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A LIOUVILLE THEOREM FOR ELLIPTIC SYSTEMS WITH DEGENERATE ERGODIC COEFFICIENTS

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We study the behavior of second-order degenerate elliptic systems in divergence form with random coefficients which are stationary and ergodic. Assuming moment bounds like Chiarini and Deuschel (2014) on the coefficient field a and its inverse, we prove an intrinsic large-scale $C^{1,\alpha}$-regularity estimate for a-harmonic functions and obtain a first-order Liouville theorem for a-harmonic functions.

1. Introduction and the main results. We study the behavior of second order nonuniformly elliptic equations, and more generally systems of equations, with random coefficients. The random coefficient fields a are assumed to be *stationary*, meaning that the joint probability distribution of a and $a(\cdot + x)$ are the same, and *ergodic*, meaning that every translation invariant random variable is almost surely constant. Furthermore, as in the framework of Chiarini and Deuschel [12], rather than assuming the field is uniformly elliptic we assume only moment bounds from above and below.

More precisely, if $\langle \cdot \rangle$ denotes the expectation with respect to the probability measure on the space of coefficient fields, which will be denoted Ω, we define the scalar random variables $0 < \lambda, \mu < \infty$ via

$$\lambda := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot a\xi}{|\xi|^2} \quad \text{and} \quad \mu := \sup_{\xi \in \mathbb{R}^d} \frac{|a\xi|^2}{\xi \cdot a\xi}, \tag{1}$$

where in the scalar symmetric case $\lambda^{-1} = |a^{-1}|$ and $\mu = |a|$ are the spectral norms of a and its inverse. Our assumption is that

$$\langle \mu^p \rangle^{\frac{1}{p}} + \langle \lambda^{-q} \rangle^{\frac{1}{q}} =: K < \infty \qquad \text{where } \frac{1}{p} + \frac{1}{q} < \frac{2}{d}. \tag{2}$$

Here $d \geq 2$ denotes the dimension. Notice that (2) coincides with the integrability condition considered in [12].

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We require this condition most essentially in two steps of the proof. First, the strict inequality appearing in (2) is used in the foremost stochastic element of the argument, and guarantees the compactness of a certain Sobolev embedding which is used in Lemma 2 to establish the sublinearity of the large-scale averages of the corrector and flux correction, which are defined in (3), (4), and (5). The deterministic elements of the argument, namely the Caccioppoli inequality (Lemma 3) and the large-scale $C^{1,\alpha}$-regularity (Theorem 2), require only $1/p + 1/q \leq 2/d$.

The primary result of this paper is a *first-order* Liouville theorem for degenerate coefficient fields satisfying (2). This means that for $\langle\cdot\rangle$-a.e. environment a, every subquadratically-growing a-harmonic function u on the whole space belongs to the $(d+1)$-dimensional space of a-affine functions. Namely, the space spanned by functions of the form $c + \xi \cdot x + \phi_\xi$ where, for every $\xi \in \mathbb{R}^d$, the corrector ϕ_ξ denotes the whole-space solution

$$-\nabla \cdot a(\xi + \nabla\phi_\xi) = 0, \tag{3}$$

whose gradient $\nabla\phi_\xi$ is *stationary*, by which we understand $\nabla\phi_\xi(a; x+z) = \nabla\phi_\xi(a(\cdot + z); x)$ for any shift vector $z \in \mathbb{R}^d$, has vanishing average $\langle\nabla\phi_\xi\rangle = 0$, and finite $\frac{2q}{q+1}$-moment $\langle|\nabla\phi_\xi|^{\frac{2q}{q+1}}\rangle$. The following theorem summarizes the result.

THEOREM 1. *Let $\langle\cdot\rangle$ be stationary and ergodic, and assume it satisfies* (2). *Then $\langle\cdot\rangle$-a.e. coefficient field $a \in \Omega$ has the following Liouville property: if u is an a-harmonic function in the whole space, that is, it solves $-\nabla \cdot a\nabla u = 0$ in $\mathbb{R}^d$ with $d \geq 2$, and in addition u is subquadratic in the sense that, for some $\alpha < 1$,*

$$\lim_{R\to\infty} R^{-(1+\alpha)} \left(\fint_{B_R} |u|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{2p}} = 0,$$

then there necessarily exists $c \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$ for which $u(x) = c + \xi \cdot x + \phi_\xi(x)$.

We remark that, although the perspective of this paper is primarily analytic, Theorem 1 has implications concerning the underlying stochastic process $\{X_t = (X_{1,t}, \ldots, X_{d,t})\}_{t\geq 0}$ on $\mathbb{R}^d$ associated to the generator

$$\mathcal{L}_a := \nabla \cdot a\nabla.$$

Namely, it is a new and immediate consequence of the theorem that, for $\langle\cdot\rangle$-a.e. a and for each $i \in \{1, \ldots, d\}$, solutions to equation (3) are unique up to an additive constant. It therefore follows that, for $\langle\cdot\rangle$-a.e. a, there exists a unique (up to the addition of a constant vector) function $\phi = (\phi_1, \ldots, \phi_d)$ on $\mathbb{R}^d$ for which the process

$$M_t = X_t + \phi(X_t)$$

is a Martingale on $\mathbb{R}^d$ with respect to the quenched measure associated to the generator $\mathcal{L}_a$. The sublinearity of the corrector proven in Lemma 2 below then suggests, at least formally, that the laws of the rescaled processes $\varepsilon X_{t/\varepsilon^2}$ are converging like those of the rescaled Martingales $\varepsilon M_{t/\varepsilon^2}$, where the limit of the latter may be understood using standard Martingale convergence theorems. See [12], Section 4, for a complete and rigorous discussion of these ideas in the scalar case.

Theorem 1 amounts to an extension of the first-order Liouville property for uniformly elliptic coefficient fields obtained by Gloria, Neukamm, and the third author ([21], Corollary 1) to the case of degenerate environments satisfying (2). For this, like in [21], we will also need, for each $i \in \{1, \dots, d\}$, the skew-symmetric matrix field σ_i, which can be viewed as a vector potential for the harmonic coordinates and provides the correction of the ith component of the flux

$$q_i := a(\nabla\phi_i + e_i),$$

by satisfying the equation

$$q_i - \langle q_i\rangle =: \nabla \cdot \sigma_i. \tag{4}$$

Here the divergence of a tensor field is defined as

$$(\nabla \cdot \sigma_i)_j := \sum_{k=1}^{d} \partial_k \sigma_{ijk}.$$

The linearity of the equation allows for the consideration of only the fluxes corresponding to the canonical basis $\{e_i\}_{i\in\{1,\dots,d\}}$, where

$$\langle q_i\rangle = \langle a(\nabla\phi_i + e_i)\rangle =: a_{\mathrm{hom}} e_i$$

defines the constant, possibly nonsymmetric, homogenized coefficients a_{hom}. The uniform ellipticity of a_{hom} was established in [12], Proposition 4.1.

In the setting of uniformly elliptic random coefficient fields the vector corrector σ was introduced and constructed in [21], Lemma 1. Since the definition of σ_i is underdetermined, taking motivation from the analogous periodic framework, they made the specific choice of gauge

$$-\Delta\sigma_{ijk} = \partial_j q_{ik} - \partial_k q_{ij}. \tag{5}$$

A principle difference between the degenerate and uniformly elliptic cases is that, in the latter the fluxes q_i belong to $L^2(\Omega)$, and therefore so too do the gradients $\nabla\sigma_{ijk} \in L^2(\Omega)$, whereas in the degenerate setting this is no longer true. In fact, Hölder's inequality and the moment bound (2) imply only that $q_i \in L^{\frac{2p}{p+1}}(\Omega)$. To see this, observe that definition (1) and condition (2) imply that $ae_i \in L^p(\Omega) \subset L^{\frac{2p}{p+1}}(\Omega)$, and definition (1) combined with Hölder's inequality in the form

$$\langle |a\nabla\phi_i|^{\frac{2p}{p+1}}\rangle^{\frac{p+1}{2p}} \leq \langle \mu^{\frac{p}{p+1}} (\nabla\phi_i \cdot a\nabla\phi_i)^{\frac{p}{p+1}}\rangle^{\frac{p+1}{2p}} \leq \langle \mu^p\rangle^{\frac{1}{2p}} \langle (\nabla\phi_i \cdot a\nabla\phi_i)\rangle^{\frac{1}{2}}$$

guarantee that $a\nabla\phi_i \in L^{\frac{2p}{p+1}}(\Omega)$. For this reason, it is necessary to replace the L^2-theory for the construction of σ used in [21], Lemma 1, with an approximation argument and a Calderón–Zygmund estimate to first construct the stationary, mean zero gradients of σ in $L^{\frac{2p}{p+1}}(\Omega)$, and thereby define the generally nonstationary flux corrections uniquely up to an additive random constant.

The properties of the correctors and flux corrections are summarized by the following lemma, where we remark that the construction of the scalar corrector ϕ in the degenerate ergodic setting under the weaker assumptions $p = q = 1$ has been carried out in [12], Section 4. Loosely speaking, and for the construction of both the corrector and the flux corrections, the definitions (3), (4) and (5) are lifted to the probability space in order to construct their gradients as stationary and mean-zero random fields with finite moments determined by (2).

LEMMA 1. *Let $\langle\cdot\rangle$ be stationary and ergodic, and let (2) be satisfied. Then there exist $C = C(d) > 0$ and two random tensor fields $\{\phi_i\}_{i=1,\dots,d}$ and $\{\sigma_{ijk}\}_{i,j,k=1,\dots,d}$ with the following properties: The gradient fields are stationary, have bounded moments, and are of vanishing expectation:*

$$\sum_{i=1}^{d}\langle\nabla\phi_i\cdot a\nabla\phi_i\rangle+\sum_{i=1}^{d}\big\langle|\nabla\phi_i|^{\frac{2q}{q+1}}\big\rangle^{\frac{q+1}{2q}}+\sum_{i,j,k=1}^{d}\big\langle|\nabla\sigma_{ijk}|^{\frac{2p}{p+1}}\big\rangle^{\frac{p+1}{2p}}\le CK, \tag{6}$$
$$\langle\nabla\phi_i\rangle=\langle\nabla\sigma_{ijk}\rangle=0.$$

Moreover, the field σ is skew-symmetric in its last two indices, that is

$$\sigma_{ijk}=-\sigma_{ikj}.$$

Furthermore, for $\langle\cdot\rangle$-a.e. a we have

$$q_i=a(\nabla\phi_i+e_i)=a_{\mathrm{hom}}e_i+\nabla\cdot\sigma_i.$$

Finally, the homogenized coefficient field a_{hom} is uniformly elliptic in the sense that, for each $\xi\in\mathbb{R}^d$,

$$\frac{1}{K}|\xi|^2\le\xi\cdot a_{\mathrm{hom}}\xi\quad\text{and}\quad|a_{\mathrm{hom}}\xi|\le K|\xi|.$$

Furthermore, owing to the fact that the gradients of the corrector ϕ and flux-correction σ have zero average, the ergodicity of the coefficient field guarantees by standard arguments that their large-scale averages are sublinear in the sense of the following lemma.

LEMMA 2. *Let $\langle\cdot\rangle$ be stationary and ergodic, and let (2) be satisfied. Then, the large-scale averages of the random tensor fields $\{\phi_i\}_{i=1,\dots,d}$ and $\{\sigma_{ijk}\}_{i,j,k=1,\dots,d}$ are sublinear in the sense that, for $\langle\cdot\rangle$-a.e. a,*

$$\lim_{R\to\infty}\frac{1}{R}\bigg(\fint_{B_R}\Big|\phi-\fint_{B_R}\phi\Big|^{\frac{2p}{p-1}}\bigg)^{\frac{p-1}{2p}}=0,$$
$$\lim_{R\to\infty}\frac{1}{R}\bigg(\fint_{B_R}\Big|\sigma-\fint_{B_R}\sigma\Big|^{\frac{2q}{q-1}}\bigg)^{\frac{q-1}{2q}}=0.$$

We remark that an alternate construction of the flux correction is presented in the Appendix, and an ingredient of this argument requires a small modification of Lemma 1. Indeed, the proof of sublinearity follows from the integrability of the gradient fields $\nabla\phi$ and $\nabla\sigma$ and does not use any properties of the underlying equation.

The large-scale $C^{1,\alpha}$-regularity first obtained in [21] asserts that whenever u is an a-harmonic function, then its deviation from the space of a-harmonic affine functions, as defined for each $r > 0$ by the excess

$$\mathrm{Exc}(r) = \inf_{\xi\in\mathbb{R}^d} \fint_{B_r} \big(\nabla u - (\xi + \nabla\phi_\xi)\big)\cdot a\big(\nabla u - (\xi + \nabla\phi_\xi)\big),$$

decays, for any $\alpha \in (0,1)$ and for all sufficiently large radii $r < R$ depending on α, as a power law

$$\mathrm{Exc}(r) \le C(r/R)^{2\alpha}\,\mathrm{Exc}(R).$$

The proof is purely deterministic and is based on estimating the homogenization error determined by an a-harmonic function u and an a_{hom}-harmonic function v, as defined by

$$u - (v + \phi_i\partial_i v),$$

for ϕ_i the first-order corrector defined in (3) corresponding to the ith standard basis vector. An essential observation of [21] was that the homogenization error satisfied a divergence-form equation with right-hand side in divergence-form. We use this fact to estimate its energy in the intrinsic $L^2(a)$-norm, where the regularity of the a_{hom}-harmonic function v plays an essential role, and to ultimately prove the excess decay and large-scale $C^{1,\alpha}$-regularity.

In this setting, the construction of the appropriate a_{hom}-harmonic function v differs considerably from the uniformly elliptic case. To estimate the homogenization error on the ball B_R, the idea is to exploit the best integrability of the coefficient field by separating

$$\text{the “Dirichlet case” } q \ge p \text{ and the “Neumann case” } p \ge q, \tag{7}$$

where in the Dirichlet case, we define v via the boundary condition

$$v = u_\varepsilon \qquad \text{on } \partial B_R,$$

and, in the Neumann case, we impose

$$\nu\cdot a_{\mathrm{hom}}\nabla v = (\nu\cdot a\nabla u)_\varepsilon \qquad \text{on } \partial B_R,$$

where the subscript ε denotes a smoothing by convolution on the boundary of the ball. Then, like in [21], the energy of the corresponding homogenization error is controlled by introducing a cutoff η vanishing near the boundary and estimating the intrinsic energy of the quantity

$$u - (v + \eta\phi_i\partial_i v),$$

where it will be necessary to use the aforementioned divergence-form equation satisfied by the homogenization error, as modified by the introduction of the cutoff, and to control the subsequent boundary terms arising from the case distinction (7). The result is summarized by the following deterministic theorem, where the constants C_0 and C_1 depend upon K from (2) through the ellipticity of a_{hom} appearing in Lemma 1.

THEOREM 2. *Let the Hölder exponent $\alpha \in (0,1)$ and $\Lambda > 0$ be given. Then there exist constants $C_0, C_1 = C_0, C_1(d, \alpha, K, \Lambda)$ with the following property*:

If $r < R$ are two radii such that for any $\rho \in [r, R]$ we have

$$\left(\fint_{B_\rho} \mu^p\right)^{\frac{1}{p}} + \left(\fint_{B_\rho} \lambda^{-q}\right)^{\frac{1}{q}} \le \Lambda, \tag{8}$$

with the exponents p and q satisfying

$$\frac{1}{p} + \frac{1}{q} \le \frac{2}{d}, \tag{9}$$

and

$$\begin{aligned} &\frac{1}{\rho}\left(\fint_{B_\rho}\left|\phi - \fint_{B_\rho}\phi\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} \le \frac{1}{C_0}, \\ &\frac{1}{\rho}\left(\fint_{B_\rho}\left|\sigma - \fint_{B_\rho}\sigma\right|^{\frac{2q}{q-1}}\right)^{\frac{q-1}{2q}} \le \frac{1}{C_0}, \end{aligned} \tag{10}$$

then any a-harmonic function u in B_R, that is, weak solution of $-\nabla \cdot a\nabla u = 0$ in B_R, satisfies

$$\mathrm{Exc}(r) \le C_1 \left(\frac{r}{R}\right)^{2\alpha} \mathrm{Exc}(R),$$

where the excess

$$\mathrm{Exc}(\rho) := \inf_{\xi \in \mathbb{R}^d} \fint_{B_\rho} \big(\nabla u - (\xi + \nabla\phi_\xi)\big) \cdot a\big(\nabla u - (\xi + \nabla\phi_\xi)\big)$$

measures in the $L^2(a)$-sense deviations of u from the set of a-affine functions.

We remark that the assumptions of Theorem 2 will be satisfied for $\langle\cdot\rangle$-a.e. environment, provided the radius r is chosen sufficiently large. Indeed, for any $\alpha \in (0,1)$ and any $C_0 > 0$, the ergodic theorem asserts that for $\langle\cdot\rangle$-a.e. environment a there exists a random radius $r_1 = r_1(a)$ such that (8) is achieved for $\Lambda = 2(\langle \mu^p \rangle^{\frac{1}{p}} + \langle \lambda^{-q} \rangle^{\frac{1}{q}})$ whenever $r \ge r_1$ and Lemma 2 guarantees the existence of $r_2 = r_2(a)$ such that (10) is satisfied for every $r \ge r_2$.

A version of the the Caccioppoli inequality adapted to the degenerate setting will be used in the proofs of Theorems 1 and 2. In the uniformly elliptic case, the statements may be used to bound the L^2-norm of the gradient of an a-harmonic function on ball by the L^2-norm of the function itself on a somewhat larger ball. A straightforward modification yields the analogous statement for elliptic systems with nonsymmetric degenerate coefficients.

LEMMA 3. *Suppose that u is an a-harmonic function in B_R, and that for some exponents $p \in (1, \infty)$, $q \in [1, \infty)$ we have*

$$\left(\fint_{B_R} \mu^p\right)^{\frac{1}{p}} + \left(\fint_{B_R} \lambda^{-q}\right)^{\frac{1}{q}} \leq \Lambda. \tag{11}$$

Then there exists $C_1, C_2 = C_1, C_2(d) > 0$ such that for any $0 < \rho < \frac{R}{2}$ and any $c \in \mathbb{R}$,

$$\begin{aligned}\left(\fint_{B_{R-\rho}} |\nabla u|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{q}} &\leq C_1 \Lambda \fint_{B_{R-\rho}} \nabla u \cdot a \nabla u \\ &\leq C_2 \frac{\Lambda^2}{\rho^2}\left(\fint_{B_R \setminus B_{R-\rho}} |u - c|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}.\end{aligned} \tag{12}$$

Additionally, while the purpose of this paper is to establish the almost sure large-scale $C^{1,\alpha}$-regularity of a-harmonic functions, as contained in Theorem 2, and the corresponding almost sure Liouville property, as contained in Theorem 1, we expect to obtain more quantitative information for ensembles $\langle\cdot\rangle$ that satisfy a stronger mixing condition and plan to make this the subject of future work. For instance, by assuming the ensemble satisfies a logarithmic Sobolev inequality of the type used in [21], Theorem 1, we expect to obtain similar stretched exponential moments as [21], Theorem 1, for the minimal radius r_* defined, for simplicity in the case $p = q$, for $C_0 > 0$ as defined in Theorem 2, by

$$r_* = \inf\left\{r \geq 1 \,\middle|\, \text{for all } R \geq r, \left(\fint_{B_R} \left|(\phi, \sigma) - \fint_{B_R} (\phi, \sigma)\right|^{\frac{2p}{p-1}}\right)^{\frac{2p}{p-1}} \leq \frac{1}{C_0}\right\},$$

which effectively defines the initial scale on which the $C^{1,\alpha}$-regularity of Theorem 2 begins to take effect. Furthermore, again assuming a logarithmic Sobolev inequality, it should be possible to obtain a quantitative two-scale expansion for a-harmonic functions like [21], Corollary 3. Lastly, following the methods of Fischer and the last author [18], we expect to obtain the existence of higher-order correctors for degenerate ensembles and corresponding higher-order Liouville statements under a mild quantification of the ergodicity; see [18], Theorem 3, Corollary 4.

In the uniformly elliptic framework, the Caccioppoli inequality (12) can be viewed as a version of a reversed Poincaré inequality, meaning that we gain one derivative in the estimate at the expense of increasing the radius of the ball. With the assumption of uniform ellipticity replaced by a weaker moment bound condition on a from below and above, one has to replace the integrability exponents in (12) on both sides. Hence, in this case, one trades a derivative for a possible loss in the integrability. While in Lemma 3 we did not assume condition (9), which appeared in Theorem 2, it has a direct relation to (12). Indeed, if one uses Sobolev embedding on the right-hand side of (12) to trade one derivative for better integrability, it is exactly condition (9) which ensures that in the end we get the same exponent as the one we started with on the left-hand side of (12). In the case of a condition on p and q with strict inequality (2), the above combination of Caccioppoli and Sobolev inequalities gives a gain in the integrability—a fact that allowed Chiarini and Deuschel [12] (see also [11]), in the case of a scalar equation, to perform Moser iteration. The condition (2) first appeared in the paper by Andres, Deuschel, and Slowik [2] (see also [3]), and was recently generalized to study invariance principles for environments with time-dependent coefficients [1, 15].

First-order Liouville statements for a-harmonic functions are a compact way to express regularity on large scales. In fact, an easy post-processing of the excess decay in Theorem 2 yields a large-scale $C^{1,\alpha}$-estimate for a-harmonic functions, see [21], Corollary 2. We thus speak of a $C^{1,\alpha}$-Liouville property. A further post-processing yields large-scale $C^{1,\alpha}$-Schauder estimates for the operator $-\nabla \cdot a\nabla$; see, for instance, [19], Theorem 5.20. In the case of constant-coefficient (and thus scale-invariant) equations, this relation between $C^{1,\alpha}$-Liouville principles and $C^{2,\alpha}$-Liouville principles on the one hand, namely that subcubic harmonic functions must be quadratic harmonic polynomials, and a $C^{1,\alpha}$- or $C^{2,\alpha}$-Schauder theory on the other hand is classical: An indirect argument by Simon [28] allows to directly pass from the Liouville property to the corresponding Schauder estimate.

For a general nonconstant coefficient field a, we call $C^{k,\alpha}$-Liouville property the fact that the linear space of a-harmonic functions that grow at most of the order $|x|^{k+\alpha}$ (say, in an averaged sense as in Theorem 1) has the same dimension as in the case of constant-coefficient equations (where it is spanned by all harmonic polynomials of order at most k). Without further structural conditions, it is almost folkloric knowledge that this equality already fails for $k = 0$ and any $\alpha > 0$ even in the case of uniformly elliptic coefficients (which may even be smooth [17], Proposition 21). The work of Yau [30], drawing a connection to curvature of the metric given by a, popularized the question of determining whether the dimensions are *asymptotically* equal for $k \uparrow \infty$, as shown by Colding and Minicozzi [13] and Li [23] for uniformly elliptic equations.

In the case of uniformly elliptic *periodic* coefficient fields, the full hierarchy of Liouville properties was established by Avellaneda and Lin [8], based on earlier ideas developed by those authors on a large-scale regularity theory in Hölder and L^p-spaces [6, 7] via a Campanato iteration, which is also used in Theorem 2. Marahrens and the last author [24], Corollary 4, derived a $C^{0,\alpha}$-Liouville property in the case of stationary *random* coefficient fields with integrable correlation tails (i.e., integrable in a sufficiently strong sense so as to allow for a Logarithmic Sobolev Inequality). Benjamini, Duminil-Copin, Kozma, and Yadin [9] derived a $C^{0,\alpha}$-Liouville property under the mere assumption of ergodicity. Their result applies to degenerate coefficient fields for which sufficient a priori Green's function estimates are available, and they also formulated the question of higher-order Liouville properties in the random case [9], Theorems 4, 5.

Armstrong and Smart [5] adapted the approach of [7] to obtain a large-scale $C^{1,0}$-regularity theory in the case of uniformly elliptic coefficient fields with a finite-range condition, which was a major step because it required a new quantitative substitute for the compactness argument, and which was later extended by Armstrong and Mourrat [4] to very general mixing conditions. Gloria, Neukamm and the last author [21] derived the $C^{1,\alpha}$-Liouville property under the mere assumption of ergodicity in the uniformly elliptic case; the main new ingredients being 1) the usage of an *intrinsic* excess decay, that is, measuring the energy distance to the space of *intrinsically* affine functions (i.e., the harmonic coordinates) and 2) the construction of the vector potential σ (which allows to bring the residuum in the two-scale expansion into divergence form). Fischer and the last author extended the uniformly elliptic version of Theorem 2 to the case of an excess of order k under a mild quantification of the sublinear growth of the corrector in [17] to obtain a full hierarchy of Liouville properties, and showed in [18] that the quantified sublinear growth of the corrector is satisfied under a mild quantification of ergodicity in a certain class of Gaussian environments. Additionally, there has recently been a lot of activity aimed at understanding the space of harmonic functions on infinite percolation clusters with specified polynomial growth. For instance, Sapozhnikov [27] recently proved the finite-dimensionality of these spaces for a large class of percolation models.

Finally, we believe these results are very likely extendable to the discrete case. Indeed, Deuschel, Nguyen and Slowik [14] have established an invariance principle for random walks in a degenerate environments under similar integrability assumptions on the coefficient field, and the techniques of the present paper are expected to be adaptable to their setting. To this end, we include two estimates which we believe will be useful in adapting our arguments to their framework.

LEMMA 4. *Let $Q=[0,1]^d$ and let w_D and w_N be solutions of the Dirichlet and Neumann problems, respectively*:

$$\begin{cases} -\Delta w_D = 0 & \text{in } Q_d, \\ w_D = v & \text{on } \partial Q_d, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w_N = 0 & \text{in } Q_d, \\ \nabla^{\text{norm}} w_N = v & \text{on } \partial Q_d, \end{cases} \tag{13}$$

where in the Neumann case we also assume $\int_{\partial Q_d} v \, \mathrm{d}x = 0$. Then for each $1 < p < \infty$, there exists $C = C(d,p) > 0$ such that

$$\|\nabla^{\text{norm}} w_D\|_{L^p(\partial Q_d)} \le C \|\nabla^{\tan} v\|_{L^p(\partial Q_d)},$$

$$\|\nabla^{\tan} w_N\|_{L^p(\partial Q_d)} \le C \|v\|_{L^p(\partial Q_d)}.$$

While the arguments of this paper do not rely upon this fact (since they use the smooth domain B_1 instead of the cube Q_d), we believe that our method, which reduces the statement of Lemma 4 on the cube Q_d to the analogous statement for harmonic functions on slab domains of the form $[0,1]\times\mathbb{R}^{d-1}$ with periodic boundary data, can be easily translated to the discrete setting.

Organization and notation. The remainder of the paper presents the proofs of Theorem 1, Lemmas 1 and 2, Theorem 2 and Lemma 3 in the order of their appearance in the Introduction. In addition, the Appendix contains an alternative argument for the construction of the flux corrector σ in Lemma 1 as well as the proof of Lemma 4. We remark that, in order to simplify the notation, the statements and proofs are written for the nonsymmetric *scalar* setting. However, at the cost of increasing some constants, all of the arguments carry through unchanged for nonsymmetric systems. Throughout, $\lesssim$ is used to denote a constant whose dependencies are specified in every case by the statement of the respective lemma or theorem.

2. The proof of Theorem 1.

Suppose that u is an a-harmonic function on the whole space, that is

$$-\nabla\cdot a\nabla u=0 \qquad \text{in } \mathbb{R}^d,$$

which is strictly subquadratic in the sense that, for some $\alpha<1$,

$$\lim_{r\to\infty} r^{-(1+\alpha)}\left(\fint_{B_r}|u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}}=0.$$

For $\langle\cdot\rangle$-a.e. a it follows from the ergodic theorem and the integrability assumption (2) that there exists $r_1=r_1(a)$ such that, for all $r\geq r_1$,

$$\left(\fint_{B_r}\mu^p\right)^{\frac{1}{p}}+\left(\fint_{B_r}\lambda^{-q}\right)^{\frac{1}{q}}\leq 2\left(\langle\mu^p\rangle^{\frac{1}{p}}+\langle\lambda^{-q}\rangle^{\frac{1}{q}}\right)=:\Lambda.$$

Let $C_0=C_0(d,\alpha,K,\Lambda)$ be as in Theorem 2, and choose $r_2=r_2(a)\geq r_1$ so that, in view of Lemma 2, for all $r\geq r_2$,

$$\frac{1}{r}\left(\fint_{B_r}\left|\phi-\fint_{B_r}\phi\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}}\leq\frac{1}{C_0}\quad\text{and}\quad \frac{1}{r}\left(\fint_{B_r}\left|\sigma-\fint_{B_r}\sigma\right|^{\frac{2q}{q-1}}\right)^{\frac{q-1}{2q}}\leq\frac{1}{C_0}.$$

In order to conclude, observe that Lemma 3 and the definition of the excess imply by the choice of r_1 that, for each $r\geq r_1$, for $C_1=C_1(d)>0$,

$$\operatorname{Exc}(r)\leq\fint_{B_r}\nabla u\cdot a\nabla u\leq\frac{C_1\Lambda}{r^2}\left(\fint_{B_{2r}}|u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}.$$

This implies that, in view of Theorem 2 and the choice of r_2, for every of $r > \rho > r_2$, for $C_2, C_3 = C_2, C_3(d, \alpha, K, \Lambda) > 0$,

$$\operatorname{Exc}(\rho) \le C_2\left(\frac{\rho}{r}\right)^{2\alpha} \operatorname{Exc}(r) \le C_3 \rho^{2\alpha} r^{-(2+2\alpha)} \left(\fint_{B_{2r}} |u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}.$$

Therefore, owing to the choice of α we have, for each $\rho > r_2$,

$$\operatorname{Exc}(\rho) \le C_3 \rho^{2\alpha} \limsup_{r\to\infty}\left(r^{-(1+\alpha)}\left(\fint_{B_{2r}} |u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}}\right)^2 = 0.$$

By definition of $\operatorname{Exc}(\rho)$ this implies existence of $\xi \in \mathbb{R}^d$ and $c \in \mathbb{R}$ s.t. $u(x) = c + \xi \cdot x + \phi_\xi(x)$ for a.e. $x \in B_\rho$. Since the values ξ and c are independent of the choice of ρ, we obtain the statement of the theorem.

3. Proof of Lemma 1. The construction of the corrector ϕ in the case of degenerate and unbounded, stationary and ergodic coefficients was performed in [12], Section 4.1.

For the construction of the flux corrector σ, we combine an approximation argument with a version of the existence result for σ from [21]. Given $n \in \mathbb{N}$ let us consider the random variable $q_n := I(|q_n| \le n)q$. Here, I stands for the characteristic (indicator) function. We will prove the existence of a random tensor field σ_n which satisfies:

- $\nabla\sigma_{n,ijk} \in L^2(\Omega)$ is stationary, $\langle\nabla\sigma_{n,ijk}\rangle = 0$, and σ_n is skew-symmetric in its last two indices;
- for $\langle\cdot\rangle$-a.e. a we have

$$\nabla \cdot \sigma_{n,i} = q_{n,i} - \langle q_{n,i}\rangle; \tag{14}$$

- for $\langle\cdot\rangle$-a.e. a we have

$$-\Delta\sigma_{n,ijk} = \partial_j q_{n,ik} - \partial_k q_{n,ij}. \tag{15}$$

This fact follows from the argument of Gloria, Neukamm, and the third author [21], Lemma 1; for the reader's convenience we outline here an alternative approach which follows the reasoning presented by the third author at the September, 2015 Oberwolfach workshop on stochastic homogenization.

Fix $n > 0$. The argument first constructs the gradient of the expected approximate flux correction $\sigma_n = \{\sigma_{n,ijk}\}_{i,j,k=1,\dots,d}$ by considering the single component $\sigma_{n,i} = \{\sigma_{n,ijk}\}$ for each $i \in \{1,\dots,d\}$ separately. For this, the equation will be lifted to the probability space, and phrased in terms of the "horizontal gradient" with respect to shifts of the coefficient field.

Precisely, for each $i \in \{1,\dots,d\}$, the horizontal derivative of a random variable ζ along the ith coordinate direction is defined by the infinitesimal generator of the corresponding translation in the probability space, and is given by the limit

$$D_i\zeta(a) := \lim_{h\to 0}\frac{\zeta(a(\cdot + he_i)) - \zeta(a)}{h}.$$

We remark that the operators D_i are closed, and densely defined on $L^2(\Omega)$. We write $\mathcal{D}(D_i)$ for their respective domains and define the Hilbert space

$$\mathcal{H}^1 = \bigcap_{i=1}^{d} \mathcal{D}(D_i) \subset L^2(\Omega),$$

equipped with the inner product

$$(f, g)_{\mathcal{H}^1} := \langle fg\rangle + \sum_{i=1}^{d} \langle D_i f D_i g\rangle.$$

The space $\mathcal{H}^1$ will be used to lift the weak formulations of (14) and (15) to the probability space and ultimately to construct the horizontal gradient of the approximate flux correction.

Henceforth, we fix $i \in \{1, \dots, d\}$ and $n \geq 0$, and to simplify notation suppress the dependence on both indices in the argument to follow. Consider the closed subspace of $L^2(\Omega)$ defined by

$$X = \big\{\{S_{ljk}\}_{l,j,k=1,\dots,d} \in L^2\big(\Omega; \mathbb{R}^{d^3}\big) \mid S_{ljk} + S_{lkj} = 0,\\ \partial_m S_{ljk} = \partial_l S_{mjk} \text{ and } \langle S_{ijk}\rangle = 0\big\},$$

where for a random variable ζ and for every $i \in \{1, \dots, d\}$, the notation $\partial_i \zeta$ denotes the distributional derivative of ζ defined by

$$\langle \partial_i \zeta \chi\rangle = -\langle \zeta D_i \chi\rangle \qquad \text{for every } \chi \in \mathcal{H}^1.$$

We observe that X is a Hilbert space with respect to the standard inner product on $L^2(\Omega; \mathbb{R}^{d^3})$ and that, formally, we expect the gradient $\{\partial_l \sigma_{ijk}\}$ to be an element of X where $S_{ljk} := \partial_l \sigma_{ijk}$.

Interpreting equation (15) on the space X, Riesz' representation theorem and the boundedness of q yield a unique element $\{\overline{S}_{ljk}\} \in X$ satisfying

$$\langle \overline{S}_{jkl} S_{jkl}\rangle = -2\langle q_k S_{jjk}\rangle \qquad \text{for every } \{S_{ljk}\}_{l,j,k=1,\dots,d} \in X, \tag{16}$$

where we have employed Einstein's summation convention and $\langle\cdot\rangle$ denotes the standard inner product on $L^2(\Omega)$.

In order to verify (15), it is necessary to prove that, in the sense of distributions, and again employing Einstein's summation convention, for each $j, k \in \{1, \dots, d\}$,

$$-\partial_l \overline{S}_{ljk} = \partial_j q_k - \partial_k q_j. \tag{17}$$

As mentioned above, for any skew-symmetric $\{\eta_{jk}\}_{j,k=1,\dots,d} \in \mathcal{H}^1$ the corresponding gradient satisfies $\{D_l \eta_{jk}\}_{l,j,k=1,\dots,d} \in X$. Therefore, for an arbitrary such $\{\eta_{jk}\}$, equation (16) implies that

$$\begin{aligned}\langle \overline{S}_{ljk} D_l \eta_{jk}\rangle &= -2\langle q_k D_j \eta_{jk}\rangle = -\langle q_k D_j \eta_{jk}\rangle - \langle q_j D_k \eta_{kj}\rangle\\ &= -\langle q_k D_j \eta_{jk}\rangle + \langle q_j D_k \eta_{jk}\rangle,\end{aligned}$$

which, since the skew-symmetric $\{\eta_{jk}\}_{j,k=1,\dots,d}$ was arbitrary and such functions are dense in X, completes the proof of (17).

It remains to prove (14) which, when interpreted on the space X turns for each $j \in \{1, \dots, d\}$ into

$$\overline{S}_{kjk} = q_j - \langle q_j \rangle. \tag{18}$$

And for this, since $\langle \overline{S}_{ljk} \rangle = 0$ for every $l, k, j \in \{1, \dots, d\}$, the ergodicity implies that it is sufficient to prove that, in the sense of distributions,

$$\partial_l \partial_l (\overline{S}_{kjk} - q_j) = 0. \tag{19}$$

But this follows immediately from the properties of X and (17), which provide the distributional equality

$$\partial_l \partial_l \overline{S}_{kjk} = \partial_l \partial_k \overline{S}_{ljk} = \partial_k \partial_l \overline{S}_{ljk} \overset{(17)}{=} \partial_k \partial_k q_j - \partial_k \partial_j q_k = \partial_k \partial_k q_j - \partial_j \partial_k q_k = \partial_l \partial_l q_j,$$

where the final inequality is obtained using the fact that q is divergence free. This completes the argument for (19) and therefore (18) as well.

To conclude, recalling that $i \in \{1, \dots, d\}$ was fixed throughout, the gradient is defined for each $l, j, k \in \{1, \dots, d\}$ as

$$\partial_l \sigma_{n,ijk} := \overline{S}_{ljk},$$

which in turn defines each component of the flux correction $\sigma_{n,i}$ and therefore the flux correction σ_n itself uniquely up to a random but spatially constant, skew-symmetric vector. This finishes the proof of the existence.

To complete the proof of the lemma, it suffices to prove the uniform in n estimates for the expectation $\langle |\nabla \sigma_{n,ijk}|^{\frac{2p}{p+1}} \rangle$. The result then follows by taking the limit $n \to \infty$. More generally, given two random fields $f \in L^\infty(\Omega; \mathbb{R}^d)$ and σ, such that $\langle \nabla \sigma \rangle = 0$, $\nabla \sigma \in L^2(\Omega)$ is stationary, σ is skew-symmetric, and σ and f are related through

$$-\Delta \sigma = -\nabla \cdot f, \tag{20}$$

it is enough to show a Calderón–Zygmund-type estimate

$$\langle |\nabla \sigma|^r \rangle \le C(d, r) \langle |f|^r \rangle, \tag{21}$$

for general $1 < r < \infty$.

For $R, T > 0$ we consider $\sigma_{T,R}$, an approximation of σ, defined as a unique finite energy solution of

$$\frac{1}{T} \sigma_{T,R} - \Delta \sigma_{T,R} = -\nabla \cdot (\eta_R f),$$

where η_R is a radial cut-off function for $\{|x| < R\}$ in $\{|x| < 2R\}$. The additional term $\frac{1}{T} \sigma_{T,R}$ is called massive term, and localizes (up to an exponentially decay) the spatial dependence of the solution on the right-hand side. In the physics community, the above equation is called a *screened Poisson equation*. By the standard Calderón–Zygmund theory of singular integral operators, applied to the massive Green's function (in fact its second mixed derivative), we get an estimate, independently of T:

$$\int_{\mathbb{R}^d} |\nabla \sigma_{T,R}|^r \le C(d,r) \int_{\mathbb{R}^d} |\eta_R f|^r \le C(d,r) \int_{B_{2R}} |f|^r. \tag{22}$$

We fix $T > 0$, and for $R' \ge R \gg \sqrt{T}$ we consider the difference $\sigma_{T,R}(x) - \sigma_{T,R'}(x)$ for points $x \in B_{R/2}(0)$. From the pointwise estimates on the massive Green's function G_T (see, e.g., [20], Section 2.3) of the form

$$\left|\nabla G_T(x,y)\right| \le C \frac{e^{-c\frac{1}{\sqrt{T}}|x-y|}}{|x-y|^{d-1}},$$

$$\left|\nabla_x \nabla_y G_T(x,y)\right| \le C \frac{e^{-c\frac{1}{\sqrt{T}}|x-y|}}{|x-y|^{d}},$$

we get that for $x \in B_{R/2}(0)$

$$\begin{aligned} &R^{-1}\left|(\sigma_{T,R} - \sigma_{T,R'})(x)\right| + \left|\nabla(\sigma_{T,R} - \sigma_{T,R'})(x)\right| \\ &\quad \le C \frac{\sqrt{T}}{R} e^{-cR/\sqrt{T}} \|f\|_{L^\infty} \le C e^{-cR/\sqrt{T}\|f\|_{L^\infty}}. \end{aligned} \tag{23}$$

In particular, in the limit $R \to \infty$ we have that $\sigma_{T,R}$ converges (pointwise) to σ_T, where σ_T is stationary and satisfies

$$\frac{1}{T}\sigma_T - \Delta \sigma_T = -\nabla \cdot f.$$

Moreover, estimate (23) with $\sigma_{T,R'}$ replaced by σ_T (estimate (23) does not depend on R', and so we are allowed to perform the limit $R' \to \infty$) in particular implies

$$\int_{B_{R/2}(0)} |\nabla \sigma_{T,R} - \nabla \sigma_T|^r \le C R^d e^{-crR/\sqrt{T}} \|f\|_{L^\infty}^r.$$

We combine this estimate with (22) to arrive at

$$\fint_{B_{R/2}(0)} |\nabla \sigma_T|^r \le C e^{-crR/\sqrt{T}} \|f\|_{L^\infty}^r + C \fint_{B_{2R}(0)} |f|^r.$$

Then by the ergodic theorem, as $R \to \infty$, the left-hand side converges to $\langle |\nabla \sigma_T|^r \rangle$ while the right-hand side converges to $\langle |f|^r \rangle$, that is, we obtain

$$\langle |\nabla \sigma_T|^r \rangle \le C \langle |f|^r \rangle.$$

Finally, since the sequence $\nabla \sigma_T$ is bounded in $L^r(\Omega)$, we can send $T \to \infty$ and obtain in the limit $\nabla \sigma$ which satisfies (20) and (21).

It remains to establish the ellipticity of the homogenized coefficient field a_{hom}. For the lower bound, we observe that for an arbitrary $\xi \in \mathbb{R}^d$,

$$\xi \cdot a_{\mathrm{hom}}\xi = \langle \xi \cdot a(\nabla\phi_\xi + \xi)\rangle = \langle (\nabla\phi_\xi + \xi) \cdot a(\nabla\phi_\xi + \xi)\rangle,$$

where the final equality follows from the definition of the corrector ϕ_ξ. Therefore, by the definition of λ in (2) we get

$$\xi \cdot a_{\mathrm{hom}}\xi \geq \langle \lambda|\nabla\phi_\xi + \xi|^2\rangle \geq \langle \lambda^{-1}\rangle^{-1}|\xi|^2 \geq \frac{1}{K}|\xi|^2,$$

where the last but one inequality follows from Jensen's inequality used for a jointly convex function $(f,g) \mapsto f^2/g$ with the choice $(f,g) = (\nabla\phi_\xi + \xi, \lambda^{-1})$, and the fact that $\langle \nabla\phi_\xi\rangle = 0$. For the upper bound, for an arbitrary $\xi \in \mathbb{R}^d$, using the definition of μ from (2),

$$|a_{\mathrm{hom}}\xi| = \left|\langle a(\nabla\phi_\xi + \xi)\rangle\right| \leq \langle |a(\nabla\phi_\xi + \xi)|\rangle \leq \langle \mu^{\frac{1}{2}}\big((\nabla\phi_\xi + \xi) \cdot a(\nabla\phi_\xi + \xi)\big)^{\frac{1}{2}}\rangle.$$

Then, after an application of Hölder's inequality, the definition of the corrector ϕ_ξ implies that

$$\begin{aligned} |a_{\mathrm{hom}}\xi| &\leq \langle\mu\rangle^{\frac{1}{2}}\langle(\nabla\phi_\xi + \xi) \cdot a(\nabla\phi_\xi + \xi)\rangle^{\frac{1}{2}} \\ &= \langle\mu\rangle^{\frac{1}{2}}\langle \xi \cdot a(\nabla\phi_\xi + \xi)\rangle^{\frac{1}{2}} \leq K^{\frac{1}{2}}|\xi|^{\frac{1}{2}}|a_{\mathrm{hom}}\xi|^{\frac{1}{2}}. \end{aligned} \tag{24}$$

Dividing by $|a_{\mathrm{hom}}\xi|^{\frac{1}{2}}$ yields the desired upper bound, and completes the proof.

4. Proof of Lemma 2. To prove the sublinearity of the correctors ϕ and σ we will only use that their gradients are stationary fields with zero expectation and that they have bounded $\frac{2q}{q+1}$ and $\frac{2p}{p+1}$ moments, respectively. Hence, we will only show the argument for ϕ, the argument for σ being analogous (after swapping p and q).

Concerning the corrector ϕ, it is our aim to prove that

$$\lim_{R\to\infty} \frac{1}{R}\left(\fint_{B_R}\left|\phi - \fint_{B_R}\phi\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} = 0. \tag{25}$$

Our proof is a simplified version of the proof of a similar, seemingly slightly stronger, property (see [12], Lemma 5.1):

$$\lim_{R\to\infty} \frac{1}{R}\left(\fint_{B_R}|\phi|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} = 0. \tag{26}$$

Before we prove (25), we point out that in fact it is equivalent (26). Indeed, assuming (25) for any $\delta > 0$ we find $r_0 > 0$ such that for all $R \geq r_0$

$$\left(\fint_{B_R}\left|\phi - \fint_{B_R}\phi\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} \leq \delta R,$$

from where by the triangle inequality we get for any $R \ge r_0$ and $R' \in [R, 2R]$

$$\left|\fint_{B_R} \phi - \fint_{B_{R'}} \phi\right| \le \left(\fint_{B_R}\left|\phi - \fint_{B_R}\phi\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} + \left(\fint_{B_R}\left|\phi - \fint_{B_{R'}}\phi\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} \\ \le C\delta R.$$

Hence by the dyadic argument we see that $|\fint_{B_{r_0}} \phi - \fint_{B_R} \phi| \le C\delta R$, which implies for $R \ge r_0$

$$\frac{1}{R}\left(\fint_{B_R} |\phi|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} \le \frac{1}{R}\left(\fint_{B_R}\left|\phi - \fint_{B_R}\phi\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} + C\delta + \frac{1}{R}\left|\fint_{B_{r_0}} \phi\right|,$$

from where we get that $\limsup_{R\to\infty} \frac{1}{R}(\fint_{B_R} |\phi|^{\frac{2p}{p-1}})^{\frac{p-1}{2p}} \le C\delta$, and (26) immediately follows.

Let us now show the argument for (25), which is essentially an immediate consequence of the ergodic theorem, the Sobolev/Rellich-Kondrachov embedding and our assumption

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{d}. \tag{27}$$

Fix $i \in \{1, \dots, d\}$ and consider, for each $\varepsilon \in (0, 1)$, the rescaling $\phi_i^\varepsilon(\cdot) = \varepsilon\phi_i(\frac{\cdot}{\varepsilon})$. Assumption (27) and the Sobolev embedding theorem imply that, for each $\varepsilon \in (0, 1)$,

$$\left(\int_{B_1}\left|\phi_i^\varepsilon - \fint_{B_1}\phi_i^\varepsilon\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} \lesssim \|\nabla\phi_i^\varepsilon\|_{L^{\frac{2q}{q+1}}(B_1)}. \tag{28}$$

Since the estimates contained in (6) and the ergodic theorem coupled with the stationarity and ergodicity of the environment imply that, for $\langle\cdot\rangle$-a.e. a, the gradient $\nabla\phi_i^\varepsilon$ converges weakly to zero in $L^{\frac{2q}{q+1}}(B_1)$, we have for the renormalizations

$$\left(\phi_i^\varepsilon - \fint_{B_1}\phi_i^\varepsilon\right) \rightharpoonup 0 \qquad \text{weakly in } W^{1,\frac{2q}{q+1}}(B_1).$$

To see this, observe that estimate (28) and the weak convergence of the gradients imply that the sequence $\{\phi_i^\varepsilon - \fint_{B_1}\phi_i^\varepsilon\}_{\varepsilon\in(0,1)}$ is bounded in $W^{1,\frac{2q}{q+1}}(B_1)$ since $\frac{2q}{q+1} < \frac{2p}{p-1}$. Therefore, after passing to a subsequence, the $(\phi_i^\varepsilon - \fint_{B_1}\phi_i^\varepsilon)$ converge weakly to some $f \in W^{1,\frac{2q}{q+1}}(B_1)$ with average zero and vanishing gradient. Hence, the subsequence converges weakly to zero and, owing to the uniqueness of the limit, the convergence of the full sequence follows.

Finally, since the weak convergence and (28) imply that, for $\langle\cdot\rangle$-a.e. environment, the sequence $\{(\phi_i^\varepsilon - \fint_{B_1}\phi_i^\varepsilon)\}_{\varepsilon\in(0,1)}$ is bounded in $W^{1,\frac{2q}{q+1}}(B_1)$, the compactness of the embedding $W^{1,\frac{2q}{q+1}}(B_1) \hookrightarrow L^{\frac{2p}{p-1}}(B_1)$, owing to the strict inequality in

(27), implies for $\langle\cdot\rangle$-a.e. a, the strong convergence

$$0 = \lim_{\varepsilon\to 0}\biggl(\int_{B_1}\biggl|\phi_i^\varepsilon - \fint_{B_1}\phi_i^\varepsilon\biggr|^{\frac{2p}{p-1}}\biggr)^{\frac{p-1}{2p}} = \lim_{R\to 0}\frac{1}{R}\biggl(\fint_{B_R}\biggl|\phi_i - \fint_{B_R}\phi_i\biggr|^{\frac{2p}{p-1}}\biggr)^{\frac{p-1}{2p}}.$$

This, since $i \in \{1, \dots, d\}$ was arbitrary, completes the argument for ϕ.

5. Proof of Theorem 2. The strategy of the proof of the theorem is very similar to the proof of [21], Lemma 2. The idea is to first show decay of excess for one value θ_0 of the ratio r/R, and then iterate this estimate to show excess decay for all values r/R.

To show the decay for a fixed value of r/R, the idea is to estimate the homogenization error in B_R determined by the difference between the a-harmonic function u and a correction of an appropriately chosen a_{hom}-harmonic function to be denoted v. In the uniformly elliptic setting, following the arguments of [21], the boundary values of the a_{hom}-harmonic function can be chosen to coincide with u on a sphere with generic radius close to R. In the nonuniformly elliptic case, it is necessary, as explained in Step 2 of our arguments, to consider a v which agrees on a generic sphere with u_ε in the "Dirichlet Case" $q \ge p$ and which satisfies $\nu \cdot a_{\mathrm{hom}}\nabla v = (\nu \cdot a\nabla u)_\varepsilon$ in the "Neumann Case" $p \ge q$, where the subscript ε denotes the convolution at scale ε with a smoothing kernel on the sphere.

The corresponding augmented homogenization error will be defined for an appropriately chosen cutoff function η in B_R as $w := u - (1 + \eta\phi_i\partial_i)v$. In Step 1, as in [21], we derive the equation satisfied by w, and in Step 2 use this equation to obtain energy estimates for the homogenization error, without the cutoff η, on a smaller ball. The argument is concluded in Steps 3, 4 and 5, where the iterative argument of [21] is used to obtain the statement on excess decay.

Step 1. Let u be an a-harmonic function in B_1. In this step we consider the augmented homogenization error

$$w := u - (1 + \eta\phi_i\partial_i)v,$$

defined by a smooth function η and an a_{hom}-harmonic function v in B_1. It will be shown now that w solves the divergence-form equation

$$-\nabla\cdot a\nabla w = \nabla\cdot\bigl((1-\eta)(a - a_{\mathrm{hom}})\nabla v + (\phi_i a - \sigma_i)\nabla(\eta\partial_i v)\bigr) \qquad \text{in } B_1, \tag{29}$$

where the crucial ingredient of the proof is the skew-symmetric flux correction σ. We remark that the flux correction was used previously in the context of periodic homogenization (see, e.g., [22]), and in stochastic homogenization it was introduced only recently in [21], where (29) was first derived.

For the convenience of the reader, we repeat here the computation leading to (29), and to keep the notation lean, in this and the following steps, we will without loss of generality assume that the components of ϕ and σ have zero spatial average on B_1. Otherwise, for each $i \in \{1, \dots, d\}$, we would replace ϕ_i with $(\phi_i - \fint_{B_1}\phi_i)$, and similarly for σ.

First, compute the gradient of w to find

$$\nabla w = \nabla u - \big(\nabla v + \eta \partial_i v \nabla \phi_i + \phi_i \nabla(\eta \partial_i v)\big),$$

then we use the a-harmonicity of u to obtain

$$-\nabla \cdot a\nabla w = \nabla \cdot a\nabla v + \nabla \cdot a\big(\eta\partial_i v \nabla\phi_i + \phi_i \nabla(\eta\partial_i v)\big).$$

Since

$$\nabla \cdot a(\eta\partial_i v\nabla\phi_i) = \nabla \cdot \big(\eta\partial_i v a(\nabla\phi_i + e_i)\big) - \nabla \cdot \eta a\nabla v,$$

the vanishing divergence $-\nabla \cdot a(\nabla\phi_i + e_i) = 0$ implies that

$$-\nabla \cdot a\nabla w = \nabla \cdot (1-\eta)a\nabla v + \nabla(\eta\partial_i v) \cdot a(\nabla\phi_i + e_i) + \nabla \cdot \big(\phi_i a\nabla(\eta\partial_i v)\big).$$

Then, after observing both that

$$a_{\mathrm{hom}} e_i \cdot \nabla(\eta\partial_i v) = \nabla \cdot (\eta\partial_i v a_{\mathrm{hom}} e_i) = \nabla \cdot (\eta a_{\mathrm{hom}} \nabla v)$$

and, since $-\nabla \cdot a_{\mathrm{hom}}\nabla v = 0$, that

$$\nabla \cdot (\eta a_{\mathrm{hom}}\nabla v) = -\nabla \cdot (1-\eta)\nabla v,$$

we have

$$\begin{aligned}-\nabla \cdot a\nabla w = {}& \nabla \cdot \big((1-\eta)(a - a_{\mathrm{hom}})\nabla v\big)\\ & + \nabla(\eta\partial_i v) \cdot \big(a(\nabla\phi_i + e_i) - a_{\mathrm{hom}} e_i\big) + \nabla \cdot \big(\phi_i a\nabla(\eta\partial_i v)\big).\end{aligned}$$

The skew-symmetry of the flux correction σ now plays a role. Since

$$\nabla \cdot \sigma_i = q_i = a(\phi_i + e_i) - a_{\mathrm{hom}} e_i,$$

we have, for an arbitrary test function ζ, the distributional identity

$$\nabla\zeta \cdot (\nabla \cdot \sigma_i) = \partial_j\zeta\, \partial_k\sigma_{ijk} = \partial_k(\partial_j\zeta\sigma_{ijk}) = \partial_k(\sigma_{ijk}\partial_j\zeta) = -\nabla \cdot (\sigma_i\nabla\zeta),$$

from which (29) follows. This completes the proof of this step.

Step 2. The boundary conditions for v and the cutoff η are now specified in order to use equation (29) to obtain an energy estimate for the homogenization error. We remark that the arguments will be carried out for the unit ball B_1, and the general statement will be obtained by scaling. We assume that

$$\left(\int_{B_1} \lambda^{-q}\right)^{\frac{1}{q}} + \left(\int_{B_1} \mu^p\right)^{\frac{1}{p}} \le \Lambda \qquad \text{where } \frac{1}{p} + \frac{1}{q} \le \frac{2}{d}, \tag{30}$$

with

$$\lambda := \inf_{\xi\in\mathbb{R}^d} \frac{a\xi\cdot\xi}{|\xi|^2} \quad \text{and} \quad \mu := \sup_{\xi\in\mathbb{R}^d} \frac{|a\xi|^2}{a\xi\cdot\xi}, \tag{31}$$

and consider an a-harmonic function u in B_1, that is,

$$-\nabla \cdot a\nabla u = 0 \qquad \text{in } B_1. \tag{32}$$

We will construct an a_{hom}-harmonic function v in $B_{\frac{1}{2}}$ satisfying

$$\int_{B_{\frac{1}{2}}} \nabla v \cdot a_{\mathrm{hom}} \nabla v \lesssim \Lambda \int_{B_1} \nabla u \cdot a \nabla u =: \overline{\Lambda}, \tag{33}$$

and for which the homogenization error $w := u - (1 + \phi_i \partial_i) v$ satisfies

$$\begin{aligned} \int_{B_{\frac{1}{4}}} \nabla w \cdot a \nabla w \lesssim{}& \overline{\Lambda} \varepsilon^{1-(\frac{1}{2p}+\frac{1}{2q})(d-1)} \\ &+ \Lambda \overline{\Lambda} \rho^{\min\{\frac{p-1}{2p}, \frac{q-1}{2q}\}} \frac{1}{\varepsilon^{\min\{\frac{q+1}{q}, \frac{p+1}{p}\}(d-1)}} \\ &+ \Lambda \overline{\Lambda} \frac{1}{\rho^{d+2}} \biggl(\Bigl(\int_{B_1} |\phi|^{\frac{2p}{p-1}} \Bigr)^{\frac{p-1}{p}} + \Bigl(\int_{B_1} |\sigma|^{\frac{2q}{q-1}} \Bigr)^{\frac{q-1}{q}} \biggr), \end{aligned} \tag{34}$$

for any fudge factors $\varepsilon \in (0, 1]$ and $\rho \in (0, \frac{1}{8})$. We recall that $\lesssim$ denotes a constant depending only upon the dimension d and the constant K from (2) through the ellipticity of the homogenized coefficients.

We begin now with the construction of v which will in fact be an a_{hom}-harmonic on a somewhat larger ball B_r, for some suitably chosen radius $r \in [\frac{1}{2}, 1]$:

$$-\nabla \cdot a_{\mathrm{hom}} \nabla v = 0 \qquad \text{on } B_r. \tag{35}$$

The idea is to distinguish the two cases

$$\text{the "Dirichlet case" } q \geq p \text{ and the "Neumann case" } p \geq q. \tag{36}$$

In the Dirichlet case, we define v via the Dirichlet boundary condition

$$v = u_\varepsilon \qquad \text{on } \partial B_r, \tag{37}$$

whereas in the Neumann case, we impose

$$\nu \cdot a_{\mathrm{hom}} \nabla v = (\nu \cdot a \nabla u)_\varepsilon - \fint_{B_r} (\nu \cdot a \nabla u)_\varepsilon \qquad \text{on } \partial B_r. \tag{38}$$

Here the subscript ε stands for a convolution on ∂B_r with scale $\varepsilon > 0$.

Since Hölder's inequality, (30) and (31) imply

$$\begin{aligned} &\Bigl(\int_{B_1} |\nabla u|^{\frac{2q}{q+1}} \Bigr)^{\frac{q+1}{2q}} + \Bigl(\int_{B_1} |a \nabla u|^{\frac{2p}{p+1}} \Bigr)^{\frac{p+1}{2p}} \\ &\quad \leq \biggl(\Bigl(\int_{B_1} \lambda^{-q} \Bigr)^{\frac{1}{2q}} + \Bigl(\int_{B_1} \mu^p \Bigr)^{\frac{1}{2p}} \biggr) \Bigl(\int_{B_1} \nabla u \cdot a \nabla u \Bigr)^{\frac{1}{2}} \lesssim \overline{\Lambda}^{\frac{1}{2}}, \end{aligned}$$

we can find a radius $r \in [\frac{1}{2}, 1]$ such that both the field and the current of u have the same integrability on ∂B_r as on B_1, in the sense that

$$\left(\int_{\partial B_r} |\nabla u|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{2q}} + \left(\int_{\partial B_r} |a\nabla u|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}} \lesssim \overline{\Lambda}^{\frac{1}{2}}. \tag{39}$$

Using that both estimates are preserved by convolution, it follows that

$$\left(\int_{\partial B_r} |\nabla^{\tan} v|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{2q}} \lesssim \overline{\Lambda}^{\frac{1}{2}} \qquad \text{in the Dirichlet case}$$

and

$$\left(\int_{\partial B_r} |\nu \cdot a_{\mathrm{hom}} \nabla v|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}} \lesssim \overline{\Lambda}^{\frac{1}{2}} \qquad \text{in the Neumann case.}$$

By constant-coefficient elliptic theory applied to the Dirichlet-to-Neumann operator (see, e.g. Fabes, Jodeit and Rivière [16], Theorem 2.4, Theorem 2.6, and Stein [29], Chapter 7), see also the proof of Lemma 4 in the Appendix for the extension of these results to the cube, this yields

$$\left(\int_{\partial B_r} |\nabla v|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{2q}} \lesssim \overline{\Lambda}^{\frac{1}{2}} \qquad \text{in the Dirichlet case} \tag{40}$$

and

$$\left(\int_{\partial B_r} |\nabla v|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}} \lesssim \overline{\Lambda}^{\frac{1}{2}} \qquad \text{in the Neumann case.} \tag{41}$$

These estimates motivate the case distinction (36), which we now use to prove (33). To simplify the notation, we assume without loss of generality that v has zero average on ∂B_r, and test (35) with v to obtain

$$\int_{B_r} \nabla v \cdot a_{\mathrm{hom}} \nabla v = \int_{\partial B_r} v(\nu \cdot a_{\mathrm{hom}} \nabla v).$$

Using Hölder's inequality, the Sobolev embedding theorem and assumption (2), we have

$$\begin{aligned}\int_{B_r} \nabla v \cdot a_{\mathrm{hom}} \nabla v &\lesssim \left(\int_{\partial B_r} |\nabla v|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{2q}} \left(\int_{\partial B_r} |v|^{\frac{2q}{q-1}}\right)^{\frac{q-1}{2q}} \\ &\lesssim \left(\int_{\partial B_r} |\nabla v|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{2q}} \left(\int_{\partial B_r} |\nabla v|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}}.\end{aligned}$$

Therefore, in view of (40), (41) and the case distinction (36), we conclude that

$$\int_{B_r} \nabla v \cdot a_{\mathrm{hom}} \nabla v \lesssim \overline{\Lambda}, \tag{42}$$

which implies (33) since $r \in [\frac{1}{2}, 1]$.

We now specify precisely the cutoff function defining the augmented homogenization error. Let $0 < \rho < \frac{1}{8}$ to be fixed later and let $0 \le \eta \le 1$ be a smooth function satisfying

$$\eta = \begin{cases} 1 & \text{on } \overline{B}_{r-2\rho}, \\ 0 & \text{on } B_r \setminus B_{r-\rho}, \end{cases} \qquad \text{with } |\nabla \eta| \lesssim \frac{1}{\rho}. \tag{43}$$

The augmented homogenization error is then defined as

$$w = u - (1 + \eta \phi_i \partial_i) v,$$

and we recall from Step 1 [see (29)] the equation

$$-\nabla \cdot a \nabla w = -\nabla \cdot \big((1-\eta)(a_{\mathrm{hom}} - a)\nabla v\big) + \nabla \cdot \big((\phi_i a - \sigma_i)\nabla(\eta \partial_i v)\big) \qquad \text{in } B_r,$$

which requires both (32) and (35), and test this equation with w to obtain

$$\begin{aligned} \int_{B_r} \nabla w \cdot a \nabla w = {} & \int_{\partial B_r} (u - v)\nu \cdot (a\nabla u - a_{\mathrm{hom}} \nabla v) \\ & + \int_{B_r} (1-\eta) \nabla w \cdot (a_{\mathrm{hom}} - a) \nabla v \\ & - \int_{B_r} \nabla w \cdot (\phi_i a - \sigma_i) \nabla(\eta \partial_i v). \end{aligned}$$

Appealing to (31), we obtain by Young's inequality

$$\begin{aligned} \int_{B_r} \nabla w \cdot a \nabla w \lesssim {} & \left| \int_{\partial B_r} (u - v)\nu \cdot (a\nabla u - a_{\mathrm{hom}} \nabla v) \right| \\ & + \int_{B_r} (1-\eta)^2 \Big(\mu + \frac{1}{\lambda}\Big) |\nabla v|^2 \\ & + \int_{B_r} \Big(\mu \phi_i^2 + \frac{1}{\lambda} |\sigma_i|^2\Big) |\nabla(\eta \partial_i v)|^2. \end{aligned}$$

Then, using the properties of the cutoff η from (43),

$$\begin{aligned} \int_{B_r} \nabla w \cdot a \nabla w \lesssim {} & \left| \int_{\partial B_r} (u - v)\nu \cdot (a\nabla u - a_{\mathrm{hom}} \nabla v) \right| \\ & + \int_{B_r \setminus B_{r-2\rho}} \Big(\mu + \frac{1}{\lambda}\Big) |\nabla v|^2 \\ & + \int_{B_1} \Big(\mu |\phi|^2 + \frac{1}{\lambda} |\sigma|^2\Big) \sup_{B_{r-\rho}} \Big(|\nabla^2 v| + \frac{1}{\rho} |\nabla v|\Big)^2. \end{aligned}$$

In view of (30), this yields by Hölder's inequality

$$\int_{B_r} \nabla w \cdot a \nabla w \lesssim \left| \int_{\partial B_r} (u-v)\nu \cdot (a\nabla u - a_{\mathrm{hom}} \nabla v) \right| \tag{44}$$

$$+ \Lambda \rho^{\min\{\frac{p-1}{2p}, \frac{q-1}{2q}\}} \left(\int_{B_r} |\nabla v|^{\max\{\frac{4p}{p-1}, \frac{4q}{q-1}\}} \right)^{\min\{\frac{p-1}{2p}, \frac{q-1}{2q}\}} \tag{45}$$

$$+ \Lambda \left(\left(\int_{B_1} |\phi|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{p}} + \left(\int_{B_1} |\sigma|^{\frac{2q}{q-1}} \right)^{\frac{q-1}{q}} \right) \sup_{B_{r-\rho}} \left(|\nabla^2 v| + \frac{1}{\rho} |\nabla v| \right)^2. \tag{46}$$

For the last term, we use multiple times the Caccioppoli inequality for uniformly elliptic systems (see [19], Theorem 7.1), the uniform ellipticity of the homogenized coefficient field, the Sobolev embedding theorem, and (35) to obtain

$$\sup_{B_{r-\rho}} \left(|\nabla^2 v| + \frac{1}{\rho} |\nabla v| \right)^2 \lesssim \frac{1}{\rho^{d+2}} \int_{B_r} \nabla v \cdot a_{\mathrm{hom}} \nabla v,$$

where the full argument can be found in the proof of [21], Lemma 3. Therefore, in view of (42), for both the Dirichlet and Neumann cases we have

$$\sup_{B_{r-\rho}} \left(|\nabla^2 v| + \frac{1}{\rho} |\nabla v| \right)^2 \lesssim \frac{1}{\rho^{d+2}} \overline{\Lambda}. \tag{47}$$

We now turn to the middle right-hand side term (45). It follows from the L^p-theory for constant-coefficient elliptic systems (see [19], Theorem 7.1) and an explicit radial extension of the smooth boundary data u_ε into the ball B_r that, in the Dirichlet case,

$$\left(\int_{B_r} |\nabla v|^{\max\{\frac{4p}{p-1}, \frac{4q}{q-1}\}} \right)^{\min\{\frac{p-1}{2p}, \frac{q-1}{2q}\}} = \left(\int_{B_r} |\nabla v|^{\frac{4q}{q-1}} \right)^{\frac{q-1}{2q}} \lesssim \left(\int_{\partial B_r} |\nabla^{\tan} u_\varepsilon|^{\frac{4q}{q-1}} \right)^{\frac{q-1}{2q}} \lesssim \left(\sup_{\partial B_r} |\nabla^{\tan} u_\varepsilon| \right)^2.$$

We then use Hölder's inequality and the definition of the convolution to obtain

$$\sup_{\partial B_r} |\nabla^{\tan} u_\varepsilon| \lesssim \frac{1}{\varepsilon^{\frac{q+1}{2q}(d-1)}} \left(\int_{\partial B_r} |\nabla u|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}} \qquad \text{in the Dirichlet case,}$$

so that together with (39) we obtain

$$\left(\int_{B_r} |\nabla v|^{\max\{\frac{4p}{p-1}, \frac{4q}{q-1}\}} \right)^{\min\{\frac{p-1}{2p}, \frac{q-1}{2q}\}} \lesssim \frac{1}{\varepsilon^{\frac{q+1}{q}(d-1)}} \overline{\Lambda} \qquad \text{in the Dirichlet case.}$$

The analogous estimates for the Neumann case yield, with help of (39),

$$\left(\fint_{B_r} |\nabla v|^{\max\{\frac{4p}{p-1},\frac{4q}{q-1}\}}\right)^{\min\{\frac{p-1}{2p},\frac{q-1}{2q}\}} = \left(\fint_{B_r} |\nabla v|^{\frac{4p}{p-1}}\right)^{\frac{p-1}{2p}} \lesssim \frac{1}{\varepsilon^{\frac{p+1}{p}(d-1)}}\overline{\Lambda},$$

so that, in combination, both cases satisfy

$$\left(\fint_{B_r} |\nabla v|^{\max\{\frac{4p}{p-1},\frac{4q}{q-1}\}}\right)^{\min\{\frac{p-1}{2p},\frac{q-1}{2q}\}} \lesssim \frac{1}{\varepsilon^{\min\{\frac{q+1}{q},\frac{p+1}{p}\}(d-1)}}\overline{\Lambda}. \tag{48}$$

It remains to treat the boundary term (44). We first treat the Neumann case (38), for which we may appeal to the symmetry of the convolution operator to write

$$\int_{\partial B_r} (u-v)\nu\cdot(a\nabla u - a_{\text{hom}}\nabla v) = \int_{\partial B_r} \big((u-v)-(u-v)_\varepsilon\big)\nu\cdot a\nabla u,$$

so that we obtain by Hölder's inequality together with (39)

$$\begin{aligned}&\left|\int_{\partial B_r} (u-v)\nu\cdot(a\nabla u - a_{\text{hom}}\nabla v)\right| \\ &\quad\lesssim \overline{\Lambda}^{\frac{1}{2}}\left(\left(\fint_{\partial B_r} |u-u_\varepsilon|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} + \left(\fint_{\partial B_r} |v-v_\varepsilon|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}}\right).\end{aligned} \tag{49}$$

We then interpolate between two estimates. First, we will use the convolution estimate

$$\left(\fint_{\partial B_r} |u-u_\varepsilon|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} \lesssim \varepsilon\left(\fint_{\partial B_r} \left|\nabla^{\tan} u\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}}, \tag{50}$$

which, for an arbitrary convolution kernel $\rho^\varepsilon = \varepsilon^{-(d-1)}\rho(\frac{x}{\varepsilon})$ and function z, follows on the sphere in analogy with the Euclidean computation (the additional difficulties being solely notational)

$$\begin{aligned}&\left(\int_{\mathbb{R}^{d-1}} |z-z_\varepsilon|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} \\ &\quad= \left(\int_{\mathbb{R}^{d-1}} \left|\int_{\mathbb{R}^{d-1}} \rho(x)\big(z(y+\varepsilon x)-z(y)\big)\,\mathrm{d}x\right|^{\frac{2p}{p-1}}\mathrm{d}y\right)^{\frac{p-1}{2p}} \\ &\quad\le \varepsilon\int_0^1\int_{\mathbb{R}^{d-1}}\left(\int_{\mathbb{R}^{d-1}} \rho^{\frac{2p}{p-1}}(x)|x|^{\frac{2p}{p-1}}\left|\nabla z(y+s\varepsilon x)\right|^{\frac{2p}{p-1}}\mathrm{d}y\right)^{\frac{p-1}{2p}}\mathrm{d}x\,\mathrm{d}s \\ &\quad\lesssim \varepsilon\left(\int_{\mathbb{R}^{d-1}} |\nabla z|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}},\end{aligned} \tag{51}$$

where the first inequality is obtained using the representation of the difference as the integral of the gradient and Minkowski's integral inequality, and the final constant depends on the fixed convolution kernel ρ. Second, we will use the estimate

$$\begin{aligned}
&\Big(\int_{\partial B_r} |u-u_\varepsilon|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} \\
&\quad \le \Big(\int_{\partial B_r} \Big|u - \fint_{B_r} u\Big|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} + \Big(\int_{\partial B_r} \Big|u_\varepsilon - \fint_{B_r} u^\varepsilon\Big|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} \\
&\quad \lesssim \Big(\int_{\partial B_r} |\nabla^{\tan} u|^s\Big)^{\frac{1}{s}},
\end{aligned} \tag{52}$$

for $\frac{1}{s} = \frac{p-1}{2p} + \frac{1}{d-1}$, which follows from the triangle and Sobolev inequalities. Then, by interpolating between (50) and (52), we obtain

$$\Big(\int_{\partial B_r} |u-u_\varepsilon|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} \lesssim \varepsilon^{1-(\frac{1}{2p}+\frac{1}{2q})(d-1)} \Big(\int_{\partial B_r} |\nabla^{\tan} u|^{\frac{2q}{q+1}}\Big)^{\frac{q+1}{2q}}.$$

The analogous estimate for v reads

$$\Big(\int_{\partial B_r} |v-v_\varepsilon|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} \lesssim \varepsilon^{1-\frac{1}{p}(d-1)} \Big(\int_{\partial B_r} |\nabla^{\tan} v|^{\frac{2p}{p+1}}\Big)^{\frac{p+1}{2p}}.$$

We plug (39) and (41) into these two estimates and use our case distinction (36) to arrive at

$$\Big(\int_{\partial B_r} |u-u_\varepsilon|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} + \Big(\int_{\partial B_r} |v-v_\varepsilon|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} \lesssim \overline{\Lambda}^{\frac{1}{2}} \varepsilon^{1-(\frac{1}{2p}+\frac{1}{2q})(d-1)}.$$

Inserting this into (49) we obtain for the boundary term in the Neumann case

$$\Big|\int_{\partial B_r} (u-v)\nu\cdot(a\nabla u - a_{\mathrm{hom}}\nabla v)\Big| \lesssim \overline{\Lambda}\varepsilon^{1-(\frac{1}{2p}+\frac{1}{2q})(d-1)}. \tag{53}$$

We finally turn to the Dirichlet case (37) for the boundary term (44), which yields a simpler estimate as compared to the Neumann case because u and v are immediately comparable along the boundary ∂B_r. We use Hölder's inequality, the triangle inequality and the case distinction in combination with (39) and (40) to obtain

$$\Big|\int_{\partial B_r} (u-v)\nu\cdot(a\nabla u - a_{\mathrm{hom}}\nabla v)\Big| \lesssim \overline{\Lambda}^{\frac{1}{2}} \Big(\int_{\partial B_r} |u-u_\varepsilon|^{\frac{2q}{q-1}}\Big)^{\frac{q-1}{2q}}.$$

Appealing again to convolution estimates used to obtain (53) and the case distinction, we obtain with (39) the estimate

$$\left|\int_{\partial B_r}(u-v)\nu\cdot(a\nabla u-a_{\mathrm{hom}}\nabla v)\right|\lesssim\overline{\Lambda}^{\frac{1}{2}}\varepsilon^{1-\frac{1}{q}(d-1)}\left(\int_{\partial B_r}\left|\nabla^{\tan}u\right|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{2q}}$$
$$\lesssim\overline{\Lambda}\varepsilon^{1-(\frac{1}{2p}+\frac{1}{2q})(d-1)}, \tag{54}$$

which corresponds with the Neumann estimate (53). Inserting (54) and (53) together with (48) and (47) into (46) yields (34) which, after expanding the definition of $\overline{\Lambda}$ from (33), takes the form

$$\int_{B_{\frac{1}{4}}}\nabla w\cdot a\nabla w\lesssim\Lambda\left(\int_{B_1}\nabla u\cdot a\nabla u\right)$$
$$\cdot\left(\varepsilon^{1-(\frac{1}{2p}+\frac{1}{2q})(d-1)}+\Lambda\rho^{\min\{\frac{p-1}{2p},\frac{q-1}{2q}\}}\frac{1}{\varepsilon^{\min\{\frac{q+1}{q},\frac{p+1}{p}\}(d-1)}}\right. \tag{55}$$
$$\left.+\Lambda\frac{1}{\rho^{d+2}}\left(\left(\int_{B_1}|\phi|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}+\left(\int_{B_1}|\sigma|^{\frac{2q}{q-1}}\right)^{\frac{q-1}{q}}\right)\right).$$

Finally, to adapt (55) to an arbitrary radius $R>0$, observe that if u is an a-harmonic function on B_R, the augmented homogenization error w may be defined as

$$w(x)=u(x)-\left(1+\eta\left(\frac{x}{R}\right)\phi_i(x)\partial_i\right)v(x),$$

where v is the a_{hom}-harmonic function defined in (35) on B_r, for some $r\in[\frac{R}{2},R]$, and according to the case distinction (36), and η is defined exactly as in (43) on B_1. The general estimate

$$\fint_{B_{\frac{R}{4}}}\nabla w\cdot a\nabla w\lesssim\Lambda\left(\fint_{B_R}\nabla u\cdot a\nabla u\right)$$
$$\cdot\left(\varepsilon^{1-(\frac{1}{2p}+\frac{1}{2q})(d-1)}+\Lambda\rho^{\min\{\frac{p-1}{2p},\frac{q-1}{2q}\}}\frac{1}{\varepsilon^{\min\{\frac{q+1}{q},\frac{p+1}{p}\}(d-1)}}\right. \tag{56}$$
$$\left.+\Lambda\frac{1}{\rho^{d+2}}\left(\left(\fint_{B_R}|\phi|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}+\left(\fint_{B_R}|\sigma|^{\frac{2q}{q-1}}\right)^{\frac{q-1}{q}}\right)\right),$$

then follows from (55), the rescaling $\tilde{w}(x)=(1/R)w(Rx)$ and the scaling of the defining equations. This completes the proof of Step 2.

Step 3. In this step, we prove that whenever u is an a-harmonic function on B_R, there exists for any $\delta_0>0$ a constant $C_0=C_0(\delta_0,d,\Lambda,K)>0$ such that, whenever on the ball B_R the augmented corrector (ϕ,σ) satisfies (10) with constant C_0, we have

$$\fint_{B_{\frac{R}{4}}} \nabla w \cdot a \nabla w \le \delta_0 \fint_{B_R} \nabla u \cdot a \nabla u, \tag{57}$$

where $w := u - (1 + \phi_i \partial_i)v$ is the homogenization error. Here, we remark that the reader can take one of two approaches. To immediately apply (10), one would need first to augment the extended corrector (ϕ, σ) by subtracting their respective averages over B_R. Conversely, one could immediately apply the equivalence between (25) and (26), and work directly with the unmodified quantities. Therefore, in the three steps to follow, we assume without loss of generality that wherever encountered the corrector pair (ϕ, σ) has average zero.

Here comes the argument. Given $\delta_0 > 0$, owing to the linearity and scaling of the equation the estimate (56) from Step 2 implies existence of $C_1 = C_1(d, K)$ such that

$$\begin{aligned}\fint_{B_{\frac{R}{4}}} \nabla w \cdot a \nabla w \le C_1 \Bigg(& \Lambda \varepsilon^{1-(\frac{1}{2p}+\frac{1}{2q})(d-1)} + \Lambda^2 \frac{1}{\rho^{d+2} C_0^2} \\ & + \Lambda^2 \rho^{\min\{\frac{p-1}{2p}, \frac{q-1}{2q}\}} \frac{1}{\varepsilon^{\min\{\frac{q+1}{q}, \frac{p+1}{p}\}(d-1)}} \Bigg) \fint_{B_R} \nabla u \cdot a \nabla u.\end{aligned}$$

We first fix $\varepsilon_0 = \varepsilon_0(\delta_0, d, \Lambda, K) > 0$ small enough such that

$$C_1 \Lambda \varepsilon_0^{1-(\frac{1}{2p}+\frac{1}{2q})(d-1)} \le \frac{1}{3} \delta_0,$$

which is possible in view of (9). Second, we choose $\rho_0 = \rho_0(\delta_0, d, \Lambda, K) > 0$ small enough satisfying

$$C_1 \Lambda^2 \rho_0^{\min\{\frac{p-1}{2p}, \frac{q-1}{2q}\}} \frac{1}{\varepsilon_0^{\min\{\frac{q+1}{q}, \frac{p+1}{p}\}(d-1)}} \le \frac{1}{3} \delta_0.$$

Finally, we select $C_0 = C_0(\delta_0, d, \Lambda, K) > 0$ large enough so that

$$C_1 \Lambda^2 \frac{1}{\rho_0^{d+2} C_0^2} \le \frac{1}{3} \delta_0.$$

Since the right-hand sides in the three previous relations add to δ_0, the proof of this step is complete.

Step 4. For any $\alpha \in (0, 1)$ there exists $\theta_0 \in (0, \frac{1}{4})$ and C_0 such that the following holds: For any radius $R > 0$ with the property that the augmented corrector is small on scale R:

$$\begin{aligned} & \frac{1}{R} \left(\fint_{B_R} \left| \phi - \fint_{B_R} \phi \right|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{2p}} \le \frac{1}{C_0}, \\ & \frac{1}{R} \left(\fint_{B_R} \left| \sigma - \fint_{B_R} \sigma \right|^{\frac{2q}{q-1}} \right)^{\frac{q-1}{2q}} \le \frac{1}{C_0}, \end{aligned} \tag{58}$$

and the corrector ϕ is small on scale $r := \theta_0 R$

$$\frac{1}{r}\left(\fint_{B_r}\left|\phi - \fint_{B_r}\phi\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} \le \frac{1}{C_0}, \tag{59}$$

we get that for every a-harmonic function u on B_R its excess satisfies

$$\operatorname{Exc}(r) \le \theta_0^{2\alpha} \operatorname{Exc}(R), \tag{60}$$

where we recall

$$\operatorname{Exc}(r) = \inf_{\xi\in\mathbb{R}^d} \fint_{B_r} \left(\nabla u - (\xi + \phi_\xi)\right) \cdot a\left(\nabla u - (\xi + \phi_\xi)\right).$$

Here comes the argument. To simplify the notation, we will use $\|\cdot\|_{a,r}$ to denote the $L^2(a)$-intrinsic energy of vector fields U as defined by

$$\|U\|_{a,r} := \int_{B_r} U \cdot aU.$$

We now use the definition of the homogenization error $w = u - (1 + \phi_i \partial_i)v$, and observe that, for every $0 < r \le \frac{R}{4}$, we have

$$\|\nabla w\|_{a,r} = \left\|\nabla u - \nabla v(\mathrm{Id} + \nabla\phi) - \phi_i \nabla(\partial_i v)\right\|_{a,r}.$$

The first two terms are decomposed as

$$\nabla u - \nabla v(\mathrm{Id} + \nabla\phi) = \nabla u - \nabla v(0)(\mathrm{Id} + \nabla\phi) + \left(\nabla v(0) - \nabla v\right)(\mathrm{Id} + \nabla\phi)$$

and, by defining $\xi := \nabla v(0)$, we obtain using the triangle inequality, since $0 < r \le \frac{R}{4}$,

$$\begin{aligned} \left\|\nabla u - (\xi + \nabla\phi_\xi)\right\|_{a,r} \le \|\nabla w\|_{a,\frac{R}{4}} &+ \left(\sup_{B_r}\left|\nabla v(0) - \nabla v\right|\right)^2 \left(\|\mathrm{Id} + \nabla\phi\|_{a,r}\right) \\ &+ \left(\sup_{B_r}\left|\nabla(\partial_i v)\right|\right)^2 \|\phi\|_{a,r}, \end{aligned} \tag{61}$$

where $\phi = (\phi_1, \dots, \phi_d)$ denotes the vector of correctors.

The first term on the right-hand side of (61) is controlled using the estimate (57) from Step 3. There exists, for any $\delta_0 > 0$, a constant $C_0 = C_0(\delta_0, d, \Lambda, K) > 0$ such that, whenever the corrector and flux correction satisfy (10) with constant C_0, we have

$$\|\nabla w\|_{a,\frac{R}{4}} \le \delta_0 \int_{B_R} \nabla u \cdot a\nabla u. \tag{62}$$

It remains to control the last two terms on the right-hand side of (61). First, since the corrector satisfies

$$-\nabla \cdot a(\mathrm{Id} + \nabla\phi) = 0,$$

the Caccioppoli estimate (Lemma 3) together with (10) implies

$$\|\mathrm{Id} + \nabla\phi\|_{a,r} \lesssim r^d.$$

Then, by repeating the argument leading to (42), since $0 < r \le \frac{R}{4}$, we have the estimate

$$\sup_{B_r} |\nabla^2 v|^2 \lesssim \frac{1}{R^{d+2}} \|\nabla u\|_{a,R}. \tag{63}$$

Finally, owing to definition (1) and condition (2), and following an application of Hölder's inquality, then Young's inequality and finally the Sobolev inequality, since after subtracting a constant vector it may be assumed that $\fint_{B_r} \phi = 0$,

$$\begin{aligned} \|\phi\|_{a,r} &\lesssim \fint_{B_r} |\phi|^2 \mu \le \Big(\fint_{B_r} |\phi|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{p}} \Big(\fint_{B_r} \mu^p\Big)^{\frac{1}{p}} \\ &\lesssim \Lambda \Big(\fint_{B_r} |\nabla\phi|^{\frac{2q}{q+1}}\Big)^{\frac{q+1}{q}} \lesssim \Lambda. \end{aligned} \tag{64}$$

We insert (62), (63) and (64) into (61), use (8) and (10), and find for a constant $C_1 = C_1(d, \Lambda, K) > 0$ that

$$\big\|\nabla u - (\xi + \nabla\phi_\xi)\big\|_{a,r} \le C_1 \Big(\delta_0 + \Big(\frac{r}{R}\Big)^2 \Big(\frac{r}{R}\Big)^d\Big) \|\nabla u\|_{a,R}.$$

Then, after dividing by r^d,

$$\begin{aligned} &\fint_{B_r} \big(\nabla u - (\xi + \nabla\phi_\xi)\big) \cdot a\big(\nabla u - (\xi + \nabla\phi_\xi)\big) \\ &\quad \le C_1 \Big(\delta_0 \Big(\frac{R}{r}\Big)^d + \Big(\frac{r}{R}\Big)^2\Big) \fint_{B_R} \nabla u \cdot a \nabla u. \end{aligned} \tag{65}$$

Fix $\alpha \in (0, 1)$. The proof of Step 4 will be complete once we prove that the ratio $\frac{r}{R}$ and the constant δ_0 can fixed sufficiently small so as to guarantee the inequality

$$C_1 \Big(\delta_0 \Big(\frac{R}{r}\Big)^d + \Big(\frac{r}{R}\Big)^2\Big) \le \Big(\frac{r}{R}\Big)^{2\alpha},$$

and which is possible because α is strictly smaller than one. First, we choose $\theta_0 \in (0, \frac{1}{4})$ small enough to satisfy

$$C_1 \theta_0^2 \le \frac{1}{2} \theta_0^{2\alpha}, \tag{66}$$

and then choose δ_0 small enough to ensure

$$C_1 \delta_0 \theta_0^{-d} \le \frac{1}{2} \theta_0^{2\alpha}. \tag{67}$$

Then, for θ_0 satisfying (66) and (67), whenever $\frac{r}{R} = \theta_0$ we obtain from (65) the estimate

$$\inf_{\zeta\in\mathbb{R}^d} \fint_{B_r} (\nabla u - (\zeta + \nabla\phi_\zeta)) \cdot a(\nabla u - (\zeta + \nabla\phi_\zeta)) \le \left(\frac{r}{R}\right)^{2\alpha} \fint_{B_R} \nabla u \cdot a\nabla u.$$

After replacing u with $u - \xi \cdot (x + \phi)$, where $\xi := \operatorname{argmin}_{\xi\in\mathbb{R}^d} \fint_{B_R} a(\nabla u - (\xi + \nabla\phi_\xi)) \cdot (\nabla u - (\xi + \nabla\phi_\xi))$, we obtain (60) and complete Step 3.

Step 5. In the final step of the proof, we will prove that, for arbitrary pairs $r < R$ satisfying the hypothesis of Theorem 2, we have the excess decay

$$\operatorname{Exc}(r) \lesssim \left(\frac{r}{R}\right)^{2\alpha} \operatorname{Exc}(R). \tag{68}$$

For this, we iterate the estimate (60) to obtain, for $\theta_0 \in (0, 1)$ from Step 4,

$$\operatorname{Exc}(\theta_0 R) \le \theta_0^{2\alpha} \operatorname{Exc}(R).$$

First, we remark that, if $r/R \ge \theta_0$, then for $C = C(\theta_0) > 0$,

$$\begin{aligned}\operatorname{Exc}(r) &\le \left(\frac{R}{r}\right)^{d} \operatorname{Exc}(R) = \left(\frac{R}{r}\right)^{d+2\alpha} \left(\frac{r}{R}\right)^{2\alpha} \operatorname{Exc}(R)\\ &\le \theta_0^{-(d+2\alpha)} \left(\frac{r}{R}\right)^{2\alpha} \operatorname{Exc}(R) \le C\left(\frac{r}{R}\right)^{2\alpha} \operatorname{Exc}(R),\end{aligned}$$

which implies estimate (68). It therefore remains only to treat the case $0 < r/R < \theta_0$. For this, let n be the unique positive integer satisfying $\theta_0^{n-1} \le r/R < \theta_0^n$. Then, by induction,

$$\operatorname{Exc}(r) \lesssim \operatorname{Exc}(\theta_0^n R) \lesssim (\theta_0^n)^{2\alpha} \operatorname{Exc}(R) = \theta_0^{2\alpha} (\theta_0^{n-1})^{2\alpha} \operatorname{Exc}(R) \lesssim \left(\frac{r}{R}\right)^{2\alpha} \operatorname{Exc}(R),$$

which concludes the proof of (68) and thereby the proof of Theorem 2.

6. Proof of Lemma 3. After possibly adding a constant to u we may assume $c = 0$. Fix two radii $R > 0$ and $0 < \rho < \frac{R}{2}$, and suppose that u is an a-harmonic function in B_R. Let η denote a smooth cutoff function such that $0 \le \eta \le 1$ and

$$\eta(x) = \begin{cases} 1 & \text{in } \overline{B}_{R-\rho}, \\ 0 & \text{in } \mathbb{R}^d \setminus B_R, \end{cases}$$

and satisfying

$$|\nabla\eta| \lesssim \frac{1}{\rho}.$$

Using Hölder's inequality and (11), we have

$$\left(\fint_{B_{R-\rho}} |\nabla u|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{q}} \lesssim \Lambda \fint_{B_{R-\rho}} \nabla u \cdot a\nabla u \lesssim \Lambda \fint_{B_R} \eta^2 \nabla u \cdot a\nabla u. \tag{69}$$

Then, by testing the equation $-\nabla\cdot a\nabla u$ against $\eta^2 u$, and using the identity

$$\nabla(\eta^2 u)\cdot a\nabla u = \eta^2\nabla u\cdot a\nabla u + 2\eta u\nabla\eta\cdot a\nabla u,$$

it follows from the definition of μ in (31) that

$$\begin{aligned}\fint_{B_R}\eta^2\nabla u\cdot a\nabla u &= -2\fint_{B_R}\eta u\nabla\eta\cdot a\nabla u \le \fint_{B_R}2\eta|a\nabla u||u\nabla\eta| \\ &\le 2\fint\mu^{\frac{1}{2}}(\eta^2\nabla u\cdot a\nabla u)^{\frac{1}{2}}|u\nabla\eta|.\end{aligned}$$

Following an application of Hölder's inequality, we obtain

$$\fint_{B_R}\eta^2\nabla u\cdot a\nabla u \lesssim \left(\fint_{B_R}\eta^2\nabla u\cdot a\nabla u\right)^{\frac{1}{2}}\left(\fint_{B_R}\mu^p\right)^{\frac{1}{2p}}\left(\fint_{B_R}|u\nabla\eta|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}}.$$

Then, after dividing by the square-root of the left-hand side and using properties of the cutoff η, we have

$$\fint_{B_R}\eta^2\nabla u\cdot a\nabla u \lesssim \frac{\Lambda}{\rho^2}\left(\fint_{B_R\setminus B_{R-\rho}}|u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}. \tag{70}$$

In combination, inequalities (69) and (70) imply

$$\left(\fint_{B_{R-\rho}}|\nabla u|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{q}} \lesssim \Lambda\fint_{B_{R-\rho}}\nabla u\cdot a\nabla u \lesssim \frac{\Lambda^2}{\rho^2}\left(\fint_{B_R\setminus B_{R-\rho}}|u|^{\frac{2p}{p-1}}\right)^{\frac{p}{p-1}},$$

which completes the proof of Lemma 3.

APPENDIX

A.1. An alternate construction of the flux correction σ in Lemma 1. We claim that for any exponent in the range of

$$(2^*)' < r < 2, \tag{71}$$

where $(2^*)'$ denotes the dual exponent of 2^*, which in turn denotes the Sobolev exponent for 2, we have the following existence result: For a stationary random field g with $\langle|g|^r\rangle^{\frac{1}{r}} < \infty$ there exists a curl-free stationary random field $\nabla\sigma$ of vanishing expectation $\langle\nabla\sigma\rangle = 0$ with

$$-\triangle\sigma = \nabla\cdot g \quad\text{and}\quad \langle|\nabla\sigma|^r\rangle^{\frac{1}{r}} \lesssim \langle|g|^r\rangle^{\frac{1}{r}}. \tag{72}$$

We note that the range (71) is sufficient for our purposes: We use it for $\frac{1}{r} = \frac{1}{2} + \frac{1}{2p}$ and note that under the weaker assumption $\frac{1}{d} > \frac{1}{2p}$, as compared with $\frac{1}{p} + \frac{1}{q} \le \frac{2}{d}$, we indeed have (71) in the reciprocal form of $\frac{1}{(2^*)'} = 1 - (\frac{1}{2} - \frac{1}{d}) = \frac{1}{2} + \frac{1}{d} > \frac{1}{r} > \frac{1}{2}$.

By applying the same reasoning as in the proof of Lemma 1, we may assume that $\langle |g|^2\rangle < \infty$, so that by Riesz' representation theorem that there exists a curl-free stationary random field $\nabla\sigma$ with $\langle\nabla\sigma\rangle = 0$ and

$$-\triangle\sigma = \nabla\cdot g \quad \text{and} \quad \langle|\nabla\sigma|^2\rangle^{\frac{1}{2}} \lesssim \langle|g|^2\rangle^{\frac{1}{2}}.$$

By a standard duality argument, it is enough to establish (72) in the dual range of exponents, that is,

$$\langle|\nabla\sigma|^r\rangle^{\frac{1}{r}} \lesssim \langle|g|^r\rangle^{\frac{1}{r}} \qquad \text{for } 2 < r < 2^*. \tag{73}$$

The first ingredient is that thanks to $r < 2^*$ in conjunction with $\langle|\nabla\sigma|^2\rangle^{\frac{1}{2}} < \infty$ and $\langle\nabla\sigma\rangle = 0$ and $r < 2^*$, a repetition of the proof leading Lemma 2 we have the sublinearity

$$\lim_{R\uparrow\infty}\frac{1}{R}\left\langle \fint_{B_R}\left|\sigma - \fint_{B_R}\sigma\right|^r\right\rangle^{\frac{1}{r}} = 0. \tag{74}$$

The second ingredient is that thanks to $r > 2$ and $d \geq 2$, we have the following strengthening of Calderon–Zygmund's estimate (in physical space):

$$\left(\int|\nabla\tilde{\sigma}|^r\right)^{\frac{1}{r}} \lesssim \left(\int|\tilde{g}|^r\right)^{\frac{1}{r}} + R\left(\int|\tilde{f}|^r\right)^{\frac{1}{r}}, \tag{75}$$

provided the functions $\tilde{\sigma}$, $\tilde{f}$ and the field $\tilde{g}$ satisfy

$$-\triangle\tilde{\sigma} = \nabla\cdot\tilde{g} + \tilde{f} \quad \text{and} \quad \operatorname{supp}\tilde{\sigma}, \tilde{g}, \tilde{f} \subset B_R. \tag{76}$$

We consider $\tilde{\sigma} := \eta(\sigma - c)$ for some constant c and some smooth function η supported in B_R to be fixed soon and note that (76) is satisfied for

$$\tilde{g} := \eta g + 2(\sigma - c)\nabla\eta \quad \text{and} \quad \tilde{f} := -\nabla\eta\cdot g + (\sigma - c)\triangle\eta.$$

Choosing the cut-off of the form $\eta(x) = \hat{\eta}(\frac{x}{R})$ with $\hat{\eta}(\hat{x}) = 1$ for $|\hat{x}| \leq \frac{1}{2}$ we see that (75) turns into

$$\left(\int_{B_{\frac{R}{2}}}|\nabla\sigma|^r\right)^{\frac{1}{r}} \lesssim \left(\int_{B_R}|g|^r\right)^{\frac{1}{r}} + \frac{1}{R}\left(\int_{B_R}|\sigma - c|^r\right)^{\frac{1}{r}}.$$

We now choose $c = \fint_{B_R}\sigma$ and apply $\langle(\cdot)^r\rangle^{\frac{1}{r}}$ to the above. By stationarity of $\nabla\sigma$ and g we obtain

$$\langle|\nabla\sigma|^r\rangle^{\frac{1}{r}} \lesssim \langle|g|^r\rangle^{\frac{1}{r}} + \frac{1}{R}\left\langle\fint_{B_R}\left|\sigma - \fint_{B_R}\sigma\right|^r\right\rangle^{\frac{1}{r}}.$$

With help from (74) we obtain (73).

A.2. The proof of Lemma 4. The proof of Lemma 4 consists of two steps. In the first step, using separation of variables we prove an analogous claim for 1-*periodic* harmonic function on a slab $(0,1)\times\mathbb{R}^{d-1}$. In the second step, we decompose the solutions corresponding respectively to Dirichlet and Neumann boundary conditions into a collection of d 1-periodic harmonic functions, defined on a slab, for which Step 1 can be applied.

Step 1. Suppose that $v\in C^{\infty}(\mathbb{R}^{d-1})$ is a 1-periodic function satisfying $\int_{Q_{d-1}} v=0$, and let w denote the solution

$$\begin{cases}-\Delta w=0 & \text{in } \mathbb{R}^{d-1}\times(0,1),\\ w(\cdot,0)=v, \qquad w(\cdot,1)=0 & \text{in } \mathbb{R}^{d-1}.\end{cases} \tag{77}$$

Then for each $1<p<\infty$ there exists $C=C(d,p)$ such that

$$\|\nabla w\|_{L^p(\partial Q_d)}\le C\|\nabla v\|_{L^p(Q_{d-1})}. \tag{78}$$

Observe that (1) $\nabla v=\nabla^{\tan}w$, that is, the right-hand side measures only the *tangential* part of the gradient of ∇w; and (2) on the left-hand side the full boundary ∂Q_d (which consists of $2d$ faces) appears, compared to only the bottom face Q_{d-1} appearing on the right-hand side.

Moreover, replacing the bottom and top Dirichlet boundary conditions $w(\cdot,0)=v$ and $w(\cdot,1)=0$ by Neumann boundary conditions $\frac{\partial w}{\partial x_d}(\cdot,0)=v$ and $\frac{\partial w}{\partial x_d}(\cdot,1)=0$ in $\mathbb{R}^{d-1}$, we analogously get

$$\|\nabla w\|_{L^p(\partial Q_d)}\le C\|v\|_{L^p(Q_{d-1})}. \tag{79}$$

We first show an argument for (78). Since $\int_{Q_{d-1}} v=0$, after writing, for each $k\in\mathbb{Z}^{d-1}$,

$$\hat{v}_k=\int_{Q_{d-1}} v(x')e^{-2\pi ik\cdot x'}\,\mathrm{d}x',$$

we express v in terms of the Fourier series

$$v(x')=\sum_{k\in\mathbb{Z}^{d-1}\setminus\{0\}}\hat{v}_k e^{2\pi ik\cdot x'} \qquad \text{for } x'\in\mathbb{R}^{d-1}.$$

Using separation of variables, we see the solution w admits the representation, for $(x',x_d)\in\mathbb{R}^{d-1}\times[0,1]$,

$$w(x',x_d)=\sum_{k\in\mathbb{Z}^{d-1}\setminus\{0\}}\hat{v}_k e^{2\pi ik\cdot x'}\left(\frac{e^{-2\pi|k|x_d}-e^{-2\pi|k|(2-x_d)}}{1-e^{-4\pi|k|}}\right). \tag{80}$$

Therefore,

$$\frac{\partial w}{\partial x_d}(x',x_d)=-\sum_{k\in\mathbb{Z}^{d-1}\setminus\{0\}}(2\pi|k|)\hat{v}_k e^{2\pi ik\cdot x'}\left(\frac{e^{-2\pi|k|x_d}+e^{-2\pi|k|(2-x_d)}}{1-e^{-4\pi|k|}}\right), \tag{81}$$

and from the Mikhlin multiplier theorem (see Mikhlin [25] for the original or Mikhlin [26], Appendix, Theorem 2) we have

$$\|\nabla w\|_{L^p(Q_{d-1}\times\{0\})} \lesssim \|\nabla v\|_{L^p(Q_{d-1})}, \qquad (82)$$
$$\|\nabla w\|_{L^p(Q_{d-1}\times\{1\})} \lesssim \|\nabla v\|_{L^p(Q_{d-1})}.$$

The argument proceeds by considering each component of the gradient separately. Indeed, for the normal derivative, the Mikhlin multiplier theorem is applied to the multiplier

$$m(k) = -\left(\frac{1+e^{-4\pi|k|}}{1-e^{-4\pi|k|}}\right), \qquad (83)$$

for which an immediate computation yields, for each $0 < r < \frac{d}{2}+2$,

$$\sup_{k\in\mathbb{R}^d} |k|^r \left|\nabla^r m(k)\right| < \infty.$$

Therefore, with (81),

$$\begin{aligned}\left\|\frac{\partial w}{\partial x_d}\right\|_{L^p(Q_{d-1}\times\{0\})} &\lesssim \left\|\sum_{k\in\mathbb{Z}^{d-1}\setminus\{0\}} \big(2\pi|k|\big)\hat{v}_k e^{2\pi i k\cdot x'} m(k)\right\|_{L^p(Q_{d-1})} \\ &\lesssim \|\nabla v\|_{L^p(Q_{d-1})},\end{aligned}$$

where the final inequality is immediate for the case $p=2$ and follows generally from the vector-valued Mikhlin multiplier theorem (see Bergh and Löfström [10], Theorem 6.1.6) applied to the multiplier, considered as a map from $\mathbb{R}^{d-1}$ to $\mathbb{R}$,

$$m(k) = \left(\psi_1(k)\frac{|k|}{k_1}, \ldots, \psi_{d-1}(k)\frac{|k|}{k_{d-1}}\right) \qquad \text{for each } k=(k_1,\ldots,k_{d-1})\in\mathbb{R}^{d-1}.$$

Here, the collection $\{\psi_j\}_{j\in\{1,\ldots,d-1\}}$ forms a partition of unity defined on the sphere S^{d-2} and where, for each $j\in\{1,\ldots,d-1\}$, the function ψ_j is extended to $\mathbb{R}^{d-1}\setminus\overline{B}_{\frac{1}{2}}$ as a smooth, zero homogenous function which is nonzero on a set of the form

$$\left\{k\in\mathbb{R}^{d-1}\setminus\overline{B}_{\frac{1}{2}} \mid |k_i|\lesssim|k_j| \text{ for each } i\neq j\in\{1,\ldots,d-1\}\right\}.$$

Finally, since it is immediate from (80) that, for $j\in\{1,\ldots,d-1\}$,

$$\left\|\frac{\partial w}{\partial x_j}\right\|_{L^p(Q_{d-1}\times\{0\})} = \left\|\frac{\partial v}{\partial x_j}\right\|_{L^p(Q_{d-1})},$$

this completes the proof of (82) on the set $Q_{d-1}\times\{0\}$.

The analogous considerations prove (82) on the set $Q_{d-1}\times\{1\}$, where in this case the tangential derivatives, for $j\in\{1,\ldots,d-1\}$, are controlled using the Mikhlin multiplier theorem applied to the multiplier

$$m(k) = \left(\frac{e^{-2\pi|k|}-e^{-2\pi|k|}}{1-e^{-4\pi|k|}}\right), \qquad (84)$$

and the normal derivative is controlled following the above computation but with (83) replaced with (84). This completes the proof of (82).

The estimate for ∇w on the remaining $(2d-2)$ faces of ∂Q_d then follows from the estimate

$$\|w\|_{L^p(\partial Q_d)} \leq C\|v\|_{L^p(Q_{d-1})}, \tag{85}$$

which holds for any function w satisfying (77). Indeed, since for $i = 1, \dots, d-1$ the function $\frac{\partial w}{\partial x_i}$ solves (77) with boundary condition $\frac{\partial v}{\partial x_i}$, estimate (85) implies

$$\left\| \frac{\partial w}{\partial x_i} \right\|_{L^p(\partial Q_d)} \lesssim \left\| \frac{\partial v}{\partial x_i} \right\|_{L^p(Q_{d-1})} \leq \|\nabla v\|_{L^p(Q_{d-1})}.$$

The argument works for $\frac{\partial w}{\partial x_d}$ as well, with the small difference that the derivative in x_d does not satisfy zero boundary condition on $Q_{d-1} \times \{1\}$. Hence, we first split the harmonic extension of this derivative into two parts: the first one having 0 boundary condition at $Q_{d-1} \times \{1\}$ while keeping its value at $Q_{d-1} \times \{0\}$, the second having it reversed, and then applying (85) to each of them separately [while using (82)].

It remains to prove (85). We observe that it is enough to study harmonic functions in the half-space $\mathbb{R}^{d-1} \times (0, \infty)$, more precisely, to show that the unique decaying solution u of

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^{d-1} \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^{d-1}, \end{cases} \tag{86}$$

with u_0 being 1-periodic with zero average, satisfies, for any $1 < p < \infty$,

$$\|u(\cdot, 1)\|_{L^p(Q_{d-1})} \leq \alpha \|u_0\|_{L^p(Q_{d-1})}, \tag{87}$$

$$\|u(0, \cdot)\|_{L^p(Q_{d-2} \times (0,1))} \lesssim \|u_0\|_{L^p(Q_{d-1})}, \tag{88}$$

with $\alpha = \alpha(p) < 1$. Indeed, for the boundary data v [from (85)] we solve the half-space problem (86), denoting the solution by u_1. Let u_2 now denote the harmonic extension of $u_1(\cdot, 1)$ in $\mathbb{R}^{d-1} \times (-\infty, 1)$. Repeating this procedure [denoting u_3 harmonic extension of $u_2(\cdot, 0)$ in $\mathbb{R}^{d-1} \times (0, \infty)$ etc.] we can write the solution w of (77) with $w(\cdot, 0) = v$ as $w = \sum_{i=1}^{\infty} (-1)^{(i+1)} u_i$, where the sum is converging thanks to $\alpha < 1$. Summing estimate (88), applied to each u_i, then implies (85).

The advantage of considering the half-space over the slab as a domain is that in the half-space we have a simple representation for u using the Poisson kernel:

$$u(x, y, z) = \int_{\mathbb{R}^{d-1}} P_z(x', y') u_0(x - x', y - y') \, dx' \, dy', \tag{89}$$

where $(x, y, z) \in \mathbb{R} \times \mathbb{R}^{d-2} \times (0, \infty)$ and the Poisson kernel has an explicit form

$$P_z(x, y) = \frac{2z}{d|B_d|} \frac{1}{|x^2 + |y|^2 + z^2|^{d/2}}. \tag{90}$$

We start with (87). Writing the solution u using Fourier series (as we have done for w before), using the fact that u_0 has vanishing average and the fact that all the remaining frequencies decay exponentially, by Plancherel's equality we see that (87) holds for $p = 2$ [with $\alpha(2) < 1$]. Using (89) we get for the case $p = 1$

$$
\begin{aligned}
&\int_{Q_{d-1}} |u(x, y, 1)| \,\mathrm{d}x \,\mathrm{d}y \\
&\quad \leq \int_{Q_{d-1}} \int_{\mathbb{R}^{d-1}} |P_1(x', y')||u_0(x - x', y - y')| \,\mathrm{d}x' \,\mathrm{d}y' \,\mathrm{d}x \,\mathrm{d}y \\
&\quad = \int_{Q_{d-1}} |u_0| \,\mathrm{d}x \,\mathrm{d}y,
\end{aligned}
$$

where the final inequality follows from Fubini's theorem, the 1-periodicity of u_0 and the fact that, for any $z > 0$, the kernel $P_z > 0$ with

$$
\int_{\mathbb{R}^{d-1}} P_z(x, y) \,\mathrm{d}x \,\mathrm{d}y = 1. \tag{91}
$$

Hence, we see that (87) holds for $p = 1$ with $\alpha(1) = 1$. Combining this with the fact that $\alpha(2) < 1$, for the range $1 < p \leq 2$ estimate (87) follows by standard interpolation. Since the periodicity of u_0 and (91) imply that

$$
\begin{aligned}
&\sup_{(x,y)\in Q_{d-1}} |u(x, y, 1)| \\
&\quad \leq \sup_{(x,y)\in Q_{d-1}} \int_{\mathbb{R}^{d-1}} |P_1(x', y')||u_0(x - x', y - y')| \,\mathrm{d}x' \,\mathrm{d}y' \\
&\quad \leq \Big(\sup_{Q_{d-1}} |u_0|\Big) \int_{\mathbb{R}^{d-1}} |P_1(x', y')| \,\mathrm{d}x' \,\mathrm{d}y' \\
&\quad = \Big(\sup_{Q_{d-1}} |u_0|\Big),
\end{aligned}
$$

that is, (87) holds for $p = \infty$ with $\alpha(\infty) = 1$, we obtain (87) for the remaining range $2 \leq p < \infty$ again using interpolation.

To show (88), for $z > 0$ we will prove the following two estimates:

$$
\begin{aligned}
\|u(0, \cdot, z)\|_{L^p_y(Q_{d-2})} &\lesssim \|\nabla_{(x,y)} u(\cdot, \cdot, z)\|_{L^1_x(0,1;L^p_y(Q_{d-2}))} \\
&\lesssim \frac{1}{z} \|u_0(\cdot, \cdot)\|_{L^1_x(0,1;L^p_y(Q_{d-2}))}
\end{aligned} \tag{92}
$$

and

$$
\begin{aligned}
\|u(0, \cdot, z)\|_{L^p_y(Q_{d-2})} &\leq \|u(\cdot, \cdot, z)\|_{L^\infty_x(0,1;L^p_y(Q_{d-2}))} \\
&\lesssim \|u_0(\cdot, \cdot)\|_{L^\infty_x(0,1;L^p_y(Q_{d-2}))}.
\end{aligned} \tag{93}
$$

Considering the map $u_0 \mapsto u(0,\cdot,\cdot)$, the first estimate says that this map is weak-type from $L^1_x(0,1;L^p_y(Q_{d-2}))$ to $L^1_z(0,1;L^p_y(Q_{d-2}))$, while the second estimate says it is strong-type from $L^\infty_x(0,1;L^p_y(Q_{d-2}))$ to $L^\infty_z(0,1;L^p_y(Q_{d-2}))$. Hence, estimate (88) follows from the Banach-space valued Marcinkiewicz interpolation theorem; see [29], Chapter I, Theorem 3.

The first inequality in (92) follows from the fact that, for any $z>0$, we have

$$\int_{Q_{d-1}} u(x,y,z)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{94}$$

Indeed, for a fixed $z>0$, for each $x\in[0,1]$, denote by $(u)_x$ the average

$$(u)_x := \int_{Q_{d-2}} u(x,y,z)\,\mathrm{d}y.$$

Then, it follows that

$$\begin{aligned}\|u\|_{L^1_x(0,1;L^p_y(Q_{d-2}))} &\le \|u-(u)_x\|_{L^1_x(0,1;L^p_y(Q_{d-2}))} + \|(u)_x\|_{L^1_x(0,1;L^p_y(Q_{d-2}))}\\ &\lesssim \|\nabla_y u\|_{L^1_x(0,1;L^p_y(Q_{d-2}))} + \|\partial_x (u)_x\|_{L^1_x(0,1)}\\ &\lesssim \|\nabla_y u\|_{L^1_x(0,1;L^p_y(Q_{d-2}))} + \|\partial_x u\|_{L^1_x(0,1;L^p_y(Q_{d-2}))},\end{aligned}$$

where the first inequality follows from the triangle inequality, the second inequality follows from the Poincaré inequality, (94) and the fact that the averages are y-independent, and the final inequality follows from Hölder's inequality and the definition of the averages.

Fubini's theorem therefore implies that there exists $x_0\in[0,1]$ such that

$$\|u(x_0,\cdot,z)\|_{L^p_y(Q_{d-2})} \lesssim \|\nabla_y u\|_{L^1_x(0,1;L^p_y(Q_{d-2}))} + \|\partial_x u\|_{L^1_x(0,1;L^p_y(Q_{d-2}))}. \tag{95}$$

It remains only to transfer estimate (95) to the slice $x=0$. For this, we will use the integral version of the Minkowski inequality which, for $p\in[1,\infty)$, two measure spaces (S_1,μ_1) and (S_2,μ_2), and a measurable function $F: S_1\times S_2\to\mathbb{R}$, states that

$$\begin{aligned}&\left[\int_{S_2}\left|\int_{S_1} F(x,y)\mu_1(\mathrm{d}x)\right|^p \mu_2(\mathrm{d}y)\right]^{\frac{1}{p}}\\ &\qquad\le \int_{S_1}\left(\int_{S_2}|F(x,y)|^p\mu_2(\mathrm{d}y)\right)^{\frac{1}{p}}\mu_1(\mathrm{d}x).\end{aligned} \tag{96}$$

Precisely,

$$\begin{aligned}\|u(0,\cdot,z)\|_{L^p_y(Q_{d-2})} &\le \|u(0,\cdot,z)-u(x_0,\cdot,z)\|_{L^p_y(Q_{d-2})} + \|u(x_0,\cdot,z)\|_{L^p_y(Q_{d-2})}\\ &\lesssim \|\partial_x u\|_{L^1_x(0,x_0;L^p_y(Q_{d-2}))} + \|u(x_0,\cdot,z)\|_{L^p_y(Q_{d-2})}\\ &\lesssim \|\nabla_y u\|_{L^1_x(0,1;L^p_y(Q_{d-2}))} + \|\partial_x u\|_{L^1_x(0,1;L^p_y(Q_{d-2}))},\end{aligned}$$

where the first line uses the triangle inequality, the second relies upon an explicit computation and (96), and the final inequality relies upon $x_0 \in [0, 1]$ and (95). This completes the proof of the first inequality appearing in (92).

To prove the second inequality of (92), for $x \in (0, 1)$ we use (96) to obtain

$$\begin{aligned} &\Big[\int_{Q_{d-2}} \big|\nabla_{(x,y)} u(x, y, z)\big|^p \,\mathrm{d}y\Big]^{\frac{1}{p}} \\ &\qquad \le \int_{\mathbb{R}^{d-1}} \Big(\int_{Q_{d-2}} |u_0(x - x', y - y')|^p \,\mathrm{d}y\Big)^{\frac{1}{p}} \\ &\qquad\quad \cdot \big|\nabla_{(x',y')} P_z(x', y')\big| \,\mathrm{d}x' \,\mathrm{d}y'. \end{aligned} \tag{97}$$

Since u_0 is 1-periodic, we can write

$$\Big(\int_{Q_{d-2}} |u_0(x - x', y - y')|^p \,\mathrm{d}y\Big)^{1/p} =: \mathcal{U}(x - x')$$

for some 1-periodic $\mathcal{U}$. Integrating (97) in x over $(0, 1)$ then yields

$$\begin{aligned} &\big\|\nabla_{(x,y)} u(\cdot, \cdot, z)\big\|_{L^1_x(0,1;L^p_y(Q_{d-2}))} \\ &\quad \le \int_0^1 \int_{\mathbb{R}} \mathcal{U}(x - x') \Big(\int_{\mathbb{R}^{d-2}} \big|\nabla_{(x',y')} P_z(x', y')\big| \,\mathrm{d}y'\Big) \,\mathrm{d}x' \,\mathrm{d}x \\ &\quad = \int_{\mathbb{R}} \Big(\int_0^1 \mathcal{U}(x - x') \,\mathrm{d}x\Big) \Big(\int_{\mathbb{R}^{d-2}} \big|\nabla_{(x',y')} P_z(x', y')\big| \,\mathrm{d}y'\Big) \,\mathrm{d}x' \\ &\quad = \Big(\int_0^1 \mathcal{U}(x) \,\mathrm{d}x\Big) \Big(\int_{\mathbb{R}^{d-1}} \big|\nabla_{(x',y')} P_z(x', y')\big| \,\mathrm{d}x' \,\mathrm{d}y'\Big) \\ &\quad \lesssim \frac{1}{z} \|u_0\|_{L^1_x(0,1;L^p_y(Q_{d-2}))}, \end{aligned}$$

where the last inequality follows from (91) and $|\nabla_{(x,y)} P_z(x, y)| \lesssim \frac{1}{z} P_z(x, y)$, which in turn follows from an explicit computation.

To prove (93), for $x \in (0, 1)$, we apply the Minkowski integral inequality (96) to obtain

$$\begin{aligned} &\Big[\int_{Q_{d-2}} |u(x, y, z)|^p \,\mathrm{d}y\Big]^{\frac{1}{p}} \\ &\qquad \le \int_{\mathbb{R}^{d-1}} \Big(\int_{Q_{d-2}} |u_0(x - x', y - y')|^p \,\mathrm{d}y\Big)^{\frac{1}{p}} P_z(x', y') \,\mathrm{d}x' \,\mathrm{d}y'. \end{aligned} \tag{98}$$

We take supremum over $x \in (0,1)$ in the above relation, while using the notation $\mathcal{U}$ from above as well as (91), to get

$$\begin{aligned}\|u(\cdot,\cdot,z)\|_{L^\infty_x(0,1;L^p_y(Q_{d-2}))} &\leq \sup_{x\in(0,1)} \int_{\mathbb{R}} \mathcal{U}(x-x')\Big(\int_{\mathbb{R}^{d-2}} P_z(x',y')\,\mathrm{d}y'\Big)\,\mathrm{d}x' \\ &\leq \Big(\sup_{x\in(0,1)} \mathcal{U}(x)\Big)\Big(\int_{\mathbb{R}^{d-1}} P_z(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\Big) \\ &= \|u_0\|_{L^\infty_x(0,1;L^p_y(Q_{d-2}))}.\end{aligned}$$

Since this is exactly (93), the proof of (78) is complete.

To finish the proof of Step 1, it remains to consider the Neumann case. We need to show (79), that is, that for $1 < p < \infty$ there exists $C = C(d,p)$ such that

$$\|\nabla w\|_{L^p(\partial Q_d)} \leq C\|v\|_{L^p(Q_{d-1})},$$

where w denotes the unique solution (with zero average) of

$$\begin{cases} -\Delta w = 0 & \text{in } \mathbb{R}^{d-1}\times(0,1), \\ \dfrac{\partial w}{\partial x_d} = v & \text{at } \mathbb{R}^{d-1}\times\{0\}, \\ \dfrac{\partial w}{\partial x_d} = 0 & \text{at } \mathbb{R}^{d-1}\times\{1\}, \end{cases}$$

and v is a smooth 1-periodic function on $\mathbb{R}^{d-1}$ with zero average.

Similarly as before, we use Fourier series to write w in terms of the Fourier coefficients $\hat{v}$ of v:

$$w(x',x_d) = -\sum_{k\in\mathbb{Z}^{d-1}\setminus\{0\}} \frac{\hat{v}_k}{2\pi|k|} e^{2\pi i k\cdot x'}\Big(\frac{e^{-2\pi|k|x_d} + e^{-2\pi|k|(2-x_d)}}{1-e^{-4\pi|k|}}\Big),$$

for $x' \in \mathbb{R}^{d-1}$ and $x_d \in (0,1)$. Then after applying the Mikhlin multiplier theorem, see [25] or [26], Appendix, Theorem 2, to the maps $v \to \nabla w(\cdot,0)$ and $v \to \nabla w(\cdot,1)$, we see that

$$\|\nabla w\|_{L^p(Q_{d-1}\times\{0\})} + \|\nabla w\|_{L^p(Q_{d-1}\times\{1\})} \lesssim \|v\|_{L^p(Q_{d-1})}. \tag{99}$$

Indeed, the Mikhlin multiplier theorem is applied for each component, where for the tangential derivatives on the set $Q_{d-1}\times\{0\}$, indexed by $j \in \{1,\dots,d-1\}$, the multipliers are defined by

$$m_j(k) = -\frac{ik_j}{|k|}\Big(\frac{1+e^{-4\pi|k|}}{1-e^{-4\pi|k|}}\Big).$$

For the normal derivative on the set $Q_{d-1}\times\{0\}$, the multiplier is

$$m_d(k) = \Big(\frac{e^{4\pi|k|}-1}{1-e^{4\pi|k|}}\Big).$$

An explicit computation proves that, for each $j \in \{1, \dots, d\}$ and $0 \le r \le \frac{d}{2} + 1$,

$$\sup_{k \in \mathbb{R}^d} |k|^r \left|\nabla^r m_j(k)\right| < \infty.$$

Combining this with the analogous computation on $Q_{d-1} \times \{1\}$ and applying the Mikhlin multiplier theorem yields (99).

We will denote by w_b the solution of

$$\begin{cases} -\Delta w_b = 0 & \text{in } \mathbb{R}^{d-1} \times (0, 1), \\ w_b = w & \text{at } \mathbb{R}^{d-1} \times \{0\}, \\ w_b = 0 & \text{at } \mathbb{R}^{d-1} \times \{1\}. \end{cases}$$

Since w (and hence w_b) is 1-periodic with zero average at the bottom of the slab, estimate (78) implies that

$$\|\nabla w_b\|_{L^p(\partial Q_d)} \lesssim \|\nabla w\|_{L^p(Q_{d-1}\times\{0\})} \lesssim \|v\|_{L^p(Q_{d-1})}.$$

We similarly define w_t as a harmonic function in the slab that is zero at the bottom and equal to w at the top part of the boundary. For such function, we can use (78) after the change of coordinates $x_d \to 1 - x_d$, the triangle inequality and the identity $w = w_b + w_t$, which follows by uniqueness of w, to obtain

$$\|\nabla w\|_{L^p(\partial Q_d)} \lesssim \|v\|_{L^p(Q_{d-1})}.$$

Step 2. In this step, we decompose the given harmonic functions w in Q_d into a 2-periodic harmonic function on the slab $(0, 1) \times \mathbb{R}^{d-1}$, and then apply Step 1 to these to conclude proof of Lemma 4. We first address the Dirichlet case.

First, we will use the boundary data v to define a 2-periodic function $\tilde{v}^1$ on $\{0, 1\} \times \mathbb{R}^{d-1}$ using even reflections. To this end, for $i \in \{0, 1\}$, we consider the restriction $\tilde{v}^1$ of v to the right and left $(d-1)$-dimensional face

$$\tilde{v}^1(i, x_2, \dots, x_d) := v|_{\{0,1\}\times Q_{d-1}} = v(i, x_2, \dots, x_d) : \{0, 1\} \times Q_{d-1} \to \mathbb{R}. \tag{100}$$

We will extend $\tilde{v}^1$ to a 2-periodic function $\tilde{v}^1$ on $\{0, 1\} \times \mathbb{R}^{d-1}$ via a sequence of even reflections. To achieve this, define the even reflection on $\{i\} \times [0, 2] \times Q_{d-2}$ by the rule

$$\tilde{v}^1(i, x_2, \dots, x_d) := \begin{cases} \tilde{v}^1(i, x_2, \dots, x_d) & \text{if } x_2 \in [0, 1], \\ \tilde{v}^1(i, 2 - x_2, \dots, x_d) & \text{if } x_2 \in [1, 2]. \end{cases} \tag{101}$$

Then, for $1 \le l < d$ and $i = 0, 1$, assume inductively that $\tilde{v}^1$ has been defined on $\{i\} \times [0, 2]^l \times Q_{d-1-l}$ and define the extension to $\{i\} \times [0, 2]^{l+1} \times Q_{d-2-l}$ as

$$\begin{aligned} &\tilde{v}^1(i, \dots, x_{l+1}, \dots, x_d) \\ &\quad := \begin{cases} \tilde{v}^1(i, x_2, \dots, x_{l+1}, \dots, x_d) & \text{if } x_{l+1} \in [0, 1], \\ \tilde{v}^1(i, x_2, \dots, 2 - x_{l+1}, \dots, x_d) & \text{if } x_{l+1} \in [1, 2]. \end{cases} \end{aligned} \tag{102}$$

The function $\tilde{v}^1$ can then be extended to a 2-*periodic* function to the whole $\{0,1\} \times \mathbb{R}^{d-1}$, which completes the construction.

We then define w^1 to be the harmonic extension of the $\tilde{v}^1$ on the slab domain $[0,1] \times \mathbb{R}^{d-1}$. Namely,

$$\begin{cases} -\Delta w^1 = 0 & \text{in } (0,1) \times \mathbb{R}^{d-1}, \\ w^1 = \tilde{v}^1 & \text{on } \{0,1\} \times \mathbb{R}^{d-1}, \end{cases} \tag{103}$$

where we observe, in particular, that $v^1 := w - w^1$ solves

$$\begin{cases} -\Delta v^1 = 0 & \text{in } Q_d, \\ v^1 = 0 & \text{on } \{0,1\} \times Q_{d-1}. \end{cases} \tag{104}$$

The argument now proceeds inductively. Let us fix $1 \le k < d$, and suppose that for $i = 1, \dots, k$ we have already constructed functions w^i, which in particular satisfy the following two properties: First, for each $1 \le i \le k$,

$$v^i := w - \sum_{r=1}^{i} w^r \qquad \text{on } \partial Q_d, \tag{105}$$

with

$$v^i|_{\partial Q_i \times Q_{d-i}} = 0, \tag{106}$$

meaning v^i vanishes on the first i pairs of faces. Second, for $i = 1, \dots, k$, w^i is harmonic in Q_d [in fact in the whole slab $\mathbb{R}^{i-1} \times (0,1) \times \mathbb{R}^{d-i}$] and vanishes on $\partial Q_{i-1} \times Q_{d-i+1}$.

To define w^{k+1} (and hence v^{k+1}) we proceed similarly as before by first extending the restriction of v^k on the two faces $Q_{k-1} \times \{0,1\} \times Q_{d-k}$ by odd reflections in the first $k-1$ variables and even reflections in the remaining $d-k$ variables. More precisely, on $[0,2] \times Q_{k-2} \times \{0,1\} \times Q_{d-k}$ we use odd reflection to define

$$\begin{aligned} &\tilde{v}^{k+1}(x_1, \dots, x_{k-1}, i, x_{k+1}, \dots, x_d) \\ &\quad = \begin{cases} v^k(x_1, \dots, x_{k-1}, i, x_{k+1}, \dots, x_d) & \text{if } x_1 \in [0,1], \\ -v^k(2 - x_1, \dots, x_{k-1}, i, x_{k+1}, \dots, x_d) & \text{if } x_1 \in [1,2]. \end{cases} \end{aligned} \tag{107}$$

Observe that for the extension of $\tilde{v}^{k+1}$ to be a Sobolev function we require (106). Continuing this process, we use odd reflections in $k-1$ variables $x_2, \dots, x_k$ to extend $\tilde{v}^{k+1}$ to the set $[0,2]^k \times \{0,1\} \times Q_{d-k-1}$. To extend $\tilde{v}^{k+1}$ to $[0,2]^k \times \{0,1\} \times [0,2]^{d-k-1}$ we use even reflections in the remaining variables $x_{k+1}, \dots, x_d$. Finally, we extend $\tilde{v}^{k+1}$ to a 2-periodic function on $\mathbb{R}^k \times \{0,1\} \times \mathbb{R}^{d-k-1}$.

As before, the extension $\tilde{v}^{k+1}$ plays the role of the boundary data $\tilde{v}^1$ in (103), and is used to define w^{k+1}. Precisely, we require

$$\begin{cases} -\Delta w^{k+1} = 0 & \text{in } \mathbb{R}^k \times (0,1) \times \mathbb{R}^{d-k-1}, \\ w^{k+1} = \tilde{v}^{k+1} & \text{on } \mathbb{R}^k \times \{0,1\} \times \mathbb{R}^{d-k-1}, \end{cases} \tag{108}$$

where, owing to the odd symmetry of $\{\tilde{v}^{k+1}\}_{i\in\{0,1\}}$ in the first k-variables, it is immediate that

$$w^{k+1}|_{\partial Q_k \times Q_{d-k}} = 0. \tag{109}$$

After defining the entire collection $\{w^k\}_{k=1}^d$, owing to the definition (105) and properties (108) and (109),

$$\begin{cases} -\Delta\left(w - \sum_{k=1}^{d} w^k\right) = 0 & \text{in } Q_d, \\ \left(w - \sum_{k=1}^{d} w^k\right) = 0 & \text{on } \partial Q_d. \end{cases} \tag{110}$$

From this, we deduce that $w = w^1 + \cdots + w^d$. Hence, in order to estimate normal derivatives of w on the boundary ∂Q_d it suffices to bound the normal derivatives of the $\{w^k\}_{k=1}^d$ on the boundary ∂Q_d.

Again, proceeding by induction, we consider first the case $k = 1$. We decompose w^1 in the form

$$w^1 = w_0^1 + w_1^1 + w_a^1,$$

where, after writing $\langle \tilde{v}^{1,i}\rangle := \fint_{2Q_{d-1}} \tilde{v}^1(i, x_2, \ldots, x_d)$ for $i \in \{0, 1\}$, the functions w_0^1 and w_1^1 denote the solutions of

$$\begin{cases} -\Delta w_0^1 = 0 & \text{in } (0,1) \times \mathbb{R}^{d-1}, \\ w_0^1 = \tilde{v}^{1,0} - \langle \tilde{v}^{1,0}\rangle & \text{on } \{0\} \times \mathbb{R}^{d-1}, \\ w_0^1 = 0 & \text{on } \{1\} \times \mathbb{R}^{d-1}, \end{cases} \tag{111}$$

and

$$\begin{cases} -\Delta w_1^1 = 0 & \text{on } (0,1) \times \mathbb{R}^{d-1}, \\ w_1^1 = 0 & \text{on } \{0\} \times \mathbb{R}^{d-1}, \\ w_1^1 = \tilde{v}^{1,1} - \langle \tilde{v}^{1,1}\rangle & \text{on } \{1\} \times \mathbb{R}^{d-1}, \end{cases} \tag{112}$$

and w_a^1 is the solution

$$\begin{cases} -\Delta w_a^1 = 0 & \text{in } (0,1) \times \mathbb{R}^{d-1}, \\ w_a^1 = \langle \tilde{v}^{1,0}\rangle & \text{on } \{0\} \times \mathbb{R}^{d-1}, \\ w_a^1 = \langle \tilde{v}^{1,1}\rangle & \text{on } \{1\} \times \mathbb{R}^{d-1}, \end{cases} \tag{113}$$

which admits the explicit representation $w_a^1 = (1 - x_1)\langle \tilde{v}^{1,0}\rangle + x_1\langle \tilde{v}^{1,1}\rangle$.

We will now show how estimate (78) from Step 1 implies the first estimate in Lemma 4. Using (78) for $i = 0, 1$ we get

$$\|\nabla w_i^1\|_{L^p(\partial Q_d)} \lesssim \|\nabla^{\tan} v\|_{L^p(\partial Q_d)}. \tag{114}$$

And, by direct computation,

$$\|\nabla w_a^1\|_{L^p(\partial Q_d)} \lesssim |\langle \tilde{v}^{1,1}\rangle - \langle \tilde{v}^{1,0}\rangle| \lesssim \|\nabla^{\tan} v\|_{L^p(\partial Q_d)}. \tag{115}$$

In combination, (114) and (115) yield

$$\|\nabla w^1\|_{L^p(\partial Q_d)} \lesssim \|\nabla^{\tan} v\|_{L^p(\partial Q_d)}. \tag{116}$$

Notice here that we consider the full gradient, not merely the normal component, since it will be necessary for the induction step.

Assume inductively that for $1 < k < d$ estimate (116) is satisfied for every $\{w^i\}$, $i - 1, \ldots, k$. Then, in view of (78), (105), the definition of $\tilde{v}^{k+1}$ (107), (108) and the inductive hypothesis,

$$\begin{aligned} \|\nabla w^{k+1}\|_{L^p(\partial Q_d)} &\lesssim \|\nabla^{\tan} \tilde{v}^{k+1}\|_{L^p(Q_k\times\{0,1\}\times Q_{d-k-1})} \\ &\lesssim \|\nabla^{\tan} v\|_{L^p(\partial Q_d)} + \sum_{i=1}^{k} \|\nabla^{\tan} w^i\|_{L^p(\partial Q_d)} \\ &\lesssim \|\nabla^{\tan} v\|_{L^p(\partial Q_d)}. \end{aligned} \tag{117}$$

Since $w = w^1 + \cdots + w^d$ by (110), the proof of the lemma in the Dirichlet case is complete.

In the Neumann case, the process works verbatim with a single modification: we need to switch the role of even and odd reflections.

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