The Annals of Applied Probability
2016, Vol. 26, No. 5, 3206–3250
DOI: 10.1214/16-AAP1174

CONNECTIVITY THRESHOLDS FOR BOUNDED SIZE RULES

BY HAFSTEINN EINARSSON[*,1], JOHANNES LENGLER[*], FRANK MOUSSET[*,2], KONSTANTINOS PANAGIOTOU[†,3] AND ANGELIKA STEGER[*]

ETH Zürich[*] and University of Munich[†]

In an *Achlioptas process*, starting with a graph that has n vertices and no edge, in each round $d \geq 1$ vertex pairs are chosen uniformly at random, and using some rule exactly one of them is selected and added to the evolving graph. We investigate the impact of the rule's choice on one of the most basic properties of a graph: connectivity. In our main result we focus on the prominent class of *bounded size rules*, which select the edge to add according to the component sizes of its vertices, treating all sizes larger than some constant equally. For such rules we provide a fine analysis that exposes the limiting distribution and the expectation of the number of rounds until the graph gets connected, and we give a detailed picture of the dynamics of the formation of the single component from smaller components. Our results allow us to study the connectivity transition of all Achlioptas processes, in the sense that we identify a process that accelerates it as much as possible.

1. Introduction. Over the last decades the so-called "power of choice" paradigm has received a large amount of attention in various fields. Rather roughly, the term *power of choice* stands for the impact that an observer can have on a system even if she can influence it only by small, local choices. To give an illustration, suppose that we throw n balls uniformly at random into n bins. Then a classical result asserts that the largest number of balls in a bin, the so-called *maximum load*, is close to $\log n / \log\log n$ with high probability (w.h.p.), that is, with probability tending to one as $n \to \infty$; see, for example, [15]. If, instead, we distribute the balls one after the other, and we place each ball in the least loaded out of $d \geq 2$ randomly selected bins, then the maximum load becomes w.h.p. exponentially smaller, namely $\log\log n / \log d + \Theta(1)$ [3]. The paradigm of the power of choice has many applications and was investigated in numerous different situations; see [23, 24] for some techniques and results. In this paper, we study it in the context of the (online) formation of graphs, where the appearance of edges is driven by some random process.

Received July 2014; revised April 2015.

[1]Supported by Grant no. 200021 143337 of the Swiss National Science Foundation.
[2]Supported by Grant no. 6910960 of the Fonds National de la Recherche, Luxembourg.
[3]Supported in part by DFG Grant PA 2080/1, Germany.
MSC2010 subject classifications. Primary 05C80; secondary 60C05.
Key words and phrases. Random graphs, random graph processes, Achlioptas processes, connectivity threshold.

Let ℓ be a positive even integer. An *ℓ-Achlioptas process* is a game with a single player, Paul, who is building a graph. The game is played in rounds. It starts with a graph that has n vertices and no edge. In each round, ℓ vertices $v_1, \dots, v_\ell$ are chosen uniformly at random and shown to Paul, who can then select one of the edges $\{v_1, v_2\}, \{v_3, v_4\}, \dots, \{v_{\ell-1}, v_\ell\}$ and add it to the graph. This game defines a random sequence $(G_N)_{N\ge 0}$, where G_N is the graph after N rounds of the game.

The most prominent and well-studied instance of an Achlioptas process is when $\ell = 2$. In this case, Paul is presented only one pair of vertices per round, so he actually has no choice. This is the classical Erdős–Rényi random graph process ER, and we denote by G_N^{ER} the graph that is created after N edges have been added (where we will always ignore multiple edges and loops). The asymptotic properties of G_N^{ER} have been studied in depth. One of the most striking and intensely studied properties is the *phase transition*, which is also described as the emergence of the giant component [14]: if one parametrizes $N = tn$, then for $t < 1/2$, the largest component in G_N^{ER} contains w.h.p. $O(\log n)$ vertices, while for $t > 1/2$, there is w.h.p. a unique component with $\Theta(n)$ vertices. From today's perspective, the fine details of the phase transition in G_N^{ER} are well understood; see, for example, [10, 11, 16].

The classical Erdős–Rényi process does not give any power to our protagonist. Dimitris Achlioptas proposed a simple modification in which Paul is presented *two* randomly chosen edges per round and may select any one of them. This corresponds to a 4-Achlioptas process as defined above. Specifically, Achlioptas asked if there exists a strategy that Paul can adopt to delay the phase transition past the critical point $N = n/2$. This question was answered affirmatively by Bohman and Frieze, who gave the following rule, which we call the Bohman–Frieze rule, BF for short: of the two given pairs $\{v_1, v_2\}$ and $\{v_3, v_4\}$, select $\{v_1, v_2\}$ if and only if both v_1 and v_2 are isolated vertices in the current graph, and select $\{v_3, v_4\}$ otherwise. Bohman and Frieze proved that if $N = 0.535n$, then w.h.p. the largest component of G_N^{BF} is polylogarithmic in n [8]. The BF process has received a lot of attention since then; see [6, 18, 29] and references therein.

The BF rule is an example of a so-called *size rule*: in each round, Paul bases his decision only on the current *component sizes* of the randomly selected vertices $v_1, \dots, v_\ell$. Actually, it is even what one calls a *bounded-size rule*, since there exists an absolute constant ω such that all components that have at least ω vertices are treated in the same way (for the BF process $\omega = 2$). An example of a size rule that is not bounded-size is the infamous *product rule*, which always selects the pair of vertices for which the product of the component sizes is smallest. For many size rules, and particularly for bounded-size rules, it is by now established that they exhibit a phase transition that shares many qualitative characteristics with the transition in the Erdős–Rényi process; see [4, 5, 9, 12, 26, 27, 30].

While the study of the phase transition has attracted a large amount of attention, the typical properties of a random graph that is created by an Achlioptas process *after* the transition are far less understood. There are some results concerning the presence of small subgraphs [20, 22, 25] or Hamiltonicity [21]. However, one of the most basic properties of a graph—connectivity—has been studied only very little [19].

Before we state our results, we quickly review what is known for the Erdős–Rényi process. For G_N^{ER}, the *connectivity transition* is very well understood. If we write $T_{\text{con}}^{\mathsf{ER}}$ for the smallest N for which G_N^{ER} is connected, then it is known that w.h.p. $T_{\text{con}}^{\mathsf{ER}} = (1 + o(1))n \log n/2$. Moreover, the fine behavior of $T_{\text{con}}^{\mathsf{ER}}$ has been studied; in particular, for any $c \in \mathbb{R}$

$$\lim_{n\to\infty} \Pr\left[T_{\text{con}}^{\mathsf{ER}} \le \frac{n\log n + cn}{2}\right] = \exp\{-e^{-c}\}.$$

Actually, more can be said. Let T_1^{ER} denote the smallest N for which G_N^{ER} contains no isolated vertex. Then w.h.p. $T_1^{\mathsf{ER}} = T_{\text{con}}^{\mathsf{ER}}$, that is, the graph becomes w.h.p. connected exactly at the round in which the last isolated vertex disappears. For more details, we refer to [11] and the references therein.

In this paper, we study the fine details of the connectivity transition of all *bounded-size rules*. For such rules, we give a simple combinatorial criterion that distinguishes between "degenerate" and "nondegenerate" rules. We show that every degenerate rule needs in expectation $\Omega(n^2)$ rounds to create a connected graph, while every nondegenerate rule needs $\Theta(n \log n)$ rounds. For nondegenerate rules R, we are in fact much more precise. If $T_{\text{con}}^{\mathsf{R}}$ denotes the number of rounds until G_N^{R} first becomes connected, then we determine the expectation and the limiting distribution of $T_{\text{con}}^{\mathsf{R}}$, which is always a Gumbel distribution.

The results for nondegenerate rules are summarized in Theorem 4, the main result of this paper. Since the precise statements require some preparations, we postpone them to Section 2. The following corollary illustrates our results by stating what they imply for the Bohman–Frieze process. Its proof, a simple application of Theorem 4, can be found in Section 6.1.

COROLLARY 1. *For the* BF *process, we have that*

$$\mathbb{E}[T_{\text{con}}^{\mathsf{BF}}] = \frac{n\log n}{2} + \left(\frac{\gamma}{2} - \frac{\log\varphi}{\sqrt{5}}\right)\cdot n + o(n),$$

where $\gamma = 0.577\ldots$ *is the Euler–Mascheroni constant and* $\varphi = (1+\sqrt{5})/2$ *the golden ratio, and*

$$\lim_{n\to\infty} \Pr\left[T_{\text{con}}^{\mathsf{BF}} \le \frac{n\log n + cn}{2}\right] = \exp\{-\varphi^{-2/\sqrt{5}}e^{-c}\} \qquad \text{for all } c \in \mathbb{R}.$$

Moreover, w.h.p. G_N^{BF} *gets connected with the addition of the same edge that destroys the last isolated vertex.*

In addition, we also discover a surprising phenomenon: while the Erdős–Rényi process becomes w.h.p. connected exactly at the round in which the last isolated vertex disappears (and, as already mentioned in the previous corollary, the Bohman–Frieze process has the same property), this is not true for general bounded-size rules. In particular, depending on the rule, several different component sizes may be involved in a "race" to get extinct last, and each one of them has a positive limiting probability, which we determine explicitly, of achieving this. The next corollary gives a simple example of this phenomenon, which is a small modification of the BF process; its proof is again an application of Theorem 4 and can be found in Section 6.2.

COROLLARY 2. *The* KP *process is a* 4*-Achlioptas process from* [19] *defined as follows*: *select* $\{v_1, v_2\}$ *if* at least *one of* v_1, v_2 *is isolated in the current graph*, *and select* $\{v_3, v_4\}$ *otherwise. Let* T_k^{KP} *denote the first round in which the last component with k vertices disappears*, *that is*,

$$T_k^{\mathsf{KP}} = \min\{T \mid G_N^{\mathsf{KP}} \textit{ has no component with } k \textit{ vertices } \forall N \geq T\}.$$

Then

$$\lim_{n\to\infty} \Pr[T_1^{\mathsf{KP}} = T_{\mathrm{con}}^{\mathsf{KP}}] = 0.693\ldots \quad \textit{and} \quad \lim_{n\to\infty} \Pr[T_2^{\mathsf{KP}} = T_{\mathrm{con}}^{\mathsf{KP}}] = 0.306\ldots,$$

and the two limiting probabilities sum up to one. Analytic expressions for the limiting probabilities are given in Section 6.2.

Although our results are concerned with bounded-size rules, they enable us to study the connectivity transition of *all* ℓ-Achlioptas processes in the following sense. A fundamental question is to identify the processes that *accelerate* the connectivity transition as much as possible. We solve this problem by giving a specific bounded-size rule that is provably the *fastest* among all ℓ-Achlioptas processes. We define the *lexicographic rule* LEX_ℓ as follows: given randomly chosen vertices $v_1, v_2, \ldots, v_\ell$ whose components have sizes $s_1, s_2, \ldots, s_\ell$, respectively, choose the pair $\{v_{2i-1}, v_{2i}\}$ for which $(\min\{s_{2i-1}, s_{2i}\}, \max\{s_{2i-1}, s_{2i}\})$ is minimal with respect to the lexicographical ordering. In case of ties, choose the smallest eligible i. The following theorem states that no ℓ-Achlioptas process connects the graph faster than LEX_ℓ. The proof and some further discussion can be found in Section 6.3.

THEOREM 3. *Let* ℓ *be a positive even integer and let*

$$c_\ell := \lim_{z\to 0}\left(\log z + \ell \int_1^z \frac{1}{(1-x^2)^{\ell/2} + (1-x)^\ell - 2}\, dx\right).$$

Set $f(c) := (n \log n + cn)/\ell$. *For every* ℓ*-Achlioptas process* A *and every* $c \in \mathbb{R}$, *we have*

$$\limsup_{n\to\infty} \Pr[T_{\mathrm{con}}^{\mathsf{A}} \leq f(c)] \leq \lim_{n\to\infty} \Pr[T_{\mathrm{con}}^{\mathsf{LEX}_\ell} \leq f(c)] = \exp\{-e^{c_\ell - c}\}.$$

We believe that some of the methods used in this paper can also be modified appropriately to study the connectivity transition for general (unbounded) size rules, provided that certain mild regularity conditions are satisfied. Some conditions are certainly necessary, since, in the completely general case of a size rule, it is not even guaranteed that there exists a connectivity transition in the sense established in this paper. For example, the KP rule from Corollary 2 can either reach a round where there are two components of sizes 1 and $n-1$, or it can reach a round where there are two components of sizes 2 and $n-2$, and both cases occur with a probability bounded away from 0. If we allow completely general unbounded size rules, then Paul could play to delay the connectivity transition if he sees a component of size $n-1$, and play to hasten it if he sees a component of size $n-2$. It is clear that for such a rule, there can be no connectivity transition in the classical sense. In bounded-size rules this cannot happen, as Paul is unable to distinguish between components of sizes $n-1$ and $n-2$.

1.1. *Further related work.* The authors of [1] study a different generalization of the Erdős–Rényi process, where in every round the new edge is selected by a nonuniform distribution that gives pairs of isolated vertices a weight of 1 and all other pairs a weight of $K \in [0, \infty)$. In [2], a similar process is studied, where a pair receives weight 1 already if only one of its vertices is isolated. These processes are not Achlioptas processes as we define them here, but still they are "bounded-size" in the sense that only components of size one are treated in a special way. Indeed, the methods used in the analysis of processes of this kind seem to be quite similar to those used for bounded-size Achlioptas processes. In [2], the authors determine the asymptotic number of rounds until the graph gets connected. It is likely that one could extend their results by computing the limiting distribution of the connectivity transition, in a way similar to how we perform this for bounded-size rules in this work.

1.2. *Outline and methods.* Since the proof of our main result is long and spread over the following sections, we provide at this point a (rather informal) overview of the arguments that we will exploit and the obstacles that we have to overcome. We use BF and also sometimes the KP process for illustration. The reader who wishes to skip this subsection may do so and proceed directly to Section 2, which contains the formal definitions and our main theorem.

In order to gain some intuition for the relevant features regarding the connectivity transition, let us start with a fact that can be established rather easily for *all* Achlioptas processes; see also Lemma 11 below. For any $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that after at most $C_\varepsilon n$ rounds, the graph will typically contain a component with at least $(1-\varepsilon)n$ vertices, where n is the total number of vertices in the graph. That is, the graph has a unique largest component, and the at most εn remaining vertices are contained in small components.

Apart from the existence of a unique giant component there are also other features that determine the fine details of the connectivity transition. Particularly, the number of vertices in "small" components plays an important role. For example, for the BF process, it suffices to consider the number $Y(N)$ of isolated vertices in G_N^{BF}. This is so because—as it will turn out—the components of size 1 will be the last to disappear in the BF process. For the KP process from Corollary 2, the situation is different, and one would also need to track components of size two. At this point, due to technical reasons, we split our analysis in two stages: the so-called *early stages*, where $N \le Tn$ for some (large) constant T; and second, the case $N > Tn$, the *late stages* of the process.

Early stages. In Section 3, we study bounded-size rules in their early stages. The analysis relies on the *differential equation method* of Wormald [31], although alternative routes can be taken (see below for a discussion). In the case of the BF process, the main idea is to set up an ordinary differential equation of the form $z'(t) = f(z(t))$ such that for all $0 \le t \le T$, the value of the solution $z(t)$ is very close to $Y(tn)/n$, the fraction of isolated vertices in the tnth round. This reduces the study of the process in the early stages to the study of the analytic properties of $z(t)$.

For the BF process, the equation is given by $z(0) = 1$ and

$$z' = -2z^2 - (1 - z^2)2z. \tag{1}$$

This equation has actually a neat probabilisitc interpretation. Indeed, in the process either (a) the first two selected vertices are isolated, and thus disappear, which corresponds to the term $-2z^2$ or (b) at least one of the two first selected vertices is not isolated (this happens with probability $1 - z^2$) and then the expected number of isolated vertices among the third and the fourth randomly selected vertex is $2z$. It is then intuitively reasonable that $z(t)$ should describe the fraction of isolated vertices at time tn. The theorem of Wormald guarantees us that this is in fact true for any fixed $t \ge 0$. As an alternative, the same statement could be obtained by applying the results of [28], where differential equations describing the evolution of the component size distribution are derived.

For the specific equation (1), one can show that the limit

$$c_1 := \lim_{t\to\infty} \log z(t) - 2t$$

exists. In this particular case, this is not too difficult, but performing this task for general bounded-size rules is a cornerstone of our proof; see Lemma 10. This statement implies that for every $\varepsilon > 0$ and sufficiently large t we have

$$Y(tn)/n \in (1 \pm \varepsilon)z(t) \subseteq (1 \pm 2\varepsilon)e^{c_1 - 2t}, \tag{2}$$

that is, the fraction of isolated vertices after tn rounds is close to e^{c_1-2t}. In fact, a similar statement is true for every bounded size rule. To make the connection with Corollary 1, we just note here that for the BF process we have $c_1 = -2(\log \varphi)/\sqrt{5}$.

The use of differential equations in the context of Achlioptas processes is not new. For example, Spencer and Wormald used differential equations similar to (1) to describe the evolution of bounded-size 4-Achlioptas processes, and they proved that the analytic properties of the solution to these equations determines the critical point for the phase transition [30]. Moreover, Riordan and Warnke proved that for a large class of rules (including all bounded-size rules), the solution to certain similar differential equations also determines the evolution of the component sizes [28]. However, we need to extend the methods to also cover times much later than the emergence of the giant component.

Late stages. The analysis of the early stages is not sufficient to determine the connectivity transition, since we need information about $Y(tn)$ when $t = \omega(1)$, in particular, when $t = \Theta(\log n)$. As it turns out a statement similar to (2) remains true for much longer time scales. We will argue that

$$Y(tn)/n \in (1 \pm \varepsilon)e^{c_1 - 2t} \tag{3}$$

even for all $T \le t \le (1 - \varepsilon)(\log n)/2$. In other words, the answer given by the (asymptotic) solution to the differential equation is true almost all the way up to the connectivity transition.

The first part of Section 4 is dedicated to proving the validity of (3) for bounded-size rules (and for several component sizes); see Lemma 12. Ideally, we would like to use some general-purpose theorem that tells us that the differential equation method can be extended all the way up to $t = (1 - \varepsilon)(\log n)/2$. However, the standard deviation of the number $Y(t)$ of isolated vertices is of order $\sqrt{n}$ if we consider early stages of the process, that is, if $t = \Theta(1)$. Therefore, the (additive) error term in the description of $Y(t)$ must necessarily be at least of this order. Moreover, in a general-purpose theorem it seems that such error terms cannot decrease again, and it is unclear under what conditions the desired shrinking can be established.

With this in mind, we prove (3) directly, by developing a novel induction argument over t. The main reason why this works (for the BF process) is the following. As we have already argued, whenever T is large enough, then w.h.p. for all $t \ge T$, the graph at time tn consists of a single component containing all but εn vertices. The most likely way for an isolated vertex to disappear is then that it is merged into the giant component because it was selected as one of the two vertices v_3, v_4, which happens with probability close to $2/n$. Thus, if there are $e^{c_1 - 2t}$ isolated vertices in round tn, then in round $(t + 1)n$, we expect to have $e^{c_1 - 2t}(1 - 2/n)^{(t+1)n - tn} \approx e^{c_1 - 2(t+1)}$ isolated vertices. By a delicate analysis of several error terms and probabilities we argue that this expectation is in fact very close to the truth for all $t \le (1 - \varepsilon)(\log n)/2$.

Fine analysis of $T_{\mathrm{con}}^{\mathsf{BF}}$. After establishing (3), we are ready to prove Corollary 1. The full proof for the general case is in Section 4. Given $c \in \mathbb{R}$, let

$N_c := (n \log n + cn)/2$. The first step is to show that for every c, w.h.p. all nonisolated vertices in $G^{\mathsf{BF}}_{N_c}$ are in the same component. We will also show that w.h.p. the graph is not connected at time $(n \log n)/2 - \omega(n)$, and that it is connected at time $(n \log n)/2 + \omega(n)$; this will imply by a union bound over all relevant values of c that the graph becomes connected with the addition of the same edge that removes the last isolated vertex. For a general bounded-size rule, we will instead show that all vertices in "large" components are in the same component, for example, for the KP process there is only one component of size larger than two.

Knowing this, our ultimate goal is to determine the limiting distribution of $Y(N_c)$ as $n \to \infty$. To achieve this, we compute the (factorial) moments of $Y(N_c)$ and show that they coincide with the moments of our desired limiting distribution. More precisely, for the BF process we show that

$$\lim_{n\to\infty} \mathbb{E}\big[Y^{\underline{k}}(N_c)\big] = \big(e^{c_1 - c}\big)^k,$$

which are the factorial moments of a Poisson distribution with parameter e^{c_1-c}. Hence, by the *method of moments* (see, e.g., [17], Theorem 6.10), this implies

$$\lim_{n\to\infty} \Pr\big[Y(N_c) = x\big] = e^{-e^{c_1-c}} \big(e^{c_1-c}\big)^x / x!,$$

whence

$$\lim_{n\to\infty} \Pr\big[T^{\mathsf{BF}}_{\mathrm{con}} \le N_c\big] = \lim_{n\to\infty} \Pr\big[Y(N_c) = 0\big] = \exp\{-e^{c_1-c}\}.$$

In fact, for technical reasons, the analysis is carried out in a conditional space in which all rounds are "wellbehaved." The details are in the proof of Theorem 4.

Finally, to compute the expectation of $T^{\mathsf{BF}}_{\mathrm{con}}$, we use the standard fact that if a sequence of random variables X_n converges weakly to X, and, moreover, the sequence is uniformly integrable, then $\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$. After proving that the sequence $2T^{\mathsf{BF}}_{\mathrm{con}}(n)/n - \log n$ is, in fact, uniformly integrable, we obtain the expression

$$\lim_{n\to\infty} \mathbb{E}\big[2T^{\mathsf{BF}}_{\mathrm{con}}/n - \log n\big] = \int_{-\infty}^{\infty} c e^{-e^{c_1-c}} e^{c_1-c}\, dc = c_1 + \gamma,$$

and one can check that this is exactly the statement of Corollary 1.

2. Bounded-size rules. In this paper, we study a broad class of random graph processes that in particular include all bounded-size rules treated in [30]. We use the conventions $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $[m] = \{1, \ldots, m\}$ for $m \in \mathbb{N}$. Let $K, \ell \in \mathbb{N}$, with ℓ even. Let $S_K = \{1, 2, \ldots, K, \omega\}$, where ω stands (informally) for "larger than K." A (K, ℓ)-*rule* is a mapping

$$\mathsf{R} : S_K^{\ell} \to [\ell/2].$$

Any such mapping naturally defines a random graph process $(G_N^{\mathsf{R}})_N$ that will be defined shortly. For a given graph G, we write $C(G; v)$ for the connected component of G containing v and $|G|$ for the number of vertices in G. Set

$$c_K(G; v) := \begin{cases} |C(G; v)|, & \text{if } |C(G; v)| \leq K, \\ \omega, & \text{otherwise.} \end{cases}$$

In the following, we will often omit the subscript K and the reference to G whenever they are obvious from the context. With this notation, the R-random graph process (or R-process for short) with n vertices is defined as follows. Unless otherwise stated, we begin with G_0^{R} being the graph with vertex set $[n]$ and no edge. G_N^{R} is then obtained by choosing independently and uniformly at random ℓ vertices $v_1, \dots, v_\ell$ and adding the edge $\{v_{2i-1}, v_{2i}\}$ to G_{N-1}^{R}, where $i = \mathsf{R}(c_K(G_{N-1}^{\mathsf{R}}; v_1), \dots, c_K(G_{N-1}^{\mathsf{R}}; v_\ell))$. In words, given the vector of the (truncated) sizes of the components that contain the v_i's, R determines which of the $\ell/2$ edges determined by the v_i's is to be added to G_{N-1}^{R}. Note that we do not require R to be symmetric, that is, we allow, for example, $\mathsf{R}(1, 2, 2, \dots, 2) \neq \mathsf{R}(2, 1, 2, \dots, 2)$.

EXAMPLE 1. The BF process is given by the $(1, 4)$-rule

$$\mathsf{BF}(1, 1, x, y) = 1, \qquad \mathsf{BF}(\omega, x, y, z) = \mathsf{BF}(x, \omega, y, z) = 2, \tag{4}$$

where $x, y, z \in \{1, \omega\}$. Similarly, the KP process is given by the $(1, 4)$-rule

$$\mathsf{KP}(1, x, y, z) = \mathsf{KP}(x, 1, y, z) = 1, \qquad \mathsf{KP}(\omega, \omega, x, y) = 2,$$

where $x, y, z \in \{1, \omega\}$.

For a given (K, ℓ)-rule R, we write $T_{\mathrm{con}}^{\mathsf{R}}(n)$ for the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that G_N^{R} is connected. We will usually drop the dependence on n, unless it is necessary to make it explicit. For $k \in \mathbb{N}$, a *k-component* of a graph is a component with k vertices, a *small component* is a component with at most K vertices, and an *ω-component* is a component with more than K vertices. Given $\mu, \nu \in S_K$, we let $C_{\mu,\nu} = C_{\mu,\nu}(\mathsf{R})$ be the set of all component size vectors for which a μ-component and a ν-component are connected by an edge in a step of the R-process. That is,

$$C_{\mu,\nu}(\mathsf{R}) := \{s = (s_1, \dots, s_\ell) \in S_K^\ell \mid \{s_{2i-1}, s_{2i}\} = \{\mu, \nu\} \text{ for } i = \mathsf{R}(s)\}.$$

Note that $C_{\mu,\nu} = C_{\nu,\mu}$. For $1 \leq k \leq K$, we call

$$\mathrm{ex}_k(\mathsf{R}) := k \cdot |\{s \in C_{k,\omega}(\mathsf{R}) \mid \exists i \in [\ell] : s_i = k \text{ and } s_j = \omega \text{ for all } j \in [\ell] \setminus \{i\}\}|$$

the *extinction rate* for size k. That is, $\mathrm{ex}_k(\mathsf{R})$ equals k times the number of all component size vectors in $C_{k,\omega}(\mathsf{R})$ that contain exactly one k at some position, and all other positions are equal to ω. The role of this parameter will become clear at a later point of the analysis; here we provide an informal discussion of its actual meaning. Let us consider the R-process at a rather late point N in time, where G_N^{R}

is *almost* connected. Then G_N^{R} typically consists of one huge component (we make this statement precise in Lemma 11) that contains almost all vertices, and all other vertices are in constant-sized components; this is, for example, the situation in the Erdős–Rényi process; see, for example, [11, 17]. Then, if we select ℓ vertices uniformly at random, most likely they will all be part of the huge component. However, now and then we will also select a vertex in a small component, say with k vertices, and then the most likely event is that we select *exactly one* such vertex. So, the observed component size vector will look like $(\omega, \dots, k, \dots, \omega)$ with the "k" at a random position. Whether we actually connect the component of size k with the large component depends on whether this component vector belongs to $C_{k,\omega}(\mathsf{R})$ or not. In other words, the speed with which components of size k disappear depends on the number of such vectors in $C_{k,\omega}(\mathsf{R})$. This explains the second factor in the definition of $\mathtt{ex}_k(\mathsf{R})$. The first factor comes from the fact that a component of size k has k vertices that can be chosen in order to select this component. Indeed, as we will see during the analysis, the smaller $\mathtt{ex}_k(\mathsf{R})$, the later components of size k will disappear in the R-process. We also let

$$\mathtt{ex}(\mathsf{R}) := \min_{1 \le k \le K} \mathtt{ex}_k(\mathsf{R})$$

be the *total extinction rate* of R, and we let

$$\mathtt{slow}(\mathsf{R}) := \{k \in [K] \mid \mathtt{ex}_k(\mathsf{R}) = \mathtt{ex}(\mathsf{R})\} \quad \text{and} \quad \mathtt{fast}(\mathsf{R}) := [K] \setminus \mathtt{slow}(\mathsf{R})$$

be the sets of *slow indices* and *fast indices*, respectively. As already indicated above, we will see that the main "obstacles" that delay the point in time at which G_N^{R} becomes connected are the components of size $k \in \mathtt{slow}(\mathsf{R})$. Thus, not too surprisingly, the value of the total extinction rate essentially determines the point in time where the R-process gets connected, which is w.h.p. $(1 + o(1))n \log n / \mathtt{ex}(\mathsf{R})$ if $\mathtt{ex}(\mathsf{R}) > 0$. This already shows that the case $\mathtt{ex}(\mathsf{R}) = 0$ is special, and we call a rule *degenerate* if $\mathtt{ex}(\mathsf{R}) = 0$ and *nondegenerate* otherwise.

EXAMPLE 2. The BF and KP rules are nondegenerate. Indeed, for the BF rule, we have $\mathtt{ex}_1(\mathsf{BF}) = \mathtt{ex}(\mathsf{BF}) = 2$, and for the KP rule, we have $\mathtt{ex}_1(\mathsf{KP}) = \mathtt{ex}(\mathsf{KP}) = 4$.

The main results of this paper are summarized in the following theorem, which asserts that all nondegenerate rules belong to the same "universality class": with respect to the connectivity transition, the limiting distribution is always a Gumbel distribution, and the expected value of $T_{\mathrm{con}}^{\mathsf{R}}$ equals $(n \log n + dn)/\mathtt{ex}(\mathsf{R}) + o(n)$ for some $d = d(\mathsf{R})$. To the best of our knowledge, the latter statement has not previously been shown even for the ER-process. Finally, only a finite set of component sizes provides the main "obstacle" for the graph becoming connected.

THEOREM 4. *Let ℓ, K be positive integers, ℓ even, and let* R *be a nondegenerate* (K,ℓ)*-rule such that* $\mathtt{ex}(\mathsf{R}) < 2K+2$. *For* $1 \le k \le K$ *let* $Y_k(N)$ *denote the number of vertices in* k*-components of* G_N^{R}. *Then, for each* $k \in \mathtt{slow}(\mathsf{R})$ *there exists*[4] *a constant* $d_k = d_k(\mathsf{R})$ *such that the following statements are true*:

(a) *For any* $c \in \mathbb{R}$, *w.h.p. for all* $N \ge (n\log n + cn)/\mathtt{ex}(\mathsf{R})$ *we have for all* $k \in \mathtt{fast}(\mathsf{R})$ *that* $Y_k(N) = 0$, *and there is only one component with more than* K *vertices in* G_N^{R}.

(b) *For any* $c \in \mathbb{R}$,

$$\lim_{n\to\infty} \Pr\left[T_{\text{con}}^{\mathsf{R}} \le \frac{n\log n + cn}{\mathtt{ex}(\mathsf{R})}\right] = \prod_{k\in\mathtt{slow}(\mathsf{R})} e^{-d_k e^{-c}}.$$

(c) *Let* $\gamma = 0.577\ldots$ *be the Euler–Mascheroni constant, and let* $c_0 := \log(\sum_{k\in\mathtt{slow}(\mathsf{R})} d_k)$. *Then*

$$\mathbb{E}[T_{\text{con}}^{\mathsf{R}}] = \frac{n\log n + \gamma n + c_0 n}{\mathtt{ex}(\mathsf{R})} + o(n).$$

(d) *For* $k \in [K]$, *let* $T_k^{\mathsf{R}} := \min\{T \mid \forall N \ge T : Y_k(N) = 0\}$ *be the time at which the last* k*-component vanishes. Then* $\Pr[T_k^{\mathsf{R}} = T_{\text{con}}^{\mathsf{R}}] \stackrel{n\to\infty}{\longrightarrow} 0$ *for* $k \in \mathtt{fast}(\mathsf{R})$, *and for* $k \in \mathtt{slow}(\mathsf{R})$,

$$\Pr[T_k^{\mathsf{R}} = T_{\text{con}}^{\mathsf{R}}] \stackrel{n\to\infty}{\longrightarrow} \frac{d_k}{\sum_{i\in\mathtt{slow}(\mathsf{R})} d_i}.$$

For a better understanding of the theorem, we give two remarks.

REMARK 1. The theorem is in general *not* true if $\mathtt{ex}(\mathsf{R}) \ge 2K+2$. For example, for the KP rule as we defined it in Example 1 we have $\mathtt{ex}(\mathsf{KP}) = 4 \ge 2\cdot 1 + 2$, and indeed, the conclusions of the theorem do not hold (intuitively, because components of size two also play an important role).

However, every (K,ℓ)-rule R is also naturally a (K',ℓ)-rule R' for every $K' > K$, with extinction speeds $\mathtt{ex}_k(\mathsf{R}') = \mathtt{ex}_k(\mathsf{R})$ for $1 \le k \le K$ and $\mathtt{ex}_k(\mathsf{R}') = 2k$ for $K < k \le K'$. More precisely, let

$$\text{trunc}\colon S_{K'} \to S_K, \qquad \text{trunc}(k) = \begin{cases} k, & \text{if } k \le K, \\ \omega, & \text{otherwise.} \end{cases}$$

Then the (K',ℓ)-rule R' is defined by

$$\mathsf{R}'(s_1,\ldots,s_\ell) = \mathsf{R}\big(\text{trunc}(s_1),\ldots,\text{trunc}(s_\ell)\big).$$

[4]A formula for $d_k(\mathsf{R})$ is given in Lemma 10. Concrete values of $d_k(\mathsf{R})$ for the BF, KP and lexicographic rules can be found in the last section.

In particular, if we have a (K,ℓ)-rule R for which $\mathtt{ex}(\mathsf{R}) \geq 2K+2$, then we can as well express it as a $((\ell/2)\cdot K,\ell)$-rule R', and one checks immediately that $\mathtt{ex}(\mathsf{R}') < 2\cdot(\ell/2)\cdot K+2$. In this way, Theorem 4 is applicable to every bounded size rule.

REMARK 2. The proof of the theorem will also imply that for all $k \in \mathtt{slow}(\mathsf{R})$,

$$\lim_{n\to\infty}\Pr\left[Y_k\left(\left\lfloor\frac{n\log n+cn}{\mathtt{ex}(\mathsf{R})}\right\rfloor\right)=0\right]=e^{-d_k e^{-c}}.$$

Given this, the theorem shows an "independence in the limit" of the variables Y_k in the following sense. Since by Theorem 4(a) w.h.p. there is only one component with more than K vertices for $N=(n\log n+cn)/\mathtt{ex}(\mathsf{R})$, the graph G_N^{R} is connected if and only if $Y_k(N)=0$ for all $1\leq k\leq K$. Hence, Theorem 4(b) can also be stated as "$\lim_{n\to\infty}\Pr[Y_k(N)=0$ for all $1\leq k\leq K] = \lim_{n\to\infty}\prod_{1\leq k\leq K}\Pr[Y_k(N)=0]$."

Theorem 4 addresses only nondegenerate rules. Degenerate rules R have the unpleasant property that $T_{\mathrm{con}}^{\mathsf{R}}$ can be very large, as it is possible that components of a given fixed size $\leq K$ are never connected to other components unless the rule has no choice. In particular, assume that $\mathtt{ex}_k(\mathsf{R})=0$ for some $1\leq k\leq K$, and that there are only two components left in G_N^{R}: a component with $n-k$ vertices and a k-component. Then the k-component will not be connected to the big component unless at least two of the randomly selected vertices in the current round belong to the k-component. Since the probability that this happens is in $O(n^{-2})$, we will need to wait an expected quadratic number of rounds until the graph gets connected. In fact, a similar situation always occurs with nonnegligible probability, which is the reason for the following theorem. The proof can be found in Section 5.

THEOREM 5. *Let ℓ, K be positive integers, ℓ even, and let R be a degenerate (K,ℓ)-rule. Then* $\mathbb{E}[T_{\mathrm{con}}^{\mathsf{R}}]=\Omega(n^2)$.

Note that for certain rules $T_{\mathrm{con}}^{\mathsf{R}}$ can be even larger than n^2. Consider for example a $(1,\ell)$-rule that does not take any 1-component unless forced to, that is, the rule chooses (ω,ω)-edges whenever such an edge is available. In the proof of Theorem 5, we will show that w.h.p. there is a situation where only one or two isolated vertices remain. These last vertices will only be collected if in every edge there is at least one isolated vertex. This will eventually happen since we allow a vertex to appear several times in the same round. However, the probability of this event is $O(n^{-\ell/2})$, and thus $\mathbb{E}[T_{\mathrm{con}}^{\mathsf{R}}]=\Omega(n^{\ell/2})$.

2.1. *Further terminology and prerequisites.* For a graph G and an induced subgraph C of G, we write $C \in \mathrm{comp}_k(G)$ if C is a k-component of G. We say that an event $\mathcal{E} = \mathcal{E}(n)$ holds with high probability (w.h.p.) if $\lim_{n\to\infty} \Pr[\mathcal{E}(n)] = 1$. Without further reference, we will use for $x \in [0, 1]$ the well-known bounds

$$(1-x)^n = 1 - nx + O\bigl(n^2x^2\bigr) \quad \text{and} \quad 1 - x = e^{-x+\Theta(x^2)}.$$

In several proofs we will also exploit the following version of the *Chernoff bounds*; see, for example, [17], Section 2.1.

LEMMA 6. *Let $X_1, \dots, X_n$ be independent Bernoulli variables such that $\Pr[X_i = 1] = p$ and $\Pr[X_i = 0] = 1 - p$ for all $1 \le i \le n$, and let $X = \sum_{i=1}^n X_i$. Then for all $\delta \in [0, 1]$,*

$$\Pr\bigl[X \ge (1+\delta)np\bigr] \le e^{-\delta^2 np/3} \quad \textit{and} \quad \Pr\bigl[X \le (1-\delta)np\bigr] \le e^{-\delta^2 np/3},$$

and for all $t \ge 2enp$,

$$\Pr[X \ge t] \le 2^{-t}.$$

3. Early stages of the R-process. Let R be a (K, ℓ)-rule. In this section, we will prove several key lemmas that describe the typical structure of G_N^{R} when N is proportional to the number n of vertices, corresponding to the *early stages* in the outline given in the Introduction. Let us write $Y_k^{\mathsf{R}}(N) = Y_k(N)$ for the number of vertices in k-components in $G_N^{\mathsf{R}} = G_N$, where $k \in S_K$.

We will show in Lemma 7 that for an appropriate range of N, $Y_k(N) = (1 + o(1)) \cdot z_k(N/n) \cdot n$, where the z_k's are the unique solution of a specific system of differential equations (5). To this end, we will use a version of Wormald's differential equation method [31]. An alternative route to obtain the same statement would be to apply the results in [28]; however, we prefer to give a direct proof since it is rather short and demonstrates where the various terms in (5) stem from.

The argument for establishing the typical trajectory of the Y_k's on the basis of differential equations is rather standard. However, the main contribution of this section is to study in detail the *analytic* properties of the solution of the system (5), and in particular the case where N/n gets large; see Lemmas 9, 10. These results will be important ingredients in forthcoming arguments.

Let us begin with specifying the system of differential equations. For $s = (s_1, \dots, s_\ell) \in S_K^\ell$ and $\mu, \nu \in S_K$, we define the following polynomials in the variables $(z_k)_{k\in S_K}$:

$$P_s\bigl((z_k)_{k\in S_K}\bigr) := \prod_{k=1}^{\ell} z_{s_k} \quad \text{and} \quad P_{\mu,\nu}\bigl((z_k)_{k\in S_K}\bigr) := \sum_{s\in C_{\mu,\nu}(\mathsf{R})} P_s\bigl((z_k)_{k\in S_K}\bigr).$$

The system is given by

$$\frac{dz_k}{dt} = f_k\bigl(z_1(t), \dots, z_K(t), z_\omega(t)\bigr) \tag{5}$$

with initial conditions $z_1(0) = 1$ and $z_k(0) = 0$ for $k \in S_K \setminus \{1\}$, and where [omitting for brevity the argument $(z_1(t), \dots, z_\omega(t))$] for $k \in [K]$,

$$
\begin{aligned}
& f_k = f_k^+ - f_k^- \\
& \text{with } f_k^+ = k \sum_{\substack{1 \le \mu \le \nu \\ \mu+\nu=k}} P_{\mu,\nu}, \; f_k^- = 2kP_{k,k} + k \sum_{\mu \in S_K \setminus \{k\}} P_{\mu,k},
\end{aligned} \tag{6}
$$

and for $k = \omega$,

$$
f_\omega = \sum_{\substack{1 \le \mu \le \nu \le K \\ \mu+\nu>K}} (\mu+\nu) P_{\mu,\nu} + \sum_{\mu=1}^{K} \mu P_{\mu,\omega}. \tag{7}
$$

The idea behind these definitions is that if $Y_k(N) = nz_k(N/n)$ for all $k \in S_K$, then P_s equals the probability that s is the component size vector of K randomly selected vertices (i.e., the ith selected vertex is in an s_i-component, for all $1 \le i \le \ell$). Thus, f_k^+ is (close to) the expected number of vertices in k-components created in round $N + 1$, and f_k^- is (close to) the expected number of vertices in k-components destroyed in round $N + 1$; this will be made precise in the proof of Lemma 7. Note that these functions depend on the underlying (K, ℓ)-rule R. The following lemma justifies the specific choice of the differential equation system.

LEMMA 7. *Let $k, \ell \in \mathbb{N}$, let R be a (K, ℓ)-rule, and let $T > 0$. Let $\lambda \in \omega(n^{-1}) \cap o(1)$. Then there exists a unique solution $(z_k(t))_{k \in S_K}$ of the system (5), and with probability at least $1 - O(\frac{1}{\lambda}\exp(-n\lambda^3/8K^3))$,*

$$
Y_k(N) = nz_k(N/n) + O(\lambda n)
$$

uniformly for all $k \in S_K$ and all $0 \le N \le Tn$.

In the proof of Lemma 7, we use the following general statement that is a special case of [31], Theorem 5.1. Assume that for every $n \ge 1$ we have a Markov chain $(G_0^{(n)}, G_1^{(n)}, \dots)$, where the random variable $G_N^{(n)}$ takes values in the set $\mathcal{G}^{(n)}$ of all graphs on n vertices. When referring to the Markov chain, we usually drop the dependence on n from the notation. In our context, $G_N = G_N^{\mathsf{R}}$. Let $\mathcal{G}^{(n)+}$ be the set of valid sequences with respect to the Markov chain, that is, the set of all sequences $(G_0, G_1, \dots)$ such that $G_N \in \mathcal{G}^{(n)}$, and the transition probability from G_N to G_{N+1} is positive for all $N \ge 0$. For functions $Y_1 = Y_1^{(n)}, \dots, Y_a = Y_a^{(n)}: \mathcal{G}^{(n)} \to \mathbb{R}$, and $D \subseteq \mathbb{R}^{a+1}$, we define the *stopping time* $N_D(Y_1, \dots, Y_a)$ to be the minimum N such that

$$
\big(N/n, Y_1(G_N)/n, \dots, Y_a(G_N)/n\big) \notin D.
$$

In our context, $a = K + 1$ and $Y_{K+1} = Y_\omega$. With this notation, the following theorem holds.

THEOREM 8 (Theorem 5.1. in [31], simplified[5]). *Let $a, n \in \mathbb{N}$. For $1 \le k \le a$ let $Y_k: \mathcal{G}^{(n)} \to \mathbb{R}$ and $f_k: \mathbb{R}^{a+1} \to \mathbb{R}$ be functions such that $|Y_k(G)| \le n$ for all $G \in \mathcal{G}^{(n)}$. Let D be some bounded connected open set containing the closure of*

$$\{(0, z_1, \dots, z_a) \mid \Pr[Y_k(G_0) = z_k n \text{ for all } 1 \le k \le a] \ne 0 \text{ for some } n\}.$$

Assume the following three conditions hold.

(i) (*Boundedness hypothesis*) *There is a constant $\beta \ge 1$ such that for all $1 \le k \le a$, all $(G_0, G_1, \dots) \in \mathcal{G}^{(n)+}$, and all $N \ge 0$ we have*

$$|Y_k(G_{N+1}) - Y_k(G_N)| \le \beta.$$

(ii) (*Trend hypothesis*) *For some function $\lambda = \lambda(n) = o(1)$ and for all $1 \le k \le a$ and all $G \in \mathcal{G}$,*

$$\left|\mathbb{E}[Y_k(G_{N+1}) - Y_k(G_N) \mid G_N = G] - f_k\left(\frac{N}{n}, \frac{Y_1(G)}{n}, \dots, \frac{Y_a(G)}{n}\right)\right| \le \lambda$$

for all $N < N_D$.

(iii) (*Lipschitz hypothesis*) *Each function f_k is continuous, and satisfies a Lipschitz condition on $D \cap \{(t, z_1, \dots, z_a) \mid t \ge 0\}$.*

Then the following is true.

(a) *For $(0, \hat{z}_1, \dots, \hat{z}_a) \in D$, the system of differential equations*

$$\frac{dz_k}{dt} = f_k(t, z_1, \dots, z_a), \qquad k = 1, \dots, a$$

has a unique solution in D for $z_k : \mathbb{R} \to \mathbb{R}$ passing through $z_k(0) = \hat{z}_k$, $1 \le k \le a$ and the solution extends to points arbitrarily close to the boundary of D;

(b) *For some $C > 0$, with probability $1 - O(\frac{1}{\lambda}\exp(-n\lambda^3/\beta^3))$,*

$$Y_k(G_N) = n z_k(N/n) + o(\lambda n) \tag{8}$$

uniformly for $0 \le N \le \sigma n$ and for each k, where $z_k(t)$ is the solution in (a) *with $\hat{z}_k = \frac{1}{n} Y_k(0)$, and $\sigma = \sigma(n)$ is the supremum of those x to which the solution can be extended before reaching within ℓ^∞-distance $C\lambda$ of the boundary of D.*

PROOF OF LEMMA 7. We apply Theorem 8 as follows. As domain D, we choose (somewhat arbitrarily) $D := (-2T, 2T) \times (-1, 2)^{K+1}$. Note that D contains the set $[0, T] \times [0, 1]^{K+1}$, as required. We will verify the conditions in Theorem 8 one by one.

As already mentioned, for $k \in S_K$, $Y_k(N) = Y_k(G_N^{\mathsf{R}})$ denotes the number of vertices in k-components in G_N^{R}. Note that $\sum_{k \in S_K} Y_k(N) = n$ for all N. Then the

[5]The theorem in [31] is not restricted to Markov chains, and it is also not restricted to graphs. Moreover, the boundedness hypothesis may be satisfied only for a function $\beta = \beta(n)$ and may fail with some error probability $\gamma = \gamma(n)$.

boundedness hypothesis (i) is met with $\beta = 2K$, since any of the Y_k's, $k \in [K]$, can change by at most $2K$ when adding an edge to G_N.

The functions f_k, $k \in S_K$ are given by (6) and (7). To see that they satisfy the trend hypothesis (ii), note that the probability that $s \in S_K^\ell$ is the (truncated) component size vector of ℓ randomly selected vertices is $P_s((Y_k)_{k\in S_K})$. On the other hand, if $s \in C_{\mu,\nu}$, $\mu \neq \nu$ is the component size vector, then two components of size μ and ν are combined into a component of size $\mu + \nu$, so Y_μ and Y_ν decrease by μ and ν, respectively, and $Y_{\mu+\nu}$ (or Y_{K+1}, if $\mu + \nu > K$) increases by $\mu + \nu$. The case $\mu = \nu$ is slightly more complicated, as it might be that both components are identical, in which case only an internal edge (or a loop) is added to the component. However, this event occurs only with probability $O(n^{-1})$. Since all sums are over finitely many terms, the trend hypothesis is satisfied for a suitable function $\lambda \in O(n^{-1})$.

Finally, all the functions f_k are polynomials in $z_1, \dots, z_\omega$, so they trivially satisfy the Lipschitz condition (iii). Thus, all the assumptions of Theorem 8 are satisfied.

It remains to check that the solution of the differential equations does not come close to the boundary of D except for the first component. Observe that $z_k(N/n) \in [0, 1]$ for all N for which (8) holds, because $0 \le Y_k(N) \le n$. Thus, part (b) and the continuity of the z_k's imply the claim. $\square$

We also state a simpler but more explicit bound that will be convenient to use in the sequel.

COROLLARY 3. *Let $k, \ell \in \mathbb{N}$, let R be a (K, ℓ)-rule, and let $T > 0$. For every $\varepsilon > 0$, with probability at least $1 - O(\exp(-n^\varepsilon))$,*

$$Y_k(N) = nz_k(N/n) + o\bigl(n^{2/3+\varepsilon}\bigr),$$

uniformly for all $k \in S_K$ and all $0 \le N \le Tn$.

PROOF. Use $\lambda := 2Kn^{-1/3+\varepsilon}$ in Lemma 7. $\square$

For later reference, we first collect some basic properties of the functions z_k. The following lemma is in parts a generalization of Theorem 2.1 in [30], where the phase transition was studied in the case $\ell = 4$.

LEMMA 9. *Let $K, \ell \in \mathbb{N}$ and let R be a (K, ℓ)-rule. Then the unique solution $(z_k(t))_{k\in S_K}$ of (5) has the following properties:*

(a) *$\sum_{k\in S_K} z_k(t) = 1$ for all $t \ge 0$.*
(b) *For all $t > 0$ and all $k \in S_K$ we have $0 < z_k(t) < 1$.*
(c) *For every $1 \le i \le K$ the function $\sum_{k=1}^{i} z_k$ is strictly decreasing. Moreover, z_ω is strictly increasing.*

(d) *If* R *is nondegenerate, then there is* $t_0 > 0$ *and* $c > 0$ *such that* $1 - z_\omega(t) \leq e^{-c(t-t_0)}$ *for all* $t \geq t_0$. *In particular,* $z_\omega(t) \to 1$ *for* $t \to \infty$.

PROOF OF (a). In the sum $\sum_{k\in S_k} f_k(z_1,\dots,z_\omega)$, for $\mu \neq \nu$ and $s \in C_{\mu,\nu}$ the term $\mu \cdot P_s$ is added and subtracted exactly once. For $s \in C_{\mu,\mu}$, the term $\mu \cdot P_s$ is added and subtracted exactly twice. Hence, all terms cancel, and we have $\sum_{k\in S_k} f_k(z_1,\dots,z_\omega) = 0$. Thus, the function $\tilde{z}(t) := \sum_{k\in S_K} z_k(t)$ satisfies the differential equation $d\tilde{z}/dt = 0$, with initial condition $\tilde{z}(0) = 1$. Therefore, $\tilde{z}(t) = 1$. $\square$

PROOF OF (b). We will show $z_k(t) > 0$ for all $k \in S_K$ and all $t > 0$; the other inequality follows then directly from (a). By applying Corollary 3, we infer that $z_k(t) \geq 0$ for all $k \in S_K$. By (a), this implies $z_k(t) \leq 1$ for all $k \in S_K$.

First, we show that if there is $t_0 > 0$ and $1 \leq k \leq K$ such that $z_k(t_0) > 0$, then $z_k(t) > 0$ for all $t \geq t_0$. Note that

$$z_k'(t) \geq -f_k^- \geq -k \sum_{\mu\in S_K\setminus\{k\}} \sum_{s\in C_{\mu,k}} P_s - 2k \sum_{s\in C_{k,k}} P_s.$$

In the last expression each occurring term P_s contains a factor z_k, and all other factors are ≤ 1. Thus, by abbreviating $C_k := \sum_{\mu\in S_K\setminus\{k\}} \sum_{s\in C_{\mu,k}} 1 + 2\sum_{s\in C_{k,k}} 1$, we readily get that $z_k'(t) \geq -kC_k z_k(t)$. By integrating this from t_0 to t, we obtain that $z_k(t) \geq e^{-kC_k(t-t_0)} z_k(t_0) > 0$ for all $t \geq t_0$.

Next, note that $z_1(0) = 1 > 0$, so the previous argument implies that $z_1(t) > 0$ for all $t \geq 0$. We show by induction on k that $z_k(t) > 0$ holds for all $t > 0$ and $1 \leq k \leq K$. For some $2 \leq k \leq K$, assume there was $t_0 > 0$ with $z_k(t_0) = 0$. Since $z_k(0) = 0$, then again the previous argument implies that $z_k(t) = 0$ for all $0 \leq t \leq t_0$, and for this range (6) simplifies to

$$f_k(z_1,\dots,z_\omega) = \sum_{\substack{1\leq\mu\leq\nu\\ \mu+\nu=k}} \left((\mu+\nu) \sum_{s\in C_{\mu,\nu}} P_s \right).$$

This expression is at least $\sum_{1\leq\mu\leq\nu,\mu+\nu=k} z_\mu^{\ell/2} z_\nu^{\ell/2}$, since $(\mu,\nu,\dots,\mu,\nu) \in C_{\mu,\nu}$. The right-hand side is positive by induction hypothesis, which contradicts the fact that $f_k(z_1,\dots,z_\omega) = dz_k/dt = 0$ for all $0 \leq t \leq t_0$. This shows the claim for all $1 \leq k \leq K$ and $t > 0$.

It remains to treat the case $k = \omega$. Equation (7) implies that

$$f_\omega(z_1,\dots,z_\omega) \geq P_{1,K}$$

and since we have already shown that $z_k(t) > 0$ for all $1 \leq k \leq K$ and $t > 0$ this expression is >0 for all $t > 0$. The claim now follows with the same contradiction as for f_k. $\square$

PROOF OF (c). In the sum $\sum_{k=1}^{i} f_k(z_1, \ldots, z_\omega)$, for all $1 \le \mu \le \nu \le K$ with $\mu + \nu \le i$ and $s \in C_{\mu,\nu}$ the term $\mu \cdot P_s$ is added and subtracted exactly once if $\mu \ne \nu$, and it is added and subtracted exactly twice if $\mu = \nu$. So all these terms cancel. On the other hand, for all $1 \le \mu \le \nu \le K$ with $\mu + \nu > i$, the terms $\mu \cdot P_s$ are only subtracted (once or twice), but not added, and by (b) all these terms are >0 for $t > 0$. Hence, the function $\sum_{k=1}^{i} z_k$ has negative derivative for all $t > 0$, so it is strictly decreasing. As for the final remark, we infer directly that $z_\omega = 1 - \sum_{k=1}^{K} z_k$ is strictly increasing. $\square$

PROOF OF (d). By definition of $C_{\mu,\nu}$ the bounded size rule is nondegenerate, if and only if for each $1 \le k \le K$ there exists $s \in C_{k,\omega}$ such that $P_s = z_k \cdot z_\omega^{\ell-1}$. Thus, $f_\omega(z_1, \ldots, z_\omega) \ge (\sum_{k=1}^{K} z_k) z_\omega^{\ell-1}$. Fix any $t_0 > 0$, and let $\tilde{c} := z_\omega(t_0)$. From (b) and (c), we know that $\tilde{c} > 0$ and that $z_\omega(t) \ge \tilde{c}$ for all $t \ge t_0$, respectively. Hence,

$$\sum_{k=1}^{K} f_k = -f_\omega \le -\left(\sum_{k=1}^{K} z_k(t)\right) z_\omega(t)^{\ell-1} \le -\left(\sum_{k=1}^{K} z_k(t)\right) \tilde{c}^{\ell-1}$$

for all $t \ge t_0$. Therefore, the function $\tilde{z} := \sum_{k=1}^{K} z_k$ satisfies $d\tilde{z}/dt \le -\tilde{c}^{\ell-1}\tilde{z}$ for all $t \ge t_0$. Thus, $0 \le \tilde{z}(t) \le \tilde{z}(t_0) e^{-\tilde{c}^{\ell-1}(t-t_0)} \to 0$ for $t \to \infty$. The claim now follows with $c := \tilde{c}^{\ell-1}$ from $\tilde{z}(t_0) \le 1$ and $1 - z_\omega(t) = \tilde{z}(t)$. $\square$

We continue with a crucial ingredient for studying the fine properties of the distribution of $T_{\mathrm{con}}^{\mathsf{R}}$. We determine the limiting behavior of the fraction $z_k(t)$ of vertices in k-components in G_{tn}^{R}; in particular, for all $k \in \mathtt{slow}(\mathsf{R})$ the next lemma asserts that $z_k(t)$ approaches $C_k e^{-\mathrm{ex}(\mathsf{R})t}$, for some $C_k = C_k(\mathsf{R}) > 0$.

LEMMA 10. *Let $K, \ell \in \mathbb{N}$ and let R be a (K, ℓ)-rule.*

(a) *For every $\varepsilon > 0$, there exists a $t_0 > 0$ such that for all $t \ge t_0$*

$$\sum_{k \in \mathtt{fast}(\mathsf{R})} z_k(t) \le \varepsilon \cdot \sum_{k \in \mathtt{slow}(\mathsf{R})} z_k(t).$$

(b) *If R is nondegenerate then for $k \in \mathtt{slow}(\mathsf{R})$ the limit*

$$c_k := \lim_{t \to \infty} \bigl(\mathrm{ex}(\mathsf{R}) \cdot t + \log z_k(t)\bigr)$$

exists.[6]

PROOF. For brevity, we write $\mathtt{slow} = \mathtt{slow}(\mathsf{R})$, $\mathtt{fast} = \mathtt{fast}(\mathsf{R})$ and $\mathrm{ex} = \mathrm{ex}(\mathsf{R})$. Furthermore, we let $z_{\mathtt{slow}} := \sum_{k \in \mathtt{slow}} z_k$ and $z_{\mathtt{fast}} := \sum_{k \in \mathtt{fast}} z_k$.

[6]As will be proven later, the constant d_k from Theorem 4 equals e^{c_k}/k.

Note that, by Lemma 9, we have $z_{\mathtt{slow}}(t) + z_{\mathtt{fast}}(t) + z_\omega(t) = 1$ for all $t \geq 0$ and that $z_\omega(t)$ is increasing, while $z_{\mathtt{slow}}(t) + z_{\mathtt{fast}}(t)$ is decreasing.

Recall that $z_k' = f_k = f_k^+ - f_k^-$ for all $1 \leq k \leq K$; see (6). Here, f_k^+ consists of terms P_s, $s \in C_{\mu,\nu}$, with $1 \leq \mu \leq \nu \leq K$. Every such term contains at least two factors z_i, z_j with $1 \leq i, j \leq K$. Since by Lemma 9 we know that $z_i \leq 1$ for all $i \in S_K$ there is a $c > 0$, such that

$$0 \leq f_k^+ \leq c(z_{\mathtt{slow}} + z_{\mathtt{fast}})^2. \tag{9}$$

The term f_k^- sums up terms P_s for indices $s \in S_K^\ell$ for which at least one component equals k. Moreover, if $k \in \mathtt{slow}$ then the coefficient of the polynomial $z_k z_\omega^{\ell-1}$ in f_k^- is exactly $\mathtt{ex}$, and for $k \in \mathtt{fast}$ it is $\geq(\mathtt{ex}+1)$. All other terms in f_k^- contain at least the factor z_k and another factor z_i, $1 \leq i \leq K$. Hence, by making the constant $c > 0$ from (9) larger if necessary we obtain

$$\mathtt{ex} \cdot z_\omega^{\ell-1} \cdot z_k \leq f_k^- \leq \big(\mathtt{ex} + c(z_{\mathtt{slow}} + z_{\mathtt{fast}})\big) \cdot z_k \qquad \text{for all } k \in \mathtt{slow}, \tag{10}$$

and

$$(\mathtt{ex}+1) z_\omega^{\ell-1} \cdot z_k \leq f_k^- \qquad \text{for all } k \in \mathtt{fast}. \tag{11}$$

Consider an arbitrary $\varepsilon > 0$. By Lemma 9(d), there exists $t_0' > 0$ such that $z_\omega(t_0') \geq 1 - \varepsilon^2$, and by the monotonicity of $z_\omega(t)$,

$$z_\omega(t) \geq 1 - \varepsilon^2 \quad \text{and} \quad z_{\mathtt{slow}}(t) + z_{\mathtt{fast}}(t) \leq \varepsilon^2 \qquad \text{for all } t \geq t_0'. \tag{12}$$

Together with $z_k' = f_k^+ - f_k^-$, the lower bound in (9) and the upper bound in (10) imply $z_k'(t) \geq -(\mathtt{ex} + \varepsilon) z_k(t)$ for all $0 < \varepsilon < 1/c$ and $k \in \mathtt{slow}$. Dividing both sides by $z_k(t)$ and integrating from t' to t yields

$$z_k(t) \geq z_k(t') \cdot e^{-(\mathtt{ex}+\varepsilon)(t-t')} \qquad \text{for all } k \in \mathtt{slow} \text{ and all } t \geq t' \geq t_0'. \tag{13}$$

With the above preparations, we are ready to prove the lemma. In order to see (a), we first prove an auxiliary statement. We claim that whenever there exist $t_2 > t_1 \geq t_0'$ with $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R} \cup \{\infty\}$ such that for all $t \in [t_1, t_2)$ we have $z_{\mathtt{fast}}(t) \geq \varepsilon z_{\mathtt{slow}}(t)/2$, then

$$\frac{z_{\mathtt{fast}}(t)}{z_{\mathtt{slow}}(t)} \leq \frac{z_{\mathtt{fast}}(t_1)}{z_{\mathtt{slow}}(t_1)} \cdot e^{-(1/2-\varepsilon)(t-t_1)} \qquad \text{for all } t \in [t_1, t_2). \tag{14}$$

To prove (14), note that the assumption on t_1 and t_2, together with (9), (11) and (12), imply that for all $t \in [t_1, t_2)$

$$\begin{aligned} z_{\mathtt{fast}}'(t) &\leq \sum_{k \in \mathtt{fast}} \big(c(z_{\mathtt{slow}} + z_{\mathtt{fast}})^2 - (\mathtt{ex}+1) z_\omega^{\ell-1} \cdot z_k\big) \\ &\leq c|\mathtt{fast}|\varepsilon^2 \Big(1 + \frac{2}{\varepsilon}\Big) z_{\mathtt{fast}}(t) - (\mathtt{ex}+1)\big(1-\varepsilon^2\big)^{\ell-1} z_{\mathtt{fast}}(t). \end{aligned}$$

For $\varepsilon > 0$ sufficiently small, we thus have $z'_{\texttt{fast}}(t) \le -(\texttt{ex} + \frac{1}{2}) z_{\texttt{fast}}(t)$ and so

$$z_{\texttt{fast}}(t) \le z_{\texttt{fast}}(t_1) \cdot e^{-(\texttt{ex}+1/2)(t-t_1)} \qquad \text{for all } t \in [t_1, t_2).$$

Together with (13) (where we use $t' = t_1$), this implies (14), as claimed.

Equation (14) allows us to infer (a) by contradiction as follows. First of all, note that if for all $t \ge t'_0$ we had $z_{\texttt{fast}}(t) \ge \varepsilon z_{\texttt{slow}}(t)/2$ then we could apply (14) with t'_0 in place of t_1 and ∞ in place of t_2. Since by Lemma 9 $z_{\texttt{slow}}(t'_0) > 0$, we infer that there is a $t''_0 \ge t'_0$ such that $z_{\texttt{fast}}(t''_0) < \varepsilon z_{\texttt{slow}}(t''_0)/2$, a contradiction. So there is a $t'_1 \ge t'_0$ such that $z_{\texttt{fast}}(t'_1) < \varepsilon z_{\texttt{slow}}(t'_1)/2$. Assume for the sake of contradiction that there is a $t'_2 > t'_1$ such that $z_{\texttt{fast}}(t'_2) > \varepsilon z_{\texttt{slow}}(t'_2)$. Then by continuity of the z_k's, there would be an interval $I = [t''_1, t''_2] \subseteq [t'_1, t'_2]$ such that $z_{\texttt{fast}}(t''_1) = \varepsilon z_{\texttt{slow}}(t''_1)/2$, $z_{\texttt{fast}}(t''_2) = \varepsilon z_{\texttt{slow}}(t''_2)$ and $z_{\texttt{fast}}(t) \ge \varepsilon z_{\texttt{slow}}(t)/2$ for all $t \in I$. However, this is a contradiction since (14) implies that the ratio $z_{\texttt{fast}}(t)/z_{\texttt{slow}}(t)$ cannot increase in I. Thus, $z_{\texttt{fast}}(t) \le \varepsilon z_{\texttt{slow}}(t)$ for all $t \ge t'_1$; this establishes (a) with $t_0 = t'_1$.

In order to prove (b), by applying (9), (10) and (12) we infer that for $0 < \varepsilon < \min\{1/c, 1\}$ and $t \ge t_0$

$$\begin{aligned} z'_{\texttt{slow}}(t) &= \sum_{k \in \texttt{slow}} (f_k^+ - f_k^-) \\ &\le cK\varepsilon^2 \big(z_{\texttt{slow}}(t) + z_{\texttt{fast}}\big) - \texttt{ex} \cdot \big(1 - \varepsilon^2\big)^{\ell-1} z_{\texttt{slow}}(t). \end{aligned}$$

By using (a), we further get for $t \ge t_0$

$$z'_{\texttt{slow}}(t) \le 2cK\varepsilon^2 z_{\texttt{slow}}(t) - \texttt{ex} \cdot \big(1 - \varepsilon^2\big)^{\ell-1} z_{\texttt{slow}}(t).$$

For all $\varepsilon > 0$ small enough, we thus get $z'_{\texttt{slow}}(t) \le -(\texttt{ex} - \varepsilon) z_{\texttt{slow}}(t)$ and so

$$z_{\texttt{slow}}(t) \le z_{\texttt{slow}}(t_0) \cdot e^{-(\texttt{ex}-\varepsilon)(t-t_0)} \qquad \text{for all } t \ge t_0. \tag{15}$$

Since R is nondegenerate, we have $\texttt{ex} \ge 1$. Together with (13) (applied to $t' = t_0$) this implies that for $\varepsilon > 0$ small enough there exists a constant $C > 0$ such that

$$\frac{(z_{\texttt{slow}}(t))^2}{z_k(t)} \le Ce^{-t/2} \qquad \text{for all } k \in \texttt{slow} \text{ and all } t \ge t_0. \tag{16}$$

Next, we use (10) and (a) again to obtain for $k \in \texttt{slow}$ and $t \ge t_0$

$$\begin{aligned} z'_k(t) &\ge -f_k^- \ge -\big(\texttt{ex} + c\big(z_{\texttt{slow}}(t) + z_{\texttt{fast}}(t)\big)\big) z_k(t) \\ &\ge -\big(\texttt{ex} + 2cz_{\texttt{slow}}(t)\big) z_k(t) \end{aligned}$$

and similarly, using (9), (10) and $z_\omega(t) = 1 - z_{\texttt{fast}}(t) - z_{\texttt{slow}}(t) \ge 1 - 2z_{\texttt{slow}}(t)$

$$z'_k(t) = f_k^+ - f_k^- \le 4c \cdot \big(z_{\texttt{slow}}(t)\big)^2 - \texttt{ex} \cdot \big(1 - 2z_{\texttt{slow}}(t)\big)^{\ell-1} z_k(t).$$

Thus, for $t \ge t_0$

$$-\big(\texttt{ex} + 2cz_{\texttt{slow}}(t)\big) \le \frac{z'_k(t)}{z_k(t)} \le \frac{4c \cdot (z_{\texttt{slow}}(t))^2}{z_k(t)} - \texttt{ex} \cdot \big(1 - 2\ell z_{\texttt{slow}}(t)\big).$$

Now we integrate all three sides from t_1 to t_2. By (15) and (16), all terms on the left and the right-hand side decay exponentially, except for the constant $-\mathtt{ex}$. Therefore, there exists $\hat{t}_0 \ge t_0$ so that for $t_2 > t_1 \ge \hat{t}_0$

$$-\mathtt{ex} \cdot (t_2 - t_1) - \varepsilon \le \log z_k(t_2) - \log z_k(t_1) \le -\mathtt{ex} \cdot (t_2 - t_1) + \varepsilon.$$

That is, the sequence $\mathtt{ex} \cdot t + \log z_k(t)$ is a Cauchy sequence, and thus convergent. □

We close this section with a statement about the existence of a large component in random graph processes, which is also true if we begin with a graph that already contains some edges.

LEMMA 11. *Let $\varepsilon > 0$ and let $\ell > 0$ be an even integer. Then there exists a constant $C_\varepsilon > 0$ such that for any ℓ-Achlioptas process with arbitrary initial graph G_0 on n vertices, after at most $C_\varepsilon n$ rounds the number of vertices in the largest component is at least $(1 - \varepsilon)n$ with probability $1 - o(1/n)$.*

PROOF. We may assume that $\varepsilon < 1/3$. Then observe that as long as there is no component with $\ge(1 - \varepsilon)n$ vertices there exist two disjoint vertex sets A and B (not necessarily the same in each round) such that $|A|, |B| \ge \varepsilon n$ and such that no component contains vertices both from A and B. Note that this assumption on A and B implies that every edge between A and B connects two different components. Observe also that the probability that the ℓ-Achlioptas process will choose such an edge is at least $p_\varepsilon := \varepsilon^\ell$, as this is at least the probability that $(v_1, \dots, v_\ell) \in (A \times B)^{\ell/2}$.

Set $C_\varepsilon := 2/p_\varepsilon$ and assume that for $C_\varepsilon n$ rounds the size of the largest component is $\le(1 - \varepsilon)n$. From the previous discussion, we know that this occurs with probability at most $\Pr[\mathrm{Bin}(C_\varepsilon n, p_\varepsilon) < n]$, which is easily seen to be $o(1/n)$ by the choice of C_ε and the Chernoff bounds. □

4. Late stages of the R–process and proof of Theorem 4. As described in the proof outline given in the Introduction, the differential equation method allows us to analyze (K, ℓ)-rules in (relatively) early stages of the process, that is, when a linear number of edges is added to an initially empty graph. However, as we will see shortly, the graph will become connected much later, after roughly $n \log n/\mathtt{ex}(\mathsf{R})$ rounds. The following lemma shows that the concentration result of Lemma 7 can essentially be extended up to the point where $(1/\mathtt{ex}(\mathsf{R}) - o(1))n \log n$ edges have been added. Since it is not clear (at least to us) how such a statement can be established via a method based on differential equations, we choose a more direct route, where we prove inductively that the post-linear regime follows typically a suitably defined deterministic trajectory. In the remainder of this section, we write $a = x \pm y$, where $y > 0$, for $a \in [x - y, x + y]$.

LEMMA 12. *Let $K, \ell \in \mathbb{N}$ and let R be a nondegenerate (K, ℓ)-rule. For $1 \leq k \leq K$, let $Y_k(N)$ denote the number of vertices in components with k vertices in G_N^{R}. Moreover, for $k \in \texttt{slow}(\mathsf{R})$ let $c_k(\mathsf{R})$ be defined as in Lemma 10. Then there is an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there is a $t_0 = t_0(\varepsilon)$ such that with probability $1 - o(1/\log n)$, the following holds for all $t_0 \leq t \leq (1-\varepsilon)\log n$:*

(a) $Y_k(tn/\mathrm{ex}(\mathsf{R})) = (1 \pm \varepsilon) \cdot e^{c_k(\mathsf{R})-t} n$ *for all* $k \in \texttt{slow}(\mathsf{R})$, *and*
(b) $Y_k(tn/\mathrm{ex}(\mathsf{R})) \leq \varepsilon e^{-t} n$ *for all* $k \in \texttt{fast}(\mathsf{R})$.

PROOF. For $t \in \mathbb{R}_{\geq 0}$, let us write $N_t = tn/\mathrm{ex}(\mathsf{R})$. We use an inductive argument, where Corollary 3 provides us with the base case. Indeed, for any fixed $T > 0$, by Corollary 3, there exists $\delta > 0$ such that with probability $1 - o(1/\log n)$, we have

$$Y_k(N_t) = nz_k\big(t/\mathrm{ex}(\mathsf{R})\big) + o\big(n^{1-\delta}\big) \qquad \text{for all } 1 \leq k \leq K \text{ and } t \leq T. \tag{17}$$

By Lemma 10, there exists some positive constant T such that for all $k \in \texttt{slow}(\mathsf{R})$ and all $t \in [T, T+1]$, we have

$$z_k\big(t/\mathrm{ex}(\mathsf{R})\big) = \big(1 \pm \varepsilon^2/2\big)e^{c_k - t}$$

and such that for all $t \in [T, T+1]$, we have

$$\sum_{k \in \texttt{fast}(\mathsf{R})} z_k(t) \leq \frac{\varepsilon^2 \sum_{k \in \texttt{slow}(\mathsf{R})} z_k(t)}{2K \cdot \max_{k \in \texttt{slow}(\mathsf{R})} e^{c_k}}.$$

Together with (17), we obtain that (a) and (b) hold for all $t \in [T, T+1]$, with ε replaced by ε^2. This will serve as the base case of our induction.

The reason why we showed (a) and (b) with much smaller error terms than required (for $t \in [T, T+1]$) is that, in order to cover all cases $t_0 \leq t \leq (1-\varepsilon)\log n$, we will prove inductively a statement in which the error terms will gradually increase. Formally, let

$$\varepsilon_0 := \varepsilon^2 \quad \text{and} \quad \varepsilon_{i+1} = \varepsilon_i + c'\big(e^{-t_i} + e^{t_i/2}\sqrt{\log n/n}\big) \qquad \text{for } i \in \mathbb{N}_0 \tag{18}$$

for a constant $c' > 0$ that we will fix later (and that will not depend on ε). By expanding the recursive definition, it is easy to see that for n sufficiently large we have $\varepsilon_i \leq \varepsilon^2 + 2c'e^{-t_0} \leq c''\varepsilon^2$, by choice of t_0. Note that c'' does not depend on ε, as c' does not. If we thus choose $\varepsilon < 1/c''$, we obtain that $\varepsilon_i \leq \varepsilon$ for all $i \in \mathbb{N}_0$.

Recall that our choice of T implies that for all $t_0 \in [T, T+1]$, we have $Y_k(N_{t_0}) = (1 \pm \varepsilon_0)e^{c_k - t_0}n$. Fix some $t_0 \in [T, T+1]$ and let $t_i := t_0 + i$ for $i \in \mathbb{N}$. To simplify notation, let us write $N_i = N_{t_i}$. With this notation at hand, it suffices to prove that with probability $1 - o((n\log n)^{-1})$ we have for all $1 \leq i \leq (1-\varepsilon)\log n$ that

$$Y_k(N_i) = (1 \pm \varepsilon_i)e^{c_k - t_i}n \qquad \text{for } k \in \texttt{slow}(\mathsf{R}) \tag{19}$$

and

$$Y_k(N_i) \le \varepsilon_i e^{-t_i} n \qquad \text{for } k \in \texttt{fast}(\mathsf{R}). \tag{20}$$

Then the lemma follows by a union bound over n possible choice of t_0.

We use induction over i. We already know that the claim is true for $i = 0$. For the induction step, we will show that assuming the claim holds for some $i \ge 0$ it also holds for $i + 1$ with probability $1 - o(n^{-2})$.

A round $N_i \le N < N_{i+1}$ is called *N_i-regular* if out of the ℓ randomly selected vertices there is at most one vertex v that is contained in a small component (i.e., in a k-component for $1 \le k \le K$) in $G^{\mathsf{R}}_{N_i}$, and N_i-nonregular otherwise. Note that the definition refers to the graph $G^{\mathsf{R}}_{N_i}$, not to G^{R}_N. This seemingly strange definition has the advantage that for different rounds the events that a specific round is N_i-regular are *independent*. In particular, let I be the number of N_i-nonregular rounds. The induction assumption guarantees that in round N_i, there are only $O(e^{-t_i} n)$ vertices in small components. Thus, the probability that any succeeding round is N_i-nonregular is in $O(e^{-2t_i})$. Since $N_{i+1} - N_i = n/\mathtt{ex}(\mathsf{R})$ it follows that the expected number of nonregular rounds between N_i and N_{i+1} is at most $ce^{-2t_i} n$, for some sufficiently large constant $c > 0$. By the Chernoff bounds there is a constant $C > 0$ (not depending on ε) such that

$$\begin{aligned}\Pr\bigl[I \ge \max\{Ce^{-2t_i} n, e^{-t_i/2}\sqrt{n}\}\bigr] &\le \min\{2^{-ce^{-2t_i} n}, 2^{-e^{-t_i/2}\sqrt{n}}\} \\ &= o(n^{-2}),\end{aligned} \tag{21}$$

using $t_i \le t_0 + (1 - \varepsilon)\log n$.

In order to prove the induction step, consider some $1 \le k \le K$. Denote a position $1 \le p \le \ell$ as *k-good* if a vector $(\omega, \dots, \omega, k, \omega, \dots, \omega)$ with the k at position p results in merging the k-component with an ω-component. Recall that the number of k-good positions is exactly $\mathtt{ex}_k(\mathsf{R})/k$.

For ease of notation, we use $X_k(N)$ to denote the number of k-components in G^{R}_N. Clearly, $X_k(N) = Y_k(N)/k$ for all $k \in [K]$. To give bounds on $X_k(N_{i+1})$, let Z be the number of k-components $C \in \mathrm{comp}_k(G^{\mathsf{R}}_{N_i})$ that never appear in a k-good position during rounds $[N_i, N_{i+1})$. Observe that if a k-component appears in an N_i-regular round $N \ge N_i$, then it is merged with an ω-component if and only if its position is k-good (since the other vertices in this round belong to ω-components in N_i and in all subsequent rounds). Thus, Z counts basically the number of k-components in round N_{i+1}, miscounting only components that appear in N_i-nonregular rounds. Since there are at most ℓI such components,

$$Z - \ell I \le X_k(N_{i+1}) \le Z + \ell I. \tag{22}$$

In the sequel, we bound Z. We proceed as follows: enumerate the $X_k(N_i)$ k-components in $G^{\mathsf{R}}_{N_i}$ from 1 to $X_k(N_i)$ in an arbitrary but fixed way and let $Z_{j,s}$

be a Bernoulli random variable that is one if and only if in round j we choose a vertex from the sth component in a k-good position. Thus,

$$\Pr[Z_{j,s}=1]=1-\left(1-\frac{k}{n}\right)^{\mathtt{ex}_k(\mathsf{R})/k}=\frac{\mathtt{ex}_k(\mathsf{R})}{n}\pm\frac{c}{n^2}$$

for a constant c that depends on K and ℓ but not on ε. Clearly,

$$Z=\sum_{s=1}^{X_k(N_i)} Z_s, \qquad \text{where } Z_s \text{ is the indicator function for } \sum_{j=N_i+1}^{N_{i+1}} Z_{j,s}=0.$$

Therefore, using the induction assumption on $X_k(N_i)$ and bounding the error terms very generously, we get

$$\begin{aligned}\mathbb{E}[Z]&=X_k(N_i)\cdot\left(1-\frac{\mathtt{ex}_k(\mathsf{R})}{n}\pm\frac{c}{n^2}\right)^{n/\mathtt{ex}(\mathsf{R})}\\ &=e^{-\mathtt{ex}_k(\mathsf{R})/\mathtt{ex}(\mathsf{R})}\cdot X_k(N_i)\cdot(1\pm 2c/n)\\ &=e^{-\mathtt{ex}_k(\mathsf{R})/\mathtt{ex}(\mathsf{R})}\cdot X_k(N_i)\pm\tilde{c}e^{-2t_i}n,\end{aligned}\tag{23}$$

for some constant $\tilde{c}$ that depends on K, ℓ and $\max_{k\in\mathtt{slow}(\mathsf{R})} c_k(\mathsf{R})$ but not on ε.

Note that for $k\in\mathtt{slow}(\mathsf{R})$ [i.e., for $\mathtt{ex}_k(\mathsf{R})=\mathtt{ex}(\mathsf{R})$] the expectation agrees with the prediction of the statement: the main term is $X_k(N_i)/e$, as desired. We thus just need to show that Z is concentrated. For the alert reader, this should come as no surprise: the variables $X_{j,s}$ are defined similarly as in a balls-and-bin game where the variable Z counts the number of empty bins. It is well known that in a balls-and-bin game the variables are negatively associated and one can thus apply Chernoff bounds to the variable Z. Adapted to our scenario, we can argue as follows. For a fixed round j, the random variables $(Z_{j,s})_s$ are Bernoulli random variables that sum up to at most 1. By adding an additional variable $Z_{j,0}:=1-\sum_{s\ge 1}Z_{j,s}$, we may thus assume that they sum up to exactly one and [13], Lemma 8, thus implies that these variables are negatively associated. Moreover, for $j\neq j'$ the variables are independent and then [13], Lemma 7, implies that the whole sequence $(Z_{j,s})_{j,s}$ is also negatively associated. Finally, the functions Z_s are given by applying a decreasing function to the variables $(Z_{j,s})_j$ and so [13], Lemma 7, implies that the variables Z_s are also negatively associated. Therefore, we may apply the Chernoff bound to $Z=\sum_{s=1}^{X_k(N_i)} Z_s$ [13], Proposition 5. Using (23) and our induction assumption on $X_k(N_i)$, we thus obtain

$$\Pr\bigl[|Z-\mathbb{E}[Z]|\ge Ce^{-t_i/2}\sqrt{n\log n}\bigr]\le e^{-C\log n/3}=o(n^{-2}),\tag{24}$$

for an appropriately chosen constant $C>0$ (not depending on ε).

It remains to collect the pieces. From (22), (23) and (24), we obtain that with probability $1-o(n^{-2})$,

$$X_k(N_{i+1})=e^{-\mathtt{ex}_k(\mathsf{R})/\mathtt{ex}(\mathsf{R})}\cdot X_k(N_i)\pm\tilde{c}e^{-2t_i}n\pm Ce^{-t_i/2}\sqrt{n\log n}\pm\ell I.$$

We can bound the effect of I by using (21). We immediately observe that the terms in (21) are of the same form (or smaller) than the terms that we already have. We can thus incorporate the effect of I by just increasing the constants in the error terms. For $k \in \texttt{slow}$, we thus get from the induction assumption that

$$X_k(N_{i+1}) = \frac{1}{k} e^{c_k - t_{i+1}} n \cdot \left(1 \pm \varepsilon_i \pm c' e^{-t_i} \pm c' e^{t_i/2} \sqrt{\log n/n}\right),$$

for an appropriate constant $c' > 0$ that does not depend on ε. This proves the inductive step for $k \in \texttt{slow}(\mathsf{R})$, cf. the definition of ε_{i+1}. The claim for $k \in \texttt{fast}(\mathsf{R})$ follows similarly. $\square$

We are now ready to prove the main theorem, which we restate here in a slightly stronger form.

THEOREM 4. *Let $K, \ell \in \mathbb{N}$ and let R be a nondegenerate (K, ℓ)-rule. For $1 \leq k \leq K$, let $Y_k(N)$ denote the number of vertices in k-components in G_N^{R}. Moreover, for $k \in \texttt{slow}(\mathsf{R})$ let $d_k := e^{c_k(\mathsf{R})}/k$, where $c_k(\mathsf{R})$ is defined as in Lemma 10. Then, if $\texttt{ex}(\mathsf{R}) < 2K + 2$ the following statements are true*:

(a) *For any $c \in \mathbb{R}$, with probability*[7] *$1 - o(1/\log n)$ we have for all $N \geq (n \log n + cn)/\texttt{ex}(\mathsf{R})$ and all $k \in \texttt{fast}(\mathsf{R})$ that $Y_k(N) = 0$, and there is only one component with more than K vertices in G_N^{R}.*

(b) *For any $c \in \mathbb{R}$,*

$$\lim_{n \to \infty} \Pr\left[T_{\mathrm{con}}^{\mathsf{R}} \leq \frac{n \log n + cn}{\texttt{ex}(\mathsf{R})}\right] = \prod_{k \in \texttt{slow}(\mathsf{R})} e^{-d_k e^{-c}}.$$

(c) *Let $c_0 := \log(\sum_{k \in \texttt{slow}(\mathsf{R})} d_k)$. Then*

$$\mathbb{E}[T_{\mathrm{con}}^{\mathsf{R}}] = \frac{n \log n + \gamma n + c_0 n}{\texttt{ex}(\mathsf{R})} + o(n).$$

(d) *For $k \in [K]$, let $T_k^{\mathsf{R}} := \min\{T \mid \forall N \geq T : Y_k(N) = 0\}$ be the time at which the last k-component vanishes. Then $\Pr[T_k^{\mathsf{R}} = T_{\mathrm{con}}^{\mathsf{R}}] \overset{n \to \infty}{\longrightarrow} 0$ for $k \in \texttt{fast}(\mathsf{R})$, and for $k \in \texttt{slow}(\mathsf{R})$,*

$$\Pr[T_k^{\mathsf{R}} = T_{\mathrm{con}}^{\mathsf{R}}] \overset{n \to \infty}{\longrightarrow} \frac{d_k}{\sum_{i \in \texttt{slow}(\mathsf{R})} d_i}.$$

PROOF. Given $\delta > 0$ and $c \in \mathbb{R}$, we use the following notation throughout:

$$N_\delta := \left\lfloor \frac{(1-\delta) n \log n}{\texttt{ex}(\mathsf{R})} \right\rfloor, \qquad N_c := \left\lfloor \frac{n \log n + cn}{\texttt{ex}(\mathsf{R})} \right\rfloor, \qquad N_\infty := 2 \left\lfloor \frac{n \log n}{\texttt{ex}(\mathsf{R})} \right\rfloor.$$

---

[7] This is stronger than the statement given in the Introduction, and it is needed in the proof of part (c).

In order to avoid ambiguity, we will use the first expression *always* with a constant named δ, while the expression for N_c is also sometimes used with other subscripts. For example, we use $N_{c+\varepsilon}$ to mean $\lfloor (n\log n + (c+\varepsilon)n)/\mathtt{ex}(\mathsf{R})\rfloor$.

For all statements, we will make use of the following basic observation. Fix some $\delta < 1/2$. By Lemma 12, the total number of vertices in small components (in components of size k for some $1 \le k \le K$) is in $O(n^{\delta})$ with probability at least $1 - o(1/\log n)$, and this number cannot increase in succeeding rounds. Similarly, as in the proof of Lemma 12, we call a round *regular* if at least $\ell - 1$ of the randomly selected vertices belong to ω-components. Let $\mathcal{E}_\delta$ be the event that all rounds between $N_\delta + 1$ and N_∞ are regular. The probability that a round is not regular is in $O(n^{2\delta-2})$, and so $\Pr[\mathcal{E}_\delta] = 1 - O(N_\infty n^{2\delta-2}) = 1 - o(1/\log n)$. Note that $\mathcal{E}_\delta$ implies that no new small component is created between rounds N_δ and N_∞.

We will make frequent use of this observation in the following way. Let $1 \le k \le K$. As in the proof of Lemma 12, we denote a position $1 \le p \le \ell$ as *k-good* if a vector $(\omega, \dots, \omega, k, \omega, \dots, \omega)$ with the k at position p results in merging the k-component with an ω-component. Recall that the number of k-good positions equals $\mathtt{ex}_k(\mathsf{R})/k$. Now assume that $C \in \mathrm{comp}_k(G^{\mathsf{R}}_{N_\delta})$ is a k-component in round N_δ. Then the probability that in some fixed round $N \ge N_\delta$, the component C does not appear in a k-good position is exactly $(1 - k/n)^{\mathtt{ex}_k(\mathsf{R})/k}$.

Note that a k-component that appears at a k-good position is merged with an ω-component, unless the round is not regular. Since with probability $1 - o(1/\log n)$ there are no nonregular rounds between rounds N_δ and N_∞, Markov's inequality gives that for all $0 < \delta < 1/2$, $M \in \mathbb{N}$ and $N \in [N_\delta, N_\infty)$,

$$
\begin{aligned}
&\Pr\bigl[Y_k(N) > 0 \mid \bigl|\mathrm{comp}_k\bigl(G^{\mathsf{R}}_{N_\delta}\bigr)\bigr| \le M\bigr] \\
&\qquad \le M \cdot (1 - k/n)^{(N - N_\delta)\mathtt{ex}_k(\mathsf{R})/k} + o(1/\log n).
\end{aligned} \tag{25}
$$

With these preparations, we come to the proof of the specific statements.

PROOF OF (a). We first prove the statement for all $N \in [N_c, N_\infty)$. Let $k \in \mathtt{fast}(\mathsf{R})$. Then the right-hand side of (25), applied for $N = N_c$ and $M = n^{\delta}$, is $o(1/\log n)$, since $\mathtt{ex}_k(\mathsf{R}) \ge \mathtt{ex}(\mathsf{R}) + 1$ and $N_c - N_\delta = n(\delta\log n + c)/\mathtt{ex}(\mathsf{R})$. As with probability $1 - o(1/\log n)$, all rounds between N_δ and N_∞ are regular, and since a small component can only be created in a nonregular round, this shows that

$$\Pr\bigl[\forall N \in [N_c, N_\infty), k \in \mathtt{fast}(\mathsf{R}):\ Y_k(N) = 0\bigr] = 1 - o(1/\log n).$$

Actually, we can say a little more. We interpret R as a (K', ℓ)-rule R' for $K' := 24(K+1)$, as outlined in Remark 1(a). Note that $K' > 12\mathtt{ex}(\mathsf{R})$, since by assumption $\mathtt{ex}(\mathsf{R}) < 2K + 2$. Then $\{K+1, \dots, K'\} \subseteq \mathtt{fast}(\mathsf{R}')$ and $\mathtt{ex}(\mathsf{R}') = \mathtt{ex}(\mathsf{R})$, so by the same argument as before, with probability $1 - o(1/\log n)$ all components of these sizes [i.e., in $\mathtt{fast}(\mathsf{R}) \cup \{K+1, \dots, K'\}$] will be extinct at round N_c.

Next, we show that in round N_c, with probability $1 - o(1/\log n)$ all vertices that are in an ω-component are actually contained in the same component. Fix $\delta = 1/3$, and note that the event $\mathcal{E}_\delta$ guarantees that no new ω-component (i.e., with more than K' vertices) is created between round N_δ and N_c. We apply the same idea as before, but now instead of regular rounds we consider ω-rounds, that is, rounds in which all ℓ randomly selected vertices are in ω-components. Let $\mathcal{X}_\omega(N)$ be the set of components in G_N^{R} with more than K' and less than $n/2$ vertices, and set $X_\omega(N) = |\mathcal{X}_\omega(N)|$. Moreover, let $\varepsilon > 0$ be so small that $(1-\varepsilon)^{\ell-1} > 1/2$. Let $\mathcal{E}_{\text{giant}}$ denote the event that $G_{N_\delta}^{\mathsf{R}}$ contains a giant component of size at least $(1-\varepsilon)n$. By Lemma 11, we have $\Pr[\mathcal{E}_{\text{giant}}] = 1 - o(1/\log n)$. Then it suffices to show that with probability at least $1 - o(1/\log n)$, for every $C \in \mathcal{X}_\omega(N_\delta)$, there is some round $N_\delta \le N \le N_c$ in which an edge between C and the giant component is added to the graph.

Fix a component $C \in \mathcal{X}_\omega(N_\delta)$, and consider some $N \in [N_\delta, N_c]$. If all chosen vertices $v_1, \dots, v_\ell$ of the Nth round are in ω-components, then R will select some edge, say $\{v_{2i-1}, v_{2i}\}$ for some $1 \le i \le \ell/2$. So, if one of v_{2i-1} and v_{2i} is in C, and the other $\ell - 1$ vertices are in the giant, then C will be connected to the giant. The probability that this happens is at least $2K'(1-\varepsilon)^{\ell-1}/n > K'/n$. In a regular round, no new large component is created. Hence, by the same argument as for (25), and using $N_c - N_\delta = \frac{n\log n/3 + cn}{\mathrm{ex}(\mathsf{R})} \ge \frac{n \log n}{6\mathrm{ex}(\mathsf{R})}$ for sufficiently large n and $X_\omega(N_\delta) \le n$, we may bound

$$\begin{aligned}\Pr\bigl[X_\omega(N_c) > 0 \mid \mathcal{E}_{\text{giant}}\bigr] &\le n\left(1 - \frac{K'}{n}\right)^{N_c - N_\delta} + o(1/\log n) \\ &\le n e^{-K'\log n/6\mathrm{ex}(\mathsf{R})} + o(1/\log n) = o(1/\log n).\end{aligned}$$

Since all rounds between N_c and N_∞ are regular with probability $1 - o(1/\log n)$, this proves (a) for all rounds $N \in [N_c, N_\infty)$. To see the claim for $N \ge N_\infty$, recall that with probability $1 - o(1/\log n)$, there are $O(n^\delta)$ components with at most K vertices in $G_{N_\delta}^{\mathsf{R}}$. Moreover, since R is nondegenerate, we have $\mathrm{ex}_k(\mathsf{R}) \ge \mathrm{ex}(\mathsf{R}) \ge 1$ for all $1 \le k \le K$. Together with (25), applied for $N = N_\infty = 2\lfloor n \log n/\mathrm{ex}(\mathsf{R})\rfloor$ and any $0 < \delta < 1/2$, this implies that

$$\Pr\bigl[\forall 1 \le k \le K:\ Y_k(N_\infty) = 0\bigr] = 1 - o(1/\log n).$$

Thus, $G_{N_\infty}^{\mathsf{R}}$ is connected with probability at least $1 - o(1/\log n)$. Hence, the claim also follows for $N \ge N_\infty$. $\square$

PROOF OF (b). We will resort to the so-called *method of moments*. Suppose that we have r sequences $Z_i(1), Z_i(2), \dots$ of random variables, $1 \le i \le r$, with support on $\mathbb{N}_0$. Suppose further that there are $\lambda_1, \dots, \lambda_k > 0$ such that for all $e_1, \dots, e_r \in \mathbb{N}_0$

$$\mathbb{E}\bigl[Z_1^{e_1}(N) \cdot Z_2^{e_2}(N) \cdots Z_r^{e_r}(N)\bigr] \to \lambda_1^{e_1} \cdots \lambda_r^{e_r} \qquad \text{as } N \to \infty,$$

where $n^{\underline{x}} := n(n-1)\cdots(n-x+1)$. Then the joint distribution of the $Z_k(N)$ converges to the joint distribution of independent Poisson random variables with parameters $\lambda_1, \dots, \lambda_r$, that is, for all $z_1, \dots, z_r \in \mathbb{N}_0$ we have $\Pr[Z_1(N) = z_1 \wedge \cdots \wedge Z_r(N) = z_r] \to \prod_{1 \le k \le r} e^{-\lambda_k} \lambda_k^{z_k}/z_k!$ as $N \to \infty$, see e.g. [17], Theorem 6.10.

Let $\delta > 0$ be sufficiently small and recall that $N_\delta = \lfloor (1-\delta) n \log n / \mathtt{ex}(\mathsf{R}) \rfloor$. For any $\varepsilon = \varepsilon(n) > 0$, let $\mathcal{E}(\varepsilon)$ be the event that for $k \in \mathtt{slow}(\mathsf{R})$ we have $|\mathrm{comp}_k(G^{\mathsf{R}}_{N_\delta})| = (1 \pm \varepsilon) d_k n^\delta$. By Lemma 12, $\Pr[\mathcal{E}(\varepsilon_0)] = 1 - o(1)$ for every $\varepsilon_0 > 0$. By a standard argument there exists also a (possibly very slowly converging) sequence $\varepsilon = \varepsilon(n) = o(1)$ such that $\Pr[\mathcal{E}(\varepsilon)] = 1 - o(1)$ for $n \to \infty$.

For every $k \in \mathtt{slow}(\mathsf{R})$, every $N \ge N_\delta$, and every $C \in \mathrm{comp}_k(G^{\mathsf{R}}_{N_\delta})$, let $Z(C)$ be a Bernoulli random variable that is 1 if C does not appear in a k-good position between rounds N_δ and N (recall that a position is called *k-good* if a k-component that appears in this position in a regular round is merged into an ω-component). Moreover, for every $k \in \mathtt{slow}(\mathsf{R})$ and $N \ge N_\delta$ let $Z_k(N) := \sum_{C \in \mathrm{comp}_k(G^{\mathsf{R}}_{N_\delta})} Z(C)$. We will apply the method of moments to the random variables $Z_k(N_c)$ in the conditional space in which $\mathcal{E}(\varepsilon)$ occurs. More precisely, we will show that for every vector $e \in \mathbb{N}_0^K$ we have for all large enough n that

$$\mathbb{E}\Bigg[\prod_{k \in \mathtt{slow}(\mathsf{R})} Z_k^{\underline{e_k}}(N_c) \,\Bigg|\, \mathcal{E}(\varepsilon)\Bigg] = \prod_{k \in \mathtt{slow}(\mathsf{R})} \big((1 \pm 3\varepsilon) d_k e^{-c}\big)^{e_k}. \tag{26}$$

This implies the claim as follows: recall that $\mathcal{E}_\delta$ is the event that all rounds between N_δ and N_∞ are regular. Then by (a) and since $\Pr[\mathcal{E}_\delta] = 1 - o(1)$, we have

$$\Pr\big[T_{\mathrm{con}} \le N_c \mid \mathcal{E}(\varepsilon)\big] = \Pr\big[\forall k \in \mathtt{slow}(\mathsf{R}) : Z_k(N_c) = 0 \mid \mathcal{E}(\varepsilon)\big] + o(1),$$

where the error term does not depend on ε. Since $\Pr[\mathcal{E}(\varepsilon)] = 1 - o(1)$, for $n \to \infty$ the left-hand side converges to $\Pr[T_{\mathrm{con}} \le N_c]$, while the right-hand side converges to $\prod_{k \in \mathtt{slow}(\mathsf{R})} \exp\{-d_k e^{-c}\}$ by the method of moments, thus proving the claim.

It remains to prove (26). For this, let H be the set of all ordered tuples $((C_{k,i})_{i=1}^{e_k})_{k \in \mathtt{slow}(\mathsf{R})}$ of pairwise distinct components $C_{k,i} \in \mathrm{comp}_k(G^{\mathsf{R}}_{N_\delta})$. For $h \in H$, we write $Z_h(N) = 1$ if $\prod_{C \in H} Z(C) = 1$ or, in other words, if none of the components $C_{k,i}$ of h occur in a k-good position between rounds N_δ and N. Note that an elementary counting argument implies

$$\prod_{k \in \mathtt{slow}(\mathsf{R})} Z_k^{\underline{e_k}}(N) = \sum_{h \in H} Z_h(N).$$

Consider any tuple $h = ((C_{k,i})_{i=1}^{e_k})_{k \in \mathtt{slow}(\mathsf{R})} \in H$. The probability that for a fixed $k \in \mathtt{slow}(\mathsf{R})$ none of the components $(C_{k,i})_{i=1}^{e_k}$ appears in a k-good position in a given round N is $(1 - ke_k/n)^{\mathtt{ex}(\mathsf{R})/k}$ [even if we condition on $\mathcal{E}(\varepsilon)$]. Similarly, using the fact that $\Pr[\forall i : \mathcal{A}_i] = \prod_i \Pr[\mathcal{A}_i \mid \forall j < i : \mathcal{A}_j]$, we deduce that the probability that none of the components $\{C_{k,i}\}_{1 \le i \le e_k}$ appears in a k-good position in a given round N is

$$F := \prod_{k \in \mathtt{slow}(\mathsf{R})} \left(1 - \frac{ke_k}{n - O(1)}\right)^{\mathtt{ex}(\mathsf{R})/k},$$

where the $O(1)$ term depends only on $e_1, \dots, e_k$ and R. Thus, for every $N \geq N_\delta$,

$$\mathbb{E}\Big[\prod_{k\in\texttt{slow}(\mathsf{R})} Z_k^{e_k}(N+1) \,\Big|\, \mathcal{E}(\varepsilon)\Big] = F \cdot \mathbb{E}\Big[\prod_{k\in\texttt{slow}(\mathsf{R})} Z_k^{e_k}(N) \,\Big|\, \mathcal{E}(\varepsilon)\Big].$$

Moreover, by definition of $\mathcal{E}(\varepsilon)$ we have

$$\mathbb{E}\Big[\prod_{k\in\texttt{slow}(\mathsf{R})} Z_k^{e_k}(N_\delta) \,\Big|\, \mathcal{E}(\varepsilon)\Big] = \prod_{k\in\texttt{slow}(\mathsf{R})} \big((1\pm 2\varepsilon)d_k n^\delta\big)^{e_k},$$

for all large enough n, and so by induction we get for any $c \in \mathbb{R}$

$$\mathbb{E}\Big[\prod_{k\in\texttt{slow}(\mathsf{R})} Z_k^{e_k}(N_c) \,\Big|\, \mathcal{E}(\varepsilon)\Big] = F^{N_c-N_\delta} \cdot \big((1\pm 2\varepsilon)d_k n^\delta\big)^{e_k}.$$

Note that $1 - \frac{ke_k}{n-O(1)} = (e^{-k/n+O(n^{-2})})^{e_k}$. Thus,

$$\mathbb{E}\Big[\prod_k Z_k^{e_k}(N_c) \,\Big|\, \mathcal{E}(\varepsilon)\Big] = \prod_k \big((1\pm 2\varepsilon)\big(e^{-(k/n)+O(n^{-2})}\big)^{\mathtt{ex}(\mathsf{R})(N_c-N_\delta)/k} d_k n^\delta\big)^{e_k},$$

where the products are over all $k \in \texttt{slow}(\mathsf{R})$. Since $\mathtt{ex}(\mathsf{R})(N_c - N_\delta) = cn + \delta n \log n$, the claim in (26) follows.

For later reference (and omitting the details), we note that a slight variation on this argument shows the following: for every $\varepsilon > 0$ and $c \in \mathbb{R}$, and for all $k \in \texttt{slow}(\mathsf{R})$, we have, as $n \to \infty$

$$\Pr\big[Y_k(N_{c+\varepsilon}) = 0 \wedge \forall i \neq k : Y_i(N_c) = 0\big] \to e^{-d_k e^{-(c+\varepsilon)}} \prod_{i\in\texttt{slow}(\mathsf{R})\setminus\{k\}} e^{-d_i e^{-c}} \tag{27}$$

and

$$\Pr\big[Y_k(N_c) = 0 \wedge \forall i \neq k : Y_i(N_{c+\varepsilon}) = 0\big] \to e^{-d_k e^{-c}} \prod_{i\in\texttt{slow}(\mathsf{R})\setminus\{k\}} e^{-d_i e^{-(c+\varepsilon)}}. \tag{28}$$

□

PROOF OF (c). We consider $T'_{\mathrm{con}}(n) = \mathtt{ex}(\mathsf{R}) \cdot T^{\mathsf{R}}_{\mathrm{con}}(n)/n - \log n$. Let D be a random variable such that $\Pr[D \leq c] = F(c) := \prod_{k\in\texttt{slow}(\mathsf{R})} \exp\{-d_k e^{-c}\}$, where $c \in \mathbb{R}$. Then, by (b), $T'_{\mathrm{con}}(n)$ converges in distribution to D. In the following, we will prove that the sequence $T'_{\mathrm{con}}(n)$ is uniformly integrable, that is,

$$\limsup_{n\in\mathbb{N}} \big(\mathbb{E}\big[\big|T'_{\mathrm{con}}(n)\big|_{\geq\alpha}\big]\big) \to 0 \qquad \text{as } \alpha \to \infty, \tag{29}$$

where $X_{\geq\alpha} = X$ if $X \geq \alpha$, and $X_{\geq\alpha} = 0$ otherwise.

First, we show how (29) implies the statement of (c). Convergence in distribution together with uniform integrability implies convergence of the means, that is, $\mathbb{E}[T'_{\mathrm{con}}(n)] \to \mathbb{E}[D]$ (see, e.g., [7]). By elementary calculus and the change of variables $u = \sum_{k\in\texttt{slow}(\mathsf{R})} d_k e^{-c} = e^{c_0-c}$, we get

$$\mathbb{E}[D] = \int_{-\infty}^{\infty} c\Bigl(\prod_{k\in\texttt{slow}(\mathsf{R})} e^{-d_k e^{-c}}\Bigr)\Bigl(\sum_{k\in\texttt{slow}(\mathsf{R})} d_k e^{-c}\Bigr)\,dc$$
$$= \int_{\infty}^{0} (-c_0 + \log u)e^{-u}\,du = c_0 + \gamma,$$

where we used the well-known identity $\gamma = -\int_0^\infty (\log u)e^{-u}\,du$ for the Euler–Mascheroni constant. Thus, $\mathbb{E}[T'_{\mathrm{con}}(n)] = \gamma + c_0 + o(1)$, and

$$\mathbb{E}\bigl[T_{\mathrm{con}}(n)\bigr] = \frac{n}{\texttt{ex}(\mathsf{R})}\bigl(\mathbb{E}\bigl[T'_{\mathrm{con}}(n)\bigr] + \log n\bigr) = \frac{n\log n + \gamma n + c_0 n}{\texttt{ex}(\mathsf{R})} + o(n).$$

Thus, it suffices to prove (29). We define the following events, where $\delta = 1/3$ [and, as usual, $Y_k(N)$ is the number of vertices in k-components of G_N^{R}]:

(i) for $N_0 = n\log n/\texttt{ex}(\mathsf{R})$ we have $Y_k(N_0) = 0$ for all $k \in \texttt{fast}(\mathsf{R})$, and there is only one ω-component in $G_{N_0}^{\mathsf{R}}$,

(ii) $e^{c_0}n^{\delta}/2 \le \sum_{k\in\texttt{slow}(\mathsf{R})} Y_k(N_\delta)$ and $\sum_{1\le k\le K} Y_k(N_\delta) \le 2e^{c_0}Kn^{\delta}$, and

(iii) all rounds between N_δ and N_∞ are regular.

Then by part (a) of Theorem 4, by Lemma 12, and by the properties of N_∞, respectively, the events (i), (ii) and (iii) each have probability $1 - o(1/\log n)$, where for (ii) we also use $\sum_{k\in\texttt{slow}(\mathsf{R})} e^{c_k} \le Ke^{c_0}$. For the proof, we also need the following claim, whose justification we postpone to a later point: there exists a constant $\eta > 0$ such that

$$\mathbb{E}\bigl[\bigl|T'_{\mathrm{con}}(n)\bigr|_{\ge\alpha} \mid T'_{\mathrm{con}}(n) > 2\lfloor\log\log n\rfloor\bigr] \le \eta\log n. \tag{30}$$

Our next goal is to give bounds for $\Pr[T'_{\mathrm{con}}(n) > c]$ that are uniform in c [in (b) we calculated the limit of this probability only for constant c]. For every $1 \le k \le K$ and $N \ge N_\delta$, write $X_k(N)$ for the number of components in $\mathrm{comp}_k(N_\delta)$ that never appear at a k-good position between rounds N_δ and N. Furthermore, let $X_{\texttt{slow}}(N) := \sum_{k\in\texttt{slow}(\mathsf{R})} X_k(N)$. Note that for every c such that $N_0 \le N_c \le N_\infty$ we have

$$\Pr\bigl[T'_{\mathrm{con}}(n) > c\bigr] \le \Pr\bigl[X_{\texttt{slow}}(N_c) > 0\bigr] + o(1/\log n), \tag{31}$$

since $T'_{\mathrm{con}}(n) > c$ implies that at least one of the events $\neg$(i), $\neg$(iii) or $X_{\texttt{slow}}(N_c) > 0$ occurs. Conversely, for every c such that $N_\delta \le N_c \le N_\infty$ [note that the range for c in (31) is different] we have

$$\Pr\bigl[T'_{\mathrm{con}}(n) > c\bigr] \ge \Pr\bigl[X_{\texttt{slow}}(N_c) > 0\bigr] - o(1/\log n), \tag{32}$$

since $X_{\texttt{slow}}(N_c) > 0$ implies that at least one of the events $\neg$(iii) or $T'_{\mathrm{con}}(n) > c$ occurs.

Observe that in every round a k-component C fails to appear at a k-good position with probability $(1-k/n)^{\mathtt{ex}_k(\mathsf{R})/k} \le (1-k/n)^{\mathtt{ex}(\mathsf{R})/k}$. Therefore, for every $M \in \mathbb{N}$ and $N \ge N_\delta$,

$$\mathbb{E}\big[X_k(N) \mid \big|\mathrm{comp}_k\big(G^{\mathsf{R}}_{N_\delta}\big)\big| \le M\big] \le M \cdot \left(1 - \frac{k}{n}\right)^{(N-N_\delta)\mathtt{ex}(R)/k}.$$

Thus, by Markov's inequality and the fact that (ii) occurs with probability $1 - o(1/\log n)$, we get from (31) uniformly for c such that $N_0 \le N_c \le N_\infty$

$$\begin{aligned}\Pr\big[T'_{\mathrm{con}}(n) > c\big] &\le \Pr\big[X_{\mathtt{slow}}(N_c) > 0 \mid \text{(ii)}\big] + o(1/\log n)\\ &\le 2K^2 e^{c_0} n^\delta \max_{k \in \mathtt{slow}(\mathsf{R})} \left(1 - \frac{k}{n}\right)^{(N_c - N_\delta)\mathtt{ex}(\mathsf{R})/k} + o\left(\frac{1}{\log n}\right)\\ &\le 2K^2 e^{c_0 - c} + o(1/\log n).\end{aligned} \tag{33}$$

On the other hand, a k-component, where $k \in \mathtt{slow}(\mathsf{R})$, fails to appear at a k-good position with probability $(1-k/n)^{\mathtt{ex}(\mathsf{R})/k} \ge 1 - \mathtt{ex}(\mathsf{R})/n$. Therefore, in every given round $N \ge N_\delta$, the probability that $X_{\mathtt{slow}}(N)$ decreases is at most $X_{\mathtt{slow}}(N)\mathtt{ex}(\mathsf{R})/n$, by the union bound. This allows us to couple the number of rounds until $X_{\mathtt{slow}}(N)$ decreases with geometrically distributed random variables. Indeed, for every $i \le n/\mathtt{ex}(\mathsf{R})$, let T_i be geometrically distributed with mean $n/(i\,\mathtt{ex}(\mathsf{R}))$, and let $T = \sum_{i=1}^{e^{c_0} n^\delta/(2K)} T_i$. Then, for every $N_\delta \le N_c \le N_\infty$ and $x \in \mathbb{N}$

$$\Pr\big[X_{\mathtt{slow}}(N_c) = 0 \mid \text{(ii)}\big] \le \Pr[T \le N_c - N_\delta].$$

It is not difficult to bound $\Pr[T \le N_c - N_\delta]$. Straightforward calculation shows that for a suitable constant $\zeta > 0$ we have that $\mathbb{E}[T] \ge (n \log n - \zeta n)/3\mathtt{ex}(\mathsf{R})$ and $\mathrm{Var}[T] \le \zeta n^2/\mathtt{ex}(\mathsf{R})^2$. Hence, by (32) and Chebyshev's inequality, and since (ii) occurs with probability $1 - o(1/\log n)$, for all $-\delta \log n < c < -\zeta/3$

$$\begin{aligned}\Pr\big[T'_{\mathrm{con}}(n) \le c\big] &\le \Pr\big[X_{\mathtt{slow}}(N_c) = 0\big] + o(1/\log n)\\ &\le \Pr[T \le N_c - N_\delta] + o(1/\log n)\\ &= \Pr\left[T \le \frac{n \log n + 3cn}{3\mathtt{ex}(\mathsf{R})}\right] + o(1/\log n) \qquad [\text{as } \delta = 1/3]\\ &\le 9\zeta(\zeta + 3c)^{-2} + o(1/\log n).\end{aligned} \tag{34}$$

We are now ready to complete the proof of (29). For $\alpha > 0$, we write

$$\mathbb{E}\big[\big|T'_{\mathrm{con}}(n)\big|_{\ge\alpha}\big] = \mathbb{E}\big[T'_{\mathrm{con}}(n)_{\ge\alpha}\big] - \mathbb{E}\big[T'_{\mathrm{con}}(n)_{\le -\alpha}\big]. \tag{35}$$

We will consider each term separately. For the second term, observe that by the definition of $T'_{\mathrm{con}}(n)$ the inequality $T'_{\mathrm{con}}(n) \ge -\log n$ holds. Thus, for $X = (-T'_{\mathrm{con}})_{\ge\alpha}$ the general formula $\mathbb{E}[X] = \int_0^\infty \Pr[X \ge c]\,dc$ simplifies to

$$-\mathbb{E}\big[T'_{\mathrm{con}}(n)_{\le -\alpha}\big] = \alpha \Pr\big[T'_{\mathrm{con}}(n) \le -\alpha\big] + \int_\alpha^{\log n} \Pr\big[T'_{\mathrm{con}}(n) \le -c\big]\,dc.$$

Since the integrand is nonincreasing (as a function of c), we get that

$$-\mathbb{E}\big[T'_{\text{con}}(n)_{\le -\alpha}\big] \le \alpha \Pr\big[T'_{\text{con}}(n) \le -\alpha\big] + \sum_{c=\lfloor \alpha \rfloor}^{\lceil \log n \rceil} \Pr\big[T'_{\text{con}}(n) \le -c\big].$$

Then, for all sufficiently large $\alpha > \zeta/3$, (34) gives

$$-\mathbb{E}\big[T'_{\text{con}}(n)_{\le -\alpha}\big] \le \frac{10\zeta}{\alpha} + \sum_{c=\lfloor \alpha \rfloor}^{\lceil \log n \rceil} \frac{9\zeta}{(\zeta - 3c)^2} + o(1) \le \frac{20\zeta}{\alpha} + o(1).$$

This establishes that $\limsup_{n\to\infty} -\mathbb{E}[T'_{\text{con}}(n)_{\le -\alpha}] \to 0$ as $\alpha \to \infty$. For the first term in (35), by (30) and (33),

$$\begin{aligned}\mathbb{E}\big[T'_{\text{con}}(n)_{\ge \alpha}\big] \le & \sum_{c=\lfloor \alpha \rfloor}^{2\lfloor \log\log n \rfloor} (c+1)\big(2K^2 e^{c_0 - c} + o(1/\log n)\big) \\ & + \big(2K^2 e^{c_0 - 2\lfloor \log\log n \rfloor} + o(1/\log n)\big)(\eta + 3)\log n.\end{aligned}$$

The second summand is in $O(e^{-2\log\log n}\log n) + o(1) = o(1)$. Moreover, the first term is a partial sum of a converging series, and becomes arbitrarily small as $\alpha \to \infty$. We obtain

$$\limsup_{n\to\infty} \mathbb{E}\big[T'_{\text{con}}(n)_{\ge \alpha}\big] \to 0 \qquad \text{as } \alpha \to \infty.$$

This completes the proof of (c), assuming (30), and it only remains to prove this auxiliary claim. So let $N \in \mathbb{N}$, and let G_0 be a nonempty graph on n vertices. We will bound the conditional expectation $\mathbb{E}[|T'_{\text{con}}(n)| \mid G^{\mathsf{R}}_N = G_0]$. If G_0 is connected, then $T_{\text{con}}(n) \le N$, so assume otherwise. Fix some $\varepsilon > 0$ with the property $(1-\varepsilon)^{\ell-1} > 1/2$. Let $\mathcal{E}_{\text{giant}}(N')$ be the event that in round N' there is a giant component with $(1-\varepsilon)n$ vertices. Then by Lemma 11 there is a constant $\rho > 0$ such that uniformly over all G_0 we have $\Pr[\mathcal{E}_{\text{giant}}(N_\rho) \mid G^{\mathsf{R}}_N = G_0] \ge 1 - o(1/n)$, where $N_\rho := N + \rho n$.

Assuming that $\mathcal{E}_{\text{giant}}(N_\rho)$ occurred, let S be the vertex set of the giant. Let $Z(N)$, $N \ge N_\delta$, be the set of vertices in $V \setminus S$ that have no neighbor in S in round N. Fix some $v \in Z(N)$. Then the probability that in round $N+1$ the vertex v appears at position i (among the ℓ randomly selected vertices), while at all other positions there are vertices of S is at least $1/n \cdot (1-\varepsilon)^{\ell-1} > 1/(2n)$. If v is in a component of size $k \le K$, then there is a k-good position, so with probability at least $1/(2n)$ the vertex v is joined to S by an edge. Similarly, if v is in a component of size larger than K, then it is also joined to S with probability at least $1/(2n)$. Since $Z(N_\delta) \le n/2$,

$$\mathbb{E}\big[\big|Z(N_\rho + \Delta)\big| \mid \mathcal{E}_{\text{giant}}(N_\rho) \text{ and } G^{\mathsf{R}}_N = G_0\big] \le \frac{n}{2}\Big(1 - \frac{1}{2n}\Big)^{\Delta}$$

for every $\Delta \in \mathbb{N}$. In particular, for $\Delta = 4n \log n$ the right-hand side is at most $1/(2n)$. Note that $|Z(N)| = 0$ implies $T_{\mathrm{con}}(n) \le N$. Thus, by Markov's inequality,

$$\Pr[T_{\mathrm{con}}(n) > N_\rho + 4n \log n \mid \mathcal{E}_{\mathrm{giant}}(N_\rho) \text{ and } G_N^{\mathsf{R}} = G_0] \le 1/2n.$$

Note that $N_\rho + 4n \log n < N + 5n \log n$ for sufficiently large n. Therefore, we get for sufficiently large n (but uniformly for all N and G_0):

$$\begin{aligned}
&\Pr[T_{\mathrm{con}}(n) > N + 5n \log n \mid G_N^{\mathsf{R}} = G_0] \\
&\quad\le \Pr[\neg\mathcal{E}_{\mathrm{giant}}(N_\rho) \mid G_N^{\mathsf{R}} = G_0] \\
&\qquad + \Pr[T_{\mathrm{con}}(n) > N_\rho + 4n \log n \mid \mathcal{E}_{\mathrm{giant}}(N_\rho) \text{ and } G_N^{\mathsf{R}} = G_0] \\
&\quad\le 1/n.
\end{aligned}$$

Applying this bound iteratively, we find that for sufficiently large n we have for all $N > 0$, all graphs G_0, and all $i \in \mathbb{N}$,

$$\Pr[T_{\mathrm{con}}(n) > N + 5in \log n \mid G_N^{\mathsf{R}} = G_0] \le n^{-i}. \tag{36}$$

In particular, $\mathbb{E}[T_{\mathrm{con}}(n) \mid G_N^{\mathsf{R}} = G_0] \le N + n \log n \cdot O(\sum_{i=0}^{\infty} i \cdot n^{-i}) = N + O(n \log n)$. Recalling the definition $T'_{\mathrm{con}}(n) = \mathtt{ex}(\mathsf{R}) T_{\mathrm{con}}(n)/n - \log n$, we get

$$\mathbb{E}[T'_{\mathrm{con}}(n) \mid G_N^{\mathsf{R}} = G_0] \le \mathtt{ex}(\mathsf{R}) \cdot N/n + O(\log n).$$

On the other hand, $T'_{\mathrm{con}}(n) \ge -\log n$ always, since $T_{\mathrm{con}}(n) \ge 0$. Summarizing, there exists $\eta' > 0$ such that for all $n \ge 2$, all nonempty graphs G_0, and all $N > 0$,

$$\mathbb{E}[|T'_{\mathrm{con}}(n)| \mid G_N^{\mathsf{R}} = G_0] \le \frac{\mathtt{ex}(\mathsf{R}) \cdot N}{n} + \eta' \log n. \tag{37}$$

Let us denote by $\mathcal{L}$ the event that $T'_{\mathrm{con}}(n) > 2 \log \log n$, and let $N' = \lceil (n \log n + 2n \lfloor \log \log n \rfloor)/\mathtt{ex}(\mathsf{R}) \rceil$. Then

$$\begin{aligned}
\mathbb{E}[|T'_{\mathrm{con}}(n)|_{\ge\alpha} \mid \mathcal{L}] &= \sum_{G_0} \Pr[G_{N'}^{\mathsf{R}} = G_0 \mid \mathcal{L}] \cdot \mathbb{E}[|T'_{\mathrm{con}}(n)|_{\ge\alpha} \mid G_{N'}^{\mathsf{R}} = G_0] \\
&\le \left(\frac{\mathtt{ex}(\mathsf{R}) \cdot N'}{n} + \eta' \log n \right) \sum_{G_0} \Pr[G_{N'}^{\mathsf{R}} = G_0 \mid \mathcal{L}] \\
&= \mathtt{ex}(\mathsf{R}) \cdot N'/n + \eta' \log n,
\end{aligned}$$

and (30) follows. $\square$

PROOF OF (d). For a $C > 0$, let $\mathcal{A} = \mathcal{A}(C)$ be the event that:

- all rounds from N_{-C} to N_C are regular (where N_{-C} and N_C are defined in the beginning of the proof),
- there is only one ω-component in $G_{N_{-C}}^{\mathsf{R}}$ and
- for all $k \in \mathtt{fast}(\mathsf{R})$ we have $Y_k(N_{-C}) = 0$.

Fix $k \in \texttt{slow}(\mathsf{R})$ and $\varepsilon > 0$ and define for all $-C \le c \le C$ three events

$$\mathcal{E}_1(c) := \mathcal{A} \wedge Y_k(N_{c+\varepsilon}) = 0 \wedge Y_k(N_c) > 0 \wedge \forall i \in \texttt{slow}(\mathsf{R}) \setminus \{k\} : Y_i(N_c) = 0,$$

and

$$\mathcal{E}_2(c) := \mathcal{A} \wedge T_k^{\mathsf{R}} = T_{\text{con}}^{\mathsf{R}} \wedge T_{\text{con}}^{\mathsf{R}} \in (N_c, N_{c+\varepsilon}],$$
$$\mathcal{E}_3(c) := \mathcal{A} \wedge Y_k(N_c) > 0 \wedge \forall i \in \texttt{slow}(\mathsf{R}) : Y_i(N_{c+\varepsilon}) = 0.$$

Since $\mathcal{A}$ guarantees that all rounds from N_{-C} to N_C are regular, we have that $\mathcal{E}_1(c)$ implies $\mathcal{E}_2(c)$ which in turn implies $\mathcal{E}_3(c)$. That is, we have

$$\Pr[\mathcal{E}_1(c)] \le \Pr[\mathcal{E}_2(c)] \le \Pr[\mathcal{E}_3(c)] \qquad \text{for all } -C \le c \le C - \varepsilon. \tag{38}$$

From (27) and (28), and the fact that $\Pr[\mathcal{A}] = 1 - o(1)$, we infer that for all $-C \le c \le C - \varepsilon$ we have

$$\begin{aligned} \lim_{n\to\infty} \Pr[\mathcal{E}_1(c)] &= \left(e^{-d_k e^{-(c+\varepsilon)}} - e^{-d_k e^{-c}}\right) \prod_{i \in \texttt{slow}(\mathsf{R})\setminus\{k\}} e^{-d_i e^{-c}}, \\ \lim_{n\to\infty} \Pr[\mathcal{E}_3(c)] &= \left(e^{-d_k e^{-(c+\varepsilon)}} - e^{-d_k e^{-c}}\right) \prod_{i \in \texttt{slow}(\mathsf{R})\setminus\{k\}} e^{-d_i e^{-(c+\varepsilon)}}. \end{aligned} \tag{39}$$

Let $f_d(c) = e^{-de^{-c}}$. Then we infer, with $S_k := \sum_{i \in \texttt{slow}(\mathsf{R})\setminus\{k\}} d_i$, by applying Taylor's theorem

$$\lim_{n\to\infty} \Pr[\mathcal{E}_1(c)] = \left(\varepsilon f'_{d_k}(c) + O(\varepsilon^2)\right) \cdot f_{S_k}(c),$$

and

$$\lim_{n\to\infty} \Pr[\mathcal{E}_3(c)] = \left(\varepsilon f'_{d_k}(c) + O(\varepsilon^2)\right) \cdot f_{S_k}(c+\varepsilon).$$

Note that since f_d is smooth, and since we will be considering only values of f_d and its derivatives in a compact interval $[-C, C]$ for a $C > 0$ independent of ε, there exists is universal constant C' (depending on C only) such that all error terms are in absolute value at most $(C'-1)\varepsilon^2$. Moreover, let $S_{C,\varepsilon} := \{j \cdot \varepsilon \mid j \in \mathbb{N}, -C \le j \cdot \varepsilon \le C - \varepsilon\}$. Since $|S_{C,\varepsilon}|$ is a constant, for sufficiently large n the probabilities $\Pr[\mathcal{E}_1(c)]$ and $\Pr[\mathcal{E}_3(c)]$ are within distance at most ε^2 from their respective limits for all $c \in S_{C,\varepsilon}$. Therefore, together with (38) we obtain that there is $C' > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and all $c \in S_{C,\varepsilon}$

$$\begin{aligned} \varepsilon f'_{d_k}(c) f_{S_k}(c) + C'\varepsilon^2 &\le \Pr[\mathcal{E}_1(c)] \le \Pr[\mathcal{E}_3(c)] \\ &\le \varepsilon f'_{d_k}(c) f_{S_k}(c+\varepsilon) + C'\varepsilon^2. \end{aligned} \tag{40}$$

With those preparations at hand, let $\mathcal{E}^*$ be the event that $T_k^{\mathsf{R}} = T_{\text{con}}^{\mathsf{R}}$ and $T_{\text{con}}^{\mathsf{R}} \in (N_{-C}, N_C]$ and that $\mathcal{A}$ holds. We may assume that C is a multiple of ε. Then

$$\Pr[\mathcal{E}^*] = \sum_{j=-C/\varepsilon}^{(C-\varepsilon)/\varepsilon} \Pr[\mathcal{E}_2(j\varepsilon)].$$

For any $\varepsilon' > 0$ we have, by choosing $C = C(\varepsilon')$ large enough, that

$$\left|\Pr\left[T_k^{\mathsf{R}} = T_{\mathrm{con}}^{\mathsf{R}}\right] - \Pr\left[\mathcal{E}^*\right]\right| \le \Pr\left[T_{\mathrm{con}}^{\mathsf{R}} \notin (N_{-C}, N_C]\right] + \Pr[\neg\mathcal{A}].$$

However, the last expression is at most ε', due to part (b) for $C(\varepsilon')$ and $n = n(\varepsilon')$ large enough. In particular, this derivation, combined with (38), implies that

$$\sum_{j=-C/\varepsilon}^{(C-\varepsilon)/\varepsilon} \Pr\left[\mathcal{E}_1(j\varepsilon)\right] \le \Pr\left[T_k^{\mathsf{R}} = T_{\mathrm{con}}^{\mathsf{R}}\right] \le \varepsilon' + \sum_{j=-C/\varepsilon}^{(C-\varepsilon)/\varepsilon} \Pr\left[\mathcal{E}_3(j\varepsilon)\right].$$

Thus, (40) guarantees that for sufficiently large n,

$$\left(\sum_{j=-C/\varepsilon}^{(C-\varepsilon)/\varepsilon} \varepsilon f'_{d_k}(j\varepsilon) f_{S_k}(j\varepsilon)\right) + C'C\varepsilon \le \Pr\left[T_k^{\mathsf{R}} = T_{\mathrm{con}}^{\mathsf{R}}\right]$$

and

$$\Pr\left[T_k^{\mathsf{R}} = T_{\mathrm{con}}^{\mathsf{R}}\right] \le \varepsilon' + \left(\sum_{j=-C/\varepsilon}^{(C-\varepsilon)/\varepsilon} \varepsilon f'_{d_k}(j\varepsilon) f_{S_k}\big((j+1)\varepsilon\big)\right) + C'C\varepsilon.$$

Since the statement holds for any choice of $\varepsilon > 0$, we have

$$\int_{-C}^{C} f_{S_k}(x) \cdot f'_{d_k}(x)\,dx \le \lim_{n\to\infty} \Pr\left[T_k^{\mathsf{R}} = T_{\mathrm{con}}^{\mathsf{R}}\right] \le \varepsilon' + \int_{-C}^{C} f_{S_k}(x) \cdot f'_{d_k}(x)\,dx.$$

Again this statement holds for any choice of $\varepsilon' > 0$ if $C = C(\varepsilon')$ is large enough. Hence,

$$\lim_{n\to\infty} \Pr\left[T_k^{\mathsf{R}} = T_{\mathrm{con}}^{\mathsf{R}}\right] = \lim_{C\to\infty} \int_{-C}^{C} f_{S_k}(x) \cdot f'_{d_k}(x)\,dx.$$

Setting $S = S_k + d_k = \sum_{i\in\mathtt{slow}(\mathsf{R})} d_i$, the integral can be computed as follows:

$$\begin{aligned}
\lim_{C\to\infty} \int_{-C}^{C} f_{S_k}(x) f'_{d_k}(x)\,dx &= \lim_{C\to\infty} \int_{-C}^{C} e^{-S_k e^{-x}} \cdot e^{-d_k e^{-x}} \cdot d_k e^{-x}\,dx \\
&= \lim_{C\to\infty} d_k \int_{-C}^{C} e^{-Se^{-x}-x}\,dx \\
&= \lim_{C\to\infty} \frac{d_k}{S}\left(e^{-Se^{-C}} - e^{-Se^{C}}\right) = \frac{d_k}{S},
\end{aligned}$$

as claimed. □ □

5. Degenerate rules. In this section, we will discuss lower bounds for degenerate rules and we will prove Theorem 5. As an auxiliary result, we need the following lemma implying that the typical behavior of the process is to end up with fast components gone and only a constant number of slow components left.

For brevity, we will denote $\texttt{fast}(\mathsf{R})$ and $\texttt{slow}(\mathsf{R})$ by $\texttt{fast}$ and $\texttt{slow}$, respectively. Moreover, we will denote $\sum_{k\in\texttt{fast}} Y_k(N)$ by $Y_{\texttt{fast}}(N)$, $\sum_{k\in\texttt{fast}} z_k(t)$ by $z_{\texttt{fast}}(t)$, $\sum_{k\in\texttt{slow}} Y_k(N)$ by $Y_{\texttt{slow}}(N)$ and $\sum_{k\in\texttt{slow}} z_k(t)$ by $z_{\texttt{slow}}(t)$. Recall that a component is *small* if it has size $\leq K$.

LEMMA 13. *Let $K,\ell \in \mathbb{N}$ and let R be a degenerate (K,ℓ)-rule. Then for every $\varepsilon > 0$ there is a $C > 0$ and $t_0 > 0$ such that w.h.p. $Y_{\texttt{fast}}(tn) < \varepsilon Y_{\texttt{slow}}(tn) + C\log n$ for all $t_0 < t < n$.*

PROOF. Let $\varepsilon > 0$. Since the statement becomes stronger for smaller $\varepsilon > 0$ we may assume $\varepsilon < (32\ell^2 K)^{-1}$ and $(1-\varepsilon^2)^{\ell-1} \geq 1/2$. By Lemma 10 and Lemma 11, there is $t_0 > 0$ such that $z_{\texttt{fast}}(t) < \varepsilon/2 \cdot z_{\texttt{slow}}(t)$ and $z_{\texttt{fast}}(t) + z_{\texttt{slow}}(t) < \varepsilon^2/2$ for all $t \in [t_0, t_0+1]$, and by Corollary 3, w.h.p.

$$Y_{\texttt{fast}}(tn) < \varepsilon \cdot Y_{\texttt{slow}}(tn) \quad \text{and} \quad Y_{\texttt{fast}}(tn) + Y_{\texttt{slow}}(tn) < \varepsilon^2 n \tag{41}$$

for all $t \in [t_0, t_0+1]$.

We show by induction on t that (41) holds w.h.p. for all $t' \in [t_0, t]$ as long as $t < n$ and $Y_{\texttt{slow}}(tn) \geq 64K/\varepsilon \cdot \log n$. More precisely, we show that if (41) holds for some $t \geq t_0$, and if $Y_{\texttt{slow}}(tn) \geq 64K/\varepsilon \cdot \log n$, then with probability $1 - o(1/n)$ it also holds for $t+1$. The idea of the inductive step is similar as in the proof for Lemma 10(a), but here we need to work with Y_k instead of z_k, which forces us to split the proof into small steps.

Since $Y_{\texttt{fast}}(N) + Y_{\texttt{slow}}(N)$ is nonincreasing, we only need to show the first inequality of (41). Note that if $Y_{\texttt{slow}}(tn) + Y_{\texttt{fast}}(tn) < C\log n$ for some $C > 0$ then the statement is trivial. So assume that (41) holds for some $t \geq t_0$, and that $Y_{\texttt{slow}}(tn) \geq 64K/\varepsilon \cdot \log n$. We first give a lower bound for $Y_{\texttt{slow}}((t+1)n)$. As R is degenerate, small components can only be removed in rounds for which at least two of the ℓ vertices belong to small components. Let I denote the number of such rounds in the interval $(tn, (t+1)n]$. The probability that a single round contains at least two vertices in small components is at most

$$\binom{\ell}{2}\left(\frac{Y_{\texttt{slow}}(tn) + Y_{\texttt{fast}}(tn)}{n}\right)^2 \leq \ell^2 \frac{Y_{\texttt{slow}}(tn)^2}{n^2}. \tag{42}$$

Since the number of small components is nonincreasing, using the induction assumption (41) for tn guarantees that

$$\mathbb{E}[I] \leq n \cdot \ell^2 Y_{\texttt{slow}}(tn)^2/n^2 \leq \ell^2\varepsilon^2 Y_{\texttt{slow}}(tn) < \varepsilon Y_{\texttt{slow}}(tn)/(32K). \tag{43}$$

Note that the right-hand side is $>2\log n$ by our assumption on $Y_{\texttt{slow}}(tn)$. By the Chernoff bounds, with probability $1 - o(1/n)$ the actual number of nonregular rounds is at most $I \leq \varepsilon Y_{\texttt{slow}}(tn)/(16K)$. Since in each round at most two slow components can be merged, each having at most K vertices, with probability $1 - o(1/n)$ we have with room to spare

$$Y_{\texttt{slow}}\big((t+1)n\big) \geq Y_{\texttt{slow}}(tn) - 2K \cdot I > \tfrac{15}{16} \cdot Y_{\texttt{slow}}(tn). \tag{44}$$

Next, we derive an upper bound for the fast components. A new fast component can only be created by merging two small components. Hence, the number of vertices in fast components that are created between time t and $t+1$ is at most $K \cdot I$. To use this fact, we distinguish two cases. First, assume that there exists a round $N \in (tn, (t+1)n]$ such that $Y_{\mathtt{fast}}(N) \le \varepsilon Y_{\mathtt{slow}}(tn)/2$. In this case, we can directly bound

$$Y_{\mathtt{fast}}\big((t+1)n\big) \le Y_{\mathtt{fast}}(N) + K \cdot I < \tfrac{15}{16} \cdot \varepsilon Y_{\mathtt{slow}}(tn), \tag{45}$$

where we used the bound $I \le \varepsilon Y_{\mathtt{slow}}(tn)/(16K)$, which holds with probability $1 - o(1/n)$.

Now let us turn to the second case. So assume that for all $N \in (tn, (t+\delta)n]$ we have $Y_{\mathtt{fast}}(N) \ge \varepsilon Y_{\mathtt{slow}}(tn)/2$. Recall that for each $k \in \mathtt{fast}$ we have $\mathtt{ex}_k(\mathsf{R}) > 0$. In other words, for each fast component C there exists at least one "good" position so that if C appears in this position (and all other positions are filled with vertices from ω-components) then C is merged with an ω-component. Therefore, the probability that in a fixed round N a fast component is merged with an ω-component is at least

$$\frac{Y_{\mathtt{fast}}(N)}{n}\big(1-\varepsilon^2\big)^{\ell-1} \ge \frac{\varepsilon Y_{\mathtt{slow}}(tn)}{4n}.$$

Again we apply the Chernoff bounds and use that the right-hand side is $>2\log n/n$. Thus, with probability $1 - o(1/n)$ the number Z of fast components that are merged with an ω-component between time t and $t+1$ is at least $Z \ge \varepsilon Y_{\mathtt{slow}}(tn)/8$. Therefore, with probability at least $1 - o(1/n)$,

$$\begin{aligned} Y_{\mathtt{fast}}\big((t+1)n\big) &\le Y_{\mathtt{fast}}(tn) - Z + K \cdot I \\ &\le \varepsilon Y_{\mathtt{slow}}(tn) - \frac{\varepsilon Y_{\mathtt{slow}}(tn)}{8} + \frac{\varepsilon Y_{\mathtt{slow}}(tn)}{16} \\ &\le \frac{15}{16} \cdot \varepsilon Y_{\mathtt{slow}}(tn), \end{aligned} \tag{46}$$

so we get the same bound as in the first case; cf. (45).

In either case, together with (44) the inductive conclusion follows since with probability $1 - o(1/n)$,

$$\varepsilon Y_{\mathtt{slow}}\big((t+1)n\big) \ge \tfrac{15}{16}\varepsilon Y_{\mathtt{slow}}(tn) \ge Y_{\mathtt{fast}}\big((t+\delta)n\big).$$

This concludes the induction step and the proof of the lemma.

PROOF OF THEOREM 5. Choose any $0 < \varepsilon < 1$, and let $C, t_0 > 0$ be as in Lemma 13. From Lemma 9 and Corollary 3, we know that at time t_0 there is a linear number of slow components. We distinguish two cases. First, if the number of vertices in small components is larger than $Y_0 := (C + 1 + \varepsilon) \cdot \log n$ after n^2 rounds, then there is nothing to show. Second, suppose that the number drops below Y_0 at some round N_0. Then Lemma 13 implies $Y_{\mathtt{slow}}(N_0) \geq \log n$, while $Y_{\mathtt{fast}}(N_0) \leq Y_0 = O(\log n)$. We wait a bit further until in some round N_1 the number of slow components has dropped to 1 or 2. (In each round, it can decrease by at most 2.) Then the number of fast components is still at most Y_0.

Now we wait for $\Delta := n^{3/2}$ further rounds. The probability that in a fixed round at least two vertices in small components are chosen is in $O((Y_0/n)^2) = O(\log^2 n/n^2)$. Thus, the probability that there exists a round between N_1 and $N_1 + \Delta$ in which two vertices in small components are chosen is in $O(\log^2 n/n^2 \cdot \Delta) = o(1)$. In particular, w.h.p. the set of slow components remains unchanged.

On the other hand, in each round the probability that a particular fast component is merged with an ω-component is $\Omega(1/n)$. Thus, the expected number of fast components that will remain after Δ rounds is $(1 - \Omega(1/n))^{\Delta} = o(1)$. By Markov's inequality, the probability that there are no fast components left is $1 - o(1)$. Thus, w.h.p. after $N_1 + \Delta$ rounds there is no fast component left, and only one or two slow components. Then we only merge the slow components if at least two of their vertices are selected in some round. The expected time until this happens is $\Omega(n^2)$, which proves the theorem. $\square$

6. Examples and applications. In this section, we illustrate Theorem 4 by performing the analysis of the BF, KP and LEX_ℓ rules defined in the Introduction. In Section 6.3, we also prove Theorem 3 by showing that the lexicographic rule is connects asymptotically at least as fast as any other Achlioptas processes.

6.1. *The BF process.* The BF process is defined by the $(1, 4)$-rule

$$\mathsf{BF}(1, 1, *, *) = 1 \quad \text{and} \quad \mathsf{BF}(\omega, *, *, *) = \mathsf{BF}(*, \omega, *, *) = 2,$$

where we use $*$ as a placeholder for either 1 or ω. Components can be combined in three different ways given by

$$\begin{aligned} C_{1,1} &= \{(1, 1, *, *), (\omega, *, 1, 1), (1, \omega, 1, 1)\}, \\ C_{1,\omega} &= \{(\omega, *, 1, \omega), (\omega, *, \omega, 1), (1, \omega, 1, \omega), (1, \omega, \omega, 1)\} \quad \text{and} \\ C_{\omega,\omega} &= \{(*, \omega, \omega, \omega), (\omega, 1, \omega, \omega)\}. \end{aligned}$$

The extinction rate for 1-components is

$$\mathtt{ex}(\mathsf{BF}) = \mathtt{ex}_1(\mathsf{BF}) = 1 \cdot \left|\{(\omega, \omega, \omega, 1), (\omega, \omega, 1, \omega)\}\right| = 2.$$

Since $\mathtt{ex}_1(\mathsf{BF}) < 2K + 2 = 4$, Theorem 4 is applicable. As $K = 1$, we have $z_1 + z_\omega = 1$, so we can express the functions f_k in the differential equations (5) in terms of z_1 only. By writing z instead of z_1, they are given by

$$z' = -2z - 2z^2 + 2z^3 \quad \text{and} \quad z_\omega' = 2z + 2z^2 - 2z^3.$$

Integrating, we get

$$-2T = \int_0^T \frac{z'(t)}{z(t)+z(t)^2-z(t)^3}\,dt \overset{x=z(t)}{=} \int_1^{z(T)} \frac{1}{x+x^2-x^3}\,dx$$
$$= \left[\log x - \frac{5-\sqrt{5}}{10}\log(1+\sqrt{5}-2x) - \frac{5+\sqrt{5}}{10}\log(-1+\sqrt{5}+2x)\right]_{x=1}^{z(T)}.$$

For $T \to \infty$, we know $z(T) \to 0$, so $2T + \log(z(T))$ converges to

$$c_1 = \frac{\log(5-\sqrt{5}) - \log(5+\sqrt{5})}{\sqrt{5}} = \frac{-2\log\varphi}{\sqrt{5}} = -0.43040894\ldots,$$

where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. Hence, by Theorem 4, since $c_0 = c_1$, the expected time until the graph is connected is

$$\mathbb{E}[T_{\mathrm{con}}^{\mathsf{BF}}] = \frac{n\log n + \gamma n + c_1 n}{\mathtt{ex}(\mathsf{BF})} + o(n) = \frac{n\log n + 0.1468067\ldots n}{2} + o(n),$$

and for all $c \in \mathbb{R}$, since $d_1 = e^{c_1} = 0.6502431\ldots$,

$$\lim_{n\to\infty} \Pr\left[T_{\mathrm{con}}^{\mathsf{BF}} \le \frac{n\log n + cn}{2}\right] = e^{-d_1 e^{-c}} = e^{-0.6502431\ldots e^{-c}}.$$

6.2. *The KP process.* The KP process is defined by the $(1,4)$-rule

$$\mathsf{KP}(\omega,\omega,*,*) = 2 \quad \text{and} \quad \mathsf{KP}(1,*,*,*) = \mathsf{KP}(*,1,*,*) = 1,$$

where we use $*$ as a placeholder for either 1 or ω. We have $\mathtt{ex}(\mathsf{KP}) = 4$, as

$$\mathtt{ex}_1(\mathsf{KP}) = 1 \cdot \left|\{(1,\omega,\omega,\omega),(\omega,1,\omega,\omega),(\omega,\omega,1,\omega),(\omega,\omega,\omega,1)\}\right| = 4.$$

Moreover, $\mathtt{slow}(\mathsf{KP}) = \{1\}$. Note that Theorem 4 is not directly applicable, since $\mathtt{ex}(\mathsf{KP}) \ge 2K+2$. However, we can instead study a $(2,4)$-rule KP' as described in Remark 1, by setting

$$\mathsf{KP}'(\bullet,\bullet,*,*) = 2 \quad \text{and} \quad \mathsf{KP}'(1,*,*,*) = \mathsf{KP}'(*,1,*,*) = 1,$$

where $*$ is a placeholder for 1, 2 or ω and $\bullet$ is a placeholder for 2 or ω. Then we have $\mathtt{ex}_1(\mathsf{KP}) = \mathtt{ex}_1(\mathsf{KP}') = 4$ and

$$\mathtt{ex}_2(\mathsf{KP}') = 2 \cdot \left|\{(\omega,\omega,2,\omega),(\omega,\omega,\omega,2)\}\right| = 4,$$

and thus $\mathtt{ex}(\mathsf{KP}') = 4$ and $\mathtt{slow}(\mathsf{KP}') = \{1,2\}$. Since $\mathtt{ex}(\mathsf{KP}') < 2\cdot 2 + 2$ Theorem 4 is applicable to KP'. Moreover,

$$\begin{aligned}
C_{1,1} &= \{(1,1,*,*),(\bullet,\bullet,1,1)\},\\
C_{1,2} &= \{(1,2,*,*),(2,1,*,*),(\bullet,\bullet,1,2),(\bullet,\bullet,2,1)\},\\
C_{1,\omega} &= \{(1,\omega,*,*),(\omega,1,*,*),(\bullet,\bullet,1,\omega),(\bullet,\bullet,\omega,1)\},\\
C_{2,2} &= \{(\bullet,\bullet,2,2)\},\\
C_{2,\omega} &= \{(\bullet,\bullet,2,\omega),(\bullet,\bullet,\omega,2)\} \quad \text{and}\\
C_{\omega,\omega} &= \{(\bullet,\bullet,\omega,\omega)\}.
\end{aligned}$$

Recall that $z_1 + z_2 + z_\omega \equiv 1$. We can express f_k in (5) for $k \in \{1, 2, \omega\}$ in terms of z_1 and z_2 only. The differential equations are given by

$$z_1' = -4z_1 + 4z_1^2 - 2z_1^3,$$
$$z_2' = 2z_1^4 - 4z_1^3 - 4z_1^2 z_2 + 4z_1^2 + 4z_1 z_2 - 4z_2.$$

Like in the Bohman–Frieze process we get for z_1

$$\begin{aligned} -4T &= \int_0^T \frac{z_1'(t)}{z_1(t) - z_1(t)^2 + z_1(t)^3/2}\,dt \overset{x=z_1(t)}{=} \int_1^{z_1(T)} \frac{dx}{x - x^2 + x^3/2} \\ &= \left[-\frac{1}{2}\log\bigl(x^2 - 2x + 2\bigr) + \log(x) - \tan^{-1}(1-x)\right]_{x=1}^{z_1(T)}. \end{aligned} \tag{47}$$

For $T \to \infty$, we know $z_1(T) \to 0$, so the expression $4T + \log(z_1(T))$ converges to

$$c_1 = \bigl(\pi + \log(4)\bigr)/4 = 1.13197\ldots.$$

We can also compute the value of c_2. Note that the differential equation for z_2 is linear and we can rewrite it as

$$z_2' = f + g z_2, \qquad \text{where } f = 2z_1^4 - 4z_1^3 + 4z_1^2 \quad \text{and} \quad g = -4z_1^2 + 4z_1 - 4.$$

Thus, we can solve explicitly for z_2 in terms of z_1, and since $z_2(0) = 0$

$$\begin{aligned} z_2(T) &= \exp\left\{\int_0^T g(t)\,dt\right\} \cdot \int_0^T f(t) \exp\left\{-\int_0^t g(y)\,dy\right\} dt \\ &= \exp\left\{-4T + 4\int_0^T \bigl(z_1(t) - z_1^2(t)\bigr)\,dt\right\} \\ &\quad \times \int_0^T f(t) \exp\left\{4t + 4\int_0^t \bigl(z_1(y)^2 - z_1(y)\bigr)\,dy\right\} dt. \end{aligned} \tag{48}$$

In order to simplify this expression, note that

$$\int_0^T z_1(t)\,dt \overset{(x=z_1(t))}{=} \int_1^{z_1(T)} \frac{x}{-4x + 4x^2 - 2x^3}\,dx = \frac{1}{2}\tan^{-1}\bigl(1 - z_1(T)\bigr),$$

and the same change of variables yields

$$\begin{aligned} \int_0^T z_1(t)^2\,dt &= \int_1^{z_1(T)} \frac{x^2}{-4x + 4x^2 - 2x^3}\,dx \\ &= -\frac{1}{4}\log\bigl(z_1(T)^2 - 2z_1(T) + 2\bigr) + \frac{1}{2}\tan^{-1}\bigl(1 - z_1(T)\bigr). \end{aligned}$$

By plugging this into (48) and using that $z_1(T) \to 0$, we infer that as $T \to \infty$

$$4T + \log z_2(T) \to \log(2) + \log \int_0^\infty f(t) \exp\bigl\{4t - \log\bigl(z_1(t)^2 - 2z_1(t) + 2\bigr)\bigr\}\,dt.$$

Using once more the change of variables $x = z_1(t)$ and (47), we infer that

$$c_2 = \log(2) + \log\left[\int_0^1 \frac{e^{\tan^{-1}(1-x)}}{\sqrt{x^2 - 2x + 2}}\,dx\right].$$

The last integral can be approximated numerically and we get $c_2 = 1.008\ldots$. Hence, by Theorem 4, for all $c \in \mathbb{R}$, since

$$d_1 = e^{c_1} = 3.1017\ldots \quad \text{and} \quad d_2 = e^{c_2}/2 = 1.3700\ldots, \tag{49}$$

we obtain

$$\lim_{n\to\infty} \Pr\left[T_{\mathrm{con}}^{\mathsf{KP}} \le \frac{n\log n + cn}{4}\right] = e^{-(d_1+d_2)e^{-c}} = e^{-4.47\ldots e^{-c}},$$

and moreover,

$$\mathbb{E}[T_{\mathrm{con}}^{\mathsf{KP}}] = \frac{n\log n + \gamma n + \log(d_1+d_2)n}{4} + o(n) = \frac{n\log n + 2.075\ldots n}{4} + o(n).$$

Finally, we obtain that with some probability that is bounded away from zero and from one, the graph gets connected when the last isolated vertex/isolated edge disappears. More precisely, with d_1, d_2 as in (49)

$$\lim_{n\to\infty} \Pr[T_1^{\mathsf{KP}} = T_{\mathrm{con}}^{\mathsf{KP}}] = \frac{d_1}{d_1+d_2} = 0.693\ldots$$

and $\lim_{n\to\infty} \Pr[T_2^{\mathsf{KP}} = T_{\mathrm{con}}^{\mathsf{KP}}] = d_1/(d_1+d_2) = 0.306\ldots$. Note that the same statements are also true for KP'.

6.3. *The lexicographic rule.* Fix some even $\ell \ge 0$, and let $K \ge \ell/2$. The (K,ℓ)-rule LEX_ℓ is defined as follows. We begin with mapping the component size vector to one in which the sizes are ordered, that is,

$$\begin{aligned}(s_1,\ldots,s_\ell) &\mapsto (s_1',\ldots,s_\ell') \\ &:= \bigl(\min\{s_1,s_2\},\max\{s_1,s_2\},\ldots,\min\{s_{\ell-1},s_\ell\},\max\{s_{\ell-1},s_\ell\}\bigr).\end{aligned}$$

We then choose that $i \in \{1,\ell/2\}$ for which (s_{2i-1}', s_{2i}') is minimal with respect to the lexicographical ordering. In case of ties, we choose the smallest eligible i. As an example, consider $\ell = 4$ and $K = 2$. In this case, we almost get the KP-rule, except that we choose the second edge (instead of the first) for the two vectors $(1,\omega,1,1)$ and $(\omega,1,1,1)$.

Note that for all component size vectors of the form $(\omega,\ldots,\omega,k,\omega,\ldots,\omega)$, where $k \in [K]$, LEX_ℓ selects the edge with the component of size k, thus we have that $\texttt{ex}(\mathsf{LEX}_\ell) = \ell$ and $\texttt{slow}(\mathsf{LEX}_\ell) = \{1\}$. Note that the condition on K ensures that we may apply Theorem 4.

We abbreviate again $z := z_1$. Then by using $\sum_{k\in S_K} z_k = 1$, we will be able to express the differential equation (5) for z without reference to the other functions z_k,

$k \in S_K$. The differential equation for z is given by $\frac{dz}{dt} = -f_1^-(z)$. Recall that $P_{1,1}$ corresponds to the probability of adding an edge that joins two isolated vertices conditioned on the fraction of isolated vertices being z. Similarly, $\sum_{\mu \in S_k \setminus \{1\}} P_{\mu,1}$ corresponds to the probability to add an edge that joins an isolated vertex to a component with at least two vertices. Thus,

$$\begin{aligned} z_1' &= -f_1^-(z) = -2P_{1,1} - \sum_{\mu \in S_k \setminus \{1\}} P_{\mu,1} \\ &= -2\bigl(1 - (1 - z^2)^{\ell/2}\bigr) - \bigl(1 - (1 - z)^{\ell} - \bigl(1 - (1 - z^2)^{\ell/2}\bigr)\bigr) \\ &= -2 + (1 - z^2)^{\ell/2} + (1 - z)^{\ell}. \end{aligned}$$

By the same means as in the Bohman–Frieze process, an explicit expression for $c_\ell := \lim_{t\to\infty}(\ell \cdot t + \log z(t))$ is given by

$$\lim_{z\to 0}\left(\log z + \ell \int_1^z \frac{1}{-2 + (1 - x^2)^{\ell/2} + (1 - x)^{\ell}}\, dx\right).$$

In general, this integral can be expressed as a rational function in the roots of the polynomial f_1^- and their logarithms. Since the rational functions give little insight, we only give Table 1 with the numerical values.

Mind that the table only gives second-order terms. Since the dominating term of $\mathbb{E}[T_{\mathrm{con}}^{\mathrm{lex}}]$ is $n \log n/\ell$, the lexicographic rules become faster to connect the graph as ℓ increases.

PROOF OF THEOREM 3. Consider any ℓ-Achlioptas process A, that is, a process in which ℓ vertices are drawn uniformly at random, and then any strategy may be used to choose between the $\ell/2$ edges. We claim that for every $N \geq 0$ the number of isolated vertices after N rounds of A stochastically dominates the number of isolated vertices after N rounds of LEX_ℓ. Formally, if $Y_1^{\mathrm{lex}}(N)$ and $Y_1^{\mathsf{A}}(N)$ denote the number of isolated vertices after N rounds of LEX_ℓ and of A, respectively, then for every $N \geq 0$ and every $\mu \in \mathbb{N}_0$,

$$\Pr\bigl[Y_1^{\mathrm{lex}}(N) \leq \mu\bigr] \geq \Pr\bigl[Y_1^{\mathsf{A}}(N) \leq \mu\bigr]. \tag{50}$$

In order to show (50), let I_N^{lex} and I_N^{A} denote the sets of isolated vertices in G_N^{lex} and G_N^{A}, respectively. We will show by induction on N that it is possible to couple

TABLE 1
Numerical expressions for c_ℓ

ℓ	2	4	6	8	10	12	14	16
c_1	0	0.935...	1.910...	2.905...	3.912...	4.927...	5.948...	6.972...
d_1	1	2.549...	6.756...	18.275...	50.03...	138.07...	383.0...	$1.06\ldots \cdot 10^3$

G_N^{lex} and G_N^{A} such that there is a permutation π_N of the vertex set with the property that $I_N^{\mathrm{lex}} \subseteq \pi_N(I_N^{\mathsf{A}})$; this immediately establishes (50).

The claim is trivial for $N = 0$ (with π_0 being the identity map). For the induction step, let $N \in \mathbb{N}$ and let π_N be a permutation with the required property. Let $v_1, \dots, v_\ell$ be the ℓ random vertices selected at the beginning of round $N + 1$. Then we create G_{N+1}^{A} as usual, that is, by adding to G_N^{A} the edge that we choose according to A when $v_1, \dots, v_\ell$ (and G_N^{A}) are presented. The crucial idea of the coupling is that we may assume that LEX_ℓ is presented the vertices $\pi_N(v_1), \dots, \pi_N(v_\ell)$. More formally, we create a second graph G that includes all edges in G_N^{lex} and an additional edge e, which is the edge that LEX_ℓ would choose when presented the images of the v_i's under π_N. That is, $e = \{\pi_N(v_i), \pi_N(v_{i+1})\}$ and $i = \mathsf{LEX}_\ell(c(\pi_N(v_1)), \dots, c(\pi_N(v_\ell)))$, where $c(u)$ denotes the number of vertices in the component that contains u in G_N^{lex}. Since the v_i's are uniformly random and π_N is a permutation of the vertices we infer that G is distributed like G_{N+1}^{lex}, and so this construction is indeed a coupling for G_N^{lex} and G_N^{A}.

It remains to show the existence of a permutation π_{N+1} such that $I_{N+1}^{\mathrm{lex}} \subseteq \pi_{N+1}(I_{N+1}^{\mathsf{A}})$. Note that it suffices to show that $|I_{N+1}^{\mathrm{lex}}| \le |I_{N+1}^{\mathsf{A}}|$. However, this is a consequence of the fact that LEX_ℓ favours 1-components. For example, suppose that A selects the edge $\{u, v\}$ such that $u, v \in I_N^{\mathsf{A}}$ and, moreover, $\pi_N(u), \pi_N(v) \in I_N^{\mathrm{lex}}$. Then both $|I_N^{\mathrm{lex}}|$ and $|I_N^{\mathsf{A}}|$ decrease by two (albeit LEX_ℓ might select a different edge joining two isolated vertices in G_N^{lex}), and the induction hypothesis implies $|I_{N+1}^{\mathrm{lex}}| \le |I_{N+1}^{\mathsf{A}}|$. More generally, in the case $u, v \in I_N^{\mathsf{A}}$ set $s = |\{x \in \{u, v\} : \pi_N(x) \notin I_N^{\mathrm{lex}}\}| \in \{0, 1, 2\}$; we just handled the case $s = 0$. Since $I_N^{\mathrm{lex}} \subseteq \pi_N(I_N^{\mathsf{A}})$, this definition implies $|I_N^{\mathrm{lex}}| \le |I_N^{\mathsf{A}}| - s$. Moreover, $|I_N^{\mathsf{A}}|$ will decrease by 2, and $|I_N^{\mathrm{lex}}|$ will decrease by at least $2 - s$. Again the hypothesis guarantees $|I_{N+1}^{\mathrm{lex}}| \le |I_{N+1}^{\mathsf{A}}|$. The other cases (i.e., when $u \notin I_N^{\mathsf{A}}$ or $v \notin I_N^{\mathsf{A}}$) follow by completely analogous arguments, so we leave them as an easy exercise to the reader.

To make use of (50), recall that Theorem 4(b) implies that for every $c \in \mathbb{R}$,

$$\lim_{n\to\infty} \Pr\left[Y_1^{\mathrm{lex}}\left(\left\lfloor \frac{n \log n + cn}{\ell} \right\rfloor\right) = 0\right] = \exp\{-e^{c_\ell - c}\}.$$

Thus, by (50) we also have

$$\begin{aligned}\limsup_{n\to\infty} \Pr\left[T_{\mathrm{con}}^{\mathsf{A}} \le \frac{n \log n + cn}{\ell}\right] &\le \limsup_{n\to\infty} \Pr\left[Y_1^{\mathsf{A}}\left(\left\lfloor \frac{n \log n + cn}{\ell} \right\rfloor\right) = 0\right] \\ &\le \lim_{n\to\infty} \Pr\left[Y_1^{\mathrm{lex}}\left(\left\lfloor \frac{n \log n + cn}{\ell} \right\rfloor\right) = 0\right] \\ &= \exp\{-e^{c_\ell - c}\},\end{aligned}$$

as required. $\square$

REFERENCES

[1] AMIR, G., GUREL-GUREVICH, O., LUBETZKY, E. and SINGER, A. (2010). Giant components in biased graph processes. *Indiana Univ. Math. J.* **59** 1893–1929. MR2919740

[2] AMIR, G. and LUBETZKY, E. (2013). On two biased graph processes. Unpublished manuscript.

[3] AZAR, Y., BRODER, A. Z., KARLIN, A. R. and UPFAL, E. (1999). Balanced allocations. *SIAM J. Comput.* **29** 180–200. MR1710347

[4] BHAMIDI, S., BUDHIRAJA, A. and WANG, X. (2014). The augmented multiplicative coalescent, bounded size rules and critical dynamics of random graphs. *Probab. Theory Related Fields* **160** 733–796. MR3278920

[5] BHAMIDI, S., BUDHIRAJA, A. and WANG, X. (2014). Bounded-size rules: The barely subcritical regime. *Combin. Probab. Comput.* **23** 505–538. MR3217358

[6] BHAMIDI, S., BUDHIRAJA, A. and WANG, X. (2015). Aggregation models with limited choice and the multiplicative coalescent. *Random Structures Algorithms* **46** 55–116. MR3291294

[7] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. MR1700749

[8] BOHMAN, T. and FRIEZE, A. (2001). Avoiding a giant component. *Random Structures Algorithms* **19** 75–85. MR1848028

[9] BOHMAN, T. and KRAVITZ, D. (2006). Creating a giant component. *Combin. Probab. Comput.* **15** 489–511. MR2238042

[10] BOLLOBÁS, B. (1984). The evolution of random graphs. *Trans. Amer. Math. Soc.* **286** 257–274. MR0756039

[11] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge Univ. Press, Cambridge. MR1864966

[12] DRMOTA, M., KANG, M. and PANAGIOTOU, K. (2013). Universality results for stochastic coalescence models with restricted choice. Unpublisched manuscript.

[13] DUBHASHI, D. and RANJAN, D. (1998). Balls and bins: A study in negative dependence. *Random Structures Algorithms* **13** 99–124. MR1642566

[14] ERDŐS, P. and RÉNYI, A. (1960). On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **5** 17–61. MR0125031

[15] GONNET, G. H. (1981). Expected length of the longest probe sequence in hash code searching. *J. Assoc. Comput. Mach.* **28** 289–304. MR0612082

[16] JANSON, S., KNUTH, D. E., ŁUCZAK, T. and PITTEL, B. (1993). The birth of the giant component. *Random Structures Algorithms* **4** 231–358. MR1220220

[17] JANSON, S., ŁUCZAK, T. and RUCINSKI, A. (2000). *Random Graphs*. Wiley, New York. MR1782847

[18] JANSON, S. and SPENCER, J. (2012). Phase transitions for modified Erdős–Rényi processes. *Ark. Mat.* **50** 305–329. MR2961325

[19] KANG, M. and PANAGIOTOU, K. (2014). On the connectivity threshold of Achlioptas processes. *J. Comb.* **5** 291–304. MR3274958

[20] KRIVELEVICH, M., LOH, P.-S. and SUDAKOV, B. (2009). Avoiding small subgraphs in Achlioptas processes. *Random Structures Algorithms* **34** 165–195. MR2478543

[21] KRIVELEVICH, M., LUBETZKY, E. and SUDAKOV, B. (2010). Hamiltonicity thresholds in Achlioptas processes. *Random Structures Algorithms* **37** 1–24. MR2674619

[22] KRIVELEVICH, M. and SPÖHEL, R. (2012). Creating small subgraphs in Achlioptas processes with growing parameter. *SIAM J. Discrete Math.* **26** 670–686. MR2967491

[23] MITZENMACHER, M. (2009). Some open questions related to cuckoo hashing. In *Algorithms—ESA* 2009. *Lecture Notes in Computer Science* **5757** 1–10. Springer, Berlin. MR2557734

[24] MITZENMACHER, M., RICHA, A. W. and SITARAMAN, R. (2001). The power of two random choices: A survey of techniques and results. In *Handbook of Randomized Computing*. *Comb. Optim.* **9** 255–312. Kluwer Academic, Dordrecht. MR1966907

[25] MÜTZE, T., SPÖHEL, R. and THOMAS, H. (2011). Small subgraphs in random graphs and the power of multiple choices. *J. Combin. Theory Ser. B* **101** 237–268. MR2788094

[26] PANAGIOTOU, K., SPÖHEL, R., STEGER, A. and THOMAS, H. (2013). Explosive percolation in Erdős–Rényi-like random graph processes. *Combin. Probab. Comput.* **22** 133–145. MR3002578

[27] RIORDAN, O. and WARNKE, L. (2012). Achlioptas process phase transitions are continuous. *Ann. Appl. Probab.* **22** 1450–1464. MR2985166

[28] RIORDAN, O. and WARNKE, L. (2016). Convergence of Achlioptas processes via differential equations with unique solutions. *Combin. Probab. Comput.* **25** 154–171. MR3438291

[29] SEN, S. (2013). On the largest component in the subcritical regime of the Bohman–Frieze process. Preprint. Available at arXiv:1307.2041.

[30] SPENCER, J. and WORMALD, N. (2007). Birth control for giants. *Combinatorica* **27** 587–628. MR2375718

[31] WORMALD, N. C. (1999). The differential equation method for random graph processes and greedy algorithms. In *Lectures on Approximation and Randomized Algorithms* 73–155. PWN, Warsaw.

H. EINARSSON
J. LENGLER
F. MOUSSET
A. STEGER
INSTITUTE OF THEORETICAL COMPUTER SCIENCE
ETH ZÜRICH
8092 ZÜRICH
SWITZERLAND
E-MAIL: hafsteinn.einarsson@inf.ethz.ch
johannes.lengler@inf.ethz.ch
frank.moussetf@inf.ethz.ch
steger@inf.ethz.ch

K. PANAGIOTOU
MATHEMATICAL INSTITUTE
UNIVERSITY OF MUNICH
THERESIENSTR. 39
80333 MUNICH
GERMANY
E-MAIL: kpanagio@math.lmu.de