{t+s'} - B^i_t\bigr) < -s^{1/4}\Bigr\},\end{aligned} \tag{7.10}$$

let $\iota \in [0, (\frac{S_{t-}-S_{t+s}}{S_{t-}}(1+2s^{1/4}) - 2s^{1/4})^+]$, and choose k as the integer part of $\frac{\iota+s^{1/4}}{1+s^{1/4}} S_{t-}$, so that $0 \le k \le S_{t-} - S_{t+s}$. Moreover, on $A^{i,1}(k) \cap (A^{i,2})^c$:

$$\begin{aligned} \widetilde{Y}^i_{t-} + C\log(1-\iota) &\le \widetilde{Y}^i_{t-} + C\log\biggl(1 - \frac{k}{S_{t-}}(1+s^{1/4}) + s^{1/4}\biggr) \\ &\le \widetilde{Y}^i_{t-} + C\log\biggl(1 - \frac{k}{S_{t-}}\biggr) + s^{1/4} \le 2s^{1/4}. \end{aligned} \tag{7.11}$$

Consequently, on A,

$$\begin{aligned} &\frac{1}{S_{t-}} \sum_{i=1}^N \mathbf{1}_{\{\tau^i \ge t, \widetilde{Y}^i_{t-} + C\log(1-\iota) \le 2s^{1/4}\}} \\ &\quad \ge \frac{1}{S_{t-}} \sum_{i=1}^N \mathbf{1}_{A^{i,1}(k)} - \frac{1}{S_{t-}} \sum_{i=1}^N \mathbf{1}_{A^{i,2}} \\ &\quad \ge \frac{k}{S_{t-}} - s \ge \frac{\iota + s^{1/4}}{1+s^{1/4}} - \frac{1}{Nr} - s \ge \frac{\iota}{1+s^{1/4}} \end{aligned} \tag{7.12}$$

for $N \ge r^{-1}s^{-1}$, provided ε is smaller than an appropriate uniform constant. Finally, we obtain $\mathbb{P}(A^c) \le C_0 s$ for some $C_0 < \infty$ by conditioning on the information up to time t and using Markov's inequality in conjunction with a standard estimate for Brownian motion. □

Next, we recall the space $\mathcal{D}([0, T+1])$ of càdlàg functions on $[0, T+1]$ that are continuous at $T+1$, endowed with the Skorokhod M1 topology (see, e.g., [10, 19, 27, 31] for a detailed discussion of the M1 topology). We write $\mathcal{P}(\mathcal{D}([0, T+1]))$ for the space of probability measures on $\mathcal{D}([0, T+1])$, endowed with the topology of weak convergence.

PROPOSITION 7.3. *The sequence* $\widetilde{\mu}^N$, $N \in \mathbb{N}$ *is tight on* $\mathcal{P}(\mathcal{D}([0, T+1]))$.

PROOF. We first claim that the sequence of $\widetilde{Y}^1$, indexed by $N \in \mathbb{N}$, is tight on $\mathcal{D}([0, T+1])$. To this end, we decompose $\widetilde{Y}^1$ into the sum of its continuous and jump parts. Notice that, between the jump times, $\widetilde{Y}^1$ is given by

$$\widetilde{Y}^1_t = \frac{\widetilde{Y}^1_0 + \alpha t + \sigma B^1_t + \mathbf{1}_{\{\widetilde{Y}^1_t \ge -1\}} C\log(1 - S_t/N)}{1 - \mathbf{1}_{\{-1 \le \widetilde{Y}^1_t \le 0\}} C\log(1 - S_t/N)}. \tag{7.13}$$

Hence, the modulus of continuity of the continuous part of $\widetilde{Y}^1$ is bounded above by the modulus of continuity of a Brownian motion with drift, started from $\widetilde{Y}^1_0$. The same is true for the supremum of the absolute value of the continuous part of $\widetilde{Y}^1$. Moreover, the jump part of $\widetilde{Y}^1$ is nonincreasing, and the supremum of its absolute value is bounded above by the supremum of the continuous part plus 1, since $\widetilde{Y}^1$ does not jump across -1. The tightness of the sequence of $\widetilde{Y}^1$, indexed by $N \in \mathbb{N}$, can be now deduced as in the proof of [10], Lemma 5.4, by invoking our Lemma 7.1. To obtain the proposition from this, it remains to use a standard argument from the theory of propagation of chaos; cf. [29], Proposition 2.2. □

We proceed with the proof of Theorem 2.4. Let us denote by ω the canonical process in $\mathcal{D}([0, T+1])$ and introduce

$$m_t := \mathbf{1}_{\{\inf_{s\in[0,t]} \omega_s \le 0\}}, \qquad t \in [0, T+1]. \tag{7.14}$$

By Proposition 7.3, the sequence $\widetilde{\mu}^N$, $N \in \mathbb{N}$ is tight, and we write Π_∞ for the law of an arbitrary limit point. In the remainder of the proof, we assume that all limits are taken along the corresponding convergent subsequence of $\widetilde{\mu}^N$, $N \in \mathbb{N}$. By repeating the first part of the proof of [10], Theorem 4.4, we find a countable set $J \subset [0, T+1]$ such that for any $t \in J^c$ it holds $\langle \mu, m_{t-}\rangle = \langle \mu, m_t\rangle$ and $\mu(\omega_{t-} = \omega_t) = 1$ for Π_∞-almost every $\mu \in \mathcal{P}(\mathcal{D}([0, T+1]))$. Hereby, $\langle \cdot, \cdot\rangle$ stands for the integral of a function of the canonical process with respect to a given measure. The following lemma is the analogue of [10], Lemma 5.9, and its proof is postponed to Section 7.1.

LEMMA 7.4. *For Π_∞-almost every $\mu \in \mathcal{P}(\mathcal{D}([0, T+1]))$ and any μ_n, $n \in \mathbb{N}$ converging weakly to μ, we have*

$$\lim_{n\to\infty} \langle \mu_n, m_t\rangle = \langle \mu, m_t\rangle, \qquad t \in J^c. \tag{7.15}$$

Next, we fix a rational $T' \in J^c \cap [0, T]$, an integer $\ell \ge 1$, elements $0 = s_0 < s_1 < \cdots < s_\ell < T'$ of J^c, and uniformly continuous bounded functions $g_1, g_2, \dots, g_\ell : \mathbb{R} \to \mathbb{R}$. In addition, for any uniformly continuous bounded function $G : \mathbb{R} \to \mathbb{R}$ we let

$$Q^N := \mathbb{E}\left[\widetilde{G}\left(\left\langle \widetilde{\mu}^N, \prod_{j=1}^{\ell} g_j\big(\omega_{s_j} - \omega_0 - L_{s_j}(\omega, \widetilde{\mu}^N)\big)\right\rangle\right)\big(1 - \langle \widetilde{\mu}^N, m_{T'}\rangle\big)\right], \tag{7.16}$$

where

$$L_t(\omega, \mu) := \big((\omega_t + 1)^+ \wedge 1\big) C \log\big(1 - \langle \mu, m_t\rangle\big), \tag{7.17}$$

$$\widetilde{G}(y) := \left(G(y) - G\left(\mathbb{E}\left[\prod_{j=1}^{\ell} (\alpha s_j + \sigma \overline{B}_{s_j})\right]\right)\right)^2, \tag{7.18}$$

and we use the convention that the expression inside the expectation in the definition of Q^N is 0 whenever $\langle \widetilde{\mu}^N, m_{T'}\rangle = 1$. Note

$$Q^N = \mathbb{E}\left[\widetilde{G}\left(\frac{1}{N}\sum_{i=1}^{N} \prod_{j=1}^{\ell} (\alpha s_j + \sigma B^i_{s_j})\right)\big(1 - \langle \widetilde{\mu}^N, m_{T'}\rangle\big)\right], \tag{7.19}$$

so that $\lim_{N\to\infty} Q^N = 0$ by the strong law of large numbers. The last ingredient in the proof of Theorem 2.4 is the following lemma, which is the analogue of [10], Lemma 5.10.

LEMMA 7.5. *The functional*

$$\mathcal{P}(\mathcal{D}([0, T+1])) \ni \mu \mapsto \widetilde{G}\left(\left\langle \mu, \prod_{j=1}^{\ell} g_j(\omega_{s_j} - \omega_0 - L_{s_j}(\omega, \mu))\right\rangle\right)(1 - \langle \mu, m_{T'}\rangle) \tag{7.20}$$

is continuous at Π_∞*-almost every* μ.

PROOF. Lemma 7.4 implies that the mappings $\mu \mapsto \langle \mu, m_{s_j}\rangle$, $j = 1, 2, \dots, \ell$ and $\mu \mapsto \langle \mu, m_{T'}\rangle$ are continuous at Π_∞-almost every μ. Pick such a μ and μ_n, $n \in \mathbb{N}$ converging to μ. If $\langle \mu, m_{s_\ell}\rangle = 1$, then $\langle \mu, m_{T'}\rangle = 1$, and the value of the functional at μ is 0. At the same time, $\lim_{n\to\infty}\langle \mu_n, m_{T'}\rangle = 1$, and the boundedness of $\widetilde{G}$ implies that the values of the functional at μ_n, $n \in \mathbb{N}$ converge to 0, yielding the desired continuity.

If $\langle \mu, m_{s_\ell}\rangle < 1$, then for all $n \in \mathbb{N}$ sufficiently large $\langle \mu_n, m_{s_\ell}\rangle$ is bounded away from 1 by a constant (recall the continuity of $\langle \cdot, m_{s_\ell}\rangle$ at μ). In particular, no discontinuity can arise from the logarithm in the definition of $L_t(\omega, \mu)$. Consequently, we can repeat the proof of [10], Lemma 5.10, to conclude that the values of the functional at μ_n, $n \in \mathbb{N}$ converge to its value at μ. $\square$

Lemma 7.5 gives

$$\begin{aligned} &\int_{\mathcal{P}(\mathcal{D}([0,T+1]))} \widetilde{G}\left(\left\langle \mu, \prod_{j=1}^{\ell} g_j(\omega_{s_j} - \omega_0 - L_{s_j}(\omega, \mu))\right\rangle\right) \\ &\qquad \times (1 - \langle \mu, m_{T'}\rangle)\Pi_\infty(\mathrm{d}\mu) = \lim_{N\to\infty} Q^N = 0, \end{aligned} \tag{7.21}$$

and hence,

$$G\left(\left\langle \mu, \prod_{j=1}^{\ell} g_j(\omega_{s_j} - \omega_0 - L_{s_j}(\omega, \mu))\right\rangle\right) = G\left(\mathbb{E}\left[\prod_{j=1}^{\ell} (\alpha s_j + \sigma \overline{B}_{s_j})\right]\right) \tag{7.22}$$

for Π_∞-almost every μ with $\langle \mu, m_{T'}\rangle < 1$. The standard arguments in the proof of [10], Lemma 5.4, allow to deduce from (7.22) that the process $\omega_t - \omega_0 - L_t(\omega, \mu)$, $t \in [0, T']$ is a Brownian motion with drift and $\omega_0 \stackrel{d}{=} \widetilde{Y}_0^1$, under Π_∞-almost every μ with $\langle \mu, m_{T'}\rangle < 1$. Since the set of possible T' is countable and dense in $[0, T]$, we conclude that the canonical process satisfies the condition (2.12) in Definition 2.1 of a physical solution, under Π_∞-almost every μ.

To see condition (2.15) in Definition 2.1, we cast the estimate (7.8) of Lemma 7.2 as

$$\begin{aligned}
&\mathbb{P}\biggl(\langle\widetilde{\mu}^N, m_{t-}\rangle \le 1-r, \forall\iota \le \biggl(\frac{\langle\widetilde{\mu}^N, m_{t+s}\rangle - \langle\widetilde{\mu}^N, m_{t-}\rangle}{1-\langle\widetilde{\mu}^N, m_{t-}\rangle} - 2s^{1/4}\biggr)^+ : \\
&\qquad \frac{\widetilde{\mu}^N(m_{t-}=0, \omega_{t-} + C\log(1-\iota) \le 2s^{1/4})}{1-\langle\widetilde{\mu}^N, m_{t-}\rangle} \ge \frac{\iota}{1+s^{1/4}}\biggr) \qquad (7.23)\\
&\qquad \ge \mathbb{P}(\langle\widetilde{\mu}^N, m_{t-}\rangle \le 1-r) - C_0 s.
\end{aligned}$$

By following the last part of the proof of [10], Theorem 4.4, we obtain from this for all $r \in (0,1)$ sufficiently small and all $t \in J \cap [0,T]$:

$$\begin{aligned}
&\Pi_\infty\biggl(\langle\mu, m_t\rangle \le 1-r/2, \forall\iota < \frac{\langle\mu, m_t - m_{t-}\rangle}{1-\langle\mu, m_{t-}\rangle} : \\
&\qquad \frac{\mu(m_{t-}=0, \omega_{t-} + C\log(1-\iota) \le 0)}{1-\langle\mu, m_{t-}\rangle} \ge \iota\biggr) \qquad (7.24)\\
&\qquad \ge \Pi_\infty(\langle\mu, m_t\rangle \le 1-r),
\end{aligned}$$

which in the limit $r \downarrow 0$ yields

$$\begin{aligned}
&\Pi_\infty\biggl(\langle\mu, m_t\rangle < 1, \forall\iota < \frac{\langle\mu, m_t - m_{t-}\rangle}{1-\langle\mu, m_{t-}\rangle} : \\
&\qquad \frac{\mu(m_{t-}=0, \omega_{t-} + C\log(1-\iota) \le 0)}{1-\langle\mu, m_{t-}\rangle} \ge \iota\biggr) \qquad (7.25)\\
&\qquad \ge \Pi_\infty(\langle\mu, m_t\rangle < 1).
\end{aligned}$$

Consequently,

$$\forall\iota < \frac{\langle\mu, m_t - m_{t-}\rangle}{1-\langle\mu, m_{t-}\rangle} : \frac{\mu(m_{t-}=0, \omega_{t-} + C\log(1-\iota) \le 0)}{1-\langle\mu, m_{t-}\rangle} \ge \iota \qquad (7.26)$$

for all $t \in J \cap [0,T]$ and Π_∞-almost every μ with

$$\overline{\tau}^0(\mu) := \inf\{s \in [0,T] : \langle\mu, m_s\rangle = 1\} > t. \qquad (7.27)$$

Since the set J is countable, the latter statement holds for Π_∞-almost every μ and all $t \in J \cap [0,T] \cap [0, \overline{\tau}^0(\mu))$. This implies, for all such μ and t,

$$\frac{\langle\mu, m_t - m_{t-}\rangle}{1-\langle\mu, m_{t-}\rangle} \le \frac{\mu(m_{t-}=0, \omega_{t-} \le \bar{y})}{1-\langle\mu, m_{t-}\rangle}, \qquad (7.28)$$

with

$$\bar{y} := \inf\biggl\{y > 0 : y + C\log\biggl(1 - \frac{\mu(m_{t-}=0, \omega_{t-} \le y)}{1-\langle\mu, m_{t-}\rangle}\biggr) > 0\biggr\}, \qquad (7.29)$$

which yields the desired upper bound on the jumps of the canonical process. Proposition 2.5 yields the lower bound, and we conclude that the canonical process satisfies the condition (2.15) in Definition 2.1 of a physical solution, under Π_∞-almost every μ. The proof of Theorem 2.4 is complete.

7.1. *Proof of Lemma* 7.4. Let us fix an arbitrary $T_1 \in (0, \infty)$ and, as before, denote by $\mathcal{D}([0, T_1])$ the space of real-valued càdlàg functions on $[0, T_1]$ that are continuous at T_1, endowed with the Skorokhod M1 topology. The following two lemmas are the analogues of [10], Lemma 5.6, and [10], Proposition 5.8, respectively, and we omit their proofs, since they constitute very minor modifications of the proofs in [10].

LEMMA 7.6. *Consider any $\omega \in \mathcal{D}([0, T_1])$ satisfying the crossing property*

$$\forall s > 0: \qquad \tau < T_1 \Longrightarrow \inf_{t \in [\tau, (\tau+s) \wedge T_1]} (\omega_t - \omega_\tau) < 0, \tag{7.30}$$

where $\tau := (\inf\{t \in [0, T_1] : \omega_t \le 0\}) \wedge T_1$. Then, for any ω^n, $n \in \mathbb{N}$ converging to ω in $\mathcal{D}([0, T_1])$ there exists a countable set $J \subset [0, T_1]$ such that

$$\lim_{n \to \infty} m_t^n := \lim_{n \to \infty} \mathbf{1}_{\{\inf_{s \in [0,t]} \omega_s^n \le 0\}} = \mathbf{1}_{\{\inf_{s \in [0,t]} \omega_s \le 0\}} =: m_t,$$
$$t \in [0, T_1] \setminus J \tag{7.31}$$

and all points t of continuity of ω satisfying $\inf_{s \in [0,t]} \omega_s \ne 0$ are contained in $[0, T_1] \setminus J$.

LEMMA 7.7. *Consider any $\mu \in \mathcal{P}(\mathcal{D}([0, T_1]))$ satisfying*

$$\forall s > 0: \qquad \mu\Big(\tau < T_1, \inf_{t \in [\tau, (\tau+s) \wedge T_1]} (\omega_t - \omega_\tau) = 0\Big) = 0, \tag{7.32}$$

where $\tau := (\inf\{t \in [0, T_1] : \omega_t \le 0\}) \wedge T_1$. Then, for any μ_n, $n \in \mathbb{N}$ converging weakly to μ we have

$$\lim_{n \to \infty} \langle \mu_n, m_t \rangle = \langle \mu, m_t \rangle \tag{7.33}$$

for all points t of continuity of the mapping $t \mapsto \langle \mu, m_t \rangle$.

The next lemma shows that the canonical process satisfies the crossing property under Π_∞-almost every μ. It is the analogue of [10], Lemma 5.9.

LEMMA 7.8. *For Π_∞-almost every μ, it holds*

$$\forall s > 0: \qquad \mu\Big(\tau < T + 1, \inf_{t \in [\tau, (\tau+s) \wedge (T+1)]} (\omega_t - \omega_\tau) = 0\Big) = 0, \tag{7.34}$$

where $\tau := (\inf\{t \in [0, T + 1] : \omega_t \le 0\}) \wedge (T + 1)$.

The proof of Lemma 7.8 is essentially the same as the proof of [10], Lemma 5.9, with Step 2 therein allowing for a simplification, since the drift coefficient α is constant in the present setting. It is also worth mentioning the typo in the seventh displayed equation in the proof of [10], Lemma 5.9: "$\max(\zeta^N_{r^N_t -}, \zeta^N_{r^N_t}) - \min(\zeta^N_{r^N_s -}, \zeta^N_{r^N_s})$" should be replaced by "$\min(\zeta^N_{r^N_t -}, \zeta^N_{r^N_t}) - \max(\zeta^N_{r^N_s -}, \zeta^N_{r^N_s})$."

Lastly, we observe that Lemma 7.4 is a direct consequence of Lemmas 7.7 and 7.8.

REFERENCES

[1] ALICHI, A., RYOO, S. C. and HONG, C. (2012). Managing non-core liabilities and leverage of the banking system: A building block for macroprudential policy making in Korea. Technical report, International Monetary Fund, Washington, DC.

[2] BATTISTON, S., GATTI, D. D., GALLEGATI, M., GREENWALD, B. C. and STIGLITZ, J. E. (2009). Liaisons dangereuses: Increasing connectivity, risk sharing, and systemic risk. Technical report, National Bureau of Economic Research, Cambridge, MA.

[3] BUSH, N., HAMBLY, B. M., HAWORTH, H., JIN, L. and REISINGER, C. (2011). Stochastic evolution equations in portfolio credit modelling. *SIAM J. Financial Math*. **2** 627–664. MR2836495

[4] CÁCERES, M. J., CARRILLO, J. A. and PERTHAME, B. (2011). Analysis of nonlinear noisy integrate & fire neuron models: Blow-up and steady states. *J. Math. Neurosci*. **1** Art. 7, 33. MR2853216

[5] CÁCERES, M. J. and PERTHAME, B. (2014). Beyond blow-up in excitatory integrate and fire neuronal networks: Refractory period and spontaneous activity. *J. Theoret. Biol*. **350** 81–89. MR3190511

[6] CARMONA, R., FOUQUE, J.-P. and SUN, L.-H. (2015). Mean field games and systemic risk. *Commun. Math. Sci*. **13** 911–933. MR3325083

[7] CARRILLO, J. A., PERTHAME, B., SALORT, D. and SMETS, D. (2015). Qualitative properties of solutions for the noisy integrate and fire model in computational neuroscience. *Nonlinearity* **28** 3365–3388. MR3403402

[8] DAI PRA, P., RUNGGALDIER, W. J., SARTORI, E. and TOLOTTI, M. (2009). Large portfolio losses: A dynamic contagion model. *Ann. Appl. Probab*. **19** 347–394. MR2498681

[9] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Global solvability of a networked integrate-and-fire model of McKean–Vlasov type. *Ann. Appl. Probab*. **25** 2096–2133. MR3349003

[10] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Particle systems with a singular mean-field self-excitation. Application to neuronal networks. *Stochastic Process. Appl*. **125** 2451–2492. MR3322871

[11] DEMBO, A., DEUSCHEL, J.-D. and DUFFIE, D. (2004). Large portfolio losses. *Finance Stoch*. **8** 3–16. MR2022976

[12] DEMBO, A. and TSAI, L.-C. (2017). Criticality of a randomly-driven front. ArXiv:1705.10017.

[13] DUFFIE, D. and ZHU, H. (2011). Does a central clearing counterparty reduce counterparty risk? *Rev. Asset Pricing Stud*. **1** 74–95.

[14] GAI, P. and KAPADIA, S. (2010). Contagion in financial networks. Technical report, Bank of England, London.

[15] GIESECKE, K., SPILIOPOULOS, K., SOWERS, R. B. and SIRIGNANO, J. A. (2015). Large portfolio asymptotics for loss from default. *Math. Finance* **25** 77–114. MR3312197

[16] GLASSERMAN, P. (2015). Contagion in financial networks. Technical report, Office of Financial Research.

[17] HORST, U. (2007). Stochastic cascades, credit contagion, and large portfolio losses. *J. Econ. Behav. Organ*. **63** 25–54.

[18] INGLIS, J. and TALAY, D. (2015). Mean-field limit of a stochastic particle system smoothly interacting through threshold hitting-times and applications to neural networks with dendritic component. *SIAM J. Math. Anal*. **47** 3884–3916. MR3411723

[19] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877

[20] KESTEN, H. and SIDORAVICIUS, V. (2008). A problem in one-dimensional diffusion-limited aggregation (DLA) and positive recurrence of Markov chains. *Ann*. *Probab*. **36** 1838–1879. MR2440925

[21] KRYLOV, N. V. (2009). *Controlled Diffusion Processes*. *Stochastic Modelling and Applied Probability* **14**. Springer, Berlin. Translated from the 1977 Russian original by A. B. Aries, Reprint of the 1980 edition. MR2723141

[22] LADYŽENSKAYA, O. A. (1954). On solvability of the fundamental boundary problems for equations of parabolic and hyperbolic type. *Dokl*. *Akad*. *Nauk SSSR* **97** 395–398. MR0073834

[23] LADYŽENSKAYA, O. A., SOLONNIKOV, V. A. and URAL'CEVA, N. N. (1968). *Linear and Quasilinear Equations of Parabolic Type*. *Translations of Mathematical Monographs* **23**. Amer. Math. Soc., Providence, RI. Translated from the Russian by S. Smith. MR0241822

[24] LORENZ, J., BATTISTON, S. and SCHWEITZER, F. (2009). Systemic risk in a unifying framework for cascading processes on networks. *Eur. Phys*. *J*. *B* **71** 441–460. MR2591887

[25] MAY, R. M. and ARINAMINPATHY, N. (2010). Systemic risk: The dynamics of model banking systems. *J*. *R*. *Soc*. *Interface* **7** 823–838.

[26] SHIN, H. S., HAHM, J.-H. and SHIN, K. (2011). Non-core bank liabilities and financial vulnerability. Technical report, Princeton Univ., Princeton, NJ.

[27] SKOROHOD, A. V. (1956). Limit theorems for stochastic processes. *Teor. Veroyatn*. *Primen*. **1** 289–319. MR0084897

[28] SLY, A. (2016). On one-dimensional multi-particle diffusion limited aggregation. ArXiv:1609.08107.

[29] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math*. **1464** 165–251. Springer, Berlin. MR1108185

[30] WATTS, D. J. (2002). A simple model of global cascades on random networks. *Proc*. *Natl*. *Acad*. *Sci*. *USA* **99** 5766–5771. MR1896072

[31] WHITT, W. (2002). *Stochastic-Process Limits*: *An Introduction to Stochastic-Process Limits and Their Application to Queues*. Springer, New York. MR1876437

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