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ERGODIC THEORY FOR CONTROLLED MARKOV CHAINS WITH STATIONARY INPUTS[1]

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Consider a stochastic process $\boldsymbol{X}$ on a finite state space $\mathsf{X} = \{1, \dots, d\}$. It is conditionally Markov, given a real-valued "input process" $\boldsymbol{\zeta}$. This is assumed to be small, which is modeled through the scaling,

$$\zeta_t = \varepsilon \zeta_t^1, \qquad 0 \le \varepsilon \le 1,$$

where $\boldsymbol{\zeta}^1$ is a bounded stationary process. The following conclusions are obtained, subject to smoothness assumptions on the controlled transition matrix and a mixing condition on $\boldsymbol{\zeta}$:

(i) A stationary version of the process is constructed, that is coupled with a stationary version of the Markov chain $\boldsymbol{X}^\bullet$ obtained with $\boldsymbol{\zeta} \equiv 0$. The triple $(\boldsymbol{X}, \boldsymbol{X}^\bullet, \boldsymbol{\zeta})$ is a jointly stationary process satisfying

$$\mathsf{P}\{X(t) \ne X^\bullet(t)\} = O(\varepsilon).$$

Moreover, a second-order Taylor-series approximation is obtained:

$$\mathsf{P}\{X(t) = i\} = \mathsf{P}\{X^\bullet(t) = i\} + \varepsilon^2 \pi^{(2)}(i) + o(\varepsilon^2), \qquad 1 \le i \le d,$$

with an explicit formula for the vector $\pi^{(2)} \in \mathbb{R}^d$.

(ii) For any $m \ge 1$ and any function $f : \{1, \dots, d\} \times \mathbb{R} \to \mathbb{R}^m$, the stationary stochastic process $Y(t) = f(X(t), \zeta(t))$ has a power spectral density S_f that admits a second-order Taylor series expansion: A function $\mathrm{S}_f^{(2)} : [-\pi, \pi] \to \mathbb{C}^{m \times m}$ is constructed such that

$$\mathrm{S}_f(\theta) = \mathrm{S}_f^\bullet(\theta) + \varepsilon^2 \mathrm{S}_f^{(2)}(\theta) + o(\varepsilon^2), \qquad \theta \in [-\pi, \pi]$$

in which the first term is the power spectral density obtained with $\varepsilon = 0$. An explicit formula for the function $\mathrm{S}_f^{(2)}$ is obtained, based in part on the bounds in (i).

The results are illustrated with two general examples: mean field games, and a version of the timing channel of Anantharam and Verdu.

1. Introduction.

This paper concerns second-order ergodic theory for a controlled Markov chain. Consider for the sake of illustration a stochastic process $\boldsymbol{X}$

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on a finite state space $\mathsf{X} = \{1, \dots, d\}$ which evolves together with a real-valued stationary sequence $\boldsymbol{\zeta}$ and an i.i.d. sequence $\boldsymbol{N}$ according to the recursion

$$X_{t+1} = \varphi(X_t, \zeta_t, N_{t+1}), \qquad t \in \mathbb{Z}, \tag{1}$$

where $\varphi : \mathsf{X} \times \mathbb{R}^2 \to \mathsf{X}$ is Borel measurable. The solution is denoted $\boldsymbol{X}^\bullet$ in the special case $\boldsymbol{\zeta} \equiv 0$: a time-homogeneous Markov chain.

In the generality of this paper, we cannot expect to compute exact statistics of $\boldsymbol{X}$, such as the marginal distribution. We instead obtain conditions under which a stationary solution to (1) exists, and obtain approximations of the statistics of $\boldsymbol{X}$. Under the assumptions imposed in this paper, the joint stationary process $(\boldsymbol{X}, \boldsymbol{X}^\bullet, \boldsymbol{\zeta})$ is constructed on the same probability space. This makes it possible to compare the statistics of $\boldsymbol{X}$ with the stationary Markov chain $\boldsymbol{X}^\bullet$.

It is assumed that $\boldsymbol{\zeta}$ is small: It is simplest to consider a family of processes, parameterized by a small constant $\varepsilon > 0$,

$$\zeta_t = \varepsilon \zeta_t^1,$$

where $\boldsymbol{\zeta}^1 = \{\zeta_t^1\}$ is a bounded sequence. The construction of $(\boldsymbol{X}, \boldsymbol{X}^\bullet, \boldsymbol{\zeta})$ is obtained so that

$$\mathsf{P}\{X_t \neq X_t^\bullet\} = O(\varepsilon). \tag{2}$$

In order to apply techniques from second-order statistics, the process is lifted to the simplex in $\mathbb{R}^d$ through the following notational convention:

$$\Gamma_t = \big[e^j\big]^T \qquad \text{when } X_t = j, \tag{3}$$

where e^j denotes the jth standard basis element in $\mathbb{R}^d$, and hence Γ_t is a row vector. This is a standard construction; it is useful since the evolution of $\{\Gamma_t\}$ can be expressed as a linear state space model driven by an uncorrelated "noise process" [see (8) below]. This linear representation is used in [14] to construct a Kalman filter for a time-homogeneous Markov chain (without the input $\boldsymbol{\zeta}$), and these results are extended to a class of controlled Markov chains in [6].

The initial motivation for [6], as well as the research described here, is application to distributed control for the purposes of "demand dispatch" using distributed resources in a power grid. The results of the present paper are applied in [4, 5] to obtain performance approximations in the same power grid model. Similar bounds were previously obtained in [7], but this is the first paper to obtain an exact second-order Taylor series approximation for second-order statistics.

The main contribution of this paper is to obtain tight approximations for the joint auto-correlation function for $(\boldsymbol{\Gamma}, \boldsymbol{\zeta})$, and hence also its power spectral density. To obtain these results requires the coupling bound (2), a second-order Taylor series expansion for $\pi_\varepsilon = \mathsf{E}[\Gamma_t]$ in steady-state, and surprisingly complex calculations for a linearized model.

The goals of the present work are similar to elements of singular perturbation theory for Markov chains (see [11, 21] and their references). In some of our approximations, we borrow one technique from [19]—the use of the fundamental matrix appears in the approximation of π_ε; see (19) for a definition, and further explanation following this equation.

The main results are summarized in Section 2, with all of the technical proofs contained in the Appendices. Application to mean-field games is discussed in Section 3, and Section 4 contains numerical results for an application to information theory—a variant of the timing channel introduced in [2]. Conclusions and directions for future research are contained in Section 5.

2. Model and main results. Consider an irreducible and aperiodic Markov chain $\boldsymbol{X}^\bullet$ evolving on a finite state space $\mathsf{X} = \{1, \dots, d\}$, with transition matrix P_0. This admits a stationary realization on the two sided time-interval $\mathbb{Z}$, whose marginal distribution π_0 is the invariant probability mass function (p.m.f.) for P_0, satisfying $\pi_0 P_0 = \pi_0$. The goal of this paper is to investigate how the statistics change when the dynamics are subject to an exogenous disturbance.

2.1. *Controlled Markov model.* The stochastic process $\boldsymbol{X}$ considered in this paper also evolves on the finite state space X. The "disturbance" in the controlled model is a one-dimensional stationary process denoted $\boldsymbol{\zeta} = \{\zeta_t : -\infty < t < \infty\}$. A controlled transition matrix $\{P_\zeta : \zeta \in \mathbb{R}\}$ describes the dynamics of the process

$$\mathsf{P}\{X_{t+1} = k | \zeta_s, X_s : s \le t\} = P_\zeta(j,k) \qquad \text{when } \zeta_t = \zeta,\ X_t = j. \tag{4}$$

It is assumed that P_ζ is a smooth function of ζ, and that P_0 is the transition matrix for $\boldsymbol{X}^\bullet$.

Since $\boldsymbol{X}$ is no longer Markovian, we cannot apply standard Markov chain theory to investigate properties of a stationary version of $\boldsymbol{X}$. Instead we apply linear systems theory, and for this we require a linear systems representation for the controlled stochastic process.

This is obtained by embedding the process in $\mathbb{R}^d$ through the indicator process Γ_t defined in (3). Linear dynamics are obtained by considering a specific realization of the model. We assume that there is a $d \times d$ matrix-valued function $\mathcal{G}$ and an i.i.d. sequence $\boldsymbol{N}$ for which

$$\Gamma_{t+1} = \Gamma_t G_{t+1}, \qquad G_{t+1} = \mathcal{G}(\zeta_t, N_{t+1}). \tag{5}$$

It is assumed moreover that $\boldsymbol{N}$ is independent of $\boldsymbol{\zeta}$, and that the entries of $\mathcal{G}$ are zero or one, with

$$\sum_{k=1}^{d} \mathcal{G}_{j,k}(z,n) = 1 \qquad \text{for all } j, z, n.$$

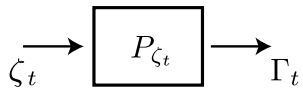

FIG. 1. *Markov model.*

We have $\mathcal{G}_{j,k}(z,n) = \mathbb{I}\{\varphi(j,z,n) = k\}$ for the nonlinear state space model (1). In general, it follows from (4) that, for each t,

$$\mathsf{E}\big[\mathcal{G}(\zeta_t, N_{t+1})|\zeta_{-\infty}^{\infty}\big] = P_{\zeta_t}. \tag{6}$$

The random linear system (5) illustrated in Figure 1 is the focus of study in this paper. The sequence $\mathbf{\Gamma} = \{\Gamma_t\}$ is viewed as a state process, that is driven by the disturbance (or "input") $\boldsymbol{\zeta}$. The state process evolves on the extreme points of the simplex in $\mathbb{R}^d$. We let $\mathbf{\Gamma}^\bullet = \{\Gamma_t^\bullet\}$ denote the stationary Markov chain obtained with $\boldsymbol{\zeta} \equiv 0$.

The main assumptions are summarized in the following:

A1: The transition matrix P_0 is irreducible and aperiodic. The matrix valued function P_ζ is twice continuously differentiable (C^2) in a neighborhood of $\zeta = 0$, and the second derivative is Lipschitz continuous.

A2: $\zeta_t = \varepsilon\zeta_t^1$ where $\boldsymbol{\zeta}^1 = \{\zeta_t^1 : t \in \mathbb{Z}\}$ is a real-valued stationary stochastic process with zero mean. The following additional assumptions are imposed:

(i) It is bounded, $|\zeta_t^1| \le 1$ for all t with probability one. Hence $\sigma^2_{\zeta^1} = \mathsf{E}[(\zeta_t^1)^2] \le 1$.

(ii) Its auto-covariance is absolutely summable:

$$\sum_{t=0}^{\infty} \big|R_{\zeta^1}(t)\big| < \infty.$$

The power spectral density S_{ζ^1} exists and is continuous under Assumption A2(ii). It also admits a spectral factor, denoted H_{ζ^1}:

$$\mathrm{S}_{\zeta^1}(\theta) = H_{\zeta^1}\big(e^{j\theta}\big)H_{\zeta^1}\big(e^{-j\theta}\big), \qquad -\pi \le \theta \le \pi. \tag{7}$$

See [3] for background.

Assumption A1 is used to obtain the approximation of (5) by an LTI (linear time invariant) system. The following intermediate step is an extension of Lemma 1 in [14], which is used to derive a Kalman filter for an uncontrolled Markov chain.

PROPOSITION 2.1. *The random linear system* (5) *can be represented as*

$$\Gamma_{t+1} = \Gamma_t P_{\zeta_t} + \Delta_{t+1}, \tag{8}$$

where $\Delta_{t+1} = \Gamma_t(G_{t+1} - P_{\zeta_t})$. *This is a martingale difference sequence, with covariance matrix*

$$\Sigma^\Delta = \mathsf{Cov}\big(\Gamma_t[G_{t+1} - P_{\zeta_t}]\big) = \mathsf{E}\big[\Lambda^\Gamma_{t+1} - P_{\zeta_t}^T \Lambda_t^\Gamma P_{\zeta_t}\big], \tag{9}$$

where Λ_t^Γ *is the diagonal matrix with diagonal entries* $\{\Gamma_t(i) : 1 \le i \le d\}$. *Moreover,*

$$R_{\Delta,\zeta}(t) = 0 \qquad \textit{for all } t. \tag{10}$$

That is, $\boldsymbol{\Delta}$ *and* $\boldsymbol{\zeta}$ *are uncorrelated.*

We next apply the second-order Taylor series approximation:

$$P_{\zeta_t} = P_0 + \mathcal{E}\zeta_t + \frac{1}{2}\mathcal{W}\zeta_t^2 + O(\varepsilon^3),$$

where $\mathcal{E}$ and $\mathcal{W}$ denote the first and second derivatives of P_ζ, evaluated at $\zeta = 0$:

$$\frac{d}{d\zeta}P_\zeta\Big|_{\zeta=0} = \mathcal{E}, \qquad \frac{d^2}{d\zeta^2}P_\zeta\Big|_{\zeta=0} = \mathcal{W}.$$

The $O(\varepsilon^3)$ bound holds under the Lipschitz condition for the second derivative of P_ζ. The following identities will be useful: $\mathcal{E}1 = \mathcal{W}1 = 0$. This follows from the definitions and the fact that $P_\zeta 1 = 1$ for all ζ. In particular,

$$\mathcal{E}1 = \frac{d}{d\zeta}P_\zeta\Big|_{\zeta=0} 1 = \frac{d}{d\zeta}P_\zeta 1\Big|_{\zeta=0} = \frac{d}{d\zeta}1 = 0. \tag{11}$$

The recursion (8) can be approximated as

$$\Gamma_{t+1} = \Gamma_t\Big(P_0 + \mathcal{E}\zeta_t + \frac{1}{2}\mathcal{W}\zeta_t^2\Big) + \Delta_{t+1} + O(\varepsilon^3).$$

This is the LTI approximation.

PROPOSITION 2.2. *The recursion* (5) *can be approximated as follows*:

$$\Gamma_{t+1} = \Gamma_t P_0 + D_{t+1} + O(\varepsilon^3), \tag{12}$$

where $D_{t+1} = B_t^T\zeta_t + V_t^T\zeta_t^2 + \Delta_{t+1}$, *with*

$$B_t^T = \Gamma_t\mathcal{E}, \qquad V_t^T = \frac{1}{2}\Gamma_t\mathcal{W}. \tag{13}$$

Applying the LTI approximation (12), an approximation for the auto-correlation of $(\boldsymbol{\Gamma}, \boldsymbol{\zeta})$ is obtained from an approximation for the pair process $(\boldsymbol{D}, \boldsymbol{\zeta})$. Since $\boldsymbol{D}$ is taken as a row vector, we use the following notation for the auto-correlation of $(\boldsymbol{D}, \boldsymbol{\zeta})$:

$$R(t) = \begin{bmatrix} R_D(t) & R_{D,\zeta}(t) \\ R_{D,\zeta}(-t)^T & R_\zeta(t) \end{bmatrix}, \tag{14}$$

where $R_D(t) = \mathsf{E}[D(t)^T D(0)]$, $R_{D,\zeta}(t) = \mathsf{E}[D(t)^T\zeta(0)]$, and the expectations are taken in steady-state.

The *existence* of a steady-state solution is established in Proposition 2.3 that follows.

2.2. *Correlation formulae and approximations*. Under Assumptions A1 and A2, we obtain a coupling result, which plays a crucial role in the approximations that follow. We write $\Gamma_t = \Gamma_t^\bullet + \widetilde{O}(\varepsilon)$ if

$$\mathsf{E}\big[\|\Gamma_t - \Gamma_t^\bullet\|\big] = O(\varepsilon),$$

which implies that (2) also holds. We adopt similar notation for other random variables. The following result is proven in Appendix A.

PROPOSITION 2.3. *Under Assumptions* A1 *and* A2, *there exists* $\varepsilon_0 > 0$ *such that the following holds for each* $\varepsilon \in (0, \varepsilon_0]$: *the two process* $\boldsymbol{\Gamma}$ *and* $\boldsymbol{\Gamma}^\bullet$ *can be constructed so that* $(\boldsymbol{\Gamma}, \boldsymbol{\Gamma}^\bullet, \boldsymbol{\zeta})$ *is jointly stationary on the two-sided time interval* $\mathbb{Z}$, $\Gamma_t^\bullet$ *is independent of* $\boldsymbol{\zeta}$, *and moreover*

$$\Gamma_t = \Gamma_t^\bullet + \widetilde{O}(\varepsilon), \tag{15}$$

$$\mathsf{E}[\Gamma_t \zeta_t] = O\big(\varepsilon^2\big). \tag{16}$$

Consequently, for the stationary process,

$$B_t = B_t^\bullet + \widetilde{O}(\varepsilon), \qquad V_t = V_t^\bullet + \widetilde{O}(\varepsilon), \qquad \Delta_t = \Delta_t^\bullet + \widetilde{O}(\varepsilon). \tag{17}$$

The following strengthening of Assumption A2 is useful in computations:

A3: The transfer function H_{ζ^1} in (7) is rational, with distinct poles $\{\rho_1, \ldots, \rho_{n_z}\}$ satisfying $|\rho_i| < 1$ for each i.

Under A2 and A3 the auto-covariance function for $\boldsymbol{\zeta}$ can be expressed as a sum of geometrically decaying terms,

$$R_\zeta(t) = \varepsilon^2 \sum_{k=1}^{n_z} a_k \rho_k^{|t|}, \tag{18}$$

where the $\{a_k\}$ can be determined from H_{ζ^1}. Approximations for the auto-correlation functions $R_{D,\zeta}(t)$ and $R_D(t)$ in (14) are given in Theorem 2.4.

As in the perturbation theory of [19], one component in these approximations is based on the *fundamental matrix*,

$$U_1 = [I - P_0 + 1 \otimes \pi_0]^{-1}, \tag{19}$$

where $1 \otimes \pi_0$ denotes the matrix whose rows are identical, and equal to π_0. Because the chain is irreducible and aperiodic, this can be expressed as a power series expansion,

$$U_1 = I + \sum_{k=1}^{\infty} [P_0 - 1 \otimes \pi_0]^k.$$

The summand can also be expressed $[P_0 - 1 \otimes \pi_0]^k = [P_0^k - 1 \otimes \pi_0]$, $k \geq 1$. Hence convergence of the sum follows from the mean ergodic theorem,

$$\lim_{k\to\infty} P_0^k = 1 \otimes \pi_0, \tag{20}$$

where the rate of convergence is geometric.

THEOREM 2.4. *Suppose that Assumptions* A1 *and* A2 *hold, and consider the stationary process* $(\mathbf{\Gamma}, \mathbf{\Gamma}^\bullet, \boldsymbol{\zeta})$ *constructed in Proposition* 2.3, *with* $\varepsilon \in (0, \varepsilon_0]$. *Then, for each* t,

$$R_{D,\zeta}(t) = BR_\zeta(t-1) + O(\varepsilon^3), \tag{21}$$

$$R_D(t) = R_{B\zeta}(t) \tag{22a}$$

$$+ R_\Delta(t) \tag{22b}$$

$$+ R_{B\zeta,\Delta}(t-1) + R^T_{B\zeta,\Delta}(-t-1) \tag{22c}$$

$$+ R_{V\zeta^2,\Delta}(t-1) + R^T_{V\zeta^2,\Delta}(-t-1) \tag{22d}$$

$$+ O(\varepsilon^3)$$

in which $B^T = \pi_0 \mathcal{E}$ *in* (21), *and each component shown on the right-hand side of* (22a)–(22d) *is given below*:

(a) *The auto-correlation* $R_{B\zeta}(t) = \mathsf{E}[B_t \zeta_t B_0^T \zeta_0]$ *in* (22a) *admits the approximation*,

$$R_{B\zeta}(t) = \big(P_0^t \mathcal{E}\big)^T \Pi_0 \mathcal{E} R_\zeta(t) + O(\varepsilon^3), \qquad t \geq 0, \tag{23}$$

where $\Pi_0 = \operatorname{diag}(\pi_0)$.

(b) *The covariance for the martingale-difference sequence* $\mathbf{\Delta}$ *is given by* $R_\Delta(t) = 0$ *for* $t \neq 0$.

When $\varepsilon = 0$, *we have*

$$R^{\Delta^\bullet}(0) = \Sigma^{\Delta^\bullet} = \Pi_0 - P_0^T \Pi_0 P_0, \tag{24}$$

and for nonzero ε *this admits the approximation*

$$\begin{aligned} R_\Delta(0) &= \Sigma^\Delta \\ &= \Pi_\varepsilon - P_0^T \Pi_\varepsilon P_0 \\ &\quad - \big[P_0^T \operatorname{diag}\big(R_{\Gamma,\zeta}(0)\big)\mathcal{E} + \mathcal{E}^T \operatorname{diag}\big(R_{\Gamma,\zeta}(0)\big) P_0\big] \\ &\quad - \frac{1}{2} R_\zeta(0)\big[P_0^T \Pi_0 \mathcal{W} + 2\mathcal{E}^T \Pi_0 \mathcal{E} + \mathcal{W}^T \Pi_0 P_0\big] + O(\varepsilon^3), \end{aligned} \tag{25}$$

where $\Pi_\varepsilon = \operatorname{diag}(\pi_\varepsilon)$, *with* $\pi_\varepsilon = \mathsf{E}[\Gamma_t]$, *and*

$$R_{\Gamma,\zeta}(t) = \mathsf{E}\big[\Gamma(t)^T \zeta(0)\big], \qquad t \in \mathbb{Z}. \tag{26}$$

(c) *The cross-covariance* $R_{B\zeta,\Delta}(t) = \mathsf{E}[B_t\zeta_t\Delta_0]$ *admits the approximation*

$$R_{B\zeta,\Delta}(t) = \begin{cases} 0 & t<0, \\ \mathcal{E}^T R_{\Delta^2,\zeta}(0) + O(\varepsilon^3) & t=0, \\ \mathcal{E}^T A^t R_{\Delta^2,\zeta}(-t) & \\ \qquad + \mathcal{E}^T \sum_{i=0}^{t-1} A^{t-1-i}\mathcal{E}^T A^i R_\zeta(t-i)\Sigma^{\Delta^\bullet} + O(\varepsilon^3) & t\ge 1, \end{cases} \tag{27}$$

where $A = P_0^T$ *and*

$$R_{\Delta^2,\zeta}(t) = \mathsf{E}\big[\Delta_t^T\Delta_t\zeta_0\big]. \tag{28}$$

(d) *The cross-covariance* $R_{V\zeta^2,\Delta}(t) = \mathsf{E}[V_t\zeta_t^2\Delta_0]$ *admits the approximation*

$$R_{V\zeta^2,\Delta}(t) = \begin{cases} O(\varepsilon^3) & t<0, \\ \frac{1}{2}\sigma_\zeta^2\big(P_0^t\mathcal{W}\big)^T\Sigma^{\Delta^\bullet} + O(\varepsilon^3) & t\ge 0. \end{cases} \tag{29}$$

The derivation of Theorem 2.4 is given in Appendix E.

Theorem 2.4 leaves out an approximation for π_ε that is required in (25). It also leaves out an approximation for $R_{\Gamma,\zeta}(0)$ required in (25), and approximations for $\{R_{\Delta^2,\zeta}(t) : t\le 0\}$ in (27). These are obtained in the following.

PROPOSITION 2.5. *The following hold under Assumptions* A1 *and* A2:

(i) *The steady state mean admits the approximation*

$$\pi_\varepsilon = \pi_0 + \xi U_1 + O(\varepsilon^3), \tag{30}$$

where U_1 *is the fundamental matrix* (19), *and*

$$\xi = \big(R_{\Gamma,\zeta}(0)\big)^T\mathcal{E} + \frac{1}{2}\sigma_\zeta^2\pi_0\mathcal{W}. \tag{31}$$

(ii) *For* $t\ge 0$ *we have*

$$\begin{aligned} R_{\Delta^2,\zeta}(-t) &= \operatorname{diag}\big(R_{\Gamma,\zeta}(-t-1)P_0\big) \\ &\quad - P_0^T\operatorname{diag}\big(R_{\Gamma,\zeta}(-t-1)\big)P_0 + R_\zeta(t+1)\mathsf{E}\big[\mathcal{X}_\bullet^{(1)}\big] + O(\varepsilon^3), \end{aligned} \tag{32}$$

where $\mathsf{E}[\mathcal{X}_\bullet^{(1)}] := \operatorname{diag}(\pi_0\mathcal{E}) - (P_0^T\Pi_0\mathcal{E} + [P_0^T\Pi_0\mathcal{E}]^T)$.

(iii) *The correlation* $R_{\Gamma,\zeta}$ *is approximated as the infinite sum*

$$R_{\Gamma,\zeta}(t) = \varepsilon^2\sum_{i=1}^{\infty}\big(B^T P_0^{i-1}\big)^T R_{\zeta^1}(t-i) + O(\varepsilon^3), \qquad t\in\mathbb{Z}, \tag{33}$$

in which $\|B^T P_0^{i-1}\| \to 0$ *geometrically fast as* $i\to\infty$.

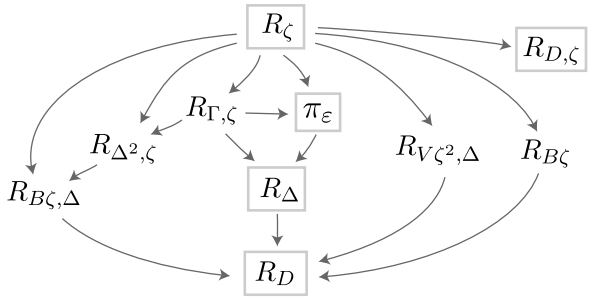

FIG. 2. *Dependency of autocorrelation functions involved in the approximations of* $R(t)$ *in* (14).

The proposition shows that the Taylor-series coefficient $\pi^{(2)} := \varepsilon^{-2}\xi U_1$ depends on the entire auto-covariance sequence R_ζ, along with the first and second derivatives of P_ζ.

The proof of (30) is given in Appendix C, (32) is given in Appendix D.2 and (33) is established in Appendix D.4. The geometric bound on the limit $\|B^T P_0^{i-1}\| \to 0$ follows from the ergodic limit (20), and the formula $B^T 1 = \pi_0 \mathcal{E} 1 = 0$ [see (11)].

The directed graph shown in Figure 2 summarizes the dependency between all of these terms. For example, the approximation of $R_{D,\zeta}$ only requires R_ζ, and the covariance Σ^Δ that defines R_Δ is a function of $R_{\Gamma,\zeta}$ and π_ε. The approximation of R_D is a function of the four correlation functions shown [as can also be seen from (22a)–(22d)]. The five boxed terms are those that are of interest to us directly; the remaining five terms are introduced only to obtain a closed set of algebraic equations.

Closed-form expressions for the approximations in Proposition 2.5 are possible under A3. The proof of Proposition 2.6 is given in Appendix D.4.

PROPOSITION 2.6. *Under* A1–A3, *the row vector* ξ *in* (31) *has the approximation*

$$\xi = \varepsilon^2 B^T \sum_{k=1}^{n_z} a_k \rho_k [I - \rho_k P_0]^{-1} \mathcal{E} + \frac{1}{2}\varepsilon^2 \sigma_{\zeta^1}^2 \pi_0 \mathcal{W} + O(\varepsilon^3). \tag{34}$$

2.3. *Power spectral density approximations.* Theorem 2.4 provides a second-order approximation of the auto-covariance function $\{R(t)\}$ defined in (14), which we denote $\{\widehat{R}(t)\}$. In particular, $\widehat{R}_D(t)$ is defined as the sum of (22a)–(22d). Based on this and Proposition 2.2, we obtain a second-order approximation $\{\widehat{R}^{\text{tot}}(t)\}$ of the auto-covariance function $\{R^{\text{tot}}(t)\}$ for the triple $(\boldsymbol{\Gamma}, \boldsymbol{D}, \boldsymbol{\zeta})$.

The power spectral density (PSD) of a stationary process is the Fourier transform of its *auto-covariance*. This matrix-valued function is denoted

$$\mathrm{S}(\theta) = \sum_{t=-\infty}^{\infty} \Sigma^{\text{tot}}(t) e^{-j\theta t}, \qquad \theta \in \mathbb{R},$$

in which $\Sigma^{\text{tot}}(t) := R^{\text{tot}}(t) - \mu\mu^T$, $t \geq 0$, with $\mu^T = \mathsf{E}[(\Gamma_t, D_t, \zeta_t)]$.

To define an approximation for S we must obtain an approximation $\{\widehat{\Sigma}(t)\}$ that is summable. It turns out that this is obtained from $\{\widehat{R}(t)\}$ without normalization. For each t, the $(2d+1)\times(2d+1)$ matrix is decomposed as follows:

$$\widehat{\Sigma}^{\mathrm{tot}}(t)=\begin{bmatrix}\widehat{\Sigma}_{\Gamma}(t) & \widehat{\Sigma}_{\Gamma,(D,\zeta)}(t)\\ \widehat{\Sigma}_{(D,\zeta),\Gamma}(t) & \widehat{\Sigma}(t)\end{bmatrix}$$

in which $\widehat{\Sigma}(t)=\widehat{R}(t)$, and the remaining terms are what would be obtained by ignoring the $O(\varepsilon^3)$ error term appearing in (12), and replacing P_0 by its deviation $P_0-1\otimes\pi_0$. Denote $\widetilde{A}=(P_0-1\otimes\pi_0)^T$, and

$$\widehat{\Sigma}_{\Gamma}(t)=\sum_{i,j=0}^{\infty}\widetilde{A}^i\,\widehat{R}_D(t-i+j)\big(\widetilde{A}^j\big)^T.$$

The matrix $\widehat{\Sigma}_{\Gamma,(D,\zeta)}(t)$ is the $(d+1)$-dimensional column vector whose first d components are

$$\widehat{\Sigma}_{\Gamma,D}(t)=\sum_{i=0}^{\infty}\widetilde{A}^i\,\widehat{R}_D(t-i)$$

and the final component is defined by the right-hand side of (33), ignoring the approximation error. This can be equivalently expressed,

$$\widehat{\Sigma}_{\Gamma,\zeta}(t)=\varepsilon^2\sum_{i=1}^{\infty}\widetilde{A}^{i-1}B\,R_{\zeta^1}(t-i),$$

where we have used the fact that $\widetilde{A}^kB=(P_0^T)^kB$ since $1^TB=\sum B_i=0$. Finally, $\widehat{\Sigma}_{(D,\zeta),\Gamma}(t)=\widehat{\Sigma}_{\Gamma,(D,\zeta)}(-t)^T$.

Denote

$$\widehat{\mathrm{S}}(\theta)=\sum_{t=-\infty}^{\infty}\widehat{\Sigma}^{\mathrm{tot}}(t)e^{-j\theta t},\qquad\theta\in\mathbb{R}.$$

It can be shown that the sequence $\{\widehat{R}^{\mathrm{tot}}(t)\}$ is absolutely summable, so that the approximation $\widehat{\mathrm{S}}$ is a continuous bounded function of θ. The following is an immediate corollary to Theorem 2.4.

PROPOSITION 2.7. *The approximation of the power spectral density of the stationary sequence* $\{D_{t+1}=\Delta_{t+1}+B_t^T\zeta_t+V_t^T\zeta_t^2\}$ *can be expressed as the sum*:

$$\begin{aligned}\widehat{\mathrm{S}}_D(\theta)=\widehat{\mathrm{S}}_{\Delta}(\theta)+\widehat{\mathrm{S}}_{B\zeta}(\theta)+e^{-j\theta}\widehat{\mathrm{S}}_{B\zeta,\Delta}(\theta)+\big[e^{-j\theta}\widehat{\mathrm{S}}_{B\zeta,\Delta}(\theta)\big]^*\\ +e^{-j\theta}\widehat{\mathrm{S}}_{V\zeta^2,\Delta}(\theta)+\big[e^{-j\theta}\widehat{\mathrm{S}}_{V\zeta^2,\Delta}(\theta)\big]^*\end{aligned}\tag{35}$$

in which each approximation on the right-hand side is obtained from the Fourier transform of the corresponding approximations (23)–(29) *in Theorem* 2.4, *and where "*" denotes complex-conjugate transpose*.

The power spectral density approximation for $\mathbf{\Gamma}$ *is given by*

$$\widehat{\mathrm{S}}_{\Gamma}(\theta) = \big[Ie^{-j\theta} - \widetilde{A}\big]^{-1}\widehat{\mathrm{S}}_{D}(\theta)\big[Ie^{j\theta} - \widetilde{A}^{T}\big]^{-1}$$

and the cross-power spectral density approximations are

$$\widehat{\mathrm{S}}_{\Gamma,D}(\theta) = \big[Ie^{-j\theta} - \widetilde{A}\big]^{-1}\widehat{\mathrm{S}}_{D}(\theta),$$

$$\widehat{\mathrm{S}}_{\Gamma,\zeta}(\theta) = \varepsilon^{2}\sum_{t=-\infty}^{\infty}\sum_{i=1}^{\infty}\widetilde{A}^{i-1}BR_{\zeta^{1}}(t-i)e^{-j\theta t}.$$

Under slightly stronger assumptions, we obtain a uniform bound for this approximation. The proof of Proposition 2.8 is given in Section E.4.

PROPOSITION 2.8. *Suppose that Assumptions* A1 *and* A2 *hold, and in addition* $R_{\zeta}(t) \to 0$ *geometrically fast as* $t \to \infty$. *Then the uniform approximation holds*: *For any* $\varrho \in (0,1)$,

$$\widehat{\mathrm{S}}(\theta) = \mathrm{S}(\theta) + O\big(\varepsilon^{2+\varrho}\big), \qquad \theta \in \mathbb{R}.$$

3. Example: Individual in a mean-field limit. The theory of mean field games is an active area of research today. While some of the basic concepts can be found in statistical physics, much of the current research has been inspired by more recent contributions [1, 12, 13, 15, 20] with applications to control and economics, and applications to power systems in [16, 17].

The initial motivation for the research reported in this paper is the mean-field model of [18]; as in [17], the setting is not a game since no local optimization is assumed. Instead, each "agent" (an electric load) responds to a global command signal, and at each discrete time t, it changes its state at time $t+1$ based on this information and its local state. The goal of [17, 18] is *demand dispatch*: power consumption from the collection of loads is varied automatically and continuously to provide service to the grid, without impacting QoS (quality of service) to the consumers.

Figure 3 illustrates the control architecture, in which $\boldsymbol{r} = \{r_t\}$ is a reference signal that the normalized aggregate power deviation $\boldsymbol{y}^N = \{y_t^N\}$ is intended to

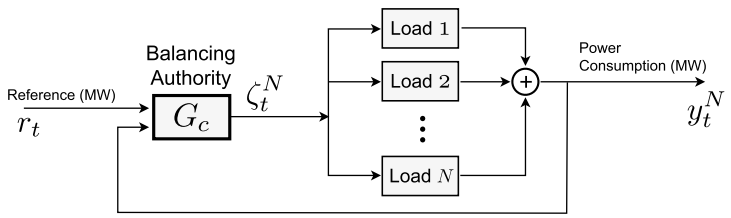

FIG. 3. *Demand Dispatch model of* [5, 18].

track (perhaps scaled by a constant depending on N). It is assumed that y_t^N is a linear function of the histogram of states μ_t^N: for a function $\mathcal{U} : \mathsf{X} \to \mathbb{R}$,

$$y_t^N = \langle \mu^N, \mathcal{U} \rangle := \sum_{x \in \mathsf{X}} \mathcal{U}(x) \mu_t^N(x),$$

where

$$\mu_t^N(x) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}\{X^i(t) = x\}, \qquad x \in \mathsf{X}.$$

The pair $(\boldsymbol{r}, \boldsymbol{y}^N)$ is transformed to create a signal $\boldsymbol{\zeta}^N$ via the block G_c shown in the figure, which is assumed to be a causal linear transfer function in [4, 5, 18]. Each load evolves probabilistically according to a controlled transition matrix. For each load, the only departure from Figure 1 is that the signal $\boldsymbol{\zeta}^N$ depends on N since it is a functional of $\boldsymbol{r}$ and $\boldsymbol{y}^N$.

It is shown under general conditions that a limiting model exists:

$$\mu_t = \lim_{N \to \infty} \mu_t^N, \qquad \zeta_t = \lim_{N \to \infty} \zeta_t^N \qquad t \geq 0, \text{ a.s.},$$

in which the row vectors $\{\mu_t\}$ evolve according to the mean-field equation

$$\mu_{t+1} = \mu_t P_{\zeta_t}, \qquad t \geq 0.$$

It follows that $\boldsymbol{\zeta}$ is a deterministic functional of $\boldsymbol{r}$ and the initial distribution μ_0. This limiting model is illustrated in Figure 4.

In the aforementioned work on mean field games, the limiting model is used to understand aggregate behavior such as $\boldsymbol{y}$ and $\boldsymbol{\mu}$. Here and in [5], the mean field model is used to approximate the statistics of an individual load in this demand dispatch architecture.

There is no space here to present any detailed numerical examples, so we show results from one numerical experiment from the dissertation [4]; full details can be found in [4], Section 5.3.2. In these experiments, the stochastic process $\boldsymbol{r}$ is assumed stationary and scaled by $\varepsilon > 0$. A stationary realization of $\boldsymbol{\zeta}$ is defined consistently so that it is scaled by the same factor.

In these experiments, QoS for an individual load is defined to be a discounted sum of the deviation of power consumption from its nominal mean value $\bar{\mathcal{U}}$:

$$\text{QoS}(t) = \sum_{k=0}^{\infty} \beta^k \{\mathcal{U}(X(t-k)) - \bar{\mathcal{U}}\}, \qquad t \in \mathbb{Z},$$

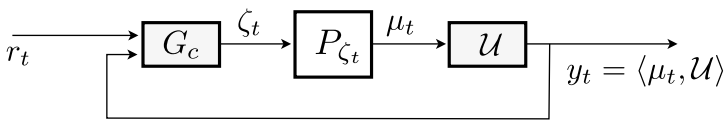

FIG. 4. *Mean field model obtained from Figure 3 as the limit as N tends to infinity.*

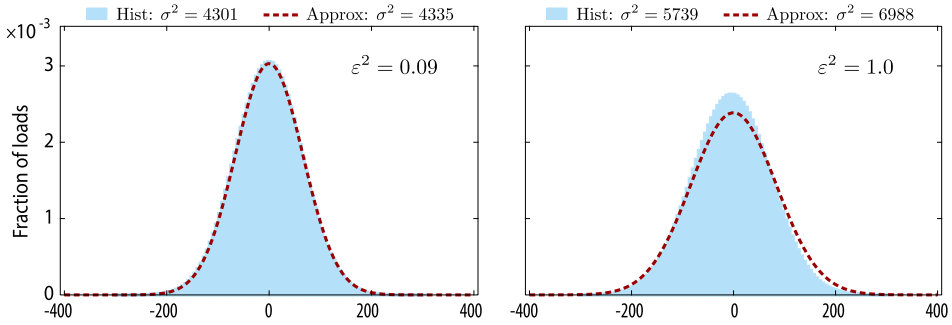

FIG. 5. *Histogram of QoS and its approximation obtained from Proposition* 2.8 (*plots based on Figures* 5–8 *of* [4]).

with $\beta < 1$. An approximation for the power spectral density of $\{\mathcal{U}(X(t)) : t \in \mathbb{Z}\}$ can be obtained from Proposition 2.8 and based on this we obtain an approximation for the power spectral density of QoS. The resulting variance approximations are illustrated in Figure 5: the histograms of QoS are based on 10^4 independent experiments.

Even with $\varepsilon = 1$, the approximation of the mean and variance obtained from Proposition 2.8 is remarkably accurate. The approximations are nearly exact with $\varepsilon = 0.3$. The histograms appear Gaussian because β is close to unity in these experiments.

4. Example: Bits through queues. The following example is motivated by the communication model of [2]. There is a sender that wishes to send data to a receiver. Neither has access to a communication channel in the usual sense. Instead, the sender manipulates the timing of packets to a queue, and the receiver gathers data through observations of the timing of departures from the queue.

4.1. *Timing channel model.* To obtain a finite state-space model, it is assumed that the queue size is bounded by $\bar{q}$, and arrivals are rejected if they cause an overflow. The dynamics of the queue are described as a reflected random walk:

$$Q_{t+1} = \min\{\bar{q}, \max(0, Q_t - S_{t+1} + A_{t+1})\}, \qquad t \ge 0. \tag{36}$$

In the nominal model in which $\boldsymbol{\zeta} \equiv 0$, the pair process $(\boldsymbol{S}, \boldsymbol{A})$ is i.i.d. on $\mathbb{Z}_+^2$. The sender wishes to manipulate the arrival process $\boldsymbol{A}$, and the receiver observes the departure process $\boldsymbol{S}$. This manipulation is modeled through a scalar input sequence $\boldsymbol{\zeta}$.

For simplicity, for the nominal model we restrict to the M/M/1 queue: The usual model evolves in continuous time, but after sampling using *uniformization* one obtains (36), in which $\boldsymbol{A}$ a Bernoulli sequence, and $S_t = 1 - A_t$ for each t. For each

integer $n \in \mathsf{X} = \{0, 1, \ldots, \bar{q}\}$, denote $n^+ = \min(n+1, \bar{q})$ and $n^- = \max(n-1, 0)$. If $0 < \lambda < \frac{1}{2}$ is the probably of success for $\boldsymbol{A}$, we then have

$$\mathsf{P}\big\{Q(t+1) = n^+ | Q(t) = n\big\} = 1 - \mathsf{P}\big\{Q(t+1) = n^- | Q(t) = n\big\} = \lambda.$$

Its steady-state p.m.f. is given by

$$\pi_0^Q(n) = \kappa \rho^n,$$

where $\rho = \lambda/(1-\lambda)$, and $\kappa > 0$ is a normalizing factor.

Recall that the receiver observes departures from the queue, which is equivalent to observations of the sequence $\boldsymbol{S}$. To estimate joint statistics, we expand the state space to $X(t) = (Q(t), S(t))$, which evolves on the state space $\mathsf{X} = \{0, 1, \ldots, \bar{q}\} \times \{0, 1\}$. The nominal transition matrix is defined as follows:

$$P_0\big((n, s), (n^+, 0)\big) = \lambda,$$
$$P_0\big((n, s), (n^-, 1)\big) = 1 - \lambda.$$

The first identity holds because a transition from (n, s) to $(n^+, 0)$ means that $A_{t+1} = 1$, in which case $S_{t+1} = 1 - A_{t+1} = 0$. The justification for the second identity is symmetrical. The transition matrix is sparse: $P_0(x, x') = 0$ for all but at most two values of x', regardless of x.

The sender wishes to the manipulate timing of arrivals, which motivates the following formulation for the controlled transition matrix:

$$P_\zeta\big((n, s), (n^+, 0)\big) = \lambda(1+\zeta),$$
$$P_\zeta\big((n, s), (n^-, 1)\big) = 1 - \lambda(1+\zeta)$$

in which ζ is constrained to the interval $[-1, 1]$. The state process evolves as the nonlinear state space model (1), with

$$Q_{t+1} = \min\big\{\bar{q}, \max\big(0,\, Q_t - 1 + 2\mathbb{I}\big\{N_{t+1} \le \lambda(1+\zeta_t)\big\}\big)\big\},$$
$$S_{t+1} = \mathbb{I}\big\{N_{t+1} > \lambda(1+\zeta_t)\big\}, \qquad t \ge 0,$$

in which $\boldsymbol{N}$ is i.i.d., with marginal equal to the uniform distribution on $[0, 1]$.

Figure 6 shows a comparison of the steady-state mean queue length as a function of ε^2 for a numerical example. The linear approximation is obtained from the approximation of π_ε given in Proposition 2.5. Other statistics are shown in Figure 7. Details can be found in Section 4.3.

4.2. *Second-order bound for mutual information.* The mutual information rate $I(\boldsymbol{S}, \boldsymbol{\zeta})$ between $\boldsymbol{S}$ and $\boldsymbol{\zeta}$ defines the capacity of this channel. Letting χ^N denote the joint distribution of $\{S_1, \ldots, S_N, \zeta_1, \ldots, \zeta_N\}$, and denoting the marginals $\{S_1, \ldots, S_N\} \sim \chi_S^N$, $\{\zeta_1, \ldots, \zeta_N\} \sim \chi_\zeta^N$, the mutual information rate is defined as the limit

$$I(\boldsymbol{S}, \boldsymbol{\zeta}) = \lim_{N\to\infty} \frac{1}{N} \mathrm{D}(\chi^N \,\|\, \chi_S^N \times \chi_\zeta^N),$$

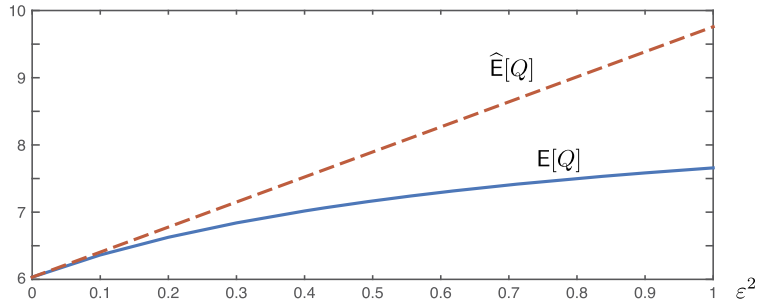

FIG. 6. *Mean queue length grows approximately linearly in* ε^2, *for* $\varepsilon^2 \le 0.2$. *The error for* $\varepsilon = 1$ *is approximately* 25%.

where D denotes relative entropy (i.e., K-L divergence) [8]. In the following, an approximation $\widehat{\mathsf{D}}$ is introduced, and based on this an approximation to mutual information,

$$\hat{I}(\boldsymbol{S}, \boldsymbol{\zeta}) = \lim_{N\to\infty} \frac{1}{N} \widehat{\mathsf{D}}(\chi^N \parallel \chi_S^N \times \chi_\zeta^N). \tag{37}$$

The approximation of relative entropy is given here in a general setting. Let ψ_a and ψ_b be probability measures on an abstract measurable space $(\mathcal{E}, \mathcal{B})$. For a measurable function $f : \mathcal{E} \to \mathbb{R}$, we let $\psi_a(f)$ denote the mean $\int f(x)\psi_a(dx)$.

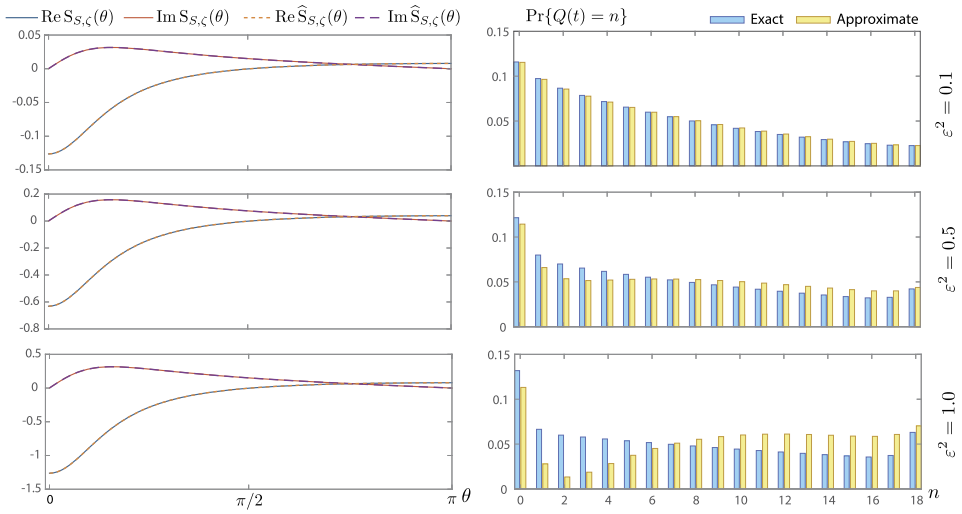

FIG. 7. *Experiments using* $\gamma = 0.4$. *The approximation for the cross power spectral density* $\mathrm{S}_{S,\zeta}$ *appears to be exact for the entire range of* ε. *The approximation for the steady-state distribution of* $\boldsymbol{Q}$ *is accurate for* $\varepsilon^2 \le 0.5$, *but is very poor for* $\varepsilon = 1$.

The approximation is the nonnegative functional defined as follows:

$$\widehat{\mathrm{D}}(\psi_a \parallel \psi_b) = \frac{1}{2}\sup\Big\{ \frac{\psi_a(f)^2}{\psi_b(f^2)} : \psi_b(f) = 0, \psi_a(|f|) < \infty, \text{ and } 0 < \psi_b(f^2) < \infty \Big\}. \tag{38}$$

The proof of Proposition 4.1 is contained in Appendix F.

PROPOSITION 4.1. *The following hold for any two probability measures ψ_a and ψ_b on an abstract measurable space $(\mathcal{E}, \mathcal{B})$. Let $f^* = \log(d\psi_a/d\psi_b)$ denote the log-likelihood ratio*:

(i) *The maximum in (38) is given by $\hat{f}^* = e^{f^*} - 1 = d\psi_a/d\psi_b - 1$, whenever $\widehat{\mathrm{D}}(\psi_a \parallel \psi_b)$ is finite.*

(ii) *There is a convex, increasing function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ that vanishes only at the origin, and such that the following bound holds for any two probability measures with bounded log-likelihood ratio*:

$$\big|\widehat{\mathrm{D}}(\psi_a \parallel \psi_b) - \mathrm{D}(\psi_a \parallel \psi_b)\big| \le \kappa\big(\|f^*\|_\infty^3\big),$$

where $\|f^*\|_\infty$ denotes the supremum norm.

Returning to the stochastic process setting, in the context of (37), we have for fixed N the following correspondences:

$$\psi_a = \chi^N, \qquad \psi_b = \chi_S^N \times \chi_\zeta^N.$$

Consider for $0 \le n < N$ the function

$$f(S_1, \dots, S_N, \zeta_1, \dots, \zeta_N) = \sum_{k=1}^{N-n} \widetilde{S}_{k+n}\zeta_k$$

in which $\widetilde{S}_t = S_t - \mathsf{E}[S_t]$. This has mean zero under ψ_b, and its mean under ψ_a is

$$\psi_a(f) = \mathsf{E}_{\chi^N}\big[f(S_1, \dots, S_N, \zeta_1, \dots, \zeta_N)\big] = (N-n)\Sigma_{S,\zeta}(n).$$

The second moment is also expressed in terms of autocorrelation functions:

$$\begin{aligned}\psi_b(f^2) &= \mathsf{E}_{\chi_S^N \times \chi_\zeta^N}\big[f^2(S_1, \dots, S_N, \zeta_1, \dots, \zeta_N)\big] \\ &= \sum_{k=1}^{N-n}\sum_{\ell=1}^{N-n} \mathsf{E}_{\chi_S^N \times \chi_\zeta^N}\mathsf{E}[\widetilde{S}_{k+n}\widetilde{S}_{\ell+n}\zeta_k\zeta_\ell] \\ &= \sum_{k=1}^{N-n}\sum_{\ell=1}^{N-n} \Sigma_S(k-\ell)\Sigma_\zeta(k-\ell).\end{aligned}$$

For fixed n and $N \gg n$, this admits the approximation $\psi_b(f^2) \approx (N-n)\mathrm{S}_{S\times\zeta}(0)$, where

$$\mathrm{S}_{S\times\zeta}(0) = \sum_{m=-\infty}^{\infty} \Sigma_S(m)\Sigma_\zeta(m).$$

This gives the large-N approximation,

$$\widehat{\mathrm{D}}(\psi_a \parallel \psi_b) \geq \frac{1}{2}\frac{\psi_a(f)^2}{\psi_b(f^2)} = \frac{1}{2}\frac{\Sigma_{S,\zeta}(n)^2}{\mathrm{S}_{S\times\zeta}(0)}N + O(1).$$

While the derivation was performed for $n \geq 0$, similar arguments establish the same bound for any integer n. The approximation for mutual information rate is thus lower bounded:

$$\hat{I}(\boldsymbol{S}, \boldsymbol{\zeta}) \geq \frac{1}{2} \sup_{-\infty<n<\infty} \frac{\Sigma_{S,\zeta}(n)^2}{\mathrm{S}_{S\times\zeta}(0)}. \tag{39}$$

This function class is of course highly restrictive. A larger class of functions can be obtained by defining for each n and each $\alpha, \beta \in \mathbb{R}^{n+1}$,

$$f(S_1, \ldots, S_N, \zeta_1, \ldots, \zeta_N) = \sum_{k=1}^{N-n} S_k^\alpha \zeta_k^\beta,$$

with

$$S_k^\alpha = \sum_{m=0}^{n} \alpha_m \widetilde{S}_{k+m}, \qquad \zeta_k^\beta = \sum_{m=0}^{n} \beta_m \zeta_{k+m}.$$

Formulae for $\psi_a(f)$ and $\psi_b(f^2)$ can be obtained as in the foregoing, yielding

$$\hat{I}(\boldsymbol{S}, \boldsymbol{\zeta}) \geq \frac{1}{2}\frac{\Sigma_{S^\alpha,\zeta^\beta}(0)^2}{\mathrm{S}_{S^\alpha\times\zeta^\beta}(0)}.$$

4.3. *Numerical experiments.* In all of the numerical examples described here, λ is chosen so that $\rho = \lambda/(1-\lambda) = 0.9$, and the upper bound appearing in (36) is $\bar{q} = 18$.

A Markovian model was chosen for ζ^1 so that exact computations can be obtained for the larger Markov chain. A simple model was chosen, in which $\boldsymbol{\zeta}^1$ evolves on the three states $\{-1, 0, 1\}$. The larger Markov chain $\Psi_t = (Q_t, S_t, \zeta_t^1)$ then evolves on a state space of size $6(1+\bar{q})$.

The three states are labeled $\{z^i : i = 1, 2, 3\} = \{-1, 0, 1\}$. For a fixed parameter $\gamma \in (0, \frac{1}{2})$, the transition matrix K is defined as follows. First, $\mathsf{P}\{\zeta_{t+1} = z^j \mid \zeta_t = z^i\} = \gamma$ whenever $|z^j - z^i| = 1$:

$$K_{1,2} = K_{2,1} = K_{2,3} = K_{3,2} = \gamma.$$

The remaining transition probabilities are $K_{1,1} = K_{3,3} = 1 - \gamma$, and $K_{2,2} = 1 - 2\gamma$. The steady-state p.m.f. μ_0 is uniform, so the steady-state variance is

$$\sigma_\zeta^2 = \big((-1)^2 + 0^2 + 1^2\big)/3 = 2/3.$$

Its autocorrelation is equal to its autocovariance: $R_\zeta(m) = \sigma_\zeta^2(1-\gamma)^{|m|}$.

Unless explicitly stated otherwise, the results that follow use $\gamma = 0.4$, so that the asymptotic variance (the variance appearing in the central limit theorem) is

$$\text{asym. variance} = \mathrm{S}_\zeta(0) = \sum_{k=-\infty}^{\infty} R_\zeta(k) = \Big(\frac{2}{\gamma} - 1\Big)\sigma_\zeta^2 = 4\sigma_\zeta^2.$$

Let $\hat{\pi}_\varepsilon = \pi_0 + \xi U_1$ [with ξ and U_1 defined in (2.5)]. The approximate p.m.f. illustrated in the plots on the right-hand side of Figure 7 are defined by the first marginal, $\hat{\pi}_\varepsilon^{Q}(n) = \sum_{s=0,1} \hat{\pi}_\varepsilon(n,s)$, for $n = 0, \dots, 18$. The approximate steady-state queue length plotted in Figure 6 is defined by

$$\widehat{\mathsf{E}}[Q] = \sum_{n=0}^{18} n\hat{\pi}_\varepsilon^{Q}(n).$$

The steady state p.m.f. for $\boldsymbol{\Psi}$ was computed to obtain the exact steady-state mean $\mathsf{E}[Q]$, which is the concave plot shown in Figure 6. The approximations are accurate for $\varepsilon \le 0.7$.

The approximations for the cross power spectral density shown on the left-hand side of Figure 7 are remarkable.

The statistics of $\boldsymbol{Q}$ and its approximations are highly sensitive to the parameter γ. For $\gamma = 0.8$, the approximation of the steady-state mean $\mathsf{E}[Q]$ and approximations of $\pi_\varepsilon^{Q}(n)$ are nearly exact for the entire range of ε. For $0 < \gamma \le 0.2$, the approximations are accurate only for a very narrow range of $\varepsilon \sim 0$. Results for $\gamma = 0.2$ and $\gamma = 0.8$ are shown in Figure 8.

The approximation for mutual information in (39) is defined as a maximum, which was achieved at $n = 1$ in each experiment:

$$\hat{I}(\boldsymbol{S}, \boldsymbol{\zeta}) \ge \frac{1}{2} \max_{-\infty < n < \infty} \frac{\Sigma_{S,\zeta}(n)^2}{\mathrm{S}_{S\times\zeta}(0)} = \frac{1}{2}\frac{\Sigma_{S,\zeta}(1)^2}{\mathrm{S}_{S\times\zeta}(0)}.$$

Plots for four different values of γ are shown in Figure 9.

The plots in Figure 9 use the approximations obtained in Section 2. However, plots obtained using the exact values of $\Sigma_{S,\zeta}(1)$ and $\mathrm{S}_{S\times\zeta}(0)$ are indistinguishable.

Unfortunately, we cannot compare $\hat{I}(\boldsymbol{S}, \boldsymbol{\zeta})$ with the true mutual information rate. This is a computational challenge that is beyond the scope of this paper.

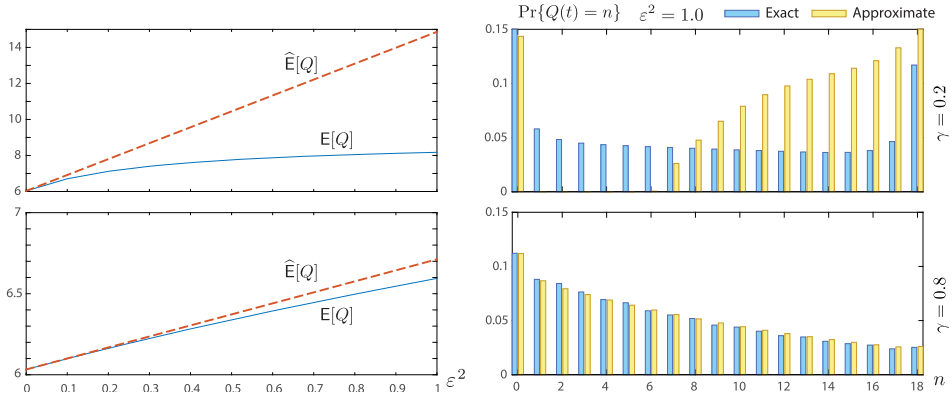

FIG. 8. *The top row shows results from numerical experiments with* $\gamma = 0.2$, *and the bottom row* $\gamma = 0.8$. *The approximation for* $\pi_\varepsilon^Q(n)$ *is nearly perfect in the latter case, even with* $\varepsilon = 1$. *With* $\gamma = 0.2$, *the second-order approximation is poor for* $\varepsilon \geq 0.5$.

5. Conclusions. It is very surprising to obtain an exact second-order Taylor series expansion for these second-order statistics with minimal assumptions on the controlled Markov model. The accuracy of the approximations obtained in numerical examples is also fortunate. The companion paper [5] and dissertation [4] contain more numerical work related to distributed control. Further work is needed to see if this will lead to useful bounds in applications to information theory.

APPENDIX A: COUPLING

We present here the proof of Proposition 2.3.

We first obtain a recursion for the joint process $\boldsymbol{\Psi} := (\boldsymbol{\Gamma}, \boldsymbol{\Gamma}^\bullet)$, driven by $\boldsymbol{\zeta}$, and two i.i.d. sequences $\boldsymbol{N}^\circ$ and $\boldsymbol{N}^\bullet$, each with marginal distribution uniform on $[0, 1]$. The three sequences $\boldsymbol{\zeta}$, $\boldsymbol{N}^\circ$, $\boldsymbol{N}^\bullet$ are mutually independent.

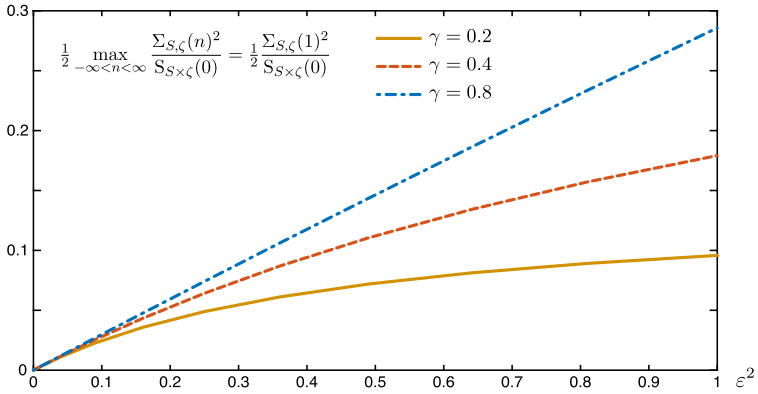

FIG. 9. *Lower bound for* $\hat{I}(\boldsymbol{S}, \boldsymbol{\zeta})$ *as a function of* ε^2 *for three values of* γ.

Letting $\boldsymbol{W}$ denote the 3-dimensional stationary stochastic process $(\boldsymbol{\zeta}, \boldsymbol{N}^\circ, \boldsymbol{N}^\bullet)$, we construct a function F for which

$$\Psi_{t+1} = F(\Psi_t, W_{t+1}). \tag{40}$$

Since $\boldsymbol{\Psi}$ evolves on a finite set, the existence of a stationary solution follows from [10] (see Theorem 5 and the discussion that follows).

To construct the function F, it is enough to define the matrix sequence $\boldsymbol{G}$ that appears in (5), and also the sequence $\boldsymbol{G}^\bullet$ that defines the dynamics of $\boldsymbol{\Gamma}^\bullet$. Each are based on the following definition: for $\zeta \in \mathbb{R}$ and $s \in [0, 1]$, denote

$$\mathcal{G}_{i,j}(\zeta, s) = \mathbb{I}\left\{\sum_{k=1}^{j-1} P_\zeta(i,k) \le s < \sum_{k=1}^{j} P_\zeta(i,k)\right\}$$

with the convention that "$\sum_{k=1}^{0}[\cdot]_k = 0$". We then take, for any t,

$$G_t^\bullet = \mathcal{G}\big(0, N_t^\bullet\big).$$

A third i.i.d. sequence $\boldsymbol{N}$ is obtained by sampling:

$$N_{t+1} = \begin{cases} N_{t+1}^\bullet & \text{if } \Gamma_t = \Gamma_t^\bullet, \\ N_{t+1}^\circ & \text{else.} \end{cases} \tag{41}$$

We then take $G_t = \mathcal{G}(\zeta_{t-1}, N_t)$.

Based on these definitions, the evolution equation (40) holds for some F; we now focus analysis on a stationary solution defined for all $t \in \mathbb{Z}$.

Choose $T_0 \ge 1$, $\delta_0 > 0$, $\varepsilon_0 > 0$, so that

$$\begin{aligned} \mathsf{P}\big\{X_{t+1} = X_{t+1}^\bullet \,|\, X_t = X_t^\bullet\big\} &\ge 1 - \delta_0\varepsilon, \\ \mathsf{P}\big\{X_{t+T_0} = X_{t+T_0}^\bullet \,|\, X_t \ne X_t^\bullet\big\} &\ge \delta_0, \qquad t \in \mathbb{Z}, \varepsilon \le \varepsilon_0. \end{aligned}$$

This is possible by the construction of the joint evolution equations, and the assumption that $\boldsymbol{X}^\bullet$ is irreducible and aperiodic. The first bound may be extended to obtain

$$\mathsf{P}\big\{X_{t+T} = X_{t+T}^\bullet \,|\, X_t = X_t^\bullet\big\} \ge (1 - \delta_0\varepsilon)^T \ge 1 - T\delta_0\varepsilon, \qquad T \ge 1.$$

We then have by stationarity

$$\begin{aligned} \mathsf{P}\big\{X_0 = X_0^\bullet\big\} &= \mathsf{P}\big\{X_{T_0} = X_{T_0}^\bullet\big\} \\ &= \mathsf{P}\big\{X_{T_0} = X_{T_0}^\bullet \,|\, X_0 = X_0^\bullet\big\}\mathsf{P}\big\{X_0 = X_0^\bullet\big\} \\ &\quad + \mathsf{P}\big\{X_{T_0} = X_{T_0}^\bullet \,|\, X_0 \ne X_0^\bullet\big\}\mathsf{P}\big\{X_0 \ne X_0^\bullet\big\}. \end{aligned}$$

Now, substitute the prior bounds, giving

$$\begin{aligned} \mathsf{P}\big\{X_0 = X_0^\bullet\big\} &\ge (1 - T_0\delta_0\varepsilon)\mathsf{P}\big\{X_0 = X_0^\bullet\big\} \\ &\quad + \delta_0\mathsf{P}\big\{X_0 \ne X_0^\bullet\big\}. \end{aligned}$$

Substituting $\mathsf{P}\{X_0 \neq X_0^\bullet\} = 1 - \mathsf{P}\{X_0 = X_0^\bullet\}$ and rearranging terms gives

$$\mathsf{P}\{X_0 = X_0^\bullet\} \geq \frac{1}{1 + T_0\varepsilon} \geq 1 - T_0\varepsilon$$

which completes the proof of (15).

The approximation (16) follows from (15) and independence of $\boldsymbol{\zeta}$ and $\boldsymbol{\Gamma}^\bullet$:

$$\mathsf{E}[\Gamma_t \zeta_t] = \mathsf{E}\big[\zeta_t \Gamma_t^\bullet\big] + \mathsf{E}\big[\zeta_t (\Gamma_t - \Gamma_t^\bullet)\big] = \mathsf{E}[\zeta_t]\mathsf{E}\big[\Gamma_t^\bullet\big] + O(\varepsilon^2) = O(\varepsilon^2).$$

The remaining bounds in (17) follow directly from (15) and the smoothness assumptions on P_ζ. □

APPENDIX B: MARTINGALE DIFFERENCE SEQUENCE

This appendix contains the proof of Proposition 2.1.

Define the σ-algebra $\mathcal{F}_t = \sigma\{\zeta_{-\infty}^{\infty}, N_{-\infty}^{t}\}$. The random vector Δ_t is $\mathcal{F}_t$-measurable for each t, and it follows from (6) that

$$\mathsf{E}[\Delta_{t+1} | \mathcal{F}_t] = \mathsf{E}\big[\Gamma_t \{\mathcal{G}(\zeta_t, N_{t+1}) - P_{\zeta_t}\} | \mathcal{F}_t\big] = 0,$$

where $\mathcal{G}(\zeta_t, N_{t+1}) = G_{t+1}$. This proves that Δ_t is a martingale difference sequence. Moreover, using the smoothing property of conditional expectation, for any t and τ,

$$\begin{aligned}\mathsf{E}[\Delta_{t+1}\zeta_\tau] &= \mathsf{E}\big[\Gamma_t \big[\mathcal{G}(\zeta_t, N_{t+1}) - P_{\zeta_t}\big]\zeta_\tau\big] \\ &= \mathsf{E}\big[\Gamma_t \zeta_\tau \mathsf{E}\big[\{\mathcal{G}(\zeta_t, N_{t+1}) - P_{\zeta_t}\} | \mathcal{F}_t\big]\big] = 0.\end{aligned}$$

This establishes $R_{\Delta,\zeta}(k) = 0$ for any k, which is (10).

The covariance of $\boldsymbol{\Delta}$ is obtained by applying the representation $\Gamma_t^T \Gamma_t = \operatorname{diag}(\Gamma_t) := \Lambda_t^\Gamma$. This follows from the fact that Γ_t^T is a standard basis vector for each t. Consequently,

$$\begin{aligned}\mathsf{Cov}(\Delta_{t+1}) &= \mathsf{E}\big[(\Gamma_t G_{t+1})^T (\Gamma_t G_{t+1})\big] - \mathsf{E}\big[(\Gamma_t P_{\zeta_t})^T (\Gamma_t P_{\zeta_t})\big] \\ &= \mathsf{E}\big[\Gamma_{t+1}^T \Gamma_{t+1}\big] - \mathsf{E}\big[(P_{\zeta_t})^T \Gamma_t^T \Gamma_t P_{\zeta_t}\big] \\ &= \mathsf{E}\big[\Lambda_{t+1}^\Gamma\big] - \mathsf{E}\big[(P_{\zeta_t})^T \Lambda_t^\Gamma P_{\zeta_t}\big]\end{aligned}$$

which is (9). □

APPENDIX C: APPROXIMATING THE STEADY-STATE MEAN

The approximation (30) is obtained here, starting with the approximate evolution equation that was used to obtain (12):

$$\begin{aligned}\Gamma_{t+1} &= \Gamma_t P_{\zeta_t} + \Delta_{t+1} \\ &= \Gamma_t \left[P_0 + \zeta_t \mathcal{E} + \frac{1}{2}\zeta_t^2 \mathcal{W} \right] + \Delta_{t+1} + O(\varepsilon^3).\end{aligned}$$

Taking the mean of each side, and using stationarity,

$$\begin{aligned}\mathsf{E}[\Gamma_t] &= \mathsf{E}[\Gamma_{t+1}] \\ &= \mathsf{E}[\Gamma_t]P_0 \\ &\quad + \mathsf{E}[\zeta_t\Gamma_t]\mathcal{E} \\ &\quad + \frac{1}{2}\mathsf{E}[\zeta_t^2\Gamma_t]\mathcal{W} + O(\varepsilon^3).\end{aligned} \tag{42}$$

To approximate $\mathsf{E}[\zeta_t^2\Gamma_t]$, we use $\Gamma_t = \Gamma_t^\bullet + \widetilde{O}(\varepsilon)$. This combined with independence of $\mathbf{\Gamma}^\bullet, \boldsymbol{\zeta}$ gives

$$\begin{aligned}\mathsf{E}[\zeta_t^2\Gamma_t] &= \mathsf{E}[\zeta_t^2\Gamma_t^\bullet] + \mathsf{E}[\zeta_t^2(\Gamma_t - \Gamma_t^\bullet)] \\ &= \mathsf{E}[\zeta_t^2]\mathsf{E}[\Gamma_t^\bullet] + O(\varepsilon^3) \\ &= \sigma_\zeta^2\pi_0 + O(\varepsilon^3).\end{aligned}$$

Substituting this into (42) gives the approximate fixed-point equation,

$$\tilde{\pi}_\varepsilon = \tilde{\pi}_\varepsilon P_0 + \xi + O(\varepsilon^3), \tag{43}$$

where $\tilde{\pi}_\varepsilon = \pi_\varepsilon - \pi_0$, and ξ is defined in (31).

The matrix $I - P_0$ is not invertible since it has eigenvalue at 0. To obtain an invertible matrix, we note that $\tilde{\pi}_\varepsilon[1 \otimes \pi_0] = 0$ for any ε, and hence (43) is equivalent to the vector equation,

$$\tilde{\pi}_\varepsilon[I - P_0 + 1 \otimes \pi_0] = \tilde{\pi}_\varepsilon[I - P_0] = \xi\varepsilon^2 + O(\varepsilon^3).$$

The desired result (30) is obtained by inversion:

$$\tilde{\pi}_\varepsilon = \xi[I - P_0 + 1 \otimes \pi_0]^{-1} + O(\varepsilon^3) = \xi U_1 + O(\varepsilon^3). \qquad \square$$

APPENDIX D: CROSS-COVARIANCE WITH INPUT

Approximations for $R_{D,\zeta}$ and $R_{B\zeta}$ are relatively simple because $\zeta_t = O(\varepsilon)$.

D.1. Cross-covariance between D and $\boldsymbol{\zeta}$. Using the coupling result, we obtain an approximation for the cross-correlation function,

$$\begin{aligned}R_{D,\zeta}(t) &= \mathsf{E}\big[D_{t+1}^T\zeta_1\big] \\ &= \mathsf{E}\big[\big(B_t^T\zeta_t + \Delta_{t+1}\big)^T\zeta_1\big] + O(\varepsilon^3) \\ &= \mathsf{E}\big[B_t^\bullet\zeta_t\zeta_1\big] + O(\varepsilon^3) \\ &= BR_\zeta(t-1) + O(\varepsilon^3).\end{aligned} \tag{44}$$

D.2. Cross-covariance between $\boldsymbol{\Delta\Delta^T}$ and $\boldsymbol{\zeta}$. Recall the σ-algebra $\mathcal{F}_s = \sigma\{\zeta_{-\infty}^{\infty}, N_{-\infty}^{s}\}$ introduced in Appendix B. Taking $s = -1$, we obtain from the smoothing property of the conditional expectation:

$$R_{\Delta^2,\zeta}(-t) = \mathsf{E}\big[\Delta_0^T \Delta_0 \zeta_t\big] = \mathsf{E}\big[\zeta_t \mathsf{E}\big[\Delta_0^T \Delta_0 | \mathcal{F}_{-1}\big]\big], \qquad t \in \mathbb{Z}.$$

The conditional expectation is a matrix that is denoted

$$\mathcal{X} = \mathsf{E}\big[\Delta_0^T \Delta_0 | \mathcal{F}_{-1}\big] = \operatorname{diag}(\Gamma_{-1} P_{\zeta_{-1}}) - P_{\zeta_{-1}}^T \Lambda_{-1}^{\Gamma} P_{\zeta_{-1}}. \tag{45}$$

LEMMA D.1. *For $t \geq 0$, we have*

$$\begin{aligned} R_{\Delta^2,\zeta}(-t) &= \mathsf{E}[\zeta_t \mathcal{X}] \\ &= \operatorname{diag}\big(R_{\Gamma,\zeta}(-t-1) P_0\big) - P_0^T \operatorname{diag}\big(R_{\Gamma,\zeta}(-t-1)\big) P_0 \\ &\quad + R_{\zeta}(t+1) \mathsf{E}\big[\mathcal{X}_{\bullet}^{(1)}\big] + O\big(\varepsilon^3\big), \end{aligned}$$

where $\mathsf{E}[\mathcal{X}_{\bullet}^{(1)}]$ is defined below (32).

PROOF. A first-order Taylor series approximation gives $\mathcal{X} = \mathcal{X}^{(0)} + \zeta_{-1}\mathcal{X}^{(1)} + O(\varepsilon^2)$, where

$$\begin{aligned} \mathcal{X}^{(0)} &= \operatorname{diag}(\Gamma_{-1} P_0) - P_0^T \operatorname{diag}(\Gamma_{-1}) P_0, \\ \mathcal{X}^{(1)} &= \operatorname{diag}(\Gamma_{-1}\mathcal{E}) - \big(P_0^T \operatorname{diag}(\Gamma_{-1})\mathcal{E} + \big[P_0^T \operatorname{diag}(\Gamma_{-1})\mathcal{E}\big]^T\big). \end{aligned}$$

Hence, for $t \geq 0$,

$$\begin{aligned} \mathsf{E}[\zeta_t \mathcal{X}] &= \mathsf{E}\big[\zeta_t \big(\mathcal{X}^{(0)} + \zeta_{-1}\mathcal{X}^{(1)} + O\big(\varepsilon^2\big)\big)\big] \\ &= \mathsf{E}\big[\zeta_t \mathcal{X}^{(0)}\big] + \mathsf{E}\big[\zeta_t \zeta_{-1} \mathcal{X}_{\bullet}^{(1)}\big] + O\big(\varepsilon^3\big) \\ &= \mathsf{E}\big[\zeta_t \mathcal{X}^{(0)}\big] + R_{\zeta}(t+1) \mathsf{E}\big[\mathcal{X}_{\bullet}^{(1)}\big] + O\big(\varepsilon^3\big), \end{aligned}$$

where, in the second equality we used $\mathcal{X}^{(1)} = \mathcal{X}_{\bullet}^{(1)} + \widetilde{O}(\varepsilon)$ with

$$\mathcal{X}_{\bullet}^{(1)} = \operatorname{diag}(\Gamma_{-1}^{\bullet}\mathcal{E}) - \big(P_0^T \operatorname{diag}(\Gamma_{-1}^{\bullet})\mathcal{E} + \big[P_0^T \operatorname{diag}(\Gamma_{-1}^{\bullet})\mathcal{E}\big]^T\big),$$

and also used the fact that $\Gamma_{-1}^{\bullet}$ is independent of $\boldsymbol{\zeta}$. We have by the definitions:

$$\mathsf{E}\big[\zeta_t \mathcal{X}^{(0)}\big] = \operatorname{diag}\big(R_{\Gamma,\zeta}(-t-1) P_0\big) - P_0^T \operatorname{diag}\big(R_{\Gamma,\zeta}(-t-1)\big) P_0.$$

Substitution into the previous approximation for $\mathsf{E}[\zeta_t \mathcal{X}]$ completes the proof. □

D.3. Auto-correlation of $B\zeta$. Applying the coupling result (17) in Proposition 2.3 gives $B_t^T = B^{\bullet}{}_t^T + \widetilde{O}(\varepsilon)$, where $B_t^T = \Gamma_t \mathcal{E}$. Hence,

$$R_{B\zeta}(t) = \mathsf{E}\big[B_t \zeta_t B_0^T \zeta_0\big] = \mathsf{E}\big[B_t^{\bullet} \zeta_t \big(B_0^{\bullet} \zeta_0\big)^T\big] + O\big(\varepsilon^3\big) = R_{B^{\bullet}\zeta}(t) + O\big(\varepsilon^3\big). \tag{46}$$

Independence of $\boldsymbol{\zeta}$ and $\Gamma_t^{\bullet}$ implies a formula for the simpler auto-correlation:

$$\begin{aligned} R_{B^{\bullet}\zeta}(t) &:= \mathsf{E}\big[B_t^{\bullet} \zeta_t \big(B_0^{\bullet} \zeta_0\big)^T\big] \\ &= \mathsf{E}\big[\big(\Gamma_t^{\bullet} \mathcal{E}\big)^T \big(\Gamma_t^{\bullet} \mathcal{E}\big)\big] \mathsf{E}[\zeta_t \zeta_0] \\ &= \mathcal{E}^T R_{\Gamma^{\bullet}}(t) \mathcal{E} R_{\zeta}(t). \end{aligned} \tag{47}$$

A formula for $R_{\Gamma^{\bullet}}(t)$ is given next: For $t \geq 0$,

$$R_{\Gamma^{\bullet}}(t) = \mathsf{E}\big[\big(\Gamma_t^{\bullet}\big)^T \Gamma_0^{\bullet}\big] = \mathsf{E}\big[\big(\Gamma_0^{\bullet} P_0^t\big)^T \Gamma_0^{\bullet}\big] = \big(P_0^T\big)^t \mathsf{E}\big[\big(\Gamma_0^{\bullet}\big)^T \Gamma_0^{\bullet}\big] = \big(P_0^T\big)^t \Pi_0, \tag{48}$$

where the last equality is from the fact that Γ_t has binary entries and $\mathsf{E}[\Gamma_t] = \pi_0$.

Combining (46)–(48) completes the proof of (23). □

D.4. Cross-covariance between $\boldsymbol{\Gamma}$ and $\boldsymbol{\zeta}$. The derivation of (33) is obtained via a recursion, similar to the calculation in Section C. It is simplest to work with the row vectors $\nu_k = (R_{\Gamma,\zeta}(k))^T = \mathsf{E}[\Gamma_k \zeta_0]$. For any $k \in \mathbb{Z}$,

$$\begin{aligned} \nu_{k+1} &= \mathsf{E}[\Gamma_{k+1} \zeta_0] \\ &= \mathsf{E}\big[(\Gamma_k P_{\zeta_k} + \Delta_{k+1}) \zeta_0\big] \\ &= \mathsf{E}[\Gamma_k P_{\zeta_k} \zeta_0] \\ &= \mathsf{E}\big[\Gamma_k \big(P_0 + \mathcal{E}\zeta_k + O\big(\varepsilon^2\big)\big) \zeta_0\big] \\ &= \nu_k P_0 + \mathsf{E}[\Gamma_k \mathcal{E} \zeta_k \zeta_0] + O\big(\varepsilon^3\big), \end{aligned}$$

where in the third equation we used the fact that the sequences $\boldsymbol{\Delta}$ and $\boldsymbol{\zeta}$ are uncorrelated. Recalling the definition $B^T = \pi \mathcal{E} = \mathsf{E}[\Gamma_k^{\bullet}] \mathcal{E}$, and applying the coupling result $\Gamma_k = \Gamma_k^{\bullet} + \widetilde{O}(\varepsilon)$ gives

$$\mathsf{E}[\Gamma_k \mathcal{E} \zeta_k \zeta_0] = \mathsf{E}\big[\Gamma_k^{\bullet} \mathcal{E} \zeta_k \zeta_0\big] + O\big(\varepsilon^3\big) = B^T R_{\zeta}(k) + O\big(\varepsilon^3\big).$$

Hence, $\nu_{k+1} = \nu_k P_0 + B^T R_{\zeta}(k) + O(\varepsilon^3)$; that is, there is a bounded sequence of row vectors $\{\gamma_k\}$ such that

$$\nu_{k+1} = \nu_k P_0 + B^T R_{\zeta}(k) + \varepsilon^3 \gamma_k.$$

It follows from (11) that $B^T 1 = \pi_0 \mathcal{E} 1 = 0$. Moreover, since $\{\Gamma_t\}$ are p.m.f.s, we have for any ℓ,

$$\nu_{\ell} 1 = \mathsf{E}[\zeta_0 \Gamma_{\ell}] 1 = \mathsf{E}[\zeta_0] = 0. \tag{49}$$

It then follows that $\gamma_k 1 = 0$ for each k.

On iterating, we obtain for any integer $n \ge 1$

$$\nu_{k+n} = \nu_k P_0^n + \sum_{i=1}^{n} \bigl[B^T + \varepsilon^3 \gamma_i\bigr] P_0^{i-1} R_\zeta(k+n-i).$$

Now, substitute $t = k+n$, where $t \in \mathbb{Z}$ is a fixed integer, and n is a large positive integer:

$$\nu_t = \nu_{t-n} P_0^n + \sum_{i=1}^{n} \bigl[B^T + \varepsilon^3 \gamma_i\bigr] P_0^{i-1} R_\zeta(t-i). \tag{50}$$

For large n, we have $P_0^n = 1 \otimes \pi_0 + o_e(1)$, where $o_e(1) \to 0$ geometrically fast as $n \to \infty$. Consequently,

$$\nu_{t-n} P_0^n = \mathsf{E}[\zeta_0 \Gamma_{t-n}] 1 \otimes \pi_0 + o_e(1),$$
$$\gamma_i P_0^{i-1} = \gamma_i 1 \otimes \pi_0 + o_e(1).$$

We have seen that $\mathsf{E}[\zeta_0 \Gamma_{t-n}]1 = 0$ and $\gamma_i 1 = 0$, from which we conclude that

$$\nu_{t-n} P_0^n = o_e(1) \quad \textit{and} \quad \sum_{i=1}^{\infty} \bigl\| \gamma_i P_0^{i-1} R_\zeta(t-i) \bigr\| < \infty.$$

This justifies letting $n \to \infty$ in (50) to obtain

$$\nu_t = B^T \sum_{i=1}^{\infty} P_0^{i-1} R_\zeta(t-i) + O\bigl(\varepsilon^3\bigr),$$

which is equivalent to (33) on substituting the definition $\nu_k = (R_{\Gamma,\zeta}(k))^T$. □

Based on this result, we now prove Proposition 2.6. It is sufficient to establish the following approximation:

$$R_{\Gamma,\zeta}(0) = \varepsilon^2 \sum_{k=1}^{n_z} a_k \rho_k \bigl[I - \rho_k P_0^T\bigr]^{-1} B + O\bigl(\varepsilon^3\bigr). \tag{51}$$

The representation (34) for ξ then follows immediately from the original definition (31).

Recall that $\nu_k = \mathsf{E}[\Gamma_k \zeta_0]$, $k \in \mathbb{Z}$. Under A3 we apply (18) to conclude that for $t \le 1$ and $i \ge 1$,

$$R_\zeta(t-i) = \varepsilon^2 \sum_{k=1}^{n_z} a_k \rho_k^{i-t}.$$

For these values of t and i, we have $i - t = i + |t|$, and hence the approximation (33) gives

$$\nu_t = \varepsilon^2 \sum_{k=1}^{n_z} \sum_{i=1}^{\infty} B^T P_0^{i-1} a_k \rho_k^{i+|t|} + O\bigl(\varepsilon^3\bigr).$$

Rearranging terms, and letting $j = i - 1$ gives

$$\nu_t = \varepsilon^2 B^T \sum_{k=1}^{n_z} a_k \rho_k^{1+|t|} \sum_{j=0}^{\infty} P_0^j \rho_k^j + O(\varepsilon^3).$$

On setting $t = 0$ and taking transposes, this becomes (51). □

APPENDIX E: PROOF OF THEOREM 2.4

The representation of $R_D(t)$ as the sum of the terms (22a)–(22d), plus an error of order ε^3, follows immediately from the arguments preceding the theorem. The proof of (21) is given in (44).

Next, we consider each of the terms (22a)–(22d) separately.

The approximation for $R_{B\zeta}(t)$ that appears in (22a) was given in Section D.3. Consider next the covariance Σ^Δ in (22b).

E.1. Approximation of Σ^Δ in (25). Equation (9) gives

$$\Sigma^\Delta = \Pi_\varepsilon - \mathsf{E}\big[P_{\zeta_t}^T \Lambda_t^\Gamma P_{\zeta_t}\big], \tag{52}$$

where $\Pi_\varepsilon = \operatorname{diag}(\pi_\varepsilon)$; recall that π_ε is approximated in Proposition 2.5.

The random variable $P_{\zeta_t}^T \Lambda_t^\Gamma P_{\zeta_t}$ is approximated using a second-order Taylor series expansion. The random matrix Λ_t^Γ has binary entries, so we leave it fixed in this approximation. For any scalar ζ and matrix Λ, we have

$$P_\zeta^T \Lambda P_\zeta = P_0^T \Lambda P_0 + \zeta M^{(1)} + \frac{1}{2}\zeta^2 M^{(2)} + O(|\zeta|^3), \tag{53}$$

where $M^{(i)}$ is the ith derivative of $P_\zeta^T \Lambda P_\zeta$ with respect to ζ, evaluated at $\zeta = 0$.

To obtain $M^{(1)}$, first differentiate using the product rule:

$$\frac{d}{d\zeta} P_\zeta^T \Lambda P_\zeta = P_\zeta^T \Lambda P_\zeta' + \big[P_\zeta^T \Lambda P_\zeta'\big]^T.$$

Evaluating at $\zeta = 0$ gives

$$M^{(1)} = P_0^T \Lambda \mathcal{E} + \big[P_0^T \Lambda \mathcal{E}\big]^T.$$

Similarly, given the second derivative,

$$\frac{d^2}{d\zeta^2} P_\zeta^T \Lambda P_\zeta = \big(P_\zeta'^T \Lambda P_\zeta' + P_\zeta^T \Lambda P_\zeta''\big) + \big(P_\zeta'^T \Lambda P_\zeta' + P_\zeta^T \Lambda P_\zeta''\big)^T$$

we obtain

$$M^{(2)} = P_0^T \Lambda \mathcal{W} + 2\mathcal{E}^T \Lambda \mathcal{E} + \mathcal{W}^T \Lambda P_0.$$

Substituting $\zeta_t = \zeta$ and $\Lambda_t = \Lambda$ in (53) gives the approximation

$$\begin{aligned}\mathsf{E}\big[P_{\zeta_t}^T \Lambda_t^\Gamma P_{\zeta_t}\big] &= \mathsf{E}\big[P_0^T \Lambda_t^\Gamma P_0\big] \\ &\quad + \mathsf{E}\big[(P_0^T \Lambda_t^\Gamma \mathcal{E} + \mathcal{E}^T \Lambda_t^\Gamma P_0)\zeta_t\big] \\ &\quad + \frac{1}{2}\mathsf{E}\big[(P_0^T \Lambda_t^\Gamma \mathcal{W} + 2\mathcal{E}^T \Lambda_t^\Gamma \mathcal{E} + \mathcal{W}^T \Lambda_t^\Gamma P_0)\zeta_t^2\big] + O(\varepsilon^3).\end{aligned} \tag{54}$$

The first term on the right-hand side can be approximated using Proposition 2.5. The second expectation involves $\mathsf{E}[\zeta_t \Gamma_t]$, which is approximated in Lemma 2.6. For the third term, we replace Γ_t by $\Gamma_t^\bullet + \widetilde{O}(\varepsilon)$, where $\Gamma_t^\bullet$ has mean π_0 and is independent of $\boldsymbol{\zeta}$. Combining all of these approximations gives the following approximation for the second term in (52):

$$\begin{aligned}\mathsf{E}\big[P_{\zeta_t}^T \Lambda_t^\Gamma P_{\zeta_t}\big] &= P_0^T \Pi_\varepsilon P_0 \\ &\quad + P_0^T \operatorname{diag}\big(\mathsf{E}[\Gamma_t \zeta_t]\big)\mathcal{E} + \mathcal{E}^T \operatorname{diag}\big(\mathsf{E}[\Gamma_t \zeta_t]\big) P_0 \\ &\quad + \frac{1}{2}(P_0^T \Pi_0 \mathcal{W} + 2\mathcal{E}^T \Pi_0 \mathcal{E} + \mathcal{W}^T \Pi_0 P_0)\mathsf{E}[\zeta_t^2] + O(\varepsilon^3),\end{aligned} \tag{55}$$

where $\Pi_0 = \operatorname{diag}(\pi_0)$ and $\Pi_\varepsilon = \operatorname{diag}(\pi_\varepsilon)$. This gives (25) since $R_{\Gamma,\zeta}(0) = \mathsf{E}[\Gamma_t^T \zeta_t]$. □

E.2. Computation of $R_{B\zeta,\Delta}(t)$ in (27). This is the most complex part of the proof.

We consider three cases separately: For $t < 0$, we have demonstrated that $R_{B\zeta,\Delta}(t) = 0$. The second case is $t = 0$. The approximation for $R_{B\zeta,\Delta}(0)$ is used as an initial condition in a recursive approximation for $R_{B\zeta,\Delta}(t)$, $t \geq 1$.

An approximation for $R_{B\zeta,\Delta}(0)$ is obtained from Lemma D.1. Using $\mathsf{E}[\Delta_0 | \mathcal{F}_{-1}] = 0$, we obtain

$$\begin{aligned}R_{B\zeta,\Delta}(0) &= \mathsf{E}\big[(\Gamma_0 \mathcal{E} \zeta_0)^T \Delta_0\big] \\ &= \mathcal{E}^T \mathsf{E}\big[\zeta_0 (\Gamma_{-1} P_{\zeta_{-1}} + \Delta_0)^T \Delta_0\big] \\ &= \mathcal{E}^T \mathsf{E}\big[\zeta_0 (\Gamma_{-1} P_{\zeta_{-1}})^T \Delta_0\big] + \mathcal{E}^T R_{\Delta^2,\zeta}(0) \\ &= \mathcal{E}^T \mathsf{E}\big[\zeta_0 (\Gamma_{-1} P_{\zeta_{-1}})^T \mathsf{E}[\Delta_0 | \mathcal{F}_{-1}]\big] + \mathcal{E}^T R_{\Delta^2,\zeta}(0) \\ &= \mathcal{E}^T R_{\Delta^2,\zeta}(0).\end{aligned} \tag{56}$$

Lemma D.1 provides an approximation for $R_{\Delta^2,\zeta}(0)$. Substituting this into (56) gives the approximation for $R_{B\zeta,\Delta}(0)$ shown in (27). □

In the remainder of this subsection, we consider $t \geq 1$. Iterating (5) gives

$$B_t^T = \Gamma_t \mathcal{E} = \Gamma_0 G_1 G_2 \cdots G_t \mathcal{E}.$$

Each term is conditionally independent of Δ_0, given $\boldsymbol{\zeta}$, except for $\Gamma_0 = \Gamma_{-1}G_0 = \Gamma_{-1}P_{\zeta_{-1}} + \Delta_0$. Using the fact that $\Gamma_{-1}P_{\zeta_{-1}}$ is also conditionally independent of Δ_0, we obtain

$$\begin{aligned} R_{B\zeta,\Delta}(t) &= \mathsf{E}\big[\zeta_t \mathcal{E}^T P_{\zeta_{t-1}}^T P_{\zeta_{t-2}}^T \cdots P_{\zeta_0}^T [\Gamma_{-1}P_{\zeta_{-1}} + \Delta_0]^T \Delta_0\big] \\ &= \mathcal{E}^T \mathsf{E}\big[\zeta_t P_{\zeta_{t-1}}^T \cdots P_{\zeta_0}^T \mathsf{E}\big[\Delta_0^T \Delta_0 | \mathcal{F}_{-1}\big]\big] \\ &= \mathcal{E}^T \mathsf{E}\big[\zeta_t P_{\zeta_{t-1}}^T \cdots P_{\zeta_0}^T \mathcal{X}\big], \end{aligned} \tag{57}$$

where $\mathcal{X}$ was introduced in (45).

Denote $A_i = P_{\zeta_i}^T$, $A = P_0^T$, and the matrix product $\Omega_t = A_t A_{t-1} \cdots A_0$, for $t \geq 0$. We set $\Omega_t = I$ for $t < 0$. The product is approximated in the following.

LEMMA E.1. *For $t \geq 0$,*

$$\Omega_t = A^{t+1} + \sum_{i=0}^{t} A^{t-i} \mathcal{E}^T A^i \zeta_i + O(\varepsilon^2). \tag{58}$$

The proof of (58) is postponed to the end of this subsection.

Once we have established this lemma, we then have the complete cross-correlation.

PROOF OF APPROXIMATION (27) FOR $R_{B\zeta,\Delta}(t)$, $t \geq 1$. We return to (57), which can be expressed

$$R_{B\zeta,\Delta}(t) = \mathcal{E}^T \mathsf{E}[\zeta_t \Omega_{t-1} \mathcal{X}], \qquad t \geq 1.$$

Substituting the bound in Lemma E.1,

$$\begin{aligned} R_{B\zeta,\Delta}(t) &= \mathcal{E}^T \mathsf{E}[\zeta_t A^t \mathcal{X}] + \mathcal{E}^T \mathsf{E}\left[\zeta_t \sum_{i=0}^{t-1} A^{t-1-i} \mathcal{E}^T A^i \zeta_i \mathcal{X}\right] + O(\varepsilon^3) \\ &= \mathcal{E}^T A^t \mathsf{E}[\zeta_t \mathcal{X}] + \mathcal{E}^T \mathsf{E}\left[\zeta_t \sum_{i=0}^{t-1} A^{t-1-i} \mathcal{E}^T A^i \zeta_i (\mathcal{X}^\bullet + \widetilde{O}(\varepsilon))\right] + O(\varepsilon^3) \\ &= \mathcal{E}^T A^t \mathsf{E}[\zeta_t \mathcal{X}] + \mathcal{E}^T \sum_{i=0}^{t-1} A^{t-1-i} \mathcal{E}^T A^i R_\zeta(t-i) \Sigma^{\Delta^\bullet} + O(\varepsilon^3). \end{aligned}$$

This establishes (27) since $\mathsf{E}[\zeta_t \mathcal{X}] = R_{\Delta^2,\zeta}(-t)$. □

The proof of Lemma E.1 uses a Taylor series approximation for Ω_t:

$$\Omega_t = A_t \Omega_{t-1} = A\Omega_{t-1} + W_t,$$

where $W_t = E_t + F_t$; $E_t = \mathcal{E}^T \zeta_t \Omega_{t-1}$ is the first-order term in the Taylor series approximation, and F_t is the approximation error whose norm is bounded by $O(\varepsilon^2)$.

The following result provides a uniform bound on $A^{t-i}F_i$ for each t and i, and also $A^{t-i}\mathcal{E}^T$ [which appears in (58)].

LEMMA E.2. *There exists* $0 < \kappa < \infty$ *and* $0 < \varrho < 1$ *such that*

$$\|A^n \mathcal{E}^T\| = \|\mathcal{E} P_0^n\| \le \kappa \varrho^n,$$
$$\|A^n F_i\| \le \kappa \varrho^n \varepsilon^2.$$

PROOF. The proof that $\mathcal{E}1 = 0$ was established previously; see (11). Next, we apply the ergodic limit (20), recalling that $P_0^n \to 1 \otimes \pi_0$ geometrically fast as $n \to \infty$: there exists $0 < \kappa_0 < \infty$ and $0 < \varrho < 1$ such that

$$\|e_n\| \le \kappa_0 \varrho^n \qquad \text{with } e_n = P_0^n - 1 \otimes \pi_0. \tag{59}$$

Consequently, $\mathcal{E} P_0^n = \mathcal{E} P_0^n 1 \otimes \pi_0 + \mathcal{E} e_n = \mathcal{E} e_n$, which implies the desired bound $\|\mathcal{E} e_n\| \le \kappa \varrho^n$, with $\kappa = \kappa_0 \|\mathcal{E}\|$.

The proof of the second bound is similar: For each i, we have

$$1^T W_i = 1^T (A_i - A)\Omega_{i-1} = 0^T,$$
$$1^T E_i = 1^T \mathcal{E}^T \Omega_{i-1}\zeta_i = (\mathcal{E}1)^T \Omega_{i-1}\zeta_i = 0^T.$$

We then have $1^T F_i = 1^T (W_i - E_i) = 0^T$, from which we obtain as before

$$F_i^T P_\zeta^n = F_i^T (1 \otimes \pi_0 + e_n) = F_i^T e_n.$$

Applying (59), we arrive at the desired bound:

$$\|A^n F_i\| = \|F_i^T P_0^n\| = \|F_i^T e^n\| \le \|F_i\| \|e^n\| \le \varepsilon^2 \kappa \varrho^n,$$

where $\varepsilon^2 \kappa$ is equal to κ_0 times a worst-case bound on $\|F_i\|$. □

PROOF OF LEMMA E.1. Applying Lemma E.2,

$$\sum_{i=1}^t \|A^{t-i} F_i\| = \sum_{i=1}^t \|F_i^T P_0^{t-i}\| \le \varepsilon^2 \frac{\kappa}{1-\varrho} = O(\varepsilon^2).$$

This implies the following approximation for Ω_t:

$$\begin{aligned}
\Omega_t &= A_t \Omega_{t-1} \\
&= (A + \mathcal{E}^T \zeta_t)\Omega_{t-1} + F_t \\
&= A\Omega_{t-1} + \mathcal{E}^T \Omega_{t-1}\zeta_t + F_t \\
&= A^t \Omega_0 + \sum_{i=1}^t A^{t-i} \mathcal{E}^T \Omega_{i-1}\zeta_i + \sum_{i=1}^t A^{t-i} F_i \\
&= A^t (A + \mathcal{E}^T \zeta_0 + O(\varepsilon^2)) + \sum_{i=1}^t A^{t-i} \mathcal{E}^T \Omega_{i-1}\zeta_i + O(\varepsilon^2) \\
&= A^{t+1} + \sum_{i=0}^t A^{t-i} \mathcal{E}^T \Omega_{i-1}\zeta_i + O(\varepsilon^2).
\end{aligned}$$

In particular, this shows that $\Omega_t = A^{t+1} + O(\varepsilon)$. Moreover, Lemma E.2 gives the geometric bound $\|A^{t-i}\mathcal{E}^T\| \le \kappa\varrho^{t-i}$. This justifies substitution $\Omega_{i-1}\zeta_i = A^i\zeta_i + O(\varepsilon^2)$ to obtain the desired result (58). □

E.3. Approximation of $R_{V\zeta^2,\Delta}(t)$ in (29). Recall that $V_t^T = \frac{1}{2}\Gamma_t\mathcal{W}$, and denote $R_{\Gamma^\bullet,\Delta^\bullet}(t) = \mathsf{E}[(\Gamma_t^\bullet)^T\Delta_0^\bullet]$.

Applying the coupling result Proposition 2.3, the cross-correlation is approximated as follows:

$$\begin{aligned} R_{V\zeta^2,\Delta}(t) &= \mathsf{E}\Big[\frac{1}{2}(\Gamma_t\mathcal{W})^T\zeta_t^2\Delta_0\Big] \\ &= \frac{1}{2}\mathcal{W}^T\mathsf{E}[\zeta_t^2]\mathsf{E}\big[(\Gamma_t^\bullet)^T\Delta_0^\bullet\big] + O(\varepsilon^3) \\ &= \frac{1}{2}\varepsilon^2\sigma_{\zeta^1}^2\mathcal{W}^T R_{\Gamma^\bullet,\Delta^\bullet}(t) + O(\varepsilon^3). \end{aligned}$$

We have $R_{\Gamma^\bullet,\Delta^\bullet}(t) = 0$ for $t < 0$, and thus $R_{V\zeta^2,\Delta}(t) = O(\varepsilon^3)$ for $t < 0$.

We also have $R_{\Gamma^\bullet,\Delta^\bullet}(0) = \mathsf{E}[(\Gamma_0^\bullet)^T\Delta_0^\bullet] = \Sigma^{\Delta^\bullet} = \Pi_0 - P_0^T\Pi_0 P_0$, and for $t \ge 1$,

$$\begin{aligned} R_{\Gamma^\bullet,\Delta^\bullet}(t) &= \mathsf{E}\big[(\Gamma_0^\bullet P_0^t)^T\Delta_0^\bullet\big] \\ &= \mathsf{E}\big[((\Gamma_{-1}^\bullet P_0 + \Delta_0^\bullet)P_0^t)^T\Delta_0^\bullet\big] \\ &= \mathsf{E}\big[(P_0^t)^T(\Delta_0^\bullet)^T\Delta_0^\bullet\big] = (P_0^t)^T\Sigma^{\Delta^\bullet}. \end{aligned}$$

Substituting $R_{\Gamma^\bullet,\Delta^\bullet}(t) = (P_0^T)^t\Sigma^{\Delta^\bullet}$ into the previous expression for $R_{V\zeta^2,\Delta}(t)$ gives (29) for $t \ge 0$.

E.4. Proof of Proposition 2.8. The proof begins with the uniform bound

$$\big\|\mathrm{S}(\theta) - \widehat{\mathrm{S}}(\theta)\big\| \le \sum_{t=-\infty}^{\infty}\big\|\Sigma^{\mathrm{tot}}(t) - \widehat{\Sigma}^{\mathrm{tot}}(t)\big\|, \qquad \theta\in\mathbb{R},$$

where $\|\cdot\|$ is any matrix norm. The right-hand side is finite under Assumptions A1 and A2. It remains to obtain an estimate that is $O(\varepsilon^{2+\varrho})$. We establish a slightly stronger bound:

$$\sum_{t=0}^{\infty}\big\|\Sigma^{\mathrm{tot}}(t) - \widehat{\Sigma}^{\mathrm{tot}}(t)\big\| = O\big(\varepsilon^2\log(1/\varepsilon)\big). \tag{60}$$

Under the assumption that $R_\zeta(t) \to 0$ geometrically fast, it follows that the same is true for $\widehat{\Sigma}^{\mathrm{tot}}(t)$ and $\Sigma^{\mathrm{tot}}(t)$: for some $b_0 < \infty$ and $\delta > 0$,

$$\big\|\Sigma^{\mathrm{tot}}(t)\big\| + \big\|\widehat{\Sigma}^{\mathrm{tot}}(t)\big\| \le \exp\big(b_0 - \delta|t|\big), \qquad t\in\mathbb{Z}.$$

Moreover, $\mu = \mu_0 + O(\varepsilon^2)$, where the first d components of μ_0 coincide with π_0, and the remaining are zero. It follows that $\Sigma^{\text{tot}}(t) = R(t) - \mu_0\mu_0^T + O(\varepsilon^4)$. Consequently, Theorem 2.4 implies that for some $b_1 < \infty$,

$$\|\Sigma^{\text{tot}}(t) - \widehat{\Sigma}^{\text{tot}}(t)\| \le \exp(b_1)\varepsilon^3, \qquad t \in \mathbb{Z}.$$

To establish (60), we decompose the sum into two parts. Denote

$$N(\varepsilon) = \min\{t \ge 0 : \exp(b_0 - \delta t) \le \exp(b_1)\varepsilon^3\}.$$

This implicit definition can be solved to give $N(\varepsilon) = [b_0 + 3b_1 \log(1/\varepsilon)]\delta^{-1}$.

From this, we obtain

$$\begin{aligned}\sum_{t=0}^{\infty} \|\Sigma^{\text{tot}}(t) - \widehat{\Sigma}^{\text{tot}}(t)\| &\le \exp(b_1)\varepsilon^3 N(\varepsilon) + \sum_{t>N(\varepsilon)} \exp(b_0 - \delta t) \\ &\le \exp(b_1)\varepsilon^3 N(\varepsilon) + \exp\big(b_0 - \delta N(\varepsilon)\big)\frac{1}{1-\exp(-\delta)} \\ &\le \exp(b_1)\varepsilon^3 N(\varepsilon) + \exp(b_1)\varepsilon^3 \frac{1}{1-\exp(-\delta)}.\end{aligned}$$

This together with the formula for $N(\varepsilon)$ immediately gives the bound (60). □

APPENDIX F: PROOF OF PROPOSITION 4.1

Recall the representation of relative entropy as the convex dual of the log-moment generating function: For any probability measure ψ_b on $\mathcal{E}$ and measurable function $f : \mathcal{E} \to \mathbb{R}$, the log-moment generating function is denoted

$$\Lambda_b(f) = \log\big(\psi_b\big(e^f\big)\big).$$

For any other probability measure ψ_a on $(\mathcal{E}, \mathcal{B})$, we have

$$\mathrm{D}(\psi_a \parallel \psi_b) = \sup\{\psi_a(f) - \Lambda_b(f)\},$$

where the supremum is over all measurable functions f for which $\psi_a(f)$ is defined [9], Theorem 3.1.2.

Provided the relative entropy is finite, the supremum is achieved uniquely with the log-likelihood function $f^* = \log(d\psi_a/d\psi_b)$. The error bound (ii) in the proposition is vacuous unless $\|f^*\|_\infty < \infty$. Consequently, to establish the error bound we can restrict to functions f for which $\|f\|_\infty < \infty$.

Apply the second-order Taylor-series expansion:

$$\Lambda_b(f) = \log\left(1 + \psi_b(f) + \frac{1}{2}\psi_b\big(f^2\big)\right) = \psi_b(f) + \frac{1}{2}\psi_b\big(f^2\big) + O\big(\|f\|_\infty^3\big).$$

A quadratic approximation to relative entropy is obtained on dropping the error term. To complete the proof, we establish the following alternate expression for (38):

$$\widehat{\mathrm{D}}(\psi_a \parallel \psi_b) := \sup\left\{\psi_a(f) - \psi_b(f) - \frac{1}{2}\psi_b\big(f^2\big)\right\},$$

where the supremum is over all functions f whose mean is defined with respect to both ψ_a and ψ_b. Without loss of generality, we may assume that the maximum is over all functions f for which $\psi_b(f) = 0$. It is not difficult to show that the maximizing function is $\hat{f}^* = e^{f^*} - 1 = d\psi_a/d\psi_b - 1$ whenever $\widehat{\mathrm{D}}(\psi_a \parallel \psi_b)$ is finite. This establishes (i).

We can scale by a constant θ to obtain

$$\widehat{\mathrm{D}}(\psi_a \parallel \psi_b) = \max_f \max_\theta \left\{ \theta \psi_a(f) - \frac{1}{2}\theta^2 \psi_b(f^2) \right\},$$

where the first maximum is over measurable functions f satisfying $\psi_b(f) = 0$ and $\psi_a(|f|) + \psi_b(f^2) < \infty$. The optimizing value is $\theta^* = \psi_a(f)/\psi_b(f^2)$. Substitution leads to the formula (38). □

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