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FROM FEYNMAN–KAC FORMULAE TO NUMERICAL STOCHASTIC HOMOGENIZATION IN ELECTRICAL IMPEDANCE TOMOGRAPHY

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In this paper, we use the theory of symmetric Dirichlet forms to derive Feynman–Kac formulae for the forward problem of electrical impedance tomography with possibly anisotropic, merely measurable conductivities corresponding to different electrode models on bounded Lipschitz domains. Subsequently, we employ these Feynman–Kac formulae to rigorously justify stochastic homogenization in the case of a stochastic boundary value problem arising from an inverse anomaly detection problem. Motivated by this theoretical result, we prove an estimate for the speed of convergence of the projected mean-square displacement of the underlying process which may serve as the theoretical foundation for the development of new scalable stochastic numerical homogenization schemes.

1. Introduction. Electrical impedance tomography (EIT) aims to reconstruct the unknown conductivity κ in the conductivity equation

$$\nabla \cdot (\kappa \nabla u) = 0 \qquad \text{in } D \tag{1}$$

from current and voltage measurements on the boundary of the domain D. This inverse conductivity problem is known to be severely ill-posed, that is, its solution is extremely sensitive with respect to measurement and modeling errors. As a result, EIT suffers from inherent low resolution and due to this limitation, many practical applications focus on the detection of conductivity anomalies in a known background conductivity rather than conductivity imaging. In the mathematical modeling of such inverse anomaly detection problems, randomness typically reflects a lack of precise information about the meso- and micro-structure of the heterogeneous background conductivity, which may fluctuate on many scales. Recently, the second author has proposed a novel method for the detection of conductivity

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anomalies in a random background conductivity which is based on homogenization of the underlying stochastic boundary value problem; cf. [48, 49].

Although the homogenization theory for elliptic divergence form operators is well-developed (cf., e.g., [6, 28, 41, 42]), the numerical approximation of the *effective conductivity* in the random setting still poses major challenges. The commonly used deterministic methods based on a discretization of the so-called *auxiliary problem* have two main drawbacks. First, the auxiliary problem is formulated on the whole space $\mathbb{R}^d$, and second, it has to be solved for almost every realization of the random medium. That is, a truncated version of the auxiliary problem has to be considered and choosing an appropriate spatial truncation with appropriate boundary conditions is a delicate issue; cf. [8]. Moreover, in practically relevant cases, such as high contrast digitized random media, it is extremely difficult to solve the corresponding variational problems by usual deterministic methods, such as the finite element or the finite difference method, due to the behavior of the solutions near the corner points. Therefore, numerical approximation of the effective conductivity can be prohibitively expensive in terms of computation time. As a matter of fact, practitioners often choose to avoid these computations at all and rather content themselves with theoretical bounds; cf., for example, [55]. However, it has been reported in the physics literature that the shortcomings of the standard deterministic methods can be circumvented by using continuum micro-scale Monte Carlo simulation of certain diffusion processes evolving in random media instead; cf., for example, [31–33, 50, 56]. In this work, we give a rigorous mathematical justification for homogenizing the EIT forward problem using such methods by studying the interconnection between reflecting diffusion processes and certain boundary value problems for the conductivity equation. More precisely, we derive *Feynman–Kac formulae* for solutions of the deterministic conductivity equation (1) posed on a bounded domain $D \subset \mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary ∂D and possibly anisotropic uniformly elliptic and uniformly bounded conductivity subject to different boundary conditions modeling electrode measurements. Subsequently, we employ these Feynman–Kac formulae to prove a homogenization result for the corresponding stochastic boundary value problem which justifies the use of stochastic numerical homogenization schemes based on simulation of the underlying diffusion processes in order to approximate the effective conductivity. Finally, we prove an estimate for the speed of convergence of the projected mean-square displacement of the underlying diffusion processes. The main advantage of the presented approach to numerical homogenization, beside its inherent parallelism, is that its convergence rate is dimension-independent and its computational cost grows only linearly with the dimension.

It is well known that reflecting diffusion processes generated by nondivergence form operators with smooth coefficients on bounded, smooth domains are *Feller processes* satisfying Skorohod-type stochastic differential equations. The construction in the case of divergence form operators with merely measurable coefficients requires the theory of *symmetric Dirichlet forms* which has its origin in the *energy* *method* used by Dirichlet to address the boundary value problem in classical electrostatics that was subsequently named after him. When D is a bounded Lipschitz domain, Bass and Hsu [4] constructed the reflecting Brownian motion living on $\overline{D}$ by showing that the so-called *Martin–Kuramochi boundary* coincides with the Euclidean boundary in this case. A general diffusion process on a bounded Lipschitz domain, even allowing locally a finite number of Hölder cusps, was first constructed by Fukushima and Tomisaki [23]. In this work, we use such a Dirichlet form construction in order to derive Feynman–Kac representation formulae for the solutions of Neumann, respectively Robin, boundary value problems modeling EIT measurements. Probabilistic approaches to both parabolic and elliptic boundary value problems for second-order differential operators have been studied by many authors, starting with Feynman's Princeton thesis [17] and the article [29] by Kac. The probabilistic approach to the Dirichlet problem for a general class of second-order elliptic operators with merely measurable coefficients, even allowing singularities of a certain type, was elaborated by Chen and Zhang [12]; see also Zhang's paper [58]. However, there are only few works that treat Feynman–Kac representation formulae for Neumann- or Robin-type boundary conditions. Moreover, the approaches existing in the literature consider either the Laplacian; see, for example, [4, 9, 27], or nondivergence form operators with smooth coefficients, see, for example, [5, 19, 43]. For the particular case of the conductivity equation on a bounded Lipschitz domain, we generalize both, the Feynman–Kac formula for the Robin problem on domains with boundary of class C^3 for an isotropic $C^{2,\gamma}$-smooth conductivity, $\gamma > 0$, obtained by Papanicolaou [43] as well as the representation obtained by Benchérif–Madani and Pardoux [5] for the Neumann problem under similar regularity assumptions. While both of the aforementioned approaches use stochastic differential equations and Itô calculus, our approach is based on the theory of symmetric Dirichlet forms, following the pioneering work [4] for the reflecting Brownian motion by Bass and Hsu. We derive in this work Feynman–Kac formulae for both the Robin boundary value problem, corresponding to the so-called *complete electrode model*, as well as the Neumann boundary value problem, corresponding to the so-called *continuum model*. Both formulae are valid for possibly anisotropic, uniformly elliptic and uniformly bounded conductivities with merely measurable coefficients on bounded Lipschitz domains. During the preparation of this work, we became aware of the paper [13] by Chen and Zhang, where a probabilistic approach to some mixed boundary value problems with singular coefficients is derived. In contrast to our setting, however, the mixed boundary condition studied there results from a singular lower-order term of the differential operator.

Homogenization of stochastic differential equations with reflection and partial differential equations with Neumann boundary conditions, respectively, in half-space type domains have been studied for periodic coefficients in [3, 6, 54] and for random divergence form operators with smooth coefficients in [46]. In contrast to boundary value problems with homogeneous Dirichlet boundary conditions, these problems are nontranslation invariant, which excludes the standard stochastic homogenization approach via the so-called *environment as viewed from the particle*. Employing the Feynman–Kac formula in conjunction with a recently obtained invariance principle for reflecting diffusion processes associated with random divergence form operators with merely measurable coefficients due to Chen, Croydon and Kumagai [11], we provide a homogenization result for a stochastic forward problem built on the complete electrode model. Clearly, such a result motivates the derivation of stochastic numerical homogenization schemes for the approximation of the effective conductivity which are based on simulation of the underlying diffusion processes. However, the convergence analysis of such a method requires a quantitative convergence result that is stronger than the usual qualitative results obtained from the central limit theorem for martingales. As in the case of a discrete random walk in random environment (cf. Gloria and Mourrat [24]), it turns out that the behavior at the bottom of the spectrum of the infinitesimal generator of the environment as viewed from the particle process, projected on a suitably chosen function, yields bounds on the approximation error. This spectral behavior has been the subject of recent interest. Most notably, Gloria, Neukamm and Otto [25] have obtained optimal estimates in the discrete case which have been carried over to the continuum case by Gloria and Otto [26]. The main difficulty in obtaining such estimates for diffusion processes evolving in random media arises from the lack of a Poincaré inequality for the horizontal derivative in the space of square integrable functions on the probability space which corresponds to the random medium. Therefore, in contrast to the periodic case, where the Poincaré inequality on the torus is available, one cannot expect a spectral gap in the random case. Still, it has been shown that the bottom of the spectrum is sufficiently "thin." Using these estimates together with a classical argument due to Kipnis and Varadhan [34], we obtain an estimate for the speed of convergence of the projected mean-square displacement of the underlying diffusion process in a random medium to its limit. Qualitative results of this kind have been obtained by Kipnis and Varadhan in the case of discrete random walks in random environments and by De Masi, Ferrari, Goldstein and Wick [14] in the continuum case, whereas quantitative results in the case of discrete random walks have been proved more recently by Gloria and Mourrat [24] and Egloffe, Gloria, Mourrat and Nguyen [15]. Finally, we refer to the paper [38] by Mourrat which initiated the idea of using the Kipnis and Varadhan argument in order to obtain quantitative results.

The rest of the paper is structured as follows: We start in Section 2 by briefly introducing our notation. In Section 3, we recall the modeling of electrode measurements in EIT as well as the modeling of random heterogeneous media. Moreover, we introduce the stochastic forward problem we are interested in. In Section 4, we describe the construction of reflecting diffusion processes via Dirichlet form theory. Subsequently, in Section 5, the Feynman–Kac formulae for the deterministic boundary value problems will be derived. Then in Section 6 we study the interconnection between Feynman–Kac formulae, stochastic homogenization and stochastic numerics. Finally, we conclude with a brief summary of our results. In Appendix A we recall key concepts from the theory of Dirichlet forms and in Appendix B, we provide Skorohod decompositions for two practically relevant classes of conductivities.

2. Notation. Let D denote a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with connected complement and Lipschitz parameters (r_D, c_D), that is, for every $x \in \partial D$ we have after rotation and translation that $\partial D \cap B(x, r_D)$ is the graph of a Lipschitz function in the first $d-1$ coordinates with Lipschitz constant no larger than c_D and $D \cap B(x, r_D)$ lies above the graph of this function. Moreover, we set $\mathbb{R}^d_- := \{x \in \mathbb{R}^d : x \cdot \nu < 0\}$, with $\nu = e_d$ the outward unit normal on $\mathbb{R}^{d-1}$, where we identify the boundary of $\mathbb{R}^d_-$ with $\mathbb{R}^{d-1}$, with straightforward abuse of notation.

For Lipschitz domains, there exists a unique outward unit normal vector ν a.e. on ∂D so that the real Lebesgue spaces $L^p(D)$ and $L^p(\partial D)$ can be defined in the standard manner with the usual L^p norms $\|\cdot\|_p$, $p = 1, 2, \infty$. The standard L^2 inner-products are denoted by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\partial D}$, respectively. The $(d-1)$-dimensional Lebesgue surface measure is denoted by σ and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$.

By $(\Gamma, \mathcal{G}, \mathcal{P})$, we always mean a complete probability space corresponding to a random medium. We use the notation ω for an arbitrary element of Γ and $\mathbb{M}$ for the expectation with respect to the probability measure $\mathcal{P}$. We use bold letters to denote functions on $(\Gamma, \mathcal{G}, \mathcal{P})$, while we use italic letters for the corresponding realizations on $\mathbb{R}^d \times \Gamma$. The canonical probability space corresponding to diffusion processes evolving in a deterministic medium starting in x is denoted $(\Omega, \mathcal{F}, \mathbb{P}_x)$ and the expectation with respect to $\mathbb{P}_x$ is denoted $\mathbb{E}_x$. If the process is evolving in a random medium, we indicate this with a superscript ω for the probability measure, that is, the measure $\mathbb{P}_x^\omega$ corresponds to the particular realization ω of the medium. Finally, the product probability space corresponding to the *annealed measure* $\overline{\mathbb{P}} := \mathcal{P}\mathbb{P}_0^\omega$ on $\overline{\Omega} := \Gamma \times \Omega$, which is obtained by integrating with respect to the measure $\mathbb{P}_0^\omega$ and subsequent averaging over the realizations of the random medium, is denoted $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$. The expectation with respect to $\overline{\mathbb{P}}$ is denoted $\overline{\mathbb{E}}$.

All functions in this work will be real-valued and derivatives are understood in distributional sense. We use a diamond subscript to denote subspaces of the standard spaces containing functions with vanishing mean and interpret integrals over ∂D as dual evaluations with a constant function, if necessary. For example, we will frequently use the spaces

$$H_\diamond^{\pm 1/2}(\partial D) := \{\phi \in H^{\pm 1/2}(\partial D) : \langle \phi, 1 \rangle_{\partial D} = 0\}$$

and

$$H_\diamond^1(D) := \{\phi \in H^1(D) : \langle \phi, 1 \rangle = 0\}.$$

Moreover, we will frequently assume that ∂D is partitioned into two disjoint parts, $\partial_1 D$ and $\partial_2 D$. We denote by $H_0^1(D \cup \partial_1 D)$ the closure of $C_c^\infty(D \cup \partial_1 D)$, the linear subspace of $C^\infty(\overline{D})$ consisting of functions ϕ such that $\operatorname{supp}(\phi)$ is a compact subset of $D \cup \partial_1 D$, in $H^1(D)$. Moreover, we define the Bochner space

$$L^2(\Gamma; H_0^1(D \cup \partial_1 D)) := \left\{ \boldsymbol{\phi} : \Gamma \to H_0^1(D \cup \partial_1 D) : \int_\Gamma \|\boldsymbol{\phi}(\cdot, \omega)\|^2_{H_0^1(D \cup \partial_1 D)} \, \mathrm{d}\mathcal{P}(\omega) < \infty \right\},$$

see, for example, [2] for properties of this space.

For the reason of notational compactness, we use the Iverson brackets: Let S be a mathematical statement, then

$$[S] = \begin{cases} 1, & \text{if } S \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

We also use the Iverson brackets $[x \in B]$ to denote the indicator function of a set B, which we abbreviate by $[B]$ if there is no danger of confusion.

In what follows, all unimportant constants are denoted c, sometimes with additional subscripts, and they may vary from line to line.

3. Electrical impedance tomography.

3.1. *Modeling of electrode measurements.* Throughout this work, we assume that the possibly anisotropic conductivity is defined by a symmetric, matrix-valued function $\kappa : D \to \mathbb{R}^{d \times d}$ with components in $L^\infty(D)$ such that κ is uniformly bounded and uniformly elliptic, that is, there exists some constant $c > 0$ such that

$$c^{-1}|\xi|^2 \le \xi \cdot \kappa(x)\xi \le c|\xi|^2, \qquad \text{for every } \xi \in \mathbb{R}^d \text{ and a.e. } x \in D. \tag{2}$$

The forward problem of electrical impedance tomography can be described by different measurement models. In the so-called *continuum model*, the conductivity equation (1) is equipped with a co-normal boundary condition

$$\partial_{\kappa\nu} u := \kappa\nu \cdot \nabla u|_{\partial D} = f \qquad \text{on } \partial D, \tag{3}$$

where f is a measurable function modeling the signed density of the outgoing current. The boundary value problem (1), (3) has a solution if and only if

$$\langle f, 1 \rangle_{\partial D} = 0. \tag{4}$$

Physically speaking, this means that the current must be conserved. Given an appropriate function f, the solution to (1), (3) is unique up to an additive constant, which physically corresponds to the choice of the ground level of the potential. If $f \in H_\diamond^{-1/2}(\partial D)$, then there exists a unique equivalence class of functions $u \in H^1(D)/\mathbb{R}$ that satisfies the weak formulation of the boundary value problem

$$\int_D \kappa \nabla u \cdot \nabla v \, \mathrm{d}x = \langle f, v|_{\partial D} \rangle_{\partial D} \qquad \text{for all } v \in H^1(D)/\mathbb{R},$$

where $v|_{\partial D} := \gamma v$ and $\gamma : H^1(D)/\mathbb{R} \to H^{1/2}(\partial D)/\mathbb{R} = (H_\diamond^{-1/2}(\partial D))'$ is the standard trace operator. Note that we occasionally write v instead of $v|_{\partial D}$ for the sake of readability.

In practical EIT measurements, a number of electrodes, denoted $E_1, \ldots, E_N \subset \partial D$, are attached on the boundary of the object D. These electrodes are modeled by disjoint surface patches given by simply connected subsets of ∂D, each having a Lipschitz boundary curve. The most accurate forward model for real-life EIT is the so-called *complete electrode model* which takes into account the fact that during electrode measurements there is a *contact impedance* caused by a thin, highly resistive layer at the electrode object interface. It was demonstrated experimentally that the complete electrode model can correctly predict measurements up to instrument precision; cf. [51]. For a given voltage pattern $U \in \mathbb{R}^N$, the boundary conditions for the complete electrode model are given by

$$\kappa \nu \cdot \nabla u|_{\partial D} + g u|_{\partial D} = f \qquad \text{on } \partial D, \tag{5}$$

where the functions $f, g : \partial D \to \mathbb{R}$ are defined by

$$f(x) := \frac{1}{z(x)} \sum_{l=1}^{N} U_l [E_l], \qquad g(x) := \frac{1}{z(x)} \sum_{l=1}^{N} [E_l] \tag{6}$$

and the contact impedance $z : \partial D \to \mathbb{R}$ is assumed to be a piecewise continuous function, with interfaces that are of zero surface measure, satisfying

$$0 < c_0 \leq z \leq c_1 \qquad \text{a.e. on } \partial D.$$

For a given voltage pattern $U \in \mathbb{R}^N$ satisfying the grounding condition

$$\sum_{l=1}^{N} U_l = 0, \tag{7}$$

equations (1) and (5) define the electric potential $u \in H^1(D)$ uniquely (cf. [51] and the variational form of the boundary value problem), (1), (5) reads as follows: Given $U \in \mathbb{R}^N$ satisfying (7), find $u \in H^1(D)$ such that

$$\int_D \kappa \nabla u \cdot \nabla v \, \mathrm{d}x + \langle g u|_{\partial D}, v|_{\partial D} \rangle_{\partial D} = \langle f, v|_{\partial D} \rangle_{\partial D} \qquad \text{for all } v \in H^1(D). \tag{8}$$

Knowledge of u yields the corresponding electrode current vector $J \in \mathbb{R}^N$ via

$$J_l = \int_{E_l} \partial_{\kappa \nu} u \, \mathrm{d}\sigma(x), \qquad 1 \leq l \leq N. \tag{9}$$

3.2. *The stochastic problem.* The basic geometric setting of the stochastic problem we are interested in is as follows: Assume that the model domain is given by the lower hemisphere

$$D := B(0, R) \cap \mathbb{R}^d_-, \qquad R > 0$$

and that ∂D is partitioned into two disjoint parts, namely the *accessible boundary* $\partial_1 D := \partial D \cap \mathbb{R}^{d-1}$ and the *inaccessible boundary* $\partial_2 D := \partial D \setminus \partial_1 D$, respectively. Such a setting is found for instance in geophysical applications, where measurements can only be taken on the surface; cf., for example, [48, 49].

Let $(\Gamma, \mathcal{G}, \mathcal{P})$ be a probability space and let $\boldsymbol{\Theta} : \Gamma \to \Gamma$ denote an ergodic d-dimensional *dynamical system*, that is, a family of automorphisms $\{\boldsymbol{\Theta}_x, x \in \mathbb{R}^d\}$ which satisfies the following conditions:

(i) The family $\{\boldsymbol{\Theta}_x, x \in \mathbb{R}^d\}$ is a group, that is, $\boldsymbol{\Theta}_0 = \text{id}$ and

$$\boldsymbol{\Theta}_{x+y} = \boldsymbol{\Theta}_x \boldsymbol{\Theta}_y \qquad \text{for all } x, y \in \mathbb{R}^d;$$

(ii) the mappings $\boldsymbol{\Theta}_x : \Gamma \to \Gamma$, $x \in \mathbb{R}^d$, preserve the measure $\mathcal{P}$ on Γ, that is, for every $B \in \mathcal{G}$, $\boldsymbol{\Theta}_x B$ is $\mathcal{P}$-measurable and

$$\mathcal{P}(\boldsymbol{\Theta}_x B) = \mathcal{P}(B);$$

(iii) for every measurable function $\boldsymbol{\phi}$ on $(\Gamma, \mathcal{G}, \mathcal{P})$, the function $(x, \omega) \mapsto \boldsymbol{\phi}(\boldsymbol{\Theta}_x \omega)$ is a measurable function on $(\mathbb{R}^d \times \Gamma, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}, \text{Leb} \times \mathcal{P})$, where $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$ denotes the sigma-algebra generated by the measurable rectangles and Leb is the d-dimensional Lebesgue measure;

(iv) the family $\{\boldsymbol{\Theta}_x, x \in \mathbb{R}^d\}$ is ergodic, that is, $\boldsymbol{\phi}(\boldsymbol{\Theta}_x \omega) = \boldsymbol{\phi}(\omega)$ for all $x \in \mathbb{R}^d$ and $\mathcal{P}$-a.e. $\omega \in \Gamma$ implies $\boldsymbol{\phi} = \text{const}$ $\mathcal{P}$-a.e.

Throughout this work, we assume that the conductivity random field

$$\{\kappa(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$$

is the *stationary extension* on $\mathbb{R}^d \times \Gamma$ of some function $\boldsymbol{\kappa} \in L^2(\Gamma; \mathbb{R}^{d \times d})$, that is,

$$(x, \omega) \mapsto \kappa(x, \omega) = \boldsymbol{\kappa}(\boldsymbol{\Theta}_x \omega). \tag{10}$$

Note that if κ can be written in the form (10) with a dynamical system $\{\boldsymbol{\Theta}_x, x \in \mathbb{R}^d\}$ which satisfies (i)–(iii), then it is automatically *stationary* with respect to $\mathcal{P}$, that is, for every finite collection of points $x^{(i)}$, $i = 1, \dots, k$, and any $h \in \mathbb{R}^d$ the joint distribution of

$$\kappa(x^{(1)} + h, \omega), \dots, \kappa(x^{(k)} + h, \omega)$$

under $\mathcal{P}$ is the same as that of

$$\kappa(x^{(1)}, \omega), \dots, \kappa(x^{(k)}, \omega).$$

Even if it is not explicitly stated, we always assume that the conductivity random field may be written in the form (10), where the underlying dynamical system

$\{\boldsymbol{\Theta}_x, x \in \mathbb{R}^d\}$ satisfies conditions (i)–(iv). Moreover, we will explicitly state if the conductivity random field $\{\kappa(x,\omega), (x,\omega) \in \mathbb{R}^d \times \Gamma\}$ satisfies one of the following assumptions:

(A1) $\boldsymbol{\kappa} \in L^2(\Gamma; L^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d}))$ and the random field is strictly positive and uniformly bounded, that is, there exists a constant $c > 0$ such that for every $\xi \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$

$$\mathcal{P}\{\omega \in \Gamma : c^{-1}|\xi|^2 \le \xi \cdot \kappa(x,\omega)\xi \le c|\xi|^2\} = 1.$$

(A2) $\{\kappa(x,\omega), (x,\omega) \in \mathbb{R}^d \times \Gamma\}$ satisfies the *spectral gap property*; cf. [26]: There exist constants $\rho > 0$ and $r < \infty$ such that for all measurable functions on $\{\kappa : \mathbb{R}^d \to \{\kappa_0 \in \mathbb{R}^{d\times d} : |\kappa_0\xi| \le |\xi|, c|\xi|^2 \le \xi \cdot \kappa_0 \xi \text{ for all } \xi \in \mathbb{R}^d\}\}$ the inequality

$$\mathbb{V}\boldsymbol{\phi} \le \frac{1}{\rho}\mathbb{M}\int_{\mathbb{R}^d} (\operatorname{osc}_{\kappa|_{B(x,r)}} \boldsymbol{\phi})^2 \, \mathrm{d}x$$

holds, where we have set

$$\begin{aligned}(\operatorname{osc}_{\kappa|_{B(x,r)}} \boldsymbol{\phi}) := {} & \sup\{\boldsymbol{\phi}(\tilde{\kappa}) : \tilde{\kappa} \in \Omega, \tilde{\kappa}|_{\mathbb{R}^d \setminus B(x,r)} = \kappa|_{\mathbb{R}^d \setminus B(x,r)}\} \\ & - \inf\{\boldsymbol{\phi}(\tilde{\kappa}) : \tilde{\kappa} \in \Omega, \tilde{\kappa}|_{\mathbb{R}^d \setminus B(x,r)} = \kappa|_{\mathbb{R}^d \setminus B(x,r)}\}.\end{aligned}$$

To account for the highly heterogeneous properties of the background medium, the latter is modeled using the conductivity random field with appropriate scaling by a small parameter $\varepsilon > 0$, that is,

$$\kappa_\varepsilon(\cdot,\cdot) : \mathbb{R}^d \times \Gamma \to \mathbb{R}^{d\times d}, \qquad \kappa_\varepsilon(x,\omega) := \kappa(x/\varepsilon, \omega).$$

If the correlation length of the conductivity random field κ is, say 1, then the correlation length of the scaled version κ_ε is of order ε and for $\varepsilon \ll 1$ we obtain thus a rapidly oscillating random field.

Let us introduce a stochastic forward model based on the complete electrode model: We search for a random field $\{\mathbf{u}_\varepsilon(x,\omega), (x,\omega) \in \overline{D} \times \Gamma\}$ for the electrical potential with $u_\varepsilon \in L^2(\Gamma; H_0^1(D \cup \partial_1 D))$ such that the stochastic conductivity equation

$$\nabla \cdot (\kappa_\varepsilon \nabla u_\varepsilon) = 0 \qquad \text{in } D \times \Gamma \tag{11}$$

subject to the boundary conditions

$$\begin{aligned} \kappa_\varepsilon \nu \cdot \nabla u_\varepsilon|_{\partial_1 D} + g u_\varepsilon|_{\partial_1 D} = f \qquad & \text{on } \partial_1 D \times \Gamma, \\ u_\varepsilon|_{\partial_2 D} = 0 \qquad & \text{on } \partial_2 D \times \Gamma \end{aligned} \tag{12}$$

is satisfied $\mathcal{P}$-a.s. The variational formulation of the forward problem is to find $\mathbf{u}_\varepsilon \in L^2(\Gamma; H_0^1(D \cup \partial_1 D))$ such that

$$\mathbb{M}\left\{\int_D \kappa_\varepsilon \nabla u_\varepsilon \cdot \nabla v \, \mathrm{d}x + \langle g u_\varepsilon, v\rangle_{\partial_1 D}\right\} = \mathbb{M}\langle f, v\rangle_{\partial_1 D} \tag{13}$$

for all $\mathbf{v} \in L^2(\Gamma; H_0^1(D \cup \partial_1 D))$. For a given voltage pattern $U \in \mathbb{R}^N$, the corresponding measurement data is given by the random current pattern $J(\varepsilon, \omega) = (J_1(\varepsilon, \omega), \ldots, J_N(\varepsilon, \omega))^T$, defined for $\mathcal{P}$-a.e. $\omega \in \Gamma$ by

$$J_l(\varepsilon, \omega) = \frac{1}{|E_l|} \int_{E_l} \kappa_\varepsilon(x, \omega) \nu \cdot \nabla u_\varepsilon(x, \omega)|_{\partial_1 D} \, \mathrm{d}\sigma(x), \qquad l = 1, \ldots, N. \tag{14}$$

Due to the assumption (A1), the well-posedness of the variational formulation follows from a straightforward application of the Lax–Milgram theorem. Moreover, standard arguments from measure theory show that the solution to the stochastic forward problem (11), (12) also solves (13); cf. [2].

4. Reflecting diffusion processes. In his seminal paper [20] from 1971, Fukushima established a one-to-one correspondence between regular symmetric Dirichlet forms and symmetric Hunt processes, which is the foundation for the construction of stochastic processes via Dirichlet form techniques. Therefore, we assume that the reader is familiar with the theory of symmetric Dirichlet forms, as elaborated for instance in the monograph [22].

Let us consider the following symmetric bilinear forms on $L^2(D)$:

$$\mathcal{E}(v, w) := \int_D \kappa \nabla v(x) \cdot \nabla w(x) \, \mathrm{d}x, \qquad v, w \in \mathcal{D}(\mathcal{E}) := H^1(D) \tag{15}$$

and for the particular case $\kappa \equiv 1/2$, which is of special importance, we set

$$\mathcal{E}^{\mathrm{BM}}(v, w) := \frac{1}{2} \int_D \nabla v(x) \cdot \nabla w(x) \, \mathrm{d}x, \qquad v, w \in \mathcal{D}(\mathcal{E}^{\mathrm{BM}}) := H^1(D). \tag{16}$$

The pair $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ defined by (15) is a strongly local, regular symmetric Dirichlet form on $L^2(D)$. In particular, there exist an $\mathcal{E}$-exceptional set $\mathcal{N} \subset \overline{D}$ and a conservative diffusion process $X = (\Omega, \mathcal{F}, \{X_t, t \geq 0\}, \mathbb{P}_x)$, starting from $x \in \overline{D} \setminus \mathcal{N}$ such that X is associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Without loss of generality, let us assume that X is defined on the *canonical sample space* $\Omega = C([0, \infty); \overline{D})$. It is well known that the symmetric Hunt process associated with (16) is the reflecting Brownian motion. Therefore, we call the symmetric Hunt process associated with (15) a *reflecting diffusion process*.

Let us briefly recall the concept of the *boundary local time* of reflecting diffusion processes (see, e.g., [5, 27, 43]), which will be crucial for the subsequent derivation of the Feynman–Kac formulae. If the diffusion process is the solution to a stochastic differential equation, say the reflecting Brownian motion, then the boundary local time is given by the one-dimensional process L in the Skorohod decomposition, which prevents the sample paths from leaving $\overline{D}$, that is,

$$X_t = x + W_t - \frac{1}{2} \int_0^t \nu(X_s) \, \mathrm{d}L_s, \tag{17}$$

$\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$. This boundary local time is a continuous nondecreasing process which increases only when $X_t \in \partial D$, namely for all $t \geq 0$ and q.e. $x \in \overline{D}$

$$L_t = \int_0^t [\partial D](X_s)\,\mathrm{d}L_s,$$

$\mathbb{P}_x$-a.s. and

$$\mathbb{E}_x \int_0^t [\partial D](X_s)\,\mathrm{d}s = 0.$$

Although the reflecting diffusion process associated with (15) does in general not admit a Skorohod decomposition of the form (17), we may still define a continuous one-dimensional process having the properties of the local time process mentioned above. More precisely, we employ the Revuz correspondence from Lemma A.2 to establish a rigorous definition of the boundary local time of X generalizing the classical definition for solutions to an SDE where a Skorohod decomposition exists. For convenience and in order to make the text flow more naturally, the concepts that appear in the following definition are collected in the Appendix A.

DEFINITION 4.1. We call the positive continuous additive functional L of X whose Revuz measure is the Lebesgue surface measure σ on ∂D the *boundary local time* of the reflecting diffusion process X.

So far, both X and L have been defined up to an $\mathcal{E}$-exceptional set and the rest of this section is devoted to showing that this set is actually empty. Therefore, let us consider the nonpositive definite self-adjoint operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. That is, for $v \in \mathcal{D}(\mathcal{L})$ we have

$$\langle -\mathcal{L}v, w\rangle = \mathcal{E}(v, w) \qquad \text{for all } w \in \mathcal{D}(\mathcal{E}) \tag{18}$$

and the domain of $\mathcal{L}$ is given by

$$\mathcal{D}(\mathcal{L}) = \left\{v \in \mathcal{D}(\mathcal{E}) : \exists \phi \in L^2(D) \text{ s.t. } \mathcal{E}(v, w) = \int_D \phi w \,\mathrm{d}x \ \forall w \in \mathcal{D}(\mathcal{E})\right\}.$$

In order to *refine* the reflecting diffusion process X to start from every $x \in \overline{D}$, we exploit the connection between the strongly continuous sub-Markovian contraction semigroup $\{T_t, t \geq 0\}$ on $L^2(D)$ and the evolution system corresponding to $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$; see, for example, the monograph [44]. Namely, for every $v_0 \in L^2(D)$, the trajectory $v : (0, T) \to H^1(D)$, $v(t) = T_t v_0$ belongs to the function space

$$\{\phi \in L^2((0, T); H^1(D)) : \dot{\phi} \in L^2((0, T); H^{-1}(D))\}$$

and is the unique mild solution to the parabolic *abstract Cauchy problem*

$$\begin{aligned} \dot{v} + \mathcal{L}v &= 0 \qquad \text{in } (0, T),\\ v(0) &= v_0. \end{aligned} \tag{19}$$

This is equivalent to the variational formulation

$$(20)\qquad -\int_0^T \langle v(t), w\rangle \dot{\varphi}(t)\,\mathrm{d}t + \int_0^T \langle \mathcal{L}v(t), w\rangle \varphi(t)\,\mathrm{d}t - \langle v_0, w\rangle \varphi(0) = 0$$

for all $w \in H^1(D)$ and all $\varphi \in C_c^\infty([0, T))$. Moreover, T_t is known to be a bounded operator from $L^1(D)$ to $L^\infty(D)$ for every $t > 0$. Therefore, by the Dunford–Pettis theorem, it can be represented as an integral operator for every $t > 0$,

$$(21)\qquad T_t\phi(x) = \int_D p(t, x, y)\phi(y)\,\mathrm{d}y \qquad \text{for every } \phi \in L^1(D),$$

where for all $t > 0$ we have $p(t, \cdot, \cdot) \in L^\infty(D \times D)$ and $p(t, \cdot, \cdot) \geq 0$ a.e. We call the function p the *transition kernel density* of X.

The following proposition adapts a well-known result for diffusion processes on $\mathbb{R}^d$ (cf. [52]), which follows from the famous De Giorgi–Nash–Moser theorem, to the case of reflecting diffusion processes on $\overline{D}$. The key idea of the proof is the following *extension by reflection* technique from [57], Section 2.4.3: We extend the solution to a parabolic problem by reflection at the boundary. Then we show that this extension again solves a parabolic problem so that we can apply the interior regularity result due to De Giorgi, Nash and Moser. See also the article [40] by Nittka, where such a technique is applied to elliptic boundary value problems.

PROPOSITION 4.2. *$p \in C^{0,\delta}((0, T] \times \overline{D} \times \overline{D})$ for some $\delta \in (0, 1)$, that is, for each fixed $0 < t_0 \leq T$, there exists a positive constant c such that*

$$(22)\qquad \big|p(t_2, x_2, y_2) - p(t_1, x_1, y_1)\big| \leq c\big(\sqrt{t_2 - t_1} + |x_2 - x_1| + |y_2 - y_1|\big)^\delta$$

for all $t_0 \leq t_1 \leq t_2 \leq T$ and all $(x_1, y_1), (x_2, y_2) \in \overline{D} \times \overline{D}$. Moreover, the mapping $t \mapsto p(t, \cdot, \cdot)$ is analytic from $(0, \infty)$ to $C^{0,\delta}(\overline{D} \times \overline{D})$.

PROOF. First, note that Nash's inequality holds for the underlying Dirichlet form $(\mathcal{E}, H^1(D))$, that is, there exists a constant $c_1 > 0$ such that

$$\|v\|_2^{2+4/d} \leq c_1\big(\mathcal{E}(v, v) + \|v\|_2^2\big)\|v\|_1^{4/d} \qquad \text{for all } v \in H^1(D).$$

This is a direct consequence of the uniform ellipticity (2) and [4], Corollary 2.2, where Nash's inequality is shown to hold for the Dirichlet form $(\mathcal{E}^{\mathrm{BM}}, H^1(D))$ for a bounded Lipschitz domain D. Analogously to the proof of [4], Theorem 3.1, it follows thus from [10], Theorem 3.25, that the transition kernel density satisfies an Aronson-type Gaussian upper bound

$$(23)\qquad p(t, x, y) \leq c_1 t^{-d/2} \exp\left(-\frac{|x - y|^2}{c_2 t}\right)$$

for all $t \leq 1$ and all $(x, y) \in \overline{D} \times \overline{D}$. In particular, $\sup_{0<t\leq 1} \|p(t, \cdot, \cdot)\|_\infty$ is finite, and hence by the interior Hölder continuity obtained from the De Giorgi–Nash–Moser theorem (cf. [39, 52]), the estimate (22) is true for all (x_1, y_1), (x_2, y_2)

satisfying $d(x_i, \partial D), d(y_i, \partial D) > c_3$, $i = 1, 2$, for some constant $c_3 > 0$ and all $t_0 \le t_1 \le t_2 \le 1$. Note that by the semigroup property the Chapman–Kolmogorov equation holds, that is,

$$p(t_1 + t_2, x, y) = \int_D p(t_1, x, z) p(t_2, z, y)\, \mathrm{d}z \tag{24}$$

for every pair $t_1, t_2 \ge 0$ and a.e. $x, y \in \overline{D}$. In particular, for fixed $y \in \overline{D}$ the function $v := p(\cdot, \cdot, y)$ is the unique solution to (19) with initial value $v_0 := p(0, \cdot, y) \in L^2(D)$. Now let $z \in \partial D$ so that by the Lipschitz property of ∂D we have after translation and rotation $B(z, r_D) \cap \overline{D} = \{(\tilde{x}, x_d) \in B(z, r_D) : x_d \ge \gamma(\tilde{x})\}$ and $B(z, r_D) \cap \partial D = \{\tilde{x} \in B(z, r_D) : x_d = \gamma(\tilde{x})\}$, where we have introduced the notation $\tilde{x} = (x_1, \dots, x_{d-1})^T$. Let us furthermore introduce the one-to-one transformation $\Psi(x) := (\tilde{x}, x_d - \gamma(\tilde{x}))$ which straightens the boundary $B(z, r_D) \cap \partial D$. Ψ is a bi-Lipschitz transformation and the Jacobians of both Ψ and Ψ^{-1} are bounded with bounds that depend only on the Lipschitz constant c_D. Since v is the solution to (19) with appropriate initial condition, the function $\hat{v} := v(\cdot, \Psi^{-1}(\cdot))$ must satisfy the variational formulation of the following parabolic problem in $\hat{D}(z, r_D) := \Psi(B(z, r_D) \cap \overline{D})$:

$$\begin{aligned} \int_0^T \dot{\varphi}(t) \int_{\hat{D}(z,r_D)} \hat{v}(t) w \,\mathrm{d}x\,\mathrm{d}t = &- \sum_{i,j=1}^d \int_0^T \varphi(t) \int_{\hat{D}(z,r_D)} \hat{\kappa}_{ij} \partial_i \hat{v}(t) \partial_j w \,\mathrm{d}x\,\mathrm{d}t \\ &- \varphi(0) \int_{\hat{D}(z,r_D)} \hat{v}_0 w \,\mathrm{d}x \end{aligned}$$

for all $w \in C_c^\infty(\hat{D}(z, r_D))$ and all $\varphi \in C_c^\infty([0, T))$. The coefficient $\hat{\kappa}$ is obtained via change of variables and it is bounded and uniformly elliptic by the boundedness of the Jacobians of Ψ and Ψ^{-1}, respectively. Now we use reflection at the hyperplane $\{(\tilde{y}, 0)\}$ via the mapping $\rho(x) := (\tilde{x}, -x_d)$ which yields that the function $\hat{v}(\cdot, \rho(\cdot))$ satisfies the variational formulation of a parabolic problem on $\rho(\hat{D}(z, r_D))$. Summing up both variational formulations on $\hat{D}(z, r_D)$ and on $\rho(\hat{D}(z, r_D))$, respectively, we obtain that the function

$$\check{v}(t, x) := \begin{cases} \hat{v}(t, x), & x \in \hat{D}(z, r_D), \\ \hat{v}(t, \rho(x)), & x \in \rho(\hat{D}(z, r_D)) \end{cases}$$

satisfies the variational formulation of a parabolic problem in $\hat{D}(z, r_D) \cup \rho(\hat{D}(z, r_D))$. By the interior Hölder estimate for $\check{v}$, together with the fact that we may choose $c_3 = r_D/4c_D$, we obtain thus

$$\left| p(t_2, x_2, \Psi^{-1}(y_2)) - p(t_1, x_1, \Psi^{-1}(y_1)) \right| \le c_1 \left(\sqrt{t_2 - t_1} + |y_2 - y_1| \right)^{c_2}$$

for all $t_0 \le t_1 \le t_2 \le 1$ and $y_1, y_2 \in \{(\tilde{x}, x_d) : |\tilde{x}| < c_3, x_d \in (0, r_D/4)\}$. As Ψ is bi-Lipschitz, for fixed x, the mapping $(t, y) \mapsto p(t, x, y)$ is Hölder continuous in $(t_0, 1] \times (B(z, c_3) \cap \overline{D})$ and by symmetry of the transition kernel density the same holds true for the mapping $(t, x) \mapsto p(t, x, y)$ for fixed y. Finally, the first assertion on $(t_0, 1] \times \overline{D} \times \overline{D}$ follows due to compactness of ∂D and its generalization to arbitrary $T > 0$ is obtained after repeatedly applying the Chapman–Kolmogorov equation.

The second assertion follows by the fact that the semigroup $\{T_t, t \geq 0\}$ extrapolates to a holomorphic semigroup on $L^2(D)$. More precisely, the semigroup possesses a holomorphic extension to the sector $\Sigma_\theta := \{re^{i\alpha} : r > 0, |\alpha| < \theta\}$ for some $\theta \in (0, \frac{\pi}{2}]$, cf., for example, [44]. Let $0 < t_0 \leq T$ and set

$$\Sigma_\theta(t_0, T) := \big\{z \in \mathbb{C} : z - t_0 \in \Sigma_\theta, |z| < T\big\}.$$

By the Hölder continuity of p, the set $\{p(z, \cdot, \cdot) : z \in \Sigma_\theta(t_0, T)\}$ is a bounded subset of $C^{0,\delta}(\overline{D} \times \overline{D})$. Moreover, the family of functionals obtained form integration against the functions $[B_1](x)[B_2](y)$ for measurable $B_1, B_2 \subset \overline{D}$ form a separating subspace of $(C^{0,\delta}(\overline{D}, \overline{D}))'$, that is, for $k \in C^{0,\delta}(\overline{D} \times \overline{D})$

$$\int_{D\times D} k(x, y)[B_1](x)[B_2](y)\, \mathrm{d}x\, \mathrm{d}y = 0 \qquad \text{for all measurable } B_1, B_2 \subset \overline{D}$$

implies that $k \equiv 0$. As the mapping

$$z \mapsto \langle T_z[B_1], [B_2]\rangle = \int_{D\times D} p(z, x, y)[B_1](y)[B_2](x)\, \mathrm{d}x\, \mathrm{d}y$$

is holomorphic for all $z \in \Sigma_\theta$, the mapping $z \mapsto p(z, \cdot, \cdot)$ is holomorphic from $\Sigma_\theta(t_0, T)$ to $C^{0,\delta}(\overline{D} \times \overline{D})$ by [1], Theorem 3.1. Since t_0 and T were arbitrary, the assertion is proved. □

By [21], Theorem 2, the existence of a Hölder continuous transition kernel density ensures that we may refine the process X to start from *every* $x \in \overline{D}$ by identifying the strongly continuous semigroup $\{T_t, t \geq 0\}$ with the transition semigroup $\{P_t, t \geq 0\}$.

It remains to show that L can be refined as well. Note that by the Lipschitz property of ∂D, we have that $D \cap B(x, r_D) = \{(\tilde{x}, x_d) : x_d > \gamma(\tilde{x})\} \cap B(x, r_D)$ and the Lipschitz function γ is differentiable a.e. with a bounded gradient. In particular, we have for every Borel set $B \subset \partial D \cap B(x, r_D)$ that

$$\sigma(B) = \int_{\{\tilde{x}:(\tilde{x},\gamma(\tilde{x}))\in B\}} \big(1 + \big|\nabla\gamma(\tilde{x})\big|^2\big)^{1/2}\, \mathrm{d}\tilde{x}$$

and a straightforward computation yields that the Lebesgue surface measure σ is a smooth measure with respect to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ having finite energy. Thus, by the Revuz correspondence from Lemma A.2 the boundary local time L exists as a positive continuous additive functional in the strict sense, that is, without an exceptional set.

5. Feynman–Kac formulae. In this section, we derive the Feynman–Kac formulae for both the continuum model and the complete electrode model. Afterward, we will obtain, as a corollary, a Feynman–Kac formula for the mixed boundary value problem corresponding to the stochastic problem introduced in Section 3.2. Compared to the earlier works [5, 27, 43] on Feynman–Kac formulae, the main difficulty in deriving these formulae in our particular setting comes from the lack of Itô's formula for general reflecting diffusion processes.

The rest of this subsection is devoted to providing some auxiliary lemmata.

LEMMA 5.1. *The transition kernel density p approaches the stationary distribution uniformly and exponentially fast, that is, there exist positive constants c_1 and c_2 such that for all $(x, y) \in \overline{D} \times \overline{D}$ and every $t \geq 0$,*

$$\big|p(t, x, y) - |D|^{-1}\big| \leq c_1 \exp(-c_2 t). \tag{25}$$

For the proof of this lemma, we refer to the monograph [49].

LEMMA 5.2. *Let $\kappa \in C^\infty(\overline{D}; \mathbb{R}^{d\times d})$. Then the set*

$$V_\kappa(D) := \{\phi \in H^1(D) \cap C^2(D) : \partial_{\kappa\nu}\phi(x) = 0 \text{ for } \sigma\text{-a.e. } x \in \partial D\} \tag{26}$$

is dense in $H^1(D)$.

PROOF. Diagonalizing the operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ corresponding to the conductivity κ, we obtain an orthonormal basis $\{\phi_k, k \in \mathbb{N}\}$ of $L^2(D)$ and an increasing sequence of real numbers $(\lambda_k)_{k\in\mathbb{N}}$ such that $\lambda_k \uparrow \infty$ and for every $k \in \mathbb{N}$, the function $\phi_k \in H^1(D)$ satisfies $\mathcal{E}(\phi_k, \psi) = \lambda_k\langle\phi_k, \psi\rangle$ for every $\psi \in H^1(D)$. Note that the inner product $\mathcal{E}_1(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + \langle\cdot, \cdot\rangle$ is equivalent to the standard inner product on $H^1(D)$ and by interior elliptic regularity theorem [16], Theorems 6.1–6.3, we have $\phi_k \in V_\kappa(D)$ for every $k \in \mathbb{N}$. In particular, it is enough to show the density of the linear span of $\{\phi_k\}$ in $H^1(D)$. Therefore, let $\psi \in H^1(D)$ such that $\mathcal{E}_1(\phi_k, \psi) = 0$ for every $k \in \mathbb{N}$, then

$$0 = \mathcal{E}_1(\phi_k, \psi) = \mathcal{E}(\phi_k, \psi) + \langle\phi_k, \psi\rangle = (\lambda_k + 1)\langle\phi_k, \psi\rangle.$$

Hence, it follows $\langle\phi_k, \psi\rangle = 0$ for every $k \in \mathbb{N}$ and the fact that $\{\phi_k, k \in \mathbb{N}\}$ is an orthonormal basis of $L^2(D)$ implies $\psi \equiv 0$ which proves the assertion. □

LEMMA 5.3. *For every $x \in \overline{D}$ and every bounded Borel function ϕ on ∂D, we have*

$$\mathbb{E}_x \int_0^t \phi(X_s)\,\mathrm{d}L_s = \int_0^t \int_{\partial D} p(s, x, y)\phi(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}s \qquad \textit{for all } t \geq 0. \tag{27}$$

PROOF. First, the expression (27) is well-defined since the boundary local time of X exists as a positive continuous additive functional in the strict sense. Without loss of generality, we may assume that ϕ is nonnegative. It follows from [22], Theorem 5.1.3, that the Revuz correspondence (66) is equivalent to

$$\begin{aligned}\int_D \psi(x)\mathbb{E}_x \int_0^t \phi(X_s)\,\mathrm{d}L_s\,\mathrm{d}x &= \int_0^t \int_{\partial D} \phi(y) T_s \psi(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}s \\ &= \int_D \psi(x) \int_0^t \int_{\partial D} p(s,y,x)\phi(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}s\,\mathrm{d}x\end{aligned}$$

for every $t > 0$ and all nonnegative Borel functions ψ and ϕ, where we have used Fubini's theorem in the second line. As this holds for every nonnegative Borel function ψ, we deduce

$$\mathbb{E}_x \int_0^t \phi(X_s)\,\mathrm{d}L_s = \int_0^t \int_{\partial D} p(s,x,y)\phi(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}s \qquad \text{a.e. in } \overline{D}.$$

To obtain the assertion everywhere in $\overline{D}$, fix an arbitrary $x_0 \in \overline{D}$ and consider for $t_0 > 0$ the integral

$$\begin{aligned}\mathbb{E}_{x_0} \int_{t_0}^t \phi(X_s)\,\mathrm{d}L_s &= \int_D p(t_0,x_0,x)\mathbb{E}_x\left\{\int_0^{t-t_0} \phi(X_s)\,\mathrm{d}L_s\right\}\mathrm{d}x \\ &= \int_D p(t_0,x_0,x)\left(\int_0^{t-t_0}\int_{\partial D} p(s,x,y)\phi(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}s\right)\mathrm{d}x \\ &= \int_{t_0}^t \int_{\partial D} p(s,x_0,y)\phi(z)\,\mathrm{d}\sigma(y)\,\mathrm{d}s,\end{aligned}$$

where we have used the Markov property of X. Now let $(t_k)_{k\in\mathbb{N}}$ denote a positive sequence which monotonically decreases to zero as $k \to \infty$. By the computation from above, we have for every $x \in \overline{D}$

$$\mathbb{E}_x \int_0^t \phi(X_s)\,\mathrm{d}L_s = \int_{t_k}^t \int_{\partial D} p(s,x,y)\phi(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}s + \mathbb{E}_x \int_0^{t_k} \phi(X_s)\,\mathrm{d}L_s.$$

The claim follows by the facts that ϕ is bounded and $\mathbb{E}_x L_{t_k}$ goes to zero as $k \to \infty$ which follows from monotonicity and continuity of the local time and the property $L_0 = 0$ $\mathbb{P}_x$-a.s. for every $x \in \overline{D}$. □

5.1. *Continuum model.* The main result for the continuum model (1), (3) is the following theorem.

THEOREM 5.4. *Let f be a bounded Borel function satisfying $\langle f, 1\rangle_{\partial D} = 0$. Then there is a unique weak solution $u \in C(\overline{D}) \cap H^1_\diamond(D)$ to the boundary value problem* (1), (3). *This solution admits the Feynman–Kac representation*

$$u(x) = \lim_{t\to\infty} \mathbb{E}_x \int_0^t f(X_s)\,\mathrm{d}L_s \qquad \textit{for all } x \in \overline{D}. \tag{28}$$

PROOF. The existence of a unique normalized weak solution u to (1), (3) is guaranteed by the standard theory of linear elliptic boundary value problems. Let us set

$$u_t(x) := \mathbb{E}_x \int_0^t f(X_s)\,\mathrm{d}L_s \quad \text{and} \quad u_\infty(x) := \lim_{t\to\infty} u_t(x), \qquad x \in \overline{D},$$

respectively. Note that u_∞ represents the right-hand side of (28) and proving the claim is equivalent to verifying that $u = u_\infty$ with the required properties.

The proof consists of four steps. First, we verify that $u_\infty \in C_\diamond(\overline{D})$. Then we show the claim for smooth approximations $\kappa^{(k)}$ of κ that yield a sequence $(u^{(k)})_{k\in\mathbb{N}} \subset H^1_\diamond(D) \cap C(\overline{D})$. In the third step, we show that $u^{(k)} \to u$ in $H^1(D)$ as $k \to \infty$. The last step is to show that $u^{(k)} \to u_\infty$ uniformly on compact sets which yields the claim.

From the occupation formula (27) and the compatibility condition (4), we obtain an integral representation of u_t for every $t > 0$ and since the convergence toward the stationary distribution is uniform over $\overline{D}$, by Lemma 5.1, we may deduce that

$$u_\infty(x) = \int_0^\infty \int_{\partial D} \bigl(p(s,x,y) - |D|^{-1}\bigr) f(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}s \qquad \text{for all } x \in \overline{D}. \tag{29}$$

The first step follows from (29) together with the Hölder continuity shown in Proposition 4.2 and the Aronson-type upper bound (23), which give that $u_\infty \in C(\overline{D})$. Moreover, Fubini's theorem yields that u_∞ has zero mean, that is, $u_\infty \in C_\diamond(\overline{D})$.

Next, we show the claim for an approximative sequence $(\kappa^{(k)})_{k\in\mathbb{N}}$ of conductivities with components in $C^\infty(\overline{D})$ such that $\kappa^{(k)} \to \kappa$ componentwise a.e. as $k \to \infty$. Let us consider the Dirichlet forms $(\mathcal{E}^{(k)}, H^1(D))$, $k \in \mathbb{N}$, with

$$\mathcal{E}^{(k)}(v,w) := \int_D \kappa^{(k)} \nabla v \cdot \nabla w \,\mathrm{d}x$$

and the associated reflecting diffusion processes $X^{(k)}$. By Proposition B.1, we obtain the Skorohod decomposition

$$X_t^{(k)} = x + \int_0^t a^{(k)}\bigl(X_s^{(k)}\bigr)\,\mathrm{d}s + \int_0^t B^{(k)}\bigl(X_s^{(k)}\bigr)\,\mathrm{d}W_s - \int_0^t \kappa^{(k)}\bigl(X_s^{(k)}\bigr)\nu\bigl(X_s^{(k)}\bigr)\,\mathrm{d}L_s^{(k)},$$

where W is a standard d-dimensional Brownian motion, $a_i^{(k)} := \sum_{j=1}^d \partial_j \kappa_{ij}^{(k)}$, $i = 1,\dots,d$, and the matrix $B^{(k)}$ satisfies $2\kappa^{(k)} = (B^{(k)})^2$. Let us define $u_t^{(k)}$ in the same manner as u_t and $u^{(k)}(x) := \lim_{t\to\infty} u_t^{(k)}(x)$, $x \in \overline{D}$.

The second step follows, therefore, after we have shown that $u^{(k)}$ is the unique weak solution [in the Sobolev space $H^1_\diamond(D)$] of the elliptic boundary value problem obtained from (1) and (3) by replacing κ with $\kappa^{(k)}$ which, by Lemma 5.2, is equivalent to

$$\forall v \in V_{\kappa^{(k)}}(D): \qquad \bigl\langle u^{(k)}, \nabla \cdot \bigl(\kappa^{(k)} \nabla v\bigr)\bigr\rangle = -\langle f, v\rangle_{\partial D}. \tag{30}$$

For $v \in V_{\kappa^{(k)}}(D)$ we may apply Itô's formula for semimartingales to obtain

$$\mathbb{E}_x v(X_t^{(k)}) = v(x) + \mathbb{E}_x \int_0^t \nabla \cdot (\kappa^{(k)} \nabla v(X_s^{(k)}))\, \mathrm{d}s.$$

By Fubini's theorem, this is equivalent to

$$T_t^{(k)} v(x) - v(x) = \int_0^t \int_D p^{(k)}(s, x, y) \nabla \cdot (\kappa^{(k)} \nabla v(y))\, \mathrm{d}y\, \mathrm{d}s,$$

where we have used the superscript "(k)" for the semigroup and transition kernel density, respectively, corresponding to $\kappa^{(k)}$. Multiplication with f, integration over ∂D and another change of the orders of integration finally yield

$$\int_{\partial D} f(y)\big(T_t^{(k)} v(y) - v(y)\big)\, \mathrm{d}\sigma(y) = \langle u_t^{(k)}, \nabla \cdot (\kappa^{(k)} \nabla v)\rangle.$$

Since $u_t^{(k)} \to u^{(k)}$ and $T_t^{(k)} v \to |D|^{-1} \int_D v\, \mathrm{d}x$, both uniformly on $\overline{D}$, as $t \to \infty$, we obtain (30).

Let us prove the third step, that is, the convergence of the sequence $(u^{(k)})_{k\in\mathbb{N}}$ as $k \to \infty$ toward $u \in H_\diamond^1(D)$, the unique solution to (1), (3). Consider the variational form of the Neumann problem for $u^{(k)} \in H_\diamond^1(D)$, that is, $\mathcal{E}^{(k)}(u^{(k)}, v) = \langle f, v\rangle_{\partial D}$ for all $v \in H_\diamond^1(D)$. As $u^{(k)} \in H_\diamond^1(D)$ for all $k \in \mathbb{N}$, we have by the Poincaré inequality that the sequence $(u^{(k)})_{k\in\mathbb{N}}$ is bounded in $H_\diamond^1(D)$. Therefore, by weak compactness, we may without loss of generality assume that $u^{(k)} \rightharpoonup \tilde{u}$ in $H_\diamond^1(D)$, as $k \to \infty$. Hence, it remains to show the convergence of the flows

$$\kappa^{(k)} \nabla u^{(k)} \rightharpoonup \kappa \nabla \tilde{u} \qquad \text{in } L^2(D; \mathbb{R}^d), \qquad \text{as } k \to \infty.$$

Since $u^{(k)} \rightharpoonup \tilde{u}$, this is equivalent to showing that

$$\lim_{k\to\infty} \langle (\kappa^{(k)} - \kappa) \nabla u^{(k)}, \phi\rangle = 0$$

for every $\phi \in L^2(D; \mathbb{R}^d)$. By Egorov's theorem, we have for every $\varepsilon > 0$ a set $D_\varepsilon \subset D$ such that $\kappa^{(k)}$ converges uniformly to κ on D_ε and the Lebesgue measure of the complement is $|D \setminus D_\varepsilon| < \varepsilon$. Since $\|\nabla u^{(k)}\|_2 \le M$ is uniformly bounded by the Poincaré inequality, we obtain with the Cauchy–Schwarz inequality that

$$|\langle (\kappa^{(k)} - \kappa) \nabla u^{(k)}, \phi\rangle| \le 2M \|\phi\|_2 \sqrt{\varepsilon}$$

for large enough k. Hence, we deduce the convergence of the flows and we may pass to the limit in the variational formulation to see that $\tilde{u} \equiv u$ is the unique solution to the Neumann problem.

The last step follows by [47], Lemma 2.2, together with the Hölder continuity up to the boundary of both $p^{(k)}$, $k \in \mathbb{N}$, and p. These imply that for fixed $x \in \overline{D}$, $p^{(k)}(\cdot, x, \cdot) \to p(\cdot, x, \cdot)$, as $k \to \infty$, uniformly on compacts in $(0, T] \times \overline{D}$ for all $T > 0$. Therefore, by (29) we have that $u^{(k)} \to u_\infty$ pointwise, as $k \to \infty$, and the claim follows. □

REMARK 5.5. Note that the regularization technique we employed in the proof of Theorem 5.4 may be easily modified to prove the Feynman–Kac formula

$$u(x) = \mathbb{E}_x \phi(X_{\tau(D)}), \qquad x \in D$$

for the conductivity equation (1) with Dirichlet boundary condition $u|_{\partial D} = \phi$, where $\phi \in H^{1/2}(D)$ and

$$\tau(D) := \inf\{t \geq 0 : X_t \in \mathbb{R}^d \setminus D\}$$

denotes the *first exit time* from the domain D. Such a proof requires the fact that

$$X^{(k)}_{\cdot \wedge \tau^{(k)}(D)} \to X_{\cdot \wedge \tau(D)} \qquad \text{in law on } C\big([0,\infty); \overline{D}\big)$$

as $k \to \infty$ for every $x \in D$. This follows from the Lipschitz property of ∂D, implying that all points of ∂D are *regular* for the reflecting diffusion process X in the sense of [30], Chapter 4.2. Indeed, it is well known that every Lipschitz domain is regular for standard Brownian motion; this property holds more generally for domains that satisfy the exterior cone condition at every point. For the reflecting diffusion process, an analogous result follows from a reflection argument (as in proof of Proposition 4.2) in the neighborhood of a given point $x \in \partial D$ to extend the process to a small neighborhood which now contains x as an interior point. We may then use an Aronson-type lower bound for the so-obtained transition kernel density $\hat{p}$ which is similar to the upper bound we deduced in Proposition 4.2; see [53], Chapter 4.2. This allows transferring the usual proof for sufficiency of the cone condition for regularity of Brownian motion to the extended process $\overline{X}$. Therefore, $x \in \partial D$ is regular for $\overline{X}$, and hence also for the original reflecting diffusion.

A slight modification of the arguments from above yields the following result which is in fact a corollary rather to the proof of Theorem 5.4 than to its actual statement.

COROLLARY 5.6. *Let f be a bounded Borel function and let $\alpha > 0$. Then there is a unique weak solution $u \in C(\overline{D}) \cap H^1_\diamond(D)$ to the boundary value problem*

$$\nabla \cdot (\kappa \nabla u) - \alpha u = 0 \qquad \text{in } D$$

subject to the boundary condition (3). *This solution admits the Feynman–Kac representation*

$$u(x) = \mathbb{E}_x \int_0^\infty e^{-\alpha t} f(X_t)\,\mathrm{d}L_t \qquad \text{for all } x \in \overline{D}. \tag{31}$$

PROOF. Repeat the proof of Theorem 5.4, however, substituting $\{T_t, t \geq 0\}$ with the *Feynman–Kac semigroup* $\{\widetilde{T}_t, t \geq 0\}$, $\widetilde{T}_t v(x) := \mathbb{E}_x e^{-\alpha t} v(X_t)$. Note that in contrast to the Neumann problem without the zero-order term, the *gauge* $\mathbb{E}_x \int_0^\infty e^{-t}\,\mathrm{d}L_t$ is finite $\mathbb{P}_x$-a.s. for every $x \in \overline{D}$. □

5.2. *Complete electrode model*. The main result for the complete electrode model (1), (5) is the following theorem.

THEOREM 5.7. *For given functions f, g defined by (6) and a voltage pattern $U \in \mathbb{R}^N$ satisfying (7), there is a unique weak solution $u \in C(\overline{D}) \cap H^1(D)$ to the boundary value problem (1), (5). This solution admits the Feynman–Kac representation*

$$u(x) = \mathbb{E}_x \int_0^\infty e_g(t) f(X_t)\,\mathrm{d}L_t \qquad \text{for all } x \in \overline{D}, \tag{32}$$

with the gauge function

$$e_g(t) := \exp\left(-\int_0^t g(X_s)\,\mathrm{d}L_s\right), \qquad t \geq 0. \tag{33}$$

Before we are ready to give a proof of Theorem 5.7, let us introduce the *Feynman–Kac semigroup* of the complete electrode model, that is, the one-parameter family of operators $\{T_t^g, t \geq 0\}$ defined by

$$T_t^g v(x) := \mathbb{E}_x e_g(t) v(X_t), \qquad x \in \overline{D} \text{ and } t \geq 0. \tag{34}$$

Moreover, let us define the *perturbed* Dirichlet form $(\mathcal{E}^g, \mathcal{D}(\mathcal{E}^g))$ by a perturbation of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with the measure $g \cdot \sigma$, that is,

$$\mathcal{E}^g(v, w) := \mathcal{E}(v, w) + \langle gv, w\rangle_{\partial D}, \qquad v, w \in \mathcal{D}(\mathcal{E}^g), \tag{35}$$

where $\mathcal{D}(\mathcal{E}^g) = H^1(D)$ by the standard trace theorem.

The following lemma yields a continuity result *up to the boundary* which is not so easy to obtain by standard regularity arguments using Sobolev embeddings.

LEMMA 5.8. *Let the function u be defined by the Feynman–Kac formula (32), then $u \in C(\overline{D})$.*

PROOF. Let us define a $\mathbb{P}_x$-martingale by

$$\mathbb{E}_x\left\{\int_0^\infty e_g(s) f(X_s)\,\mathrm{d}L_s \middle| \mathcal{F}_t\right\} = \int_0^t e_g(s) f(X_s)\,\mathrm{d}L_s + e_g(t) u(X_t),$$

where the right-hand side is obtained using the Markov property of X together with the fact that the gauge function e_g is a multiplicative functional of X. Obviously,

$$e_g(t) u(X_t) - u(x) + \int_0^t e_g(s) f(X_s)\,\mathrm{d}L_s$$

is a $\mathbb{P}_x$-martingale as well, and hence we have for all $0 \leq s \leq t$

$$e_g(s) u(X_s) = e_g(s)\mathbb{E}_{X_s} e_g(t-s) u(X_{t-s}) + e_g(s)\mathbb{E}_{X_s} \int_0^{t-s} e_g(r) f(X_r)\,\mathrm{d}L_r.$$

Setting $s = 0$ thus yields

$$u(x) = T_t^g u(x) + \mathbb{E}_x \int_0^t e_g(r) f(X_r)\,\mathrm{d}L_r \qquad \text{for all } t \geq 0. \tag{36}$$

One can prove that the Feynman–Kac semigroup $\{T_t^g, t \geq 0\}$ is a strong Feller semigroup on $L^2(D)$ (see, e.g., [49], Proposition 2.17), in particular $T_t^g u$ is continuous on $\overline{D}$. To prove that u is continuous on $\overline{D}$, it is sufficient to show that the second term on the right-hand side of (36) tends to zero uniformly in x, as $t \to 0$. This is, however, clear since we may estimate

$$\sup_{x\in\overline{D}} \left\{ \mathbb{E}_x \int_0^t e_g(s) f(X_s)\,\mathrm{d}L_s \right\} \leq z^{-1} \max_{l=1,\dots,N} \{U_l\} \sup_{x\in\overline{D}} \{\mathbb{E}_x L_t\},$$

where the right-hand side tends to zero as $t \to 0$ by Lemma 5.3. □

The following lemma yields a semimartingale decomposition for the composite process $u(X_t)$ which compensates for the lack of Itô's formula in the proof of Theorem 5.7.

LEMMA 5.9. *Let $u \in H^1(D)$ denote the weak solution of the boundary value problem* (1), (5). *Then for all $t \geq 0$*

$$u(X_t) = u(x) + M^u - \int_0^t f(X_s)\,\mathrm{d}L_s + \int_0^t g(X_s) u(X_s)\,\mathrm{d}L_s, \tag{37}$$

$\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$, where M^u is the martingale additive functional given by Lemma A.3.

PROOF. The Fukushima decomposition from Lemma A.3 gives

$$u(X_t) = u(x) + M_t^u + N_t^u$$

$\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$. Therefore, we only need to show that N^u has the claimed representation. We exploit the fact that the identity (34) implies that there exists a nonconservative Hunt process X^g associated with $(\mathcal{E}^g, H^1(D))$ which is related to X by a random time change, namely

$$X_s^g = \begin{cases} X_s, & s < \zeta^g, \\ \partial, & s \geq \zeta^g, \end{cases}$$

where the lifetime ζ^g is given by

$$\zeta^g := \zeta^{g,Z}, \qquad \text{where } \zeta^{g,\alpha} := \inf\left\{ t : \int_0^t g(X_s)\,\mathrm{d}L_s > \alpha \right\}$$

and the random variable Z is exponentially distributed with parameter 1 and is independent of the σ-algebra generated by the process X. The Fukushima decomposition from Lemma A.3 applied to X^g gives

$$u(X_t^g) = u(x) + M_t^{g,u} + N_t^{g,u}$$

$\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$.

In order to prove the claim, we study the relation between the continuous additive functionals N^v and $N^{g,v}$. We will first show that the identity

$$N_t^v = N_t^{g,v} + \int_0^t v(X_s^g)g(X_s^g)\,\mathrm{d}L_s \qquad \text{for all } t < \zeta^g \tag{38}$$

holds for every $v \in H^1(D)$ $\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$. Then we verify that

$$N_t^{g,u} = -\int_0^t f(X_s^g)\,\mathrm{d}L_s, \qquad \mathbb{P}_x\text{-a.s. for q.e. } x \in \overline{D} \tag{39}$$

for the weak solution $u \in H^1(D)$ to the boundary value problem (1), (5). Therefore,

$$N_t^u = -\int_0^t f(X_s)\,\mathrm{d}L_s + \int_0^t u(X_s)g(X_s)\,\mathrm{d}L_s \qquad \text{for all } t < \zeta^g \tag{40}$$

holds $\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$. All in all, after showing (38) and (39) we have

$$[t < \zeta^g]\left(u(X_t) - u(x) - M_t^u + \int_0^t \bigl(f(X_s) - g(X_s)u(X_s)\bigr)\,\mathrm{d}L_s\right) = 0 \tag{41}$$

for every $t \geq 0$ $\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$. By independence of Z and $\mathcal{F}_t$, we obtain that $\mathbb{P}_x$-a.s. it holds that

$$\mathbb{E}_x\{[t < \zeta^g]|\mathcal{F}_t\} = \int_0^\infty e^{-\alpha}[t < \zeta^{g,\alpha}]\,\mathrm{d}\alpha \geq \int_{c(t,\omega)}^\infty e^{-\alpha}\,\mathrm{d}\alpha > 0,$$

where the existence of $c(t,\omega) < \infty$ follows from the fact that $\zeta^{g,\alpha} \uparrow \infty$ for fixed ω. This implies that the claim follows by taking the conditional expectation of the identity (41) with respect to $\mathcal{F}_t$.

Let $\{G_\alpha, \alpha > 0\}$ denote the resolvent on $L^2(D)$ corresponding to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ given by the Laplace transform

$$G_\alpha v = \int_0^\infty e^{-\alpha t} T_t v\,\mathrm{d}t,$$

and let $\{G_\alpha^g, \alpha > 0\}$ be defined analogously for the perturbed Dirichlet form $(\mathcal{E}^g, \mathcal{D}(\mathcal{E}^g))$. We will verify the identity (38) by first showing it for every $v \in G_1^g(L^2(D)) \subset H^1(D)$ and then for every $v \in H^1(D)$.

When $v \in G_1^g(L^2(D))$, say, when $v = G_1^g(\phi)$ for some $\phi \in L^2(D)$, then by the resolvent property (cf. [22], Lemma 1.3.3), and the definition of $\mathcal{E}_1^g$ we have that

$$\mathcal{E}^g(v,w) = \mathcal{E}_1^g(G_1^g\phi, w) - \langle v,w\rangle = \langle \phi, w\rangle - \langle v, w\rangle. \tag{42}$$

Thus, the Fukushima decomposition from Lemma A.3 and the Revuz correspondence from Lemma A.2 imply that

$$N_t^{g,v} = -\int_0^t \bigl(\phi(X_s^g) - v(X_s^g)\bigr)\,\mathrm{d}s$$

$\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$. Similarly,

$$\mathcal{E}(G_1\phi, w) = \langle \phi - G_1\phi, w\rangle \quad \text{and} \quad N_t^{G_1\phi} = -\int_0^t \bigl(\phi(X_s) - G_1\phi(X_s)\bigr)\,\mathrm{d}s$$

holds $\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$. The resolvent property [22], Lemma 1.3.3, and the definition of the perturbed Dirichlet form give that

$$\mathcal{E}_1\bigl(G_1\phi - G_1^g\phi, w\bigr) = \mathcal{E}_1(G_1\phi, w) - \mathcal{E}_1^g\bigl(G_1^g\phi, w\bigr) + \bigl\langle gG_1^g\phi, w\bigr\rangle_{\partial D} = \langle gv, w\rangle_{\partial D}.$$

Moreover, the previous identity is equivalent to

$$\mathcal{E}(G_1\phi - v, w) = \langle gv, w\rangle_{\partial D} - \langle G_1\phi - v, w\rangle$$

and by linearity, we obtain thus

$$\mathcal{E}(v, w) = \mathcal{E}(G_1\phi, w) - \mathcal{E}(G_1\phi - v, w) = \langle \phi - v, w\rangle + \langle gv, w\rangle_{\partial D}.$$

Invoking both the Revuz correspondence and the Fukushima decomposition once more, we see that

$$N_t^v = -\int_0^t \bigl(\phi(X_s) - v(X_s)\bigr)\,\mathrm{d}s + \int_0^t v(X_s)g(X_s)\,\mathrm{d}L_s \tag{43}$$

and since X^g is related to X by a random time change we obtain the identity (38) for $v \in G_1^g(L^2(D))$. This equality may be generalized to the case of an arbitrary $v \in H^1(D)$ not necessarily in the range of the resolvent using an approximation argument. Namely, we consider the sequence $(v^{(k)})_{k\in\mathbb{N}}$ with $v^{(k)} := kG_{k+1}^g v = G_1^g\phi^{(k)}$, $\phi^{(k)} := k(v - kG_{k+1}^g v)$. Then $v^{(k)} \in G_1^g(L^2(D))$ for all $k \in \mathbb{N}$ and the sequence $(v^{(k)})_{k\in\mathbb{N}}$ satisfies both,

$$\lim_{k\to\infty} \mathcal{E}^g\bigl(v^{(k)} - v, v^{(k)} - v\bigr) = 0 \quad \text{and} \quad \lim_{k\to\infty} \mathcal{E}\bigl(v^{(k)} - v, v^{(k)} - v\bigr) = 0$$

so that by [22], Corollary 5.2.1, there exists a subsequence, for convenience still denoted $(v^{(k)})_{k\in\mathbb{N}}$, such that $v^{(k)}(X_t^g) \to v(X_t^g)$, $N_t^{g,v^{(k)}} \to N_t^{g,v}$ and $N_t^{v^{(k)}} \to N_t^v$ uniformly on each finite time interval, $\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$. In particular, it follows that (38) holds for arbitrary $v \in H^1(D)$.

The last step is to verify the identity (39). Since u solves the boundary value problem (1), (5) we have that

$$\mathcal{E}^g(u, v) = \langle f, v\rangle_{\partial D} \qquad \text{for all } v \in H^1(D) \cap C(\overline{D}).$$

Since the perturbed Dirichlet form $(\mathcal{E}^g, H^1(D))$ is regular, we may apply Lemmata A.3 and A.2 and then the identity (39) follows. This proves the claim. □

PROOF OF THEOREM 5.7. There exists a weak solution $u \in H^1(D)$ of the boundary value problem (1), (5) so that with regard to Lemma 5.8, it remains to show that this weak solution u admits the Feynman–Kac representation (32). Note first that the gauge

$$\mathbb{E}_x \int_0^\infty e_g(t)\,\mathrm{d}L_t \tag{44}$$

is finite $\mathbb{P}_x$-a.s. for every $x \in \overline{D}$, hence the expression on the right-hand side of (32) is well-defined. The finiteness of the gauge follows along the lines of Papanicolaou [43]. Indeed, it is not difficult to verify that a version of the gauge theorem developed there also holds in our situation, that is, either the gauge is infinite or it is bounded a.s. If it is infinite, then $T_t^g 1 = 1$ for every $t > 0$ which in turn means that

$$0 = \int_0^t g(X_s)\,\mathrm{d}L_s \geq c_1^{-1} \int_0^t \Big[X_s \in \bigcup E_l\Big]\,\mathrm{d}L_s$$

for every $t > 0$ almost surely. However, this would only be possible if the electrodes were polar for the reflecting diffusion process X for a.e. $x \in \overline{D}$.

Lemma 5.9 yields the semimartingale decomposition

$$u(X_t) = u(x) + M_t^u - \int_0^t f(X_s)\,\mathrm{d}L_s + \int_0^t g(X_s)u(X_s)\,\mathrm{d}L_s,$$

$\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$. Since the gauge function e_g is continuous, adapted to $\{\mathcal{F}_t, t \geq 0\}$ and of bounded variation we may use integration by parts and we thus obtain for q.e. $x \in \overline{D}$ the identity

$$u(X_t)e_g(t) = u(x) + \mathcal{M}_t + \int_0^t e_g(s) f(X_s)\,\mathrm{d}L_s,$$

$\mathbb{P}_x$-a.s., where $\mathcal{M}$ is a local $\mathbb{P}_x$-martingale. That is, there exists a sequence $(\tau_k)_{k\in\mathbb{N}}$ of stopping times such that $\tau_k \uparrow \infty$ and that $\mathcal{M}_{\tau_k \wedge \cdot}$ is a $\mathbb{P}_x$-martingale. Hence, by taking expectations and letting $k \to \infty$ we get

$$u(x) = \mathbb{E}_x \int_0^t e_g(s) f(X_s)\,\mathrm{d}L_s + \mathbb{E}_x u(X_t)e_g(t) \qquad \text{for q.e. } x \in \overline{D}.$$

Fatou's lemma implies that $T_1^g u$ is essentially bounded, and hence, the finiteness of the gauge implies that $\|T_{t+1}^g u\|_\infty = \|T_t^g T_1^g u\|_\infty \to 0$, as $t \to \infty$. Therefore, together with dominated convergence, the boundedness of f and the finiteness of the gauge imply that

$$u(x) = \mathbb{E}_x \int_0^\infty e_g(t) f(X_t)\,\mathrm{d}L_t \qquad \text{for q.e. } x \in \overline{D}.$$

The right-hand side is continuous in $C(\overline{D})$ by Lemma 5.8 and, therefore, the claim follows. □

REMARK 5.10. Note that the technique we used to prove Theorem 5.7 fails for the Neumann problem corresponding to the continuum model. This comes from the fact that in this case the gauge (44) becomes infinite. For the same reason Theorem 1.2 from [12], specialized to a zero lower-order term, does not yield the desired Feynman–Kac formula for the continuum model either.

5.3. *Mixed boundary value problem.* Now we can directly deduce the desired Feynman–Kac formula for the mixed boundary value problem corresponding to the stochastic problem introduced in Section 3.2. Recall that in this setting ∂D consists of two disjoint parts $\partial_1 D$ and $\partial_2 D$ and that measurements can be taken only on the *accessible boundary* $\partial_1 D$ while the electric potential vanishes on the *inaccessible boundary* $\partial_2 D$. The deterministic EIT forward problem for the complete electrode model is then given by the conductivity equation (1) subject to the mixed boundary conditions

$$\begin{aligned} \kappa\nu\cdot\nabla u|_{\partial D} + gu|_{\partial D} &= f \qquad \text{on } \partial_1 D,\\ u|_{\partial D} &= 0 \qquad \text{on } \partial_2 D. \end{aligned} \tag{45}$$

The following result is a corollary to the line of arguments that led to the proof of Theorem 5.7 rather than to its actual statement.

COROLLARY 5.11. *For given functions f, g defined by* (6) *and a voltage pattern $U \in \mathbb{R}^N$ satisfying* (7), *there is a unique weak solution $u \in C(\overline{D}) \cap H^1(D)$ to the boundary value problem* (1), (45). *This solution admits the Feynman–Kac representation*

$$u(x) = \mathbb{E}_x \int_0^\tau e_g(t) f(X_t)\,\mathrm{d}L_t \qquad \text{for all } x \in \overline{D}, \tag{46}$$

where $\tau := \inf\{t \geq 0 : X_t \in \partial_2 D\}$ denotes the first hitting time of $\partial_2 D$.

PROOF. Repeat the computations from Section 5.2 with the Feynman–Kac semigroup $\{\widetilde{T}_t^g, t \geq 0\}$, where $\widetilde{T}_t^g v(x) := \mathbb{E}_x\{[t \leq \tau] e_g(t) v(X_t)\}$ instead of $\{T_t^g, t \geq 0\}$. □

6. Stochastic homogenization. In this section, we are going to employ the Feynman–Kac formula from Corollary 5.11 to show that the stochastic EIT forward problem may be homogenized both theoretically and numerically by homogenization of the underlying diffusion process on the whole space $\mathbb{R}^d$. Moreover, we provide a continuum version of a quantitative estimate which has been obtained recently for the discrete random walk in random environment in [15, 24].

6.1. *Preliminaries*. For convenience of the reader, let us recall some standard concepts from homogenization theory. Let $\phi := (\phi_1, \dots, \phi_d)$, $\phi_i \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, $i = 1, \dots, d$, denote a vector field. We say that ϕ is a *gradient field* if for every $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \phi_i \partial_j \psi - \phi_j \partial_i \psi \, \mathrm{d}x = 0, \qquad i, j = 1, \dots, d.$$

Moreover, we say that ϕ is *divergence-free* if for every $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$\sum_{i=1}^d \int_{\mathbb{R}^d} \phi_i \partial_i \psi \, \mathrm{d}x = 0.$$

Now let us consider a conductivity random field $\{\kappa(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$ and let $\{\boldsymbol{\Theta}_x, x \in \mathbb{R}^d\}$ denote the underlying dynamical system which is assumed to satisfy the assumptions (i)–(iv) from Section 3.2. A vector field $\boldsymbol{\phi} \in L^2(\Gamma; \mathbb{R}^d)$ is called a *gradient field*, respectively *divergence-free*, if its realizations $\phi(\cdot, \omega) : \mathbb{R}^d \mapsto \mathbb{R}^d$, $x \mapsto \boldsymbol{\phi}(\boldsymbol{\Theta}_x \omega)$ are gradient fields, respectively divergence-free, for $\mathcal{P}$-a.e. $\omega \in \Gamma$. We define the function spaces

$$L^2_{\mathrm{pot}}(\Gamma) := \{\boldsymbol{\phi} \in L^2(\Gamma; \mathbb{R}^d) : \phi(\cdot, \omega) \text{ is a gradient field } \mathcal{P}\text{-a.s.}\},$$
$$L^2_{\mathrm{sol}}(\Gamma) := \{\boldsymbol{\phi} \in L^2(\Gamma; \mathbb{R}^d) : \phi(\cdot, \omega) \text{ is divergence-free } \mathcal{P}\text{-a.s.}\}.$$

If $\boldsymbol{\phi} \in L^2_{\mathrm{pot}}(\Gamma)$, we can find a function $\eta : \mathbb{R}^d \times \Gamma \to \mathbb{R}$ such that $\eta(\cdot, \omega) \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$\nabla \eta(\cdot, \omega) = \boldsymbol{\phi}(\boldsymbol{\Theta}.\omega) \qquad \text{a.e. in } \mathbb{R}^d \text{ for } \mathcal{P}\text{-a.e. } \omega \in \Gamma. \tag{47}$$

In particular, (47) defines a stationary random field with respect to the measure $\mathcal{P}$. We call η the *potential* corresponding to $\boldsymbol{\phi}$.

REMARK 6.1. Note that while $\boldsymbol{\phi} \in L^2_{\mathrm{pot}}(\Gamma)$ implies that $\nabla\eta$ is a stationary random field, it does not imply that η is a stationary random field with respect to $\mathcal{P}$. In fact, it can be shown that this is not true for $d = 1$. For example, consider $\Gamma = [0, 1]$ with Borel sets and $\mathcal{P}$ as the Lebesgue measure. Then $\boldsymbol{\phi}(x) = [x \in (0, \frac{1}{2})]$ is in $L^2_{\mathrm{pot}}(\Gamma)$ and the corresponding $\nabla\eta = \eta'$ is a stationary random field, while η which is an integral function of η' is not.

We define another function space

$$\mathcal{V}^2_{\mathrm{pot}} := \{\boldsymbol{\phi} \in L^2_{\mathrm{pot}}(\Gamma) : \mathbb{M}\boldsymbol{\phi} = 0\},$$

so that one obtains an orthogonal Weyl decomposition of $L^2(\Gamma; \mathbb{R}^d)$, namely

$$L^2(\Gamma; \mathbb{R}^d) = \mathcal{V}^2_{\mathrm{pot}}(\Gamma) \oplus L^2_{\mathrm{sol}}(\Gamma);$$

cf., for example, [28]. Let $\xi \in \mathbb{R}^d$ denote a direction vector, that is, $|\xi| = 1$. The so-called *auxiliary problem* for the direction ξ reads as follows: Find $\boldsymbol{\chi}^\xi \in \mathcal{V}^2_{\mathrm{pot}}(\Gamma)$ such that $\boldsymbol{\kappa}(\xi + \boldsymbol{\chi}^\xi) \in L^2_{\mathrm{sol}}(\Gamma)$ or equivalently,

$$\mathbb{M}\{\boldsymbol{\kappa}(\xi + \boldsymbol{\chi}^\xi) \cdot \boldsymbol{\phi}\} = 0 \qquad \text{for all } \boldsymbol{\phi} \in \mathcal{V}^2_{\mathrm{pot}}(\Gamma). \tag{48}$$

For a proof of existence and uniqueness of the solution to the auxiliary problem, we refer the reader to the seminal paper [42] by Papanicolaou and Varadhan.

We can now bring the underlying diffusion processes evolving in the random medium into play by recalling a stochastic homogenization result which was obtained by Lejay [36]. Let $\{\kappa_\varepsilon(x,\omega), (x,\omega) \in \mathbb{R}^d \times \Gamma\}$ denote the scaled conductivity random field (see Section 3.2) and let $X^{\omega,\varepsilon}$ denote the diffusion process on $\mathbb{R}^d$ which is associated with the regular symmetric Dirichlet form

$$\mathcal{E}^{\omega,\varepsilon}(v,w) := \int_{\mathbb{R}^d} \kappa_\varepsilon(\cdot,\omega)\nabla v \cdot \nabla w \,\mathrm{d}x, \qquad v,w \in \mathcal{D}(\mathcal{E}^{\omega,\varepsilon}) := H^1(\mathbb{R}^d)$$

on $L^2(\mathbb{R}^d)$. It has been shown in [36] that, under assumption (A1) from Section 3.2, for $\mathcal{P}$-a.e. $\omega \in \Gamma$

$$X^{\omega,\varepsilon}_\cdot \to X^*_\cdot \qquad \text{in law on } C([0,\infty);\mathbb{R}^d) \text{ as } \varepsilon \to 0, \tag{49}$$

where X^* denotes the diffusion process on $\mathbb{R}^d$ which is associated with the *homogenized Dirichlet form*

$$\mathcal{E}^*(v,w) := \int_{\mathbb{R}^d} \kappa^*\nabla v \cdot \nabla w \,\mathrm{d}x, \qquad v,w \in \mathcal{D}(\mathcal{E}^*) := H^1(\mathbb{R}^d)$$

on $L^2(\mathbb{R}^d)$ and the constant, symmetric and positive definite matrix κ^* satisfies the equation

$$\xi \cdot \kappa^* \xi = \mathbb{M}\{(\xi + \boldsymbol{\chi}^\xi) \cdot \boldsymbol{\kappa}(\xi + \boldsymbol{\chi}^\xi)\}, \tag{50}$$

where $\boldsymbol{\chi}^\xi \in \mathcal{V}^2_{\mathrm{pot}}(\Gamma)$ denotes the solution to the auxiliary problem (48) for the direction $\xi \in \mathbb{R}^d$.

6.2. *Homogenization of the EIT forward problem.* The following theorem is our main result. Its proof relies on an invariance principle for reflecting diffusion processes obtained recently by Chen, Croydon and Kumagai [11] and the Feynman–Kac formula (46) from Corollary 5.11.

THEOREM 6.2. *Let* $\{\kappa_\varepsilon(x,\omega), (x,\omega) \in \mathbb{R}^d \times \Gamma\}$ *be a stationary random field satisfying assumption* (A1) *from Section* 3.2 *and assume that the trajectories satisfy* $\kappa(\cdot,\omega) \in C^{0,1}_{\mathrm{loc}}(\overline{D};\mathbb{R}^{d\times d})$ *or* $\kappa(\cdot,\omega)$ *piecewise constant for* $\mathcal{P}$*-a.e.* $\omega \in \Gamma$; *let* $\Sigma = \varnothing$. *Then, for a given voltage pattern* $U \in \mathbb{R}^N$, *we have for the potentials in the stochastic boundary value problem* (11), (12)

$$u_\varepsilon(x,\omega) \to u^*(x), \qquad x \in \overline{D} \text{ for } \mathcal{P}\text{-a.e. } \omega \in \Gamma, \text{ as } \varepsilon \to 0, \tag{51}$$

and the corresponding electrode currents satisfy

$$\lim_{\varepsilon\to 0} J_l(\varepsilon,\omega) = \frac{1}{|E_l|}\int_{E_l} \kappa^* \nu \cdot \nabla u^*(x)|_{\partial_1 D}\, \mathrm{d}\sigma(x) \qquad \text{for } \mathcal{P}\text{-a.e. } \omega\in\Gamma, \tag{52}$$

$l = 1,\dots,N$, *where the function* $u^* \in H_0^1(D\cup\partial_1 D)\cap C(\overline{D})$ *is the unique solution to the deterministic forward problem*

$$\nabla\cdot\big(\kappa^*\nabla u^*\big) = 0 \qquad \text{in } D \tag{53}$$

subject to the boundary conditions

$$\begin{aligned} \kappa^*\nu\cdot\nabla u^*|_{\partial_1 D} + g u^*|_{\partial_1 D} &= f \qquad \text{on } \partial_1 D,\\ u^*|_{\partial_2 D} &= 0 \qquad \text{on } \partial_2 D \end{aligned} \tag{54}$$

with a constant, *symmetric and positive definite matrix* κ^* *given by* (50).

PROOF. The proof consists of two top level parts, namely the proofs of the limit relations (51) and (52). Let us subdivide the first part into the following steps: Let X^* denote the reflecting diffusion process on the half-space associated with the homogenized regular symmetric Dirichlet form $(\mathcal{E}^*, H^1(\mathbb{R}^d_-\cup\mathbb{R}^{d-1}))$ on $L^2(\mathbb{R}^d_-\cup\mathbb{R}^{d-1})$. The constant, symmetric and positive definite matrix κ^* is given by (50). Moreover, let $(\varepsilon_k)_{k\in\mathbb{N}}$ be an arbitrary monotone decreasing null sequence and let $X^{\omega,\varepsilon}$ denote the reflecting diffusion process on the half-space corresponding to the regular symmetric Dirichlet form $(\mathcal{E}^{\omega,\varepsilon}, H^1(\mathbb{R}^d_-\cup\mathbb{R}^{d-1}))$ on $L^2(\mathbb{R}^d_-\cup\mathbb{R}^{d-1})$. We show that for $\mathcal{P}$-a.e. $\omega\in\Gamma$ and for a.e. $x\in\overline{D}$ we have for $k\to\infty$:

(i) $X^{\omega,\varepsilon_k}\to X^*$ in law in $C([0,\infty);\mathbb{R}^d_-\cup\mathbb{R}^{d-1})$,

(ii) $X^{\omega,\varepsilon_k}_{\cdot\wedge\tau^{\omega,\varepsilon_k}}\to X^*_{\cdot\wedge\tau^*}$ in law in $C([0,\infty);\mathbb{R}^d_-\cup\mathbb{R}^{d-1})$,

(iii) $(X^{\omega,\varepsilon_k}_{\cdot\wedge\tau^{\omega,\varepsilon_k}}, L^{\omega,\varepsilon_k}_{\cdot\wedge\tau^{\omega,\varepsilon_k}})\to(X^*_{\cdot\wedge\tau^*}, L^*_{\cdot\wedge\tau^*})$ in law in $C([0,\infty);\mathbb{R}^d_-\cup\mathbb{R}^{d-1}\times\mathbb{R}_+)$,

(iv)

$$\limsup_{j\to\infty}\left|\mathbb{E}_x\int_0^{\tau^{\omega,\varepsilon_j}} e_g^{\omega,\varepsilon_j}(t) f\big(X_t^{\omega,\varepsilon_j}\big)\,\mathrm{d}L_t^{\omega,\varepsilon_j} - \mathbb{E}_x\int_0^{\tau^*} e_g^*(t) f\big(X_t^*\big)\,\mathrm{d}L_t^*\right| = 0,$$

(v) $u_{\varepsilon_k}(x,\omega)\to u^*(x)$.

By assumption (A1) from Section 3.2, we can deduce (i) directly from [11], Section 4. In order to prove (ii), let us consider the functional $F: C([0,\infty);\mathbb{R}^d_-\cup\mathbb{R}^{d-1})\to[0,\infty]$ and a related mapping Φ on continuous functions, namely

$$F(\phi) := \inf\{t\ge 0 : |\phi(t)| = R\} \quad\text{and}\quad \Phi(\phi) := t\mapsto \phi\big(t\wedge F(\phi)\big).$$

Note that Φ maps the process without stopping to the stopped variant. Moreover, note that the part (ii) of the claim follows from the continuous mapping theorem, cf. [7] if Φ is continuous up to a negligible set with respect to the measures $\mathbb{P}_x^{\omega,\varepsilon_k}$ and $\mathbb{P}_x^*$. This, in turn, follows if F is continuous up to a negligible set.

Let ϕ be a fixed continuous function and let $(\phi_k)_{k\in\mathbb{N}}$ be a sequence of continuous functions that converges uniformly toward ϕ on compacts in $[0,\infty)$. If $F(\phi)=\infty$, the continuity of F at ϕ follows by compactness. Without a loss of generality, we may therefore assume that $F(\phi)<\infty$ and $\sup F(\phi_k)<\infty$. We have

$$\left|\phi(\lambda)-\phi_k\big(F(\phi_k)\big)\right| \le \left|\phi(\lambda)-\phi\big(F(\phi_k)\big)\right| + \left|\phi\big(F(\phi_k)\big)-\phi_k\big(F(\phi_k)\big)\right|$$

for every $k\in\mathbb{N}$ where we denoted $\lambda := \liminf_{j\to\infty} F(\phi_j)$. Note that the right-hand side vanishes as $k\to\infty$ for a suitable subsequence and, therefore, by closedness of $\partial_2 D$, we have that $\phi(\lambda)$ is on $\partial_2 D$ which implies that F is lower semi-continuous. Note further that semi-continuity of F implies that if ϕ is a discontinuity point of F, then there exists a $\delta>0$ such that $\phi(t)\in\overline{D}$ for all $t\in[F(\phi),\liminf_{k\to\infty}F(\phi_k))$. However, the boundary ∂D is regular in the sense of [30], Chapter 4.2[3] for X^* as well as for all X^{ω,ε_k} and, therefore, the part (ii) follows.

In order to prove (iii), we employ the fact that for every $y\in\overline{D}$ and every $\rho>0$ the transition kernel densities corresponding to the processes $X^{\omega,\varepsilon_k}_{\cdot\wedge\tau^{\omega,\varepsilon_k}}$ and $X^*_{\cdot\wedge\tau^*}$, respectively, satisfy for $k\to\infty$

$$p^{(k)}(\cdot,\cdot,y)\to p^*(\cdot,\cdot,y) \qquad \text{uniformly on compact subsets in } (0,\rho]\times\overline{D}.$$

This is a consequence of the convergence (ii), a reflection argument, the De Giorgi–Nash–Moser theorem and the Arzela–Ascoli theorem: Let $\hat{p}^{(k)}$ and $\hat{p}^*$ denote the transition kernel densities that are obtained by reflection with respect to x^d-axis. That is, they are defined in $\overline{B}(0,R)$ with killing boundary on the sphere $S(0,R)$. Let $\delta>0$, then $\hat{p}^{(k)}(\cdot,x,\cdot)$ [resp., $\hat{p}^*(\cdot,x,\cdot)$] as mappings on $(\delta,\rho]\times\overline{B}(0,R)$ are solutions to heat equations equipped with Dirichlet boundary conditions corresponding to the extended coefficients $\hat{\kappa}^{\omega}_{\varepsilon_k}$ (resp., $\hat{\kappa}^*$) and initial values $\hat{p}^{(k)}(\cdot,\delta,\cdot)$ [resp., $\hat{p}^*(\cdot,\delta,\cdot)$].

Therefore, by the De Giorgi–Nash–Moser theorem $\{\hat{p}^{(k)}(\cdot,x,\cdot)\}$ is equicontinuous and uniformly bounded in the open ball $B(0,r)$ for $r<R$, since we know that the conductivities satisfy assumption (A1). Hence, by Arzela–Ascoli, every subsequence of $(\hat{p}^{(k)}(\cdot,x,\cdot))_{k\in\mathbb{N}}$ has a subsequence, say, $(q^{(k)})_{k\in\mathbb{N}}$ that converges uniformly on compact subsets of $(\delta,\rho]\times B(0,r)$, say, to some continuous function q. On the other hand, since we have proved (ii), we may deduce that for the transition kernel densities it holds that $p^{(k)}(t,x,y)\to p^*(t,x,y)$ as $k\to\infty$ for every $t>0$, for every $x\in\overline{D}$ and a.e. $y\in\overline{D}$. This implies, however, that $q(t,x,y)=\hat{p}^*(t,x,y)$ for every $t>\delta$, for every $x\in B(0,r)$ and for a.e. $y\in B(0,r)$. Since both functions $\hat{p}^*$ and q are continuous, we have obtained that $\hat{p}^{(k)}(\cdot,x,\cdot)\to\hat{p}^*(\cdot,x,\cdot)$, as $k\to\infty$, uniformly on compact subsets

[3] See Remark 5.5.

$(\delta, \rho] \times B(0, r)$. Since $\delta > 0$ is arbitrary and $r < R$ is arbitrary, the convergence extends to compact subsets of $(0, \rho] \times B(0, R)$. Moreover, we can extend the convergence to the boundary by the compactness of the closed ball $\overline{B}(0, R)$ and the fact that $\hat{p}^{(k)}(\cdot, x, y) = \hat{p}^*(\cdot, x, y) = 0$ for every $y \in S(0, R)$. Therefore, we obtain that for the original transition kernel densities the required convergence holds on compact subsets of $(\delta, \rho] \times \overline{D}$ since $p^{(k)}$ and p^* are symmetric with respect to x and y.

Now, notice that for every finite collection of times $t^{(1)} < t^{(2)} < \cdots < t^{(n)}$ the finite dimensional marginals satisfy

$$(L^{\omega,\varepsilon_k}_{t_1\wedge\tau^{\omega,\varepsilon_k}}, \dots, L^{\omega,\varepsilon_k}_{t_n\wedge\tau^{\omega,\varepsilon_k}}) \to (L^*_{t_1\wedge\tau^*}, \dots, L^*_{t_n\wedge\tau^*}) \qquad \text{as } k \to \infty \tag{55}$$

which follows by induction from the special case $L^{\omega,\varepsilon_k}_{t\wedge\tau^{\omega,\varepsilon_k}} \to L^*_{t\wedge\tau^*}$ weakly, (i) and the Markov property. This special case in turn follows by the convergence of all the moments $\mathbb{E}_x(L^{\omega,\varepsilon_k}_{t\wedge\tau^{\omega,\varepsilon_k}})^m \to \mathbb{E}_x(L^*_{t\wedge\tau^*})^m$ as $k \to \infty$.

Moreover, for every monotone null sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ and $\mathcal{P}$-a.e. $\omega \in \Gamma$, for a.e. $x \in \overline{D}$ and for every $\phi \in C^2(\overline{D})$ we have that

$$\big(X^{\omega,\varepsilon_k}_{\cdot\wedge\tau^{\omega,\varepsilon_k}}, M^{(k),\phi}, N^{(k),\phi}\big) \to \big(X^*_{\cdot\wedge\tau^*}, M^{*,\phi}, N^{*,\phi}\big) \qquad \text{in law, as } k \to \infty$$

in $C([0, \infty); (\mathbb{R}^d_- \cup \mathbb{R}^{d-1})^3)$, where $(M^{(k),\phi}, N^{(k),\phi})$ and $(M^{*,\phi}, N^{*,\phi})$ correspond to the Fukushima decomposition of $\phi(X^{\omega,\varepsilon_k}_{\cdot\wedge\tau^{\omega,\varepsilon_k}})$ and $\phi(X^*_{\cdot\wedge\tau^*})$, respectively; see Lemma A.3. This follows from combining the Fukushima decomposition and the convergence result for the transition kernel densities with a result by Rozkosz and Słomiński [47], Theorem 6.1. To be precise, with the Fukushima decompositions

$$\phi\big(X^{\omega,\varepsilon_k}_{t\wedge\tau^{\omega,\varepsilon_k}}\big) = \phi\big(X^{\omega,\varepsilon_k}_{0\wedge\tau^{\omega,\varepsilon_k}}\big) + M^{(k),\phi}_t + N^{(k),\phi}_t,$$
$$\phi\big(X^*_{t\wedge\tau^*}\big) = \phi\big(X^*_{0\wedge\tau^*}\big) + M^{*,\phi}_t + N^{*,\phi}_t$$

corresponding to ϕ we have fulfilled all the assumptions of [47], Theorem 6.1, once we have also verified that $\{\phi(X^{\omega,\varepsilon_k}_{\cdot\wedge\tau^{\omega,\varepsilon_k}})\}$ satisfies UTD; see [47], page 170, for the definition of this property. That is, once we have verified that

$$\Big\{\sup_{0\le t\le\rho} N^{(k),\phi}_t\Big\} \qquad \text{is tight}$$

and for every $\varepsilon > 0$ it holds that

$$\lim_{m\to\infty} \sup_{k\ge 1} \mathbb{P}^k_x\bigg(\sum_{t_{n,m}\in\Pi_m} \big(N^{(k),\phi}_{t_{n+1,m}} - N^{(k),\phi}_{t_{n,m}}\big)^2 > \varepsilon\bigg) = 0,$$

where Π_m is an increasing partition of $[0, \rho)$ such that the maximum distance between consecutive points in Π_m goes to zero as $m \to \infty$. It is straightforward to check that (55) and (ii) imply the UTD property. The claim (iii) finally follows since we may first marginalize the martingale part away, and thus,

$$\big(X^{\omega,\varepsilon_k}_{\cdot\wedge\tau^{\omega,\varepsilon_k}}, N^{(k),\phi}\big) \to \big(X^*_{\cdot\wedge\tau^*}, N^{*,\phi}\big) \qquad \text{in law, as } k \to \infty$$

in $C([0,\infty);(\mathbb{R}^d_- \cup \mathbb{R}^{d-1})^2)$. With partition of unity, we can localize the problem to a small neighborhood of a point $x \in \mathbb{R}^d \times \{0\} \cap \overline{D}$ and by approximating the renormalized indicator function of the ε-neighborhood of the boundary close to the point x with $\nabla\phi$ the claim (iii) follows.

With regard to the Feynman–Kac formula (46), the claim (v) follows from claim (iv). We first verify the assertion (iv) for continuous f. We estimate the left-hand side of (iv) with a difference of truncated Riemann sums

$$\left| \sum_{k=0}^{\lfloor N/h \rfloor} \mathbb{E}_x \int_{t_k}^{t_{k+1}} e_g^{\omega,\varepsilon_j}(t_k) f\big(X_{t_k}^{\omega,\varepsilon_j}\big)[t_{k+1} < \tau_j]\, \mathrm{d}L_t^{\omega,\varepsilon_j} \right. \\ \left. - \mathbb{E}_x \int_{t_k}^{t_{k+1}} e_g(t_k) f(X_{t_k})[t_{k+1} < \tau]\, \mathrm{d}L_t \right|,$$

where $t_k = kh$ and h is a step size. The difference $S_{N,h}$ of truncated Riemann sums for fixed N and h goes to zero as $j \to \infty$ if we assume that $(X^{\omega,\varepsilon_j}, L^{\omega,\varepsilon_j}, A^{\omega,\varepsilon_j})$ converge weakly to (X, L, A), where $A^{\omega,\varepsilon_j} := \log e_g^{\omega,\varepsilon_j}$ and $A := \log e_g$. The error terms for X^{ω,ε_j} and X are nearly analogous, so it is enough to consider just X^{ω,ε_j} in detail. The increments[4] are of the following form:

$$\begin{aligned} &\int_{t_k}^{t_{k+1}} \big(e_g^{\omega,\varepsilon_j}(t) f\big(X_t^{\omega,\varepsilon_j}\big) - e_g^{\omega,\varepsilon_j}(t_k) f\big(X_{t_k}^{\omega,\varepsilon_j}\big)\big)\, \mathrm{d}L_t^{\omega,\varepsilon_j} \\ &\quad = \int_{t_k}^{t_{k+1}} \big(e_g^{\omega,\varepsilon_j}(t) - e_g^{\omega,\varepsilon_j}(t_k)\big) f\big(X_t^{\omega,\varepsilon_j}\big)\, \mathrm{d}L_t^{\omega,\varepsilon_j} \\ &\qquad + e_g^{\omega,\varepsilon_j}(t_k) \int_{t_k}^{t_{k+1}} \big(f\big(X_t^{\omega,\varepsilon_j}\big) - f\big(X_{t_k}^{\omega,\varepsilon_j}\big)\big)\, \mathrm{d}L_t^{\omega,\varepsilon_j}. \end{aligned}$$

The latter term can be handled with the continuity of the paths of X together with the uniform continuity of f on the compact set $\overline{D}$. This is since

$$\begin{aligned} &\left| e_g^{\omega,\varepsilon_j}(t_k) \int_{t_k}^{t_{k+1}} \big(f\big(X_t^{\omega,\varepsilon_j}\big) - f\big(X_{t_k}^{\omega,\varepsilon_j}\big)\big)\, \mathrm{d}L_t^{\omega,\varepsilon_j} \right| \\ &\quad \le \big(2\big(1 - \psi_\delta\big(\theta_h\big(X^{\omega,\varepsilon_j}\big)\big)\big)\|f\|_\infty + \theta_{2\delta}(f)\psi_\delta\big(\theta_h\big(X^{\omega,\varepsilon_j}\big)\big)\big)\big(L_{t_{k+1}}^{\omega,\varepsilon_j} - L_{t_k}^{\omega,\varepsilon_j}\big), \end{aligned}$$

where $\theta_\delta(x)$ is the maximum variation of the function x on the interval $[0, N]$

$$\theta_\delta(x) := \sup\{|x(t) - x(s)|; 0 \le t, s \le N, |t - s| < \delta\}$$

and $\psi_\delta(t)$ is the continuous approximation of the indicator function $[|t| < \delta]$ with support in $[-2\delta, 2\delta]$. Therefore, after taking the limit $j \to \infty$, the latter terms give that the corresponding total approximation error can be bounded by

$$4\|f\|_\infty \mathbb{E}_x\{[\theta_h(X) \ge \delta] L_{\tau\wedge N}\} + 2\theta_{2\delta}(f)\mathbb{E}_x\{L_{\tau\wedge N}\}$$

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[4]Excluding the edge case where $t_k < \tau_j < t_{k+1}$ which we will omit but which can be treated in the same way.

which goes to zero for fixed N if we first let $h \to 0$ and then let $\delta \to 0$.

The first terms can be estimated by

$$\int_{t_k}^{t_{k+1}} \big(e_g^{\omega,\varepsilon_j}(t) - e_g^{\omega,\varepsilon_j}(t_k)\big) f\big(X_t^{\omega,\varepsilon_j}\big)\,\mathrm{d}L_t^{\omega,\varepsilon_j} \le \|f\|_\infty \|g\|_\infty \big(L_{t_{k+1}}^{\omega,\varepsilon_j} - L_{t_k}^{\omega,\varepsilon_j}\big)^2$$

since $g \ge 0$. Therefore, after taking $j \to \infty$ the first terms give the total approximation error that is bounded by

$$2\|f\|_\infty \|g\|_\infty \big(\mathbb{E}_x\big\{[\theta_h(L) \ge \delta] L_{\tau\wedge N}\big\} + 2\delta \mathbb{E}_x\{L_{\tau\wedge N}\}\big)$$

which goes to zero for fixed N if we first let $h \to 0$ and then let $\delta \to 0$.

We are still left with the truncation which can, however, be removed since τ_j and τ are a.s. finite and moreover, $L_{\tau_j}^{\omega,\varepsilon_j}$ converges weakly to L_τ as $j \to \infty$ and thus, we get a uniform estimate

$$\begin{aligned}\limsup_{j\to\infty} &\left|\mathbb{E}_x \int_0^{\tau_j} e_g^{\omega,\varepsilon_j}(t) f\big(X_t^{\omega,\varepsilon_j}\big)\,\mathrm{d}L_t^{\omega,\varepsilon_j} - \int_0^\tau e_g(t) f(X_t)\,\mathrm{d}L_t\right| \\ &\le 2\|f\|_\infty \mathbb{E}_x\{L_\tau - L_{\tau\wedge N}\}\end{aligned}$$

which gives the claimed convergence for continuous f by the assumption of weak convergence of A^{ω,ε_j} toward A as $j \to \infty$.

We will next verify the joint weak convergence of the family A^{ω,ε_j} together with $(X^{\omega,\varepsilon_j}, L^{\omega,\varepsilon_j})$. We can assume that $X^{\omega,\varepsilon_j} \to X$ and $L^{\omega,\varepsilon_j} \to L$ almost surely in $C(0,T)$. Suppose that $g = \sum_{l=1}^n g_l[E_l]$ where $g_l \ge 0$ and continuous, $E_l \cap E_k = \varnothing$ and $\sigma(\partial E_l) = 0$. Therefore, it is enough to show the convergence for $g = [E_l]$ for E_l open in ∂D. Since E_l is open, it can be approximated from below by an increasing sequence of continuous functions converging pointwise to g. Moreover, since X^{ω,ε_j} and X will not hit ∂E_k in $[0,T]$ almost surely, we can similarly approximate from above by approximating $1 - g$ from below. This implies pointwise convergence $A^{\omega,\varepsilon_j} \to A$ as $j \to \infty$ on a countable dense set almost surely and with monotonicity we deduce the almost sure convergence of A^{ω,ε_j} to A in $C(0,T)$, and hence the weak convergence of the original versions follows. The same technique allows us to extend the assertion to the case of discontinuous f. Therefore, we have shown (iv) and since (v) follows from (iv) with the Feynman–Kac formula (46), we have shown (51).

For the proof of (52), note that the boundary condition (12) allows us to write

$$J_l(\varepsilon_k,\omega) = \frac{1}{|E_l|}\int_{E_l} \big(f - g u_{\varepsilon_k}(\cdot,\omega)|_{\partial_1 D}\big)\,\mathrm{d}\sigma(x) \qquad \text{for } \mathcal{P}\text{-a.e. } \omega \in \Gamma \tag{56}$$

and that (12) may be written in the form $(\Lambda_{\kappa_\varepsilon} + gI)u_\varepsilon = f$. We deduce from the well-posedness of the forward problem that $(\Lambda_{\kappa_\varepsilon} + gI)^{-1}$, $L^2(\partial D) \to L^2(\partial D)$, is bounded. Since, due to assumption (A1), the corresponding constant does not depend on ε, the sequence $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ is bounded in $L^2(\partial D)$ for $\mathcal{P}$-a.e. $\omega \in \Gamma$ implying its uniform integrability; see, for example, [35]. As we already know the pointwise convergence $u_{\varepsilon_k}(x,\omega) \to u^*(x)$, $x \in \overline{D}$, as $k \to \infty$, an application of Egorov's theorem yields convergence in $L^1(\partial D)$ so that the assertion follows by the triangle inequality and taking the limit $k \to \infty$ in (56). $\square$

REMARK 6.3. We would like to point out that the effective conductivity κ^* is determined by the invariance principle on the whole space $\mathbb{R}^d$.

6.3. *Continuum approximation of the effective conductivity.* In this subsection, we provide the theoretical foundation for the convergence analysis of numerical homogenization methods based on simulation of the underlying diffusion process. More precisely, a rigorous convergence analysis of such a method requires a quantitative estimate that is stronger than the qualitative result (49), which was obtained in [36] using merely the central limit theorem for martingales. We provide such a quantitative result in the following theorem by generalizing a classical argument due to Kipnis and Varadhan [34]. The proof relies on new spectral bounds which were obtained recently by Gloria and Otto [26]. We refer the reader to the recent papers [15, 24] for an analogous estimate for the discrete lattice random walk in random environment as well as to the paper [38] which was the first one to use the Kipnis and Varadhan argument in order to obtain quantitative results.

THEOREM 6.4. *Let* $\{\kappa_\varepsilon(x,\omega), (x,\omega) \in \mathbb{R}^d \times \Gamma\}$ *be a stationary random field satisfying assumptions* (A1) *and* (A2) *from Section* 3.2. *Then for every direction vector* $\xi \in \mathbb{R}^d$ *there exist positive constants* c_1, c_2 *such that*

$$\left|\frac{\overline{\mathbb{E}}(X_t^{\cdot} \cdot \xi)^2}{2t} - \xi \cdot \kappa^* \xi\right| \le c_1 \begin{cases} |\log t|^{c_2} t^{-1}, & d = 2, \\ t^{-1}, & d \ge 3. \end{cases} \tag{57}$$

PROOF. For fixed $\omega \in \Gamma$, let us consider the diffusion process X^ω on $\mathbb{R}^d$ which is associated with the symmetric regular Dirichlet form $(\mathcal{E}^{\omega,1}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ under the measure $\mathbb{P}_0^\omega$. Following [34], we search for a decomposition of the form

$$X_t^\omega = M_t^\omega + R_t^\omega, \tag{58}$$

where M_t^ω is a $\mathbb{P}_0^\omega$-martingale and for every direction $\xi \in \mathbb{R}^d$ the projected remainder $R_t^\omega \cdot \xi$ converges to zero in $L^2(\overline{\Gamma})$ as $t \to \infty$.

Once we have found a suitable decomposition (58), we first show that

$$t^{-1}\overline{\mathbb{E}}\{(X_t^{\cdot} \cdot \xi)^2 - (M_t^{\cdot} \cdot \xi)^2\} = t^{-1}\overline{\mathbb{E}}(R_t^{\cdot} \cdot \xi)^2. \tag{59}$$

Then we will use spectral calculus to estimate the right-hand side of (59) in order to obtain the claimed inequality (57). More precisely, we show that the right-hand side of (59) admits a spectral representation

$$t^{-1}\overline{\mathbb{E}}(R_t^{\cdot} \cdot \xi)^2 = 2t^{-1}\int_0^\infty (1 - e^{-\lambda t})\lambda^{-2}\,\mathrm{d}\mu(\lambda), \tag{60}$$

with a positive Radon measure μ which we will specify later. By combining the representations (60), (59) with the estimates from [25, 26], we will then deduce the claimed estimate (57).

In order to obtain the decomposition (58), we recall that the auxiliary problem (48) is equivalent to the following stochastic elliptic equation in the physical space $\mathbb{R}^d$: Find $\boldsymbol{\chi}^\xi \in \mathcal{V}^2_{\mathrm{pot}}(\Gamma)$ such that for $\mathcal{P}$-a.e. $\omega \in \Gamma$, the corresponding potential $\eta^\xi : \mathbb{R}^d \times \Gamma \to \mathbb{R}$ is in $C(\mathbb{R}^d) \cap H^1_{\mathrm{loc}}(\mathbb{R}^d)$ as a function of x, satisfies $\eta^\xi(0,\omega) = 0$ and

$$-\nabla \cdot \kappa(\cdot,\omega)\bigl(\xi + \nabla \eta^\xi(\cdot,\omega)\bigr) = 0 \qquad \text{in } \mathbb{R}^d \tag{61}$$

for $\mathcal{P}$-a.e. $\omega \in \Gamma$. Let us define the function

$$\phi : \mathbb{R}^d \times \Gamma \to \mathbb{R}^d, \qquad \phi(x,\omega) := x + \eta(x,\omega) - \eta(0,\omega),$$

where $\eta := (\eta^{e_1}, \dots, \eta^{e_d})^T$ and η^{e_i}, $i = 1, \dots, d$, denotes the potential corresponding to the solution to the auxiliary problems for the coordinate directions. As the transition density kernel of X^ω is jointly Hölder-continuous and $\phi_i(\cdot,\omega) \in C(\mathbb{R}^d) \cap H^1_{\mathrm{loc}}(\mathbb{R}^d)$, $i = 1, \dots, d$, for $\mathcal{P}$-a.e. $\omega \in \Gamma$, the Fukushima decomposition of $\phi_i(X^\omega_t, \omega)$ holds for every starting point $x \in \mathbb{R}^d$ rather than quasi-every $x \in \mathbb{R}^d$. A straightforward computation using the fact that η is defined via the auxiliary problem yields that the continuous additive functional of zero energy in the Fukushima decomposition vanishes so that

$$\phi_i\bigl(X^\omega_t, \omega\bigr) = M_t^{\phi_i(\cdot,\omega)}, \qquad i = 1, \dots, d.$$

We set

$$M^\omega_t := \bigl(M_t^{\phi_1(\cdot,\omega)}, \dots, M_t^{\phi_d(\cdot,\omega)}\bigr)^T \quad \text{and} \quad R^\omega_t := -\eta\bigl(X^\omega_t,\omega\bigr) + \eta(0,\omega)$$

which therefore provides the decomposition (58) we were searching for. In order to prove (59), we consider the quantity

$$\overline{\mathbb{E}}\bigl(X^\cdot_t \cdot \xi\bigr)^2 = \overline{\mathbb{E}}\bigl(M^\cdot_t \cdot \xi\bigr)^2 + \overline{\mathbb{E}}\bigl(R^\cdot_t \cdot \xi\bigr)^2 + 2\overline{\mathbb{E}}\bigl(M^\cdot_t \cdot \xi\bigr)\bigl(R^\cdot_t \cdot \xi\bigr). \tag{62}$$

By computing the predictable quadratic variation of the martingale additive functional, we obtain for all $t \geq 0$ and a.e. $x \in \mathbb{R}^d$

$$\mathbb{E}^\omega_x\bigl(M^\omega_t \cdot \xi\bigr)^2 = \mathbb{E}^\omega_x\biggl(\int_0^t 2\bigl(\xi + \nabla\eta^\xi(X_s,\omega)\bigr) \cdot \kappa(X_s,\omega)\bigl(\xi + \nabla\eta^\xi(X_s,\omega)\bigr)\,\mathrm{d}s\biggr).$$

By the stationarity of $\nabla\eta^\xi$ with respect to $\mathcal{P}$, we have thus

$$\overline{\mathbb{E}}\bigl(M^\cdot_t \cdot \xi\bigr)^2 = 2\xi \cdot \kappa^* \xi t \qquad \text{for all } t \geq 0.$$

Moreover, as in [14], it follows that the last term on the right-hand side of (62) vanishes. This can be seen by studying the so-called *environment seen by the particle* process, that is, the stochastic process defined by

$$Y^\omega_t := [t > 0]\boldsymbol{\Theta}_{X^\omega_t}\omega + [t = 0]\omega.$$

The process $Y^{\cdot}$ is a stationary process with respect to the annealed measure $\overline{\mathbb{P}}$, that is, for every finite collection of times $t^{(i)}$, $i = 1, \dots, k$, the joint distribution of $Y_{t^{(1)}+h}, \dots, Y_{t^{(k)}+h}$ under $\overline{\mathbb{P}}$ does not depend on $h \geq 0$. It is well known that the underlying dynamical system $\{\mathbf{\Theta}_x, x \in \mathbb{R}^d\}$ defines a d-parameter group $\{\mathbf{S}_x, x \in \mathbb{R}^d\}$ of unitary operators on $L^2(\Gamma)$ by $\mathbf{S}_x \boldsymbol{\psi}(\omega) := \boldsymbol{\psi}(\mathbf{\Theta}_x \omega)$ and this group is strongly continuous; cf. [36]. Its d infinitesimal generators $(\mathbf{D}_1, \mathcal{D}(\mathbf{D}_1)), \dots, (\mathbf{D}_d, \mathcal{D}(\mathbf{D}_d))$ are given by

$$\mathbf{D}_i \boldsymbol{\psi} = \lim_{h \to 0+} \frac{\mathbf{S}_{he_i} \boldsymbol{\psi} - \boldsymbol{\psi}}{h}, \qquad i = 1, \dots, d,$$

for all $\boldsymbol{\psi} \in L^2(\Gamma)$ such that the limit exists. These operators are closed and densely defined. We denote $\mathbf{D} := (\mathbf{D}_1, \dots, \mathbf{D}_d)^T$ and introduce the infinitesimal generator $(\boldsymbol{\mathcal{L}}, \mathcal{D}(\boldsymbol{\mathcal{L}}))$ on $L^2(\Gamma)$ of the environment viewed by the particle process, that is, the nonnegative definite self-adjoint operator $\boldsymbol{\mathcal{L}} := -\mathbf{D} \cdot \boldsymbol{\kappa} \mathbf{D}$ on $L^2(\Gamma)$. By the self-adjoinedness of $\boldsymbol{\mathcal{L}}$, the law of the environment as viewed from the particle process under $\overline{\mathbb{P}}$ is invariant with respect to time reversal and M^{ω} is odd by [18], Corollary 2.1, that is, it changes its sign under time reversal, whereas R^{ω}, which is the zero energy part of the Fukushima decomposition (58), is even by [18], Theorem 2.1. Thus, we notice that the identity (59) holds, as claimed.

Before showing the identity (60) with measure μ given by the nondecreasing function $\lambda \mapsto \mathbb{M}(\mathbf{v}^{\xi} E_{\lambda} \mathbf{v}^{\xi})$ where $\{E_{\lambda}, \lambda \in \mathbb{R}\}$ is the unique spectral family given by the spectral theorem such that $\boldsymbol{\mathcal{L}} = \int_0^{\infty} \lambda \, \mathrm{d}E_{\lambda}$ (see [37], theorem, page 199) and the function $\mathbf{v}^{\xi} := \mathbf{D} \cdot \boldsymbol{\kappa} \xi \in L^2(\Gamma)$, we show that the estimate (57) follows from the spectral representation (60). Indeed, given the formula (60) for the projected remainder and due to the assumption (A2), we can now exploit the following optimal estimate from [25, 26]: For all $0 < \gamma \leq 1$, there exists a positive constant c such that

$$\int_0^{\gamma} \mathrm{d}\mu(\lambda) = \int_0^{\gamma} \mathrm{d}\big(E_{\lambda} \mathbf{v}^{\xi}, \mathbf{v}^{\xi}\big)_{\Gamma} \leq c \gamma^{d/2+1},$$

where we denoted the inner product $(\mathbf{v}, \mathbf{w})_{\Gamma} := \mathbb{M}(\mathbf{v}\mathbf{w})$.

More precisely, we split the integral (60) into three parts, the first ranging from 0 to t^{-1}, the second from t^{-1} to 1 and the third from 1 to ∞, respectively when $t > 1$. For the latter, we have the trivial estimate

$$\int_1^{\infty} \mathrm{d}\mu(\lambda) \leq \int_0^{\infty} \mathrm{d}\mu(\lambda) = \mathbb{M}\big\{\big(\mathbf{v}^{\xi}\big)^2\big\},$$

where the last equality follows by the spectral theorem [37], theorem, page 199. The first part is bounded by a positive constant as well, namely by

$$\begin{aligned} \int_0^{t^{-1}} t\lambda^{-1} \, \mathrm{d}\mu(\lambda) &= t \int_0^{t^{-1}} \int_{\lambda}^{\infty} \alpha^{-2} \, \mathrm{d}\alpha \, \mathrm{d}\mu(\lambda) \\ &= t \int_0^{\infty} \alpha^{-2} \int_0^{\alpha \wedge t^{-1}} \mathrm{d}\mu(\lambda) \, \mathrm{d}\alpha \leq ct \int_0^{\infty} \alpha^{-2} \big(\alpha \wedge t^{-1}\big)^{d/2+1} \, \mathrm{d}\alpha. \end{aligned}$$

Similarly, the second part can be estimated by

$$\int_{t^{-1}}^{1} \lambda^{-2}\,\mathrm{d}\mu(\lambda) = 2\int_{t^{-1}}^{1}\int_{\lambda}^{\infty} \alpha^{-3}\,\mathrm{d}\alpha\,\mathrm{d}\mu(\lambda) \le 2c\int_{t^{-1}}^{\infty} \alpha^{-3}(\alpha\wedge 1)^{d/2+1}\,\mathrm{d}\alpha,$$

which diverges logarithmically for $d = 2$ and is bounded by a positive constant for $d \ge 3$. Therefore, combining these computations with the identities (59) and (60), the estimate (57) follows.

It remains to prove the spectral representation (60). In order to take advantage of the spectral theorem, we would like to express the projected remainder in the form $\mathbb{M}\psi_1(\boldsymbol{\mathcal{L}})\mathbf{v}^\xi\psi_2(\boldsymbol{\mathcal{L}})\mathbf{v}^\xi$ for some bounded continuous functions ψ_1 and ψ_2. This is desirable since by spectral theorem

$$\mathbb{M}\psi_1(\boldsymbol{\mathcal{L}})\mathbf{v}^\xi\psi_2(\boldsymbol{\mathcal{L}})\mathbf{v}^\xi = \big(\psi_1(\boldsymbol{\mathcal{L}})\mathbf{v}^\xi, \psi_2(\boldsymbol{\mathcal{L}})\mathbf{v}^\xi\big)_\Gamma = \int_0^\infty \psi_1(\lambda)\psi_2(\lambda)\mu(\mathrm{d}\lambda) \tag{63}$$

for every bounded continuous functions ψ_1 and ψ_2. Because of the nonstationarity of η and inspiration we got from the computations in [42], we consider a modified function

$$R_t^{\omega,\delta} := -\eta_\delta(X_t^\omega,\omega) + \eta_\delta(0,\omega),$$

where η_δ is defined in analogy to η with the difference that it corresponds to a different auxiliary problem, modified by a zero-order term. This modified auxiliary problem reads as follows: Find $\eta_\delta^\xi(\cdot,\omega) \in C(\mathbb{R}^d)\cap H^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\delta > 0$ such that for $\mathcal{P}$-a.e. $\omega\in\Gamma$, the random field $\{\eta_\delta^\xi(x,\omega), (x,\omega)\in\mathbb{R}^d\times\Gamma\}$ is stationary with respect to $\mathcal{P}$ with $\mathbb{M}\eta_\delta^\xi(x,\cdot) = 0$ for every $x\in\mathbb{R}^d$ and satisfies

$$\delta\eta_\delta^\xi(\cdot,\omega) - \nabla\cdot\kappa(\cdot,\omega)\big(\xi + \nabla\eta_\delta^\xi(\cdot,\omega)\big) = 0 \qquad \text{in } \mathbb{R}^d$$

for $\mathcal{P}$-a.e. $\omega\in\Gamma$. We refer to [42] for the proof of existence and uniqueness of η_δ^ξ, $\delta > 0$. Note that $\eta_\delta^\xi(0,\omega) \ne 0$ in general so that we have for the projected modified remainder the expression

$$\overline{\mathbb{E}}(R_t^{\cdot,\delta}\cdot\xi)^2 = \overline{\mathbb{E}}\big(\eta_\delta^\xi(X_t^\cdot,\cdot)\big)^2 - 2\overline{\mathbb{E}}\eta_\delta^\xi(X_t^\cdot,\cdot)\eta_\delta^\xi(0,\cdot) + \overline{\mathbb{E}}\big(\eta_\delta^\xi(0,\cdot)\big)^2. \tag{64}$$

The equivalent formulation on $L^2(\Gamma)$ of the modified auxiliary problem for the direction $\xi\in\mathbb{R}^d$ reads as follows: Find the unique solution $\boldsymbol{\eta}_\delta^\xi \in L^2(\Gamma)$ of the elliptic equation

$$\delta\boldsymbol{\eta}_\delta^\xi - \mathbf{D}\cdot\boldsymbol{\kappa}\big(\xi + \mathbf{D}\boldsymbol{\eta}_\delta^\xi\big) = 0 \qquad \text{in } \Gamma.$$

In particular, the function $\mathbf{v}^\xi$ was chosen such that $\mathbf{v}^\xi = (\delta + \boldsymbol{\mathcal{L}})\boldsymbol{\eta}_\delta^\xi = \boldsymbol{\mathcal{L}}\boldsymbol{\eta}^\xi$.

We will now only need to show that the modified remainder for fixed δ and fixed t can be written in the form $\mathbb{M}\psi_1(\boldsymbol{\mathcal{L}})\mathbf{v}^\xi\psi_2(\boldsymbol{\mathcal{L}})\mathbf{v}^\xi$, where $\psi_1(x) = 2(x+\delta)^{-1}$ and $\psi_2(x) = 2e^{-tx}(x+\delta)^{-1}$. In fact, the introduction of the zero-order perturbation removes the singularity coming from the term x^{-1}.

The first term on the right-hand side of (64) may equivalently be written as

$$\overline{\mathbb{E}}\big(\eta_\delta^\xi(X_t^\cdot,\cdot)\big)^2 = \overline{\mathbb{E}}\big(\eta_\delta^\xi(0,Y_t^\cdot)\big)^2 = \mathbb{M}\big(\eta_\delta^\xi(0,\cdot)\big)^2,$$

where we have used the stationarity of the environment as viewed from the particle with respect to $\overline{\mathbb{P}}$ and the stationarity of the random field $\{\eta_\delta^\xi(x,\omega),(x,\omega)\in\mathbb{R}^d\times\Gamma\}$ with respect to $\mathcal{P}$, respectively. Therefore, treating the second term on the right-hand side of (64) we obtain

$$\overline{\mathbb{E}}\big(R_t^{\cdot,\delta}\cdot\xi\big)^2 = 2\mathbb{M}\big(\eta_\delta^\xi(0,\cdot)\big)^2 - 2\mathbb{M}\eta_\delta^\xi(0,\cdot)T_t^\cdot\eta_\delta^\xi(0,\cdot), \tag{65}$$

where $\{T_t^\omega, t\ge 0\}$ denotes the strongly continuous semigroup on $L^2(\mathbb{R}^d)$ associated with $(\mathcal{E}^{\omega,1}, H^1(\mathbb{R}^d))$ which satisfies

$$\mathbb{E}_0^\omega\eta_\delta^\xi(X_t^\omega,\omega) = T_t^\omega\eta_\delta^\xi(0,\omega).$$

Going back to (65), respectively the corresponding identity on $L^2(\Gamma)$, we have thus for the first term on the right-hand side

$$2\mathbb{M}\big(\boldsymbol{\eta}_\delta^\xi\big)^2 = 2\mathbb{M}(\delta+\boldsymbol{\mathcal{L}})^{-1}\mathbf{v}^\xi(\delta+\boldsymbol{\mathcal{L}})^{-1}\mathbf{v}^\xi = 2\int_0^\infty(\delta+\lambda)^{-2}\,\mathrm{d}\mu(\lambda),$$

whereas the second term may be written in the form

$$2\mathbb{M}(\delta+\boldsymbol{\mathcal{L}})^{-1}\mathbf{v}^\xi e^{-t\boldsymbol{\mathcal{L}}}(\delta+\boldsymbol{\mathcal{L}})^{-1}\mathbf{v}^\xi = 2\int_0^\infty(\delta+\lambda)^{-2}e^{-t\lambda}\,\mathrm{d}\mu(\lambda),$$

where we used (63) with corresponding ψ_1 and ψ_2. Altogether we have obtained

$$t^{-1}\overline{\mathbb{E}}\big(R_t^{\cdot,\delta}\cdot\xi\big)^2 = 2t^{-1}\int_0^\infty(1-e^{-\lambda t})(\delta+\lambda)^{-2}\,\mathrm{d}\mu(\lambda)$$

and by letting $\delta\to 0$ the spectral representation (60) follows. □

7. Conclusion. We have derived Feynman–Kac formulae for the forward problem of electrical impedance tomography and studied the interconnection between these formulae and stochastic homogenization. Using the properties of the underlying diffusion processes and some new spectral estimates from [24, 25], we have then obtained a bound on the speed of convergence of the projected mean-square displacement of the processes. These results provide the theoretical foundation for the development of new scalable continuum Monte Carlo homogenization schemes.

Both the homogenization of the forward model for the complete electrode model and the stochastic numerical approximation of the effective conductivity have direct applications in EIT anomaly detection problems for random heterogeneous background media; cf. [48, 49].

APPENDIX A: REVUZ CORRESPONDENCE AND FUKUSHIMA DECOMPOSITION

For the convenience of the reader, we recall definitions, results and notation relating to the Revuz correspondence and to the Fukushima decomposition.

DEFINITION A.1 ([22], pages 74–79). We say that a Radon measure μ is *of finite energy* with respect to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, if there exists a constant $c > 0$ such that

$$\int_{\overline{D}} |v(x)| \, \mathrm{d}\mu(x) \le c\|v\|_{\mathcal{E}_1} \qquad \text{for all } v \in \mathcal{D}(\mathcal{E}) \cap C(\overline{D}),$$

where the norm $\|\cdot\|_{\mathcal{E}_1}$ is induced by the inner product $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + \langle\cdot,\cdot\rangle$.

LEMMA A.2 ([22], Theorem 5.1.3(iii), Theorem 5.1.7(ii)). *Let X be associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Then there is a one-to-one correspondence between the family of all equivalence classes of positive continuous additive functionals of X and the family of all smooth measures with respect to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$,*[5] *say between $A \leftrightarrow \mu$, given by*

$$\lim_{t\to 0+} \frac{1}{t} \int_{\overline{D}} \mathbb{E}_x\left\{\int_0^t \phi(X_s)\,\mathrm{d}A_s\right\}\psi(x)\,\mathrm{d}x = \int_{\overline{D}} \phi(x)\psi(x)\,\mathrm{d}\mu(x) \tag{66}$$

for all nonnegative Borel functions ϕ and all α-excessive functions ψ. In this case, we say that A admits the Revuz measure μ. Furthermore, every smooth measure μ with respect to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ having finite energy is the Revuz measure of a positive continuous additive functional A in strict sense, that is, without an exceptional set.

LEMMA A.3. *Let X be associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and suppose $u \in H^1(D)$. Then there is a unique Fukushima decomposition into a martingale additive functional M^u and a continuous additive functional N^u of X having zero energy such that*

$$u(X_t) - u(X_0) = M_t^u + N_t^u$$

for every $t \ge 0$ $\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$. Moreover, if there are Revuz measures μ_1 and μ_2 such that

$$\mathcal{E}(u, v) = -\int_{\overline{D}} v(x)(\mu_1 - \mu_2)(\mathrm{d}x)$$

for every $v \in H^1(D)$, then $N^u = A^{(1)} - A^{(2)}$ where $A^{(j)}$ admits the Revuz measure μ_j, $j = 1, 2$.

PROOF. This follows from [22], Theorem 5.2.2, Theorem 5.2.4 and Revuz correspondence Lemma A.2. □

[5] Cf. [22], page 80 for the definition.

APPENDIX B: SKOROHOD DECOMPOSITIONS

For convenience of the reader, let us state Skorohod decompositions of the reflecting diffusion process X for two practically relevant special cases, namely local Lipschitz conductivities and isotropic piecewise constant conductivities.

The assertion of the following proposition is covered by [23], Theorem 2.3.

PROPOSITION B.1. *Let $\kappa \in C^{0,1}_{\mathrm{loc}}(\overline{D};\mathbb{R}^{d\times d})$ be a symmetric, uniformly bounded and uniformly elliptic conductivity. Then the reflecting diffusion process X admits the following Skorohod decomposition*:

$$X_t = x + \int_0^t B(X_s)\,\mathrm{d}W_s + \int_0^t \nabla\kappa(X_s)\,\mathrm{d}s - \int_0^t \kappa(X_s)\nu(X_s)\,\mathrm{d}L_s, \tag{67}$$

$\mathbb{P}_x$-*a.s., where $B:\overline{D}\to\mathbb{R}^{d\times d}$ denotes the positive definite diffusion matrix satisfying $B^2 = 2\kappa$, W is a standard d-dimensional Brownian motion and L is the boundary local time of X.*

Now let us turn to the case of isotropic piecewise constant conductivities and for simplicity of the presentation let us consider a simplistic two-phase medium, where

$$\kappa(x) = \begin{cases} \kappa_1, & x \in D_1, \\ \kappa_2, & x \in D_2, \end{cases} \tag{68}$$

with constants $\kappa_1, \kappa_2 > 0$ and D is a simply connected bounded Lipschitz domain which consists of two disjoint subdomains such that $D_1 = D \setminus \overline{D}_2$. We assume that D_2 is a simply connected Lipschitz domain. ν is the outer unit normal vector on ∂D and the outer unit normal vector on ∂D_2 with respect to D_2. The positive continuous additive functional L^0 of X whose Revuz measure is given by the scaled Lebesgue surface measure $(\kappa_1+\kappa_2)\sigma$ on ∂D_2 is called the *symmetric local time* of the reflecting diffusion process X at ∂D_2. The term "symmetric" comes from the fact that in the one-dimensional case L^0 is the local time defined by the Tanaka formula with the convention $\operatorname{sign}(0) = 0$, which is called the symmetric local time; see [45]. In this case, we have

$$L^0_t = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t [[-\varepsilon,\varepsilon]](X_s)\,\mathrm{d}s.$$

For the proof of the following result, we refer to [49].

PROPOSITION B.2. *Let κ be given by* (68). *Then the reflecting diffusion process X admits the following Skorohod decomposition*:

$$X_t = x + \int_0^t \sqrt{2\kappa(X_s)}\,\mathrm{d}W_s + \frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\int_0^t \nu(X_s)\,\mathrm{d}L^0_s - \kappa_1\int_0^t \nu(X_s)\,\mathrm{d}L_s, \tag{69}$$

$\mathbb{P}_x$-*a.s., where W is a standard d-dimensional Brownian motion, L^0 is the symmetric local time of X at ∂D_2 and L is the boundary local time.*

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