ON DIRECTIONAL DERIVATIVES OF SKOROKHOD MAPS IN CONVEX POLYHEDRAL DOMAINS

BY DAVID LIPSHUTZ[1] AND KAVITA RAMANAN[2]

Brown University

The study of both sensitivity analysis and differentiability of the stochastic flow of a reflected process in a convex polyhedral domain is challenging due to the abrupt change in the nature of the dynamics at the boundary and is further complicated because the boundary is not smooth. These difficulties can be addressed by studying directional derivatives of an associated extended Skorokhod map, which is a deterministic mapping that takes an unconstrained path to a suitably reflected or constrained version. In this work, we develop an axiomatic framework for the analysis of directional derivatives of a large class of Lipschitz continuous extended Skorokhod maps in convex polyhedral domains with oblique directions of reflection. We establish existence of directional derivatives at a path whose reflected version satisfies a certain boundary jitter property, and also show that the right-continuous regularization of such a directional derivative can be characterized as the unique solution to a Skorokhod-type problem, where both the domain and directions of reflection vary (discontinuously) depending on the state of the reflected path. A key step in the analysis is the proof of certain contraction properties for a family of (oblique) derivative projection operators. The results of this paper are used in subsequent work to study differentiability of stochastic flows and sensitivity analysis for a large class of reflected diffusions in convex polyhedral domains.

CONTENTS

1. Introduction.

1.1. *Overview.* Reflected stochastic processes that are constrained to lie in the closure of a convex polyhedral domain arise in many contexts, including as diffusion approximations of stochastic networks [11, 40], in the study of interacting diffusions and limits of interacting particle systems [10, 44, 46] and in mathematical finance [4]. In these applications, it is of interest to analyze the sensitivity of the reflected process to the parameters that define the process. The analysis of processes with state constraints is challenging due to the fact that the dynamics are often discontinuous on the boundary of the domain, and is further complicated when the boundary is not smooth. In many cases, the extended Skorokhod problem (ESP), which is a generalization of the Skorokhod problem (SP), provides a convenient tool for the pathwise analysis of such processes [13, 17, 23, 29, 38, 39, 41, 43]. Roughly speaking, the ESP provides an axiomatic framework to constrain a path taking values in J-dimensional Euclidean space $\mathbb{R}^J$ to the closure of a domain in $\mathbb{R}^J$ with a constraining function that "pushes" in prescribed directions on the boundary of the domain, referred to as the directions of reflection, to ensure that the path remains within the closure of the domain. The constrained path can often be represented as the image of an unconstrained path, which is typically analytically more tractable, under the associated extended Skorokhod map (ESM), a generalization of the Skorokhod map (SM). The analysis of the constrained path is then largely reduced to the study of properties of the ESM. We show that the study

of sensitivity to perturbations of the constrained path is greatly facilitated when the ESM possesses so-called directional derivatives (see Definition 2.15 below). The main goal of this work is to establish existence and provide a useful characterization of directional derivatives of a large class of ESMs in convex polyhedral domains. These results are used in [31] to establish pathwise differentiability of a large class of reflected diffusions in convex polyhedral domains, and are potentially useful for the sensitivity analysis of a broader class of reflected stochastic processes.

The study of the differentiability of the stochastic flow associated with a stochastic process or, more broadly speaking, sensitivity of the stochastic process to perturbations in the initial condition and other parameters that define the process, is a classical topic in stochastic analysis. For example, there is a substantial body of work that studies these questions for (unconstrained) diffusions in $\mathbb{R}^J$, with contributions from Elworthy [22], Bismut [6], Ikeda and Watanabe [25], Kunita [27], Metivier [35] and others. The book by Kunita [28] contains a summary of many of these results. In contrast, there are relatively few results for reflected processes or even reflected Brownian motions, especially in the context of oblique reflection and nonsmooth domains which, as mentioned above, is relevant in applications. Two exceptions include the work of Andres [1], which establishes differentiability of the stochastic flow of a reflected diffusion with identity covariance in a convex polyhedral domain, but only until the first time the reflected diffusion hits a nonsmooth part of the boundary, thus avoiding having to deal with the effects of the nonsmooth part of the boundary; and the work of Dieker and Gao [16], which looks at sensitivities of reflected diffusions in the nonnegative orthant (with reflection matrices that are $\mathcal{M}$-matrices; see [34], Definition 1.2) with respect to a particular perturbation of the drift, namely in the direction $-\mathbf{1}$, the vector with negative one in each component. In addition to these works, Deuschel and Zambotti [15] considered differentiability of stochastic flows for normally reflected diffusions with identity covariance in the orthant; Pilipenko (see [37] and references therein) studied differentiability properties of stochastic flows of reflected diffusions in the half space with normal reflection; and Burdzy [7] and Andres [2] characterized derivatives of stochastic flows for normally reflected Brownian motions and reflected diffusions with identity covariance, respectively, in domains with smooth boundaries, where geometric challenges arise due to the curvature of the boundary. It should be emphasized that each of works above considers the sensitivity of a reflected process with respect to a single parameter. An advantage of our approach is that directional derivatives of SMs or ESMs provide a framework to simultaneously consider perturbations to a variety of parameters that describe a reflected process.

Our approach to studying pathwise differentiability of reflected stochastic processes leverages properties of directional derivatives of the associated SM or ESM. Directional derivatives of the one-dimensional SM were first introduced

by Mandelbaum and Massey [33] to analyze diffusion approximations of time-inhomogeneous queues (see also [45], Chapter 9, for a generalization). In the multidimensional setting, Mandelbaum and Ramanan [34] established existence and obtained a characterization of directional derivatives of SMs in the nonnegative orthant with reflection matrices that are $\mathcal{M}$-matrices. This was used by Cudina and Ramanan [14] to study asymptotically optimal controls for fluid limits of time-inhomogeneous queueing networks; by Chen [12] to develop an algorithm for unbiased estimators of sensitivities of a stochastic fluid network; by Dieker and Gao in their work [16] mentioned above; and by Honnappa, Jain and Ward [24] to study the behavior of transient queueing networks. The proof in [34] relies on such SMs having a fairly explicit representation and satisfying a certain monotonicity property, which does not hold for more general classes of multidimensional SMs or ESMs. In particular, any SM considered in [34] can be viewed as a system of coupled one-dimensional SMs, and so its analysis is greatly facilitated by the known explicit expression for the one-dimensional SM and the monotonicity property of the SM. These SMs arise as diffusion approximations of open single-class queueing networks [40]. However, SMs associated with more general multiclass queueing networks fall outside this class and do not possess these nice properties. For example, the monotonicity property can fail to hold even for SMs that lie in a slightly larger class of SMs (see Example 2.14 below), which arise in the study of feedforward networks. Thus, the approach in [34] does not yield existence of directional derivatives even for SMs in this slightly larger class.

We develop a completely different approach from [34] to studying directional derivatives of an ESM—we adopt an axiomatic framework much in the spirit of the ESP. Specifically, given an ESM and a continuous path that is constrained by the ESM, we characterize the derivative of the ESM in the direction of any continuous perturbation in terms of a constrained version of this perturbation that satisfies a Skorokhod-type problem whose domain and directions of reflection vary (discontinuously) depending on the state of the constrained version of the path. This formulation is made precise via the so-called derivative problem (DP) introduced in Definition 3.4. We show that under general conditions on the domain and directions of reflection that ensure that the ESM is well defined and Lipschitz continuous, directional derivatives of the ESM exist and are uniquely characterized via the DP when the ESM is evaluated at a continuous path whose constrained version (i.e., image under the ESM) satisfies a certain boundary jitter property (see Definition 3.1 below). While the boundary jitter property is not necessary to prove existence of directional derivatives (see the discussion following Theorem 3.11), this property enables a nice characterization of directional derivatives via the DP. Furthermore, consideration of such paths is sufficient for many stochastic applications. In [31], we show that the boundary jitter property is satisfied by a large class of reflected diffusions in convex polyhedral domains. The pathwise nature of our analysis allows for our results to be applied in principle to a larger class

of continuous reflected processes that can be shown to satisfy the boundary jitter property.

In summary, the main contributions of this paper are as follows:

- Definition and analysis of the boundary jitter property (Section 3.1 and Section 4).
- Formulation and analysis of the DP (Section 3.2 and Section 5).
- Existence of directional derivatives of the ESM evaluated at paths satisfying the boundary jitter property and their characterization via the DP (Section 3.3 and Sections 6–9).

Our results are used in [31] to study pathwise differentiability of reflected diffusions and to obtain probabilistic representations for derivatives of expectations of certain functionals of reflected diffusions. Our framework allows for the simultaneous consideration of pathwise derivatives with respect to the dispersion coefficient and directions of reflection, in addition to the initial condition and drift coefficient. The consideration of perturbations with respect to all parameters, including the directions of reflection, is strongly motivated by applications—for example, this is typically necessary for the computation of sensitivities of the (reflected) diffusion approximation of a queueing network with respect to physically relevant parameters such as the mean arrival rate or service rate at a queue (see, e.g., [40]). To the best of our knowledge, sensitivities of a reflected diffusion with respect to its dispersion coefficient or directions of reflection have not been considered in any prior work. Our results are also used in [32] to develop a Monte Carlo method for estimating sensitivities of reflected diffusions.

1.2. *Outline of the paper.* The paper is organized as follows. In Section 2, a precise statement of the ESP with a convex polyhedral domain is given and its associated ESM is introduced. Moreover, assumptions guaranteeing that the ESM is well defined are stated and the notion of a directional derivative of the ESM is introduced. Our main results on directional derivatives of an ESM are presented in Section 3. In Section 4, some important consequences of the boundary jitter property are shown. A discussion of the DP and its properties is given in Section 5. The proof of our main result on existence of directional derivatives and their characterization via the DP is given in Sections 6–9. Proofs of some useful lemmas are relegated to Appendices A–B.

1.3. *Notation.* We now collect some notation that will be used throughout this work. We use $\mathbb{N} = \{1, 2, \ldots\}$ to denote the set of positive integers, and let $\mathbb{N}_\infty \doteq \mathbb{N} \cup \{\infty\}$. For $J \in \mathbb{N}$, let $\mathbb{R}^J_+$ denote the closed nonnegative orthant in $\mathbb{R}^J$. When $J = 1$, we suppress J and simply write $\mathbb{R}$ for $(-\infty, \infty)$ and $\mathbb{R}_+$ for $[0, \infty)$. Given $r, s \in \mathbb{R}$, we let $r \vee s \doteq \max(r, s)$ and $r \wedge s \doteq \min(r, s)$. For a column vector $x \in \mathbb{R}^J$, let x^j denote the jth component of x. We let $\{e_1, \ldots, e_J\}$ denote the standard orthonormal basis for $\mathbb{R}^J$. We use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the usual Euclidean inner

product and Euclidean norm, respectively. We let $\mathbb{S}^{J-1}$ denote the unit sphere in $\mathbb{R}^J$. For $J, K \in \mathbb{N}$, let $\mathbb{R}^{J \times K}$ denote the set of real-valued matrices with J rows and K columns. For a matrix $M \in \mathbb{R}^{J \times K}$, let M_k^j denote the entry of the jth row and kth column, M^j denote the jth row and M_k denote the kth column. We write M' to denote the transpose of M.

Given a subset $A \subseteq \mathbb{R}^J$, let A° and ∂A denote the interior and boundary, respectively, of A. We let $\mathrm{cone}(A)$ denote the convex cone generated by A; that is,

$$\mathrm{cone}(A) \doteq \left\{ \sum_{k=1}^K r_k x_k : K \in \mathbb{N}, x_k \in A, r_k \geq 0 \right\},$$

with the convention that $\mathrm{cone}(\varnothing) \doteq \{0\}$. We let $\mathrm{span}(A)$ denote the set of all possible finite linear combinations of vectors in A with the convention that $\mathrm{span}(\varnothing) \doteq \{0\}$. We let $A^\perp$ denote the orthogonal complement of $\mathrm{span}(A)$ in $\mathbb{R}^J$.

Given $T \in (0, \infty]$ and a closed, convex subset $E \subseteq \mathbb{R}^J$, we let $\mathbb{D}_{\mathrm{l,r}}([0, T) : E)$ denote the set of functions on $[0, T)$ taking values in E that have finite left limits at all $t \in (0, T)$, finite right limits at all $t \in [0, T)$, and are left continuous and/or right continuous at each $t \in (0, \infty)$. We let $\mathbb{D}_{\mathrm{r}}([0, T) : E)$ denote the subset of right continuous functions with finite left limits in $\mathbb{D}_{\mathrm{l,r}}([0, T) : E)$ and let $\mathbb{C}([0, T) : E)$ denote the further subset of continuous functions in $\mathbb{D}_{\mathrm{r}}([0, T) : E)$. Given a subset $A \subseteq E$, we use $\mathbb{C}_A([0, T) : E)$ to denote the subset of continuous functions $f \in \mathbb{C}([0, T) : E)$ with $f(0) \in A$. When $T = \infty$, $E = \mathbb{R}^J$ and $A \subseteq \mathbb{R}^J$, we simply write $\mathbb{D}_{\mathrm{l,r}}$, $\mathbb{D}_{\mathrm{r}}$, $\mathbb{C}$ and $\mathbb{C}_A$ for $\mathbb{D}_{\mathrm{l,r}}([0, \infty) : \mathbb{R}^J)$, $\mathbb{D}_{\mathrm{r}}([0, \infty) : \mathbb{R}^J)$, $\mathbb{C}([0, \infty) : \mathbb{R}^J)$ and $\mathbb{C}_A([0, \infty) : \mathbb{R}^J)$, respectively. We endow $\mathbb{D}_{\mathrm{r}}([0, T) : E)$ and its subsets with the topology of uniform convergence on compact intervals in $[0, T)$. For $f \in \mathbb{D}_{\mathrm{l,r}}([0, T) : E)$ and $t \in [0, T)$, define the supremum norm of f over $[0, t]$ by

$$\|f\|_t \doteq \sup_{s \in [0,t]} |f(s)| < \infty.$$

Let $|f|(t) \in [0, \infty]$ denote the total variation of f over the interval $[0, t]$. We let $f(t-) \doteq \lim_{s \uparrow t} f(s)$ for all $t \in (0, T)$ and $f(t+) \doteq \lim_{s \downarrow t} f(s)$ for all $t \in [0, T)$. We call the function $g \in \mathbb{D}_{\mathrm{r}}([0, T) : E)$ defined by $g(t) \doteq f(t+)$ for all $t \in [0, T)$ the *right continuous regularization of f*.

We abbreviate "such that" as "s.t."

2. The extended Skorokhod reflection problem.

In this section, we introduce the ESP and directional derivatives of the associated ESM. In Section 2.1, we describe the class of convex polyhedral domains that we consider and give a precise definition of a solution to the ESP. In Section 2.2, we provide sufficient conditions for the associated ESM to satisfy a Lipschitz continuity condition. In Section 2.3, we present further conditions under which the ESM is well defined on all of $\mathbb{C}$. In Section 2.4, we define a directional derivative of the ESM.

2.1. *Statement of the extended Skorokhod reflection problem.* Let G be the closure of a nonempty convex polyhedral domain in $\mathbb{R}^J$, which can be expressed as the intersection of a finite number of closed half spaces; that is,

$$(2.1) \qquad G \doteq \bigcap_{i=1,\ldots,N} \{x \in \mathbb{R}^J : \langle x, n_i \rangle \geq c_i\},$$

for some positive integer $N \in \mathbb{N}$, unit vectors $n_i \in \mathbb{S}^{J-1}$ and constants $c_i \in \mathbb{R}$, for $i = 1, \ldots, N$. To each face $F_i \doteq \{x \in \partial G : \langle x, n_i \rangle = c_i\}$ of the polyhedron is associated a reflection vector $d_i \in \mathbb{R}^J$ that points into the interior G° of G; that is, $\langle d_i, n_i \rangle > 0$. Without loss of generality, the reflection vectors are assumed to be normalized so that $\langle d_i, n_i \rangle = 1$ for $i = 1, \ldots, N$. For notational convenience, we let $\mathcal{I} \doteq \{1, \ldots, N\}$ and for $x \in G$, we write

$$(2.2) \qquad \mathcal{I}(x) \doteq \{i \in \mathcal{I} : x \in F_i\}$$

to denote the (possibly empty) set of indices associated with the faces that intersect at x. For $x \in G$, we let $|\mathcal{I}(x)|$ denote the cardinality of the set $\mathcal{I}(x)$. In the following lemma, we state an upper semicontinuity property of the set-valued function $\mathcal{I}(\cdot)$ on G.

LEMMA 2.1 ([26], Lemma 2.1). *For each $x \in G$, there is an open neighborhood U_x of x in $\mathbb{R}^J$ such that*

$$(2.3) \qquad \mathcal{I}(y) \subseteq \mathcal{I}(x) \qquad \text{for all } y \in U_x \cap G.$$

For $x \in \partial G$, we let $d(x)$ denote the cone generated by the permissible directions of reflection at x. In other words, for $x \in \partial G$,

$$(2.4) \qquad d(x) \doteq \text{cone}(\{d_i, i \in \mathcal{I}(x)\}).$$

For convenience, we extend the definition of $d(x)$ to all of G by setting $d(x) \doteq \{0\}$ for all $x \in G^\circ$.

We now give a precise formulation of the ESP for continuous paths.

DEFINITION 2.2. Suppose that $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ and $X \in \mathbb{C}$ are given. Then $(Z, Y) \in \mathbb{C} \times \mathbb{C}$ solves the ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ for X if $Y(0) \in d(Z(0))$ and if for all $t \in [0, \infty)$, the following conditions hold:

1. $Z(t) = X(t) + Y(t)$;
2. $Z(t) \in G$;
3. for all $s \in [0, t)$,

$$(2.5) \qquad Y(t) - Y(s) \in \text{cone}\left[\bigcup_{u \in (s,t]} d(Z(u)) \right].$$

If there exists a unique solution (Z, Y) to the ESP for X, then we write $Z = \bar{\Gamma}(X)$ and we refer to $\bar{\Gamma}$ as the extended Skorokhod map (ESM).

REMARK 2.3. Given $T \in (0, \infty)$ and $X \in \mathbb{C}([0, T) : \mathbb{R}^J)$, we say that $(Z, Y) \in \mathbb{C}([0, T) : \mathbb{R}^J) \times \mathbb{C}([0, T) : \mathbb{R}^J)$ solves the ESP for X on $[0, T)$ if $Y(0) \in d(Z(0))$ and conditions 1–3 of the ESP hold for all $t \in [0, T)$.

REMARK 2.4. Given $(Z, Y) \in \mathbb{C} \times \mathbb{C}$, the inclusion (2.5) holds for all $0 \le s < t < \infty$ if and only if the following inclusion holds for all $0 \le s < t < \infty$:

$$(2.6) \qquad Y(t) - Y(s) \in \text{cone}\left[\bigcup_{u \in (s,t)} d(Z(u))\right].$$

Here, the "only if" direction is immediate and the "if" direction follows from the continuity of Y and because (2.6) implies that

$$Y(t-) - Y(s) \in \bigcup_{r < t} \text{cone}\left[\bigcup_{u \in (s,r)} d(Z(u))\right] \subseteq \text{cone}\left[\bigcup_{u \in (s,t)} d(Z(u))\right].$$

REMARK 2.5. The formulation of the ESP in Definition 2.2 appears slightly different from the one originally given in [38], Definition 1.2, since the ESP in [38] was formulated for paths $X \in \mathbb{D}_r$ that satisfy $X(0) \in G$. In particular, [38], Definition 1.2, requires that $Y(0) = 0$ and $Y(t) - Y(t-) \in \text{cone}[d(Z(t))]$ for all $t \in (0, \infty)$. Here, we only consider continuous paths, so the jump condition holds automatically. In addition, we allow input paths X that start outside G; that is, $X(0) \notin G$, but instead allow $Y(0) \ne 0$ as long as $Y(0) \in d(Z(0))$. This mild generalization is useful when considering directional derivatives of the ESM, where if $X(0) \in \partial G$, an ε-perturbation of X in the direction $\psi \in \mathbb{C}$ may result in the perturbed initial condition $X(0) + \varepsilon\psi(0)$ lying outside of G. When $X \in \mathbb{C}_G$, the conditions of Definition 2.2 ensure that any solution (Z, Y) of the ESP for X must satisfy $Y(0) = 0$, so Definition 2.2 coincides with [38], Definition 1.2.

REMARK 2.6. The ESP is a strict generalization of the SP that does not require that the constraining term Y have finite variation on compact intervals. However, it was shown in [38] that when the domain and directions of reflection satisfy a certain geometric condition, the constraining term has finite variation on compact intervals and the ESM coincides with the SM. To emphasize this point, when referring to specific examples of ESPs that satisfy this geometric condition, we will use SP and SM in place of ESP and ESM.

We close this section with a useful time-shift property of the ESP. Given a solution (Z, Y) of the ESP for $X \in \mathbb{C}$ and $S \in [0, \infty)$, define $X^S, Y^S, Z^S \in \mathbb{C}$ by

$$(2.7) \qquad X^S(\cdot) \doteq Z(S) + X(S + \cdot) - X(S),$$

$$(2.8) \qquad Z^S(\cdot) \doteq Z(S + \cdot),$$

$$(2.9) \qquad Y^S(\cdot) \doteq Y(S + \cdot) - Y(S).$$

LEMMA 2.7. *Suppose (Z, Y) solves the ESP for $X \in \mathbb{C}$. Let $S \in [0, \infty)$ and define X^S, Z^S, Y^S as in (2.7)–(2.9). Then (Z^S, Y^S) solves the ESP for X^S. Moreover, if (Z, Y) is the unique solution to the ESP for X, then for any $0 \le S < T < \infty$, $Z(T)$ depends only on $Z(S)$ and $\{X(S + t) - X(S), t \in [0, T - S]\}$.*

PROOF. In [38], Lemma 2.3, this result was shown in the case $X(0) \in G$. The same argument can be applied when $X(0) \notin G$. □

2.2. *Lipschitz continuity.* We now provide sufficient conditions on the ESP for the associated ESM to be Lipschitz continuous on its domain of definition. The conditions, stated in Assumption 2.8 below, are expressed in terms of the existence of a convex set B whose inward normals satisfy certain geometric properties expressed in terms of the data $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$. Given a convex set B and $z \in \partial B$, we let $\nu_B(z)$ denote the set of unit inward normals to the set at the point z. In other words,

$$\nu_B(z) \doteq \big\{\nu \in \mathbb{S}^{J-1} : \langle \nu, y - z \rangle \ge 0 \text{ for all } y \in B\big\}.$$

ASSUMPTION 2.8. There exists $\delta > 0$ and a compact, convex, symmetric set B with $0 \in B^\circ$ such that for $i \in \mathcal{I}$,

$$(2.10) \qquad \begin{Bmatrix} z \in \partial B \\ |\langle z, n_i \rangle| < \delta \end{Bmatrix} \quad \Rightarrow \quad \langle \nu, d_i \rangle = 0 \qquad \text{for all } \nu \in \nu_B(z).$$

REMARK 2.9. Suppose $\delta > 0$ and B are such that (2.10) holds for $i \in \mathcal{I}$. Then given any $c > 0$, (2.10) holds with $c\delta$ and $cB \doteq \{cz : z \in B\}$ in place of δ and B, respectively. In particular, if $z \in \partial(cB)$ for some $c > 0$ and $|\langle z, n_i \rangle| = 0$, then $\langle \nu, d_i \rangle = 0$ for all $\nu \in \nu_{cB}(z)$.

This assumption was first introduced as [17], Assumption 2.1, and was shown in [17], Theorem 2.2, to imply Lipschitz continuity of the associated SM on its domain of definition. In [38], Theorem 3.3, it was shown that Assumption 2.8 is a sufficient condition for Lipschitz continuity of the ESM as well. An analogue of Assumption 2.8 also serves as a sufficient condition for Lipschitz continuity of the map associated with the so-called constrained discontinuous media problem (see [3], Theorem 2.9). A dual condition on the data $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ that implies the existence of a set B that satisfies Assumption 2.8 was introduced in [19, 20]. As demonstrated in [18, 21], the dual condition is often more convenient to use in practice.

We now give a precise statement of the Lipschitz continuity property that follows from Assumption 2.8.

THEOREM 2.10. *Given an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$, suppose Assumption 2.8 holds. Then there exists $\kappa_{\bar{\Gamma}} < \infty$ such that if (Z_1, Y_1) solves the ESP for $X_1 \in \mathbb{C}$ and (Z_2, Y_2) solves the ESP for $X_2 \in \mathbb{C}$, then for all $T \in [0, \infty)$,*

$$(2.11) \qquad \|Z_1 - Z_2\|_T \le \kappa_{\bar{\Gamma}} \|X_1 - X_2\|_T.$$

PROOF. By [38], Theorem 3.3, there exists $\tilde{\kappa} < \infty$ such that whenever $X_1(0), X_2(0) \in G$, (2.11) holds with $\tilde{\kappa}$ in place of $\kappa_{\bar{\Gamma}}$ for all $T \in [0, \infty)$. Now suppose $X_1, X_2 \in \mathbb{C}$ are arbitrary. For $i = 1, 2$, define X_i^0, Z_i^0 as in (2.7)–(2.8), with $S = 0$ and X_i, Z_i in place of X, Z, respectively, so that $Z_i^0(0) = X_i^0(0) \in G$. By (2.8), the time-shift property of the ESP (Lemma 2.7), [38], Theorem 3.3, (2.7) and condition 1 of the ESP, for all $T \in [0, \infty)$,

$$\|Z_1 - Z_2\|_T = \|Z_1^0 - Z_2^0\|_T \le \tilde{\kappa}\|X_1^0 - X_2^0\|_T$$
$$\le \tilde{\kappa}\|X_1 - X_2\|_T + \tilde{\kappa}|Y_1(0) - Y_2(0)|.$$

Therefore, it suffices to show there exists $\hat{\kappa} < \infty$ [depending only on the data $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$] such that $|Y_1(0) - Y_2(0)| \le \hat{\kappa}|X_1(0) - X_2(0)|$. Then (2.11) will hold with $\kappa_{\bar{\Gamma}} = \tilde{\kappa}(1 + \hat{\kappa})$. The existence of $\hat{\kappa}$ can be shown using an argument that is related to the one used in the proof of [17], Theorem 2.2. To avoid redundancy, we omit the argument here. $\square$

2.3. *Existence and uniqueness of solutions.* In this section, we summarize results on existence and uniqueness of solutions to the ESP. We start by assuming the existence of a certain map that projects points in $\mathbb{R}^J$ onto G in a way that is compatible with the directions of reflection $d(\cdot)$.

ASSUMPTION 2.11. There is a map $\pi : \mathbb{R}^J \mapsto G$ satisfying $\pi(x) = x$ for all $x \in G$ and $\pi(x) - x \in d(\pi(x))$ for all $x \notin G$.

For general results on the existence of such a map π, see [19], Section 4.

THEOREM 2.12. *Given an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$, suppose Assumptions 2.8 and 2.11 hold. Then there exists a unique solution (Z, Y) of the ESP for each $X \in \mathbb{C}$ and $Z(0) = \pi(X(0))$.*

PROOF. Define $Z(0) \doteq \pi(X(0))$ and $X^0 \in \mathbb{C}_G$ as in (2.7), with $S = 0$. By Theorem 2.10, Assumption 2.11 and [38], Lemma 2.6, there exists a unique solution (Z^0, Y^0) of the ESP for X^0 and $Z^0(0) = X^0(0) = \pi(X(0)) \in G$. Define $Z(\cdot) \doteq Z^0(\cdot)$ and $Y(\cdot) \doteq \pi(X(0)) - X(0) + Y^0(\cdot)$. According to Assumption 2.11, $Y(0) = \pi(X(0)) - X(0) \in d(Z(0))$. It is readily verified that (Z, Y) satisfies conditions 1–3 of the ESP for X, so (Z, Y) is a solution to the ESP for X. Uniqueness of the solution then follows from the Lipschitz continuity property established in Theorem 2.10. $\square$

We close this section with some examples of SPs that satisfy Assumptions 2.8 and 2.11. Additional examples of SPs and ESPs can be found in [17–21, 30, 38].

EXAMPLE 2.13.   For $J = 1$. Consider the one-dimensional SP $\{(e_1, e_1, 0)\}$, which was first formulated by Skorokhod [42] to construct pathwise reflected diffusions on $\mathbb{R}_+$. As is well known [see, e.g., [38], equation (1.1)], given $X \in \mathbb{C}$ such that $X(0) \geq 0$, the one-dimensional SM, which we denote by Γ_1, admits the following explicit representation:

$$(2.12) \qquad \Gamma_1(X)(t) = X(t) + \sup_{s \in [0,t]} (-X(s)) \vee 0, \qquad t \in [0, \infty).$$

It is readily verified that the above expression is also valid when $X(0) < 0$. Consequently, if (Z, Y) is the solution to the one-dimensional SP for $X \in \mathbb{C}$, then due to the property $Z(t) = X(t) + Y(t)$ for $t \in [0, \infty)$, it follows that

$$(2.13) \qquad Y(t) = \sup_{s \in [0,t]} (-X(s)) \vee 0, \qquad t \in [0, \infty).$$

EXAMPLE 2.14.   Consider an SP $\{(d_i, n_i, c_i), i = 1, \ldots, J\}$ with linearly independent directions of reflection $\{d_i, i = 1, \ldots, J\}$ (normalized so that $\langle d_i, n_i \rangle = 1$ for $i = 1, \ldots, J$) and define the matrix $Q \in \mathbb{R}^{J \times J}$ by

$$(2.14) \qquad Q_i^j = \begin{cases} |\langle d_i, n_j \rangle| & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Suppose $\varrho(Q)$, the spectral radius of Q, satisfies $\varrho(Q) < 1$. Then, according to the results in [20], Section 2, the SP satisfies Assumptions 2.8 and 2.11. These SPs are a natural generalization of the SPs considered by Harrison and Reiman [23], and whose directional derivatives were characterized in [34]. Figure 1 depicts an example of such an SP along with its associated set B. This SP does not fall into the class of SPs considered in [34] because it violates the requirement that $\langle d_i, e_j \rangle \leq 0$ for $i \neq j$.

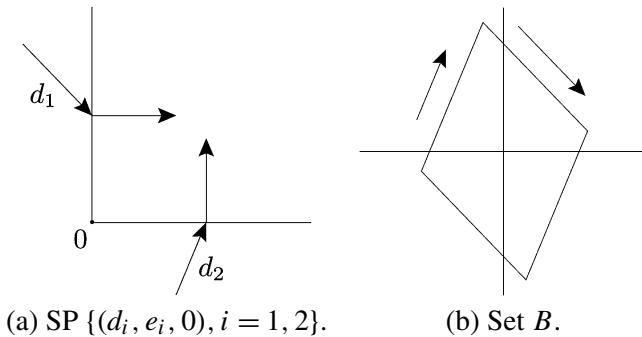(a) SP $\{(d_i, e_i, 0), i = 1, 2\}$.   (b) Set B.

FIG. 1.   (a) *The domain and directions of reflection for the SP on* $\mathbb{R}_+^2$ *with directions of reflection* $d_1 = (1, -1)'$ *and* $d_2 = (1/2, 1)'$. (b) *An associated set B satisfying the conditions in Assumption* 2.8. *The two arrows about the set B correspond to the directions of reflection* d_1 *and* d_2 *depicted in* (a).

2.4. *Directional derivatives of the extended Skorokhod map.* Fix an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ satisfying Assumptions 2.8 and 2.11, so by Theorems 2.10 and 2.12, the associated ESM $\bar{\Gamma}$ is well defined and Lipschitz continuous on $\mathbb{C}$. Given $X, \psi \in \mathbb{C}$ and $\varepsilon > 0$, we define

$$(2.15) \qquad \nabla_\psi^\varepsilon \bar{\Gamma}(X) \doteq \frac{\bar{\Gamma}(X + \varepsilon\psi) - \bar{\Gamma}(X)}{\varepsilon}.$$

DEFINITION 2.15. Given $X, \psi \in \mathbb{C}$, the directional derivative of the ESM $\bar{\Gamma}$ along the direction ψ evaluated at X, denoted $\nabla_\psi \bar{\Gamma}(X)$, is the function from $[0, \infty)$ into $\mathbb{R}^J$ that is equal to the pointwise limit of $\nabla_\psi^\varepsilon \bar{\Gamma}(X)$ as $\varepsilon \downarrow 0$; that is, for $t \in [0, \infty)$,

$$(2.16) \qquad \nabla_\psi \bar{\Gamma}(X)(t) \doteq \lim_{\varepsilon \downarrow 0} \nabla_\psi^\varepsilon \bar{\Gamma}(X)(t).$$

REMARK 2.16. Let $X, \psi \in \mathbb{C}$. Given $t \in [0, \infty)$, if the limit (2.16) exists we say that $\nabla_\psi \bar{\Gamma}(X)(t)$ exists. Given $T \in (0, \infty)$, if the limit (2.16) exists for all $t \in [0, T)$ we say that $\nabla_\psi \bar{\Gamma}(X)$ exists on $[0, T)$. If the limit (2.16) exists for all $t \in [0, \infty)$, we simply say that $\nabla_\psi \bar{\Gamma}(X)$ exists.

The following proposition provides sufficient conditions for pointwise convergence of $\nabla_{\psi_\varepsilon}^\varepsilon \bar{\Gamma}(X)$ as $\varepsilon \downarrow 0$, where $X \in \mathbb{C}$ and $\{\psi_\varepsilon\}_{\varepsilon > 0}$ is a family in $\mathbb{C}$.

PROPOSITION 2.17. *Given an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$, suppose Assumptions 2.8 and 2.11 hold. Let $X, \psi \in \mathbb{C}$ and $T \in (0, \infty]$ be such that $\nabla_\psi \bar{\Gamma}(X)(t)$ exists on $[0, T)$, and let $\{\psi_\varepsilon\}_{\varepsilon > 0}$ be a family in $\mathbb{C}$ such that $\psi_\varepsilon \to \psi$ in $\mathbb{C}$ as $\varepsilon \downarrow 0$. Then for all $t \in [0, T)$,*

$$(2.17) \qquad \lim_{\varepsilon \downarrow 0} \nabla_{\psi_\varepsilon}^\varepsilon \bar{\Gamma}(X)(t) = \nabla_\psi \bar{\Gamma}(X)(t).$$

PROOF. Let $t \in [0, T)$. By (2.15) and Theorem 2.10,

$$\left| \nabla_{\psi_\varepsilon}^\varepsilon \bar{\Gamma}(X)(t) - \nabla_\psi^\varepsilon \bar{\Gamma}(X)(t) \right| = \frac{|\bar{\Gamma}(X + \varepsilon\psi_\varepsilon)(t) - \bar{\Gamma}(X + \varepsilon\psi)(t)|}{\varepsilon}$$

$$\leq \kappa_{\bar{\Gamma}} \|\psi_\varepsilon - \psi\|_t,$$

which converges to zero as $\varepsilon \downarrow 0$. This, along with the triangle inequality and (2.16), implies (2.17). $\square$

3. Main results. In this section, we present our main results. We consider directional derivatives of the ESM evaluated at paths whose images under the ESM satisfy a certain boundary jitter property, which we define in Section 3.1. In Section 3.2, we introduce the DP. In Section 3.3, we present our main result on the existence of directional derivatives of the ESM and their characterization via solutions to the DP.

3.1. *Definition of the boundary jitter property.* In order to prove existence of directional derivatives of the ESM evaluated at $X \in \mathbb{C}_G$, we require that the solution (Z, Y) of the ESP for X satisfies certain conditions at the boundary ∂G. We collectively refer to these conditions as the *boundary jitter property*. Recall that $|\mathcal{I}(x)|$ denotes the cardinality of the set $\mathcal{I}(x)$ defined in (2.2). Let $\mathcal{S}$ and $\mathcal{N}$ respectively denote the smooth and nonsmooth parts of the boundary ∂G; that is,

$$(3.1) \qquad\qquad \mathcal{S} \doteq \big\{ x \in \partial G : |\mathcal{I}(x)| = 1 \big\}$$

and

$$(3.2) \qquad\qquad \mathcal{N} \doteq \partial G \setminus \mathcal{S} = \big\{ x \in \partial G : |\mathcal{I}(x)| \geq 2 \big\}.$$

DEFINITION 3.1. Given an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$, $T \in (0, \infty]$ and $(Z, Y) \in \mathbb{C}([0, T) : G) \times \mathbb{C}([0, T) : \mathbb{R}^J)$. We say that (Z, Y) satisfies the boundary jitter property on $[0, T)$ if the following hold:

1. If $t \in [0, T)$ is such that $Z(t) \in \mathcal{S}$, then for all $s < t < u < T$, Y is nonconstant on $(s \vee 0, u)$.
2. On the interval $[0, T)$, Z does not spend positive Lebesgue time in $\mathcal{N}$; that is,

$$\int_0^T 1_{\mathcal{N}}\big(Z(t)\big) \, dt = 0.$$

3. If $t \in (0, T)$ is such that $Z(t) \in \mathcal{N}$, then for each $i \in \mathcal{I}(Z(t))$ and every $\delta \in (0, t)$, there exists $s \in (t - \delta, t)$ such that $\mathcal{I}(Z(s)) = \{i\}$.
4. If $Z(0) \in \mathcal{N}$, then for each $i \in \mathcal{I}(Z(0))$ and every $\delta \in (0, T)$, there exists $u \in (0, \delta)$ such that $\mathcal{I}(Z(u)) = \{i\}$.

REMARK 3.2. When $T = \infty$, we omit the interval $[0, \infty)$ and write (Z, Y) satisfies the boundary jitter property to mean conditions 1–4 hold on $[0, \infty)$. When the pair (Z, Y) is defined on an interval that contains $[0, T)$, we write (Z, Y) satisfies the boundary jitter property on $[0, T)$ to mean the restriction of (Z, Y) to $[0, T)$ satisfies the boundary jitter property on $[0, T)$. The boundary jitter property depends on the ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$; however, we omit this dependence since the ESP will be clear from the context. Since conditions 2, 3 and 4 of the boundary jitter property only depend on Z, we often write Z satisfies condition 2, 3 or 4 of the boundary jitter property to mean (Z, Y) satisfies condition 2, 3 or 4, respectively, of the boundary jitter property.

REMARK 3.3. Given an ESP on the half space (i.e., when $N = 1$), the set $\mathcal{N}$ is empty and so conditions 2–4 of the boundary jitter property hold automatically.

Condition 1 can be interpreted to mean that whenever Z lies on the smooth part of the boundary, it must be actively constrained to remain in G. Condition 2 is self-explanatory. Condition 3 states that whenever the path Z is on a nonsmooth part of

the boundary at time $t \in (0, \infty)$, it must hit the smooth part of each face intersecting the point $Z(t)$ infinitely often immediately before time t. See Figure 2(a) (in Section 8.3) for an illustration of a path Z that satisfies condition 3 of the boundary jitter property. Condition 4 states that if the path Z starts on the nonsmooth part of the boundary, then it must hit the smooth part of each face intersecting the point $Z(0)$ infinitely often immediately after time zero.

By imposing the boundary jitter property on (Z, Y), we will be able characterize the right continuous regularizations of directional derivatives of the ESM as solutions to the DP along Z, which we introduce in the next section. In [31], we show that a large class of reflected diffusions in polyhedral domains satisfy the boundary jitter property; see also [30], Section 4.1, for a simpler proof of this property for a class of reflected Brownian motions in the nonnegative quadrant.

3.2. *Statement of the derivative problem.* In this section, we introduce the DP along a solution to the ESP. The DP provides a useful axiomatic framework that will be used to characterize directional derivatives of the ESM. Let $X \in \mathbb{C}$ and $Z \doteq \bar{\Gamma}(X)$. The DP along Z is a certain time-inhomogeneous Skorokhod-type problem where both the domain and directions of reflection vary (discontinuously) in time. Other works that have considered SPs or ESPs in domains that vary in time include [8, 9] in the one-dimensional setting, and [36] in the multidimensional setting with time-varying domains and oblique directions of reflection.

In order to state the DP, recall the definitions of $\mathcal{I}(x)$ and $d(x)$ given in (2.2) and (2.4). For $x \in \partial G$, define the linear subspace

$$(3.3) \qquad H_x \doteq \bigcap_{i \in \mathcal{I}(x)} \{y \in \mathbb{R}^J : \langle y, n_i \rangle = 0\},$$

and for $x \in G^\circ$, set $H_x \doteq \mathbb{R}^J$. We now give a precise formulation of the DP.

DEFINITION 3.4. Given an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ and $X \in \mathbb{C}$, suppose (Z, Y) is a solution to the ESP for X. Let $\psi \in \mathbb{D}_r$. Then $(\phi, \eta) \in \mathbb{D}_r \times \mathbb{D}_r$ solves the DP along Z for ψ if $\eta(0) \in \text{span}[d(Z(0))]$ and for all $t \in [0, \infty)$, the following conditions hold:

1. $\phi(t) = \psi(t) + \eta(t)$;
2. $\phi(t) \in H_{Z(t)}$;
3. for all $s \in [0, t)$,

$$(3.4) \qquad \eta(t) - \eta(s) \in \text{span}\left[\bigcup_{u \in (s, t]} d(Z(u))\right].$$

If there exists a unique solution (ϕ, η) to the DP for ψ, we write $\phi = \Lambda_Z(\psi)$ and refer to Λ_Z as the derivative map (DM) along Z.

REMARK 3.5. When there is no confusion regarding Z, we omit the phrase "along Z" and simply say (ϕ, η) solves the DP for ψ.

REMARK 3.6.   Given $T \in (0, \infty)$, we say $(\phi, \eta) \in \mathbb{D}_r([0, T) : \mathbb{R}^J) \times \mathbb{D}_r([0, T) : \mathbb{R}^J)$ solves the DP for ψ on $[0, T)$ if $\eta(0) \in \text{span}[d(Z(0))]$ conditions 1–3 hold for $t \in [0, T)$. If the pair (ϕ, η) is defined on an interval that strictly contains $[0, T)$, we say (ϕ, η) solves the DP for ψ on $[0, T]$ if the restriction of (ϕ, η) to $[0, T)$ solves the DP for ψ on $[0, T)$.

REMARK 3.7.   If η is discontinuous at $t \in (0, \infty)$, then by condition 3 of the DP, the definition of $d(\cdot)$ given in (2.4), the continuity of Z and the upper semi-continuity of $\mathcal{I}(\cdot)$ (Lemma 2.1),

$$(3.5) \qquad \eta(t) - \eta(t-) \in \bigcap_{s < t} \text{span}\left[\bigcup_{u \in (s,t]} d(Z(u)) \right] = \text{span}[d(Z(t))].$$

REMARK 3.8.   The definition of the DP has many similarities to the definition of the ESP. In particular, conditions 1–3 of the DP correspond to conditions 1–3 of the ESP, but with ϕ, ψ, η, $H_{Z(t)}$ and "span" in place of Z, X, Y, G and "cone." Here, $H_{Z(t)}$ is time-varying and depends on the position of $Z(t)$, and, for each $t \in [0, \infty)$, is equal to the intersection of finitely many hyperplanes, whereas G is fixed (in time) and equal to the intersection of finitely many half spaces.

In Section 5, we establish properties of the DP and the associated DM.

3.3. *Existence and characterization of directional derivatives.*   Throughout this section, we fix an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ satisfying Assumptions 2.8 and 2.11.

Provided the solution (Z, Y) of the ESP for X satisfies the boundary jitter property, we prove existence of directional derivatives of the ESM evaluated at $X \in \mathbb{C}_G$ up until the time τ given by

$$(3.6) \qquad\qquad\qquad \tau \doteq \inf\{t \in [0, \infty) : Z(t) \in \mathcal{W}\},$$

where $\mathcal{W}$ is the (possibly empty) subset of $\mathcal{N}$, the nonsmooth part of the boundary, given by

$$(3.7) \qquad\qquad \mathcal{W} \doteq \{x \in \mathcal{N} : \text{span}(H_x \cup d(x)) \neq \mathbb{R}^J\}.$$

The restriction to the interval $[0, \tau]$ cannot in general be removed because the example in [30], Appendix D.2, shows that the directional derivative of the ESM may not exist at $t = \tau$ even when the pair (Z, Y) satisfies the jitter property. However, this is not a serious restriction because, as shown in Lemma 8.2, if a mild linear independence assumption on the directions of reflection holds, then the set $\mathcal{W}$ is empty, so $\tau = \infty$ holds trivially.

The following lemma will be used to establish the existence of $\nabla_\psi \bar{\Gamma}(X)(t)$ at $t = 0$.

LEMMA 3.9. *The following limit exists for all $(x, v) \in G \times \mathbb{R}^J$:*

$$(3.8) \qquad \nabla_v \pi(x) \doteq \lim_{\varepsilon \downarrow 0} \frac{\pi(x + \varepsilon v) - \pi(x)}{\varepsilon}.$$

Furthermore, $\nabla_v \pi(x) - v \in \mathrm{cone}[d(\pi(x))] = \mathrm{cone}[d(x)]$.

REMARK 3.10. When $(x, v) \in G \times \mathbb{R}^J$ is such that $x + \varepsilon v \in G$ for all $\varepsilon > 0$ sufficiently small, the existence of the limit (3.8) is immediate and we simply have $\nabla_v \pi(x) = v$.

PROOF OF LEMMA 3.9. According to the discussion in [17], Section 5.3, the limit (3.8) exists. Due to the convergence $\pi(x + \varepsilon v) \to \pi(x) = x$ as $\varepsilon \downarrow 0$ and the upper semicontinuity of $\mathcal{I}(\cdot)$ (Lemma 2.1), $\mathcal{I}(\pi(x + \varepsilon v)) \subseteq \mathcal{I}(\pi(x)) = \mathcal{I}(x)$ for $\varepsilon > 0$ sufficiently small. This, along with Assumption 2.11, implies that for all $\varepsilon > 0$ sufficiently small,

$$\frac{\pi(x + \varepsilon v) - \pi(x)}{\varepsilon} - v = \frac{\pi(x + \varepsilon v) - (x + \varepsilon v)}{\varepsilon}$$
$$\in \mathrm{cone}[d(\pi(x))] = \mathrm{cone}[d(x)].$$

The final assertion of the lemma then follows from taking limits as $\varepsilon \downarrow 0$ because $\mathrm{cone}[d(x)]$ is a closed set. $\quad \square$

The second part of our main result is to relate the directional derivative $\nabla_\psi \bar{\Gamma}(X)$ to the unique solution (ϕ, η) of the DP along Z for ψ. In order to state this result, we define a functional

$$\Theta_Z : \mathbb{D}_r([0, \tau) : \mathbb{R}^J) \mapsto \mathbb{D}_{l,r}([0, \tau) : \mathbb{R}^J)$$

so that $\Theta_Z(\phi)$ and $\nabla_\psi \bar{\Gamma}(X)$ are equal on $(0, \tau)$. To this end, for each $x \in \mathcal{S}$, the smooth part of the boundary, let $i_x \in \mathcal{I}$ denote the unique index such that $\mathcal{I}(x) = \{i_x\}$ and define

$$(3.9) \qquad G_x \doteq \{y \in \mathbb{R}^J : \langle y, n_{i_x} \rangle \geq 0\}.$$

Given $f \in \mathbb{D}_r([0, \tau) : \mathbb{R}^J)$, define $\Theta_Z(f)$ as follows: for each $t \in [0, \tau)$,

$$(3.10) \qquad \Theta_Z(f)(t) \doteq \begin{cases} f(t) & \text{if } Z(t) \in G \setminus \mathcal{S}, \\ f(t) & \text{if } Z(t) \in \mathcal{S}, f(t-) \notin G_{Z(t)} \\ f(t-) & \text{if } Z(t) \in \mathcal{S}, f(t-) \in G_{Z(t)}. \end{cases}$$

We can now state our main result on directional derivatives of the ESM.

THEOREM 3.11. *Fix an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ satisfying Assumptions 2.8 and 2.11. Given $X \in \mathbb{C}_G$, let (Z, Y) denote the solution to the ESP for X and define τ as in (3.6). Suppose (Z, Y) satisfies the boundary jitter property (Definition 3.1) on $[0, \tau)$. Then for all $\psi \in \mathbb{C}$:*

1. $\nabla_\psi \bar\Gamma(X)$ *exists on* $[0, \tau)$ *and lies in* $\mathbb{D}_{l,r}([0, \tau) : \mathbb{R}^J)$;
2. *there exists a unique solution* (ϕ, η) *to the DP along* Z *for* ψ *on* $[0, \tau)$;
3. ϕ *is equal to the right continuous regularization of* $\nabla_\psi \bar\Gamma(X)$ *on* $[0, \tau)$, *that is,* $\phi(t) = \nabla_\psi \bar\Gamma(X)(t+)$ *for all* $t \in [0, \tau)$;
4. $\nabla_\psi \bar\Gamma(X)(0) = \nabla_{\psi(0)}\pi(X(0))$ *and* $\nabla_\psi \bar\Gamma(X) = \Theta_Z(\phi)$ *on* $(0, \tau)$;
5. $\nabla_\psi \bar\Gamma(X)$ *and* ϕ *are continuous at all* $t \in (0, \tau)$ *such that* $Z(t) \in G^\circ \cup \mathcal{N}$.

Since the proof of Theorem 3.11, which is deferred to Section 9.1, is rather involved, we provide a brief outline. Given $X \in \mathbb{C}_G$, let $Z \doteq \bar\Gamma(X)$. In Section 7, for any given ψ, we prove existence of and characterize the directional derivative $\nabla_\psi \bar\Gamma(X)$ up until the first time Z reaches the nonsmooth part of the boundary $\mathcal{N}$. We denote this time by θ_2. Roughly speaking, given an interval such that Z hits at most a single face F_i, we can exploit prior results on directional derivatives of the one-dimensional SM, which are reviewed in Section 6, to prove existence of and characterize $\nabla_\psi \bar\Gamma(X)$ on the interval. We then patch together these results to prove existence of and characterize $\nabla_\psi \bar\Gamma(X)$ on $[0, \theta_2)$. The proof of existence of $\nabla_\psi \bar\Gamma(X)$ on $[0, \theta_2)$ does not require that the boundary jitter property hold; however condition 1 of the boundary jitter property is needed to characterize the right continuous regularization of $\nabla_\psi \bar\Gamma(X)$ as the solution to the DP on $[0, \theta_2)$.

In Section 9, we prove existence of and characterize $\nabla_\psi \bar\Gamma(X)$ on $[0, \tau)$. The key challenge is to characterize $\nabla_\psi \bar\Gamma(X)(t)$ at times $t \in [0, \tau)$ that $Z(t) \in \mathcal{N}$. We first show that it suffices to consider ψ that lie in a dense subset of $\mathbb{C}$ consisting of paths that are constant about times that the path Z lies in $\mathcal{N}$. We then use the boundary jitter property, along with properties of certain (oblique) derivative projection operators, which are introduced in Section 8.1, to characterize $\nabla_\psi \bar\Gamma(X)(t)$ at such times. In particular, given $t \in (0, \infty)$ such that $Z(t) \in \mathcal{N}$, the boundary jitter property implies that Z hits the relative interior of each face that intersects $Z(t)$ infinitely often in any left neighborhood of t. Roughly speaking, each time Z reaches the relative interior of a face F_i, $\nabla_\psi \bar\Gamma(X)$ is projected onto the hyperplane $H_i \doteq \{x \in \mathbb{R}^J : \langle x, n_i \rangle = 0\}$ associated with F_i along a direction that lies in the span of d_i. As a consequence, understanding $\nabla_\psi \bar\Gamma(X)(t)$ when $Z(t)$ lies in $\mathcal{N}$ is largely reduced to the analysis of countable sequences of derivative projection operators, which is carried out in Section 8.2.

4. The boundary jitter property. In this section, we discuss some useful ramifications of the boundary jitter property (see Definition 3.1) that are used in the proof that the directional derivative of the ESM exists at times $t \in [0, \infty)$ that $Z(t) \in \mathcal{N}$, the nonsmooth part of the boundary.

LEMMA 4.1. *Given* $U \in (0, \infty]$ *and* $Z \in \mathbb{C}([0, U) : G)$, *suppose* Z *satisfies condition* 3 *of the boundary jitter property (Definition* 3.1) *on* $[0, U)$ *and* $0 \le S <$

$T < U$ are such that $Z(T) \in \mathcal{N}$ and $\mathcal{I}(Z(t)) \subsetneq \mathcal{I}(Z(T))$ for all $t \in [S, T)$. Then there is a nested increasing sequence

$$(4.1) \qquad S \doteq \xi_0 < s_1 \leq \xi_1 < \cdots < s_j \leq \xi_j < \cdots < T$$

such that $\xi_j \to T$ as $j \to \infty$ and for each $j \in \mathbb{N}$, $Z(\xi_j) \in \partial G$ and

$$(4.2) \qquad \bigcup_{t \in [\xi_{j-1}, s_j)} \mathcal{I}(Z(t)) \subseteq \mathcal{I}(Z(\xi_{j-1})) \quad and \quad \bigcup_{t \in [s_j, \xi_j]} \mathcal{I}(Z(t)) \subseteq \mathcal{I}(Z(\xi_j)).$$

See Figure 2(a) (in Section 8.3) for an illustration of a path Z that satisfies condition 3 of the boundary jitter property with the sequence of times in (4.1) marked.

PROOF OF LEMMA 4.1. To construct the nested increasing sequences (4.1), recursively define, for $j \in \mathbb{N}$, s_j to be the first time after ξ_{j-1} such that $Z(s_j) \in F_i$ for some $i \notin \mathcal{I}(Z(\xi_{j-1}))$; that is,

$$(4.3) \qquad s_j \doteq \inf\{t \in (\xi_{j-1}, T] : \mathcal{I}(Z(t)) \nsubseteq \mathcal{I}(Z(\xi_{j-1}))\},$$

and let

$$(4.4) \qquad \xi_j \doteq \sup\{t \in [s_j, T) : \mathcal{I}(Z(s)) \subseteq \mathcal{I}(Z(t)) \,\forall s \in [s_j, t]\}.$$

We claim that for each $j \in \mathbb{N}$,

$$(4.5) \qquad S \leq \xi_{j-1} < s_j \leq \xi_j < T \quad and \quad Z(\xi_j) \in \partial G.$$

This, along with (4.3) and (4.4), will establish (4.1) and (4.2).

To prove the claim (4.5), we use the principle of mathematical induction. By definition, $\xi_0 \doteq S \in [S, T)$. Now suppose $\xi_{j-1} \in [S, T)$ for some $j \in \mathbb{N}$. By (4.3), the continuity of Z and the upper semicontinuity of $\mathcal{I}(\cdot)$ (Lemma 2.1), we have $s_j > \xi_{j-1}$. To see that $s_j < T$, first choose $i \in \mathcal{I}(Z(T)) \setminus \mathcal{I}(Z(\xi_{j-1}))$, where the set is nonempty because, by assumption, the strict inclusion $\mathcal{I}(Z(t)) \subsetneq \mathcal{I}(Z(T))$ holds for all $t \in [S, T)$. According to condition 3 of the boundary jitter property, since $Z(T) \in \mathcal{N}$, there exists $t \in (\xi_{j-1}, T)$ such that $\mathcal{I}(Z(t)) = \{i\}$. Thus, (4.3) implies that $s_j \leq t < T$. This, together with the previously established lower bound $s_j > \xi_{j-1}$, implies $s_j \in (\xi_{j-1}, T)$. By definition (4.4), $s_j \leq \xi_j$.

Proceeding, we show that $\xi_j < T$. Since $Z(T) \in \mathcal{N}$, condition 3 of the boundary jitter property, with $t = T$ and $\delta = T - s_j$, implies that for $i \in \mathcal{I}(Z(T))$,

$$t_0^i \doteq \inf\{t \in [s_j, T) : \mathcal{I}(Z(t)) = \{i\}\} < T.$$

Consequently,

$$(4.6) \qquad t_0 \doteq \max_{i \in \mathcal{I}(Z(T))} t_0^i < T \quad and \quad \bigcup_{u \in [s_j, t_0]} \mathcal{I}(Z(u)) = \mathcal{I}(Z(T)).$$

Combining (4.6) with the strict inclusion $\mathcal{I}(Z(t)) \subsetneq \mathcal{I}(Z(T))$ for all $t \in [t_0, T)$, and the definition (4.4) for ξ_j, we have $\xi_j < t_0 < T$. To see that $Z(\xi_j) \in \partial G$, first observe that (4.3) clearly implies that $Z(s_j) \in \partial G$. Since (4.4) and the continuity of Z imply that $\mathcal{I}(Z(s_j)) \subseteq \mathcal{I}(Z(\xi_j))$, this ensures $Z(\xi_j) \in \partial G$. Thus, we have proved the claim in (4.5), and hence, that (4.1) and (4.2) hold.

We are left to show that $\xi_j \to T$ as $j \to \infty$. Since (4.5) implies ξ_j is increasing and bounded above by T, there exists $\xi_\infty \leq T$ such that $\xi_j < \xi_\infty$ for all $j \in \mathbb{N}$ and $\xi_j \to \xi_\infty$ as $j \to \infty$. By the continuity of Z and the upper semicontinuity of $\mathcal{I}(\cdot)$, there exists $j_0 \in \mathbb{N}$ such that $\mathcal{I}(Z(\xi_j)) \subseteq \mathcal{I}(Z(\xi_\infty))$ holds for all $j \geq j_0$. The inclusions in (4.2) imply that for each $j \in \mathbb{N}$, $\mathcal{I}(Z(t)) \subseteq \mathcal{I}(Z(\xi_j))$ for all $t \in [s_j, s_{j+1})$. Combining these properties, we have

$$\bigcup_{t \in [s_{j_0}, \xi_\infty]} \mathcal{I}(Z(t)) = \bigcup_{j \geq j_0} \bigcup_{t \in [s_j, s_{j+1})} \mathcal{I}(Z(t)) = \bigcup_{j \geq j_0} \mathcal{I}(Z(\xi_j)) \subseteq \mathcal{I}(Z(\xi_\infty)).$$

Thus, $\mathcal{I}(Z(t)) \subseteq \mathcal{I}(Z(\xi_\infty))$ for all $t \in [s_0, \xi_\infty]$. If $\xi_\infty < T$, then (4.4) would imply the contradiction $\xi_\infty \leq \xi_{j_0}$. Therefore, $\xi_\infty = T$. $\quad\square$

The following result describes the behavior of a path that starts at the nonsmooth part of the boundary and satisfies the boundary jitter property. The proof uses a time-reversal argument in conjunction with Lemma 4.1.

LEMMA 4.2. *Given $Z \in \mathbb{C}([0, \infty) : G)$, suppose Z satisfies conditions* 3 *and* 4 *of the boundary jitter property* (*Definition* 3.1), $Z(0) \in \mathcal{N}$ *and $T \in (0, \infty)$ is such that $\mathcal{I}(Z(t)) \subsetneq \mathcal{I}(Z(0))$ for all $t \in (0, T)$. Then there is a nested decreasing sequence*

$$(4.7) \qquad T > \chi_0 > u_1 \geq \chi_1 > \cdots > u_j \geq \chi_j > \cdots > 0$$

such that $Z(\chi_0) \in G^\circ$, $\chi_j \to 0$ as $j \to \infty$ and for each $j \in \mathbb{N}$, $Z(\chi_j) \in \partial G$ and

$$(4.8) \qquad \bigcup_{t \in [\chi_j, u_j]} \mathcal{I}(Z(t)) \subseteq \mathcal{I}(Z(\chi_j)) \quad \text{and} \quad \bigcup_{t \in (u_j, \chi_{j-1}]} \mathcal{I}(Z(t)) \subseteq \mathcal{I}(Z(\chi_{j-1})).$$

PROOF.   Let $x \in G^\circ$ be arbitrary. Define the path $\widetilde{Z} \in \mathbb{C}([0, \infty) : G)$ to be the time reversal of Z on $[0, T]$ concatenated with the line segment connecting $Z(0)$ and x as follows:

$$(4.9) \qquad \widetilde{Z}(t) \doteq \begin{cases} Z(T - t) & \text{for } t \in [0, T], \\ Z(0)e^{T-t} + (x - Z(0))(1 - e^{T-t}) & \text{for } t \in (T, \infty). \end{cases}$$

Note that the definition of $\widetilde{Z}(t)$ for $t \in [T, \infty)$ and the fact that $x \in G^\circ$ and G is convex together ensure that $\widetilde{Z}(t)$ remains in G° for all $t \in (T, \infty)$.

We now show that $\widetilde{Z}$ satisfies condition 3 of the boundary jitter property. Suppose $t \in (0, T]$ is such that $\mathcal{I}(\widetilde{Z}(t)) = \mathcal{I}(Z(T - t)) \in \mathcal{N}$. By condition 4 of the

boundary jitter property, for each $i \in \mathcal{I}(\widetilde{Z}(t)) = \mathcal{I}(Z(T-t))$ and every $\delta \in (0, t)$, there exists $u \in (T-t, T-t+\delta) \subsetneq (T-t, T)$ such that $\mathcal{I}(Z(u)) = \{i\}$, which implies that $\tilde{u} \doteq T - u \in (t-\delta, t)$ and $\mathcal{I}(\widetilde{Z}(\tilde{u})) = \{i\}$. Since $\widetilde{Z}(t) \in G^\circ$ for all $t > T$, this proves that $\widetilde{Z}$ satisfies condition 3 of the boundary jitter property.

By condition 4 of the boundary jitter property, there exist $0 < s < t < T$ and $i, j \in \mathcal{I}(Z(0))$ such that $i \neq j$, $\mathcal{I}(Z(s)) = \{i\}$ and $\mathcal{I}(Z(t)) = \{j\}$. Then by condition 3 of the boundary jitter property, the continuity of Z and the upper semicontinuity of $\mathcal{I}(\cdot)$, there exists $S \in (0, T)$ such that $T - S \in (s, t)$ and $Z(T-S) \in G^\circ$. It follows that $\widetilde{Z}(t) = Z(T-t) \subsetneq \mathcal{I}(Z(0)) = \mathcal{I}(\widetilde{Z}(T))$ for all $t \in [S, T)$. By Lemma 4.1, there is a nested increasing sequence as in (4.1) such that $\xi_j \to T$ as $j \to \infty$ and for each $j \in \mathbb{N}$, $\widetilde{Z}(\xi_j) \in \partial G$ and (4.2) holds with $\widetilde{Z}$ in place of Z. For each $j \in \mathbb{N}$, set $\chi_j \doteq T - \xi_j$ and $u_j \doteq T - s_j$. It is then a straightforward to verify that these properties along with the definition (4.9) of $\widetilde{Z}$ imply that (4.7) holds, $\chi_j \to 0$ as $j \to \infty$ and for each $j \in \mathbb{N}$, $Z(\chi_j) = \widetilde{Z}(\xi_j) \in \partial G$ and (4.8) holds. $\square$

5. The derivative problem.

In this section, we establish some useful properties of the DP and the associated DM, which were introduced in Definition 3.4. Some of these properties are analogous to properties satisfied by the ESP that are stated in [38], Section 2.1 and Section 3.1. Due to the similarity between the axiomatic framework for the DP and the axiomatic framework for the ESP, we are able to leverage arguments used to prove properties of the ESP to prove analogous properties for the DP. Throughout this section, fix an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$.

5.1. *Basic properties of solutions to the DP.*   Throughout this section, we fix a solution (Z, Y) of the ESP for $X \in \mathbb{C}_G$. Our first result states that the DM is linear. The result is a consequence of the fact that, for $x \in G$, H_x and $\text{span}[d(x)]$ are (closed) linear subspaces of $\mathbb{R}^J$. Since the proof is a straightforward verification argument, we omit it.

LEMMA 5.1.   *Suppose (ϕ_1, η_1) solves the DP along Z for $\psi_1 \in \mathbb{D}_r$ and (ϕ_2, η_2) solves the DP along Z for $\psi_2 \in \mathbb{D}_r$. Then for all $\alpha, \beta \in \mathbb{R}$, $(\alpha\phi_1 + \beta\phi_2, \alpha\eta_1 + \beta\eta_2)$ solves the DP along Z for $\alpha\psi_1 + \beta\psi_2$.*

Given a solution (ϕ, η) to the DP for $\psi \in \mathbb{D}_r$ and $S \in [0, \infty)$, define $\psi^S, \phi^S, \eta^S \in \mathbb{D}_r$ by

$$(5.1) \qquad \psi^S(\cdot) \doteq \phi(S) + \psi(S + \cdot) - \psi(S),$$

$$(5.2) \qquad \phi^S(\cdot) \doteq \phi(S + \cdot),$$

$$(5.3) \qquad \eta^S(\cdot) \doteq \eta(S + \cdot) - \eta(S).$$

The following lemma states a useful time-shift property of the DM that is analogous to time-shift property of the ESP stated in Lemma 2.7.

LEMMA 5.2. *Suppose (ϕ, η) solves the DP along Z for $\psi \in \mathbb{D}_r$. For $S \in [0, \infty)$, define Z^S as in* (2.8) *and ψ^S, ϕ^S, η^S as in* (5.1)–(5.3). *Then (ϕ^S, η^S) solves the DP along Z^S for ψ^S. Moreover, if (ϕ, η) is the unique solution to the DP along Z for ψ, then for any $0 \leq S < T < \infty$, $\phi(T)$ depends only on $\{Z(t), t \in [S, T]\}$, $\phi(S)$ and $\{\psi(S + t) - \psi(S), t \in [0, T - S]\}$.*

PROOF. Fix $S \in [0, \infty)$ and let $Z^S, \psi^S, \phi^S, \eta^S$ be as in the statement of the lemma. Fix $t \in [0, \infty)$. We first show that (ϕ^S, η^S) satisfies condition 1 of the DP along Z^S. By (5.2), the fact that (ϕ, η) satisfies condition 1 of the DP along Z, (5.1) and (5.3), we have for $t \geq 0$,

$$\phi^S(t) = \psi(S + t) + \eta(S + t) + \phi(S) - \psi(S) - \eta(S) = \psi^S(t) + \eta^S(t).$$

Next, we show that (ϕ^S, η^S) satisfies condition 2 of the DP along Z^S, which is equivalent, by (5.2) and (2.8), to showing that $\langle \phi(S + t), n_i \rangle = 0$ for all $i \in \mathcal{I}(Z(S + t))$, which follows because (ϕ, η) satisfies condition 2 of the DP along Z. We now turn to the proof that (ϕ^S, η^S) satisfies condition 3 of the DP along Z^S. Fix $s \in [0, t)$. By (5.3), the fact that (ϕ, η) satisfies condition 3 of the DP along Z and (2.8),

$$\eta^S(t) - \eta^S(s) \in \text{span}\left[\bigcup_{u \in (S+s, S+t]} d(Z(u))\right] = \text{span}\left[\bigcup_{u \in (s, t]} d(Z^S(u))\right].$$

This completes the proof of the lemma. $\quad\square$

5.2. *Lipschitz continuity and closure property of the derivative map.* We first state a useful consequence of Assumption 2.8.

LEMMA 5.3. *Under Assumption* 2.8,

(5.4) $\qquad z \in \partial B, v \in v_B(z) \quad \Rightarrow \quad \langle z, n_i \rangle \langle v, d_i \rangle \leq 0 \qquad \forall i \in \mathcal{I}(x).$

In particular, for all $i \in \mathcal{I}$,

(5.5) $\qquad\qquad z \in \partial B, v \in v_B(z), \langle v, d_i \rangle < 0 \quad \Rightarrow \quad \langle z, n_i \rangle \geq 1.$

PROOF. The implication (5.4) follows from [17], Lemma 2.1, and (5.5) is a straightforward consequence of (2.10) and (5.4). $\quad\square$

We now state the Lipschitz continuity that follows from Assumption 2.8.

THEOREM 5.4. *Suppose the ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ satisfies Assumption* 2.8. *Then there exists $\kappa_\Lambda \in (0, \infty)$ such that if (Z, Y) is a solution to the ESP for $X \in \mathbb{C}$, (ϕ_1, η_1) solves the DP along Z for $\psi_1 \in \mathbb{D}_r$ and (ϕ_2, η_2) solves the DP along Z for $\psi_2 \in \mathbb{D}_r$, then for all $T \in [0, \infty)$,*

(5.6) $\qquad\qquad\qquad \|\phi_1 - \phi_2\|_t \leq \kappa_\Lambda \|\psi_1 - \psi_2\|_T.$

As a consequence, for every $\psi \in \mathbb{D}_r$, there is at most one solution to the DP along Z for ψ.

PROOF. For $f = \psi, \phi, \eta$, define $\Delta f \doteq f_1 - f_2$. Fix $T \in [0, \infty)$ and let

$$(5.7) \qquad c \doteq \|\Delta \psi\|_T.$$

We will show that

$$(5.8) \qquad \eta(t) \in cB \qquad \text{for all } t \in [0, T].$$

Since B is compact and $\phi_j = \psi_j + \eta_j$ for $j = 1, 2$, (5.8) implies the desired Lipschitz continuity result. To show (5.8), it suffices to show that for any $a > c$,

$$(5.9) \qquad \Delta \eta(t) \in aB \qquad \text{for all } t \in [0, T].$$

Fix $a > c$ and define

$$\tau \doteq \inf\{t \geq 0 : \Delta \eta(t) \notin (aB)^\circ\},$$

where τ is possibly infinite. Then

$$(5.10) \qquad \Delta \eta(t) \in (aB)^\circ \qquad \text{for all } t \in [0, \tau).$$

Note that if $\tau > T$ then (5.9) follows. We will now argue by contradiction to show that (5.9) must hold. Suppose (5.9) is false. Then $\tau \leq T$. We consider two mutually exclusive and exhaustive cases. In the following, we adopt the convention $\Delta \eta(0-) \doteq 0$.

Case 1: $\tau \in (0, T]$, $\Delta \eta(\tau-) \in \partial(aB)$.

For this case, let $z \doteq \Delta \eta(\tau-)$ and fix $v \in v(z/a)$. By (5.10) and the fact that $v(z/a)$ is the set of inward normals to aB at $z \in \partial(aB)$, it follows that for all $t \in (0, \tau)$,

$$\langle z - \Delta \eta(t), v \rangle = \langle \eta_1(\tau-) - \eta_1(t), v \rangle - \langle \eta_2(\tau-) - \eta_2(t), v \rangle < 0.$$

This implies there exists a sequence $\{t_k\}_{k \in \mathbb{N}}$ with $t_k \uparrow \tau$ such that either

$$(5.11) \qquad \langle \eta_1(\tau-) - \eta_1(t_k), v \rangle < 0 \qquad \text{for all } k \in \mathbb{N}$$

or

$$(5.12) \qquad \langle \eta_2(\tau-) - \eta_2(t_k), v \rangle > 0 \qquad \text{for all } k \in \mathbb{N}.$$

Suppose (5.11) holds. By condition 3 of the DP and the definition of $d(x)$ given in (2.4), there exists $i \in \mathcal{I}$ and a sequence $\{u_k\}_{k \in \mathbb{N}}$ in $(0, \tau)$ with $u_k \uparrow \tau$ as $k \to \infty$ such that

$$\langle d_i, v \rangle \neq 0 \quad \text{and} \quad \langle Z(u_k), n_i \rangle = c_i \qquad \text{for all } k \in \mathbb{N}.$$

Then, since $v \in v(z/a)$ and $z/a \in \partial B$, it follows from condition 1 of the DP and the geometric property (2.10) of the set B that

$$(5.13) \qquad \left\langle \frac{z}{a}, n_i \right\rangle = \frac{1}{a} \langle \Delta \phi(\tau-) - \Delta \psi(\tau-), n_i \rangle \notin (-1, 1).$$

Condition 2 of the DP, (3.3) and the fact that $\langle Z(u_k), n_i \rangle = c_i$ imply that $\langle \Delta\phi(u_k), n_i \rangle = 0$ for all $k \in \mathbb{N}$. Taking limits as $k \to \infty$ yields

$$(5.14) \qquad \langle \Delta\phi(\tau-), n_i \rangle = 0.$$

When combined with (5.13), it follows that $|\langle \Delta\psi(\tau-), n_i \rangle| \geq a$. Since $a > c$ and $\tau \leq T$, this contradicts (5.7). Therefore, (5.11) does not hold. In an analogous fashion, it can be shown that (5.12) cannot hold and, therefore, Case 1 cannot hold.

Case 2: $\tau \in [0, T]$, $\Delta\eta(\tau-) \in (aB)^\circ$ and $\Delta\eta(\tau) \notin (aB)^\circ$.

The proof in this case is similar, with minor differences, but we fill in the details for completeness. For this case, let $z \doteq \Delta\eta(\tau)$. Then there exists $r \geq a$ such that $z \in \partial(rB)$. Fix $\nu \in \nu(z/r)$. Since $\nu(z/r)$ is the set of inward normals to rB at $z \in \partial(rB)$ and $\Delta\eta(\tau-) \in (rB)^\circ$,

$$\langle z - \Delta\eta(\tau-), \nu \rangle = \langle \eta_1(\tau) - \eta_1(\tau-), \nu \rangle - \langle \eta_2(\tau) - \eta_2(\tau-), \nu \rangle < 0.$$

This implies that either

$$(5.15) \qquad \langle \eta_1(\tau) - \eta_1(\tau-), \nu \rangle < 0$$

or

$$(5.16) \qquad \langle \eta_2(\tau) - \eta_2(\tau-), \nu \rangle > 0.$$

Suppose that (5.15) holds. By (3.5), there exists $i \in \{1, \ldots, N\}$ such that

$$\langle d_i, \nu \rangle \neq 0 \quad \text{and} \quad \langle Z(\tau), n_i \rangle = c_i.$$

Since $\langle d_i, \nu \rangle \neq 0$, $\nu \in \nu(z/r)$ and $z/r \in \partial B$, it follows from the geometric property (2.10) of the set B that

$$(5.17) \qquad \left\langle \frac{z}{r}, n_i \right\rangle = \frac{1}{r} \langle \Delta\phi(\tau) - \Delta\psi(\tau), n_i \rangle \notin (-1, 1).$$

Now by condition 1 of the DP and the fact that $\langle Z(\tau), n_i \rangle = c_i$, $\langle \Delta\phi(\tau), n_i \rangle = 0$ and so

$$(5.18) \qquad |\langle \Delta\psi(\tau), n_i \rangle| \geq r,$$

which contradicts (5.7) and the fact that $c < r$. Therefore, (5.15) does not hold. In an analogous fashion, it can be shown that (5.16) does not hold and, therefore, Case 2 cannot hold. $\quad\square$

The closure property of the DM is similar to the closure property of the ESM (see [38], Lemma 2.5). As shown below, it follows from the Lipschitz continuity of the DM and the fact that for each $x \in G$, H_x and $\text{span}(d(x))$ are closed subsets of $\mathbb{R}^J$.

LEMMA 5.5. *Suppose the ESP* $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ *satisfies Assumption* 2.8. *Let* $\{\psi_k\}_{k \in \mathbb{N}}$ *be a sequence in* $\mathbb{D}_r$ *such that* ψ_k *converges to* $\psi \in \mathbb{D}_r$ *as* $k \to \infty$. *Fix* $T \in (0, \infty)$. *Suppose that for each* $k \in \mathbb{N}$, (ϕ_k, η_k) *solves the DP along* Z *for* ψ_k *on* $[0, T)$. *Then there exists* $(\phi, \eta) \in \mathbb{D}_r([0, T) : \mathbb{R}^J) \times \mathbb{D}_r([0, T) : \mathbb{R}^J)$ *such that* $(\phi_k, \eta_k) \to (\phi, \eta)$ *in* $\mathbb{D}_r([0, T) : \mathbb{R}^J) \times \mathbb{D}_r([0, T) : \mathbb{R}^J)$ *as* $k \to \infty$ *and* (ϕ, η) *solves the DP along* Z *for* ψ.

PROOF. Fix $T \in (0, \infty)$. By the Lipschitz continuity property stated in Theorem 5.4, $\{(\phi_k, \eta_k)\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{D}_r([0, T) : \mathbb{R}^J) \times \mathbb{D}_r([0, T) : \mathbb{R}^J)$. Since $\mathbb{D}_r([0, T) : \mathbb{R}^J)$ is a complete metric space under the topology of uniform convergence (see, e.g., [5], Chapter 3), there exists $(\phi, \eta) \in \mathbb{D}_r([0, T) : \mathbb{R}^J) \times \mathbb{D}_r([0, T) : \mathbb{R}^J)$ such that $(\phi_k, \eta_k) \to (\phi, \eta)$ in the uniform norm as $k \to \infty$. We are left to show that (ϕ, η) solves the DP for ψ on $[0, T)$.

Let $t \in [0, T)$. Taking limits as $k \to \infty$ in $\phi_k(t) = \psi_k(t) + \eta_k(t) \in H_{Z(t)}$, we see that $\phi(t) = \psi(t) + \eta(t) \in H_{Z(t)}$, where we have used the fact that $H_{Z(t)}$ is a closed linear subspace. Thus, conditions 1 and 2 of the DP hold. Now let $0 \le s < t < T$. By condition 3 of the DP, for each $k \in \mathbb{N}$,

$$\eta_k(t) - \eta_k(s) \in \text{span}\left[\bigcup_{u \in (s, t]} d(Z(u)) \right].$$

Since the right-hand side is a closed linear subspace, letting $k \to \infty$ in the above, we see that condition 3 of the DP holds. This completes the proof of the lemma. □

REMARK 5.6. The closure property stated in Lemma 5.5 requires that Z be fixed. Indeed, the closure property above does not generally hold if Z is replaced by a convergent sequence $\{Z_k\}_{k \in \mathbb{N}}$. For example, let $J = 1$ and consider the one-dimensional SP $\{(e_1, e_1, 0)\}$. Define $Z, \psi, \phi, \eta \in \mathbb{C}$ by $Z(t) \doteq 0$, $\psi(t) \doteq 1$, $\phi(t) \doteq 1$ and $\eta(t) \doteq 0$ for all $t \in [0, \infty)$. For each $k \in \mathbb{N}$, define $Z_k(t) \doteq 1/k$ for all $t \in [0, \infty)$. Then Z_k converges to Z in $\mathbb{C}$ as $k \to \infty$ and it is readily verified that for each $k \in \mathbb{N}$, (ϕ, η) solves the DP along Z_k for ψ. However, (ϕ, η) does not solve the (one-dimensional) DP along Z for ψ [since $\phi(t) = 1 \notin \{0\} = H_{Z(t)}$ for all $t \in [0, \infty)$].

6. Directional derivatives: The one-dimensional setting. In this section, we review prior results on directional directions of the one-dimensional SM (i.e., when $J = 1$) and also present new results that relate directional derivatives to solutions to the DP. Mandelbaum and Massey [33], Lemma 5.2, were the first to establish existence of and obtain an explicit characterization for directional derivatives $\nabla_\psi \Gamma_1(X)$ when $X, \psi \in \mathbb{C}$, $X(0) = 0$ and the explicit representation for $\nabla_\psi \Gamma_1(X)$ has a finite number of discontinuities in any compact interval of $[0, \infty)$. The last two restrictions were removed and the result generalized to $X, \psi \in \mathbb{D}_r$ and

$X, \psi \in \mathbb{D}_{l,r}$ by Whitt [45], Corollary 9.5.1, and Mandelbaum and Ramanan [34], Theorem 3.2, respectively. In the following proposition, we summarize the results of [33, 34, 45] when X and ψ are continuous.

For $f, g \in \mathbb{C}$, define $F(f, g) : [0, \infty) \to \mathbb{R}$ by

$$(6.1) \qquad F(f, g)(t) \doteq \begin{cases} 0 & \text{if } \sup_{s \in [0,t]} (-f(s)) < 0, \\ \sup_{s \in \Phi_{-f}(t)} (-g(s)) \vee 0 & \text{if } \sup_{s \in [0,t]} (-f(s)) = 0, \\ \sup_{s \in \Phi_{-f}(t)} (-g(s)) & \text{if } \sup_{s \in [0,t]} (-f(s)) > 0, \end{cases}$$

where

$$(6.2) \qquad \Phi_{-f}(t) \doteq \left\{ u \in [0, t] : -f(u) = \sup_{s \in [0,t]} (-f(s)) \right\}.$$

PROPOSITION 6.1. *Given $X, \psi \in \mathbb{C}$, the directional derivative $\nabla_\psi \Gamma_1(X)$ exists, is upper semicontinuous, lies in $\mathbb{D}_{l,r}$ and is given by*

$$(6.3) \qquad \nabla_\psi \Gamma_1(X)(t) = \psi(t) + F(X, \psi)(t), \qquad t \in [0, \infty).$$

Consequently, $F(X, \psi)$ is upper semicontinuous and lies in $\mathbb{D}_{l,r}$.

PROOF. By [34], Theorem 1.1, $\nabla_\psi \Gamma_1(X)$ exists and is upper semicontinuous. By (2.12) and [34], Theorem 3.2, it follows that $\nabla_\psi \Gamma_1(X)$ is given by (6.3). By [34], Theorem 1.2, and because there are no *chains* (see [34], Definition 1.5) in the one-dimensional setting, $\nabla_\psi \Gamma_1(x)$ lies in $\mathbb{D}_{l,r}$. $\square$

In the next proposition, we characterize directional derivatives of Γ_1 via solutions to the (one-dimensional) DP when (Z, Y) satisfies condition 1 of the boundary jitter property (conditions 2–4 of the boundary jitter property are automatic in the one-dimensional setting). Recall from Example 2.13 that $G = \mathbb{R}_+$ and $\pi_1(x) = x \vee 0$. It is straightforward to check that $\nabla_v \pi_1(x)$, defined as in (3.8) for $(x, v) \in \mathbb{R}_+ \times \mathbb{R}$, is given by

$$(6.4) \qquad \nabla_v \pi_1(x) = \begin{cases} v & \text{if } x > 0, \\ v \vee 0 & \text{if } x = 0. \end{cases}$$

In addition, observe that $\mathcal{S} = \partial G = \{0\}$ and G_0, defined as in (3.9) with $x = 0$, is given by $G_0 = \mathbb{R}_+$; H_x, defined as in (3.3), is given by

$$(6.5) \qquad H_x = \begin{cases} \mathbb{R} & \text{if } x > 0, \\ \{0\} & \text{if } x = 0, \end{cases}$$

and $d(x)$, defined as in (2.4), is given by $d(0) = \mathbb{R}_+$ and $d(x) = 0$ for all $x > 0$.

PROPOSITION 6.2. *Given $X \in \mathbb{C}_G$, let (Z, Y) denote the solution to the one-dimensional SP for X. Then for all $\psi \in \mathbb{C}$:*

1. *$\nabla_\psi \Gamma_1(X)$ exists and lies in $\mathbb{D}_{l,r}$;*
2. *$\nabla_\psi \Gamma_1(X)(0) = \nabla_{\psi(0)} \pi_1(X(0))$ and if $\nabla_\psi \Gamma_1(X)$ is discontinuous at $t \in (0, \infty)$, then $Z(t) \in \mathcal{S}$ and $\nabla_\psi \Gamma_1(X)$ is left continuous at t if and only if $\nabla_\psi \Gamma_1(X)(t-) \in G_{Z(t)}$;*
3. *if (Z, Y) satisfies condition 1 of the boundary jitter property (Definition 3.1), then there is a unique solution (ϕ, η) of the DP along Z for ψ and ϕ is equal to the right continuous regularization of $\nabla_\psi \Gamma_1(X)$.*

Before proving Proposition 6.2, we first prove the following useful lemma.

LEMMA 6.3. *Given $f, g \in \mathbb{C}$, $F(f, g)$ is upper semicontinuous and lies in $\mathbb{D}_{l,r}$. Moreover, if $t \in (0, \infty)$ is a discontinuity point of $F(f, g)$, then the following properties hold:*

(i) *$\Gamma_1(f)(t) = 0$;*
(ii) *$F(f, g)$ is left continuous at t if and only if $F(f, g)(t-) \geq -g(t)$.*

PROOF. By Proposition 6.1, $F(f, g)$ is upper semicontinuous and lies in $\mathbb{D}_{l,r}$. Fix a discontinuity point $t \in (0, \infty)$ of $F(f, g)$. By (6.1)–(6.2), t lies at an endpoint of the closed interval

$$I_f \doteq \left\{ t \in [0, \infty) : \sup_{s \in [0,t]} (-f(s)) = 0 \right\}$$

and/or $\sup_{s \in \Phi_{-f}(\cdot)}(-g(s))$ is discontinuous at t. In either case, $t \in \Phi_{-f}(t)$, which along with the explicit formula for Γ_1 given in (2.12) implies that $\Gamma_1(f)(t) = 0$, so (i) holds. Suppose $F(f, g)$ is left continuous at t. Then by (6.1) and the fact that $t \in \Phi_{-f}(t)$, $F(f, g)(t-) = F(f, g)(t) \geq -g(t)$. Alternatively, suppose $F(f, g)(t-) \geq -g(t)$. If $t < a \doteq \min I_f$, then (6.1) implies $F(f, g)$ is continuous at t. Next, if $t = a$, then $F(f, g)(t-) = 0$, $\Phi_{-f}(t) = \{t\}$ and $F(f, g)(t) = (-g(t)) \vee 0 = 0$, so $F(f, g)$ is left continuous at t. Proceeding, if $t \in I_f \setminus \{a\}$, then (6.1) implies

$$F(f, g)(t-) = \sup_{s \in [0,t) \cap I_f} (-g(s)) \vee 0, \qquad F(f, g)(t) = \sup_{s \in [0,t] \cap I_f} (-g(s)) \vee 0,$$

so the fact that $F(f, g)(t-) \geq -g(t)$ implies $F(f, g)$ is left continuous at t. Finally, suppose $t > \max I_f$. If $\Phi_{-f}(t) = \{t\}$, then the continuity of f implies there is a sequence $\{s_k\}_{k \in \mathbb{N}}$ such that $s_k \uparrow t$ as $k \to \infty$ and for each $k \in \mathbb{N}$, $s_k > \max I_f$ and $\Phi_{-f}(s_k) = \{s_k\}$. In this case,

$$F(f, g)(t-) = \lim_{k \to \infty} F(f, g)(s_k) = \lim_{k \to \infty} (-g(s_k)) = -g(t-) = F(f, g)(t).$$

On the other hand, if $\Phi_{-f}(t) \neq \{t\}$, then set $\Phi_{-f}(t-) \doteq \Phi_{-f}(t) \setminus \{t\}$. Then for all $u \in [0, t)$ sufficiently large, $\Phi_{-f}(u) = \Phi_{-f}(t-) \cap [0, u]$. Thus, by the continuity of g,

$$F(f, g)(t-) = \lim_{u \uparrow t} \sup_{s \in \Phi_{-f}(t-) \cap [0,u]} (-g(s)) = \sup_{s \in \Phi_{-f}(t-)} (-g(s)),$$

and, using that $F(f, g)(t-) \geq -g(t)$, we have

$$F(f, g)(t) = \sup_{s \in \Phi_{-f}(t-)} (-g(s)) \vee (-g(t)) = F(f, g)(t-).$$

The completes the proof of (ii). $\square$

PROOF OF PROPOSITION 6.2.   Fix $\psi \in \mathbb{C}$. By [34], Theorem 3.2, $\nabla_\psi \Gamma_1(X)$ exists, lies in $\mathbb{D}_{l,r}$ and is characterized by (6.3)–(6.2). It follows from (6.4) that $\nabla_\psi \Gamma_1(X)(0) = \nabla_{\psi(0)} \pi_1(X(0))$. Let $t \in (0, \infty)$ be a discontinuity point of $\nabla_\psi \Gamma_1(X)$. Then by (6.3), the continuity of ψ and Lemma 6.3, $Z(t) = \Gamma_1(X)(t) = 0$, or equivalently, $Z(t) \in \mathcal{S}$, and $\nabla_\psi \Gamma_1(X)$ is left continuous at t if and only if $\nabla_\psi \Gamma_1(X)(t) \geq 0$, or equivalently, $\nabla_\psi \Gamma_1(X)(t-) \in G_{Z(t)}$. This proves of parts 1 and 2 of Proposition 6.2.

Now suppose (Z, Y) satisfies condition 1 of the boundary jitter property. We show that $(\hat{\phi}, \hat{\phi} - \psi)$, where $\hat{\phi}(t) \doteq \nabla_\psi \Gamma_1(X)(t+)$ for all $t \in [0, \infty)$, solves the DP along Z for ψ. Since solutions to the DP are unique under Assumption 2.8, by Theorem 5.4, this will complete the proof of part 3 of the proposition. Condition 1 of the DP holds automatically. Let $t \in [0, \infty)$. In view of (6.5), we need to show that if $Z(t) = 0$, then $\hat{\phi}(t) = 0$. Suppose $Z(t) = 0$. By condition 1 of the boundary jitter property and the fact that Y is nondecreasing, *at least* one of the following holds:

(i) $t \geq 0$ and $Y(u) > Y(t)$ for all $u > t$;
(ii) $t > 0$ and $Y(s) < Y(t)$ for all $s < t$.

First, consider case (i). Since Y is nondecreasing, there is a sequence $\{u_\ell\}_{\ell \in \mathbb{N}}$ such that $u_\ell \downarrow t$ as $\ell \to \infty$ and for each $\ell \in \mathbb{N}$, $0 \leq Y(s) < Y(u_\ell)$ for all $s < u_\ell$. Then for each $\ell \in \mathbb{N}$, (2.13) and (6.3)–(6.2) imply that $\Phi_{-X}(u_\ell) = \{u_\ell\}$ and $\nabla_\psi \Gamma_1(X)(u_\ell) = 0$. Letting $\ell \to \infty$ yields $\hat{\phi}(t) = 0$. Next, suppose case (ii) holds and case (i) does *not* hold. Then $-X(t) = Y(t) > 0$ and by (2.13) and (6.3)–(6.2), $\Phi_{-X}(t) = \{t\}$ and $\nabla_\psi \Gamma_1(X)(t) = 0$. Since case (i) does not hold, (6.2) implies $\Phi_{-X}(u) \subseteq [t, u]$ for all $u \in [t, t+\delta)$ for $\delta > 0$ sufficiently small. Upon substituting the last relation into (6.1), we see that $F(X, \psi)(u) = \sup_{r \in [t,u]} (-\psi(r))$ for all $u \in [t, t + \delta)$. Since ψ is continuous, this implies that $F(X, \psi)$ is right continuous at t, so $\hat{\phi}(t) = 0$. This proves that $\hat{\phi}$ satisfies condition 2 of the DP.

We are left to show that $(\hat{\phi}, \hat{\phi} - \psi)$ satisfies condition 3 of the DP. By (6.3), $\hat{\phi}(t) - \psi(t) = F(X, \psi)(t+)$ for all $t \in [0, \infty)$. Fix $0 \leq s < t < \infty$. In order to prove condition 3, due to the fact that $\text{span}[d(0)] = \mathbb{R}$, it suffices to show that if Z

is positive on $(s, t]$, then $F(X, \psi)(t+) - F(X, \psi)(s+) = 0$. By the continuity of Z, if Z is positive on $(s, t]$, there exists $u > t$ such that Z is positive on (s, u). By (2.12)–(2.13) and (6.2), $\Phi_X(\cdot)$ must be constant on (s, u), which, along with (6.1), implies the desired conclusion $F(X, \psi)(t+) - F(X, \psi)(s+) = 0$. $\square$

7. Directional derivatives: Up to the first hitting time of the nonsmooth part of the boundary. In this section, we prove existence of and characterize directional derivatives of the ESM up until the first time that the constrained path reaches $\mathcal{N}$, the nonsmooth part of the boundary. In order to prove our main result, we introduce the following statement, which will be referred to multiple times for different values of T. Recall the definition of G_x, for $x \in \mathcal{S}$, given in (3.9) and that $\nabla_v \pi(x)$, defined in (3.8), was shown in Lemma 3.9 to exist for all $(x, v) \in G \times \mathbb{R}^J$.

STATEMENT 7.1. For all $\psi \in \mathbb{C}$, the following hold:

1. $\nabla_\psi \bar{\Gamma}(X)$ exists on $[0, T)$ and lies in $\mathbb{D}_{l,r}([0, T) : \mathbb{R}^J)$.
2. $\nabla_\psi \bar{\Gamma}(X)(0) = \nabla_{\psi(0)} \pi(X(0))$ and if $t \in (0, T)$ is a discontinuity point of $\nabla_\psi \bar{\Gamma}(X)$, then $Z(t) \in \mathcal{S}$ and $\nabla_\psi \bar{\Gamma}(X)$ is left continuous at t if and only if $\nabla_\psi \bar{\Gamma}(X)(t-) \in G_{Z(t)}$.
3. If (Z, Y) satisfies condition 1 of the boundary jitter property on $[0, T)$, then there exists a unique solution (ϕ, η) to the DP along Z for ψ on $[0, T)$ and $\phi(t) = \nabla_\psi \bar{\Gamma}(X)(t+)$ for $t \in [0, T)$.

Given a solution (Z, Y) to the ESP for $X \in \mathbb{C}_G$, let θ_2 be the first time Z reaches the nonsmooth part of the boundary $\mathcal{N}$; that is,

$$(7.1) \qquad \theta_2 \doteq \inf\{t \in [0, \infty) : Z(t) \in \mathcal{N}\}.$$

The following proposition is the main result of this section.

PROPOSITION 7.2. *Fix an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ satisfying Assumptions 2.8 and 2.11. Given $X \in \mathbb{C}_G$, let (Z, Y) denote the solution to the ESP for X and define θ_2 as in (7.1). Then Statement 7.1 holds with $T = \theta_2$.*

In Section 7.1, we introduce some standard notation that will be used in this section and present some useful lemmas. In Section 7.2, we prove Proposition 7.2.

7.1. *Preliminary setup and results.* Fix an ESP $\{(d_i, n_i, c_i)\}$ satisfying Assumptions 2.8 and 2.11. Given $X \in \mathbb{C}_G$ and $\psi \in \mathbb{C}$, let $Z \doteq \bar{\Gamma}(X)$ and for $\varepsilon > 0$, let $Z_\varepsilon \doteq \bar{\Gamma}(X + \varepsilon\psi)$. Given $S \in (0, \infty)$, define X^S, Z^S as in (2.7)–(2.8), so by the time-shift property of the ESP (Lemma 2.7), $Z^S = \bar{\Gamma}(X^S)$ and for $\varepsilon > 0$, define X_ε^S, $Z_\varepsilon^S \in \mathbb{C}$ by

$$(7.2) \qquad X_\varepsilon^S(\cdot) = Z_\varepsilon(S) + X(S + \cdot) + \varepsilon\psi(S + \cdot) - X(S) - \varepsilon\psi(S)$$

$$(7.3) \qquad Z_\varepsilon^S(\cdot) = Z_\varepsilon(S + \cdot),$$

so by the time-shift property of the ESP, $Z_\varepsilon^S = \bar{\Gamma}(X_\varepsilon^S)$. By (7.2) and (2.7),

$$
\begin{aligned}
(7.4) \qquad X_\varepsilon^S(\cdot) &= X^S(\cdot) + Z_\varepsilon(S) - Z(S) + \varepsilon \psi(S + \cdot) - \varepsilon \psi(S) \\
&= X^S(\cdot) + \varepsilon \hat{\psi}_\varepsilon^S(\cdot),
\end{aligned}
$$

where $\hat{\psi}_\varepsilon^S \in \mathbb{C}$ is given by

$$
\begin{aligned}
(7.5) \qquad \hat{\psi}_\varepsilon^S(\cdot) &\doteq \frac{Z_\varepsilon(S) - Z(S)}{\varepsilon} + \psi(S + \cdot) - \psi(S) \\
&= \nabla_\psi^\varepsilon \bar{\Gamma}(X)(S) + \psi(S + \cdot) - \psi(S),
\end{aligned}
$$

where the last equality uses (2.15) and the definitions of Z_ε and Z. Suppose that $T \in (S, \infty)$ and $\nabla_\psi \bar{\Gamma}(X)$ exists on $[0, T)$. Then $\hat{\psi}_\varepsilon^S \to \hat{\psi}^S$ uniformly on $[0, \infty)$ as $\varepsilon \downarrow 0$, where $\hat{\psi}^S \in \mathbb{C}$ is given by

$$
(7.6) \qquad \hat{\psi}^S(\cdot) \doteq \nabla_\psi \bar{\Gamma}(X)(S) + \psi(S + \cdot) - \psi(S).
$$

The proof of Proposition 7.2 will proceed by showing the existence of and characterizing $\nabla_\psi \bar{\Gamma}(X)$ on intervals where Z only hits a single face. The following lemma will allow us to piece together these results to establish existence on $[0, \theta_2)$.

LEMMA 7.3. *Fix $0 \le S < T < U < \infty$. Let (Z, Y) be the solution to the ESP for $X \in \mathbb{C}_G$. Assume that Statement 7.1 holds. Define $X^S, Z^S, Y^S, \hat{\psi}^S$ as in (2.7)–(2.9) and (7.6), and assume that Statement 7.1 holds with $X^S, Z^S, Y^S, \hat{\psi}^S, U - S$ in place of X, Z, Y, ψ, T, respectively. Then Statement 7.1 holds with U in place of T. Furthermore, for $t \in [S, U)$, $\nabla_\psi \bar{\Gamma}(X)(t) = \nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)(t - S)$.*

PROOF. For $\varepsilon > 0$, let $Z_\varepsilon \doteq \bar{\Gamma}(X + \varepsilon \psi)$ and define $X_\varepsilon^S, Z_\varepsilon^S, \hat{\psi}_\varepsilon^S$ as in (7.2), (7.3) and (7.5). Since Statement 7.1 holds by assumption, $\nabla_\psi \bar{\Gamma}(X)$ exists on $[0, T)$, which, along with the fact that $S \in [0, T)$, implies $\hat{\psi}_\varepsilon^S \to \hat{\psi}^S$ uniformly on $[0, \infty)$ as $\varepsilon \downarrow 0$. We show that $\nabla_\psi \bar{\Gamma}(X)$ exists on $[T, U)$. By our assumption that Statement 7.1 holds with $X^S, \hat{\psi}^S, Z^S, Y^S, U - S$ in place of X, ψ, Z, Y, T, respectively, Proposition 2.17, and the fact that $\hat{\psi}_\varepsilon^S \to \hat{\psi}^S$ uniformly on $[0, \infty)$ as $\varepsilon \downarrow 0$, we have

$$
(7.7) \qquad \lim_{\varepsilon \downarrow 0} \nabla_{\hat{\psi}_\varepsilon^S}^\varepsilon \bar{\Gamma}(X^S)(t) = \nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)(t), \qquad t \in [0, U - S).
$$

By (2.16), (2.15), the time-shift property of the ESP (Lemma 2.7), (7.4) and (7.7), for $t \in [S, U)$, we have

$$
\begin{aligned}
\nabla_\psi \bar{\Gamma}(X)(t) &= \lim_{\varepsilon \downarrow 0} \frac{\bar{\Gamma}(X + \varepsilon \psi) - \bar{\Gamma}(X)}{\varepsilon} \\
(7.8) \qquad &= \lim_{\varepsilon \downarrow 0} \frac{\bar{\Gamma}(X^S + \varepsilon \hat{\psi}_\varepsilon^S)(t - S) - \bar{\Gamma}(X^S)(t - S)}{\varepsilon} \\
&= \nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)(t - S).
\end{aligned}
$$

This establishes the existence of $\nabla_\psi \bar{\Gamma}(X)$ on $[0, U)$ as well as the final assertion of the lemma.

Proceeding, by assumption, $\nabla_\psi \bar{\Gamma}(X) : [0, T) \mapsto \mathbb{R}^J$ lies in $\mathbb{D}_{l,r}([0, T) : \mathbb{R}^J)$ and $\nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S) : [0, U - S) \mapsto \mathbb{R}^J$ lies in $\mathbb{D}_{l,r}([0, U - S) : \mathbb{R}^J)$. Since $S \in [0, T)$, it follows from (7.8) that $\nabla_\psi \bar{\Gamma}(X) : [0, U) \mapsto \mathbb{R}^J$ lies in $\mathbb{D}_{l,r}([0, U) : \mathbb{R}^J)$. By assumption, if $t \in (0, T)$ is a discontinuity point of $\nabla_\psi \bar{\Gamma}(X)$ then $Z(t) \in \mathcal{S}$ and $\nabla_\psi \bar{\Gamma}(X)$ is left continuous at t if and only if $\nabla_\psi \bar{\Gamma}(X)(t-) \in G_{Z(t)}$. Now suppose $t \in [T, U)$ is a discontinuity point of $\nabla_\psi \bar{\Gamma}(X)$. Then by (7.8), $t - S$ is a discontinuity point of $\nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)$. Since Statement 7.1 holds with $U - S$ in place of T for the time shifted paths and $S < T$, it follows from (7.8) that $\nabla_\psi \bar{\Gamma}(X)(t) = \nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)(t - S) \in \mathcal{S}$ and $\nabla_\psi \bar{\Gamma}(X)(\cdot) = \nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)(\cdot - S)$ is left continuous at t if and only if

$$\nabla_\psi \bar{\Gamma}(X)(t-) = \nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)\big((t - S)-\big) \in G_{Z^S(t-S)} = G_{Z(t)},$$

where the last equality uses (2.8). This proves that part 2 of Statement 7.1 holds with U in place of T.

We are left to show part 3 of Statement 7.1 with U in place of T. Suppose (Z, Y) satisfies condition 1 of the boundary jitter property on $[0, U)$. Then it is readily verified [using the relations (2.8)–(2.9)] that (Z^S, Y^S) satisfies condition 1 of the boundary jitter property on $[0, U - S)$. Therefore, by assumption, there exists a unique solution (ϕ, η) of the DP along Z for ψ on $[0, T)$, $\phi(t) = \nabla_\psi \bar{\Gamma}(X)(t+)$ for all $t \in [0, T)$, there exists a unique solution (ϕ^S, η^S) of the DP along Z^S for ψ^S on $[0, U - S)$ and $\phi^S(t) = \nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)(t+)$ for all $t \in [0, U - S)$. Let $\hat{\phi}(t) \doteq \nabla_\psi \bar{\Gamma}(X)(t+)$ on $[0, U)$. Note that by condition 2 of the DP,

$$(7.9) \qquad \hat{\phi}(t) = \phi(t) \in H_{Z(t)}, \qquad t \in [0, T),$$

and, by (7.8), condition 2 of the DP and (2.8), for all $t \in [S, U)$,

$$(7.10) \qquad \begin{aligned} \hat{\phi}(t) &\doteq \nabla_\psi \bar{\Gamma}(X)(t+) = \nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)\big((t - S)+\big) \\ &= \phi^S(t - S) \in H_{Z^S(t-S)} = H_{Z(t)}. \end{aligned}$$

Let $\hat{\eta} \doteq \hat{\phi} - \psi$ on $[0, U)$. We prove that $(\hat{\phi}, \hat{\eta})$ solves the DP along Z for ψ on $[0, U)$, which along with the uniqueness of solutions implied by the Lipschitz continuity of the DM (Theorem 5.4) will prove part 3 of Statement 7.1.

Condition 1 of the DP holds automatically. Condition 2 of the DP follows from (7.9) and (7.10). It remains to prove that $\hat{\eta}$ satisfies condition 3 of the DP on $[0, U)$. Let $0 \le s < t < U$. If $0 \le s < t < T$, then by (7.9), (7.6) and because (ϕ, η) solves the DP for ψ on $[0, T)$,

$$(7.11) \qquad \hat{\eta}(t) - \hat{\eta}(s) = \phi(t) - \psi(t) - (\phi(s) - \psi(s)) \in \text{span}\left[\bigcup_{u \in (s,t]} d\big(Z(u)\big)\right].$$

If $S \leq s < t < U$, then by (7.10) and because (ϕ^S, η^S) solves the DP for $\hat{\psi}^S$ on $[0, U - S)$,

$$\hat{\eta}(t) - \hat{\eta}(s) \doteq \hat{\phi}^S(t - S) - \hat{\psi}^S(t) - (\hat{\phi}^S(s - S) - \hat{\psi}^S(s))$$

(7.12)
$$\in \text{span}\left[\bigcup_{u \in (s-S, t-S]} d(Z^S(u - S))\right]$$

$$= \text{span}\left[\bigcup_{u \in (s, t]} d(Z(u))\right].$$

That leaves the remaining case when $0 \leq s < S < T \leq t < U$. By (7.11)–(7.12),

$$\hat{\eta}(t) - \hat{\eta}(s) = \hat{\eta}(t) - \hat{\eta}(S) + \hat{\eta}(S) - \hat{\eta}(s)$$

$$\in \text{span}\left[\left\{\bigcup_{u \in (S, t]} d(Z(u))\right\} \cup \left\{\bigcup_{u \in (s, S]} d(Z(u))\right\}\right]$$

$$= \text{span}\left[\bigcup_{u \in (s, t]} d(Z(u))\right],$$

which completes the proof of condition 3. □

7.2. *Existence and characterization.* In this section, we prove Proposition 7.2. If $\theta_2 = 0$, the lemma is trivial. Alternatively, if $Z(t) \in G^\circ$ for all $t \in [0, \infty)$, then it is readily verified that for all $\psi \in \mathbb{C}$, $\nabla_\psi \bar{\Gamma}(X) = \psi$ and $(\psi, 0)$ solves the DP along Z for ψ. For the remainder of this section, we assume that $\theta_2 \in (0, \infty]$ and $Z(t) \in \partial G$ for some $t \in [0, \infty)$.

Recursively define the increasing sequence $\{t_k\}_{k=1,\dots,K}$, $K \in \mathbb{N}_\infty$, in $[0, \theta_2)$ as follows: first, set

(7.13) $t_1 \doteq \inf\{t \in [0, \theta_2) : Z(t) \in \partial G\}.$

Since $Z(t) \in \partial G$ for some $t \in [0, \infty)$ by assumption, $t_1 < \infty$. Given $k \in \mathbb{N}$ for which t_k is defined, if $t_k = \theta_2$, set $K = k$, whereas if $t_k < \theta_2$, then $Z(t_k)$ lies in the relative interior of some $(J - 1)$-dimensional face F_{i_k} of ∂G and we recursively define t_{k+1} to be the first time after t_k that Z hits $\bigcup_{j \neq i_k} F_j$; that is,

(7.14) $t_{k+1} \doteq \inf\{t \in (t_k, \theta_2] : Z(t) \in \partial G, \mathcal{I}(Z(t)) \neq \mathcal{I}(Z(t_k))\}.$

If $t_k < \theta_2$ for all $k \in \mathbb{N}$, set $K = \infty$. In other words,

(7.15) $K \doteq \inf\{k \in \mathbb{N} : t_k = \theta_2\}.$

If $K = \infty$, then (7.14) and the continuity of Z imply that $t_k \to \theta_2$ as $k \to \infty$.

If $t_1 = \theta_2$, then $Z(t) \in G^\circ$ for all $t \in [0, \theta_2)$ and it is straightforward to prove that for all $\psi \in \mathbb{C}$, $\nabla_\psi \bar{\Gamma}(X) = \psi$ and $(\psi, 0)$ solves the DP along Z for ψ. We assume that $t_1 \in [0, \theta_2)$. Using induction, we prove that for $2 \leq k < K + 1$, Statement 7.1

holds with $T = t_k$. Since $t_K = \theta_2$ if $K < \infty$ and $t_k \to \theta_2$ if $K = \infty$, Proposition 7.2 will then follow. In the next lemma, we establish the base case of the induction hypothesis. Since $Z(t)$ lies on at most one face of G for $t \in [0, t_2)$, we are able to reduce the problem to the one-dimensional setting and invoke Proposition 6.2.

LEMMA 7.4. *Given a solution (Z, Y) to the ESP for $X \in \mathbb{C}_G$, define t_2 as in* (7.14). *Then Statement* 7.1 *holds with $T = t_2$.*

PROOF. Fix $\psi \in \mathbb{C}$. It suffices to prove that Statement 7.1 holds for all $T \in (t_1, t_2)$. Fix $T \in (t_1, t_2)$. Let $i \in \mathcal{I}$ denote the unique index such that $Z(t_1) \in F_i$, or equivalently, $\mathcal{I}(Z(t_1)) = \{i\}$. Define $f, g \in \mathbb{C}([0, \infty) : \mathbb{R})$ by

$$f(t) \doteq \langle X(t), n_i \rangle - c_i, \qquad t \in [0, \infty),$$

$$g(t) \doteq \langle \psi(t), n_i \rangle, \qquad t \in [0, \infty).$$

Since $X(0) \in G$, $f(0) \geq 0$ holds. According to (7.13)–(7.14), $Z(t) \notin \bigcup_{j \neq i} F_j$ for all $t \in [0, T]$, so by condition 3 of the ESP and the normalization $\langle d_i, n_i \rangle = 1$,

$$(7.16) \qquad Y(t) = \langle Y(t), n_i \rangle d_i, \qquad t \in [0, T],$$

and $\langle Y(\cdot), n_i \rangle$ is nondecreasing on $[0, T]$ and can only increase when $\langle Z(t), n_i \rangle = c_i$. When combined with conditions 1 and 2 of the ESP, it is readily verified that

$$(7.17) \qquad \langle Z(t), n_i \rangle - c_i = \Gamma_1(f)(t), \qquad t \in [0, T].$$

As usual, let $Z_\varepsilon \doteq \bar{\Gamma}(X + \varepsilon \psi)$. By (2.11), (2.3) and the fact that Z and ψ are bounded on $[0, T]$, we have, for $\varepsilon > 0$ sufficiently small, $Z_\varepsilon(t) \notin \bigcup_{j \neq i} F_j$ for all $t \in [0, T]$. For such $\varepsilon > 0$, we can follow an argument analogous to the one above to obtain that

$$(7.18) \qquad Y_\varepsilon(t) = \langle Y_\varepsilon(t), n_i \rangle d_i, \qquad t \in [0, T],$$

and, with f and g defined as above,

$$(7.19) \qquad \langle Z_\varepsilon(t), n_i \rangle - c_i = \Gamma_1(f + \varepsilon g)(t), \qquad t \in [0, T].$$

Therefore, by Proposition 6.1, for $t \in [0, T)$,

$$(7.20) \qquad \langle \nabla_\psi \bar{\Gamma}(X)(t), n_i \rangle = \nabla_g \Gamma_1(f)(t),$$

which along with condition 1 of the ESP implies

$$(7.21) \qquad \lim_{\varepsilon \downarrow 0} \frac{\langle Y_\varepsilon(t), n_i \rangle - \langle Y(t), n_i \rangle}{\varepsilon} = F(f, g)(t),$$

where $F(f, g)$ is the function defined in (6.1). By condition 1 of the ESP, (7.16), (7.18) and (7.21), for all $t \in [0, T)$,

$$(7.22) \qquad \nabla_\psi \bar{\Gamma}(X)(t) = \psi(t) + \lim_{\varepsilon \downarrow 0} \frac{Y_\varepsilon(t) - Y(t)}{\varepsilon} = \psi(t) + F(f, g)(t) d_i.$$

By Lemma 6.3, $F(f, g)$ lies in $\mathbb{D}_{l,r}([0, T) : \mathbb{R})$, and so, ψ being continuous, $\nabla_\psi \bar{\Gamma}(X)$ also lies in $\mathbb{D}_{l,r}([0, T) : \mathbb{R}^J)$. This proves part 1 of Statement 7.1.

The fact that $\nabla_\psi \bar{\Gamma}(X)(0)$ exists and is equal to $\nabla_{\psi(0)} \pi(X(0))$ follows from Theorem 2.12 and Lemma 3.9. Now suppose $t \in (0, T)$ is a discontinuity point of $\nabla_\psi \bar{\Gamma}(X)$. Then (7.22) and the continuity of ψ imply that $F(f, g)$ is discontinuous. By part (i) of Lemma 6.3 and (7.17), we have $\langle Z(t), n_i \rangle = c_i$, and since $t < \theta_2$, $Z(t) \in F_i \cap \mathcal{S}$. By (3.9), $G_{Z(t)} = \{x \in \mathbb{R}^J : \langle x, n_i \rangle \geq 0\}$. By (7.20) and Proposition 6.2, $\langle \nabla_\psi \bar{\Gamma}(X)(\cdot), n_i \rangle$ is left continuous at t if and only if $\langle \nabla_\psi \bar{\Gamma}(X)(t-), n_i \rangle \geq 0$, or equivalently, $\nabla_\psi \bar{\Gamma}(X)(t-) \in G_{Z(t)}$. This proves part 2 of Statement 7.1.

Proceeding, suppose (Z, Y) satisfies condition 1 of the boundary jitter property. Then, using (7.17), it is readily verified that $(\Gamma_1(f), \Gamma_1(f) - f)$ satisfies condition 1 of the (one-dimensional) boundary jitter property on $[0, T)$. Define $\hat{\phi} \in \mathbb{D}_r([0, T) : \mathbb{R}^J)$ by $\hat{\phi}(t) \doteq \nabla_\psi \bar{\Gamma}(X)(t+)$ for $t \in [0, T)$. We show that $(\hat{\phi}, \hat{\phi} - \psi)$ solves the DP for ψ on $[0, T)$. Condition 1 of the DP holds automatically. By (7.20), for $t \in [0, T)$,

$$(7.23) \qquad h(t) \doteq \langle \hat{\phi}(t), n_i \rangle = \nabla_g \Gamma_1(f)(t+).$$

Then by Proposition 6.2, $(h, h - g)$ solves the (one-dimensional) DP along $\Gamma_1(f)$ for g on $[0, T)$. Let $t \in [0, T)$. If $Z(t) \in G^\circ$, then $H_{Z(t)} = \mathbb{R}^J$ so $\hat{\phi}(t) \in H_{Z(t)}$ clearly holds. Alternatively, if $Z(t) \in F_i$, then by (3.3) and (7.17), $H_{Z(t)} = \{y \in \mathbb{R}^J : \langle y, n_i \rangle = 0\}$ and $\Gamma_1(f)(t) = 0$. This combined with condition 2 of the (one-dimensional) DP, implies that $h(t) = 0$; or equivalently, by (7.23), $\hat{\phi}(t) \in H_{Z(t)}$. This proves that $\hat{\phi}$ satisfies condition 2 of the DP. Now let $0 \leq s < t < T$. By the definition of $\hat{\phi}$, (7.22) and the continuity of ψ,

$$(7.24) \qquad \begin{aligned} \hat{\phi}(t) - \psi(t) - (\hat{\phi}(s) - \psi(s)) &= (F(f, g)(t+) - F(f, g)(s+))d_i \\ &\in \operatorname{span}(d_i). \end{aligned}$$

By (3.3) and the fact that $\mathcal{I}(Z(u)) \subseteq \{i\}$ for all $u \in (s, t]$, we have

$$\operatorname{span}\left[\bigcup_{u \in (s,t]} d(Z(u)) \right] = \begin{cases} \operatorname{span}(d_i) & \text{if } \bigcup_{u \in (s,t]} \mathcal{I}(Z(u)) = \{i\}, \\ \{0\} & \text{if } \bigcup_{u \in (s,t]} \mathcal{I}(Z(u)) = \varnothing. \end{cases}$$

In view of (7.24), we are left to show that if $\bigcup_{u \in (s,t]} \mathcal{I}(Z(u)) = \varnothing$, or equivalently, $Z(u) \in G^\circ$ for all $u \in (s, t]$, then $F(f, g)(t+) - F(f, g)(s+) = 0$. Suppose $Z(u) \in G^\circ$ for all $u \in (s, t]$. By (7.17), $\Gamma_1(f)(u) > 0$ for all $u \in (s, t]$. Then, by (7.23), (6.3) and because $(h, h - g)$ satisfies condition 3 of the (one-dimensional) DP along $\Gamma_1(f)$ for g,

$$F(f, g)(t+) - F(f, g)(s+) = h(t) - g(t) - (h(s) - g(s)) = 0.$$

This completes the proof that $(\hat{\phi}, \hat{\phi} - \psi)$ solves the DP along Z for ψ. Thus, part 3 of Statement 7.1 holds. $\square$

In the following lemma, we establish the induction step. The proof relies on performing a certain time shift and then applying Lemma 7.4.

LEMMA 7.5. *Given a solution* (Z, Y) *to the ESP for* $X \in \mathbb{C}_G$, *define* $\{t_k\}_{k=1,\ldots,K}$, $K \in \mathbb{N}_\infty$, *as in* (7.14)–(7.15). *Let* $2 \le k < K$. *Assume that Statement 7.1 holds with* $T = t_k$. *Then Statement 7.1 holds with* $T = t_{k+1}$.

PROOF. Fix $2 \le k < K$ and $\psi \in \mathbb{C}$. By the continuity of Z, the upper semi-continuity of $\mathcal{I}(\cdot)$ (Lemma 2.1) and (7.14), we can choose $S \in [0, t_k)$ such that $Z(t) \in G^\circ$ for all $t \in [S, t_k)$. Define X^S, Z^S, $\hat{\psi}^S$ as in (2.7)–(2.8) and (7.6). Define θ_2^S, t_1^S, t_2^S as in (7.1), (7.13)–(7.14), but with Z^S, θ_2^S, t_1^S, t_2^S in place of Z, θ_2, t_1, t_2, respectively. Since $Z(t) \in G^\circ$ for all $t \in [S, t_k)$ and $S < \theta_2$, we have $t_1^S = t_k - S$ and $t_2^S = t_{k+1} - S$. By the time-shift property of the ESP, $Z^S = \bar{\Gamma}(X^S)$. Thus Lemma 7.4 implies that Statement 7.1 holds with X^S, Z^S, $\hat{\psi}^S$, t_2^S in place of X, Z, ψ, T, respectively. Therefore, by Lemma 7.3 (with t_k and t_{k+1} in place of T and U, resp.), Statement 7.1 holds with $T = t_{k+1}$. □

PROOF OF PROPOSITION 7.2. By Lemma 7.4, Lemma 7.5 and the principle of mathematical induction, for $k = 1, \ldots, K$, Statement 7.1 holds with $T = t_k$. Since either $K < \infty$ and $t_K = \theta_2$, or $K = \infty$ and $t_k \to \theta_2$ as $k \to \infty$, Statement 7.1 holds with $T = \theta_2$. □

8. Derivative projection operators. In Section 9.1, we show that when $X \in \mathbb{C}_G$ satisfies the boundary jitter property, in order to characterize directional derivatives of the ESM at X, it suffices to only consider perturbations ψ that are constant in neighborhoods of times that Z lies in $\mathcal{N}$, the nonsmooth part of the boundary (see Lemma 9.1 and the proof of Theorem 3.11). The study of directional derivatives for such perturbations ψ is largely governed by properties of a family of (oblique) projection operators, which we introduce in this section.

Fix an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ satisfying Assumption 2.8. *For the remainder of this section*, *fix a compact*, *convex*, *symmetric set* B *with* $0 \in B^\circ$ *satisfying* (2.10) *as in Assumption* 2.8. A useful interpretation of B is in terms of an associated norm on $\mathbb{R}^J$, denoted $\|\cdot\|_B$, defined as follows:

$$(8.1) \qquad \|y\|_B \doteq \min\{r \ge 0 : y \in rB\}, \qquad y \in \mathbb{R}^J.$$

For more on this norm, as well as an in-depth discussion of the set B, see [19], Section 2. We will write $(\mathbb{R}^J, \|\cdot\|_B)$ to denote $\mathbb{R}^J$ equipped with the norm $\|\cdot\|_B$.

Recall that $\mathcal{W} \doteq \{x \in \mathcal{N} : \text{span}(H_x \cup d(x)) \ne \mathbb{R}^J\}$. In the following lemma, we identify a certain decomposition of $\mathbb{R}^J$ that is associated with each $x \in \partial G \setminus \mathcal{W}$.

LEMMA 8.1. *For each* $x \in \partial G$, $H_x \cap \text{span}[d(x)] = \{0\}$. *In addition*, *if* $x \in \partial G \setminus \mathcal{W}$, *then for each* $y \in \mathbb{R}^J$, *there exists a unique pair of vectors* $v_y \in H_x$ *and* $w_y \in \text{span}[d(x)]$ *such that* $y = v_y + w_y$.

PROOF. Suppose $H_x \neq \{0\}$ and $z \in H_x \setminus \{0\}$, so that $\langle z, n_i \rangle = 0$ for $i \in \mathcal{I}(x)$. Since $0 \in B^\circ$, there exist $r > 0$ such that $z \in \partial(rB)$ and $\nu \in \nu_{rB}(z)$ such that $\langle z, \nu \rangle < 0$. Then (2.10) and Remark 2.9 imply that $\langle \nu, d_i \rangle = 0$ for all $i \in \mathcal{I}(x)$, so $z \notin \text{span}[d(x)]$. Thus, $H_x \cap \text{span}[d(x)] = \{0\}$. The last assertion of the lemma follows because $x \in \partial G \setminus \mathcal{W}$ by assumption, so $\text{span}(H_x \cup d(x)) = \mathbb{R}^J$. $\square$

For the next lemma, given a linear subspace A of $\mathbb{R}^J$, let $\dim(A)$ denote the dimension of A.

LEMMA 8.2. *Suppose that for each $x \in \partial G$, $\{d_i, i \in \mathcal{I}(x)\}$ is a set of linearly independent set vectors. Then $\mathcal{W} = \varnothing$.*

PROOF. Let $x \in \mathcal{N}$ and $K \doteq |\mathcal{I}(x)|$. Since H_x, defined in (3.3), is equal to the intersection of K $(J-1)$-dimensional hyperplanes, it follows that $\dim(H_x) \geq J - K$. Since $\{d_i, i \in \mathcal{I}(x)\}$ are linearly independent, it follows from (2.4) that $\dim(\text{span}[d(x)]) = K$. By Lemma 8.1, $H_x \cap \text{span}[d(x)] = \{0\}$, which implies that $\dim(\text{span}[H_x \cup d(x)]) = \dim(H_x) + \dim(\text{span}[d(x)]) \geq J$, so $x \notin \mathcal{W}$. $\square$

8.1. *Derivative projection operator and its adjoint operator.* In the following lemma, we associate with each $x \in \partial G \setminus \mathcal{W}$ a linear operator $\mathcal{L}_x$, which we refer to as the *derivative projection operator at x*. The operator $\mathcal{L}_x$ projects points in $\mathbb{R}^J$ to the linear subspace H_x, defined in (3.3), along a direction that lies in the span of $d(x)$, defined in (2.4).

LEMMA 8.3. *For each $x \in \partial G \setminus \mathcal{W}$, there exists a unique operator*

$$(8.2) \qquad \mathcal{L}_x : (\mathbb{R}^J, \|\cdot\|_B) \mapsto (\mathbb{R}^J, \|\cdot\|_B)$$

such that for each $y \in \mathbb{R}^J$,

$$(8.3) \qquad \mathcal{L}_x y \in H_x \quad and \quad \mathcal{L}_x y - y \in \text{span}[d(x)].$$

Furthermore, $\mathcal{L}_x$ is linear and its operator norm, denoted $\|\mathcal{L}_x\|$, satisfies

$$(8.4) \qquad \|\mathcal{L}_x\| \doteq \sup_{y \neq 0} \frac{\|\mathcal{L}_x y\|_B}{\|y\|_B} \leq 1.$$

In other words, the derivative projection operator $\mathcal{L}_x$ is a contraction on $(\mathbb{R}^J, \|\cdot\|_B)$ that maps B into $B \cap H_x$.

REMARK 8.4. The derivative projection operator $\mathcal{L}_x$ is characterized by H_x and $d(x)$, which depend on the sets $\{n_i, i \in \mathcal{I}(x)\}$ and $\{d_i, i \in \mathcal{I}(x)\}$, respectively. Therefore, given $x, \tilde{x} \in \partial G \setminus \mathcal{W}$ such that $\mathcal{I}(x) = \mathcal{I}(\tilde{x})$, the projection operators $\mathcal{L}_x$ and $\mathcal{L}_{\tilde{x}}$ are equal.

PROOF OF LEMMA 8.3. Fix $x \in \partial G \setminus \mathcal{W}$. Given $y \in \mathbb{R}^J$, Lemma 8.1 implies that there exist unique $v_y \in H_x$ and $w_y \in \text{span}[d(x)]$ such that $y = v_y + w_y$. Set $\mathcal{L}_x y \doteq v_y$ for all $y \in \mathbb{R}^J$. The uniqueness of the decomposition $y = v_y + w_y$ implies that $\mathcal{L}_x$ is well defined on $\mathbb{R}^J$. This proves (8.3). To verify that $\mathcal{L}_x$ is linear, let $\alpha, \beta \in \mathbb{R}$ and $y, z \in \mathbb{R}^J$. Since H_x and $\text{span}[d(x)]$ are linear subspaces of $\mathbb{R}^J$, we have

$$\alpha \mathcal{L}_x y + \beta \mathcal{L}_x z \in H_x \quad \text{and} \quad \alpha(\mathcal{L}_x y - y) + \beta(\mathcal{L}_x z - z) \in \text{span}[d(x)].$$

The uniqueness of $\mathcal{L}_x$ then establishes the linear relation $\mathcal{L}_x(\alpha y + \beta z) = \alpha \mathcal{L}_x y + \beta \mathcal{L}_x z$.

We now prove that $\|\mathcal{L}_x\| \leq 1$. By the linearity of $\mathcal{L}_x$, it suffices to show that given $y \in \partial B$, $\mathcal{L}_x y$ lies in B. Fix $y \in \partial B$ and set

$$(8.5) \qquad c \doteq \|\mathcal{L}_x y\|_B = \min\{r \geq 0 : \mathcal{L}_x y \in rB\}.$$

We need to show that $c \leq 1$. If $c = 0$, then this inequality is automatically satisfied. For the remainder of the proof, assume that $c > 0$. We claim, and prove below, that the set

$$(8.6) \qquad \left\{ \mathcal{L}_x y + \sum_{i \in \mathcal{I}(x)} r_i d_i, r_i \in \mathbb{R} \right\}$$

lies in a supporting hyperplane to cB at $\mathcal{L}_x y$. Since (8.3) implies y lies in the set (8.6), y lies in a supporting hyperplane to cB. In particular, $y \notin (cB)^\circ$ and so $\|y\|_B \geq c$.

To prove the claim, first note that by (8.3) and (8.5),

$$(8.7) \qquad \mathcal{L}_x y \in H_x \cap \partial(cB).$$

Thus, the definition (3.3) of H_x implies that $\langle \mathcal{L}_x y, n_i \rangle = 0$ for all $i \in \mathcal{I}(x)$. Since (2.10) holds for the set cB, we have

$$(8.8) \qquad \langle d_i, \nu \rangle = 0 \qquad \text{for all } \nu \in \nu_{cB}(\mathcal{L}_x y), i \in \mathcal{I}(x).$$

For a proof by contradiction, suppose the set (8.6) does not lie in a supporting hyperplane to cB at $\mathcal{L}_x y$. Since the set (8.6) is an affine subspace and cB is a convex set, there exist $r_i \in \mathbb{R}$, $i \in \mathcal{I}(x)$, such that

$$\mathcal{L}_x y + \sum_{i \in \mathcal{I}(x)} r_i d_i \in (cB)^\circ.$$

The above display and (8.7) together imply that $\sum_{i \in \mathcal{I}(x)} r_i \langle d_i, \nu \rangle > 0$ for some inward normal $\nu \in \nu_{cB}(\mathcal{L}_x y)$, which contradicts (8.8). Therefore, the claim must hold. $\square$

Consider the dual closed convex set B^*, introduced in this context in [19], defined as

$$(8.9) \qquad B^* \doteq \left\{ y \in \mathbb{R}^J : \sup_{z \in B} \langle y, z \rangle \leq 1 \right\}.$$

Then B^* is compact, convex, symmetric with $0 \in (B^*)^\circ$ (see, e.g., [19], Section 3.2), so, analogous to (8.1), B^* defines a norm $\|\cdot\|_{B^*}$ on $\mathbb{R}^J$ as follows:

$$(8.10) \qquad \|y\|_{B^*} \doteq \min\{r \geq 0 : y \in rB^*\}, \qquad y \in \mathbb{R}^J.$$

Let $\mathcal{L}_x^*$ denote the linear operator that is adjoint to the derivative projection operator $\mathcal{L}_x$ defined in Lemma 8.3; that is, $\langle \mathcal{L}_x y, z \rangle = \langle y, \mathcal{L}_x^* z \rangle$ for every $y, z \in \mathbb{R}^J$. In the lemma below, we derive some important properties of $\mathcal{L}_x^*$.

LEMMA 8.5. *For each $x \in \partial G \setminus \mathcal{W}$,*

$$(8.11) \qquad \mathcal{L}_x^* : (\mathbb{R}^J, \|\cdot\|_{B^*}) \mapsto (\mathbb{R}^J, \|\cdot\|_{B^*})$$

is the unique linear operator such that for each $y \in \mathbb{R}^J$,

$$(8.12) \qquad \mathcal{L}_x^* y \in \mathrm{span}[d(x)]^\perp \quad and \quad \mathcal{L}_x^* y - y \in H_x^\perp.$$

Furthermore, the operator norm of $\mathcal{L}_x^*$, denoted $\|\mathcal{L}_x^*\|$, satisfies

$$(8.13) \qquad \|\mathcal{L}_x^*\| \doteq \sup_{y \neq 0} \frac{\|\mathcal{L}_x^* y\|_B}{\|y\|_B} \leq 1.$$

In other words, the adjoint derivative projection operator $\mathcal{L}_x^*$ is a contraction on $(\mathbb{R}^J, \|\cdot\|_{B^*})$ that maps B^* into $B^* \cap \mathrm{span}[d(x)]^\perp$.

PROOF.   Fix $x \in \partial G \setminus \mathcal{W}$. Since $d_i \in \mathrm{span}[d(x)]$ for each $i \in \mathcal{I}(x)$ and the derivative projection operator at x is uniquely defined, (8.3) implies $\mathcal{L}_x d_i = 0$ for each $i \in \mathcal{I}(x)$. Thus, for all $y \in \mathbb{R}^J$ and $i \in \mathcal{I}(x)$,

$$\langle \mathcal{L}_x^* y, d_i \rangle = \langle y, \mathcal{L}_x d_i \rangle = 0,$$

which implies that $\mathcal{L}_x^* y \in \mathrm{span}[d(x)]^\perp$. Similarly, by (8.3) and the uniqueness of $\mathcal{L}_x$, $\mathcal{L}_x z = z$ for all $z \in H_x$. Hence, given $y \in \mathbb{R}^J$ and $z \in H_x$, we have

$$\langle \mathcal{L}_x^* y - y, z \rangle = \langle y, \mathcal{L}_x z - z \rangle = 0.$$

Since this holds for all $y \in \mathbb{R}^J$ and $z \in H_x$, it follows that $\mathcal{L}_x^* y - y \in H_x^\perp$.

We now establish that $\mathcal{L}_x^*$ is the unique linear operator satisfying (8.12). Suppose $A : \mathbb{R}^J \mapsto \mathbb{R}^J$ is a linear operator that satisfies (8.12) for all $y \in \mathbb{R}^J$, but with A in place of $\mathcal{L}_x^*$. Then one can readily verify that the adjoint of A, denoted A^*, satisfies (8.3) for all $y \in \mathbb{R}^J$, but with A^* in place of $\mathcal{L}_x$. By the uniqueness of $\mathcal{L}_x$ shown in Lemma 8.3, it must hold that $A^* = \mathcal{L}_x$. Since the adjoint of $\mathcal{L}_x$ is uniquely defined, this implies that $A = \mathcal{L}_x^*$. Therefore, $\mathcal{L}_x^*$ is well defined.

We are left to show that $\|\mathcal{L}_x^*\|_{B^*} \leq 1$. By the linearity of $\mathcal{L}_x^*$, it suffices to show that $\mathcal{L}_x^* y \in B^*$ for all $y \in \partial B^*$. Let $y \in \partial B^*$. Then

$$\sup_{z \in B} \langle \mathcal{L}_x^* y, z \rangle = \sup_{z \in B} \langle y, \mathcal{L}_x z \rangle \leq \sup_{z \in B} \langle y, z \rangle = 1,$$

where the inequality follows since $\mathcal{L}_x$ maps B into B, as shown in Lemma 8.3, and the last equality follows from (8.9) and because $y \in \partial B^*$. Again recalling (8.9), we see that $\mathcal{L}_x^* y \in B^*$. Since $x \in \partial G \setminus \mathcal{W}$ was arbitrary, the proof is complete. $\square$

8.2. *Contraction properties of derivative projection operators.* We establish four lemmas that characterize the contraction properties of $\mathcal{L}_x$ and $\mathcal{L}_x^*$.

LEMMA 8.6. *For each $x \in \partial G \setminus \mathcal{W}$ and $y \in \mathbb{R}^J$,*

$$(8.14) \qquad \mathcal{L}_x^* y = y \qquad \text{if } y \in \text{span}[d(x)]^\perp$$

and

$$(8.15) \qquad \|\mathcal{L}_x^* y\|_{B^*} < \|y\|_{B^*} \qquad \text{if } y \notin \text{span}[d(x)]^\perp.$$

PROOF. Fix $x \in \partial G \setminus \mathcal{W}$. By the linearity of $\mathcal{L}_x^*$, the fact that $(B^*)^\circ$ is nonempty and the definition of $\|\cdot\|_{B^*}$, it suffices to show that (8.14) and (8.15) hold for all $y \in \partial B^*$. Let $y \in \partial B^*$. Suppose $y \in \text{span}[d(x)]^\perp$. By (8.12) and the uniqueness of $\mathcal{L}_x^*$ shown in Lemma 8.5, $\mathcal{L}_x^* y = y$, so (8.14) is satisfied. On the other hand, suppose $y \notin \text{span}[d(x)]^\perp$. We show that $\mathcal{L}_x^* y \in (B^*)^\circ$, so (8.15) holds. For a proof by contradiction, suppose that $\mathcal{L}_x^* y \in \partial B^*$. Then, in view of (8.9) and using the compactness of B, there exists $\tilde{z} \in \partial B$ such that

$$\langle \mathcal{L}_x \tilde{z}, y \rangle = \langle \tilde{z}, \mathcal{L}_x^* y \rangle = 1.$$

By Lemma 8.3, $\mathcal{L}_x \tilde{z} \in B$. This, together with the above display, the fact that $y \in \partial B^*$ and (8.9), implies that $\mathcal{L}_x \tilde{z} \in \partial B$. Furthermore, given $z \in B$,

$$\langle -y, \mathcal{L}_x \tilde{z} - z \rangle = -1 + \langle y, z \rangle \le 0,$$

so $-y \in \nu_B(\mathcal{L}_x \tilde{z})$. Since $\langle \mathcal{L}_x \tilde{z}, n_i \rangle = 0$ for all $i \in \mathcal{I}(x)$, $\mathcal{L}_x \tilde{z} \in \partial B$ and $-y \in \nu_B(\mathcal{L}_x \tilde{z})$, (2.10) implies that $\langle -y, d_i \rangle = 0$ for all $i \in \mathcal{I}(x)$. However, this contradicts the fact that $y \notin \text{span}[d(x)]^\perp$, so (8.15) must hold. $\square$

LEMMA 8.7. *There exists $\delta \in [0, 1)$ such that given $x \in \partial G \setminus \mathcal{W}$ and a finite sequence $\{x_k\}_{k=1,\dots,K}$ in $\partial G \setminus \mathcal{W}$ such that*

$$(8.16) \qquad \mathcal{I}(x) = \bigcup_{k=1,\dots,K} \mathcal{I}(x_k),$$

the following inequality holds for all $y \in H_x^\perp$:

$$(8.17) \qquad \|\mathcal{L}_{x_1}^* \cdots \mathcal{L}_{x_K}^* y\|_{B^*} \le \delta \|y\|_{B^*}.$$

Consequently, given $x \in \partial G \setminus \mathcal{W}$ and a sequence $\{x_k\}_{k \in \mathbb{N}}$ in $\partial G \setminus \mathcal{W}$ such that $\mathcal{I}(x) = \bigcup_{k \ge \ell} \mathcal{I}(x_k)$ for all $\ell \in \mathbb{N}$, it follows that for any $y \in H_x^\perp$,

$$(8.18) \qquad \lim_{K \to \infty} [\mathcal{L}_{x_1}^* \cdots \mathcal{L}_{x_K}^* y] = 0.$$

PROOF. We claim, and prove below, that for each $x \in \partial G \setminus \mathcal{W}$, there exists $\delta_x \in [0, 1)$, depending only on the set $\mathcal{I}(x)$, such that given $K \in \mathbb{N}$ and a finite sequence $\{x_k\}_{k=1,\ldots,K}$ in $\partial G \setminus \mathcal{W}$ satisfying (8.16), then for all $y \in H_x^{\perp}$,

$$(8.19) \qquad \|\mathcal{L}_{x_1}^* \cdots \mathcal{L}_{x_K}^* y\|_{B^*} \leq \delta_x \|y\|_{B^*}.$$

Then, due to the fact that δ_x depends only on $\mathcal{I}(x)$ and there are only a finite number of distinct subsets of $\mathcal{I}$, we have $\delta \doteq \sup_{x \in \partial G \setminus \mathcal{W}} \delta_x \in [0, 1)$, which will complete the proof of (8.17).

We are left to establish the claim. Fix $x \in \partial G \setminus \mathcal{W}$. Let $y \in H_x^{\perp} \cap \partial B^*$. The property $\mathrm{span}(H_x \cup d(x)) = \mathbb{R}^J$ implies that $H_x^{\perp} \cap \mathrm{span}[d(x)]^{\perp} = \{0\}$, so $y \notin \mathrm{span}[d(x)]^{\perp}$. By Lemma 8.6, given $\tilde{x} \in \partial G \setminus \mathcal{W}$ such that $y \notin \mathrm{span}[d(\tilde{x})]^{\perp}$, we have $\|\mathcal{L}_{\tilde{x}}^* y\|_{B^*} < 1$. Define

$$(8.20) \qquad \delta_{x,y} \doteq \sup\{\|\mathcal{L}_{\tilde{x}}^* y\|_{B^*} : \tilde{x} \in \partial G \setminus \mathcal{W}, \mathcal{I}(\tilde{x}) \subseteq \mathcal{I}(x), y \notin \mathrm{span}[d(\tilde{x})]^{\perp}\}.$$

Since $\mathcal{L}_{\tilde{x}}^*$ depends only on $\mathcal{I}(\tilde{x})$ and there are only a finite number of distinct subsets of $\mathcal{I}(x)$, the above supremum is in fact over a finite number of elements, so $\delta_{x,y} \in [0, 1)$. In addition, $\delta_{x,y}$ depends only on $\mathcal{I}(x)$ and y. Now let $K \in \mathbb{N}$ and $\{x_k\}_{k=1,\ldots,K}$ be a finite sequence in $\partial G \setminus \mathcal{W}$ such that (8.16) holds. Observe that (8.16) implies $\mathrm{span}[d(x)]^{\perp} = \bigcap_{k=1,\ldots,K} \mathrm{span}[d(x_k)]^{\perp}$, and since $y \notin \mathrm{span}[d(x)]^{\perp}$, this ensures that $y \notin \mathrm{span}[d(x_k)]^{\perp}$ for some $k \in \{1, \ldots, K\}$. Let $\tilde{k} \doteq \max\{1 \leq k \leq K : y \notin \mathrm{span}[d(x_k)]^{\perp}\}$. Then by the nonexpansive property of the operators $\mathcal{L}_{x_k}^*$ shown in Lemma 8.5, Lemma 8.6 and (8.20),

$$\|\mathcal{L}_{x_1}^* \cdots \mathcal{L}_{x_K}^* y\|_{B^*} \leq \|\mathcal{L}_{x_{\tilde{k}}}^* \cdots \mathcal{L}_{x_K}^* y\|_{B^*} = \|\mathcal{L}_{x_{\tilde{k}}}^* y\|_{B^*} \leq \delta_{x,y}.$$

Since $y \mapsto \|\mathcal{L}_{x_1}^* \cdots \mathcal{L}_{x_K}^* y\|_{B^*}$ is continuous and $H_x^{\perp} \cap \partial B^*$ is compact, we have

$$\delta_x \doteq \sup_{y \in H_x^{\perp} \cap \partial B^*} \delta_{x,y} \in [0, 1),$$

and δ_x depends only on $\mathcal{I}(x)$. Finally, by the linearity of the operators $\mathcal{L}_{x_k}^*$, $1 \leq k \leq K$, (8.19) holds for any $y \in H_x^{\perp}$. This proves the claim. $\square$

LEMMA 8.8. *Suppose $x \in \partial G \setminus \mathcal{W}$ and $\{x_k\}_{k \in \mathbb{N}}$ is a sequence in $\partial G \setminus \mathcal{W}$ such that*

$$(8.21) \qquad \mathcal{I}(x) = \bigcup_{k \geq \ell} \mathcal{I}(x_k) \qquad \text{for all } \ell \in \mathbb{N}.$$

Then for all $y \in \mathbb{R}^J$,

$$(8.22) \qquad \lim_{k \to \infty} [\mathcal{L}_{x_k} \cdots \mathcal{L}_{x_1} y] = \mathcal{L}_x y$$

and

$$(8.23) \qquad \lim_{k \to \infty} [\mathcal{L}_{x_1} \cdots \mathcal{L}_{x_k} y] = \mathcal{L}_x y.$$

PROOF. Let $y \in \mathbb{R}^J$. We first prove (8.22). By the nonexpansive property of $\mathcal{L}_{x_k}$ stated in Lemma 8.3, there exists a subsequence $\{k_m\}_{m \in \mathbb{N}}$ and $\bar{y} \in \mathbb{R}^J$ such that $\bar{y} = \lim_{m \to \infty}[\mathcal{L}_{x_{k_m}} \cdots \mathcal{L}_{x_1} y]$. Since $\mathcal{L}_x y$ does not depend on the subsequence, it suffices to establish that $\bar{y} = \mathcal{L}_x y$.

For each $k \in \mathbb{N}$ and $\tilde{y} \in \mathbb{R}^J$, (8.3) and (8.21) together imply $\mathcal{L}_{x_k} \tilde{y} - \tilde{y} \in \mathrm{span}[d(x_k)] \subseteq \mathrm{span}[d(x)]$. Consequently,

$$\bar{y} - y = \lim_{m \to \infty}[\mathcal{L}_{x_{k_m}} \cdots \mathcal{L}_{x_1} y] - y$$

$$(8.24) \qquad = \lim_{m \to \infty} \sum_{r=1}^{k_m} \{\mathcal{L}_{x_r}[\mathcal{L}_{x_{r-1}} \cdots \mathcal{L}_{x_1} y] - [\mathcal{L}_{x_{r-1}} \cdots \mathcal{L}_{x_1} y]\}$$

$$\in \mathrm{span}[d(x)],$$

where we adopt the convention $\mathcal{L}_{x_0} \cdots \mathcal{L}_{x_1} y \doteq y$ and we have used the fact that $\mathrm{span}[d(x)]$ is a closed linear subspace. In addition, for any $z \in H_x^{\perp}$,

$$|\langle \bar{y}, z \rangle| = \lim_{m \to \infty}|\langle \mathcal{L}_{x_{k_m}} \cdots \mathcal{L}_{x_1} y, z \rangle| = \lim_{m \to \infty}|\langle y, \mathcal{L}_{x_1}^* \cdots \mathcal{L}_{x_{k_m}}^* z \rangle| = 0,$$

where the final equality is a consequence of (8.18). Therefore, $\bar{y} \in H_x$. Combined with (8.3) and (8.24), this shows that $\bar{y} - \mathcal{L}_x y \in H_x \cap \mathrm{span}[d(x)]$. Since Lemma 8.1 implies $H_x \cap \mathrm{span}[d(x)] = \{0\}$, it follows that $\bar{y} = \mathcal{L}_x y$, which completes the proof of (8.22). The proof of (8.23) is analogous, so we omit it. $\square$

The next result is a corollary to the previous lemma and states that the convergence described above holds uniformly for y in compact subsets of $\mathbb{R}^J$.

COROLLARY 8.9. *Suppose $x \in \partial G \setminus \mathcal{W}$ and $\{x_k\}_{k \in \mathbb{N}}$ is a sequence in $\partial G \setminus \mathcal{W}$ such that (8.21) holds. Then given any compact set $C \subseteq \mathbb{R}^J$,*

$$(8.25) \qquad \lim_{k \to \infty} \sup_{y \in C}|[\mathcal{L}_{x_k} \cdots \mathcal{L}_{x_1} y] - \mathcal{L}_x y| = 0$$

and

$$(8.26) \qquad \lim_{k \to \infty} \sup_{y \in C}|[\mathcal{L}_{x_1} \cdots \mathcal{L}_{x_k} y] - \mathcal{L}_x y| = 0.$$

PROOF. Fix a compact subset $C \subseteq \mathbb{R}^J$. We prove that (8.25) holds, with the proof of (8.26) being analogous. It suffices to show that given any sequence $\{y_k\}_{k \in \mathbb{N}}$ in C,

$$(8.27) \qquad \limsup_{k \to \infty}|[\mathcal{L}_{x_1} \cdots \mathcal{L}_{x_k} y_k] - \mathcal{L}_x y_k| = 0.$$

Fix a sequence $\{y_k\}_{k \in \mathbb{N}}$ in C. By compactness, we can assume, by possibly taking a subsequence, that $\hat{y} \doteq \lim_{k \to \infty} y_k$ exists in C. Then, by the triangle inequality,

$$\|[\mathcal{L}_{x_k} \cdots \mathcal{L}_{x_1} y_k] - \mathcal{L}_x y_k\|_B \leq \|\mathcal{L}_{x_k} \cdots \mathcal{L}_{x_1}(y_k - \hat{y})\|_B$$
$$+ \|[\mathcal{L}_{x_k} \cdots \mathcal{L}_{x_1} \hat{y}] - \mathcal{L}_x \hat{y}\|_B$$
$$+ \|\mathcal{L}_x(\hat{y} - y_k)\|_B.$$

Letting $k \to \infty$ in the above display and using the nonexpansive property of $\mathcal{L}_{x_k} \cdots \mathcal{L}_{x_1}$, the definition of $\hat{y}$ and (8.22) of Lemma 8.8, which is applicable since (8.21) holds, we see that each term on the right-hand side converges to zero. Since the convergence holds for any given subsequence, (8.27) holds. $\square$

8.3. *Relation to solutions to the DP.* The main result of this section relates derivative projection operators to solutions to the DP along Z at times $T \in (0, \infty)$ such that $Z(T) \in \mathcal{N}$, provided that Z satisfies condition 3 of the boundary jitter property.

LEMMA 8.10.   *Let (Z, Y) be a solution to the ESP for $X \in \mathbb{C}_G$ and define τ as in* (3.6). *Suppose Z satisfies condition 3 of the boundary jitter property on $[0, \tau)$ and $0 \le S < T < \tau$ are such that $Z(S) \in G^\circ$, $Z(T) \in \mathcal{N}$ and $\mathcal{I}(Z(t)) \subsetneq \mathcal{I}(Z(T))$ for all $t \in [S, T]$. Suppose $\psi \in \mathbb{C}$ is constant on $[S, T]$ and (ϕ, η) solves the DP along Z for ψ on $[0, T]$. Then $\phi(T-)$ exists and*

$$(8.28) \qquad\qquad \phi(T-) = \mathcal{L}_{Z(T)}\phi(S).$$

PROOF.   By Lemma 4.1, there is a nested sequence $S = \xi_0 < s_1 \le \xi_1 < \cdots < s_j \le \xi_j < \cdots < T$ such that $\xi_j \to T$ as $j \to \infty$ and for each $j \in \mathbb{N}$, $Z(\xi_j) \in \partial G$ and (4.2) holds. We first prove that for each $j \in \mathbb{N}$,

$$(8.29) \qquad\qquad \phi(\xi_j) = \mathcal{L}_{Z(\xi_j)}\phi(\xi_{j-1}).$$

See Figure 2(b) for an illustration of ϕ when Z satisfies condition 3 of the boundary jitter property. Fix $j \in \mathbb{N}$. By condition 1 of the DP and the fact that ψ is constant on $[S, T]$, we have

$$(8.30) \qquad \phi(t) = \phi(\xi_{j-1}) + \eta(t) - \eta(\xi_{j-1}), \qquad t \in [\xi_{j-1}, s_j),$$

(a) A constrained path Z.   (b) A path $\phi = \Lambda_Z(\psi)$.

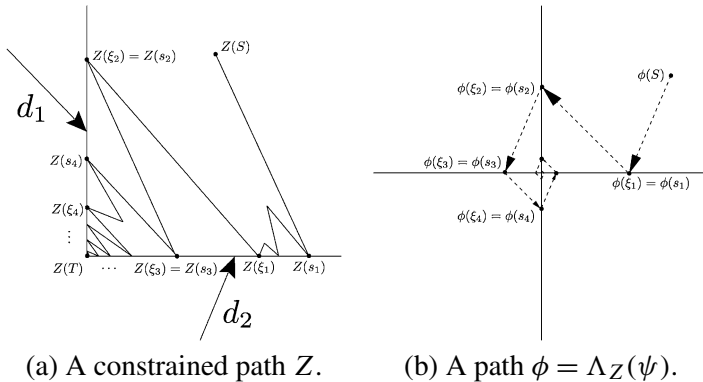FIG. 2.   (a) *A path Z that satisfies condition 3 of the boundary jitter property on the interval $[S, T]$ and $Z(T) = 0$. In both figures the times depicted correspond to the nested sequence* (4.1) *defined in Lemma* 4.1. (b) *A path $\phi = \Lambda_Z(\psi)$, where Λ_Z is the DM and ψ is constant on $[S, T]$. Here ϕ is piecewise constant with jumps at times $s_1 < s_2 < \cdots$.*

and

$$(8.31) \qquad \phi(t) = \phi(s_j-) + \eta(t) - \eta(s_j-), \qquad t \in [s_j, \xi_j].$$

We first show that (ϕ, η) is constant on $[\xi_{j-1}, s_j)$. By (4.2) and condition 2 of the DP, $\phi(\xi_{j-1}) \in H_{Z(\xi_{j-1})} \subseteq H_{Z(t)}$ for all $t \in [\xi_{j-1}, s_j)$. Thus, conditions 1–3 of the DP are satisfied with $\phi(t) = \phi(\xi_{j-1})$ and $\eta(t) = \eta(\xi_{j-1})$ for $t \in [\xi_{j-1}, s_j)$. By uniqueness of solutions to the DP, (ϕ, η) must be constant on $[\xi_{j-1}, s_j)$. Next, by (3.4), (3.5), the fact that η is constant on $[\xi_{j-1}, s_j)$ and (4.2), we have

$$(8.32) \qquad \begin{aligned} \eta(\xi_j) - \eta(\xi_{j-1}) &= \eta(\xi_j) - \eta(s_j-) \\ &\in \text{span}\left[\bigcup_{u \in [s_j, \xi_j]} d(Z(u)) \right] = \text{span}[d(Z(\xi_j))]. \end{aligned}$$

Combined with the fact that ψ is constant on $[\xi_{j-1}, \xi_j]$, this implies $\phi(\xi_j) - \phi(\xi_{j-1}) \in \text{span}[d(Z(\xi_j))]$. Moreover, by condition 2 of the DP, $\phi(\xi_j) \in H_{Z(\xi_j)}$. Relation (8.29) then follows from the characterization (8.3) of $\mathcal{L}_{Z(\xi_j)}$ established in Lemma 8.3.

Iterating the recursion relation (8.29) yields

$$(8.33) \qquad \phi(\xi_j) = [\mathcal{L}_{Z(\xi_j)} \cdots \mathcal{L}_{Z(\xi_1)}]\phi(\xi_0), \qquad j \in \mathbb{N}.$$

By condition 3 of the boundary jitter property and the fact that $\mathcal{I}(Z(t)) \subsetneq \mathcal{I}(Z(T))$ for all $t \in [S, T)$, we have $\bigcup_{t \in [\xi_j, T)} \mathcal{I}(Z(t)) = \mathcal{I}(Z(T))$ for all $j \in \mathbb{N}$. Then by (4.2) and the fact that $\xi_j \to T$ as $j \to \infty$, we have

$$\bigcup_{j \geq \ell} \mathcal{I}(Z(\xi_j)) = \bigcup_{t \in [\xi_\ell, T)} \mathcal{I}(Z(t)) = \mathcal{I}(Z(T)), \qquad \ell \in \mathbb{N}.$$

Since $T < \tau$, Lemma 4.1 implies that $Z(\xi_j) \in \partial G \setminus \mathcal{W}$ for all $j \in \mathbb{N}$. Therefore, Lemma 8.8 with $x = Z(T)$ and $x_k = Z(\xi_k)$ for $k \in \mathbb{N}$, and (8.33) imply that

$$(8.34) \qquad \lim_{j \to \infty} \phi(\xi_j) = \mathcal{L}_{Z(T)}\phi(\xi_0).$$

To see that $\phi(T-) = \mathcal{L}_{Z(T)}\phi(\xi_0)$, it suffices to show that

$$\lim_{j \to \infty} \sup_{u \in [s_j, \xi_j]} |\phi(u) - \phi(\xi_j)| = \lim_{j \to \infty} \sup_{u \in [s_j, s_{j+1})} |\phi(u) - \phi(\xi_j)| = 0,$$

where the first equality follows because ϕ is constant on $[\xi_j, s_{j+1})$.

Fix $j \in \mathbb{N}$ and define $Z^{s_j}, \psi^{s_j}, \phi^{s_j}, \eta^{s_j}$ as in (2.8) and (5.1)–(5.3), but with s_j in place of S, so by Lemma 5.2, (ϕ^{s_j}, η^{s_j}) solves the DP along Z^{s_j} for ψ^{s_j}. By condition 1 of the DP, (4.2), (3.3) and (2.8), $\phi(\xi_j) \in H_{Z(\xi_j)} \subseteq H_{Z(s_j+u)} = H_{Z^{s_j}(u)}$ for all $u \in [0, \xi_j - s_j]$. It is then readily verified that $(\tilde{\phi}, 0)$, where $\tilde{\phi} \equiv \phi(\xi_j)$, solves the DP along Z^{s_j} for $\tilde{\psi} \equiv \phi(\xi_j)$ on $[0, \xi_j - s_j]$. By (5.2), the Lipschitz

continuity of the DM (Theorem 5.4), (5.1) and the fact that ψ is constant on $[S, T]$, we have

$$
(8.35) \quad \sup_{u \in [s_j, \xi_j]} \left| \phi(u) - \phi(\xi_j) \right| = \left\| \phi^{s_j} - \tilde{\phi} \right\|_{\xi_j - s_j} \leq \kappa_\Lambda \left\| \psi^{s_j} - \tilde{\psi} \right\|_{\xi_j - s_j}
$$

$$
= \kappa_\Lambda \left| \phi(s_j) - \phi(\xi_j) \right|.
$$

By condition 2 of the DP, $\phi(s_j) \in H_{Z(s_j)}$. By conditions 1 and 3 of the DP, the facts that ϕ is constant on $[\xi_{j-1}, s_j)$ and ψ is constant on $[S, T]$, (3.5) and (4.2), we have

$$
\phi(s_j) - \phi(\xi_{j-1}) = \phi(s_j) - \phi(s_j-) = \eta(s_j) - \eta(s_j-) \in \text{span}\big[d\big(Z(s_j) \big) \big].
$$

It then follows from the characterization (8.3) of $\mathcal{L}_{Z(s_j)}$ established in Lemma 8.3 that $\phi(s_j) = \mathcal{L}_{Z(s_j)} \phi(\xi_{j-1})$. By condition 2 of the DP, (3.3) and (4.2), $\phi(\xi_j) \in H_{Z(\xi_j)} \subseteq H_{Z(s_j)}$. Again invoking the characterization (8.3) of $\mathcal{L}_{Z(s_j)}$ established in Lemma 8.3, we have $\phi(\xi_j) = \mathcal{L}_{Z(s_j)} \phi(\xi_j)$. Combining these relations with (8.35) yields

$$
\sup_{u \in [s_j, \xi_j]} \left| \phi(u) - \phi(\xi_j) \right| \leq \kappa_\Lambda \left| \mathcal{L}_{Z(s_j)} \big[\phi(\xi_{j-1}) - \phi(\xi_j) \big] \right|.
$$

Taking limits as $j \to \infty$, we see that the right-hand side converges to zero because of (8.34) and the fact that for each $x \in \partial G \setminus \mathcal{W}$, $\mathcal{L}_x$ is a linear operator. This completes the proof. $\quad\square$

9. Directional derivatives: Proof of the main result. In this section, we prove our main result, Theorem 3.11. Fix an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ satisfying Assumptions 2.8 and 2.11.

9.1. *Directional derivatives along a dense subset of paths.* Given a solution (Z, Y) of the ESP for $X \in \mathbb{C}_G$, define τ as in (3.6) and a subset of functions in $\mathbb{C}$ that are constant in neighborhoods of times $t \in [0, \infty)$ that $Z(t)$ lies in $\mathcal{N}$, the nonsmooth part of the boundary. Specifically, for $\delta > 0$, define

$$
(9.1) \qquad \mathbb{C}^{\delta, Z} \doteq \left\{ \psi \in \mathbb{C} : \begin{array}{l} \forall t \in [0, \infty), \, Z(t) \in \mathcal{N} \\ \Rightarrow \quad \psi \text{ is constant on } \big[(t - \delta) \vee 0, t + \delta \big] \end{array} \right\}.
$$

Set

$$
\mathbb{C}^Z \doteq \bigcup_{\delta > 0} \mathbb{C}^{\delta, Z}.
$$

In the next lemma, which is proved in Appendix A, we provide sufficient conditions for $\mathbb{C}^Z$ to be dense in $\mathbb{C}$.

LEMMA 9.1. *Suppose Z satisfies condition 2 of the boundary jitter property (Definition 3.1). Then $\mathbb{C}^Z$ is dense in $\mathbb{C}$.*

We now introduce the following statement, which will be called upon repeatedly throughout this section for different values of T. Recall the definition of G_x, $x \in \mathcal{S}$, given in (3.9).

STATEMENT 9.2. For all $\psi \in \mathbb{C}^{\delta, Z}$:

1. $\nabla_\psi \bar{\Gamma}(X)$ exists on $[0, T)$ and lies in $\mathbb{D}_{l,r}([0, T) : \mathbb{R}^J)$.
2. $\nabla_\psi \bar{\Gamma}(X)(0) = \nabla_{\psi(0)} \pi(X(0))$ and if $t \in (0, T)$ is a discontinuity point of $\nabla_\psi \bar{\Gamma}(X)$, then $Z(t) \in \mathcal{S}$ and $\nabla_\psi \bar{\Gamma}(X)$ is left continuous at t if and only if $\nabla_\psi \bar{\Gamma}(X)(t-) \in G_{Z(t)}$.
3. There exists a unique solution (ϕ, η) to the DP along Z for ψ and $\phi(t) = \nabla_\psi \bar{\Gamma}(X)(t+)$ for $t \in [0, T)$.

We have the following proposition.

PROPOSITION 9.3. *Given $X \in \mathbb{C}_G$, let (Z, Y) denote the solution to the ESP for X and define τ as in* (3.6). *Suppose (Z, Y) satisfies the boundary jitter property on $[0, \tau)$. Then for each $\delta > 0$, Statement 9.2 holds with $T = \tau$.*

The proof of Proposition 9.3 proceeds as follows. Given a solution (Z, Y) of the ESP for $X \in \mathbb{C}_G$, define τ as in (3.6) and θ_2 as in (7.1). If $N \geq 3$, then for each $n = 3, \ldots, N$, let $\theta_n \in [0, \tau]$ be the first time in the interval $[0, \tau)$ that Z reaches the intersection of n or more faces; that is,

$$(9.2) \qquad \theta_n \doteq \inf\{t \in [0, \tau) : |\mathcal{I}(Z(t))| \geq n\} \wedge \tau, \qquad n = 2, \ldots, N.$$

Set $\theta_{N+1} \doteq \tau$. Observe that this definition of θ_2 is consistent with (7.1). Using a proof by induction, we prove the following statement for $n = 2, \ldots, N + 1$.

STATEMENT 9.4. Given $X \in \mathbb{C}_G$, let (Z, Y) denote the solution to the ESP for X and define τ as in (3.6) and $\{\theta_n\}_{n=2,\ldots,N+1}$, as in (9.2). Suppose (Z, Y) satisfies the boundary jitter property (Definition 3.1) on $[0, \tau)$. Then Statement 9.2 holds with $T = \theta_n$.

The base case ($n = 2$) follows from Proposition 7.2. The following lemma states the induction step and is the main challenge in proving Proposition 9.3. The proof of Lemma 9.5 is given in Section 9.3.

LEMMA 9.5. *Fix $\delta > 0$. Let $2 \leq n \leq N$. Assume that Statement 9.4 holds. Then Statement 9.4 holds with θ_{n+1} in place of θ_n.*

We can now prove Proposition 9.3.

PROOF OF PROPOSITION 9.3. Let $\delta > 0$ be arbitrary. By Proposition 7.2, Lemma 9.5 and the principle of mathematical induction, Statement 9.4 holds for $n = 2, \ldots, N + 1$. In particular, since $\theta_{N+1} \doteq \tau$, the proof is complete. $\quad\square$

As we now show, the proof of Theorem 3.11 is then a consequence of Proposition 9.3, Lemma 9.1 and the closure property for the DM (Lemma 5.5).

PROOF OF THEOREM 3.11. Let $\psi \in \mathbb{C}$ be arbitrary. By Lemma 9.1 and the definition of $\mathbb{C}^Z$, there is a Cauchy sequence $\{\psi_k\}_{k \in \mathbb{N}}$ in $\mathbb{C}^Z$ such that $\psi_k \to \psi$ in $\mathbb{C}$ as $k \to \infty$ and for each $k \in \mathbb{N}$, there exists $\delta_k > 0$ such that $\psi_k \in \mathbb{C}^{\delta_k, Z}$. Thus, by Proposition 9.3, for each $k \in \mathbb{N}$:

(i) $\nabla_{\psi_k} \bar{\Gamma}(X)$ exists on $[0, \tau)$ and lies in $\mathbb{D}_{l,r}([0, \tau) : \mathbb{R}^J)$;

(ii) $\nabla_{\psi_k} \bar{\Gamma}(X)(0) = \nabla_{\psi_k(0)} \pi(X(0))$ and if $\nabla_{\psi_k} \bar{\Gamma}(X)$ is discontinuous at $t \in (0, \tau)$, then $Z(t) \in \mathcal{S}$ and $\nabla_{\psi_k} \bar{\Gamma}(X)$ is left continuous at $t \in (0, \tau)$ if and only if $\nabla_{\psi_k} \bar{\Gamma}(X)(t-) \in G_{Z(t)}$;

(iii) there is a unique solution (ϕ_k, η_k) to the DP along Z for ψ_k on $[0, \tau)$ and $\phi_k(t) = \nabla_{\psi_k} \bar{\Gamma}(X)(t+)$ for all $t \in [0, \tau)$.

Let $t \in (0, \tau)$. By (i) and (iii), we have $\nabla_{\psi_k} \bar{\Gamma}(X)(t-) = \phi_k(t-)$, $\nabla_{\psi_k} \bar{\Gamma}(X)(t) \in \{\phi_k(t-), \phi_k(t)\}$, and $\nabla_{\psi_k} \bar{\Gamma}(X)(t) = \phi_k(t)$ if and only if $\nabla_{\psi_k} \bar{\Gamma}(X)$ is right continuous at t. When combined with (ii) and the definition of the functional Θ_Z given in (3.10), this shows that $\nabla_{\psi_k} \bar{\Gamma}(X)(t) = \Theta_Z(\phi_k)(t)$.

Since $\{\psi_k\}_{k \in \mathbb{N}}$ is Cauchy in $\mathbb{C}$ and the DM is Lipschitz continuous (Theorem 5.4), there exists $(\phi, \eta) \in \mathbb{D}_r([0, \tau) : \mathbb{R}^J) \times \mathbb{D}_r([0, \tau) : \mathbb{R}^J)$ such that $(\phi_k, \eta_k) \to (\phi, \eta)$ in $\mathbb{D}_r([0, \tau) : \mathbb{R}^J) \times \mathbb{D}_r([0, \tau) : \mathbb{R}^J)$ as $k \to \infty$. The closure property of the DM (Lemma 5.5) then shows that (ϕ, η) is the unique solution to the DP along Z for ψ, so part 2 of Theorem 3.11 holds.

Let $t \in (0, \tau)$. We show that for all $k \in \mathbb{N}$ sufficiently large,

$$(9.3) \qquad |\Theta_Z(\phi_k)(t) - \Theta_Z(\phi)(t)| \le \|\phi_k - \phi\|_t.$$

First, suppose $Z(t) \in G \setminus \mathcal{S}$. Then (ii) and (iii) above imply that for each $k \in \mathbb{N}$, ϕ_k is continuous at t, and thus, ϕ is also continuous at t. This proves ϕ is continuous at all $t \in (0, \tau)$ such that $Z(t) \in G \setminus \mathcal{S}$. The fact that $\nabla_\psi \bar{\Gamma}(X)$ is continuous at all $t \in (0, \tau)$ such that $Z(t) \in G \setminus \mathcal{S}$ (namely, part 5 of Theorem 3.11) will follow once we establish part 3 of the theorem. Definition (3.10) of Θ_Z shows that $\Theta_Z(\phi_k)(t) = \phi_k(t)$ for each $k \in \mathbb{N}$ and $\Theta_Z(\phi)(t) = \phi(t)$, so (9.3) holds. Alternatively, suppose $Z(t) \in \mathcal{S}$. Let $i \in \mathcal{I}$ be the unique index such that $\mathcal{I}(Z(t)) = \{i\}$. Then (3.9) and (3.3) show that

$$\partial G_{Z(t)} = \{x \in \mathbb{R}^J : \langle x, n_i \rangle = 0\} = H_{Z(t)}.$$

If $\phi(t-) \notin \partial G_{Z(t)}$, then for all $k \in \mathbb{N}$ sufficiently large, either $\phi(t-), \phi_k(t-) \in G_{Z(t)}$ or $\phi(t-), \phi_k(t-) \notin G_{Z(t)}$. In both cases, (9.3) follows from the definition (3.10) of Θ_Z. On the other hand, if $\phi(t-) \in \partial G_{Z(t)} = H_{Z(t)}$, then by condition 2 of the DP (Definition 3.4), (3.5), condition 1 of the DP and the continuity of ψ, we have

$$\phi(t) \in H_{Z(t)} \quad \text{and} \quad \phi(t) - \phi(t-) \in \text{span}[d(Z(t))].$$

Thus, Lemma 8.3 implies that $\phi(t) = \mathcal{L}_{Z(t)}\phi(t-) = \phi(t-)$. In particular, ϕ is continuous at t so (9.3) follows from the definition (3.10) of Θ_Z. This exhausts all possible cases and so (9.3) must hold for all $t \in (0, \tau)$.

Let $t \in (0, \tau)$. By the triangle inequality, the fact that $\nabla_{\psi_k}\bar{\Gamma}(X)(t) = \Theta_Z(\phi_k)(t)$, the definition (2.15) of $\nabla^\varepsilon_{\psi_k}\bar{\Gamma}(X)$, (9.3) and the Lipschitz continuity of both the ESM (Theorem 2.10) and the DM, we have, for all $k \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}
\left|\nabla^\varepsilon_\psi\bar{\Gamma}(X)(t) - \Theta_Z(\phi)(t)\right| &\leq \left|\nabla^\varepsilon_\psi\bar{\Gamma}(X)(t) - \nabla^\varepsilon_{\psi_k}\bar{\Gamma}(X)(t)\right| \\
&\quad + \left|\nabla^\varepsilon_{\psi_k}\bar{\Gamma}(X)(t) - \nabla_{\psi_k}\bar{\Gamma}(X)(t)\right| \\
&\quad + \left|\Theta_Z(\phi_k)(t) - \Theta_Z(\phi)(t)\right| \\
&\leq \varepsilon^{-1}\left|\bar{\Gamma}(X + \varepsilon\psi)(t) - \bar{\Gamma}(X + \varepsilon\psi_k)(t)\right| \\
&\quad + \left|\nabla^\varepsilon_{\psi_k}\bar{\Gamma}(X)(t) - \nabla_{\psi_k}\bar{\Gamma}(X)(t)\right| \\
&\quad + \|\phi_k - \phi\|_t \\
&\leq \left|\nabla^\varepsilon_{\psi_k}\bar{\Gamma}(X)(t) - \nabla_{\psi_k}\bar{\Gamma}(X)(t)\right| \\
&\quad + (\kappa_\Gamma + \kappa_\Lambda)\|\psi - \psi_k\|_t.
\end{aligned}$$

Sending $\varepsilon \downarrow 0$ first and then $k \to \infty$ in the last display we see that $\lim_{\varepsilon\downarrow 0} \nabla^\varepsilon_\psi\bar{\Gamma}(X)(t) = \Theta_Z(\phi)(t)$. By Theorem 2.12 and Lemma 3.9, we have $\nabla_\psi\bar{\Gamma}(X)(0) = \nabla_{\psi(0)}\pi(X(0))$. Thus, parts 1 and 4 of Theorem 3.11 hold. By the definition of $\Theta_Z(\phi)$ in (3.10), $\phi(t) = \nabla_\psi\bar{\Gamma}(X)(t+)$ holds for $t \in [0, \tau)$. This establishes part 3, and hence part 5, of Theorem 3.11. $\quad\square$

We are left to prove Lemma 9.5. The proof is given in Section 9.3. In the next section, we state some useful lemmas that will be used in the proof of Lemma 9.5.

9.2. *Some useful lemmas.* The first lemma, which is stated without proof, is an immediate consequence of the compactness of $\{\nabla^\varepsilon_\psi\bar{\Gamma}(X)(t)\}_{\varepsilon>0}$, defined as in (2.15) for some $t \in [0, \infty)$, implied by the Lipschitz continuity of the ESM (Theorem 2.10).

LEMMA 9.6. *Given $X, \psi \in \mathbb{C}$, $t \in [0, \infty)$ and a sequence $\{\varepsilon_\ell\}_{\ell\in\mathbb{N}}$ such that $\varepsilon_\ell \downarrow 0$ as $\ell \to \infty$, there exists a subsequence, also denoted $\{\varepsilon_\ell\}_{\ell\in\mathbb{N}}$, such that $\lim_{\ell\to\infty} \nabla^{\varepsilon_\ell}_\psi\bar{\Gamma}(X)(t)$ exists.*

The following lemma will be useful for proving that Statement 9.2 holds.

LEMMA 9.7. *Fix $0 \leq S < T < U < \infty$. Let (Z, Y) be the solution to the ESP for $X \in \mathbb{C}_G$. Assume that Statement 9.2 holds. Define X^S, Z^S and $\hat{\psi}^S$ as in (2.7), (2.8) and (7.6), respectively, and assume that Statement 9.2 holds with X^S, $\hat{\psi}^S$, Z^S, $U - S$ in place of X, ψ, Z, T, respectively. Then Statement 9.2 holds with U in place of T.*

PROOF. The statement and proof of Lemma 9.7 are analogous to the statement and proof of Lemma 7.3, so to avoid redundancy, we omit the proof. The main difference is the observation that if $\psi \in \mathbb{C}^{\delta, Z}$, then $\hat{\psi}^S \in \mathbb{C}^{\delta, Z^S}$ by (2.8) and (7.6). □

The next lemma states a Lipschitz continuity property for the orthogonal projections of solutions to the ESP. Given a subset $I \subseteq \mathcal{I}$, we let $\Pi_I : \mathbb{R}^J \mapsto \text{span}(\{n_i, i \in I\})$ denote the orthogonal projection with respect to $\langle \cdot, \cdot \rangle$. Observe that when $I = \mathcal{I}(x)$ for some $x \in \partial G$, then Π_I projects onto $H_x^\perp = \text{span}(\{n_i, i \in \mathcal{I}(x)\})$. For a function $f : [0, \infty) \mapsto \mathbb{R}^J$, define $\Pi_I f : [0, \infty) \mapsto \text{span}(\{n_i, i \in I\})$ by $(\Pi_I f)(t) \doteq \Pi_I(f(t))$ for all $t \in [0, \infty)$.

LEMMA 9.8. *Given an ESP* $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ *satisfying Assumption* 2.8 *and* $I \subseteq \mathcal{I}$, *there exists* $\kappa_I < \infty$ *such that if* (Z_1, Y_1) *solves the ESP* $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ *for* $X_1 \in \mathbb{C}$, (Z_2, Y_2) *solves the ESP* $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ *for* $X_2 \in \mathbb{C}$, *and* $\mathcal{I}(Z_1(t)) \cup \mathcal{I}(Z_2(t)) \subseteq I$ *for all* $t \in [0, T]$, *then for all* $t \in [0, T]$,

$$(9.4) \qquad \qquad \|\Pi_I Z_1 - \Pi_I Z_2\|_t \leq \kappa_I \|\Pi_I X_1 - \Pi_I X_2\|_t.$$

The proof of Lemma 9.8 is given in Appendix B.

9.3. *Proof of the key induction step.* In this section, we prove Lemma 9.5. *Throughout this section, we fix* $\delta > 0$, $2 \leq n \leq N$ *and assume that Statement* 9.4 *holds.* Given $X \in \mathbb{C}_G$, let (Z, Y) be the solution to the ESP for X and define τ as in (3.6) and $\{\theta_n\}_{n=2,\dots,N+1}$ as in (9.2). We need to show that if (Z, Y) satisfies the boundary jitter property on $[0, \tau)$, then Statement 9.2 holds with $T = \theta_{n+1}$. If $\theta_{n+1} = \theta_n$, the assertion is immediate. Suppose $\theta_{n+1} > \theta_n$. Set

$$(9.5) \qquad \qquad t_1^{(n)} \doteq \theta_n.$$

Given $k \in \mathbb{N}$ for which $t_k^{(n)}$ is defined, if $t_k^{(n)} = \theta_{n+1}$, set $K_n = k$, where as if $t_k^{(n)} < \theta_{n+1}$, define $\rho_k^{(n)}$ to be the first time in the interval $(t_k^{(n)}, t_k^{(n)} + \delta]$ that Z reaches a face that is distinct from any of the faces that $Z(t_k^{(n)})$ lies on; that is,

$$(9.6) \qquad \rho_k^{(n)} \doteq \inf\{t \in (t_k^{(n)}, t_k^{(n)} + \delta] : \mathcal{I}(Z(t)) \nsubseteq \mathcal{I}(Z(t_k^{(n)}))\} \wedge (t_k^{(n)} + \delta),$$

and define $t_{k+1}^{(n)}$ to be the first time in the interval $[\rho_k^{(n)}, \theta_{n+1}]$ that Z reaches the intersection of n or more faces; that is,

$$(9.7) \qquad \qquad t_{k+1}^{(n)} = \inf\{t \in [\rho_k^{(n)}, \theta_{n+1}] : |\mathcal{I}(Z(t))| \geq n\}.$$

If $t_k^{(n)} < \theta_{n+1}$ for all $k \in \mathbb{N}$, set $K_n = \infty$ so that

$$(9.8) \qquad \qquad K_n \doteq \inf\{k \in \mathbb{N} : t_k^{(n)} = \theta_{n+1}\}.$$

If $K_n = \infty$, then (9.2), (9.6), (9.7) and the continuity of Z imply that $t_k^{(n)} \to \theta_{n+1}$ as $k \to \infty$. Therefore, in order to prove Lemma 9.5, we need to show that for all $1 \le k < K_n + 1$, Statement 9.2 holds with $T = t_k^{(n)}$. Since $t_1^{(n)} \doteq \theta_n$ and Statement 9.4 holds by assumption, Statement 9.2 holds with $T = t_1^{(n)}$. In Lemma 9.9 and Lemma 9.10 below, we show that if Statement 9.2 holds with $T = t_k^{(n)}$, then Statement 9.2 holds with $T = \rho_k^{(n)}$. In Lemma 9.11, we prove that if Statement 9.2 holds with $T = t_k^{(n)}$, then Statement 9.2 holds with $T = t_{k+1}^{(n)}$, which along with the principle of mathematical induction, will complete the proof that for all $1 \le k < K_n + 1$, Statement 9.2 holds with $T = t_k^{(n)}$.

We begin with the proof that Statement 9.2 holds with $T = \rho_k^{(n)}$. The proof is split into two lemmas. In Lemma 9.9, we consider the case that $t_k^{(n)} > 0$ and in Lemma 9.10 we consider the case that $k = 1$ and $t_1^{(n)} \doteq \theta_n = 0$. Since Lemma 9.9 is needed in the proof of Lemma 9.10, we first consider the case that $t_k^{(n)} > 0$. For the following lemma recall that $\delta > 0$, $2 \le n \le N$ are fixed and our assumption that Statement 9.4 holds.

LEMMA 9.9.   *Given $X \in \mathbb{C}_G$, let (Z, Y) denote the solution to the ESP for X. Define τ as in (3.6) and assume that (Z, Y) satisfies the boundary jitter property on $[0, \tau)$. Define θ_n and θ_{n+1} as in (9.2) and assume that $\theta_{n+1} > \theta_n$. Define $\{t_k^{(n)}\}_{k=1,\ldots,K_n}$, $\{\rho_k^{(n)}\}_{k=1,\ldots,K_n-1}$ and $K_n \in \mathbb{N}$ are as in (9.5)–(9.8). Let $1 \le k < K_n$. Assume that $t_k^{(n)} > 0$ and Statement 9.2 holds with $T = t_k^{(n)}$. Then Statement 9.2 holds with $T = \rho_k^{(n)}$.*

PROOF.   For notational convenience, we drop the superscript n notation and write t_k and ρ_k in place of $t_k^{(n)}$ and $\rho_k^{(n)}$, respectively.

Fix $\psi \in \mathbb{C}^{\delta, Z}$. By (9.6), (9.7), the continuity of Z and the upper semicontinuity of $\mathcal{I}(\cdot)$ (Lemma 2.1), we can choose $S \in [(t_k - \delta) \vee 0, t_k)$ such that

$$(9.9) \qquad \mathcal{I}(Z(t)) \subsetneq \mathcal{I}(Z(t_k)), \qquad t \in [S, t_k),$$

$$(9.10) \qquad \mathcal{I}(Z(t)) \subseteq \mathcal{I}(Z(t_k)), \qquad t \in [t_k, \rho_k).$$

By condition 3 of the boundary jitter property, the continuity of Z and the fact that $Z(t_k) \in \mathcal{N}$, S can be chosen such that $Z(S) \in G^\circ$. By (9.6), $\rho_k \le t_k + \delta$. Thus, (9.1) and the fact that $Z(t_k) \in \mathcal{N}$ imply that ψ is constant on $[S, \rho_k]$. Since Statement 9.2 holds with $T = t_k$ by assumption, $\nabla_\psi \bar{\Gamma}(X)(t)$ exists for all $t \in [0, t_k)$, there exists a unique solution (ϕ, η) of the DP for ψ on $[0, t_k)$ and $\phi(t) = \nabla_\psi \bar{\Gamma}(X)(t+)$ for $t \in [0, t_k)$. It then follows from Lemma 8.10, with $T = t_k$, that $\phi(t_k-)$ exists and $\phi(t_k-) = \mathcal{L}_{Z(t_k)} \phi(S)$. By condition 3 of the DP, the fact that $Z(S) \in G^\circ$ and the continuity of Z, $\phi - \psi$ is constant in a neighborhood of S. Since ψ is continuous, this implies ϕ is continuous at S and so $\nabla_\psi \bar{\Gamma}(X)(S) = \phi(S)$. Thus, $\nabla_\psi \bar{\Gamma}(X)(t_k-)$ exists and satisfies

$$(9.11) \qquad \nabla_\psi \bar{\Gamma}(X)(t_k-) = \mathcal{L}_{Z(t_k)} \phi(S) = \mathcal{L}_{Z(t_k)} [\nabla_\psi \bar{\Gamma}(X)(S)].$$

We claim that $\nabla_\psi \bar{\Gamma}(X)(t)$ exists for all $t \in [t_k, \rho_k)$ and satisfies

$$(9.12) \qquad \nabla_\psi \bar{\Gamma}(X)(t) = \mathcal{L}_{Z(t_k)} \phi(S), \qquad t \in [t_k, \rho_k).$$

We defer the proof of the claim and instead first derive some consequences of the claim. Parts 1 and 2 of Statement 9.2 with $T = t_k$, together with (9.11) and (9.12), imply that parts 1 and 2 of Statement 9.2 hold with $T = \rho_k$. We now turn to part 3 of Statement 9.2 holds with $T = \rho_k$. Define $\hat{\phi}(t) \doteq \nabla_\psi \bar{\Gamma}(X)(t+)$ and $\hat{\eta}(t) \doteq \hat{\phi}(t) - \psi(t)$ for $t \in [0, \rho_k)$. Observe that $\hat{\phi}(t) = \phi(t)$ for $t \in [0, t_k)$ and by (9.12), $\hat{\phi}(t) = \mathcal{L}_{Z(t_k)} \phi(S)$ for all $t \in [t_k, \rho_k)$. Due to the fact that $(\hat{\phi}, \hat{\eta})$ solves the DP for ψ on $[0, t_k)$, to prove part 3 of Statement 9.2, we are left to show that $(\hat{\phi}, \hat{\eta})$ satisfies conditions 1 and 2 of the DP for $t \in [t_k, \rho_k)$ and condition 3 of the DP for $0 \le s < t < \rho_k$ whenever $t \in [t_k, \rho_k)$. Let $t \in [t_k, \rho_k)$. Condition 1 of the DP holds by definition. Lemma 8.3, the definition (3.3) of H_x and (9.10) imply that $\mathcal{L}_{Z(t_k)} \phi(S) \in H_{Z(t_k)} \subseteq H_{Z(t)}$, so condition 2 of the DP holds. Lastly, suppose $s \in [0, t)$. Since ψ is constant on $[(t_k - \delta) \vee 0, \rho_k)$ and $\hat{\phi}(t) = \hat{\phi}(t_k-)$ for all $t \in [t_k, \rho_k)$, we have $\hat{\eta}(t) = \hat{\eta}(t_k-)$ for all $t \in [t_k, \rho_k)$. When combined with the fact that $\hat{\eta}$ satisfies condition 3 of the DP for ψ on $[0, t_k)$, we have

$$\hat{\eta}(t) - \hat{\eta}(s) = \hat{\eta}(t_k-) - \hat{\eta}(s) \in \bigcup_{r \in (s, t_k)} \mathrm{span}\left[\bigcup_{u \in (s, r]} d(Z(u)) \right]$$

$$\subseteq \mathrm{span}\left[\bigcup_{u \in (s, t]} d(Z(u)) \right].$$

Thus, condition 3 of the DP holds. This completes the proof that Statement 9.2 holds with $T = \rho_k$.

It only remains to prove the claim (9.12). Fix $t \in [t_k, \rho_k)$. By (9.9), (9.10) the upper semicontinuity of $\mathcal{I}(\cdot)$ (Lemma 2.1), the continuity of Z and ψ, and the Lipschitz continuity of the ESM, for all $\ell \in \mathbb{N}$ sufficiently large,

$$(9.13) \qquad \mathcal{I}(Z_{\varepsilon_\ell}(u)) \subseteq \mathcal{I}(Z(t_k)), \qquad u \in [S, t].$$

By Lemma 9.6, given a sequence $\{\varepsilon_\ell\}_{\ell \in \mathbb{N}}$ with $\varepsilon_\ell \downarrow 0$ as $\ell \to \infty$, there exists a subsequence, also denoted $\{\varepsilon_\ell\}_{\ell \in \mathbb{N}}$, such that $v \doteq \lim_{\ell \to \infty} \nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(t)$ exists. It suffices to show that $v = \mathcal{L}_{Z(t_k)} \phi(S)$. Due to the uniqueness of the derivative projection operators stated in Lemma 8.3, this is equivalent to showing that $v \in H_{Z(t_k)}$ and $v - \phi(S) \in \mathrm{span}[d(Z(t_k))]$.

Now (2.15), condition 1 of the ESP and the fact that ψ is constant on $[S, t]$ show that

$$\nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(t) - \nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(S) = \frac{1}{\varepsilon_\ell} \big[Y_{\varepsilon_\ell}(t) - Y_{\varepsilon_\ell}(S) - (Y(t) - Y(S)) \big].$$

When combined with condition 3 of the ESP, (9.9), (9.10) and (9.13), this implies that for all $\varepsilon > 0$ sufficiently small,

$$(9.14) \qquad \nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(t) - \nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(S) \in \mathrm{span}[d(Z(t_k))].$$

Since span$[d(Z(t_k))]$ is closed, taking limits as $\ell \to \infty$, we see that $v - \nabla_\psi \bar{\Gamma}(X)(S) \in \text{span}[d(Z(t_k))]$. Then because $\phi(t) = \nabla_\psi \bar{\Gamma}(X)(t+)$ for all $t \in [0, t_k)$ and ϕ is continuous at S, $v - \phi(S) \in \text{span}[d(Z(t_k))]$.

We are left to prove that $v \in H_{Z(t_k)}$. Let $s \in (S, t_k)$ be arbitrary. Define X^s, Z^s, $X^s_{\varepsilon_\ell}$, $Z^s_{\varepsilon_\ell}$ and $\psi^s_{\varepsilon_\ell}$ as in (2.7), (2.8), (7.2), (7.3) and (7.5), but with s and ε_ℓ in place of S and ε, respectively. Then by the time-shift property of the ESP (Lemma 2.7) and (7.4), $Z^s = \bar{\Gamma}(X^s)$ and $Z^s_{\varepsilon_\ell} = \bar{\Gamma}(X^s + \varepsilon_\ell \psi^s_{\varepsilon_\ell})$. Thus, (9.9), (9.10), (9.13), (2.8) and (7.3) imply that for $\ell \in \mathbb{N}$ sufficiently large,

$$\mathcal{I}(Z^s(u)) \cup \mathcal{I}(Z^s_{\varepsilon_\ell}(u)) \subseteq \mathcal{I}(Z(t_k)) \qquad \text{for } u \in [0, t-s].$$

Let $I \doteq \mathcal{I}(Z(t_k))$ and recall that Π_I denotes the orthogonal projection from $\mathbb{R}^J$ onto $\text{span}(\{n_i, i \in I\}) = H^\perp_{Z(t_k)}$. By Lemma 9.8, with $Z^s_{\varepsilon_\ell}$, $Y^s_{\varepsilon_\ell}$, $X^x_{\varepsilon_\ell}$, Z^s, Y^s, X^s and $t - s$ in place of Z_1, Y_1, X_1, Z_2, Y_2, X_2 and T, respectively, there is a constant $\kappa_I < \infty$ such that for all $\ell \in \mathbb{N}$ sufficiently large,

$$\frac{\|\Pi_I Z^s_{\varepsilon_\ell} - \Pi_I Z^s\|_{t-s}}{\varepsilon_\ell} \leq \frac{\kappa_I \|\Pi_I X^s_{\varepsilon_\ell} - \Pi_I X^s\|_{t-s}}{\varepsilon_\ell}$$
$$= \kappa_I \|\Pi_I \psi^s_{\varepsilon_\ell}\|_{t-s}$$
$$= \kappa_I |\Pi_I \nabla^{\varepsilon_\ell}_\psi \bar{\Gamma}(X)(s)|,$$

where the first equality uses (7.4) and the final equality uses (7.5) and the fact that ψ is constant on $[s, t]$. Hence, by (2.8), (7.3), (2.15) and the assumption that $\nabla_\psi \bar{\Gamma}(X)$ exists on $[0, t_k)$,

$$|\Pi_I v| = \lim_{\ell \to \infty} \frac{|\Pi_I Z^s_{\varepsilon_\ell}(t - s) - \Pi_I Z^s(t - s)|}{\varepsilon_\ell}$$
$$\leq \lim_{\ell \to \infty} \kappa_I |\Pi_I \nabla^{\varepsilon_\ell}_\psi \bar{\Gamma}(X)(s)|$$
$$= \kappa_I |\Pi_I \nabla_\psi \bar{\Gamma}(X)(s)|.$$

Sending $s \uparrow t_k$ in the above display, invoking the identity (9.11) and noting that $\mathcal{L}_{Z(t_k)}$ projects onto $H_{Z(t_k)}$ and Π_I denotes orthogonal projection onto $H^\perp_{Z(t_k)}$, yields

$$|\Pi_I v| \leq \kappa_I |\Pi_I \nabla_\psi \bar{\Gamma}(X)(t_k-)| = \kappa_I |\Pi_I \mathcal{L}_{Z(t_k)} \phi(S)| = 0.$$

Thus, $v \in H_{Z(t_k)}$, completing the proof. $\square$

In the following lemma, we consider the case that $\theta_n = 0$ and prove that Statement 9.2 holds with $T = \rho^{(n)}_1$. This is relevant when the path Z starts at the nonsmooth part of the boundary. The proof is much more involved than the proof of Lemma 9.9. For the following lemma, recall that $\delta > 0$, $2 \leq n \leq N$ are fixed and our assumption that Statement 9.4 holds.

LEMMA 9.10.	*Given $X \in \mathbb{C}_G$, let (Z, Y) denote the solution to the ESP for X. Define τ as in (3.6) and assume that (Z, Y) satisfies the boundary jitter property on $[0, \tau)$. Define θ_n and θ_{n+1} as in (9.2) and assume that $\theta_{n+1} > \theta_n = 0$. Define $t_1^{(n)}$ and $\rho_1^{(n)}$ as in (9.5)–(9.6). Then Statement 9.2 holds with $T = \rho_1^{(n)}$.*

PROOF.	For notational convenience, we drop the superscript n notation and write ρ_1 in place of $\rho_1^{(n)}$.

Fix $\psi \in \mathbb{C}^{\delta, Z}$. Recall the definition of $\nabla_v \pi(x)$ from (3.8), and note that by Theorem 2.12 and Lemma 3.9,

$$(9.15) \qquad v_0 \doteq \nabla_\psi \bar{\Gamma}(X)(0) \doteq \lim_{\varepsilon \downarrow 0} \nabla_\psi^\varepsilon \bar{\Gamma}(X)(0) = \nabla_{\psi(0)} \pi(X(0))$$

and $v_0 - \psi(0) \in \text{cone}[d(Z(0))]$. By the definition of ρ_1 in (9.6), we have

$$(9.16) \qquad \mathcal{I}(Z(t)) \subseteq \mathcal{I}(Z(0)) \qquad \text{for all } t \in [0, \rho_1).$$

We claim, and prove below, that for all $t \in (0, \rho_1)$, $\nabla_\psi \bar{\Gamma}(X)(t)$ exists and satisfies

$$(9.17) \qquad \nabla_\psi \bar{\Gamma}(X)(t) = \mathcal{L}_{Z(0)} v_0, \qquad t \in (0, \rho_1).$$

Given the claim, it follows that $\nabla_\psi \bar{\Gamma}(X)$ lies in $\mathbb{D}_{l,r}([0, \rho_1) : \mathbb{R}^J)$ and is continuous on $(0, \rho_1)$, so parts 1 and 2 of Statement 9.2 hold with $T = \rho_1$. Let $\hat{\phi}$ denote the right continuous regularization of $\nabla_\psi \bar{\Gamma}(X)$ on $[0, \rho_1)$; that is, $\hat{\phi} \equiv \mathcal{L}_{Z(0)} v_0$ on $[0, \rho_1)$. By (8.3), $\mathcal{L}_{Z(0)} v_0 \in H_{Z(0)}$ and

$$\mathcal{L}_{Z(0)} v_0 - \psi(0) = (\mathcal{L}_{Z(0)} v_0 - v_0) + (v_0 - \psi(0)) \in \text{span}[d(Z(0))].$$

In view of the above display and (9.16), it is readily verified that $(\hat{\phi}, \hat{\phi} - \psi)$ is the unique solution to the DP for ψ on $[0, \rho_1)$, so part 3 of Statement 9.2 holds.

We now turn to the proof of (9.17). Since $n \geq 2$ and $\theta_n = 0$, $Z(0) \in \mathcal{N}$. This, combined with the definition of $\mathbb{C}^{\delta, Z}$ in (9.1) and the fact that $\rho_k \leq \delta$ by (9.6), implies that ψ is constant on $[0, \rho_1]$. Fix $T \in (0, \rho_1)$. By Lemma 9.6, given a sequence $\{\varepsilon_\ell\}_{\ell \in \mathbb{N}}$ with $\varepsilon_\ell \downarrow 0$ as $\ell \to \infty$, there exists a subsequence, also denoted $\{\varepsilon_\ell\}_{\ell \in \mathbb{N}}$, such that the following limit exists:

$$(9.18) \qquad v_T \doteq \lim_{\ell \to \infty} \nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(T).$$

It suffices to show that $v_T = \mathcal{L}_{Z(0)} v_0$ must hold. Due to the uniqueness of the derivative projection operators stated in Lemma 8.3, this is equivalent to showing that $v_T \in H_{Z(0)}$ and $v_T - v_0 \in \text{span}[d(Z(0))]$.

We first show that $v_T - v_0 \in \text{span}[d(Z(0))]$. By (9.16), the Lipschitz continuity of the ESM and the upper semicontinuity of $\mathcal{I}(\cdot)$ (Lemma 2.1), we have, for all $\ell \in \mathbb{N}$ sufficiently large,

$$(9.19) \qquad \mathcal{I}(Z_{\varepsilon_\ell}(t)) \subseteq \mathcal{I}(Z(0)) \qquad \text{for all } t \in [0, T].$$

Due to (2.15), condition 1 of the ESP and the fact that ψ is constant on $[0, T]$, it follows that

$$\nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(T) - \nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(0) = \frac{1}{\varepsilon_\ell} \big[Y_{\varepsilon_\ell}(T) - Y_{\varepsilon_\ell}(0) - (Y(T) - Y(0)) \big].$$

When combined with condition 3 of the ESP, (9.16) and (9.19), this implies that for all $\ell \in \mathbb{N}$ sufficiently large,

$$\nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(T) - \nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(0) \in \mathrm{span}[d(Z(0))].$$

Since $\mathrm{span}[d(Z(0))]$ is a closed subspace, taking limits as $\ell \to \infty$ and using (9.15) and (9.18), we see that $v_T - v_0 \in \mathrm{span}[d(Z(0))]$.

We are left to prove that $v_T \in H_{Z(0)}$, which is the more complicated part of the proof. We consider two exhaustive and mutually exclusive cases separately. In Case 1, we show that there exists a time-shifted version of the path, which starts in $G \setminus \mathcal{N}$ and satisfies the assumptions of Lemma 9.9. We then invoke Lemma 9.9 to prove our result. In Case 2, we similarly apply Lemma 9.9 to a time-shifted version of the path; however, that case is more complicated and will make crucial use of condition 4 of the boundary jitter property.

Case 1: $\mathcal{I}(Z(t)) = \mathcal{I}(Z(0))$ for some $t \in (0, T)$.

Let $U \in (0, T)$ be such that $\mathcal{I}(Z(U)) = \mathcal{I}(Z(0))$. By (9.2) and the assumption that $\theta_{n+1} > \theta_n = 0$,

$$(9.20) \qquad |\mathcal{I}(Z(U))| = |\mathcal{I}(Z(0))| = n.$$

By condition 2 of the boundary jitter property, there exists $S \in (0, U)$ such that $Z(S) \notin \mathcal{N}$. Define X^S, Z^S, Y^S as in (2.7)–(2.9), so $X^S \in \mathbb{C}_G$ and by the time-shift property of the ESP (Lemma 2.7), (Z^S, Y^S) solves the ESP for X^S. Define

$$\tau^S \doteq \inf\{t \in [0, \infty) : Z^S \in \mathcal{W}\} = \tau - S,$$

where the last equality uses (2.8), (3.6) and the fact that $S < \tau$. Since (Z, Y) satisfies the boundary jitter property on $[0, \tau)$, it is straightforward to verify, using (2.8)–(2.9), that (Z^S, Y^S) satisfies the boundary jitter property on $[0, \tau^S)$. Define θ_n^S and θ_{n+1}^S as in (9.2), but with Z^S and τ^S in place of Z and τ, respectively. Since $Z(S) \notin \mathcal{N}$ and $S \in (0, U)$, it follows from (9.2), (2.8) and (9.20) that $\theta_n^S \in (0, U - S]$. By (2.8) and (9.16), $Z^S(t) \subseteq \mathcal{I}(Z(0))$ for all $t \in [0, \rho_1 - S)$. Together, the previous two sentences, along with the fact that $U < T < \rho_1$, imply that

$$(9.21) \qquad \mathcal{I}(Z^S(\theta_n^S)) = \mathcal{I}(Z(0)) \quad \text{and} \quad 0 < \theta_n^S \le U - S < \theta_{n+1}^S.$$

Set $t_1^S \doteq \theta_n^S$ and define

$$\rho_1^S \doteq \inf\{t \in (\theta_n^S, \theta_n^S + \delta] : \mathcal{I}(Z^S(t)) \nsubseteq \mathcal{I}(Z^S(\theta_n^S))\} \wedge (\theta_n^S + \delta)$$

$$(9.22) \qquad = (\inf\{t \in (0, \theta_n^S + \delta] : \mathcal{I}(Z(t)) \nsubseteq \mathcal{I}(Z(0))\} \wedge (\theta_n^S + \delta)) - S$$

$$\ge \rho_1 - S,$$

where the second line uses (2.8), (9.16) and (9.21), and the final line follows from
the definition (9.6) of ρ_1 along with the fact that $t_1 \doteq \theta_n = 0$. In addition, by our
assumption that Statement 9.4 holds, Statement 9.2 holds with X^S, Z^S and $t_1^S \doteq$
θ_n^S in place of X, Z and T, respectively. Therefore, the conditions in Lemma 9.9
(when $k = 1$) hold, so we can conclude that

(i) Statement 9.2 holds with X^S, Z^S and ρ_1^S in place of X, Z and T, respec-
tively.

Proceeding, by Lemma 9.6, we can choose a further subsequence of $\{\varepsilon_\ell\}_{\ell \in \mathbb{N}}$, also
denoted $\{\varepsilon_\ell\}_{\ell \in \mathbb{N}}$, such that the following limit exists:

$$(9.23) \qquad\qquad v_S \doteq \lim_{\ell \to \infty} \nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(S).$$

Define

$$(9.24) \qquad\qquad \hat{\psi}^S(\cdot) \doteq v_S + \psi(S + \cdot) - \psi(S).$$

The above definition, along with (9.1), (2.8) and the fact that $\psi \in \mathbb{C}^{\delta, Z}$, implies
that $\hat{\psi}^S \in \mathbb{C}^{\delta, Z^S}$. Thus, by (i) above, we have the following:

(ii) $\nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)$ exists on $[0, \rho_1^S)$;

(iii) there exists a unique solution (ϕ^S, η^S) of the DP along Z^S for $\hat{\psi}^S$ on
$[0, \rho_1^S)$, and $\phi^S(t) = \nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)(t+)$ for $t \in [0, \rho_1^S)$.

We show that ϕ^S is constant on $[\theta_n^S, \rho_1^S)$. By (9.21)–(9.22), we have $\mathcal{I}(Z^S(t)) \subseteq$
$\mathcal{I}(Z(0))$ for all $t \in [\theta_n^S, \rho_1^S)$. Then (3.3) implies that $H_{Z(0)} \subseteq H_{Z^S(t)}$ for all $t \in$
$[\theta_n^S, \rho_1^S)$. When combined with condition 2 of the DP and (9.21), we have

$$(9.25) \qquad \phi^S(\theta_n^S) \in H_{Z^S(\theta_n^S)} = H_{Z(0)} \subseteq H_{Z^S(t)}, \qquad t \in [\theta_n^S, \rho_1^S).$$

Since $\hat{\psi}^S \in \mathbb{C}^{\delta, Z^S}$ and $Z^S(\theta_n^S) \in \mathcal{N}$, $\hat{\psi}^S$ is constant on $[\theta_n^S, \theta_n^S + \delta]$, and in par-
ticular, by (9.22), $\hat{\psi}^S$ is constant on $[\theta_n^S, \rho_1^S]$. It is readily verified that if (ϕ^S, η^S)
are also constant on $[\theta_n^S, \rho_1^S)$, then conditions 1–3 of the DP along Z^S for $\hat{\psi}^S$ are
satisfied for all $s, t \in [\theta_n^S, \rho_1^S)$. Therefore, by the uniqueness of solutions to the
DP, ϕ^S must be constant on $[\theta_n^S, \delta - S)$. This, together with (iii) above and (9.25),
implies that

$$(9.26) \qquad \nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)(t) = \phi^S(\theta_n^S) \in H_{Z(0)}, \qquad t \in (\theta_n^S, \rho_1^S).$$

For each $\ell \in \mathbb{N}$, define $X_{\varepsilon_\ell}^S, Z_{\varepsilon_\ell}^S, \hat{\psi}_{\varepsilon_\ell}^S$ as in (7.2)–(7.3) and (7.5), but with ε_ℓ in
place of ε. Then by (7.4) and (9.23)–(9.24), the following hold:

(iv) $X_{\varepsilon_\ell}^S = X^S + \varepsilon_\ell \hat{\psi}_{\varepsilon_\ell}^S$;

(v) $\hat{\psi}_{\varepsilon_\ell}^S \to \hat{\psi}^S$ uniformly on $[0, \infty)$ as $\ell \to \infty$.

Since $0 < S < U < T < \rho_1$ by definition, $\theta_n^S \leq U - S$ by (9.21), and $\rho_1 - S < \rho_1^S$ by (9.22), it follows that $T - S \in (\theta_n^S, \rho_1^S)$. We can now conclude that

$$\begin{aligned} v_T &= \lim_{\ell \to \infty} \frac{\bar{\Gamma}(X + \varepsilon_\ell \psi)(T) - \bar{\Gamma}(X)(T)}{\varepsilon_\ell} \\ &= \lim_{\ell \to \infty} \frac{\bar{\Gamma}(X^S + \varepsilon_\ell \hat{\psi}_{\varepsilon_\ell}^S)(T - S) - \bar{\Gamma}(X^S)(T - S)}{\varepsilon_\ell} \\ &= \lim_{\ell \to \infty} \nabla_{\hat{\psi}_{\varepsilon_\ell}^S}^{\varepsilon_\ell} \bar{\Gamma}(X^S)(T - S) \\ &= \nabla_{\hat{\psi}^S} \bar{\Gamma}(X^S)(T - S) \in H_{Z(0)}, \end{aligned}$$

where the first equality follows from (9.18) and (2.15); the second equality is due to the time-shift property of the ESP and (iv); the third equality follows from (2.15); the final equality holds because of (ii), (v) and Proposition 2.17; and the inclusion is due to (9.26). This proves Case 1.

Case 2: $\mathcal{I}(Z(t)) \subsetneq \mathcal{I}(Z(0))$ for all $t \in (0, T)$.

Since Z satisfies conditions 3 and 4 of the boundary jitter property, $Z(0) \in \mathcal{N}$ and $\mathcal{I}(Z(t)) \subsetneq \mathcal{I}(Z(0))$ for all $t \in (0, T)$, there is a nested decreasing sequence

$$(9.27) \qquad T > \chi_0 > u_1 \geq \chi_1 > \cdots > u_j \geq \chi_j > \cdots > 0$$

such that the conclusions of Lemma 4.2 hold. Condition 4 of the boundary jitter property, the fact that $T < \tau$ and (4.8) imply that $\mathcal{I}(Z(0)) = \bigcup_{j \geq m} \mathcal{I}(Z(\chi_j))$ for all $m \in \mathbb{N}$. Let $\gamma > 0$ be arbitrary. Define the compact set $C \subseteq \mathbb{R}^J$ by

$$(9.28) \qquad C \doteq \left\{ y \in \mathbb{R}^J : |y| \leq \kappa_\Lambda \kappa_{\bar{\Gamma}} |\psi(0)| \right\}.$$

By (8.25) in Corollary 8.9, we can choose $m = m(\gamma, C) \in \mathbb{N}$ such that

$$(9.29) \qquad \sup_{y \in C} \left| [\mathcal{L}_{Z(\chi_m)} \cdots \mathcal{L}_{Z(\chi_1)}] y - \mathcal{L}_{Z(0)} y \right| \leq \gamma.$$

For $j \in \{0, m\}$, define $X^{\chi_j}, Z^{\chi_j}, Y^{\chi_j}$ as in (2.7)–(2.9), but with χ_j in place of S, so $X^{\chi_j} \in \mathbb{C}_G$ and by the time-shift property of the ESP, (Z^{χ_j}, Y^{χ_j}) solves the ESP for X^{χ_j}. As in Case 1, define

$$\tau^{\chi_j} \doteq \inf\{t \in [0, \infty) : Z^{\chi_j} \in \mathcal{W}\} = \tau - \chi_j,$$

where the last equality uses (2.8), (3.6) and the fact that $\chi_j < \tau$. As in Case 1, since (Z, Y) satisfies the boundary jitter property on $[0, \tau)$, (Z^{χ_j}, Y^{χ_j}) satisfies the boundary jitter property on $[0, \tau^{\chi_j})$. Define $\theta_n^{\chi_j}$ and $\theta_{n+1}^{\chi_j}$ as in (9.2), but with Z^{χ_j} and τ^{χ_j} in place of Z and τ, respectively. Since $\mathcal{I}(Z(t)) \subsetneq \mathcal{I}(Z(0))$ for all $t \in (0, T)$, $\chi_j \in (0, T)$ and $|\mathcal{I}(Z(0))| = n$ by our assumption $\theta_{n+1} > \theta_n = 0$, it follows from (2.8) that $\theta_n^{\chi_j} \geq T - \chi_j > 0$. Set $t_1^{\chi_j} \doteq \theta_n^{\chi_j}$ and define

$$\begin{aligned} \rho_1^{\chi_j} &\doteq \inf\{t \in (\theta_n^{\chi_j}, \theta_n^{\chi_j} + \delta] : \mathcal{I}(Z^{\chi_j}(t)) \nsubseteq \mathcal{I}(Z^{\chi_j}(\theta_n^{\chi_j}))\} \wedge (\theta_n^{\chi_j} + \delta) \\ &> T - \chi_j. \end{aligned}$$

By our assumption that Statement 9.4 holds, and because (Z^{χ_j}, Y^{χ_j}) solves the ESP for X^{χ_j} and satisfies the boundary jitter property on $[0, \tau^{\chi_j})$, Statement 9.2 holds with X^{χ_j}, Z^{χ_j} and $t_1^{\chi_j} \doteq \theta_n^{\chi_j}$ in place of X, Z and T, respectively. Therefore, the conditions in Lemma 9.9 (when $k = 1$) hold, so we can conclude that

(a) Statement 9.2 holds with X^{χ_j}, Z^{χ_j}, Y^{χ_j} and $\rho_1^{\chi_j}$ in place of X, Z, Y and T, respectively.

By Lemma 9.6, there is a further subsequence of $\{\varepsilon_\ell\}_{\ell \in \mathbb{N}}$, also denoted $\{\varepsilon_\ell\}_{\ell \in \mathbb{N}}$, such that the following limits exist:

$$(9.30) \qquad v_{\chi_m} \doteq \lim_{\ell \to \infty} \nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(\chi_m) \quad \text{and} \quad v_{\chi_0} \doteq \lim_{\ell \to \infty} \nabla_\psi^{\varepsilon_\ell} \bar{\Gamma}(X)(\chi_0).$$

For $j \in \{0, m\}$, define

$$(9.31) \qquad \hat{\psi}^{\chi_j}(\cdot) \doteq v_{\chi_j} + \psi(\chi_j + \cdot) - \psi(\chi_j).$$

Note that since ψ is constant on $[0, \rho_1]$ and $T \in (0, \rho_1)$,

(b) $\hat{\psi}^{\chi_j} \equiv v_{\chi_j}$ on $[0, T - \chi_j]$.

As in Case 1, the fact that $\psi \in \mathbb{C}^{\delta, Z}$ implies that $\hat{\psi}^{\chi_j} \in \mathbb{C}^{\delta, Z^{\chi_j}}$. Thus, by (a), we have the following:

(c) $\nabla_{\hat{\psi}^{\chi_j}} \bar{\Gamma}(X^{\chi_j})$ exists on $[0, \rho_1^{\chi_j})$;

(d) there exists a unique solution $(\phi^{\chi_j}, \eta^{\chi_j})$ of the DP along Z^{χ_j} for $\hat{\psi}^{\chi_j}$ on $[0, \rho_1^{\chi_j})$, and $\phi^{\chi_j}(t) = \nabla_{\hat{\psi}^{\chi_j}} \bar{\Gamma}(X^{\chi_j})(t+)$ for $t \in [0, \rho_1^{\chi_j})$.

For each $\ell \in \mathbb{N}$, define $X_{\varepsilon_\ell}^{\chi_j}$, $Z_{\varepsilon_\ell}^{\chi_j}$ and $\hat{\psi}_{\varepsilon_\ell}^{\chi_j}$ as in (7.2)–(7.3) and (7.5), but with χ_j and ε_ℓ in place of S and ε, respectively. Then by (7.4) and (9.30)–(9.31):

(e) $X_{\varepsilon_\ell}^{\chi_j} = X^{\chi_j} + \varepsilon_\ell \hat{\psi}_{\varepsilon_\ell}^{\chi_j}$;

(f) $\hat{\psi}_{\varepsilon_\ell}^{\chi_j} \to \hat{\psi}^{\chi_j}$ uniformly on $[0, \infty)$ as $\ell \to \infty$.

Let $I \doteq \mathcal{I}(Z(0))$. Recall that Π_I denotes the orthogonal projection from $\mathbb{R}^J$ onto $\text{span}(\{n_i, i \in \mathcal{I}(Z(0))\}) = H_{Z(0)}^\perp$. By (2.8), (9.16) and (9.19), for all $\ell \in \mathbb{N}$ sufficiently large, $Z^{\chi_0}(t)$, $Z_{\varepsilon_\ell}^{\chi_0}(t) \subseteq I$ for all $t \in [0, T - \chi_0]$. By (9.18), (2.15), the time-shift property of the ESP, Lemma 9.8 with $Z_1 = Z_{\varepsilon_\ell}^{\chi_0}$, $Y_1 = Y_{\varepsilon_\ell}^{\chi_0}$, $X_1 = X_{\varepsilon_\ell}^{\chi_0}$, $Z_2 = Z^{\chi_0}$, $Y_2 = Y^{\chi_0}$ and $X_2 = X^{\chi_0}$, and (b), (e) and (f) above, we have

$$|\Pi_I v_T| = \lim_{\ell \to \infty} \frac{|\Pi_I(Z_{\varepsilon_\ell}(T) - Z(T))|}{\varepsilon_\ell}$$

$$= \lim_{\ell \to \infty} \frac{\|\Pi_I(Z_{\varepsilon_\ell}^{\chi_0} - Z^{\chi_0})\|_{T - \chi_0}}{\varepsilon_\ell}$$

$$\leq \lim_{\ell \to \infty} \frac{\|\Pi_I(X_{\varepsilon_\ell}^{\chi_0} - X^{\chi_0})\|_{T - \chi_0}}{\varepsilon_\ell}$$

$$= \lim_{\ell \to \infty} \kappa_I \| \Pi_I \hat{\psi}_{\varepsilon_\ell}^{\chi_0} \|_{T - \chi_0}$$

$$= \kappa_I | \Pi_I v_{\chi_0} |.$$

Thus, to prove that $v_T \in H_{Z(0)}$, it suffices to show that $v_{\chi_0} \in H_{Z(0)}$.

We claim, and prove below, that for each $1 \le j \le m$,

(9.32) $$\phi^{\chi_m}(\chi_{j-1} - \chi_m) = \mathcal{L}_{Z(\chi_j)} \phi^{\chi_m}(\chi_j - \chi_m).$$

Iterating this recursion relation yields

(9.33) $$\phi^{\chi_m}(\chi_0 - \chi_m) = [\mathcal{L}_{Z(\chi_1)} \cdots \mathcal{L}_{Z(\chi_m)}] \phi^{\chi_m}(0).$$

By (2.8) and Lemma 4.2, $Z^{\chi_m}(\chi_0 - \chi_m) = Z(\chi_0) \in G^\circ$. By the continuity of Z^{χ_m} and the fact that G° is open, $Z^{\chi_m}(t) \in G^\circ$ for all t in a neighborhood of $\chi_0 - \chi_m$. By condition 3 of the DP, (2.2) and (3.3), this implies that η^{χ_m} is constant in a neighborhood of $\chi_0 - \chi_m$. Since $\hat{\psi}^{\chi_m}$ is constant on $[0, T - \chi_m]$ by (b) above, condition 1 of the DP implies that ϕ^{χ_m} is also constant in a neighborhood of $\chi_0 - \chi_m$. In particular, $\phi^{\chi_m}(\chi_0 - \chi_m) = \nabla_{\hat{\psi}^{\chi_m}} \bar{\Gamma}(X^{\chi_m})(\chi_0 - \chi_m)$ due to (d) above. This, along with (9.30), (2.15), the time-shift property of the ESP, (c) and (f) above, Proposition 2.17 and (9.33), implies

(9.34) $$\begin{aligned} v_{\chi_0} &= \lim_{\ell \to \infty} \nabla_{\hat{\psi}_{\varepsilon_\ell}^{\chi_m}}^{\varepsilon_\ell} \bar{\Gamma}(X^{\chi_m})(\chi_0 - \chi_m) \\ &= \nabla_{\hat{\psi}^{\chi_m}} \bar{\Gamma}(X^{\chi_m})(\chi_0 - \chi_m) \\ &= \phi^{\chi_m}(\chi_0 - \chi_m) \\ &= [\mathcal{L}_{Z(\chi_1)} \cdots \mathcal{L}_{Z(\chi_m)}] \phi^{\chi_m}(0). \end{aligned}$$

By (d) above, the Lipschitz continuity of the DM (Theorem 5.4), (9.31), (9.30), the Lipschitz continuity of the ESM (Theorem 2.10) and because ψ is constant on $[0, T]$, we have

$$|\phi^{\chi_m}(0)| \le \kappa_\Lambda |\hat{\psi}^{\chi_m}(0)| = \kappa_\Lambda |v_{\chi_m}| \le \kappa_\Lambda \kappa_{\bar{\Gamma}} |\psi(0)|.$$

Then by (9.34), (9.28) and (9.29),

$$|v_{\chi_0} - \mathcal{L}_{Z(0)} \phi^{\chi_m}(0)| \le \gamma.$$

Since $\gamma > 0$ was arbitrary and $\mathcal{L}_{Z(0)}$ projects onto $H_{Z(0)}$, we have $v_{\chi_0} \in H_{Z(0)}$.

We are left to prove that (9.32) holds. Fix $1 \le j \le m$. We first show that ϕ^{χ_m} is constant on $[\chi_j - \chi_m, u_j - \chi_m]$, where u_j is as in (9.27). By condition 2 of the DP, (2.8), (3.3) and (4.8), $\phi^{\chi_m}(\chi_j - \chi_m) \in H_{Z(\chi_j)} \subseteq H_{Z(t)}$ for all $t \in [\chi_j, u_j]$. Since $\hat{\psi}^{\chi_m}$ is constant on $[\chi_j - \chi_m, u_j - \chi_m]$, it is readily checked that if ϕ^{χ_m} is constant on $[\chi_j - \chi_m, u_j - \chi_m]$, then $(\phi^{\chi_m}, \phi^{\chi_m} - \psi^{\chi_m})$ satisfies conditions 1–3 of the DP along Z^{χ_m} on the interval $[\chi_j - \chi_m, u_j - \chi_m]$. It then follows from uniqueness of

solutions to the DP that ϕ^{χ_m} must be constant on $[\chi_j - \chi_m, u_j - \chi_m]$. Next, by condition 2 of the DP and (2.8), we have

$$(9.35) \qquad \phi^{\chi_m}(\chi_{j-1} - \chi_m) \in H_{Z(\chi_{j-1})}.$$

By condition 3 of the DP and the facts that ϕ^{χ_m} is constant on $[\chi_j - \chi_m, u_j - \chi_m]$ and $\hat{\psi}^{\chi_m}$ is constant on $[0, T - \chi_m]$ by (b) above, we have

$$\phi^{\chi_m}(\chi_{j-1} - \chi_m) - \phi^{\chi_m}(\chi_j - \chi_m) = \phi^{\chi_m}(\chi_{j-1} - \chi_m) - \phi^{\chi_m}(u_j - \chi_m)$$

$$\in \text{span}\left[\bigcup_{u \in (u_j - \chi_m, \chi_{j-1} - \chi_m]} d\big(Z^{\chi_m}(u)\big) \right].$$

Due to (2.8) and (4.8), the above display implies

$$(9.36) \qquad \phi^{\chi_m}(\chi_{j-1} - \chi_m) - \phi^{\chi_m}(\chi_j - \chi_m) \in \text{span}\big[d\big(Z(\chi_{j-1})\big)\big].$$

Thus, (9.35)–(9.36) and the uniqueness of the projection operators shown in Lemma 8.3 imply that (9.32) holds. $\square$

LEMMA 9.11.   *Given $X \in \mathbb{C}_G$, let (Z, Y) denote the solution to the ESP for X. Define τ as in (3.6) and assume that (Z, Y) satisfies the boundary jitter property on $[0, \tau)$. Define θ_n and θ_{n+1} as in (9.2) and assume that $\theta_{n+1} > \theta_n$. Define $\{t_k^{(n)}\}_{k=1,\dots,K_n}$, $\{\rho_k^{(n)}\}_{k=1,\dots,K_n-1}$, $K_n \in \mathbb{N}$, as in (9.5)–(9.8). Let $1 \le k < K_n$. Assume that Statement 9.2 holds with $T = t_k^{(n)}$. Then Statement 9.2 holds with $T = t_{k+1}^{(n)}$.*

PROOF.   For notational convenience, we drop the superscript n notation and write t_k, t_{k+1} and ρ_k in place of $t_k^{(n)}$, $t_{k+1}^{(n)}$ and $\rho_k^{(n)}$, respectively.

Fix $\psi \in \mathbb{C}^{\delta, Z}$. By Lemma 9.9 and Lemma 9.10, Statement 9.2 holds with $T = \rho_k$, so if $\rho_k = t_{k+1}$, we are done. Suppose that $\rho_k < t_{k+1}$. Let $S \in (\rho_k, t_{k+1})$. By the definition (9.7) of t_{k+1}, $|\mathcal{I}(Z(t))| \le n$ for all $t \in [S, t_{k+1})$. Since Statement 9.2 holds with $T = \rho_k$ and $S < \rho_k$, $\nabla_\psi \bar{\Gamma}(X)(S)$ exists. Define X^S, Z^S, Y^S, $\hat{\psi}^S$ as in (2.7)–(2.9) and (7.6). Define $\tau^S \doteq \inf\{t \in [0, \infty) : Z^S \in \mathcal{W}\} = \tau - S$, where we have used (2.8) and the fact that $S \in [0, \tau)$. Then, since (Z, Y) satisfies the boundary jitter property on $[0, \tau)$, it is straightforward to verify, using (2.8)–(2.9), that (Z^S, Y^S) satisfies the boundary jitter property on $[0, \tau^S)$. Using (2.8) and (7.6), it is readily verified that $\hat{\psi}^S \in \mathbb{C}^{\delta, Z^S}$. Define $\theta_n^S \doteq \inf\{t \in [0, \tau^S) : |\mathcal{I}(Z^S(t))| \ge n\} \wedge \tau^S$, so by (9.7) and the fact that $|\mathcal{I}(Z^S(t))| = |\mathcal{I}(Z(S+t))| \le n$ for all $t \in [0, t_{k+1} - S)$, we have $t_{k+1} = S + \theta_n^S$. Since Statement 9.4 holds by assumption, Statement 9.2 holds with X^S, Z^S, $\hat{\psi}^S$, ϕ^S, η^S and θ_n^S in place of X, Z, ψ, ϕ, η and T, respectively. Therefore, by Lemma 9.7 (with t_k and $S + \theta_n^S$ in place of T and U, respectively), Statement 9.2 holds with t_{k+1} in place of T. $\square$

We conclude this section with the proof of Lemma 9.5.

PROOF OF LEMMA 9.5. By assumption, Statement 9.2 holds with $T = \theta_n$. Since $t_1^{(n)} \doteq \theta_n$, Statement 9.2 holds with $T = t_1^{(n)}$. Then Lemma 9.11 and the principle of mathematical induction imply that Statement 9.2 holds with $T = t_k^{(n)}$ for $1 \leq k < K_n + 1$. Since either $K_n < \infty$ and $t_{K_n}^{(n)} = \theta_{n+1}$, or $K_n = \infty$ and $t_k^{(n)} \to \theta_{n+1}$ as $k \to \infty$, it follows that Statement 9.2 holds with $T = \theta_{n+1}$. $\square$

APPENDIX A: PROOF OF AN APPROXIMATION RESULT

We now prove that if Z satisfies condition 2 of the boundary jitter property, then $\mathbb{C}^Z$ is dense in $\mathbb{C}$.

PROOF OF LEMMA 9.1. Fix $\psi \in \mathbb{C}$. We need to show that given $T < \infty$ and $\varepsilon > 0$, there exists $\delta > 0$ and $\zeta \in \mathbb{C}^{\delta, Z}$ such that $\|\psi - \zeta\|_T < \varepsilon$. Let $T < \infty$ and $\varepsilon > 0$ be arbitrary. Since ψ is uniformly continuous on the compact interval $[0, T]$, we can choose $\gamma > 0$ such that

$$w_T(\psi, \gamma) \doteq \sup_{0 \leq s < t \leq T, |t-s| < \gamma} |\psi(t) - \psi(s)| < \varepsilon.$$

Since Z is continuous and $G \setminus \mathcal{N}$ is relatively open in G, $\{s \in (0, T) : Z(s) \in G \setminus \mathcal{N}\}$ is open and can thus be written as the countable union of disjoint open intervals $\{(s_j, t_j)\}_{j \in \mathbb{N}} \subseteq (0, T)$. By condition 2 of the boundary jitter property, we can choose $m \in \mathbb{N}$ sufficiently large so that $\sum_{j=1,\ldots,m} |t_j - s_j| \geq T - \gamma/4$. Without loss of generality, we can assume the intervals are ordered so that $t_j \leq s_{j+1}$ for $j = 1, \ldots, m - 1$. Consequently, $s_1 \leq \gamma/4$, $t_m \geq T - \gamma/4$ and

$$\{t \in [0, \infty) : Z(t) \in \mathcal{N}\} \subseteq [0, s_1] \cup \left(\bigcup_{j=1,\ldots,m-1} [t_j, s_{j+1}] \right) \cup [t_m, \infty).$$

Let $0 < \delta < \frac{\gamma}{4} \wedge \frac{1}{3} \min_{j=1,\ldots,m}(t_j - s_j)$ and define the partially linearly interpolated paths $\zeta \in \mathbb{C}$ as follows: set $\zeta(t) \doteq \psi(s_1 + \delta)$ for all $t \in [0, s_1 + \delta]$ and for $j = 1, \ldots, m - 1$, define

$$\zeta(t) \doteq \begin{cases} \psi(t), & t \in [s_j + \delta, t_j - 2\delta], \\ \psi(t_j - 2\delta) + \dfrac{\psi(s_{j+1} + \delta) - \psi(t_j - 2\delta)}{\delta} \\ \quad \times (t - t_j + 2\delta), & t \in [t_j - 2\delta, t_j - \delta], \\ \psi(s_{j+1} + \delta), & t \in [t_j - \delta, s_{j+1} + \delta]. \end{cases}$$

Set $\zeta(t) \doteq \psi(t)$ for all $t \in [s_m + \delta, t_m - \delta]$ and $\zeta(t) \doteq \psi(t_m - \delta)$ for all $t \in [t_m - \delta, \infty)$. By definition, ζ is constant on a δ-neighborhood of I in $[0, \infty)$, so $\zeta \in \mathbb{C}^{\delta, Z}$. We are left to show that $\|\psi - \zeta\|_T < \varepsilon$. Since $\delta < \gamma/4$, $s_1 < \gamma/4$, $T - t_m <$

$\gamma/4$ and $s_{j+1} - t_j < \gamma/4$ for all $j = 1, \ldots, m - 1$, we have

$$
\begin{aligned}
\|\psi - \zeta\|_T \leq{} & \sup_{0 \leq t \leq s_1 + \delta} \left| \psi(t) - \psi(s_1 + \delta) \right| \\
& \vee \max_{j=1,\ldots,m-1} \sup_{t \in [t_j - 2\delta, t_j - \delta]} \left| \psi(t) - \psi(s_{j+1} + \delta) \right| \\
& \vee \max_{j=1,\ldots,m-1} \sup_{t \in [t_j - 2\delta, t_j - \delta]} \left| \psi(t) - \psi(t_j - 2\delta) \right| \\
& \vee \max_{j=1,\ldots,m-1} \sup_{t \in [t_j - \delta, s_{j+1} + \delta]} \left| \psi(t) - \psi(s_{j+1} + \delta) \right| \\
& \vee \sup_{t_m - \delta \leq t \leq T} \left| \psi(t) - \psi(t_m - \delta) \right| \\
\leq{} & w_T(\psi, \gamma) < \varepsilon,
\end{aligned}
$$

which is our desired conclusion. □

APPENDIX B: PRESERVATION OF LIPSCHITZ CONTINUITY OF THE ESM UNDER PROJECTIONS

Recall that Lemma 9.8 states that certain orthogonal projections of solutions to the ESP satisfy Lipschitz continuity properties. In order to prove the lemma, we show that these orthogonal projections of solutions to the ESP satisfy a transformed ESP in which the directions of reflection are orthogonally projected. Fix an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$. Given a subset $I \subseteq \mathcal{I}$, recall that $\Pi_I : \mathbb{R}^J \mapsto \text{span}(\{n_i, i \in I\})$ denotes the orthogonal projection with respect to the usual Euclidean inner product $\langle \cdot, \cdot \rangle$. For $f \in \mathbb{C}$, we define $\Pi_I f \in \mathbb{C}$ by $(\Pi_I f)(t) \doteq \Pi_I(f(t))$ for all $t \in [0, \infty)$.

LEMMA B.1.   *Suppose that (Z, Y) solves the ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ for $X \in \mathbb{C}$. Given $T \in (0, \infty]$ and $I \subseteq \mathcal{I}$, suppose that $\mathcal{I}(Z(t)) \subseteq I$ for all $t \in [0, T)$. Then $(\Pi_I Z, \Pi_I Y)$ solves the ESP $\{(\Pi_I d_i, n_i, c_i), i \in I\}$ for $\Pi_I X$ on $[0, T)$.*

PROOF.   By condition 1 of the ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ and the linearity of Π_I, $(\Pi_I Z, \Pi_I Y)$ satisfies condition 1 of the ESP $\{(\Pi_I d_i, n_i, c_i), i \in I\}$ for $\Pi_I X$. Next, Π_I being an orthogonal projection onto $\text{span}(\{n_i, i \in I\})$, we have

(B.1) $\qquad \langle \Pi_I Z(t), n_i \rangle = \langle Z(t), n_i \rangle \geq c_i \qquad \text{for all } t \in [0, T) \text{ and } i \in I,$

so $\Pi_I Z$ satisfies condition 2 of the ESP $\{(\Pi_I d_i, n_i, c_i), i \in I\}$. To show $(\Pi_I Z, \Pi_I Y)$ satisfies condition 3 of the ESP $\{(\Pi_I d_i, n_i, c_i), i \in I\}$, fix $0 \leq s < t < T$. Since (Z, Y) satisfies the ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$, (2.5) implies that there exist $r_i \geq 0$, $i \in \bigcup_{u \in (s,t]} \mathcal{I}(Z(u))$, such that

$$
Y(t) - Y(s) = \sum_{i \in \bigcup_{u \in (s,t]} \mathcal{I}(Z(u))} r_i d_i = \sum_{i \in \bigcup_{u \in (s,t]} \mathcal{I}(\Pi_I Z(u))} r_i d_i,
$$

where the second equality uses the equality in (B.1) and the fact that $\mathcal{I}(Z(u)) \subseteq I$ for all $u \in (s, t]$. Then, by the linearity of Π_I,

$$\Pi_I Y(t) - \Pi_I Y(s) = \sum_{i \in \bigcup_{u \in (s,t]} \mathcal{I}(\Pi_I Z(u))} r_i \Pi_I d_i,$$

so $(\Pi_I Z, \Pi_I Y)$ satisfies condition 3 of the ESP $\{(\Pi_I d_i, n_i, c_i), i \in I\}$. Lastly, the fact that $\Pi_I Y(0) \in \text{cone}(\{d_i, i \in \mathcal{I}(\Pi_I Z(0))\})$ follows from an argument analogous to the one used to prove that condition 3 of the ESP holds, so we omit it. $\square$

The next result shows that if the ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ satisfies Assumption 2.8, then the transformed ESP $\{(\Pi_I d_i, n_i, c_i), i \in I\}$ also satisfies Assumption 2.8.

LEMMA B.2. *Given an ESP $\{(d_i, n_i, c_i), i \in \mathcal{I}\}$ satisfying Assumption 2.8 and $I \subseteq \mathcal{I}$, there is a compact, convex, symmetric set B_I with $0 \in B_I^\circ$ such that for all $i \in I$,*

$$(\text{B.2}) \qquad \begin{Bmatrix} \tilde{z} \in \partial B_I \\ |\langle \tilde{z}, n_i \rangle| < 1 \end{Bmatrix} \implies \langle \tilde{v}, \Pi_I d_i \rangle = 0 \qquad \text{for all } \tilde{v} \in \nu_{B_I}(\tilde{z}).$$

In other words, the ESP $\{(\Pi_I d_i, n_i, c_i), i \in I\}$ satisfies Assumption 2.8.

PROOF. Fix $I \subseteq \mathcal{I}$ and let $V \doteq \text{span}(\{n_i, i \in I\})$. By Assumption 2.8, there is a compact, convex, symmetric set B with $0 \in B^\circ$ such that (2.10) holds. Define B_I by

$$(\text{B.3}) \qquad B_I \doteq \{z + y : z \in B \cap V, y \in V^\perp, |y| \leq 1\}.$$

Then B_I is a compact, convex, symmetric set with $0 \in B_I^\circ$ and

$$(\text{B.4}) \qquad \begin{aligned} \partial B_I &= \{z + y : z \in \partial B \cap V, y \in V^\perp, |y| \leq 1\} \\ &\cup \{z + y : z \in B^\circ \cap V, y \in V^\perp, |y| = 1\}. \end{aligned}$$

We now prove that (B.2) holds. Suppose $\tilde{z} \in \partial B_I$ satisfies $|\langle \tilde{z}, n_i \rangle| < 1$ for some $i \in I$ and let $\tilde{v} \in \nu_{B_I}(\tilde{z})$. By (B.4), $\tilde{z} = z + y$ where either:

  (i) $z \in \partial B \cap V$ and $y \in V^\perp$ satisfies $|y| \leq 1$, or
  (ii) $z \in B^\circ \cap V$ and $y \in V^\perp$ satisfies $|y| = 1$.

In either case, since $y \in V^\perp$, we have $|\langle z, n_i \rangle| = |\langle \tilde{z}, n_i \rangle| < 1$.

Suppose (i) holds. Given $u \in B$, define $\tilde{u} \doteq \Pi_I u + y \in B_I$. Then $\tilde{u} - \tilde{z} = \Pi_I u - \Pi_I z$, and hence,

$$(\text{B.5}) \qquad \langle \Pi_I \tilde{v}, u - z \rangle = \langle \tilde{v}, \Pi_I u - \Pi_I z \rangle = \langle \tilde{v}, \tilde{u} - \tilde{z} \rangle \geq 0,$$

where we have used that Π_I is self-adjoint in the first equality and the fact that $\tilde{v} \in \nu_{B_I}(\tilde{z})$, $\tilde{z} \in \partial B_I$ and $u \in B_I$ to justify the inequality. Since (B.5) holds for all $u \in B$, it follows that $\Pi_I \tilde{v} \in \nu_B(z)$. Thus, by (i) and (2.10), $\langle \tilde{v}, \Pi_I d_i \rangle = \langle \Pi_I \tilde{v}, d_i \rangle = 0$. On the other hand, suppose (ii) holds. Then $\langle \tilde{v}, \tilde{z} + \tilde{y} - z - y \rangle \geq 0$ for all $\tilde{z} \in B \cap V$ and $\tilde{y} \in V^\perp$ satisfying $|\tilde{y}| \leq 1$. Letting $\tilde{y} = y$, we see that $\langle \tilde{v}, \tilde{z} - z \rangle \geq 0$ for all $\tilde{z} \in B \cap V$. Since $z \in B^\circ$, this implies $\tilde{v} \in V^\perp$ and so $\langle \tilde{v}, \Pi_I d_i \rangle = \langle \Pi_I \tilde{v}, d_i \rangle = 0$. In either case, $\langle \tilde{v}, \Pi_I d_i \rangle = 0$. This completes the proof of (B.2). $\square$

PROOF OF LEMMA 9.8. By Lemma B.1, Lemma B.2 and Theorem 2.10, (9.4) holds for all $t \in [0, T)$. Since Z_1, Z_2, X_1, X_2 are continuous, (9.4) holds with $t = T$ as well. $\square$

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DIVISION OF APPLIED MATHEMATICS
BROWN UNIVERSITY
182 GEORGE STREET
PROVIDENCE, RHODE ISLAND 02912
USA
E-MAIL: David_Lipshutz@brown.edu
        Kavita_Ramanan@brown.edu