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WEAKLY HARMONIC OSCILLATORS PERTURBED BY A CONSERVATIVE NOISE[1]

BY CÉDRIC BERNARDIN, PATRÍCIA GONÇALVES[2] AND MILTON JARA[3]

Université Côte d'Azur, Universidade de Lisboa and Instituto de Matemática Pura e Aplicada

We consider a chain of weakly harmonic coupled oscillators perturbed by a conservative noise. We show that by tuning accordingly the coupling constant, energy can diffuse like a Brownian motion or superdiffuse like a maximally $3/2$-stable asymmetric Lévy process. For a critical value of the coupling, the energy diffusion is described by a family of Lévy processes which interpolate between these two processes.

1. Introduction. The problem of anomalous diffusion of energy in low dimensions has been the subject of intense research in recent years; see [12] for a recent review. In [2], Hamiltonian chains of oscillators perturbed by conservative noise were introduced as a mathematically tractable model for energy superdiffusion. From the study of these models [2–4, 8, 11], the relevance of the different conservation laws in the origin of anomalous diffusion has started to be mathematically understood. These results have served as an inspiration and match perfectly the predictions of Spohn's fluctuating hydrodynamics theory [15], which allows us to make precise conjectures on the decay of correlations of these chains. Fluctuating hydrodynamics predicts several universality classes for the behavior of energy correlations in these chains, and a natural question is to understand how these different universality classes are related. In particular, we focus on the derivation of *crossovers* between different universality classes. We say that a family of equations, parametrized by some variable γ, is a *crossover* between two universality classes if the equations governing these universality classes are recovered taking the limits $\gamma \to 0$ and $\gamma \to \infty$.

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In [4, 8], it was proved that the scaling limit of the energy fluctuations of stochastic harmonic chains with at least two conserved quantities is governed by a fractional heat equation

$$\partial_t u = \mathscr{L} u, \tag{1.1}$$

where $\mathscr{L}$ is the generator of a $3/2$-stable Lévy process. Notice that there is some freedom in the choice of $\mathscr{L}$, which corresponds to the *skewness* of the corresponding Lévy process. The skewness of this operator actually depends on the sound velocity of the *mechanical* modes of the chain: volume in the case of [4] and momentum and stretch in the case of the model considered in [8]. These results correspond to the *zero-tension* universality class in the fluctuating hydrodynamics framework [15, 16].

When energy is the only conserved quantity in the chain, energy has normal diffusion, and energy fluctuations are governed by the usual heat equation, corresponding to the Edwards–Wilkinson universality class

$$\partial_t u = \Delta u.$$

In [5, 6], a crossover between these two universality classes was obtained by introducing a noise of vanishing intensity that destroys the conservation of volume. Tuning this intensity in a proper way, it was shown that energy fluctuations are governed by the evolution equation

$$\partial_t u = \mathscr{L}_\gamma u,$$

where $\mathscr{L}_\gamma$ is a nonlocal operator satisfying

$$\lim_{\gamma \to 0} \mathscr{L}_\gamma = \mathscr{L}, \qquad \lim_{\gamma \to \infty} \sqrt{\gamma} \mathscr{L}_\gamma = \Delta.$$

In this article we take a different route, which we believe is more natural[4] and provides a different interpolating operator. The stochastic chains introduced in [2, 7] are generated by an operator of the form $L_\kappa = \kappa A + S$, where A is the generator of the deterministic (Hamiltonian) part of the dynamics and S is the generator of the stochastic part of the dynamics. The operator A is antisymmetric with respect to the Gibbs measures associated to the chain, and S is symmetric. If the operator A is absent from the dynamics, namely $\kappa = 0$, energy is diffusive. If $\kappa = 1$, energy is superdiffusive [4]. Therefore, our task is to find how should κ decay in order to observe an evolution different from the cases $\kappa = 0$ and $\kappa = 1$.

This situation is very reminiscent of what happens in the so-called *weakly asymmetric*, one-dimensional exclusion process. The relevant quantity there is the current of particles through the origin. In the asymmetric case, it has been proved [10]

[4]We think it is more natural because here we only tune a parameter of the original model instead of adding some extra stochastic noise to the original model in order to link the two universality classes.

that the current of particles through the origin converges to the celebrated Tracy–Widom law. In the symmetric case, fluctuations are Gaussian [9, 13]. The crossover regime appears for $\kappa := \kappa_n = \frac{\gamma}{n^{1/2}}$, where $\frac{1}{n} \to 0$ is the space scaling, and it has been described in [1, 14]. In a more elaborated development, the *KPZ equation* serves as a crossover equation between the Edwards–Wilkinson universality class and the KPZ universality class.

In our situation, since the macroscopic model is linear, fluctuations are always Gaussian, but the covariance structure changes drastically with κ. It turns out that the crossover scaling is $\kappa = \frac{\gamma}{n^{1/3}}$, and the crossover operator is given by

$$\mathbb{L}_{\gamma,1/3} = \Delta + \gamma^{3/2}\mathscr{L}.$$

It is clear that $\mathbb{L}_{0,1/3} = \Delta$ and

$$\lim_{\gamma\to\infty} \frac{1}{\gamma^{3/2}}\mathbb{L}_{\gamma,1/3} = \mathscr{L},$$

which is the operator governing the energy fluctuations for $\kappa = 1$.

Note that the scaling $\kappa = \frac{\gamma}{n^{1/3}}$ differs from the crossover scaling of the asymmetric exclusion process. In order to derive this scaling in a heuristic way, we need to describe the results in [4] in more detail. Recall that the stochastic chains considered here have two conserved quantities: the *energy* and the so-called *volume*. It turns out that the volume serves as a fast variable for the evolution of the energy. Consider the chain generated by the operator $n^{3/2}(\kappa A + \lambda S)$. Let $P_t^n(x)$ be the energy correlation function, defined as

$$P_t^n(x) = S_{tn^{3/2}}(\lfloor nx \rfloor),$$

where S_t is defined in (2.10), and let $\psi_t^n(x, y)$ be the space-time energy-volume correlation function defined as

$$\psi_t^n(x, y) = S_{tn^{3/2}}\left(\lfloor nx \rfloor + \left\lfloor \frac{\sqrt{n}y}{2} \right\rfloor, \lfloor nx \rfloor - \left\lfloor \frac{\sqrt{n}y}{2} \right\rfloor\right),$$

where for $x, y \in \mathbb{Z}$, $S_t(x, y)$ is defined in (3.1). It turns out that these functions are well approximated by the solution of

$$\begin{cases} \partial_t P_t = -2\kappa \partial_x \psi_t|_{y=0} + \dfrac{\lambda}{\sqrt{n}}\partial_x^2 P_t, \\ 0 = -\kappa \partial_x \psi_t + \lambda \partial_y^2 \psi_t, \\ \kappa \partial_x P_t(x) = -4\lambda \partial_y \psi_t|_{y=0}. \end{cases} \tag{1.2}$$

Note the presence of the correction term $\frac{\lambda}{\sqrt{n}}\partial_x^2 P_t$. This coupled system can be solved in ψ, giving an effective equation for P_t:

$$\partial_t P_t = \frac{\kappa^{3/2}}{\lambda^{1/2}}\mathscr{L} P_t + \frac{\lambda}{\sqrt{n}}\partial_x^2 P_t,$$

where $\mathscr{L}$ is the generator of the 3/2-stable Lévy process appearing in (1.1). If we choose $\kappa = \gamma\lambda^{1/3}$, the first operator does not depend on λ. We conclude that for $\lambda \ll \sqrt{n}$, the Laplacian part of this equation vanishes in the limit, while for $\lambda = \sqrt{n}$, the Laplacian part has a finite, nontrivial contribution in the limit. Since $\kappa = \gamma\lambda^{1/3}$ and $\lambda = \sqrt{n}$, this corresponds to the microscopic generator $n^{3/2}\kappa A + n^{3/2}\lambda S = n^2(\gamma n^{-1/3}A + S)$, explaining the crossover scaling $\frac{\gamma}{n^{1/3}}$ in the diffusive scale, as well as the constant $\gamma^{3/2}$ in the definition of $\mathbb{L}_{\gamma,1/3}$.

The fact that (1.2) provides a good approximation of P_t^n and ψ_t^n is based on the approach initiated in [4]. However to control the error terms appearing in this approximation, we have to be more clever than in [4]. Indeed, one of the error terms can not be estimated by a *static* estimate and is controlled by a *dynamical* argument (see the discussion after Lemma 3.2).

The paper is organized as follows. In Section 2, we describe the model and the main result. The proof is given in Section 3 and the technical computations appear in the Appendix.

2. The model.

2.1. *Description of the model.* For $\eta : \mathbb{Z} \to \mathbb{R}$ and $\alpha > 0$, define

$$\|\|\eta\|\|_\alpha = \sum_{x\in\mathbb{Z}} |\eta(x)| e^{-\alpha|x|} \tag{2.1}$$

and let $\Omega_\alpha = \{\eta : \mathbb{Z} \to \mathbb{R}; \|\|\eta\|\|_\alpha < +\infty\}$. The normed space $(\Omega_\alpha, \|\|\cdot\|\|)$ turns out to be a Banach space. In Ω_α we consider the system of coupled ODEs

$$\frac{d}{dt}\tilde{\eta}_t(x) = \kappa[\tilde{\eta}_t(x+1) - \tilde{\eta}_t(x-1)] \qquad \text{for } t \geq 0 \text{ and } x \in \mathbb{Z}, \tag{2.2}$$

where $\kappa > 0$ is a constant.The Picard–Lindelöf theorem shows that the system (2.2) is well posed in Ω_α. We will superpose to this deterministic dynamics a stochastic dynamics as follows. To each bond $\{x, x+1\}$, with $x \in \mathbb{Z}$, we associate an exponential clock of rate one. Those clocks are independent among them. Each time the clock associated to $\{x, x+1\}$ rings, we exchange the values of $\tilde{\eta}_t(x)$ and $\tilde{\eta}_t(x+1)$. Since there is an infinite number of such clocks, the existence of this dynamics needs to be justified. If we freeze the clocks associated to bonds not contained in $\{-M, \dots, M\}$, the dynamics is easy to define, since it corresponds to a piecewise deterministic Markov process. It can be shown that for an initial data η_0 in

$$\Omega = \bigcap_{\alpha>0} \Omega_\alpha, \tag{2.3}$$

these piecewise deterministic processes stay at Ω and they converge to a well-defined Markov process $\{\eta_t; t \geq 0\}$, as $M \to \infty$, see [7] and references therein.

This Markov process is the rigorous version of the dynamics described above. Notice that Ω is a complete metric space with respect to the distance

$$d(\eta,\xi)=\sum_{\ell\in\mathbb{N}}\frac{1}{2^\ell}\min\{1,|||\eta-\xi|||_{\frac{1}{\ell}}\}. \tag{2.4}$$

Let us describe the generator of the process $\{\eta_t; t\geq 0\}$. For $x, y\in\mathbb{Z}$ and $\eta\in\Omega$ we define $\eta^{x,y}\in\Omega$ as

$$\eta^{x,y}(z)=\begin{cases}\eta(y); & z=x,\\ \eta(x); & z=y,\\ \eta(z); & z\neq x,y.\end{cases} \tag{2.5}$$

We say that a function $f:\Omega\to\mathbb{R}$ is *local* if there exists a finite set $B\subseteq\mathbb{Z}$ such that $f(\eta)=f(\xi)$ whenever $\eta(x)=\xi(x)$ for any $x\in B$. For a smooth function $f:\Omega\to\mathbb{R}$ we denote by $\partial_x f:\Omega\to\mathbb{R}$ its partial derivative with respect to $\eta(x)$. For a function $f:\Omega\to\mathbb{R}$ that is local, smooth and bounded, we define $L_\kappa f:\Omega\to\mathbb{R}$ as $L_\kappa f=Sf+\kappa Af$, where for any $\eta\in\Omega$,

$$Sf(\eta)=\sum_{x\in\mathbb{Z}}\big(f(\eta^{x,x+1})-f(\eta)\big), \tag{2.6}$$

$$Af(\eta)=\sum_{x\in\mathbb{Z}}\big(\eta(x+1)-\eta(x-1)\big)\partial_x f(\eta). \tag{2.7}$$

The process $\{\eta_t; t\geq 0\}$ has a family $\{\mu_{\rho,\beta}; \rho\in\mathbb{R}, \beta>0\}$ of invariant measures given by

$$\mu_{\rho,\beta}(d\eta)=\prod_{x\in\mathbb{Z}}\sqrt{\frac{\beta}{2\pi}}\exp\Big\{-\frac{\beta}{2}\big(\eta(x)-\rho\big)^2\Big\}\,d\eta(x). \tag{2.8}$$

It also has two conserved quantities. If one of the numbers

$$\sum_{x\in\mathbb{Z}}\eta_0(x),\qquad \sum_{x\in\mathbb{Z}}\eta_0(x)^2 \tag{2.9}$$

is finite, then its value is preserved by the evolution of $\{\eta_t; t\geq 0\}$. Following [7], we will call these conserved quantities *volume* and *energy*, respectively. Notice that $\int\eta(x)\,d\mu_{\rho,\beta}=\rho$ and $\int\eta(x)^2\,d\mu_{\rho,\beta}=\rho^2+\frac{1}{\beta}$.

REMARK 2.1. For the interpretation of the system considered here as a system of coupled harmonic oscillators, we refer the interested reader to [7].

2.2. *Description of the result.* Fix $\rho\in\mathbb{R}$ and $\beta>0$, and consider the process $\{\eta_t; t\geq 0\}$ with initial distribution $\mu_{\rho,\beta}$. Notice that $\{\eta_t+\lambda; t\geq 0\}$ has the same distribution of the process $\{\eta_t; t\geq 0\}$ with initial measure $\mu_{\rho+\lambda,\beta}$. Therefore, we can assume, without loss of generality, that $\rho=0$. We will write $\mu_\beta=\mu_{0,\beta}$ and we will denote by $\mathbb{P}$ the law of $\{\eta_t; t \geq 0\}$ and by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. The *energy correlation function* $\{S_t(x); x \in \mathbb{Z}, t \geq 0\}$ is defined as

$$S_t(x) = \frac{\beta^2}{2}\mathbb{E}\left[\left(\eta_0(0)^2 - \frac{1}{\beta}\right)\left(\eta_t(x)^2 - \frac{1}{\beta}\right)\right] \tag{2.10}$$

for any $x \in \mathbb{Z}$ and any $t \geq 0$. The constant $\frac{\beta^2}{2}$ is just the inverse of the variance of $\eta(x)^2 - \frac{1}{\beta}$ under μ_β. By translation invariance of the dynamics and the initial distribution μ_β, we see that

$$\frac{\beta^2}{2}\mathbb{E}\left[\left(\eta_0(x)^2 - \frac{1}{\beta}\right)\left(\eta_t(y)^2 - \frac{1}{\beta}\right)\right] = S_t(y - x) \tag{2.11}$$

for any $x, y \in \mathbb{Z}$. The following result was proved in [4].[5]

THEOREM 2.2 ([4]). *Assume $\kappa > 0$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions of compact support. Then*,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x,y\in\mathbb{Z}} f\left(\frac{x}{n}\right)g\left(\frac{y}{n}\right)S_{tn^{3/2}}(x - y) = \iint f(x)g(y)P_t(x - y)\,dx\,dy, \tag{2.12}$$

where $\{P_t(x); x \in \mathbb{R}, t \geq 0\}$ is the fundamental solution of the fractional heat equation

$$\partial_t u = -\frac{\kappa^{3/2}}{\sqrt{2}}\{(-\Delta)^{3/4} - \nabla(-\Delta)^{1/4}\}u. \tag{2.13}$$

It is not difficult to check that if $\kappa = 0$ then the following result holds.

THEOREM 2.3. *Assume $\kappa = 0$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions of compact support. Then*,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x,y\in\mathbb{Z}} f\left(\frac{x}{n}\right)g\left(\frac{y}{n}\right)S_{tn^2}(x - y) = \iint f(x)g(y)P_t(x - y)\,dx\,dy, \tag{2.14}$$

where $\{P_t(x); x \in \mathbb{R}, t \geq 0\}$ is the fundamental solution of the heat equation

$$\partial_t u = \Delta u. \tag{2.15}$$

We note that the previous results are obtained by looking at the system in different time scales: either in a superdiffusive time scale $tn^{3/2}$ or in the diffusive time scale tn^2. Our aim is now to investigate a crossover between these two regimes by

[5]Note that in [4], $\kappa = 1$ was assumed for simplicity but it is straightforward to extend the results there to cover the case $\kappa \neq 1$.

letting κ to be arbitrary small. To this end, we introduce a large parameter $n \in \mathbb{N}$ and take

$$\kappa := \kappa_n = \frac{\gamma}{n^b}, \tag{2.16}$$

where $\gamma > 0$ is fixed and $b \in (0, +\infty)$. Observe that the two previous theorems describe, respectively, the limiting cases $b = 0$ and $b = +\infty$. The main result of this paper is the following theorem.

THEOREM 2.4. *Assume $\kappa_n = \frac{\gamma}{n^b}$ where $b \geq 0$ and $\gamma > 0$. We define the exponent a of the time scale by $a = \inf(3/2 + 3b/2, 2)$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions of compact support. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x,y\in\mathbb{Z}} f\left(\frac{x}{n}\right) g\left(\frac{y}{n}\right) S_{tn^a}(x-y) = \iint f(x)g(y)P_t^{\gamma,b}(x-y)\,dx\,dy, \tag{2.17}$$

where $\{P_t^{\gamma,b}(x); x \in \mathbb{R}, t \geq 0\}$ is the fundamental solution of the equation

$$\partial_t u = \mathbb{L}_{\gamma,b} u \tag{2.18}$$

with $\mathbb{L}_{\gamma,b}$ being the generator of the following Lévy process:

$$\mathbb{L}_{\gamma,b} = \mathbf{1}_{b\geq 1/3}\Delta + \gamma^{3/2}\mathbf{1}_{b\leq 1/3}\mathscr{L}, \tag{2.19}$$

where $\mathscr{L} = -\frac{1}{\sqrt{2}}\{(-\Delta)^{3/4} - \nabla(-\Delta)^{1/4}\}$.

The most interesting regime is for $b = 1/3$ since the operator $\mathbb{L}_{\gamma,1/3}$ is a Lévy process connecting the Brownian motion to the totally asymmetric 3/2-stable Lévy process:

$$\mathbb{L}_{\gamma,1/3} \xrightarrow[\gamma\to 0]{} \Delta$$

and

$$\gamma^{-3/2}\mathbb{L}_{\gamma,1/3} \xrightarrow[\gamma\to\infty]{} \mathscr{L}.$$

3. Proof of Theorem 2.4. Following the method introduced in [4], the proof of this theorem will be established by a careful study of the correlation function $\{S_t(x,y); x \neq y \in \mathbb{Z}, t \geq 0\}$ given by

$$S_t(x,y) = \frac{\beta^2}{2}\mathbb{E}\left[\left(\eta_0(0)^2 - \frac{1}{\beta}\right)\eta_t(x)\eta_t(y)\right] \tag{3.1}$$

for any $t \geq 0$ and any $x \neq y \in \mathbb{Z}$. Notice that this definition makes perfect sense for $x = y$ and, in fact, we have $S_t(x,x) = S_t(x)$. For notational convenience, we define $S_t(x,x)$ as equal to $S_t(x)$. However, these quantities are of different nature, since

$S_t(x)$ is related to *energy fluctuations* and $S_t(x, y)$ is related to *volume fluctuations* (for $x \neq y$).

Let $a = \inf(3/2 + 3b/2, 2)$ which fixes the time scale in which we observe the process at. The generator $n^a L_{\kappa_n}$ is denoted by $\mathscr{L}_n$. From now on, we assume $\beta = 1$, since the general case can be recovered from this particular case by scaling.

For $d \geq 1$, denote by $\mathscr{C}_c^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ of compact support. For any function $f \in \mathscr{C}_c^\infty(\mathbb{R}^d)$, define the discrete $\ell^2(\mathbb{Z}^d)$-norm as

$$\|f\|_{2,n} = \sqrt{\frac{1}{n^d} \sum_{x \in \mathbb{Z}^d} f\Big(\frac{x}{n}\Big)^2}. \tag{3.2}$$

Let $g \in \mathscr{C}_c^\infty(\mathbb{R})$ be a fixed function. For any $n \in \mathbb{N}$, any $t \geq 0$ and any $f \in \mathscr{C}_c^\infty(\mathbb{R})$, we define the field $\{\mathscr{S}_t^n; t \geq 0\}$ as

$$\mathscr{S}_t^n(f) = \frac{1}{n} \sum_{x,y \in \mathbb{Z}} g\Big(\frac{x}{n}\Big) f\Big(\frac{y}{n}\Big) S_{tn^a}(y - x). \tag{3.3}$$

We observe that the previous field is the one which appears at the left-hand side of (2.17). Rearranging terms in a convenient way we have that

$$\begin{aligned} \mathscr{S}_t^n(f) = \frac{1}{2}\mathbb{E}\Big[&\Big(\frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} g\Big(\frac{x}{n}\Big)\big(\eta_0(x)^2 - 1\big)\Big) \\ &\times \Big(\frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}} f\Big(\frac{y}{n}\Big)\big(\eta_{tn^a}(y)^2 - 1\big)\Big)\Big]. \end{aligned} \tag{3.4}$$

By a simple application of the Cauchy–Schwarz inequality we have that

$$\big|\mathscr{S}_t^n(f)\big| \leq \|g\|_{2,n}\|f\|_{2,n}. \tag{3.5}$$

For a function $h \in \mathscr{C}_c^\infty(\mathbb{R}^2)$ we define $\{Q_t^n(h); t \geq 0\}$ as

$$\begin{aligned} Q_t^n(h) = \frac{1}{2}\mathbb{E}\Big[&\Big(\frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} g\Big(\frac{x}{n}\Big)\big(\eta_0(x)^2 - 1\big)\Big) \\ &\times \Big(\frac{1}{n} \sum_{y \neq z \in \mathbb{Z}} h\Big(\frac{y}{n}, \frac{z}{n}\Big)\eta_{tn^a}(y)\eta_{tn^a}(z)\Big)\Big]. \end{aligned} \tag{3.6}$$

Note that $Q_t^n(h)$ depends only on the symmetric part of the function h and, along this article, we will always assume, without loss of generality, that $h(x, y) = h(y, x)$ for any $x, y \in \mathbb{Z}$. We also point out that $Q_t^n(h)$ does not depend on the values of h at the diagonal $\{x = y\}$. Again, by a simple application of the Cauchy–Schwarz inequality we have that

$$\big|Q_t^n(h)\big| \leq 2\|g\|_{2,n}\|h\|_{2,n}. \tag{3.7}$$

REMARK 3.1. Observe that $\mathscr{S}_t^n(f)$ can be defined also for any function $f : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ with compact support and that by using (3.5), the definition can be extended to any function $f : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ such that $\|f\|_{2,n} < \infty$. A similar remark applies to $Q_t^n(h)$.

Note that, by an elementary computation, whose details are given in Appendix A, we have that

$$\frac{d}{dt}\mathscr{S}_t^n(f) = -2\gamma n^{a-b-3/2} Q_t^n(\nabla_n f \otimes \delta) + n^{a-2}\mathscr{S}_t^n(\Delta_n f), \tag{3.8}$$

where for a function $f \in \mathscr{C}_c^\infty(\mathbb{R})$, $\Delta_n f : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ is a discrete approximation of the second derivative of f given by

$$\Delta_n f\left(\frac{x}{n}\right) = n^2\left(f\left(\frac{x+1}{n}\right) + f\left(\frac{x-1}{n}\right) - 2f\left(\frac{x}{n}\right)\right) \tag{3.9}$$

and $\nabla_n f \otimes \delta : \frac{1}{n}\mathbb{Z}^2 \to \mathbb{R}$ is a discrete approximation of the distribution $f'(x) \otimes \delta(x = y)$, where $\delta(x = y)$ is the δ of Dirac at the line $x = y$ and it is given by

$$(\nabla_n f \otimes \delta)\left(\frac{x}{n}, \frac{y}{n}\right) = \begin{cases} \frac{n^2}{2}\left(f\left(\frac{x+1}{n}\right) - f\left(\frac{x}{n}\right)\right); & y = x+1, \\ \frac{n^2}{2}\left(f\left(\frac{x}{n}\right) - f\left(\frac{x-1}{n}\right)\right); & y = x-1, \\ 0; & \text{otherwise.} \end{cases} \tag{3.10}$$

At this point we split the proof of the theorem according to the range of the parameter b. First, we treat the case $b > 1$. For that purpose, we note that putting together (3.7) plus the fact that $\|\nabla_n f \otimes \delta\|_{2,n} = \mathscr{O}(\sqrt{n})$, we get that for $b > 1$ and $a = 2$,

$$\lim_{n\to\infty} n^{a-b-3/2} Q_t^n(\nabla_n f \otimes \delta) = 0.$$

This concludes the proof of the theorem for $b > 1$, since equation (3.8) for $\mathscr{S}_t^n(f)$ is now closed and a simple tightness argument gives the result. Now we note that applying the H_{-1}-norm argument as in the proof of Theorem 4 in [3] we get that for $b > 1/2$, the previous result still holds. We do not present this proof here since it is a reproduction of the arguments of Theorem 4 of [3] but we ask the interested reader to look particularly at the last inequality of the proof of that theorem and to note that the time scaling for this range of b is $a = 2$. At this point we still need to analyze the remaining cases where $b \leq 1/2$. We also point out that the previous arguments do not use the asymmetric part of the dynamics in order to control the problematic term $n^{a-b-3/2} Q_t^n(\nabla_n f \otimes \delta)$. The understanding of the effect of the asymmetric part is crucial to cover the case $b \leq 1/2$. From now on, we assume that this is the case (in fact, the rest of the argument is valid for $b < 1$ but not for

$b = 1$). Since from (3.8) the time evolution of $\mathscr{S}_t^n$ depends on the time evolution of Q_t^n, we need also to find an equation similar to (3.8) for Q_t^n, in order to close the equation for $\mathscr{S}_t^n$. We note that this argument has already been used in [4] when treating the case corresponding to $b = 0$. By a simple computation, whose details are given in Appendix A, we have that

$$\begin{aligned}\frac{d}{dt} Q_t^n(h) &= Q_t^n\big(n^{a-2}\Delta_n h + \gamma n^{a-b-1}\mathscr{A}_n h\big) - 2\gamma n^{a-b-3/2}\mathscr{S}_t^n(\mathscr{D}_n h) \\ &\quad + 2Q_t^n\big(n^{a-2}\widetilde{\mathscr{D}}_n h\big),\end{aligned} \tag{3.11}$$

where for $h \in \mathscr{C}_c^\infty(\mathbb{R}^2)$ the operator $\Delta_n h : \frac{1}{n}\mathbb{Z}^2 \to \mathbb{R}$ is a discrete approximation of the 2-dimensional Laplacian of h and it is given by

$$\begin{aligned}\Delta_n h\Big(\frac{x}{n}, \frac{y}{n}\Big) &= n^2\Big(h\Big(\frac{x+1}{n}, \frac{y}{n}\Big) + h\Big(\frac{x-1}{n}, \frac{y}{n}\Big) \\ &\quad + h\Big(\frac{x}{n}, \frac{y+1}{n}\Big) + h\Big(\frac{x}{n}, \frac{y-1}{n}\Big) - 4h\Big(\frac{x}{n}, \frac{y}{n}\Big)\Big),\end{aligned} \tag{3.12}$$

$\mathscr{A}_n h : \frac{1}{n}\mathbb{Z}^2 \to \mathbb{R}$ is a discrete approximation of the directional derivative $(-2, -2) \cdot \nabla h$ and is given by

$$\begin{aligned}\mathscr{A}_n h\Big(\frac{x}{n}, \frac{y}{n}\Big) &= n\Big(h\Big(\frac{x}{n}, \frac{y-1}{n}\Big) + h\Big(\frac{x-1}{n}, \frac{y}{n}\Big) \\ &\quad - h\Big(\frac{x}{n}, \frac{y+1}{n}\Big) - h\Big(\frac{x+1}{n}, \frac{y}{n}\Big)\Big),\end{aligned} \tag{3.13}$$

the operator $\mathscr{D}_n h : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ is a discrete approximation of the directional derivative of h along the diagonal $x = y$ and it is given by

$$\mathscr{D}_n h\Big(\frac{x}{n}\Big) = n\Big(h\Big(\frac{x}{n}, \frac{x+1}{n}\Big) - h\Big(\frac{x-1}{n}, \frac{x}{n}\Big)\Big) \tag{3.14}$$

and the operator $\widetilde{\mathscr{D}}_n$ is defined as follows: $\widetilde{\mathscr{D}}_n h$ is a symmetric function from $\frac{1}{n}\mathbb{Z}^2$ into $\mathbb{R}$, given by

$$\widetilde{\mathscr{D}}_n h\Big(\frac{x}{n}, \frac{y}{n}\Big) = \begin{cases} n^2\Big(\widetilde{\mathscr{E}}_n h\Big(\frac{x}{n}\Big) - \frac{1-\kappa}{2}\widetilde{\mathscr{F}}_n h\Big(\frac{x}{n}\Big)\Big); & y = x+1, \\ n^2\Big(\widetilde{\mathscr{E}}_n h\Big(\frac{y}{n}\Big) - \frac{1-\kappa}{2}\widetilde{\mathscr{F}}_n h\Big(\frac{y}{n}\Big)\Big); & y = x-1, \\ 0; & \text{otherwise,}\end{cases} \tag{3.15}$$

with

$$\widetilde{\mathscr{E}}_n h\Big(\frac{x}{n}\Big) = h\Big(\frac{x}{n}, \frac{x+1}{n}\Big) - h\Big(\frac{x}{n}, \frac{x}{n}\Big) \tag{3.16}$$

and

$$\widetilde{\mathscr{F}}_n h\left(\frac{x}{n}\right) = h\left(\frac{x+1}{n}, \frac{x+1}{n}\right) - h\left(\frac{x}{n}, \frac{x}{n}\right). \tag{3.17}$$

It is given by zero on the other points of $\frac{1}{n}\mathbb{Z}^2$.

In order to combine (3.8) and (3.11) in such a way that we obtain a simple equation for the time evolution of $\mathscr{S}_t^n$ we consider $h_n : \frac{1}{n}\mathbb{Z}^2 \to \mathbb{R}$ as the unique square-integrable solution of the Poisson equation

$$\Delta_n h\left(\frac{x}{n}, \frac{y}{n}\right) + \gamma n^{1-b} \mathscr{A}_n h\left(\frac{x}{n}, \frac{y}{n}\right) = 2\gamma n^{1/2-b} \nabla_n f \otimes \delta\left(\frac{x}{n}, \frac{y}{n}\right). \tag{3.18}$$

Note that h_n is independent of a. We get that[6]

$$\begin{aligned} \frac{d}{dt}\mathscr{S}_t^n(f) = {}& -\frac{d}{dt} Q_t^n(h_n) + \mathscr{S}_t^n\big(n^{a-2}\Delta_n f - 2\gamma n^{a-b-3/2}\mathscr{D}_n h_n\big) \\ & + 2Q_t^n\big(n^{a-2}\widetilde{\mathscr{D}}_n h_n\big). \end{aligned} \tag{3.19}$$

By integrating the last expression in time we have that for $T > 0$

$$\begin{aligned} \mathscr{S}_T^n(f) - \mathscr{S}_0^n(f) = {}& \int_0^T \mathscr{S}_t^n\big(n^{a-2}\Delta_n f - 2\gamma n^{a-b-3/2}\mathscr{D}_n h_n\big)\,dt \\ & + Q_0^n(h_n) - Q_T^n(h_n) + 2\int_0^T Q_t^n\big(n^{a-2}\widetilde{\mathscr{D}}_n h_n\big)\,dt. \end{aligned} \tag{3.20}$$

Now we need to analyze each term at the right-hand side of (3.20). Let us first observe that for any $b \geq 0$

$$\lim_{n\to\infty} \|h_n\|_{2,n}^2 = 0. \tag{3.21}$$

This is proved in Appendix C.1. We note then that by (3.7) the second and third terms at the right-hand side of (3.20) vanish, as $n \to \infty$. The contribution of the main term at the right-hand side of (3.20) is encapsulated in the following lemma which is proved in Appendix C.2.

LEMMA 3.2. *Let $f \in \mathscr{C}_c^\infty(\mathbb{R})$. If $a = \inf(3/2 + 3b/2, 2)$ and $b \in (0, +\infty)$, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}} \left| \{n^{a-2}\Delta_n f - 2\gamma n^{a-b-3/2}\mathscr{D}_n h_n\}\left(\frac{x}{n}\right) - \mathbb{L}_{\gamma,b} f\left(\frac{x}{n}\right)\right|^2 = 0, \tag{3.22}$$

where $\mathbb{L}_{\gamma,b}$ is defined in (2.19).

---

[6]Observe that $Q_t^n(h_n)$ makes sense by Remark 3.1.

The last term at the right-hand side of (3.20) is not so easy to control combining the Cauchy–Schwarz estimate with a bound on the $\mathbb{L}^2$-norm of the function involved, that is of $n^{a-2}\widetilde{\mathscr{D}}_n h_n$. Nevertheless, by repeating the same argument as above, that is, by rewriting the time evolution of the field $\mathcal{Q}_t^n$ in terms of a solution of another Poisson equation which gives us an expression for the term at the right-hand side of equation (3.20), we obtain the following result:

LEMMA 3.3. *Let $h_n : \frac{1}{n}\mathbb{Z}^2 \to \mathbb{R}$ be the solution of the Poisson equation given in* (3.18), $a = \inf(3/2 + 3b/2, 2)$ *and* $b \in (0, 1)$. *For any $T > 0$ we have that*

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big(\int_0^T \mathcal{Q}_t^n\big(n^{a-2}\widetilde{\mathscr{D}}_n h_n\big)\,dt\Big)^2\Big] = 0.$$

PROOF. As mentioned above, in order to prove the result we use again (3.11) applied to $h = v_n$ where v_n, is the solution of the Poisson equation

$$n^{a-2}\Delta_n v_n\Big(\frac{x}{n}, \frac{y}{n}\Big) + \gamma n^{a-1-b}\mathscr{A}_n v_n\Big(\frac{x}{n}, \frac{y}{n}\Big) = n^{a-2}\widetilde{\mathscr{D}}_n h_n. \tag{3.23}$$

Then by integrating in time (3.11) we have

$$\begin{aligned}
&\int_0^T \mathcal{Q}_t^n\big(n^{a-2}\widetilde{\mathscr{D}}_n h_n\big)\,dt \\
&\quad = 2\gamma n^{a-b-3/2}\int_0^T \mathscr{S}_t^n(\mathscr{D}_n v_n)\,dt - 2\int_0^T \mathcal{Q}_t^n\big(n^{a-2}\widetilde{\mathscr{D}}_n v_n\big)\,dt \\
&\qquad + \mathcal{Q}_T^n(v_n) - \mathcal{Q}_0^n(v_n).
\end{aligned}$$

Now, by using repeatedly the inequality $(x + y)^2 \le 2x^2 + 2y^2$ in order to conclude we have to show that

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big(n^{a-b-3/2}\int_0^T \mathscr{S}_t^n(\mathscr{D}_n v_n)\,dt\Big)^2\Big] = 0, \tag{3.24}$$

$$\lim_{n\to\infty} \mathbb{E}\big[\big(\mathcal{Q}_T^n(v_n) - \mathcal{Q}_0^n(v_n)\big)^2\big] = 0 \tag{3.25}$$

and

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big(\int_0^T \mathcal{Q}_t^n\big(n^{a-2}\widetilde{\mathscr{D}}_n v_n\big)\,dt\Big)^2\Big] = 0. \tag{3.26}$$

We have the following estimates on v_n which are proved in Appendix D.1.

LEMMA 3.4. *The solution v_n of* (3.23) *satisfies*

$$\lim_{n\to\infty} \|v_n\|_{2,n}^2 = 0, \tag{3.27}$$

$$\lim_{n\to\infty} \big\|n^{a-b-3/2}\mathscr{D}_n v_n\big\|_{2,n}^2 = 0. \tag{3.28}$$

The expectation in (3.24), by (3.5) and (3.28), vanishes, as $n \to \infty$. Similarly the expectation in (3.25), by (3.7) and (3.27), vanishes as $n \to \infty$. To bound the third expectation above we could be tempted to combine the a priori bound (3.7) with an estimate on the $\mathbb{L}^2$ norm of $n^{a-2}\widetilde{\mathscr{D}}_n v_n$. We leave the interested reader to check that this argument only shows (3.26) for $b > 1$ (and $a = 2$) or for $a < 2$ (i.e., $b < 1/3$). Therefore, proving (3.26) for $b > 1/3$ requires extra work. To overcome this problem, our idea is to establish the result of Lemma 3.3 with h_n replaced by v_n by using the same method used in the current lemma but with the advantage that now the a priori bound (3.7) combined with the estimates on the $\mathbb{L}^2$-norms of the functions involved, will be sufficient to conclude the proof for any $b < 1$. This is the content of Lemma E.1, from where we conclude the proof of the theorem. □

In fact, in order to complete the proof, some tightness of the fields has to be established. The arguments being similar to the ones given in [4], Section 5.2, we do not repeat them and invite the interested reader to read the proofs there.

APPENDIX A: COMPUTATIONS INVOLVING THE GENERATOR L_κ

Let $f : \mathbb{Z} \to \mathbb{R}$ be a function of finite support, and let $\mathscr{E}(f) : \Omega \to \mathbb{R}$ be defined as

$$\mathscr{E}(f) = \sum_{x \in \mathbb{Z}} f(x)\eta(x)^2.$$

A simple computation shows that

$$S\mathscr{E}(f) = \sum_{x \in \mathbb{Z}} \Delta f(x)\eta^2(x),$$

where $\Delta f(x) = f(x+1) + f(x-1) - 2f(x)$ is the discrete Laplacian on $\mathbb{Z}$. On the other hand,

$$A\mathscr{E}(f) = -2\sum_{x \in \mathbb{Z}} \nabla f(x)\eta(x)\eta(x+1),$$

where $\nabla f(x) = f(x+1) - f(x)$ is the discrete right-derivative on $\mathbb{Z}$. It follows that

$$L_\kappa \mathscr{E}(f) = -2\kappa \sum_{x \in \mathbb{Z}} \nabla f(x)\eta(x)\eta(x+1) + \sum_{x \in \mathbb{Z}} \Delta f(x)\eta^2(x). \tag{A.1}$$

Now we prove (3.8). Note that

$$\frac{d}{dt}\mathscr{S}_t^n(f) = \frac{1}{2}\mathbb{E}\Big[\Big(\frac{1}{\sqrt{n}}\sum_{x \in \mathbb{Z}} g\Big(\frac{x}{n}\Big)(\eta_0(x)^2 - 1)\Big) \times \Big(\frac{n^a}{\sqrt{n}} L_\kappa \sum_{y \in \mathbb{Z}} f\Big(\frac{y}{n}\Big)\eta_{tn^a}(y)^2\Big)\Big]$$

and by (A.1) it equals

$$\frac{1}{2}\mathbb{E}\bigg[\bigg(\frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} g\Big(\frac{x}{n}\Big)\big(\eta_0(x)^2-1\big)\bigg)\times\bigg(\frac{n^{a-2}}{\sqrt{n}}\sum_{y\in\mathbb{Z}}\Delta_n f\Big(\frac{y}{n}\Big)\eta_{tn^a}(y)^2\bigg)\bigg]$$
$$-\frac{1}{2}\mathbb{E}\bigg[\bigg(\frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} g\Big(\frac{x}{n}\Big)\big(\eta_0(x)^2-1\big)\bigg)$$
$$\times\bigg(\frac{2\gamma n^{a-b-1}}{\sqrt{n}}\sum_{y\in\mathbb{Z}}\nabla_n f\Big(\frac{y}{n}\Big)\eta_{tn^a}(y)\eta_{tn^a}(y+1)\bigg)\bigg].$$

Now, since $\eta_0(x)^2-1$ is mean zero we can remove the average of $\eta_{tn^{3/2}}(y)^2$ in the first sum above to rewrite it as $n^{a-2}\mathscr{S}_t^n(\Delta_n f)$. In the second sum above, we recall the definition of $\nabla_n f\otimes\delta$ and we rewrite it as $-2\gamma n^{a-b-3/2}Q_t^n(\nabla_n f\otimes\delta)$. From this, we recover (3.8).

Now, let $f:\mathbb{Z}^2\to\mathbb{R}$ be a symmetric function of finite support, and let $Q(f):\Omega\to\mathbb{R}$ be defined as

$$Q(f)=\sum_{\substack{x,y\in\mathbb{Z}\\ x\neq y}}\eta(x)\eta(y)f(x,y).$$

Define $\Delta f:\mathbb{Z}^2\to\mathbb{R}$ as

$$\begin{aligned}\Delta f(x,y)&=f(x+1,y)+f(x-1,y)+f(x,y+1)\\&\quad+f(x,y-1)-4f(x,y)\end{aligned}\tag{A.2}$$

for any $x,y\in\mathbb{Z}$ and $\mathscr{A}f:\mathbb{Z}^2\to\mathbb{R}$ by

$$\mathscr{A}f(x,y)=f(x-1,y)+f(x,y-1)-f(x+1,y)-f(x,y+1)\tag{A.3}$$

for any $x,y\in\mathbb{Z}$. Notice that Δf is the discrete Laplacian on the lattice $\mathbb{Z}^2$ and $\mathscr{A}f$ is a possible definition of the discrete derivative of f in the direction $(-2,-2)$. Notice that we are using the same symbol Δ for the one-dimensional and two-dimensional discrete Laplacian. From the context it will be clear which operator we will be using. We have that

$$\begin{aligned}SQ(f)&=\sum_{|x-y|\geq 2}f(x,y)\big[\eta(y)\Delta\eta(x)+\eta(x)\Delta\eta(y)\big]\\&\quad+2\sum_{x\in\mathbb{Z}}f(x,x+1)\big[\big(\eta(x-1)-\eta(x)\big)\eta(x+1)\\&\quad+\big(\eta(x+2)-\eta(x+1)\big)\eta(x)\big]\\&=\sum_{x,y\in\mathbb{Z}}\Delta f(x,y)\eta(x)\eta(y)-2\sum_{x\in\mathbb{Z}}f(x,x)\eta(x)\Delta\eta(x)\end{aligned}\tag{A.4}$$

$$
\begin{aligned}
&- 2 \sum_{x\in\mathbb{Z}} f(x, x+1)\big[\eta(x+1)\Delta\eta(x) + \eta(x)\Delta\eta(x+1)\big] \\
&+ 2 \sum_{x\in\mathbb{Z}} f(x, x+1)\big[\eta(x+1)\eta(x-1) + \eta(x+2)\eta(x) \\
&- 2\eta(x)\eta(x+1)\big].
\end{aligned}
$$

Grouping terms involving $\eta(x)^2$ and $\eta(x)\eta(x+1)$ together we get that

$$
\begin{aligned}
SQ(f) = {} & \sum_{\substack{x,y\in\mathbb{Z}\\ x\neq y}} \Delta f(x,y)\eta(x)\eta(y) \\
& + 2\sum_{x\in\mathbb{Z}} \big\{\big[f(x,x+1) - f(x,x)\big] + \big[f(x,x+1) - f(x+1,x+1)\big]\big\} \\
& \times \eta(x)\eta(x+1) \\
= {} & Q(\Delta f) \\
& + 2\sum_{x\in\mathbb{Z}} \big\{\big[f(x,x+1) - f(x,x)\big] + \big[f(x,x+1) - f(x+1,x+1)\big]\big\} \\
& \times \eta(x)\eta(x+1).
\end{aligned}
\tag{A.5}
$$

Similarly, we have that

$$
\begin{aligned}
AQ(f) = {} & \sum_{\substack{x,y\in\mathbb{Z}\\ x\neq y}} \mathscr{A} f(x,y)\eta(x)\eta(y) \\
& + 2\sum_{x\in\mathbb{Z}} \big\{\eta(x)^2\big[f(x-1,x) - f(x,x+1)\big] \\
& - \eta(x)\eta(x+1)\big[f(x,x) - f(x+1,x+1)\big]\big\} \\
= {} & Q(\mathscr{A} f) \\
& + 2\sum_{x\in\mathbb{Z}} \big\{\eta(x)^2\big[f(x-1,x) - f(x,x+1)\big] \\
& - \eta(x)\eta(x+1)\big[f(x,x) - f(x+1,x+1)\big]\big\}.
\end{aligned}
\tag{A.6}
$$

From this it follows that

$$
L_\kappa Q(f) = Q\big((\Delta + \kappa\mathscr{A})f\big) + D_\kappa(f), \tag{A.7}
$$

where the diagonal term $D_\kappa(f)$ is given by

$$
\begin{aligned}
D_\kappa(f) = {} & 2\kappa \sum_{x\in\mathbb{Z}} \left(\eta(x)^2 - \frac{1}{\beta}\right)\big(f(x-1,x) - f(x,x+1)\big) \\
& + 2\sum_{x\in\mathbb{Z}} \eta(x)\eta(x+1)\big(2f(x,x+1) - (1+\kappa)f(x,x) \\
& - (1-\kappa)f(x+1,x+1)\big).
\end{aligned} \tag{A.8}
$$

Above in D_κ, we could add the normalization constant $\frac{1}{\beta}$ for free, since $\sum_{x\in\mathbb{Z}} f(x,x+1) - f(x-1,x) = 0$. We also note that the operators $f \mapsto Q(f)$, $f \mapsto LQ(f)$ are continuous maps from $\ell^2(\mathbb{Z}^2)$ to $L^2(\mu_\beta)$ and therefore, an approximation procedure shows that the identities above are true for any $f \in \ell^2(\mathbb{Z}^2)$. Performing similar computations to those we did above for the field $\mathscr{S}_t^n$, it is quite simple to deduce (3.11).

APPENDIX B: TOOLS OF FOURIER ANALYSIS

Let $d \geq 1$ and let $x \cdot y$ denote the usual scalar product in $\mathbb{R}^d$ between x and y. The Fourier transform of a function $g : \frac{1}{n}\mathbb{Z}^d \to \mathbb{R}$ is defined by

$$
\widehat{g}_n(k) = \frac{1}{n^d} \sum_{x\in\mathbb{Z}^d} g\left(\frac{x}{n}\right) e^{\frac{2i\pi k\cdot x}{n}}, \qquad k \in \mathbb{R}^d.
$$

The function $\widehat{g}_n$ is n-periodic in all the directions of $\mathbb{R}^d$. We have the following Parseval–Plancherel identity between the ℓ^2-norm of g, weighted by the natural mesh, and the $L^2([-\frac{n}{2}, \frac{n}{2}]^d)$-norm of its Fourier transform:

$$
\|g\|_{2,n}^2 := \frac{1}{n^d} \sum_{x\in\mathbb{Z}^d} \left| g\left(\frac{x}{n}\right)\right|^2 = \int_{[-\frac{n}{2},\frac{n}{2}]^d} \left|\widehat{g}_n(k)\right|^2 dk := \|\widehat{g}_n\|_2^2. \tag{B.1}
$$

The function g can be recovered from the knowledge of its Fourier transform by the inverse Fourier transform of $\widehat{g}_n$:

$$
g\left(\frac{x}{n}\right) = \int_{[-\frac{n}{2},\frac{n}{2}]^d} \widehat{g}_n(k) e^{-\frac{2i\pi x\cdot k}{n}}\, dk. \tag{B.2}
$$

For any $p \geq 1$ let $[(\nabla_n)^p]$ denote the pth iteration of the operator ∇_n.

LEMMA B.1 ([4]). *Let $f : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ and $p \geq 1$ be such that*

$$
\frac{1}{n} \sum_{x\in\mathbb{Z}} \left| [(\nabla_n)^p] f\left(\frac{x}{n}\right)\right| < +\infty. \tag{B.3}
$$

There exists a universal constant $C := C(p)$ independent of f and n such that for any $|y| \leq 1/2$,

$$
\left|\widehat{f_n}(yn)\right| \leq \frac{C}{n^p |\sin(\pi y)|^p} \left| \frac{1}{n} \sum_{x\in\mathbb{Z}} [(\nabla_n)^p] f\left(\frac{x}{n}\right) e^{2i\pi yx}\right|.
$$

In particular, if f is in the Schwartz space $\mathscr{S}(\mathbb{R})$, then for any $p \geq 1$, there exists a constant $C := C(p, f)$ such that for any $|y| \leq 1/2$,

$$|\widehat{f_n}(yn)| \leq \frac{C}{1 + (n|y|)^p}.$$

Several times we will use the following elementary change of variable property.

LEMMA B.2 ([4]). *Let $F : \mathbb{R}^2 \to \mathbb{C}$ be a n-periodic function in each direction of $\mathbb{R}^2$. Then we have that*

$$\iint_{[-\frac{n}{2}, \frac{n}{2}]^2} F(k, \ell)\, dk\, d\ell = \iint_{[-\frac{n}{2}, \frac{n}{2}]^2} F(\xi - \ell, \ell)\, d\xi\, d\ell.$$

APPENDIX C: ESTIMATES INVOLVING h_n

Let $h_n : \frac{1}{n}\mathbb{Z}^2 \to \mathbb{R}$ be the unique solution in $\ell^2(\frac{1}{n}\mathbb{Z}^2)$ of (3.18). Observe that h_n is a symmetric function. The Fourier transform of h_n is not difficult to compute by using Appendix B. First, we not that the Fourier transform of the function $\Delta_n h$ for a given, summable function $h : \frac{1}{n}\mathbb{Z}^2 \to \mathbb{R}$ is given by

$$\widehat{(\Delta_n h)}_n(k, \ell) = -n^2 \Lambda\left(\frac{k}{n}, \frac{\ell}{n}\right) \widehat{h}_n(k, \ell), \tag{C.1}$$

where

$$\begin{aligned} \Lambda\left(\frac{k}{n}, \frac{\ell}{n}\right) &= -(e^{\frac{2\pi i k}{n}} + e^{-\frac{2\pi i k}{n}} + e^{\frac{2\pi i \ell}{n}} + e^{-\frac{2\pi i \ell}{n}} - 4) \\ &= 4\left[\sin^2\left(\frac{\pi k}{n}\right) + \sin^2\left(\frac{\pi \ell}{n}\right)\right]. \end{aligned} \tag{C.2}$$

Similarly, the Fourier transform of $\mathscr{A}_n h$ is given by

$$\widehat{(\mathscr{A}_n h)}_n(k, \ell) = i n \Omega\left(\frac{k}{n}, \frac{\ell}{n}\right) \widehat{h}_n(k, \ell), \tag{C.3}$$

where

$$\begin{aligned} i\Omega\left(\frac{k}{n}, \frac{\ell}{n}\right) &= e^{\frac{2\pi i k}{n}} + e^{\frac{2\pi i \ell}{n}} - e^{-\frac{2\pi i k}{n}} - e^{-\frac{2\pi i \ell}{n}} \\ &= 2i\left(\sin\left(\frac{2\pi k}{n}\right) + \sin\left(\frac{2\pi \ell}{n}\right)\right). \end{aligned} \tag{C.4}$$

Note in particular that $\Omega(\frac{k}{n}, \frac{\ell}{n})$ is a real number. Let us now compute the Fourier transform of the function $g_n = \nabla_n f \otimes \delta$ defined in (3.10):

$$\begin{aligned} \widehat{g}_n(k, \ell) &= \frac{1}{n^2} \sum_{x, y \in \mathbb{Z}} [\nabla_n f \otimes \delta]\left(\frac{x}{n}, \frac{y}{n}\right) e^{\frac{2i\pi(kx + \ell y)}{n}} \\ &= -\frac{in}{2} \Omega\left(\frac{k}{n}, \frac{\ell}{n}\right) \widehat{f_n}(k + \ell). \end{aligned} \tag{C.5}$$

From the previous computations, we have that[7]

$$\widehat{h}_n(k,\ell) = \frac{1}{\sqrt{n}} \frac{i\Omega(\frac{k}{n},\frac{\ell}{n})}{\gamma^{-1}n^b\Lambda(\frac{k}{n},\frac{\ell}{n}) - i\Omega(\frac{k}{n},\frac{\ell}{n})}\widehat{f}_n(k+\ell). \tag{C.6}$$

Our aim will be to study the behavior of h_n, as $n \to \infty$.

C.1. Proof of (3.21). We want to show that

$$\|h_n\|_{2,n}^2 := \frac{1}{n^2}\sum_{x,y\in\mathbb{Z}} h_n\left(\frac{x}{n},\frac{y}{n}\right)^2, \tag{C.7}$$

vanishes, as $n \to \infty$. By Plancherel–Parseval's relation, Lemma B.2 and (C.6), we have that

$$\begin{aligned}
\|h_n\|_{2,n}^2 &= \iint_{[-\frac{n}{2},\frac{n}{2}]^2} \left|\widehat{h}_n(k,\ell)\right|^2 dk\,d\ell \\
&= \frac{1}{n}\iint_{[-\frac{n}{2},\frac{n}{2}]^2} \frac{\Omega(\frac{k}{n},\frac{\ell}{n})^2|\widehat{f}_n(k+\ell)|^2}{\gamma^{-2}n^{2b}\Lambda(\frac{k}{n},\frac{\ell}{n})^2 + \Omega(\frac{k}{n},\frac{\ell}{n})^2}\,dk\,d\ell \\
&= \frac{1}{n}\iint_{[-\frac{n}{2},\frac{n}{2}]^2} \frac{\Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})^2|\widehat{f}_n(\xi)|^2}{\gamma^{-2}n^{2b}\Lambda(\frac{\xi-\ell}{n},\frac{\ell}{n})^2 + \Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})^2}\,d\xi\,d\ell.
\end{aligned}$$

Since

$$\Omega\left(\frac{\xi-\ell}{n},\frac{\ell}{n}\right)^2 \le 4\left|1-e^{\frac{2i\pi\xi}{n}}\right|^2 = 16\sin^2\left(\frac{\pi\xi}{n}\right), \tag{C.8}$$

the last expression can be bounded from above by

$$\begin{aligned}
&\frac{16}{n}\int_{-n/2}^{n/2}\sin^2\left(\frac{\pi\xi}{n}\right)\left|\widehat{f}_n(\xi)\right|^2\left[\int_{-n/2}^{n/2}\frac{d\ell}{\gamma^{-2}n^{2b}\Lambda(\frac{\xi-\ell}{n},\frac{\ell}{n})^2+\Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})^2}\right]d\xi \\
&\quad = 16n\int_{-1/2}^{1/2}\sin^2(\pi y)\left|\widehat{f}_n(ny)\right|^2\widetilde{W}_n(y)\,dy,
\end{aligned} \tag{C.9}$$

where

$$\begin{aligned}
\widetilde{W}_n(y) &= \int_{-1/2}^{1/2}\frac{dx}{\gamma^{-2}n^{2b}\Lambda(y-x,x)^2+\Omega(y-x,x)^2} \\
&\le \int_{-1/2}^{1/2}\frac{dx}{\gamma^{-2}\Lambda(y-x,x)^2+\Omega(y-x,x)^2}.
\end{aligned} \tag{C.10}$$

[7] If k and ℓ both belong to $n\mathbb{Z}$ then the denominator appearing in the definition of $\widehat{h}_n$ is zero but this is not a problem since the function $[i\Omega/(\gamma^{-1}n^b\Lambda - i\Omega)](x,y)$ converges to -1 as $(x,y)\to(k,\ell)\in\mathbb{Z}^2$.

Since by Lemma F.5 in [4], the right-hand side of (C.10) is of order $|y|^{-3/2}$ for $y \in [-\frac{1}{2}, \frac{1}{2}]$, then, from Lemma B.1 we conclude that $\|h_n\|_{2,n}^2 = O(\frac{1}{\sqrt{n}})$. Indeed, by a change of variables, (C.9) is bounded from above by a constant times

$$n\int_{-1/2}^{1/2} \frac{\sqrt{y}}{(1+(n|y|)^p)^2}\,dy \le n\int_{-1/2}^{1/2} \frac{\sqrt{y}}{1+(n|y|)^{2p}}\,dy \le \frac{1}{\sqrt{n}}\int_{\mathbb{R}} \frac{\sqrt{y}}{1+|y|^{2p}}\,dy = \mathscr{O}\left(\frac{1}{\sqrt{n}}\right),$$

as long as p is sufficiently big. This ends the proof of (3.21).

C.2. Proof of Lemma 3.2. In order to prove this lemma, let $q_n : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ be the function defined by

$$q_n\left(\frac{x}{n}\right) = n^{a-b-3/2}\mathscr{D}_n h_n\left(\frac{x}{n}\right) \tag{C.11}$$

and let $q : \mathbb{R} \to \mathbb{R}$ be given by

$$q(x) = \int_{-\infty}^{+\infty} e^{-2i\pi yx} G_0(y)\mathcal{F} f(y)\,dy,$$

where $\mathscr{F} f$ denotes the Fourier transform of f which is given by

$$(\mathscr{F} f)(\xi) = \int_{\mathbb{R}} e^{2i\pi x\xi} f(x)\,dx \tag{C.12}$$

and $G_0(y) = \sqrt{2\gamma}|\pi y|^{3/2} e^{i\,\mathrm{sgn}(y)\frac{\pi}{4}}$. Observe that the Fourier transform of q, namely $\mathscr{F} q$, coincides with $\frac{\sqrt{\gamma}}{2}\mathscr{F}\mathscr{L} f$.

Now we note that in the case $b > 1/3$ the proof of Lemma 3.2 trivially follows from the next result and also by noting that the operator $\mathbb{L}_{\gamma,b}$ is simply the usual Laplacian. In the case $b \le 1/3$, the proof of Lemma 3.2 follows from the next result and also by noting that the factor in front of the discrete Laplacian is given by $n^{a-2} = n^{\frac{1}{2}(3b-1)}$, which vanishes as $n \to \infty$ if $b < 1/3$, and is constant equal to 1 for $b = 1/3$.

LEMMA C.1. *We have that*:

1. *For* $b \le 1/3$ *and* $a = 3/2 + 3b/2$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{x\in\mathbb{Z}}\left[q\left(\frac{x}{n}\right) - q_n\left(\frac{x}{n}\right)\right]^2 = 0. \tag{C.13}$$

2. *For* $b > 1/3$ *and* $a = 2$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{x\in\mathbb{Z}} q_n^2\left(\frac{x}{n}\right) = 0. \tag{C.14}$$

PROOF. By following the proof of Lemma D.1 in [4] we have that

$$\widehat{q}_n(\xi) = -n^{a-b-3/2}\frac{in}{2}\int_{-n/2}^{n/2}\Omega\Big(\frac{\xi-\ell}{n},\frac{\ell}{n}\Big)\widehat{h}_n(\xi-\ell,\ell)\,d\ell.$$

By the explicit expression (C.6) for $\widehat{h}_n$ we obtain that

$$\widehat{q}_n(\xi) = n^{a-b-3/2}\frac{\sqrt{n}}{2}\Big[\int_{-n/2}^{n/2}\frac{\Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})^2}{\gamma^{-1}n^b\Lambda(\frac{\xi-\ell}{n},\frac{\ell}{n}) - i\Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})}\,d\ell\Big]\widehat{f}_n(\xi).$$

By the inverse Fourier transform we get that

$$q_n\Big(\frac{x}{n}\Big) = 2n^{a-b}\int_{-n/2}^{n/2}e^{-\frac{2i\pi\xi x}{n}}G_n\Big(\frac{\xi}{n}\Big)\widehat{f}_n(\xi)\,d\xi,$$

where

$$G_n(y) = \frac{1}{4}\int_{-1/2}^{1/2}\frac{\Omega(y-z,z)^2}{\gamma^{-1}n^b\Lambda(y-z,z) - i\Omega(y-z,z)}\,dz. \tag{C.15}$$

By Lemma F.1 we have that

$$\begin{aligned} G_n(y) &= \frac{\sqrt{\gamma}}{n^{b/2}\sqrt{2}}\big|\sin(\pi y)\big|^{3/2}e^{i\,\mathrm{sgn}(y)\frac{\pi}{4}} + \mathscr{O}\big(\sin^2(\pi y)\big) \qquad \text{if } b<1,\\ G_n(y) &= \mathscr{O}\big(n^{-b}\big|\sin(\pi y)\big|\big) \qquad \text{if } b \ge 1. \end{aligned} \tag{C.16}$$

Now we prove (C.14) and then (C.13). Recall that $\widehat{q}_n(\xi) = 2n^{a-b}G_n(\frac{\xi}{n})\widehat{f}_n(\xi)$. By Parseval–Plancherel's equality, we have

$$\frac{1}{n}\sum_{x\in\mathbb{Z}}\Big|q_n\Big(\frac{x}{n}\Big)\Big|^2 = \int_{-n/2}^{n/2}\big|\widehat{q}_n(\xi)\big|^2\,d\xi = 4n^{2a-2b+1}\int_{-1/2}^{1/2}\big|G_n(y)\big|^2\big|\widehat{f}_n(ny)\big|^2\,dy.$$

If $b \ge 1$, from the second equality in (C.16) and Lemma B.1, for p sufficiently big, it follows that there exists a constant $C>0$ such that

$$\frac{1}{n}\sum_{x\in\mathbb{Z}}\Big|q_n\Big(\frac{x}{n}\Big)\Big|^2 \le Cn^{2a-4b+1}\int_{-1/2}^{1/2}\frac{|y|^2}{1+(n|y|)^p}\,dy = \mathscr{O}\big(n^{2a-4b-2}\big)$$

and the right-hand side of the previous inequality goes to 0 since $a=2$ and $b \ge 1 > 1/2$. In the case $1/3 < b < 1$, we repeat the same argument as above using the function given in the first equality in (C.16) and we get a bound in the form (for p sufficiently big)

$$\begin{aligned} \frac{1}{n}\sum_{x\in\mathbb{Z}}\Big|q_n\Big(\frac{x}{n}\Big)\Big|^2 &\le C\frac{n^{2a-2b+1}}{n^b}\int_{-1/2}^{1/2}\frac{|y|^3}{1+(n|y|)^p}\,dy\\ &\quad + Cn^{2a-2b+1}\int_{-1/2}^{1/2}\frac{|y|^4}{1+(n|y|)^p}\,dy\\ &= \mathscr{O}\big(n^{2a-2b-3}\big) + \mathscr{O}\big(n^{2a-2b-4}\big). \end{aligned}$$

Since $a = 2$ and $1 > b > 1/3$ the right-hand side of the last inequality goes to 0. Finally, we look at the case $b \le 1/3$. Let us define

$$\tag{C.17} \begin{aligned} \widetilde{G}_0(y) &:= \sqrt{2\gamma}\,|\sin(\pi y)|^{3/2} e^{i\,\mathrm{sgn}(y)\frac{\pi}{4}} = \left|\frac{\sin(\pi y)}{\pi y}\right|^{3/2} G_0(y) \\ &= G_0(y) + \mathscr{O}(|y|^{9/2}) \end{aligned}$$

since $\sin(\pi y) = \pi y + \mathscr{O}(|y|^3)$ for $y \to 0$. The factor $2n^{a-b} G_n(\frac{\xi}{n})$ that appears in $\widehat{q}_n(\xi)$ can be written as $2n^{a-b} G_n(\frac{\xi}{n}) = n^{a-3b/2} \widetilde{G}_0(\frac{\xi}{n}) + \varepsilon_n(\frac{\xi}{n})$ where $\varepsilon_n(y) = n^{a-b} \mathcal{O}(|y|^2)$. Thus by using Lemma B.1 we have that

$$\int_{-n/2}^{n/2} \left|\varepsilon_n\left(\frac{\xi}{n}\right)\right|^2 |\widehat{f}_n(\xi)|^2\, d\xi = n \int_{-1/2}^{1/2} |\varepsilon_n(y)|^2 |\widehat{f}_n(ny)|^2\, dy = \mathscr{O}(n^{2a-2b-4}).$$

Since $a = 3/2 + 3b/2$ and $b < 1/3 < 1$, the corrective term involving ε_n does not contribute in (C.13) and we can assume that $2n^{a-b} G_n(\frac{\xi}{n})$ is replaced by $n^{a-3b/2} \widetilde{G}_0(\frac{\xi}{n})$ in the expression of $\widehat{q}_n(\xi)$. Since $a = 3/2 + 3b/2$ we get in front of $\widetilde{G}_0$ the factor $n^{3/2}$. Again, we can replace $\widetilde{G}_0$ by G_0 in the expression of $\widehat{q}_n(\xi)$ because, by Lemma B.1 and (C.17),

$$\begin{aligned} \int_{-n/2}^{n/2} \left| n^{3/2} \widetilde{G}_0\left(\frac{\xi}{n}\right) - n^{3/2} G_0\left(\frac{\xi}{n}\right)\right|^2 |\widehat{f}_n(\xi)|^2\, d\xi &\le C n^4 \int_{-1/2}^{1/2} |y|^9 |\widehat{f}_n(ny)|^2\, dy \\ &= \mathscr{O}(n^{-6}). \end{aligned}$$

Now, since we have been able to replace $2n^{a-b} G_n(\frac{\xi}{n})$ by $n^{3/2} G_0(\frac{\xi}{n})$ in the expression of $\widehat{q}_n(\xi)$ and that $n^{3/2} G_0(\frac{\xi}{n}) = G_0(\xi)$ the proof of Lemma B.1 in [4] can be followed. For the interested reader, we present the details below. We have that

$$\tag{C.18} \begin{aligned} q\left(\frac{x}{n}\right) - q_n\left(\frac{x}{n}\right) &= \int_{|\xi| \ge n/2} e^{-\frac{2i\pi\xi x}{n}} G_0(\xi) \mathscr{F} f(\xi)\, d\xi \\ &\quad + \int_{|\xi| \le n/2} e^{-\frac{2i\pi\xi x}{n}} G_0(\xi) [\mathscr{F} f(\xi) - \widehat{f}_n(\xi)]\, d\xi \\ &\quad + \int_{|\xi| \le n/2} e^{-\frac{2i\pi\xi x}{n}} (n^{3/2} G_0 - 2n^{a-b} G_n)\left(\frac{\xi}{n}\right) \widehat{f}_n(\xi)\, d\xi. \end{aligned}$$

Above we have used the fact that $n^{3/2} G_0(\frac{\xi}{n}) = G_0(\xi)$. From the triangular inequality together with Plancherel's theorem applied to the two last terms at the right-hand side of the previous expression and from (C.16), we have that

$$\begin{aligned}\frac{1}{n}\sum_{x\in\mathbb{Z}}\Big[q\Big(\frac{x}{n}\Big)-q_n\Big(\frac{x}{n}\Big)\Big]^2 \le{}& \frac{1}{n}\sum_{x\in\mathbb{Z}}\Big|\int_{|\xi|\ge n/2}e^{-\frac{2i\pi\xi x}{n}}G_0(\xi)(\mathscr{F}f)(\xi)\,d\xi\Big|^2\\ &+\int_{|\xi|\le n/2}\big|G_0(\xi)\big[\mathscr{F}f(\xi)-\widehat{f}_n(\xi)\big]\big|^2\,d\xi\\ &+\frac{C}{n}\int_{|\xi|\le n/2}|\xi|^4\big|\widehat{f}_n(\xi)\big|^2\,d\xi.\end{aligned}\tag{C.19}$$

Note that the last term above goes to 0, as $n\to\infty$, by Lemma B.1 for p sufficiently big. The first term above, can be estimated in the same way as we did in the case $b>1/3$ and it vanishes as $n\to\infty$. The second term term follows exactly the same steps as in the proof of Lemma B.1 of [4] and for that reason we have omitted it. This ends the proof of the lemma. □

APPENDIX D: PROOF OF LEMMA 3.4

We start by computing the Fourier transform of v_n. Recall that v_n is solution of (3.23). Applying the Fourier transform, we get that

$$\widehat{\Delta_n v_n}(k,\ell)+\gamma n^{1-b}\widehat{\mathscr{A}_n v_n}(k,\ell)=\widehat{\widetilde{\mathscr{D}}_n h_n}.$$

Note that since h_n does not depend on a it is the same for v_n. From (C.1) and (C.3) the left-hand side of the previous display is given by

$$-n^2\Lambda\Big(\frac{k}{n},\frac{\ell}{n}\Big)\widehat{v}_n(k,\ell)+i\gamma n^{2-b}\Omega\Big(\frac{k}{n},\frac{\ell}{n}\Big)\widehat{v}_n(k,\ell).$$

Now we need to compute the Fourier transform of $\widetilde{\mathscr{D}}_n h_n$. By a simple computation, we get that

$$\begin{aligned}\widehat{\widetilde{\mathscr{D}}_n h_n}(k,l)={}&\sum_{x\in\mathbb{Z}}\Big\{\widetilde{\mathscr{E}}_n h\Big(\frac{x}{n}\Big)-\Big(\frac{1-k}{2}\Big)\widetilde{\mathscr{F}}_n h\Big(\frac{x}{n}\Big)\Big\}\\ &\times\big\{e^{\frac{2\pi i(kx+\ell(x+1))}{n}}+e^{\frac{2i\pi(k(x+1)+\ell x)}{n}}\big\}\\ ={}&n\big\{e^{\frac{2i\pi\ell}{n}}+e^{\frac{2i\pi k}{n}}\big\}\Big\{\widehat{\widetilde{\mathscr{E}}_n h}(k+\ell)-\Big(\frac{1-k}{2}\Big)\widehat{\widetilde{\mathscr{F}}_n h}(k+\ell)\Big\},\end{aligned}\tag{D.1}$$

where $\widetilde{\mathscr{E}}_n h$ and $\widetilde{\mathscr{F}}_n h$ were given in (3.16) and (3.17), respectively. From the previous computations, we conclude that the Fourier transform $\widehat{v}_n$ is given by

$$\begin{aligned}\widehat{v}_n(k,\ell)={}&-\frac{1}{n}\frac{e^{\frac{2i\pi k}{n}}+e^{\frac{2i\pi\ell}{n}}}{\Lambda(\frac{k}{n},\frac{\ell}{n})-i\gamma n^{-b}\Omega(\frac{k}{n},\frac{\ell}{n})}\\ &\times\Big\{\widehat{\widetilde{\mathscr{E}}_n h}(k+\ell)-\Big(\frac{1-k}{2}\Big)\widehat{\widetilde{\mathscr{F}}_n h}(k+\ell)\Big\}\end{aligned}\tag{D.2}$$

$$= -\gamma^{-1} n^{b-1} \frac{e^{\frac{2i\pi k}{n}} + e^{\frac{2i\pi \ell}{n}}}{\gamma^{-1} n^b \Lambda(\frac{k}{n}, \frac{\ell}{n}) - i\Omega(\frac{k}{n}, \frac{\ell}{n})} \times \left\{ \widehat{\widetilde{\mathscr{E}}_n h}(k+\ell) - \left(\frac{1-k}{2}\right) \widehat{\widetilde{\mathscr{F}}_n h}(k+\ell) \right\}.$$

By Lemma B.2 and (3.16), we have that the Fourier transform of $\widetilde{\mathscr{E}}_n h$ is given by

$$\begin{aligned} \widehat{\widetilde{\mathscr{E}}_n h}(\xi) &= \frac{1}{n} \sum_{x\in\mathbb{Z}} e^{\frac{2i\pi\xi x}{n}} \left(h_n\left(\frac{x}{n}, \frac{x+1}{n}\right) - h_n\left(\frac{x}{n}, \frac{x}{n}\right)\right) \\ &= \frac{1}{n} \sum_{x\in\mathbb{Z}} e^{\frac{2i\pi\xi x}{n}} \iint_{[-\frac{n}{2},\frac{n}{2}]^2} \widehat{h}_n(k,\ell) e^{-\frac{2i\pi(k+\ell)x}{n}} \{e^{-\frac{2i\pi\ell}{n}} - 1\}\, dk\, d\ell \\ &= \frac{1}{n} \sum_{x\in\mathbb{Z}} e^{\frac{2i\pi\xi x}{n}} \int_{-n/2}^{n/2} e^{-\frac{2i\pi u x}{n}} \left\{ \int_{-n/2}^{n/2} \widehat{h}_n(u-\ell,\ell) \{e^{-\frac{2i\pi\ell}{n}} - 1\}\, d\ell \right\} du \\ &= \int_{-n/2}^{n/2} \widehat{h}_n(\xi-\ell,\ell)\{e^{-\frac{2i\pi\ell}{n}} - 1\}\, d\ell. \end{aligned} \tag{D.3}$$

In the last line, we used the inverse Fourier transform. By (C.6), we get that

$$\begin{aligned} \widehat{\widetilde{\mathscr{E}}_n h}(\xi) &= -\frac{1}{\sqrt{n}} \widehat{f}_n(\xi) \int_{-n/2}^{n/2} \frac{(1-e^{-\frac{2i\pi\ell}{n}}) i\Omega(\frac{\xi-\ell}{n}, \frac{\ell}{n})}{\gamma^{-1} n^b \Lambda(\frac{\xi-\ell}{n}, \frac{\ell}{n}) - i\Omega(\frac{\xi-\ell}{n}, \frac{\ell}{n})}\, d\ell \\ &= -\sqrt{n} I_n\left(\frac{\xi}{n}\right) \widehat{f}_n(\xi), \end{aligned} \tag{D.4}$$

where the function I_n is defined by

$$I_n(y) = \int_{-1/2}^{1/2} \frac{(1-e^{-2i\pi x}) i\Omega(y-x,x)}{\gamma^{-1} n^b \Lambda(y-x,x) - i\Omega(y-x,x)}\, dx. \tag{D.5}$$

Again, by Lemma B.2 and (3.17) the Fourier transform of $\widetilde{\mathscr{F}}_n h$ is given by

$$\begin{aligned} \widehat{\widetilde{\mathscr{F}}_n h}(\xi) &= \frac{1}{n} \sum_{x\in\mathbb{Z}} e^{\frac{2i\pi\xi x}{n}} \left(h_n\left(\frac{x+1}{n}, \frac{x+1}{n}\right) - h_n\left(\frac{x}{n}, \frac{x}{n}\right)\right) \\ &= (e^{-\frac{2i\pi\xi}{n}} - 1) \frac{1}{n} \sum_{x\in\mathbb{Z}} e^{\frac{2i\pi\xi x}{n}} h_n\left(\frac{x}{n}, \frac{x}{n}\right) \\ &= (e^{-\frac{2i\pi\xi}{n}} - 1) \frac{1}{n} \sum_{x\in\mathbb{Z}} e^{\frac{2i\pi\xi x}{n}} \iint_{[-\frac{n}{2},\frac{n}{2}]^2} \widehat{h}_n(k,\ell) e^{-\frac{2i\pi(k+\ell)x}{n}}\, dk\, d\ell \\ &= (e^{-\frac{2i\pi\xi}{n}} - 1) \frac{1}{n} \sum_{x\in\mathbb{Z}} e^{\frac{2i\pi\xi x}{n}} \int_{-n/2}^{n/2} e^{-\frac{2i\pi u x}{n}} \left\{ \int_{-n/2}^{n/2} \widehat{h}_n(u-\ell,\ell)\, d\ell \right\} du \\ &= (e^{-\frac{2i\pi\xi}{n}} - 1) \int_{-n/2}^{n/2} \widehat{h}_n(\xi-\ell,\ell)\, d\ell. \end{aligned} \tag{D.6}$$

By (C.6), we get

$$\widehat{\mathscr{F}_n h}(\xi) = -\frac{1}{\sqrt{n}}\big(1-e^{-\frac{2i\pi\xi}{n}}\big)\widehat{f_n}(\xi) \times \int_{-n/2}^{n/2} \frac{i\Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})}{\gamma^{-1}n^b\Lambda(\frac{\xi-\ell}{n},\frac{\ell}{n}) - i\Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})}\,d\ell = -\sqrt{n}\big(1-e^{-\frac{2i\pi\xi}{n}}\big)\widetilde{I}_n\Big(\frac{\xi}{n}\Big)\widehat{f_n}(\xi), \tag{D.7}$$

where the function $\widetilde{I}_n$ is defined by

$$\widetilde{I}_n(y) = \int_{-1/2}^{1/2} \frac{i\Omega(y-x,x)}{\gamma^{-1}n^b\Lambda(y-x,x) - i\Omega(y-x,x)}\,dx. \tag{D.8}$$

D.1. Proof of (3.27). In order to compute the discrete $\mathbb{L}^2$ norm of v_n we use Plancherel–Parseval's relation, (D.2), Lemma B.2 and we have that

$$\begin{aligned}
\|v_n\|_{2,n}^2 &= \iint_{[-\frac{n}{2},\frac{n}{2}]^2} \big|\widehat{v}_n(k,\ell)\big|^2\,dk\,d\ell \\
&= \frac{1}{n^2}\int_{-n/2}^{n/2}\bigg|\widehat{\mathscr{E}_n h}(\xi) - \Big(\frac{1-k}{2}\Big)\widehat{\mathscr{F}_n h}(\xi)\bigg|^2 \\
&\quad \times \int_{-n/2}^{n/2}\bigg|\frac{e^{\frac{2i\pi(\xi-\ell)}{n}} + e^{\frac{2i\pi\ell}{n}}}{\Lambda(\frac{\xi-\ell}{n},\frac{\ell}{n}) - i\gamma n^{-b}\Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})}\bigg|^2\,d\ell\,d\xi \\
&\le \frac{C}{n}\int_{-n/2}^{n/2}\bigg|\widehat{\mathscr{E}_n h}(\xi) - \Big(\frac{1-k}{2}\Big)\widehat{\mathscr{F}_n h}(\xi)\bigg|^2 W_n\Big(\frac{\xi}{n}\Big)\,d\xi,
\end{aligned}$$

where

$$W_n(y) = \int_{-1/2}^{1/2}\frac{dx}{\Lambda(y-x,x)^2 + \gamma^2 n^{-2b}\Omega(y-x,x)^2}. \tag{D.9}$$

We observe that since $b>0$, we can bound $W_n(y)$ from above by

$$W_n(y) \le n^{2b}\int_{-1/2}^{1/2}\frac{dx}{\Lambda(y-x,x)^2 + \gamma^2\Omega(y-x,x)^2}. \tag{D.10}$$

This integral has been estimated in Lemma F.5 of [4] and it is of order $|y|^{-3/2}$ for $y \to 0$. Therefore, we have that

$$\big|W_n(y)\big| \le Cn^{2b}|y|^{-3/2}. \tag{D.11}$$

By the triangular inequality, in order to finish the proof, we are reduced to show that

$$\frac{1}{n}\int_{-n/2}^{n/2}\big|\widehat{\mathscr{E}_n h}(\xi)\big|^2 W_n\Big(\frac{\xi}{n}\Big)\,d\xi \quad \text{and} \quad \frac{1}{n}\int_{-n/2}^{n/2}\big|\widehat{\mathscr{F}_n h}(\xi)\big|^2 W_n\Big(\frac{\xi}{n}\Big)\,d\xi \tag{D.12}$$

vanish as $n \to \infty$. By (D.4), the term at the left-hand side of the previous display is equal to

$$\int_{-n/2}^{n/2} |\widehat{f_n}(\xi)|^2 \left| I_n\left(\frac{\xi}{n}\right)\right|^2 W_n\left(\frac{\xi}{n}\right) d\xi = n \int_{-1/2}^{1/2} |\widehat{f_n}(ny)|^2 |I_n(y)|^2 W_n(y)\, dy.$$

By Lemma F.3 and Lemma B.1, we have

$$\begin{aligned} & n \int_{-1/2}^{1/2} |\widehat{f_n}(ny)|^2 |I_n(y)|^2 W_n(y)\, dy \\ & \quad \le Cn \int_{-1/2}^{1/2} \frac{1}{1+|ny|^p} \frac{|y|^{3/2}}{|y|+n^{-b}}\, dy = 2C \frac{1}{\sqrt{n}} \int_0^{n/2} \frac{1}{1+|z|^p} \frac{|z|^{3/2}}{|z|+n^{1-b}}\, dz \\ & \quad \le 2C \frac{1}{\sqrt{n}} \int_0^{n/2} \frac{|z|^{1/2}}{1+|z|^p}\, dz \end{aligned}$$

which goes to 0 as $n \to \infty$ for p sufficiently big. Finally, by (D.7), Lemma F.3 and Lemma B.1, the term at the right-hand side of (D.12) is bounded from above as

$$\begin{aligned} & \int_{-n/2}^{n/2} |1 - e^{\frac{2\pi i \xi}{n}}|^2 \left| \widetilde{I}_n\left(\frac{\xi}{n}\right)\right|^2 \left| \widehat{f_n}\left(\frac{\xi}{n}\right)\right|^2 W_n\left(\frac{\xi}{n}\right) d\xi \\ & \quad = n \int_{-1/2}^{1/2} |\sin(\pi y)|^2 |\widehat{f_n}(ny)|^2 |\widetilde{I}_n(y)|^2 W_n(y)\, dy \\ & \quad \le C \frac{1}{\sqrt{n}} \int_0^{n/2} \frac{|z|^{1/2}}{1+|z|^p}\, dz \end{aligned}$$

which goes to 0 as $n \to \infty$ for p sufficiently big.

D.2. Proof of (3.28). In order to compute the discrete $\mathbb{L}^2$ norm of $\mathscr{D}_n v_n$ we first note that by simple computations together with Lemma B.2 we have that (see equation (E.7) in [4])

$$\widehat{\mathscr{D}_n v_n}(\xi) = n\big(1 - e^{\frac{2i\pi\xi}{n}}\big) \int_{-\frac{n}{2}}^{\frac{n}{2}} \widehat{v}_n(\xi - \ell, \ell) e^{-\frac{2i\pi\ell}{n}}\, d\ell. \tag{D.13}$$

By (D.2), the last expression equals

$$\begin{aligned} \widehat{\mathscr{D}_n v_n}(\xi) &= -\big(1 - e^{\frac{2i\pi\xi}{n}}\big) \left\{ \widehat{\widetilde{\mathscr{E}}_n h}(\xi) - \left(\frac{1-k}{2}\right) \widehat{\widetilde{\mathscr{F}}_n h}(\xi) \right\} \\ & \quad \times \int_{-\frac{n}{2}}^{\frac{n}{2}} \frac{1 + e^{\frac{2i\pi(\xi - 2\ell)}{n}}}{\Lambda(\frac{\xi-\ell}{n}, \frac{\ell}{n}) - i\gamma n^{-b} \Omega(\frac{\xi-\ell}{n}, \frac{\ell}{n})}\, d\ell \\ &= -n\big(1 - e^{\frac{2i\pi\xi}{n}}\big) \left\{ \widehat{\widetilde{\mathscr{E}}_n h}(\xi) - \left(\frac{1-k}{2}\right) \widehat{\widetilde{\mathscr{F}}_n h}(\xi) \right\} J_n\left(\frac{\xi}{n}\right), \end{aligned} \tag{D.14}$$

where J_n is given by

$$J_n(y) = \int_{-1/2}^{1/2} \frac{1 + e^{2i\pi(y-2x)}}{\Lambda(y-x,x) - i\gamma n^{-b}\Omega(y-x,x)}\,dx. \tag{D.15}$$

Now, by using (D.4) and (D.7) we get

$$\begin{aligned}\widehat{\mathscr{D}_n v_n}(\xi) &= \frac{n^{3/2}}{2}(1 - e^{\frac{2i\pi\xi}{n}})\widehat{f}_n(\xi) I_n\Big(\frac{\xi}{n}\Big) J_n\Big(\frac{\xi}{n}\Big)\\ &\quad - 4n^{3/2}\Big(\frac{1-k}{2}\Big)\sin^2\Big(\frac{\pi\xi}{n}\Big)\widetilde{I}_n\Big(\frac{\xi}{n}\Big)\widehat{f}_n\Big(\frac{\xi}{n}\Big) J_n\Big(\frac{\xi}{n}\Big),\end{aligned} \tag{D.16}$$

where I_n is defined by (D.5). Finally, by Plancherel–Parseval's relation we have that

$$\begin{aligned}&\|n^{a-b-3/2}\mathscr{D}_n v_n\|_{2,n}^2\\ &\quad \le Cn^{2(a-b)}\int_{-n/2}^{n/2}\sin^2\Big(\pi\frac{\xi}{n}\Big)|\widehat{f}_n(\xi)|^2\Big|I_n\Big(\frac{\xi}{n}\Big)\Big|^2\Big|J_n\Big(\frac{\xi}{n}\Big)\Big|^2\,d\xi\\ &\qquad + Cn^{2(a-b)}\int_{-n/2}^{n/2}\Big|\sin\Big(\pi\frac{\xi}{n}\Big)\Big|^4|\widehat{f}_n(\xi)|^2\Big|\widetilde{I}_n\Big(\frac{\xi}{n}\Big)\Big|^2\Big|J_n\Big(\frac{\xi}{n}\Big)\Big|^2\,d\xi\\ &\quad = Cn^{2(a-b)+1}\int_{-1/2}^{1/2}|\sin(\pi y)|^2|I_n(y)|^2|J_n(y)|^2|\widehat{f}_n(ny)|^2\,dy\\ &\qquad + Cn^{2(a-b)+1}\int_{-1/2}^{1/2}|\sin(\pi y)|^4|\widetilde{I}_n(y)|^2|J_n(y)|^2|\widehat{f}_n(ny)|^2\,dy.\end{aligned} \tag{D.17}$$

By using Lemma F.4, Lemma F.3 and Lemma B.1, choosing a p sufficiently large, we can bound the first term at the right-hand side of (D.17) by a constant times

$$\begin{aligned}&n^{2a-4b+1}\int_0^{1/2}\frac{y^4}{1+(ny)^p}\frac{1}{(|y|+n^{-b})^2}\,dy\\ &\quad = n^{2a-2-4b}\int_0^{\infty}\frac{z^4}{1+z^p}\frac{1}{(z+n^{1-b})^2}\,dz.\end{aligned} \tag{D.18}$$

If $b \le 1$, the previous integral is bounded from above by

$$n^{2a-4-2b}\int_0^{\infty}\frac{z^4}{1+z^p}\,dz$$

which goes to 0 since $a = \inf(3/2 - 3/2b, 2)$. If $b > 1$, the integral (D.18) is bounded from above by

$$n^{2a-2-4b}\int_0^{\infty}\frac{z^2}{1+z^p}\,dz$$

which goes to 0 since $a = 2$ in this case.

The second term on the right-hand side of (D.17) is proved to go to 0 similarly by using Lemma F.4, Lemma F.3 and Lemma B.1. This completes the proof of (3.28).

APPENDIX E: PROOF OF (3.26)

LEMMA E.1. *Let* $v_n : \frac{1}{n}\mathbb{Z}^2 \to \mathbb{R}$ *be the solution of* (3.23) *and let* $a = \inf(3/2 + 3b/2, 2)$ *and* $b \in (0, 1)$. *For any* $T > 0$, *we have that*

$$\lim_{n\to\infty} \mathbb{E}\bigg[\bigg(\int_0^T Q_t^n(n^{a-2}\widetilde{\mathscr{D}}_n v_n)\,dt\bigg)^2\bigg] = 0.$$

PROOF. In order to prove the result we do the following. We use again (3.11) for the solution w_n of the Poisson equation

$$n^{a-2}\Delta_n w_n\bigg(\frac{x}{n}, \frac{y}{n}\bigg) + \gamma n^{a-1-b}\mathscr{A}_n w_n\bigg(\frac{x}{n}, \frac{y}{n}\bigg) = n^{a-2}\widetilde{\mathscr{D}}_n v_n. \tag{E.1}$$

Then, by integrating in time (3.11) we have

$$\begin{aligned}
&\int_0^T Q_t^n(n^{a-2}\widetilde{\mathscr{D}}_n v_n)\,dt \\
&\quad = 2\gamma n^{a-b-3/2}\int_0^T \mathscr{S}_t^n(\mathscr{D}_n w_n)\,dt - 2\int_0^T Q_t^n(n^{a-2}\widetilde{\mathscr{D}}_n w_n)\,dt \\
&\qquad + Q_T^n(w_n) - Q_0^n(w_n).
\end{aligned}$$

Now, by using repeatedly the inequality $(x+y)^2 \le 2x^2 + 2y^2$ in order to conclude we have to show that

$$\mathbb{E}\bigg[\bigg(n^{a-b-3/2}\int_0^T \mathscr{S}_t^n(\mathscr{D}_n w_n)\,dt\bigg)^2\bigg],$$
$$\mathbb{E}\big[\big(Q_T^n(w_n) - Q_0^n(w_n)\big)^2\big]$$

and

$$\mathbb{E}\bigg[\bigg(\int_0^T Q_t^n(n^{a-2}\widetilde{\mathscr{D}}_n w_n)\,dt\bigg)^2\bigg]$$

vanish as n goes to infinity. The first display above, by (3.5) and (E.3), vanishes, as $n \to \infty$. Similarly the second (resp. third display), by (3.7) and (E.2) [resp. (E.4)], vanishes as $n \to \infty$. □

Therefore the previous lemma depends on the following estimates on w_n.

LEMMA E.2. *Let $a = \inf(3/2 + 3b/2, 2)$ and $b \in (0, 1)$. The solution w_n of* (E.1) *satisfies*

$$\lim_{n\to\infty} \|w_n\|_{2,n}^2 = 0, \tag{E.2}$$

$$\lim_{n\to\infty} \|n^{a-b-3/2}\mathscr{D}_n w_n\|_{2,n}^2 = 0, \tag{E.3}$$

$$\lim_{n\to\infty} \|n^{a-2}\tilde{\mathscr{D}}_n w_n\|_{2,n}^2 = 0. \tag{E.4}$$

PROOF. We start by computing the Fourier transform of w_n. Repeating the computations done for (D.2) and recalling that w_n is solution of (E.1), we obtain

$$\begin{aligned} \widehat{w}_n(k,\ell) = &-\frac{1}{n}\frac{e^{\frac{2i\pi k}{n}} + e^{\frac{2i\pi \ell}{n}}}{\Lambda(\frac{k}{n},\frac{\ell}{n}) - i\gamma n^{-b}\Omega(\frac{k}{n},\frac{\ell}{n})} \\ &\times \left\{\widehat{\widetilde{\mathscr{E}}_n v_n}(k+\ell) - \left(\frac{1-k}{2}\right)\widehat{\widetilde{\mathscr{F}}_n v_n}(k+\ell)\right\}, \end{aligned} \tag{E.5}$$

where $\widetilde{\mathscr{E}}_n v_n$ and $\widetilde{\mathscr{F}}_n v_n$ are defined as in (3.16) and (3.17) with h replaced with v_n.

- We start by proving (E.2). As in Section D.1, we have that

$$\|w_n\|_{2,n}^2 \le \frac{C}{n}\int_{-n/2}^{n/2}\left|\widehat{\widetilde{\mathscr{E}}_n v_n}(\xi) - \left(\frac{1-k}{2}\right)\widehat{\widetilde{\mathscr{F}}_n v_n}(\xi)\right|^2 W_n\left(\frac{\xi}{n}\right)d\xi,$$

where W_n is given in (D.9). By the triangular inequality, in order to finish the proof, we are reduced to show that

$$\frac{1}{n}\int_{-n/2}^{n/2}|\widehat{\widetilde{\mathscr{E}}_n v_n}(\xi)|^2 W_n\left(\frac{\xi}{n}\right)d\xi \quad\text{and}\quad \frac{1}{n}\int_{-n/2}^{n/2}|\widehat{\widetilde{\mathscr{F}}_n v_n}(\xi)|^2 W_n\left(\frac{\xi}{n}\right)d\xi \tag{E.6}$$

vanish as $n \to \infty$. Now we compute the Fourier transform of the previous functions. As in (D.3) and using (D.2) we have that

$$\begin{aligned} \widehat{\widetilde{\mathscr{E}}_n v_n}(\xi) = &-\frac{1}{n}\int_{-n/2}^{n/2}\frac{e^{\frac{2i\pi(\xi-\ell)}{n}} + e^{\frac{2i\pi\ell}{n}}}{\Lambda(\frac{\xi-\ell}{n},\frac{\ell}{n}) - i\gamma n^{-b}\Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})}\widehat{\widetilde{\mathscr{E}}_n h_n}(\xi)\{e^{-\frac{2i\pi\ell}{n}} - 1\}\,d\ell \\ &+ \left(\frac{1-\kappa}{2}\right)\frac{1}{n}\int_{-n/2}^{n/2}\frac{e^{\frac{2i\pi(\xi-\ell)}{n}} + e^{\frac{2i\pi\ell}{n}}}{\Lambda(\frac{\xi-\ell}{n},\frac{\ell}{n}) - i\gamma n^{-b}\Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})} \\ &\times \widehat{\widetilde{\mathscr{F}}_n h_n}(\xi)\{e^{-\frac{2i\pi\ell}{n}} - 1\}\,d\ell \\ = &\sqrt{n}K_n\left(\frac{\xi}{n}\right)\left\{I_n\left(\frac{\xi}{n}\right) - \left(\frac{1-\kappa}{2}\right)(1 - e^{-\frac{2\pi i\xi}{n}})\tilde{I}\left(\frac{\xi}{n}\right)\right\}\widehat{f}_n(\xi), \end{aligned} \tag{E.7}$$

where above we used (D.4), (D.7) and where K_n is given by

$$K_n(y) = \int_{-1/2}^{1/2}\frac{(e^{-2i\pi x} - 1)(e^{2i\pi(y-x)} + e^{2i\pi x})}{\Lambda(y-x,x) - i\gamma n^{-b}\Omega(y-x,x)}\,dx. \tag{E.8}$$

Now, as in (D.6) and using (D.2) we have that

$$
\begin{aligned}
\widehat{\widetilde{\mathscr{F}}_n v_n}(\xi) &= -\frac{1}{n}(e^{-\frac{2i\pi\xi}{n}} - 1)\int_{-n/2}^{n/2} \frac{e^{\frac{2i\pi(\xi-\ell)}{n}} + e^{\frac{2i\pi\ell}{n}}}{\Lambda(\frac{\xi-\ell}{n}, \frac{\ell}{n}) - i\gamma n^{-b}\Omega(\frac{\xi-\ell}{n}, \frac{\ell}{n})} \widehat{\widetilde{\mathscr{E}}_n h_n}(\xi)\, d\ell \\
&\quad + \frac{1}{n}\left(\frac{1-\kappa}{2}\right)(e^{-\frac{2i\pi\xi}{n}} - 1)\int_{-n/2}^{n/2} \frac{e^{\frac{2i\pi(\xi-\ell)}{n}} + e^{\frac{2i\pi\ell}{n}}}{\Lambda(\frac{\xi-\ell}{n}, \frac{\ell}{n}) - i\gamma n^{-b}\Omega(\frac{\xi-\ell}{n}, \frac{\ell}{n})} \\
&\quad \times \widehat{\widetilde{\mathscr{F}}_n h_n}(\xi)\, d\ell \\
&= \sqrt{n}(e^{-\frac{2i\pi\xi}{n}} - 1)\widetilde{K}_n\left(\frac{\xi}{n}\right)\left\{I_n\left(\frac{\xi}{n}\right) - \left(\frac{1-k}{2}\right)(1 - e^{-\frac{2\pi i\xi}{n}})\widetilde{I}_n\left(\frac{\xi}{n}\right)\right\} \\
&\quad \times \widehat{f}_n(\xi),
\end{aligned}
\tag{E.9}
$$

where above we used (D.4), (D.7) and $\widetilde{K}_n$ is given by

$$
\widetilde{K}_n(y) = \int_{-1/2}^{1/2} \frac{e^{2i\pi(y-x)} + e^{2i\pi x}}{\Lambda(y-x, x) - i\gamma n^{-b}\Omega(y-x, x)}\, dx. \tag{E.10}
$$

Now we estimate the term at the left-hand side of (E.6), which, by (E.7) and the triangular inequality, can be bounded from above by the sum of the two terms below. The first one is equal to

$$
\begin{aligned}
&\int_{-n/2}^{n/2} |\widehat{f}_n(\xi)|^2 \left|I_n\left(\frac{\xi}{n}\right)\right|^2 \left|K_n\left(\frac{\xi}{n}\right)\right|^2 W_n\left(\frac{\xi}{n}\right) d\xi \\
&\qquad = n\int_{-1/2}^{1/2} |\widehat{f}_n(ny)|^2 |I_n(y)|^2 |K_n(y)|^2 W_n(y)\, dy,
\end{aligned}
$$

which by Lemma F.3, Lemma F.5, Lemma B.1 and (D.11) is bounded from above by

$$
\begin{aligned}
Cn\int_{-1/2}^{1/2} \frac{1}{1+|ny|^p} \frac{|y|^{5/2}}{(|y|+n^{-b})^2}\, dy &= 2C\frac{1}{\sqrt{n}}\int_0^{n/2} \frac{1}{1+|z|^p} \frac{|z|^{5/2}}{(|z|+n^{1-b})^2}\, dz \\
&\le 2C\frac{1}{\sqrt{n}}\int_0^{n/2} \frac{|z|^{1/2}}{1+|z|^p}\, dz
\end{aligned}
$$

and goes to 0 as $n \to \infty$ by choosing p sufficiently big. The second one is

$$
\begin{aligned}
&\int_{-n/2}^{n/2} |\widehat{f}_n(\xi)|^2 \left|\widetilde{I}_n\left(\frac{\xi}{n}\right)\right|^2 |1 - e^{-\frac{2\pi i\xi}{n}}|^2 \left|K_n\left(\frac{\xi}{n}\right)\right|^2 W_n\left(\frac{\xi}{n}\right) d\xi \\
&\qquad = n\int_{-1/2}^{1/2} |\widehat{f}_n(ny)|^2 |\widetilde{I}_n(y)|^2 |1 - e^{-2\pi iy}|^2 |K_n(y)|^2 W_n(y)\, dy,
\end{aligned}
$$

which Lemma F.3, Lemma F.5, Lemma B.1 and (D.11) is bounded from above by

$$Cn\int_{-1/2}^{1/2}\frac{1}{1+|ny|^p}\frac{|y|^{5/2}}{(|y|+n^{-b})^2}\,dy=2C\frac{1}{\sqrt{n}}\int_0^{n/2}\frac{1}{1+|z|^p}\frac{|z|^{5/2}}{(|z|+n^{1-b})^2}\,dz$$
$$\leq 2C\frac{1}{\sqrt{n}}\int_0^{n/2}\frac{|z|^{1/2}}{1+|z|^p}\,dz$$

and goes to 0 as $n\to\infty$ by choosing p sufficiently big. Therefore, we have shown that the first term in (D.12) goes to 0 as n goes to infinity. The estimate for the term at the right-hand side of (D.12) is similar and this proves (E.2).

• Now we prove (E.3). As in (D.14) together with (E.5) we have that

$$\widehat{\mathscr{D}_n w_n}(\xi)=-n\big(1-e^{\frac{2i\pi\xi}{n}}\big)\Big\{\widehat{\mathscr{E}_n}v_n(\xi)-\Big(\frac{1-k}{2}\Big)\widehat{\mathscr{F}_n}v_n(\xi)\Big\}J_n\Big(\frac{\xi}{n}\Big), \tag{E.11}$$

where J_n is given in (D.15). Now, by using (E.7) and (E.9) we get

$$\begin{aligned}\widehat{\mathscr{D}_n w_n}(\xi)&=-n^{3/2}(1-e^{\frac{2i\pi\xi}{n}})K_n\Big(\frac{\xi}{n}\Big)I_n\Big(\frac{\xi}{n}\Big)J_n\Big(\frac{\xi}{n}\Big)\widehat{f_n}(\xi)\\&\quad+4n^{3/2}\Big(\frac{1-k}{2}\Big)\sin^2\Big(\frac{\pi\xi}{n}\Big)K_n\Big(\frac{\xi}{n}\Big)\widetilde{I}_n\Big(\frac{\xi}{n}\Big)J_n\Big(\frac{\xi}{n}\Big)\widehat{f_n}(\xi)\\&\quad-4n^{3/2}\Big(\frac{1-k}{2}\Big)\sin^2\Big(\frac{\pi\xi}{n}\Big)\widetilde{K}_n\Big(\frac{\xi}{n}\Big)I_n\Big(\frac{\xi}{n}\Big)J_n\Big(\frac{\xi}{n}\Big)\widehat{f_n}(\xi)\\&\quad+4n^{3/2}\Big(\frac{1-k}{2}\Big)^2\sin^2\Big(\frac{\pi\xi}{n}\Big)(1-e^{\frac{-2i\pi\xi}{n}})\widetilde{K}_n\Big(\frac{\xi}{n}\Big)\widetilde{I}_n\Big(\frac{\xi}{n}\Big)J_n\Big(\frac{\xi}{n}\Big)\\&\quad\times\widehat{f_n}(\xi).\end{aligned} \tag{E.12}$$

Finally, by Plancherel–Parseval's relation we have that

$$\begin{aligned}&\|n^{a-b-3/2}\mathscr{D}_n w_n\|_{2,n}^2\\&\quad\leq Cn^{2(a-b)+1}\int_{-1/2}^{1/2}|\sin(\pi y)|^2|I_n(y)|^2|K_n(y)|^2|J_n(y)|^2|\widehat{f_n}(ny)|^2\,dy\\&\qquad+Cn^{2(a-b)+1}\int_{-1/2}^{1/2}|\sin(\pi y)|^4|\widetilde{I}_n(y)|^2|K_n(y)|^2|J_n(y)|^2|\widehat{f_n}(ny)|^2\,dy\\&\qquad+Cn^{2(a-b)+1}\int_{-1/2}^{1/2}|\sin(\pi y)|^4|I_n(y)|^2|\widetilde{K}_n(y)|^2|J_n(y)|^2|\widehat{f_n}(ny)|^2\,dy\\&\qquad+Cn^{2(a-b)+1}\int_{-1/2}^{1/2}|\sin(\pi y)|^6|\widetilde{I}_n(y)|^2|\widetilde{K}_n(y)|^2|J_n(y)|^2|\widehat{f_n}(ny)|^2\,dy.\end{aligned} \tag{E.13}$$

By using Lemma F.4, Lemma F.3, Lemma F.5 and Lemma B.1, choosing a p sufficiently large, we can bound the first term in on the right-hand side of (E.13) by a constant times

$$n^{2a-4b+1}\int_0^{1/2}\frac{y^5}{1+(ny)^p}\frac{1}{(|y|+n^{-b})^3}\,dy$$
$$=n^{2a-2-4b}\int_0^{\infty}\frac{z^5}{1+z^p}\frac{1}{(z+n^{1-b})^3}\,dz.$$

The previous integral is bounded from above by

$$n^{2a-5-b}\int_0^{\infty}\frac{z^5}{1+z^p}\,dz$$

which goes to 0 since $a=\inf(3/2+3/2b,2)\le 2$ and $b\ge 0$. The remaining terms at the right-hand side of (E.13) are proved to go to 0 similarly by using Lemma F.4, Lemma F.3 and Lemma B.1. This completes the proof of (E.3).

• Finally, we prove (E.4). We start by computing the Fourier transform of $\widetilde{\mathscr{E}}_n w_n$ and $\widetilde{\mathscr{F}}_n w_n$. As in (D.3), we have that

$$\widehat{\widetilde{\mathscr{E}}_n w_n}(\xi)=\int_{-n/2}^{n/2}\widehat{w}_n(\xi-\ell,\ell)\{e^{-\frac{2i\pi\ell}{n}}-1\}\,d\ell. \tag{E.14}$$

By (E.5) the last expression is equal to

$$\begin{aligned}\widehat{\widetilde{\mathscr{E}}_n w_n}(\xi)&=-\frac{1}{n}\int_{-n/2}^{n/2}\frac{e^{\frac{2i\pi(\xi-\ell)}{n}}+e^{\frac{2i\pi\ell}{n}}}{\Lambda(\frac{\xi-\ell}{n},\frac{\ell}{n})-i\gamma n^{-b}\Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})}\widehat{\widetilde{\mathscr{E}}_n v_n}(\xi)\{e^{-\frac{2i\pi\ell}{n}}-1\}\,d\ell\\
&\quad+\left(\frac{1-\kappa}{2}\right)\frac{1}{n}\int_{-n/2}^{n/2}\frac{e^{\frac{2i\pi(\xi-\ell)}{n}}+e^{\frac{2i\pi\ell}{n}}}{\Lambda(\frac{\xi-\ell}{n},\frac{\ell}{n})-i\gamma n^{-b}\Omega(\frac{\xi-\ell}{n},\frac{\ell}{n})}\\
&\quad\times\widehat{\widetilde{\mathscr{F}}_n v_n}(\xi)\{e^{-\frac{2i\pi\ell}{n}}-1\}\,d\ell\\
&=-\sqrt{n}\left(K_n\left(\frac{\xi}{n}\right)\right)^2\left\{I_n\left(\frac{\xi}{n}\right)-\left(\frac{1-\kappa}{2}\right)(1-e^{-\frac{2\pi i\xi}{n}})\widetilde{I}_n\left(\frac{\xi}{n}\right)\right\}\widehat{f}_n(\xi)\\
&\quad+\left(\frac{1-\kappa}{2}\right)\sqrt{n}K_n\left(\frac{\xi}{n}\right)\widetilde{K}_n\left(\frac{\xi}{n}\right)(e^{-\frac{2\pi i\xi}{n}}-1)\\
&\quad\times\left\{I_n\left(\frac{\xi}{n}\right)-\left(\frac{1-\kappa}{2}\right)(1-e^{-\frac{2\pi i\xi}{n}})\widetilde{I}_n\left(\frac{\xi}{n}\right)\right\}\widehat{f}_n(\xi).\end{aligned}\tag{E.15}$$

Now, as in (D.6) we have that

$$\widehat{\widetilde{\mathscr{F}}_n w_n}(\xi)=(e^{-\frac{2i\pi\xi}{n}}-1)\int_{-n/2}^{n/2}\widehat{w}_n(\xi-\ell,\ell)\,d\ell. \tag{E.16}$$

From (E.5) the last expression is equal to

$$\widehat{\widetilde{\mathscr{F}}_n w_n}(\xi) = -\frac{1}{n}\big(e^{-\frac{2i\pi\xi}{n}} - 1\big)\int_{-n/2}^{n/2} \frac{e^{\frac{2i\pi(\xi-\ell)}{n}} + e^{\frac{2i\pi\ell}{n}}}{\Lambda(\frac{\xi-\ell}{n}, \frac{\ell}{n}) - i\gamma n^{-b}\Omega(\frac{\xi-\ell}{n}, \frac{\ell}{n})} \widehat{\widetilde{\mathscr{E}}_n v_n}(\xi)\, d\ell$$

$$+ \frac{1}{n}\Big(\frac{1-\kappa}{2}\Big)\big(e^{-\frac{2i\pi\xi}{n}} - 1\big)\int_{-n/2}^{n/2} \frac{e^{\frac{2i\pi(\xi-\ell)}{n}} + e^{\frac{2i\pi\ell}{n}}}{\Lambda(\frac{\xi-\ell}{n}, \frac{\ell}{n}) - i\gamma n^{-b}\Omega(\frac{\xi-\ell}{n}, \frac{\ell}{n})}$$

$$\times \widehat{\widetilde{\mathscr{F}}_n v_n}(\xi)\, d\ell$$

$$(E.17)\qquad = -\sqrt{n}\big(e^{-\frac{2i\pi\xi}{n}} - 1\big) K_n\Big(\frac{\xi}{n}\Big)\tilde{K}_n\Big(\frac{\xi}{n}\Big)$$

$$\times \Big\{ I_n\Big(\frac{\xi}{n}\Big) - \Big(\frac{1-k}{2}\Big)\big(1 - e^{-\frac{2\pi i\xi}{n}}\big)\widetilde{I}_n\Big(\frac{\xi}{n}\Big)\Big\}\widehat{f}_n(\xi)$$

$$+ \sqrt{n}\Big(\frac{1-\kappa}{2}\Big)\big(e^{-\frac{2i\pi\xi}{n}} - 1\big)^2\Big(\tilde{K}_n\Big(\frac{\xi}{n}\Big)\Big)^2$$

$$\times \Big\{ I_n\Big(\frac{\xi}{n}\Big) - \Big(\frac{1-k}{2}\Big)\big(1 - e^{-\frac{2\pi i\xi}{n}}\big)\widetilde{I}_n\Big(\frac{\xi}{n}\Big)\Big\}\widehat{f}_n(\xi).$$

Now we note that, in order to prove that

$$\|n^{a-2}\tilde{\mathscr{D}}_n w_n\|_{2,n}^2$$

vanishes as $n \to \infty$, it is enough to show that

$$\|n^{a-1/2}\widetilde{\mathscr{E}}_n w_n\|_{2,n}^2 \quad \text{and} \quad \|n^{a-1/2}\widetilde{\mathscr{F}}_n w_n\|_{2,n}^2$$

vanish as $n \to \infty$. This is true since by (D.1) and the trivial inequality $(x+y)^2 \leq 2x^2 + 2y^2$ we have that

$$\|n^{a-2}\tilde{\mathscr{D}}_n w_n\|_{2,n}^2 = n^{2a-4}\frac{1}{n^2}\sum_{x,y\in\mathbb{Z}} \tilde{\mathscr{D}}_n w_n\Big(\frac{x}{n}, \frac{y}{n}\Big)^2$$

$$\leq 2n^{2a-2}\sum_{x\in\mathbb{Z}} \widetilde{\mathscr{E}}_n w_n\Big(\frac{x}{n}\Big)^2 + 2n^{2a-2}\sum_{x\in\mathbb{Z}} \widetilde{\mathscr{F}}_n w_n\Big(\frac{x}{n}\Big)^2$$

$$= 2\|n^{a-1/2}\widetilde{\mathscr{E}}_n w_n\|_{2,n}^2 + 2\|n^{a-1/2}\widetilde{\mathscr{F}}_n w_n\|_{2,n}^2.$$

We start with the term on the left-hand side of the last expression. From Plancherel–Parseval's relation, (E.15) and the inequality $(x_1 + \cdots + x_k)^2 \leq k[x_1^2 + \cdots + x_k^2]$, we see that we have to estimate four terms which are all of the same order. One of them is

$$n^{2a}\int_{-n/2}^{n/2}\Big|K_n\Big(\frac{\xi}{n}\Big)\Big|^4\Big|I_n\Big(\frac{\xi}{n}\Big)\Big|^2|\widehat{f}_n(\xi)|^2\, d\xi.$$

By a change of variables the last term is equal to

$$n^{2a+1}\int_{-1/2}^{1/2}|\widehat{f}_n(ny)|^2|K_n(y)|^4|I_n(y)|^2\,dy.$$

By using Lemma F.3, Lemma F.5 and Lemma B.1 and doing again a change of variables we bound the last term from above by

$$\begin{aligned}&Cn^{2a-2b-2}\int_0^\infty \frac{z^5}{1+z^p}\frac{1}{(z+n^{1-b})^3}\,dz\\ &\quad\le n^{b-1}\int_0^\infty \frac{z^5}{1+z^p}\,dz\end{aligned}\tag{E.18}$$

since $a\le 2$. This integral goes to 0 as n goes to infinity as long as $b<1$. For the remaining integrals a similar computation can be done and the proof follows. □

APPENDIX F: ASYMPTOTICS OF FEW INTEGRALS

In this section, we compute or estimate several integrals. Some quantities are going to appear many times, therefore for the sake of clarity we introduce some notations. For any $y\in[-\frac{1}{2},\frac{1}{2}]$ we denote by $w:=w(y)$ the complex number $w=e^{2i\pi y}$. We denote by $\mathscr{C}$ the unit circle positively oriented, and $z:=e^{2i\pi x}$ is the dummy variable used in the complex integrals. With these notations we have

$$\begin{aligned}\Lambda(y-x,x)&=4-z\big(w^{-1}+1\big)-z^{-1}(w+1),\\ i\Omega(y-x,x)&=z\big(1-w^{-1}\big)+z^{-1}(w-1).\end{aligned}\tag{F.1}$$

Hereafter, for any complex number z, we denote by $\sqrt{z}$ its principal square root, with positive real part. Precisely, if $z=re^{i\varphi}$, with $r\ge 0$ and $\varphi\in(-\pi,\pi]$, then the principal square root of z is $\sqrt{z}=\sqrt{r}e^{i\varphi/2}$. We introduce the degree two complex polynomial:

$$P_w(z):=z^2-\frac{4}{(1+\bar{w})+\gamma n^{-b}(1-\bar{w})}z+w=(z-z_-)(z-z_+),\tag{F.2}$$

where $|z_-|<1$ and $|z_+|>1$. The important identities are

$$z_-z_+=w,\qquad z_-+z_+=\frac{4}{(1+\bar{w})+\gamma n^{-b}(1-\bar{w})}.$$

Finally, we denote

$$\begin{aligned}a_n(w)&:=(1+\bar{w})+\gamma n^{-b}(1-\bar{w}),\\ \delta_n(w)&:=4-w\big[(1+\bar{w})+\gamma n^{-b}(1-\bar{w})\big]^2,\end{aligned}$$

so that the discriminant of P_w is $4\delta_n(w)/a_n^2(w)$ and

$$z_+=\frac{2+\sqrt{\delta_n(w)}}{a_n(w)},\qquad z_-=\frac{2-\sqrt{\delta_n(w)}}{a_n(w)}.$$

LEMMA F.1. *We have that*

$$G_n(y) = \frac{1}{n^{b/2}}\sqrt{\frac{\gamma}{2}}\big|\sin(\pi y)\big|^{3/2} e^{i\,\mathrm{sgn}(y)\frac{\pi}{4}} + \mathscr{O}\big(\sin^2(\pi y)\big) \qquad \text{if } b<1,$$
$$G_n(y) = \mathscr{O}\big(n^{-b}\big|\sin(\pi y)\big|\big) \qquad \text{if } b \geq 1. \tag{F.3}$$

REMARK F.2. The reader will remark that the first (resp. second) estimate in Lemma F.1 is in fact valid also if $b \geq 1$ (resp. $b < 1$). On the other hand, it is the first (resp. second) estimate in this lemma which is used in the case $b < 1$ (resp. $b \geq 1$).

PROOF OF LEMMA F.1. We compute the function G_n by using the residue theorem. We have that

$$G_n(y) = \frac{(1-\bar{w})^2}{8i\pi}\frac{\gamma n^{-b}}{a_n(w)}\oint_{\mathscr{C}} f_w(z)\,dz \tag{F.4}$$

with

$$f_w(z) = \frac{(z^2+w)^2}{z^2 P_w(z)} = \frac{(z^2+w)^2}{z^2(z-z_-)(z-z_+)}. \tag{F.5}$$

Therefore, we get that

$$\begin{aligned}
G_n(y) &= \frac{(1-\bar{w})^2}{4}\frac{\gamma n^{-b}}{a_n(w)}\big[\mathrm{Res}(f_w,0)+\mathrm{Res}(f_w,z_-)\big] \\
&= \frac{(1-\bar{w})^2}{4}\frac{\gamma n^{-b}}{a_n(w)}\left[z_- + z_+ + \frac{(z_-^2+w)^2}{z_-^2(z_- - z_+)}\right] \\
&= \frac{(1-\bar{w})^2}{4}\frac{\gamma n^{-b}}{a_n(w)}\left[\frac{4}{a_n(w)} - \frac{8}{a_n(w)\sqrt{\delta_n(w)}}\right] \\
&= \gamma n^{-b}\frac{(1-\bar{w})^2}{a_n^2(w)}\left[1 - \frac{2}{\sqrt{\delta_n(w)}}\right] \\
&= \frac{\gamma}{4n^b}\frac{(1-\bar{w})^2 w}{\frac{w}{4}a_n^2(w)}\left[1 - \frac{1}{\sqrt{1-\frac{w}{4}a_n^2(w)}}\right] \\
&= \frac{\gamma}{4n^b}(1-\bar{w})^2 w g\left(\frac{w}{4}a_n^2(w)\right) - \frac{\gamma}{4n^b}\frac{(1-\bar{w})^2 w}{\sqrt{1-\frac{w}{4}a_n^2(w)}},
\end{aligned} \tag{F.6}$$

where $g : u \in \mathbb{C}\backslash\{1\} \to u^{-1}(1-(1-u)^{-1/2}) + (1-u)^{-1/2}$. We have that $g(u) = \frac{1}{1+\sqrt{1-u}}$. Since $\sqrt{1-u}$ has a positive real part, we deduce that the function g is uniformly bounded. Therefore, there exists a universal constant $C > 0$ such that

$$\left|G_n(y) + \frac{\gamma}{4n^b}\frac{(1-\bar{w})^2 w}{\sqrt{1-\frac{w}{4}a_n^2(w)}}\right| \leq C n^{-b}\sin^2(\pi y).$$

Let us now observe that

$$1-\frac{w}{4}a_n^2(w)=\left(1+\frac{\gamma^2}{n^{2b}}\right)\sin^2(\pi y)-i\frac{\gamma}{n^b}\sin(2\pi y) \tag{F.7}$$

and that

$$\begin{aligned}&\operatorname{Arg}\left(1-\frac{w}{4}a_n^2(w)\right)\\&\quad=-\operatorname{sgn}(y)\frac{\pi}{2}+\arctan\left(\frac{\gamma^{-1}n^b(1+\gamma^2n^{-2b})}{2}\tan(\pi y)\right).\end{aligned} \tag{F.8}$$

Since $\cos^2(\pi y)=1-\sin^2(\pi y)$, we have that

$$\begin{aligned}\left|1-\frac{w}{4}a_n^2(w)\right|^2&=\sin^2(\pi y)\left[(1+\gamma^2n^{-2b})^2\sin^2(\pi y)+4\gamma^2n^{-2b}\cos^2(\pi y)\right]\\&=\sin^2(\pi y)\left\{\left(1-\frac{\gamma^2}{n^{2b}}\right)^2\sin^2(\pi y)+\frac{4\gamma^2}{n^{2b}}\right\}.\end{aligned} \tag{F.9}$$

It follows that

$$\begin{aligned}G_n(y)=\frac{\gamma}{n^b}&\frac{|\sin(\pi y)|^{3/2}}{[(1+\gamma^2n^{-2b})^2\sin^2(\pi y)+4\gamma^2n^{-2b}\cos^2(\pi y)]^{1/4}}e^{i\varphi_n(y)}\\&+\mathscr{O}\big(n^{-b}\sin^2(\pi y)\big)\end{aligned} \tag{F.10}$$

with

$$\varphi_n(y)=\operatorname{sgn}(y)\frac{\pi}{4}-\frac{1}{2}\arctan\left(\frac{\gamma^{-1}n^b(1+\gamma^2n^{-2b})}{2}\tan(\pi y)\right). \tag{F.11}$$

Let us look at first at the case $b<1$. Then, by (F.9), we have that

$$\begin{aligned}&\frac{\gamma}{n^b}\frac{|\sin(\pi y)|^{3/2}}{[(1+\gamma^2n^{-2b})^2\sin^2(\pi y)+4\gamma^2n^{-2b}\cos^2(\pi y)]^{1/4}}\\&\qquad=\frac{\gamma}{n^{b/2}\sqrt{2\gamma}}\frac{|\sin(\pi y)|^{3/2}}{[1+\frac{n^{2b}}{4\gamma^2}(1-\gamma^2n^{-2b})^2\sin^2(\pi y)]^{1/4}}\\&\qquad=\frac{\sqrt{\gamma}}{n^{b/2}\sqrt{2}}\left|\sin(\pi y)\right|^{3/2}+\varepsilon_n(y).\end{aligned} \tag{F.12}$$

We claim that $\varepsilon_n(y)=\mathscr{O}(\sin^2(\pi y))$. To prove it, we distinguish two cases:

- $|\sin(\pi y)|\ge n^{-b}$: Then, since $|(1+t)^{-1/4}-1|\le 2$ for $t>0$, we have

$$\left|\varepsilon_n(y)\right|\le\frac{\sqrt{2\gamma}\,|\sin(\pi y)|^{3/2}}{n^{b/2}}\le C\sin^2(\pi y).$$

- $|\sin(\pi y)| \le n^{-b}$: Then, since $|(1+t)^{-1/4} - 1| \le t$ for $t > 0$, we have

$$\begin{aligned} \left|\varepsilon_n(y)\right| &\le C \frac{|\sin(\pi y)|^{3/2}}{n^{b/2}} n^{2b} \sin^2(\pi y) \\ &= C n^{3b/2} \left|\sin(\pi y)\right|^{3/2} \sin^2(\pi y) \le C \sin^2(\pi y) \end{aligned}$$

and the claim is proved. Therefore, we have that, for any $y \in [-1/2, 1/2]$,

$$G_n(y) = \frac{\sqrt{\gamma}}{n^{b/2}\sqrt{2}} \left|\sin(\pi y)\right|^{3/2} e^{i\varphi_n(y)} + \mathcal{O}\big(\sin^2(\pi y)\big). \tag{F.13}$$

Observe also that

$$\begin{aligned} \left|e^{i\varphi_n(y)} - e^{i \operatorname{sgn}(y) \frac{\pi}{4}}\right| &\le \left|\exp\left\{-\frac{i}{2} \arctan\left(\left(\frac{\gamma^{-1} n^b (1+\gamma^2 n^{-2b})}{2}\right) \tan(\pi y)\right)\right\} - 1\right| \\ &\le \frac{1}{2}\left|\arctan\left(\frac{1}{2}\gamma^{-1} n^b (1+\gamma^2 n^{-2b}) \tan(\pi y)\right)\right| \qquad \text{(F.14)} \\ &\le \mathcal{O}\left(\frac{\pi}{2} \wedge n^b \left|\tan(\pi y)\right|\right). \end{aligned}$$

It follows that

$$\begin{aligned} G_n(y) = {} & \frac{\sqrt{\gamma}}{n^{b/2}\sqrt{2}} \left|\sin(\pi y)\right|^{3/2} e^{i \operatorname{sgn}(y) \frac{\pi}{4}} \\ & + \mathcal{O}\big(\sin^2(\pi y)\big) + \mathcal{O}\big(n^{-b/2} \left|\sin(\pi y)\right|^{3/2} \wedge n^{b/2} \left|\sin(\pi y)\right|^{5/2}\big). \end{aligned} \tag{F.15}$$

By considering the cases $|\sin(\pi y)| \le n^{-b}$ and $|\sin(\pi y)| > n^{-b}$ we see that

$$\mathcal{O}\big(n^{-b/2} \left|\sin(\pi y)\right|^{3/2} \wedge n^{b/2} \left|\sin(\pi y)\right|^{5/2}\big) = \mathcal{O}\big(\sin^2(\pi y)\big)$$

and this proves the first claim.

For the second item, assume that $b \ge 1$ and start with the expression (F.10) and we observe that, for some constant $C > 0$,

$$\begin{aligned} & \frac{\gamma}{n^b} \frac{|\sin(\pi y)|^{3/2}}{[(1+\gamma^2 n^{-2b})^2 \sin^2(\pi y) + 4\gamma^2 n^{-2b} \cos^2(\pi y)]^{1/4}} \\ & \quad \le \frac{C}{n^b} \frac{|\sin(\pi y)|^{3/2}}{|\sin(\pi y)|^{1/2}} = \frac{C}{n^b} \left|\sin(\pi y)\right|. \end{aligned} \tag{F.16}$$

□

LEMMA F.3. *Let $b \ge 0$. We have that for any $y \in [-1/2, 1/2]$*

$$\left|I_n(y)\right| \le C n^{-b} \frac{|\sin(\pi y)|^{3/2}}{\sqrt{n^{-b} + |\sin(\pi y)|}}$$

and

$$\left|\widetilde{I}_n(y)\right| \le C n^{-b} \frac{|\sin(\pi y)|^{1/2}}{\sqrt{n^{-b} + |\sin(\pi y)|}}.$$

PROOF. We have that

$$I_n(y) = -\frac{\gamma}{n^b}\frac{(1-\bar{w})}{a_n(w)}\frac{1}{2i\pi}\oint_{\mathscr{C}} g_w(z)\,dz \tag{F.17}$$

with

$$g_w(z) = \frac{(z-1)(z^2+w)}{z^2 P_w(z)} = \frac{(z-1)(z^2+w)}{z^2(z-z_-)(z-z_+)}. \tag{F.18}$$

It follows that

$$\begin{aligned}
I_n(y) &= -\frac{\gamma}{n^b}\frac{(1-\bar{w})}{a_n(w)}\{\operatorname{Res}(g_w,0)+\operatorname{Res}(g_w,z_-)\}\\
&= -\frac{\gamma}{n^b}\frac{(1-\bar{w})}{a_n(w)}\left\{1-\frac{z_-+z_+}{w}+\left(\frac{1}{z_-}-1\right)\frac{z_-+z_+}{z_+-z_-}\right\}\\
&= -\frac{\gamma}{n^b}\frac{1-\bar{w}}{a_n(w)}\left\{1-\frac{2}{w\frac{a_n(w)}{2}}+\frac{1}{\sqrt{1-\frac{w}{4}a_n^2(w)}}\frac{\frac{a_n(w)}{2}-1+\sqrt{1-\frac{w}{4}a_n^2(w)}}{1-\sqrt{1-\frac{w}{4}a_n^2(w)}}\right\}\\
&:= -\frac{\gamma(1-\bar{w})}{2n^b}\mathscr{K}_w\left(\frac{a_n(w)}{2}\right)
\end{aligned}$$

with

$$\mathscr{K}_w(u) = \frac{1}{u}\left\{1-\frac{2}{wu}+\frac{1}{\sqrt{1-wu^2}}\frac{u-1+\sqrt{1-wu^2}}{1-\sqrt{1-wu^2}}\right\}, \qquad |u|<1. \tag{F.19}$$

We observe first that uniformly in w we have by (F.7)

$$\frac{a_n(w)}{2} = \frac{1+\bar{w}}{2}+\mathscr{O}\left(\frac{|1-w|}{n^b}\right) \tag{F.20}$$

and by (F.9) that

$$c|w-1|\big(|w-1|+n^{-b}\big) \le \left|1-w\frac{a_n^2(w)}{4}\right| \le C|w-1|\big(|w-1|+n^{-b}\big). \tag{F.21}$$

It follows that if w is not close to ± 1, say $|w\pm 1|\ge\varepsilon>0$, then $\mathscr{K}_w(\frac{a_n(w)}{2})$ can be uniformly bounded by a constant C_ε independently of n.

If w is close to -1, then $a_n(w)$ is close to 0. Performing a Taylor expansion of $\mathscr{K}_w$ around $u=0$, we obtain that uniformly in w,

$$\mathscr{K}_w(u) = \mathscr{O}(1).$$

It remains thus only to consider the case where w is close to 1, say $|w-1|\le\varepsilon$, which implies that $|a_n(w)/2-1|\le\varepsilon$ for n sufficiently large. We rewrite, for

$|w-1|+|u-1|\leq\varepsilon$,

$$\begin{aligned}|\mathscr{K}_w(u)| &= \left|\frac{1}{u}\left\{1-\frac{2}{wu}+\frac{1}{1-\sqrt{1-wu^2}}+\frac{u-1}{\sqrt{1-wu^2}}\frac{1}{1-\sqrt{1-wu^2}}\right\}\right| \\ &\leq C_\varepsilon\left[1+\frac{|u-1|}{\sqrt{|1-wu^2|}}\right].\end{aligned} \tag{F.22}$$

We use now (F.20) and (F.21) to obtain

$$\left|\mathscr{K}_w\left(\frac{a_n(w)}{2}\right)\right| \leq C_\varepsilon\left[1+\frac{\sqrt{|w-1|}}{\sqrt{|w-1|+n^{-b}}}\right]\leq C_\varepsilon\frac{\sqrt{|w-1|}}{\sqrt{|w-1|+n^{-b}}}. \tag{F.23}$$

The conclusion of the first item follows.

Similarly, we have that

$$\widetilde{I}_n(y) = -\frac{\gamma}{n^b}\frac{1-\bar{w}}{a_n(w)}\frac{1}{2i\pi}\oint_{\mathscr{C}} h_w(z)\,dz \tag{F.24}$$

with

$$h_w(z)=\frac{(z^2+w)}{zP_w(z)}=\frac{(z^2+w)}{z(z-z_-)(z-z_+)}. \tag{F.25}$$

It follows that

$$\begin{aligned}\widetilde{I}_n(y) &= -\frac{\gamma}{n^b}\frac{(1-\bar{w})}{a_n(w)}\{\operatorname{Res}(h_w,0)+\operatorname{Res}(h_w,z_-)\} \\ &= \frac{\gamma(1-\bar{w})}{2n^b}\widetilde{\mathscr{K}_w}\left(\frac{a_n(w)}{2}\right),\end{aligned}$$

where

$$\widetilde{\mathscr{K}_w}(u)=\frac{1}{u}\left\{1-\frac{1}{\sqrt{1-wu^2}}\right\},\qquad |u|<1.$$

Let $\varepsilon>0$ small be fixed. If $|w\pm 1|\geq\varepsilon$, then by (F.20) we deduce that for n sufficiently large (uniformly in w in this domain)

$$\left|\widetilde{\mathscr{K}_w}\left(\frac{a_n(w)}{2}\right)\right|\leq C_\varepsilon.$$

If $|w+1|\leq\varepsilon$, then for n sufficiently large, uniformly in w in this domain, $|a_n(w)|\leq\varepsilon$. And we have that for $|u|\leq\varepsilon$, uniformly in w, $|\mathscr{K}_w(u)|\leq Cu$. Therefore, we deduce that in this case

$$\left|\widetilde{\mathscr{K}_w}\left(\frac{a_n(w)}{2}\right)\right|\leq C_\varepsilon.$$

If $|w-1| \le \varepsilon$, then for n sufficiently large, uniformly in w in this domain, $|a_n(w)/2-1| \ge \varepsilon/2$ and therefore, by (F.21) we have

$$\left|\widetilde{\mathscr{K}}_w\left(\frac{a_n(w)}{2}\right)\right| \le \frac{C_\varepsilon}{\sqrt{|w-1|^2+|w-1|n^{-b}}}.$$

The conclusion of the second item follows. □

LEMMA F.4. *Let $b \ge 0$. We have that, for any $y \in [-1/2, 1/2]$,*

$$\left|J_n(y)\right| \le C \frac{|\sin(\pi y)|^{-1/2}}{\sqrt{n^{-b}+|\sin(\pi y)|}}$$

PROOF. We have that

$$J_n(y) = -\frac{1}{(1+\bar{w})+\gamma n^{-b}(1-\bar{w})} \frac{1}{2i\pi} \oint_{\mathscr{C}} k_w(z)\, dz \tag{F.26}$$

with

$$k_w(z) = \frac{(z^2+w)}{z^2 P_w(z)}. \tag{F.27}$$

It follows that

$$\begin{aligned} J_n(y) &= -\frac{1}{(1+\bar{w})+\gamma n^{-b}(1-\bar{w})} \{\operatorname{Res}(k_w, 0) + \operatorname{Res}(k_w, z_-)\} \\ &= \frac{1}{(1+\bar{w})+\gamma n^{-b}(1-\bar{w})} \left\{ \frac{z_+ + z_-}{w} - \frac{1}{z_-} \frac{z_+ + z_-}{z_+ - z_-} \right\} \\ &= \mathscr{H}_w\left(\frac{a_n(w)}{2}\right) \end{aligned}$$

with

$$\mathscr{H}_w(u) = \frac{1}{u}\left\{ \frac{2}{wu} - \frac{1}{\sqrt{1-wu^2}} \frac{u}{1-\sqrt{1-wu^2}} \right\}, \qquad |u| < 1.$$

Since uniformly in w we have $\mathscr{H}_w(u) = \mathscr{O}(1)$ for $u \to 0$, we have only to study the behavior of $\mathscr{H}_w(\frac{a_n(w)}{2})$ for w close to 1, which implies $a_n(w)/2$ close to 1 by recalling (F.20). Therefore, for say $|w-1| \le \varepsilon$ with $\varepsilon > 0$ small, we have

$$\left|\mathscr{H}_w\left(\frac{a_n(w)}{2}\right)\right| \le C_\varepsilon \left[1 + \frac{1}{\sqrt{|1-w\frac{a_n^2(w)}{4}|}}\right]. \tag{F.28}$$

Taking into account (F.21), we get the claim. □

LEMMA F.5. *Let $b \geq 0$. We have that, for any $y \in [-1/2, 1/2]$,*

$$\left|K_n(y)\right| \leq C \frac{|\sin(\pi y)|^{1/2}}{\sqrt{n^{-b} + |\sin(\pi y)|}}$$

and

$$\left|\widetilde{K}_n(y)\right| \leq C \frac{|\sin(\pi y)|^{-1/2}}{\sqrt{n^{-b} + |\sin(\pi y)|}}.$$

PROOF. We have that

$$K_n(y) = \frac{1}{a_n(w)} \frac{1}{2i\pi} \oint_{\mathscr{C}} g_w(z)\, dz,$$

where

$$g_w(z) = \frac{(z-1)(z^2+w)}{z^2 P_w(z)}$$

has been introduced in (F.18) during the proof of Lemma F.3. We have, therefore, that

$$K_n(y) = -\frac{n^b}{\gamma} \frac{1}{1-\bar{w}} I_n(y). \tag{F.29}$$

The estimate on K_n follows from Lemma F.3.

Similarly we have that

$$\widetilde{K}_n(y) = -\frac{1}{a_n(w)} \frac{1}{2i\pi} \oint_{\mathscr{C}} h_w(z)\, dz,$$

where

$$h_w(z) = \frac{z^2+w}{z P_w(z)}$$

has been introduced in (F.25) during the proof of Lemma F.3. We have, therefore, that

$$\widetilde{K}_n(y) = \frac{n^b}{\gamma} \frac{1}{1-\bar{w}} \widetilde{I}_n(y). \tag{F.30}$$

The estimate on $\widetilde{K}_n$ follows from Lemma F.3. □

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C. BERNARDIN
UNIVERSITÉ CÔTE D'AZUR
CNRS, LJAD
PARC VALROSE
06108 NICE CEDEX 02
FRANCE
E-MAIL: cbernard@unice.fr

P. GONÇALVES
CENTER FOR MATHEMATICAL ANALYSIS,
GEOMETRY AND DYNAMICAL SYSTEMS
INSTITUTO SUPERIOR TÉCNICO
UNIVERSIDADE DE LISBOA
AV. ROVISCO PAIS
1049-001 LISBOA
PORTUGAL
E-MAIL: patricia.goncalves@math.tecnico.ulisboa.pt

M. JARA
INSTITUTO DE MATEMÁTICA PURA
E APLICADA
ESTRADA DONA CASTORINA 110
22460-320 RIO DE JANEIRO
BRAZIL
E-MAIL: mjara@impa.br