The Annals of Applied Probability
2017, Vol. 27, No. 1, 91–108
DOI: 10.1214/16-AAP1198

TWO-DIMENSIONAL VOLUME-FROZEN PERCOLATION: EXCEPTIONAL SCALES

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We study a percolation model on the square lattice, where clusters "freeze" (stop growing) as soon as their volume (i.e., the number of sites they contain) gets larger than N, the parameter of the model. A model where clusters freeze when they reach *diameter* at least N was studied in van den Berg, de Lima and Nolin [*Random Structures Algorithms* **40** (2012) 220–226] and Kiss [*Probab. Theory Related Fields* **163** (2015) 713–768]. Using volume as a way to measure the size of a cluster—instead of diameter—leads, for large N, to a quite different behavior (contrary to what happens on the binary tree van den Berg, Kiss and Nolin [*Electron. Commun. Probab.* **17** (2012) 1–11], where the volume model and the diameter model are "asymptotically the same"). In particular, we show the existence of a sequence of "exceptional" length scales.

1. Introduction.

1.1. *Frozen percolation.* Frozen percolation is a growth process on graphs that was first considered by Aldous [1] (motivated by sol-gel transitions), on the binary tree. It is a percolation-type process which can be described informally as follows. Let $G = (V, E)$ be a simple graph. Initially, all edges are closed, and they try to become open independently of each other. However, a connected component (cluster) of open edges is not allowed to grow forever: it stops growing as soon as it becomes infinite, which means that all edges along its boundary are then prevented from opening: we say that such a cluster "freezes," hence the name *frozen percolation*.

Note that it is not clear at all that such a process exists. In [1], Aldous studies the case when G is the infinite binary tree, where each vertex has degree 3 (and also the case of the planted binary tree, where all vertices have degree 3, except the root vertex which has degree 1). In this case, the tree structure allows for the derivation of recursion formulas, and it is shown in [1] that the frozen percolation process does exist. On the other hand, it was pointed out shortly afterward by Benjamini and Schramm that for the square lattice $\mathbb{Z}^2$, such a process does not exist (see also Remark (i) after Theorem 1 in [5]).

A modification of the process, for which existence follows automatically from standard results, was introduced in [4] by de Lima and the two authors. In this

Received February 2015; revised February 2016.
MSC2010 subject classifications. 60K35, 82B43.
Key words and phrases. Frozen percolation, near-critical percolation, sol-gel transitions.

modified process, we stop the growth of a cluster when it reaches a certain "size" $N < \infty$, so that frozen percolation corresponds formally to $N = \infty$. The above mentioned nonexistence result by Benjamini and Schramm motivated us to investigate what happens as $N \to \infty$. However, when N is finite, one needs to make precise what "size" means, and in [4], the diameter of a cluster is used as a way to measure its size. This case was further studied by Kiss, who provides in [10] a precise description of the process as $N \to \infty$. Roughly speaking, he shows that in a square of side length KN, for any fixed $K > 1$, only finitely many clusters freeze (with an exponential tail on their number), and they all do so in the near-critical window around the percolation threshold p_c. This implies in particular that the frozen clusters all look like near-critical percolation clusters: their density thus converges to 0 as $N \to \infty$, as well as the probability for a given vertex to be frozen. In the final configuration, one only observes macroscopic nonfrozen clusters, that is, clusters with diameter of order N, but smaller than N.

In the case of the binary tree, it is shown in [2] that the resulting configuration is completely different: one only observes frozen clusters (with diameter $\geq N$), and microscopic ones (with diameter $1, 2, 3, \ldots$), but no macroscopic nonfrozen clusters. Moreover, it is explained that, on the tree, the way to measure size does not really matter: under general conditions (see Theorem 2 in [2]), the process converges to Aldous' process as $N \to \infty$. In the present paper, we go back to the case of two-dimensional lattices, and we show that measuring the size of clusters by their volume (i.e., the number of sites that they contain) leads to a behavior which is quite different from what happens in the diameter case.

We now describe the process of interest, for some fixed parameter $N \geq 1$. We restrict ourselves to the square lattice $(\mathbb{Z}^2, \mathbb{E}^2)$, but note that our results would also hold on any two-dimensional lattice with enough symmetries, such as the triangular lattice or the honeycomb lattice. The set of vertices consists of points with integer coordinates, and two vertices x and y are connected by an edge (denoted by $x \sim y$) *iff* $\|x - y\| = 1$, where $\|\cdot\|$ refers to the usual Euclidean norm. We consider a collection of i.i.d. random variables $(\tau_e)_{e \in \mathbb{E}^2}$ indexed by the edges, where each τ_e is uniformly distributed on $[0, 1]$. We work with a graph G which is either the full lattice $\mathbb{Z}^2$, or a finite connected subgraph of it. The volume of a subset A of G is the number of sites that it contains, and we denote it by $|A|$. We start at time 0 with all edges closed, and we let time increase. Each edge e stays closed until time τ_e, when it tries to become open: it is allowed to do so if and only if its two endpoints are in clusters of volume strictly smaller than N. Then e stays in the same state up to time 1, when we can read the final configuration of the process. This process is well defined, not only on finite subgraphs but also on $\mathbb{Z}^2$: indeed, it can be seen as a finite-range interacting particle system (the rate at which an edge becomes open only depends on the configuration within distance N from that edge), so we can apply standard results about such systems (see, for instance, Chapter 1 in [14]). Moreover, we get that the resulting process is a measurable function of the family $(\tau_e)_{e \in \mathbb{E}^2}$.

We denote by $\mathbb{P}_N^{(G)}$ the probability measure governing the process. We drop the superscript G when the graph used is clear from the context. Note also that the collection $(\tau_e)_{e\in\mathbb{E}^2}$ provides a natural coupling of the processes on different subgraphs of $\mathbb{Z}^2$: this observation is used repeatedly in our proofs.

The process could be described informally as follows. Starting from a configuration where all edges are closed, we let clusters grow as long as their volume is strictly smaller than N, and they stop growing (they "freeze") when their volume becomes at least N. We thus say that a given vertex x is *frozen* if it belongs to an open cluster with volume at least N (and such a cluster is called "frozen"). We are interested in the asymptotic behavior, as $N \to \infty$, of the probability that 0 is frozen in the final configuration, that is, at time 1. We conjecture that

$$\mathbb{P}_N^{(\mathbb{Z}^2)}(0 \text{ is frozen at time } 1) \underset{N\to\infty}{\longrightarrow} 0. \tag{1}$$

We present here some results related to this question, in the case of finite subgraphs of $\mathbb{Z}^2$. We hope that the results in this paper shed some light on the full-plane process (see Remark 1 in Section 2.3).

We believe that frozen percolation provides an intriguing example of a nonmonotone process, with competing effects that make it quite challenging to study. In particular, all our proofs require to follow, in some sense, the whole dynamics. The nonmonotonicity appears clearly with the sequence of exceptional scales $(m_k)_{k\geq 1}$ in the results below.

1.2. *Statement of results.* In order to present our results, we need to introduce some more notation. We denote by $B(n) = [-n, n]^2$ the box of side length $2n$ centered at the origin (for $n \geq 0$). Our results pertain to the asymptotic behavior of

$$F_N(n) = \mathbb{P}_N^{(B(n))}(0 \text{ is frozen at time } 1) \tag{2}$$

as $N \to \infty$, for some natural choices of $n \to \infty$ as a function of N.

For $p \in [0, 1]$, $\mathbb{P}_p$ refers to independent bond percolation on $\mathbb{Z}^2$ with parameter p. It is a celebrated result of Kesten [8] that the percolation threshold is $p_c = \frac{1}{2}$ in this case, and we use the following notation for the one-arm probability at criticality:

$$\pi(n) = \mathbb{P}_{p_c}\big(0 \rightsquigarrow \partial B(n)\big). \tag{3}$$

We are now in a position to state our main results. We start with one observation, that follows almost directly from standard results about independent percolation.

PROPOSITION 1. *For every $C > 0$,*

$$F_N\big(\lceil C\sqrt{N}\rceil\big) \underset{N\to\infty}{\longrightarrow} \phi(C), \tag{4}$$

where $\phi(C) = \frac{1}{4C^2}$ for $C > \frac{1}{2}$, and $\phi(C) = 0$ for $C < \frac{1}{2}$.

Note that $\phi(C) \to 0$ as $C \to \infty$, which may tempt one to believe that for every function g with $g(N) \gg \sqrt{N}$ (i.e., such that $\frac{g(N)}{\sqrt{N}} \to \infty$ as $N \to \infty$), one has $F_N(g(N)) \to 0$ as $N \to \infty$. However, the next theorem shows that one cannot naively exchange limits to say that in the full plane process, the probability for 0 to be frozen converges to 0. Indeed, $\sqrt{N}$ corresponds to a first scale $m_1(N)$ in a sequence $(m_k(N))_{k \geq 1}$ of exceptional scales, each of them leading to a nontrivial behavior. Roughly speaking, $m_2(N)$ is such that if we start with a box of this size, then a first "giant" cluster freezes and creates "holes" (here, by a hole, we mean a maximal connected component of unfrozen sites). The time at which this cluster freezes is such that the largest holes have size roughly $m_1(N)$, and most sites are in such holes. Then inside each of these holes, the process behaves similarly to the process in a box of size $m_1(N)$, so that a "second-generation" frozen cluster is produced. One can then define in the same way $m_3(N)$, $m_4(N)$, and so on.

DEFINITION 1. We define inductively the sequence of scales $(m_k(N))_{k \geq 0}$ by: $m_0 \equiv 1$, and for all $k \geq 0$, $m_{k+1} = m_{k+1}(N)$ is given by

$$m_{k+1} = \left\lceil \left(\frac{N}{\pi(m_k)} \right)^{1/2} \right\rceil \tag{5}$$

[i.e., $m_{k+1}^2 \pi(m_k) \simeq N$].

When referring to these scales, we often omit the parameter N for the sake of brevity. Note that from the monotonicity of π, it is easy to see that $(m_k(N))_{k \geq 0}$ is nondecreasing for every fixed $N \geq 1$. The scale $m_k(N)$ is well defined for all $k \geq 0$ and $N \geq 1$, but in this paper, we are only interested in asymptotic properties as $N \to \infty$, for each fixed value of k.

Before stating our main results, let us mention some properties of the scales $(m_k(N))_{k \geq 0}$ that can be easily derived from classical percolation results. First, the definition immediately implies that

$$m_1(N) \underset{N \to \infty}{\sim} c_0 \sqrt{N}, \tag{6}$$

for a certain constant $c_0 > 0$. Also, it follows from standard estimates that $\frac{m_{k+1}}{m_k}$ is between two power laws.

LEMMA 1. *For every fixed $k \geq 0$, there exist $\alpha_k, \tilde{\alpha}_k > 0$ such that: for N large enough*,

$$N^{\alpha_k} \leq \frac{m_{k+1}(N)}{m_k(N)} \leq N^{\tilde{\alpha}_k}. \tag{7}$$

In what follows, we only use the lower bound, and knowing that $\frac{m_{k+1}}{m_k} \to \infty$ as $N \to \infty$ would actually be enough. As we explain in Section 2.3, in the particular case of site percolation on the triangular lattice, one can prove that each m_k follows a power law, with some exponent that can be explicitly computed. However, this fact is not used in our proofs.

Theorems 1 and 2 below show that the scales m_k ($k = 2, 3, \dots$) are indeed exceptional.

THEOREM 1. *Let $k \geq 2$ be fixed. For every $C \geq 1$, every function $\tilde{m}(N)$ that satisfies*

$$C^{-1} m_k(N) \leq \tilde{m}(N) \leq C m_k(N) \tag{8}$$

for N large enough, we have

$$\liminf_{N \to \infty} F_N\big(\tilde{m}(N)\big) > 0. \tag{9}$$

However, we do not expect these exceptional scales to correspond to a typical situation (i.e., sizes of holes produced in the full-plane process), so that the previous result does not contradict the conjecture. Moreover, the next theorem shows that if we start away from these unusual scales, then the probability for 0 to be frozen converges to 0 as expected.

THEOREM 2. *For every integer $k \geq 0$ and every $\varepsilon > 0$, there exists a constant $C = C(k, \varepsilon) \geq 1$ such that: for every function $\tilde{m}(N)$ that satisfies*

$$C m_k(N) \leq \tilde{m}(N) \leq C^{-1} m_{k+1}(N) \tag{10}$$

for N large enough, we have

$$\limsup_{N \to \infty} F_N\big(\tilde{m}(N)\big) \leq \varepsilon. \tag{11}$$

Note that this theorem implies in particular the following result.

COROLLARY 1. *Let $k \geq 0$ be fixed. If the function $\tilde{m}(N)$ satisfies $m_k(N) \ll \tilde{m}(N) \ll m_{k+1}(N)$ as $N \to \infty$, then*

$$F_N\big(\tilde{m}(N)\big) \underset{N \to \infty}{\longrightarrow} 0. \tag{12}$$

2. Percolation preliminaries.

2.1. *Notation.* We first introduce some standard notation from percolation theory (for a more detailed account, we refer the reader to [7, 15]). Two sets of vertices A and B are said to be connected, which we denote by $A \rightsquigarrow B$, if there exists an open path from some vertex of A to some vertex of B. For a vertex v, $v \rightsquigarrow \infty$ means that v lies in an infinite connected component, and we use

$$\theta(p) = \mathbb{P}_p(0 \rightsquigarrow \infty), \tag{13}$$

which can also be seen as the density of the (unique) infinite cluster.

For a rectangle on the lattice of the form $R = [x_1, x_2] \times [y_1, y_2]$, we denote by $\mathcal{C}_H(R)$ [resp., $\mathcal{C}_V(R)$] the existence of a horizontal (resp., vertical) crossing, and we denote by $L(p) = L_{1/4}(p)$ the usual characteristic length defined in terms of crossings of rectangles:

$$\text{(14)} \quad \text{for all } p > \frac{1}{2} \qquad L_{1/4}(p) = \inf\left\{n \geq 1 : \mathbb{P}_p\big(\mathcal{C}_H\big([0, 2n] \times [0, n]\big)\big) \geq \frac{3}{4}\right\},$$

and $L_{1/4}(p) = L_{1/4}(1 - p)$ for $p < \frac{1}{2}$.

We work with the dual graph of $\mathbb{Z}^2$, which can be seen as $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$. We adopt the convention that a dual edge e^* is open *iff* the corresponding primal edge e is closed (and we talk about dual-open and dual-closed paths).

We also denote by $A(n, n') = B(n') \setminus B(n)$ the annulus of radii $0 \leq n < n'$. Finally, if γ is a circuit (i.e., a path whose vertices are all distinct, except the starting point and the end point, which coincide) on the dual lattice, we denote by $\mathcal{D}(\gamma)$ the domain (subgraph of $\mathbb{Z}^2$) obtained by considering the vertices (on the original lattice) which lie in the interior of γ, and keeping only the edges that connect two such vertices.

2.2. *Classical results*. We now collect useful results about independent percolation, which are needed for the proofs.

(i) Uniformly over $p > p_c$,

$$\text{(15)} \qquad \theta(p) \asymp \pi\big(L(p)\big)$$

(where $\asymp$ means that the ratio between the two sides is bounded away from 0 and ∞). This result is Theorem 2 in [9] (see also Corollary 41 in [15], and the remark just below it).

(ii) There exist $\lambda_1, \lambda_2 > 0$ such that: for all $p > p_c$, for all $k \geq 1$,

$$\text{(16)} \qquad \mathbb{P}_p\big(\mathcal{C}_H\big([0, 2k] \times [0, k]\big)\big) \geq 1 - \lambda_1 e^{-\lambda_2 \frac{k}{L(p)}}$$

(see, for instance, Lemma 39 in [15]: it follows from similar arguments).

(iii) Let $(n_k)_{k \geq 1}$ be a sequence of integers, with $n_k \to \infty$ as $k \to \infty$. If $(p_k)_{k \geq 1}$, with $p_c < p_k < 1$, satisfies $L(p_k) \ll n_k$ as $k \to \infty$, then

$$\text{(17)} \quad \text{for all } \varepsilon > 0 \qquad \mathbb{P}_{p_k}\left(\frac{|C^{\max}(B(n_k))|}{\theta(p_k)|B(n_k)|} \notin (1 - \varepsilon, 1 + \varepsilon)\right) \underset{k \to \infty}{\longrightarrow} 0,$$

where $|C^{\max}(B(n_k))|$ denotes the volume of the largest open cluster in $B(n_k)$ (see Theorem 3.2 in [6]). We will also use this result with the boxes $B(n_k)$ replaced by the annuli $A(\eta n_k, n_k)$, for some fixed $\eta \in (0, 1)$: it is straightforward to adapt the proofs in [6] to this situation. Note that the condition $L(p_k) \ll n_k$ is satisfied in particular when $p_k \equiv p \in (p_c, 1)$.

(iv) Finally, the following a priori bounds on π will be needed: there exist $\alpha, c_1, c_2 > 0$ such that for all $1 \leq n_1 \leq n_2$,

$$c_1\left(\frac{n_2}{n_1}\right)^{\alpha} \leq \frac{\pi(n_1)}{\pi(n_2)} \leq c_2\left(\frac{n_2}{n_1}\right)^{\frac{1}{2}}. \tag{18}$$

The lower bound is a direct consequence of the Russo–Seymour–Welsh theorem, while the upper bound follows from the BK (van den Berg–Kesten) inequality. In particular, it yields immediately that $(n^2\pi(n))_{n\geq 1}$ is essentially increasing, in the sense that

$$\text{for all } 1 \leq n_1 \leq n_2 \qquad n_2^2\pi(n_2) \geq c_2^{-1} n_1^2 \pi(n_1). \tag{19}$$

2.3. *Exceptional scales* $(m_k)_{k\geq 1}$. We are now in a position to give a more detailed justification for the definition of the exceptional scales $(m_k)_{k\geq 1}$, and discuss the main ideas of the proofs. Our reasonings are based on the fact that we can identify, for square boxes, reasonably precisely the successive freezing times.

Let us suppose that we run the frozen percolation process in a box of side length m_{k+1}, and denote by $p_{k+1} = p_{k+1}(N)$ the time at which a first giant cluster (i.e., with volume at least N) appears. Since we expect the largest percolation cluster in the box to have a volume $\asymp \theta(p)m_{k+1}^2$ at time p [see (17)], we find that $\theta(p_{k+1})m_{k+1}^2 \asymp N$. We can then combine $\theta(p_{k+1}) \asymp \pi(L(p_{k+1}))$ [from (15)] with the inductive definition for $(m_k)_{k\geq 1}$, and get that $\pi(L(p_{k+1})) \asymp \pi(m_k)$. Hence, $L(p_{k+1}) \asymp m_k$ [using (18)], so when the first giant cluster freezes, at time p_{k+1}, it creates holes with diameter of order m_k.

Theorems 1 and 2 correspond, respectively, to two possible situations. On the one hand, if we start at scale m_k, then we expect $k-1$ successive freezings, that occur at times p_j for which $L(p_j) \asymp m_{j-1}$ ($2 \leq j \leq k$), and possibly (if the corresponding hole has a volume at least N) a kth (and last) freezing at a supercritical time p_1. This kth frozen cluster then looks like a supercritical cluster, with density $\approx \theta(p_1) > 0$, so that the probability for 0 to be frozen is bounded away from 0. On the other hand, if we start at a scale $\tilde{m}$ such that $m_k \ll \tilde{m} \ll m_{k+1}$, then we expect the next freezings to occur at times p_j for which $m_{j-1} \ll L(p_j) \ll m_j$ ($2 \leq j \leq k$), and the last freezing at a time p_1 for which $1 \ll L(p_1) \ll m_1 \asymp \sqrt{N}$. We thus obtain a time p_1 which is well after the near-critical window [defined as the values of p for which $L(p) \geq \sqrt{N}$], but that still satisfies $p_1 \to p_c$ as $N \to \infty$. Hence, the density of the last frozen cluster converges to 0, as well as the probability for the origin to be frozen.

We now prove Lemma 1, which provides a priori estimates on the scales $(m_k)_{k\geq 1}$.

PROOF OF LEMMA 1. We proceed by induction over k. As noted earlier (6), the result holds for $k = 0$. Let us assume that it holds for some $k \geq 0$. By the definition of the m_i's, we can write

$$\frac{m_{k+2}}{m_{k+1}} \underset{N\to\infty}{\sim} \left(\frac{\pi(m_k)}{\pi(m_{k+1})}\right)^{1/2}.$$

The a priori bounds (18) on π then imply

$$c_1\left(\frac{m_{k+1}}{m_k}\right)^{\alpha/2} \le \left(\frac{\pi(m_k)}{\pi(m_{k+1})}\right)^{1/2} \le c_2\left(\frac{m_{k+1}}{m_k}\right)^{1/4}.$$

Now, it suffices to plug the induction hypothesis into the left-hand side and the right-hand side. $\square$

For future reference, let us also note that

$$m_k^2\pi(m_k) = m_{k+1}^2\pi(m_k)\left(\frac{m_k}{m_{k+1}}\right)^2 \underset{N\to\infty}{\sim} N\left(\frac{m_k}{m_{k+1}}\right)^2 \tag{20}$$

so that for some strictly positive $\eta_k, \eta_k' \longrightarrow_{k\to\infty} 0$: for N large enough,

$$N^{1-\eta_k} \le m_k^2\pi(m_k) \le N^{1-\eta_k'}. \tag{21}$$

In particular, let us mention (although it is not used later) that it follows from (18) and (21) that $m_k \le N^{2/3}$.

REMARK 1. Informally speaking, (21) means that m_k approaches a scale m_∞ that satisfies $m_\infty^2\pi(m_\infty) = N^{1+o(1)}$, that is, a scale such that: for critical percolation in a box of size m_∞, the largest clusters have volume of order N [at $p = p_c$, the order of magnitude for the volume of the largest clusters in a box $B(m)$ is given by the quantity $m^2\pi(m)$]. In the full-plane process, we expect m_∞ to be the scale at which "things start to happen," that is, at which the first frozen clusters form (looking like critical clusters). Then successive clusters freeze around the origin, denser and denser, and we expect their number to tend to infinity (as $N \to \infty$).

However, we are not trying in this paper to make this heuristic argument rigorous. Instead, we are dealing with the process started in boxes of side length m_k, for fixed $k \ge 1$. We believe that in order to study the last frozen cluster around the origin in the full-plane process, and in particular to prove our conjecture (1), it is enough to start in a box of size m_k, and analyze what happens when $k \to \infty$. Roughly speaking, we expect that the full-plane process "falls" between, but not close to, two consecutive exceptional scales. And moreover, that the lower bound in Theorem 1 decreases as $k \to \infty$: because of random effects on every scale, the process gets more and more "spread out," away from the exceptional scales. For site percolation on the triangular lattice, we develop this idea further in a second paper with Demeter Kiss [3].

In the case of site percolation on the triangular lattice, where precise estimates were established thanks to the connection between critical percolation (in the scaling limit) and SLE (Schramm–Loewner Evolution) processes with parameter 6 [11, 12, 16], it is known [13] that

$$\pi(n) = n^{-\frac{5}{48}+o(1)} \qquad \text{as } n \to \infty.$$

This leads immediately to $m_k(N) = N^{\delta_k+o(1)}$ as $N \to \infty$, where the sequence of exponents $(\delta_k)_{k\geq 0}$ satisfies

$$\delta_0 = 0 \quad \text{and for all } k \geq 0 \quad \delta_{k+1} = \frac{1}{2} + \frac{5}{96}\delta_k.$$

In particular, this sequence is strictly increasing, and it converges to $\delta_\infty = \frac{48}{91}$ [note that $m_\infty(N) = N^{\delta_\infty}$ satisfies $m_\infty^2 \pi(m_\infty) = N^{1+o(1)}$]. Let us also mention that for site percolation on the triangular lattice, many other critical exponents are known: in particular, $\theta(p) = (p - p_c)^{5/36+o(1)}$ as $p \to p_c^+$, and $L(p) = |p - p_c|^{-4/3+o(1)}$ as $p \to p_c$. Here, we decided to focus on bond percolation on the square lattice in order to stress that this more sophisticated technology is not needed for our results (Theorems 1 and 2).

3. Proofs of Proposition 1 and Theorem 1. From now on, we drop the ceilings $\lceil\cdot\rceil$ for notational convenience. In what follows, an edge e is said to be p-open *iff* $\tau_e \leq p$ (hence, the configuration of p-open edges has distribution $\mathbb{P}_p$). This then leads naturally to the notions of p-open paths, p-open clusters and so on. We use $\mathbb{P}$ to denote the distribution of the $(\tau_e)_{e\in\mathbb{E}^2}$ themselves.

Here, we use the fact that the processes in various subgraphs of $\mathbb{Z}^2$ can be coupled in a natural way, since the $(\tau_e)_{e\in\mathbb{E}^2}$ govern all the processes that we are considering. For a subgraph $G = (V, E)$ of $\mathbb{Z}^2$, we call *frozen percolation process in* G the process obtained by working on G only [this process is thus completely determined by $(\tau_e)_{e\in E}$].

3.1. *Proof of Proposition* 1. We first note that the volume of the box $B(C\sqrt{N})$ is

$$\left|B(C\sqrt{N})\right| \underset{N\to\infty}{\sim} 4C^2 N, \tag{22}$$

so that the case $C < \frac{1}{2}$ is clear. We can thus assume $C > \frac{1}{2}$, and introduce $\bar{p} \in (p_c, 1)$ such that $\theta(\bar{p}) = \frac{1}{4C^2}$, that is,

$$\theta(\bar{p})\left|B(C\sqrt{N})\right| \underset{N\to\infty}{\sim} N. \tag{23}$$

For arbitrary $\hat{p}$ and $\check{p}$ with $p_c < \check{p} < \bar{p} < \hat{p} < 1$, let us consider the following events:

- $D_1 =$ {there is a $\check{p}$-open path from $B(\sqrt[3]{N})$ to $\partial B(C\sqrt{N})$, and there is a $\check{p}$-open circuit in each of the annuli $A(k\sqrt[3]{N}, (k+1)\sqrt[3]{N})$, $k \geq 1$, contained in $B(C\sqrt{N})$},
- $D_2 =$ {the largest $\check{p}$-open cluster in $B(C\sqrt{N})$ has volume $< N$},
- $D_3 =$ {the largest $\hat{p}$-open cluster in $B(C\sqrt{N})$ has volume $> N$}.

Each of these events has probability tending to 1 as $N \to \infty$. For D_1, this follows from exponential decay of connection probabilities. For D_2 and D_3, it follows from the observation below (17), and our choice of $\bar{p}$: since θ is strictly increasing on $[p_c, 1]$, we have

$$\theta(\check{p}) < \theta(\bar{p}) = \frac{1}{4C^2} < \theta(\hat{p}).$$

If all these three events D_1, D_2 and D_3 hold, then there is no freezing in $B(C\sqrt{N})$ after time $\hat{p}$ [note that each annulus appearing in the definition of D_1 has volume $\ll N$, as well as $B(\sqrt[3]{N})$], which implies

$$F_N(C\sqrt{N}) \leq \mathbb{P}_{\hat{p}}\left(0 \rightsquigarrow \partial B\left(\frac{1}{3}\sqrt{N}\right)\right) + \mathbb{P}\left(D_1^c \cup D_2^c \cup D_3^c\right), \tag{24}$$

and by letting $N \to \infty$,

$$\limsup_{N\to\infty} F_N(C\sqrt{N}) \leq \theta(\hat{p}). \tag{25}$$

On the other hand, if each of D_1, D_2 and D_3 occurs, and if there is a $\check{p}$-open path from 0 to $\partial B(C\sqrt{N})$, then 0 freezes. Hence,

$$F_N(C\sqrt{N}) \geq \mathbb{P}_{\check{p}}\left(0 \rightsquigarrow \partial B(C\sqrt{N})\right) - \mathbb{P}\left(D_1^c \cup D_2^c \cup D_3^c\right), \tag{26}$$

and by taking $N \to \infty$,

$$\liminf_{N\to\infty} F_N(C\sqrt{N}) \geq \theta(\check{p}). \tag{27}$$

Since (25) and (27) hold for all $\hat{p} > \bar{p}$ and $\check{p} < \bar{p}$, we finally get, using the continuity of θ,

$$F_N(C\sqrt{N}) \underset{N\to\infty}{\longrightarrow} \theta(\bar{p}) = \frac{1}{4C^2}, \tag{28}$$

which completes the proof of Proposition 1.

3.2. *Proof of Theorem* 1. We prove Proposition 2 below, of which Theorem 1 is clearly a particular case. In order to state it, we first need more notation: for $n_1 < n_2$, let $\Gamma_N(n_1, n_2) =$ {for every dual circuit γ in the annulus $A(n_1, n_2)$, for the process in the domain $\mathcal{D}(\gamma)$ with parameter N, 0 is frozen}.

PROPOSITION 2. *For any $k \geq 2$, and $0 < C_1 < C_2$, we have*

$$\liminf_{N \to \infty} \mathbb{P}\big(\Gamma_N\big(C_1 m_k(N), C_2 m_k(N)\big)\big) > 0. \tag{29}$$

This result also holds for $k = 1$ under the extra condition that $C_1 > (2c_0)^{-1}$, where c_0 is the constant appearing in (6).

PROOF. The proof in the case $k = 1$ uses a similar reasoning as for Proposition 1. For that, let $C_2 > C_1 > (2c_0)^{-1}$, and introduce $\check{p} = \check{p}(C_2) \in (p_c, 1)$ such that

$$\theta(\check{p}) = \frac{1}{8c_0^2 C_2^2}. \tag{30}$$

We define the following events:

- $D_1 = D_1(C_1, C_2, N) = \{$there is a $\check{p}$-open path from $B(\sqrt[3]{N})$ to $\partial B(C_2 m_1(N))$, and there is a $\check{p}$-open circuit in each of the annuli $A(k\sqrt[3]{N}, (k+1)\sqrt[3]{N})$, $k \geq 1$, contained in $B(C_2 m_1(N))\}$,
- $D_2 = D_2(C_1, C_2, N) = \{$the largest $\check{p}$-open cluster in $B(C_2 m_1(N))$ has volume $< N\}$,
- $D_3 = D_3(C_1, C_2, N) = \{$there is a $\check{p}$-open path from 0 to $\partial B(C_1 m_1(N))\}$.

For the same reasons as before (see the proof of Proposition 1), the events D_1 and D_2 have a probability tending to 1 as $N \to \infty$. If D_1, D_2 and D_3 occur, then for every dual circuit γ in $A(C_1 m_1(N), C_2 m_1(N))$, 0 freezes for the process in $\mathcal{D}(\gamma)$ with parameter N: more formally,

$$\Gamma_N\big(C_1 m_1(N), C_2 m_1(N)\big) \supseteq D_1 \cap D_2 \cap D_3. \tag{31}$$

Hence,

$$\mathbb{P}\big(\Gamma_N\big(C_1 m_1(N), C_2 m_1(N)\big)\big) \geq \mathbb{P}(D_3) - \mathbb{P}\big(D_1^c \cup D_2^c\big), \tag{32}$$

and by taking $N \to \infty$,

$$\begin{aligned}\liminf_{N \to \infty} \mathbb{P}\big(\Gamma_N\big(C_1 m_1(N), C_2 m_1(N)\big)\big) &\geq \liminf_{N \to \infty} \mathbb{P}\big(D_3(C_1, C_2, N)\big) \\ &= \theta(\check{p}) = \frac{1}{8c_0^2 C_2^2} > 0.\end{aligned} \tag{33}$$

This completes the proof in the case $k = 1$.

Now, let us consider the case $k \geq 2$. We fix some $\delta > 0$ very small (for the reasoning below, $\delta = \frac{1}{100}$ is enough, as the reader can check). For all $k \geq 2$, $0 < C_1 < C_2$, and $N \geq 1$, we define $p_2 = p_2(k, C_1, C_2, N)$ by

$$\theta(p_2)(2C_2 m_k)^2 = N(1 - \delta), \tag{34}$$

and $p_1 = p_1(k, C_1, C_2, N)$ by

$$\theta(p_1)\left(2\frac{9}{10}C_1 m_k\right)^2 = N(1+\delta). \tag{35}$$

Note that $p_2 \le p_1$, and that the associated characteristic lengths satisfy

$$L(p_1) \asymp m_{k-1} \quad \text{and} \quad L(p_2) \asymp m_{k-1}, \tag{36}$$

where the symbol $\asymp$ means that the constants depend only on C_1, C_2 and δ. Indeed, it follows from (35) and (34), and then (5), that for $i = 1, 2$,

$$\theta(p_i) \asymp \frac{N}{m_k^2} \asymp \pi(m_{k-1}),$$

so (15) implies that $\pi(L(p_i)) \asymp \pi(m_{k-1})$, which finally yields (36) [using (18)].

Let us now introduce the following events (see Figure 1 for an illustration):

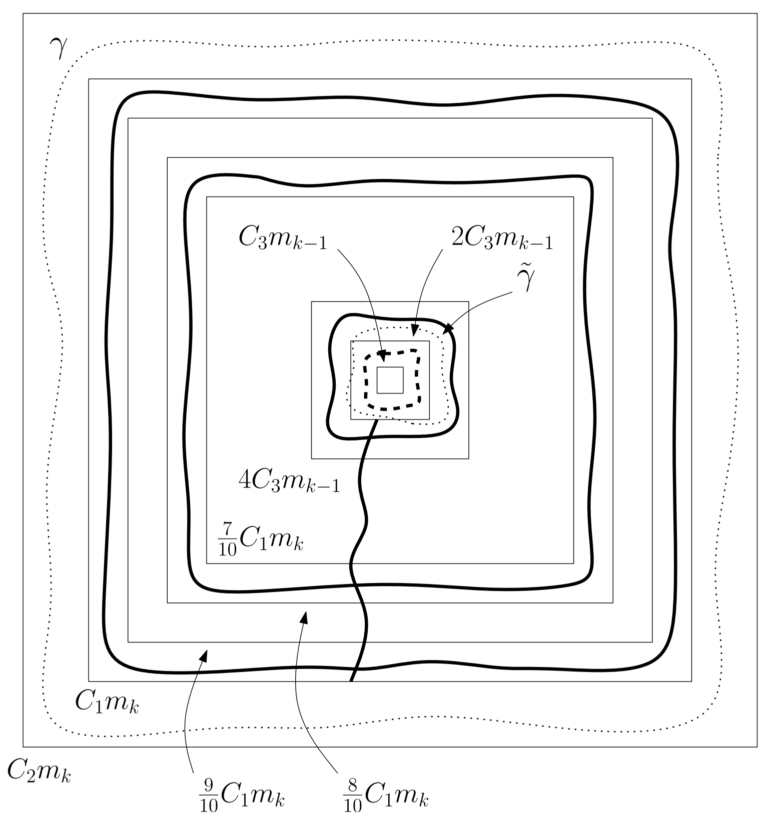

FIG. 1. *This figure depicts the events $E_i(k, \dots)$ $(1 \le i \le 5)$ used in the proof of Proposition 2: solid lines represent p_2-open paths, and the small circuit in the annulus $A(C_3 m_{k-1}, 2C_3 m_{k-1})$ is p_1-dual-open. The dual circuit $\tilde{\gamma}$ is the boundary of the hole containing* 0.

- $E_1 = E_1(k, C_1, C_2, N) = \{$there is a p_2-open circuit in the annulus $A(\frac{7}{10}C_1 m_k, \frac{8}{10}C_1 m_k)$, and in the annulus $A(\frac{9}{10}C_1 m_k, C_1 m_k)\}$,
- $E_2 = E_2(k, C_1, C_2, N) = \{$the largest p_1-open cluster in the box $B(\frac{9}{10}C_1 m_k)$ has volume $\geq N\}$,
- $E_3 = E_3(k, C_1, C_2, N) = \{$the largest p_1-open cluster in $B(\frac{8}{10}C_1 m_k)$, the largest p_1-open cluster in $A(\frac{7}{10}C_1 m_k, C_1 m_k)$, and the largest p_2-open cluster in $B(C_2 m_k)$ all have volume $< N\}$.

Note that it follows from (36) and Lemma 1 that $L(p_i) \ll m_k$ ($i \in \{1, 2\}$). Hence, (16) implies that E_1 has probability tending to 1 as $N \to \infty$, and (17) [together with (34) and (35)] implies the same for E_2 and E_3.

Now, we fix a universal constant $C_3 > (2c_0)^{-1}$: to be specific, we take $C_3 = c_0^{-1}$. In addition to the events E_1, E_2 and E_3 above, we introduce:

- $E_4 = E_4(k, C_1, C_2, C_3, N) = \{$there is a p_2-open path from $B(2C_3 m_{k-1})$ to $\partial B(C_1 m_k)$, and a p_2-open circuit in $A(2C_3 m_{k-1}, 4C_3 m_{k-1})\}$,
- $E_5 = E_5(k, C_1, C_2, C_3, N) = \{$there is a p_1-dual-open circuit in $A(C_3 m_{k-1}, 2C_3 m_{k-1})\}$.

We know from (36) that

$$\mathbb{P}\big(E_4(k, \dots) \cap E_5(k, \dots)\big) \geq \lambda, \tag{37}$$

where $\lambda = \lambda(C_1, C_2, C_3) > 0$.

We then make the following observation. If all these events $E_i(k, \dots)$ ($1 \leq i \leq 5$) happen, then for every dual circuit γ in the annulus $A(C_1 m_k, C_2 m_k)$, the frozen percolation process in $\mathcal{D}(\gamma)$ has these two properties:

(i) the first time that vertices in the box $B(\frac{9}{10}C_1 m_k)$ freeze lies in the time interval (p_2, p_1),

(ii) and if we look at the "hole containing 0" in that frozen cluster, its boundary is a dual circuit contained in $A(C_3 m_{k-1}, 4C_3 m_{k-1})$.

Hence,

$$\begin{aligned} &\Gamma_N\big(C_1 m_k(N), C_2 m_k(N)\big) \\ &\qquad \supseteq \big(E_1(k, \dots) \cap \dots \cap E_5(k, \dots)\big) \cap \Gamma_N\big(C_3 m_{k-1}(N), 4C_3 m_{k-1}(N)\big). \end{aligned} \tag{38}$$

By iterating (38), and using that for $k = 1$, (31) holds (with C_1 and C_2 replaced by C_3 and $4C_3$, resp.), we get

$$\begin{aligned} &\Gamma_N\big(C_1 m_k(N), C_2 m_k(N)\big) \\ &\quad \supseteq \big(E_1(k, C_1, C_2, N) \cap E_2(\dots) \cap E_3(\dots) \cap E_4(k, C_1, C_2, C_3, N) \cap E_5(\dots)\big) \\ &\quad \cap \Bigg(\bigcap_{j=2}^{k-1} E_1(j, C_3, 4C_3, N) \cap E_2(n) \cap E_3(\dots) \end{aligned} \tag{39}$$

$$\cap E_4(j, C_3, 4C_3, C_3, N) \cap E_5(\dots)\Big)$$

$$\cap \big(D_1(C_3, 4C_3, N) \cap D_2(C_3, 4C_3, N) \cap D_3(C_3, 4C_3, N)\big).$$

As observed before, the probabilities of all the events $D_1(C_3, 4C_3, N)$, $D_2(C_3, 4C_3, N)$, $E_i(k, C_1, C_2, N)$ $(1 \leq i \leq 3)$, and $E_i(j, C_3, 4C_3, N)$ $(1 \leq i \leq 3, 2 \leq j \leq k-1)$ tend to 1 as $N \to \infty$. Hence, it follows from (39) that

$$\begin{aligned}
&\liminf_{N\to\infty} \mathbb{P}\big(\Gamma_N\big(C_1 m_k(N), C_2 m_k(N)\big)\big) \\
&\qquad \geq \liminf_{N\to\infty} \mathbb{P}\Bigg(E_4(k, C_1, C_2, C_3, N) \cap E_5(k, C_1, C_2, C_3, N) \\
&\qquad\qquad \cap \left(\bigcap_{j=2}^{k-1} E_4(j, C_3, 4C_3, C_3, N) \cap E_5(j, C_3, 4C_3, C_3, N)\right) \\
&\qquad\qquad \cap D_3(C_3, 4C_3, N)\Bigg).
\end{aligned} \tag{40}$$

Now, we observe that in the right-hand side of (40), the k events:

- $E_4(k, C_1, C_2, C_3, N) \cap E_5(k, C_1, C_2, C_3, N)$,
- $E_4(j, C_3, 4C_3, C_3, N) \cap E_5(j, C_3, 4C_3, C_3, N)$ $(2 \leq j \leq k-1)$,
- and $D_3(C_3, 4C_3, N)$

are independent (provided N is sufficiently large so that the subsets of the lattice involved in the definitions of these events are disjoint). Using this observation, (37) and the equalities in (33), we get from (40) that

$$\begin{aligned}
&\liminf_{N\to\infty} \mathbb{P}\big(\Gamma_N\big(C_1 m_k(N), C_2 m_k(N)\big)\big) \\
&\qquad \geq \lambda(C_1, C_2, C_3)\lambda(C_3, 4C_3, C_3)^{k-2} \frac{1}{8c_0^2(4C_3)^2} > 0,
\end{aligned}$$

which completes the proof of Proposition 2. □

4. Proof of Theorem 2. We use similar constructions as for the proof of Theorem 1. Here, we proceed by induction: we show Proposition 3 below, regarding the event $\tilde{\Gamma}_N(n_1, n_2) = \{$there exists a dual circuit γ in the annulus $A(n_1, n_2)$ such that for the process in the domain $\mathcal{D}(\gamma)$ with parameter N, 0 is frozen$\}$ $(n_1 < n_2)$. This result clearly implies Theorem 2.

PROPOSITION 3. *Let $k \geq 0$, $\varepsilon > 0$, and $0 < C_1 < C_2$. Then there exists a constant $C = C(k, \varepsilon, C_1, C_2)$ such that: for every function $\tilde{m}(N)$ that satisfies*

$$C m_k(N) \leq C_1 \tilde{m}(N) \leq C_2 \tilde{m}(N) \leq C^{-1} m_{k+1}(N) \tag{41}$$

for N large enough, we have

$$\limsup_{N\to\infty} \mathbb{P}\big(\tilde{\Gamma}_N\big(C_1\tilde{m}(N), C_2\tilde{m}(N)\big)\big) \leq \varepsilon. \tag{42}$$

PROOF. We proceed by induction over k. First, we note that the case $k = 0$ is clear: we know from (6) that $C^{-1}m_1(N) \sim C^{-1}c_0\sqrt{N}$, and we just need to choose C large enough so that $C^{-1}c_0 < \frac{1}{2}$ [then the corresponding probability is 0 for N large enough, since every domain $\mathcal{D}(\gamma)$ of the prescribed form has volume $< N$, which implies in particular that 0 cannot be frozen].

Now, let us fix $k \geq 0$. We assume that Proposition 3 holds for k, and we show that it then holds for $(k + 1)$. Let us fix an arbitrary $\varepsilon > 0$, and $0 < C_1 < C_2$. Let us also consider some constant $C^{(k+1)} > 0$, and a function $\tilde{m}(N)$ satisfying

$$C^{(k+1)}m_{k+1}(N) \leq C_1\tilde{m}(N) \leq C_2\tilde{m}(N) \leq \big(C^{(k+1)}\big)^{-1}m_{k+2}(N)$$

[we will explain later how to choose $C^{(k+1)}$, as a function of k, ε, C_1, C_2 through the induction hypothesis, such that the required conclusion (42) can be drawn]. Finally, let us fix some $\delta > 0$ very small (again, $\delta = \frac{1}{100}$ works).

We define $p_2 = p_2(N)$ by

$$\theta(p_2)(2C_2\tilde{m})^2 = N(1 - \delta), \tag{43}$$

and $p_1 = p_1(N)$ by

$$\theta(p_1)\left(2\frac{9}{10}C_1\tilde{m}\right)^2 = N(1 + \delta) \tag{44}$$

(note that $p_2 \leq p_1$). We first make two observations on the associated characteristic lengths.

(i) One has

$$L(p_1) \asymp L(p_2) \tag{45}$$

(where the constants depend only on C_1, C_2 and δ). Indeed, it follows from (44) and (43) that $\theta(p_1) \asymp \theta(p_2)$, so $\pi(L(p_1)) \asymp \pi(L(p_2))$ [using (15)], and (18) finally implies (45).

(ii) Also,

$$L(p_1), L(p_2) \ll \tilde{m}. \tag{46}$$

Indeed, we know that $\tilde{m} \leq m_{k+2}$ for N large enough, so (19) and (21) imply that

$$\tilde{m}^2\pi(\tilde{m}) \leq c_2 m_{k+2}^2\pi(m_{k+2}) \leq c_2 N^{1-\eta'_{k+2}}.$$

Since we know from (44) and (15) that $\tilde{m}^2\pi(L(p_i)) \asymp N$ ($i \in \{1, 2\}$), we deduce $\pi(L(p_i)) \gg \pi(\tilde{m})$, and finally [using (18)] $L(p_i) \ll \tilde{m}$.

We now consider the following events (see Figure 2):

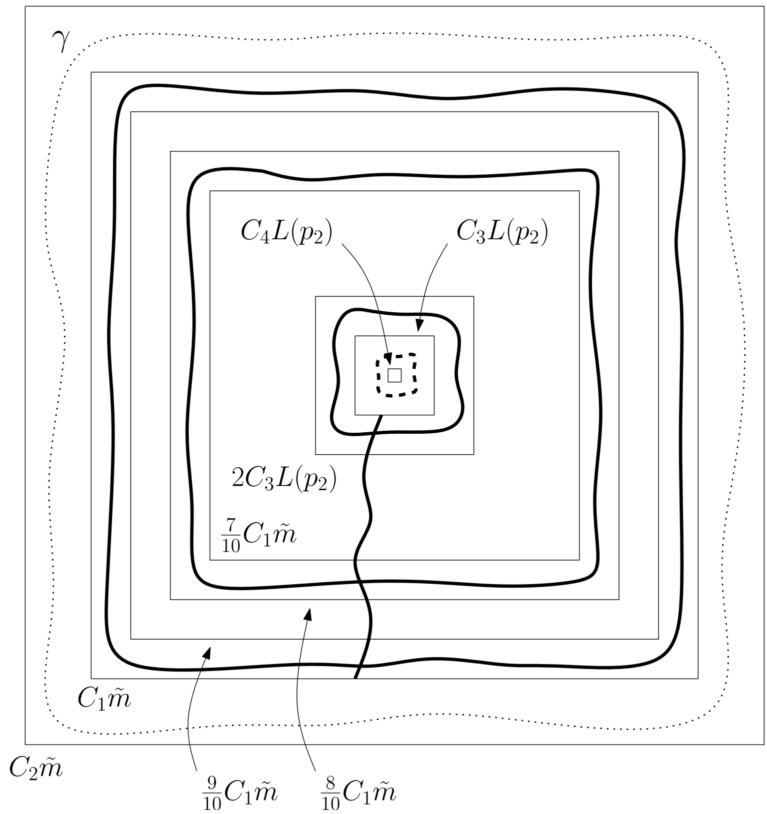

FIG. 2. *This figure presents the construction used for the proof of Proposition 3: solid lines correspond to p_2-open paths, and the small circuit in dashed line is p_1-dual-open. Here, we need the ratio C_3/C_4 to be large enough.*

- $E_1 = \{$there is a p_2-open circuit in the annulus $A(\frac{7}{10}C_1\tilde{m}, \frac{8}{10}C_1\tilde{m})$, and in the annulus $A(\frac{9}{10}C_1\tilde{m}, C_1\tilde{m})\}$,
- $E_2 = \{$the largest p_1-open cluster in the box $B(\frac{9}{10}C_1\tilde{m})$ has volume $\geq N\}$,
- $E_3 = \{$the largest p_1-open cluster in $B(\frac{8}{10}C_1\tilde{m})$, the largest p_1-open cluster in $A(\frac{7}{10}C_1\tilde{m}, C_1\tilde{m})$, and the largest p_2-open cluster in $B(C_2\tilde{m})$ all have volume $< N\}$.

Note that each of these events has probability tending to 1 as $N \to \infty$: it follows from (46), combined with (16) (for E_1) and with (17) (for E_2 and E_3).

We introduce now, for $C_3, C_4 > 0$ to be chosen later:

- $E_4 = \{$there is a p_2-open path from $B(C_3L(p_2))$ to $\partial B(C_1\tilde{m})$, and a p_2-open circuit in $A(C_3L(p_2), 2C_3L(p_2))\}$,
- $E_5 = \{$there is a p_1-dual-open circuit in $A(C_4L(p_2), C_3L(p_2))\}$.

It follows from (45) that we can fix C_3 sufficiently large so that for N large enough, E_4 occurs with probability at least $1-\varepsilon$, and then C_4 small enough to that for N large enough, E_5 occurs with probability at least $1-\varepsilon$. Note that the two constants C_3 and C_4 are universal: they can be chosen independently of N.

We then make the following observations: if all these events E_i ($1 \le i \le 5$) happen, then for every curve γ in $A(C_1\tilde{m}, C_2\tilde{m})$, the process in $\mathcal{D}(\gamma)$ satisfies:

(i) the first time that vertices in the box $B(\frac{9}{10}C_1\tilde{m})$ freeze lies in the time interval (p_2, p_1),

(ii) and if we look at the "hole containing 0" in that frozen cluster, its boundary is a dual circuit contained in $A(C_4L(p_2), 2C_3L(p_2))$.

This implies

$$\tilde{\Gamma}_N\big(C_1\tilde{m}(N), C_2\tilde{m}(N)\big) \cap \left(\bigcap_{1\le i\le 5} E_i\right) \subseteq \tilde{\Gamma}_N\big(C_4L(p_2), 2C_3L(p_2)\big).$$

Hence, using that $\limsup_{N\to\infty} \mathbb{P}((\bigcap_{1\le i\le 5} E_i)^c) \le 2\varepsilon$, we get

$$\limsup_{N\to\infty} \mathbb{P}\big(\tilde{\Gamma}_N\big(C_1\tilde{m}(N), C_2\tilde{m}(N)\big)\big) \le \limsup_{N\to\infty} \mathbb{P}\big(\tilde{\Gamma}_N\big(C_4L(p_2), 2C_3L(p_2)\big)\big) + 2\varepsilon.$$

Now, we would like to apply the induction hypothesis to the right-hand side: for that, let us denote by $C^{(k)}$ the constant associated with k, ε and $0 < C_4 < 2C_3$. In order to be in a position to use the induction hypothesis, we need to show that $C^{(k+1)}$ can be chosen so as to ensure that

$$C^{(k)}m_k(N) \le C_4L(p_2) \le 2C_3L(p_2) \le \big(C^{(k)}\big)^{-1}m_{k+1}(N) \tag{47}$$

for N large enough. Indeed, this would then imply that

$$\limsup_{N\to\infty} \mathbb{P}\big(\tilde{\Gamma}_N\big(C_4L(p_2), 2C_3L(p_2)\big)\big) \le \varepsilon,$$

and complete the proof of Proposition 3. But (47) is satisfied for $C^{(k+1)}$ large enough: it follows immediately from (43), combined with (15) and (18). □

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