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ON THE STABILITY OF PLANAR RANDOMLY SWITCHED SYSTEMS[1]

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Consider the random process $(X_t)_{t\geq 0}$ solution of $\dot{X}_t = A_{I_t} X_t$, where $(I_t)_{t\geq 0}$ is a Markov process on $\{0, 1\}$, and A_0 and A_1 are real Hurwitz matrices on $\mathbb{R}^2$. Assuming that there exists $\lambda \in (0, 1)$ such that $(1 - \lambda)A_0 + \lambda A_1$ has a positive eigenvalue, we establish that $\|X_t\|$ may converge to 0 or $+\infty$ depending on the jump rate of the process I. An application to product of random matrices is studied. This paper can be viewed as a probabilistic counterpart of the paper [*Internat. J. Control* **82** (2009) 1882–1888] by Balde, Boscain and Mason.

1. Introduction. The aim of the present paper is twofold. First, this work answers a question by Charlot about the stochastic counterpart of the work [2]. Second, the piecewise deterministic Markov processes (PDMP) under study may present a surprising blow-up when time goes to infinity.

Let $A_0, A_1 \in \mathbb{R}^{2\times 2}$ be two real matrices which admit two eigenvalues with negative real parts: A_0 and A_1 are said to be Hurwitz matrices. In [2], the authors deal with the stability problem for the planar linear switching system $\dot{x}_t = (1 - u_t)A_0 x_t + u_t A_1 x_t$, where $u : [0, \infty) \to \{0, 1\}$ is a measurable function. They provide necessary and sufficient conditions on A_0 and A_1 for the system to be asymptotically stable for arbitrary switching function u. The main hypothesis that ensures the existence of a control u such that the system is not asymptotically stable is the following.

ASSUMPTION 1.1. There exists $\lambda \in (0, 1)$ such that the matrix A_λ given by $(1-\lambda)A_0 + \lambda A_1$ has two real eigenvalues $-\lambda_- < 0 < \lambda_+$ with opposite signs. Let us denote by u_-, u_+ two associated (real, unit) eigenvectors.

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REMARK 1.2. It is shown in [2] that Assumption 1.1 is equivalent to the relation

$$\mathrm{Tr}(A_0)\,\mathrm{Tr}(A_1) - \mathrm{Tr}(A_0 A_1) < -2\sqrt{\det(A_0)\det(A_1)}. \tag{1}$$

Assumption 1.1 may hold in many different cases as is illustrated by Examples 1.3 and 1.4. The complete description of the different cases is postponed to Section 2.3.

EXAMPLE 1.3. Let us define A_0 and A_1 by

$$A_0 = \begin{pmatrix} -1 & 2b \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} -1 & 0 \\ 2b & -1 \end{pmatrix}$$

with $b > 0$. Then A_0 and A_1 are two Jordan matrices, and the eigenvalues of $A_{1/2}$ are given by $-1 \pm b$.

EXAMPLE 1.4. Let us define A_0 and A_1 by

$$A_0 = \begin{pmatrix} -1 & ab \\ -a/b & -1 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} -1 & -a/b \\ ab & -1 \end{pmatrix}$$

with $a, b > 0$. Then A_0 and A_1 have conjugate complex eigenvalues, and the eigenvalues of $A_{1/2}$ are $-1 \pm a(b - 1/b)/2$.

In the sequel, we suppose that Assumption 1.1 holds. Let us define $\lambda_0 = \lambda$ and $\lambda_1 = 1 - \lambda$. For any $\beta > 0$, consider the Markov process (X, I) on $\mathbb{R}^2 \times \{0, 1\}$ driven by the generator $\mathcal{L}_\beta$,

$$\mathcal{L}_\beta f(x, i) = \mathcal{L}_C f(x, i) + \beta \mathcal{L}_J f(x, i),$$

where

$$\mathcal{L}_C f(x, i) = A_i \nabla f(x, i) \quad \text{and} \quad \mathcal{L}_J f(x, i) = \lambda_i \big(f(x, 1 - i) - f(x, i) \big).$$

The operator $\mathcal{L}_C$ corresponds to the "continuous" part (the first component x evolves along the flow of the vector field $x \mapsto A_i x$), and $\beta \mathcal{L}_J$ gives the jumps on the second component. If ν is a probability measure on $\mathbb{R}^2 \times \{0, 1\}$, we denote by $\mathbb{P}_\nu$ the law of the process (X, I) when the law of (X_0, I_0) is ν.

REMARK 1.5. One can easily construct the process (X, I) as follows. The process $(I_t)_{t \geq 0}$ is the Markov process on $\{0, 1\}$ with jump rates $(\beta \lambda_i)_{i \in \{0,1\}}$. Then $(X_t)_{t \geq 0}$ is the solution of

$$X_t = X_0 + \int_0^t A_{I_s} X_s \, ds \qquad (t \geq 0).$$

Notice that $(I_t)_{t \geq 0}$ is a Markov process with invariant measure

$$\frac{\beta \lambda_1}{\beta \lambda_0 + \beta \lambda_1} \delta_0 + \frac{\beta \lambda_1}{\beta \lambda_0 + \beta \lambda_1} \delta_1 = (1 - \lambda) \delta_0 + \lambda \delta_1.$$

Our main result ensures that under Assumption 1.1 the norm of the continuous component X goes to zero if the jumps are rare and to $+\infty$ if the jumps are sufficiently numerous (and $X_0 \neq 0$).

THEOREM 1.6. *Under Assumption 1.1, there exists $\chi(\beta) \in \mathbb{R}$ such that, for any initial measure ν such that $\nu(\{0\} \times \{0, 1\}) = 0$,*

$$\frac{1}{t} \log \|X_t\| \xrightarrow[t\to\infty]{\mathbb{P}_\nu\text{-}a.s.} \chi(\beta). \tag{2}$$

Moreover, there exist two constants $0 < \beta_1 \leq \beta_2 < \infty$ such that:

- *if $\beta < \beta_1$, then $\chi(\beta)$ is negative and $\|X_t\| \xrightarrow[t\to\infty]{\mathbb{P}_\nu\text{-}a.s.} 0$;*
- *if $\beta > \beta_2$, then $\chi(\beta)$ is positive and $\|X_t\| \xrightarrow[t\to\infty]{\mathbb{P}_\nu\text{-}a.s.} \infty$.*

REMARK 1.7. The process $((X_t, I_t))_{t\geq 0}$ is what is called a piecewise deterministic Markov process on $\mathbb{R}^2 \times \{0, 1\}$ (see [4, 6] for details) where the continuous part is driven by two vectors fields that admit a unique stable point and are exponentially stable. In [1] it is proved that if the process is recurrent, its invariant measure is often absolutely continuous. The previous theorem shows that the recurrence may not be so easy to establish (it can depend on the jump rates).

REMARK 1.8. A similar model of switching linear evolutions is studied in [9], Chapter 8. In Section 8.4, stability results are established on a certain timescale; that is, the process is studied for high β on a time interval that depends on β. In Section 8.5 the existence of the "Lyapunov exponent" $\chi(\beta)$ is proved when the angular process defined below is ergodic. The general case and the fact that $\chi(\beta)$ may change sign with β is not considered in that work.

We prove Theorem 1.6 in Section 2. We do not know if $\beta_1 = \beta_2$ under Assumption 1.1. Nevertheless, Section 3 is dedicated to the study of Examples 1.3 and 1.4 where this "phase transition" can be established. The exponential rate of growth of the process is given by an expression analogous to Furstenberg's formula [5]. Generally it is difficult to compute the element entering the Furstenberg formula; see examples in [3, 8]. For the example of Section 3 one obtains an explicit expression of the "Lyapunov" exponent of $(X_t)_{t\geq 0}$. Finally, in Section 4, we remark that our results can be interpreted in terms of products of random matrices. We obtain examples of products of random independent matrices, with eigenvalues of modulus less than one, with a positive Lyapunov exponent (we are not in the frame of unimodular matrices studied in [3, 8]).

2. The general case. The proofs of the two parts of Theorem 1.6 use different techniques. The easy part, when β is small, follows from a martingale argument explained in Section 2.1. To study the process for large β, we use a polar decomposition, detailed in Section 2.2. The angular process is studied in Sections 2.3 and 2.4. In Section 2.5 we give the main line of the proof of Theorem 1.6; the proof of a key lemma is postponed to Section 2.6.

2.1. *Few jumps*: *Convergence to zero.* In this subsection, we suppose that β is small: the i component rarely jumps. The two flows associated to A_0 and A_1 being linear and attractive, there exists $\rho > 0$ and two norms V_0 and V_1, given by two positive symmetric matrices M_0 and M_1, such that, for $V_i(x) = \langle x, M_i x\rangle$,

$$\mathcal{L}_C V_i(x, i) \leq -\rho V_i(x).$$

Define, $V(x, i) = V_i(x)$. Since $|\mathcal{L}_J f(x, i)| \leq K(|f(x, 0)| + |f(x, 1)|)$, we get

$$\begin{aligned}\mathcal{L}_\beta V(x, i) &= \mathcal{L}_C V_i(x, i) + \beta \mathcal{L}_J V_i(x, i)\\ &\leq -\rho V_i(x) + \beta K\big(V_0(x) + V_1(x)\big)\\ &\leq -\rho V_i(x) + \beta K' V_i(x)\end{aligned}$$

by the equivalence of the norms. Therefore there exist a $\rho' > 0$ and a $\beta_1 > 0$ such that, for $\beta < \beta_1$,

$$\forall (x, i) \in \mathbb{R}^2 \times \{0, 1\} \qquad \mathcal{L}_\beta V(x, i) \leq -\rho' V(x, i).$$

Consequently, the process $(M_t)_{t\geq 0}$ defined by $M_t = e^{\rho' t} V(X_t, I_t)$ is a positive supermartingale. It converges almost surely to a random variable which is almost surely finite. Therefore $V(X_t, I_t)$ converges almost surely to zero, and $\|X_t\|$ itself converges to zero almost surely (exponentially fast).

2.2. *A polar decomposition.* We begin by decomposing the deterministic dynamics. Let A be a matrix on $\mathbb{R}^2$ and $x \in \mathbb{R}^2 \setminus \{0\}$. Consider $(x_t)_{t\geq 0}$ the solution of

$$\begin{cases} \dot{x}_t = A x_t,\\ x_0 = x.\end{cases}$$

First of all, since x is not 0, then, for any $t \geq 0$, x_t is not equal to 0. Therefore it is possible to define the polar coordinates (r_t, θ_t) of x_t. Call e_θ the unit vector $(\cos\theta, \sin\theta)$ and define $u_t = e_{\theta_t} : x_t$ may be written $r_t u_t$. Since $r_t^2 = \langle x_t, x_t\rangle$, we have

$$r_t \dot{r}_t = \langle x_t, A x_t\rangle,$$

$$A(r_t u_t) = \dot{x}_t = \dot{r}_t u_t + r_t \dot{u}_t.$$

Therefore,

$$\dot{r}_t = r_t \langle u_t, A u_t\rangle, \tag{3}$$

$$\dot{u}_t = A u_t - \langle u_t, A u_t\rangle u_t. \tag{4}$$

The evolution of u_t on the circle is autonomous. The derivative $\dot{u}_t$ vanishes when $Au_t = \langle u_t, Au_t \rangle u_t$, that is, when u_t is a eigenvector of A. As a consequence, equation (4) has:

- four stationary points if and only if A admits two different eigenvalues,
- two stationary points if and only if A is a Jordan matrix as in Example 1.3,
- no stationary points if and only if the eigenvalues of A are not real.

Let us write equation (4) in terms of the angles θ_t. Since $\dot{u}_t = \dot{\theta}_t e_{\theta_t+\pi/2}$, the scalar product of (4) with $e_{\theta_t+\pi/2}$ gives

$$
\begin{aligned}
\dot{\theta}_t &= \langle Ae_{\theta_t}, e_{\theta_t+\pi/2} \rangle \\
&= (A_{22} - A_{11}) \sin(\theta_t) \cos(\theta_t) + A_{21} \cos^2(\theta_t) - A_{12} \sin^2(\theta_t).
\end{aligned}
\tag{5}
$$

The critical points of this differential equation are related to the eigenvector of A as it is pointed out in the following lemma.

LEMMA 2.1. *For any matrix A, the function*

$$d : \theta \mapsto d(\theta) = \langle Ae_\theta, e_{\theta+\pi/2} \rangle$$

given by (5) is π-periodic and $d(\theta) = 0$ if and only if e_θ is an eigenvector of A. Finally, the function d is constant and equal to zero if and only if $A = \lambda \mathrm{I}_2$.

PROOF. If θ is changed to $\theta + \pi$, then both e_θ and $e_{\theta+\pi/2}$ are changed to their opposite, so that $\langle Ae_\theta, e_{\theta+\pi/2} \rangle$ remains unchanged. We have already seen that $d(\theta) = 0$ if and only if e_θ is an eignevector of A. □

2.3. *The angular process.* Let us use the polar decomposition to study the process $((X_t, I_t))_{t\geq 0}$. Between jumps, the process follows the deterministic dynamics described above, with $A \in \{A_0, A_1\}$. Since the evolution of the angle θ is autonomous for each dynamics, the process (Θ, I) is a Markov process on $\mathbb{R} \times \{0, 1\}$. The evolution of $(R_t)_{t\geq 0}$ is determined by the one of the process $((\Theta_t, I_t))_{t\geq 0}$, by solving equation (3) between the jumps. If we call $\mathcal{A}(\theta, i) = \langle A_i e_\theta, e_\theta \rangle$, then

$$R_t = R_0 \exp\left(\int_0^t \mathcal{A}(\Theta_s, I_s)\, ds \right) \tag{6}$$

and R_t appears as a multiplicative functional of $((\Theta_s, I_s))_{0\leq s\leq t}$.

The proof of Theorem 1.6 relies on the study of the long time behavior of (Θ, I). We will see in the sequel that this process may be ergodic (i.e., it admits a unique invariant measure) or not. Let us define, for $i \in \{0, 1\}$ and $\lambda \in (0, 1)$,

$$
\begin{aligned}
d_i(\theta) &= \langle A_i e_\theta, e_{\theta+\pi/2} \rangle, \\
d_\lambda(\theta) &= (1 - \lambda)\, d_0(\theta) + \lambda\, d_1(\theta).
\end{aligned}
$$

The generator of the Markov process (Θ, I) is given by

$$L_\beta f(\theta, i) = L_C f(\theta, i) + \beta L_J f(\theta, i),$$

where

$$L_C f(\theta, i) = d_i(\theta)\partial_\theta f(\theta, i) \quad \text{and} \quad L_J f(\theta, i) = \lambda_i\big(f(\theta, 1-i) - f(\theta, i)\big). \tag{7}$$

Once again, L_C is the continuous drift and βL_J is the jump part. Let us also introduce the averaged (deterministic) dynamic

$$L_A f(\theta, i) = d_\lambda(\theta)\partial_\theta f(\theta, i). \tag{8}$$

Under Assumption 1.1, Lemma 2.1 ensures that the vector field $F^\lambda = d_\lambda \partial_\theta$ has exactly four critical points on $[0, 2\pi)$. As d_λ is π-periodic it suffices to describe it only on an interval of length π connecting two zeros of d_λ corresponding to the negative eigenvalues of A_λ. Let $[\theta_-, \theta_- + \pi)$ this interval. The function d_λ vanishes only once on $(\theta_-, \theta_- + \pi)$ at a point θ_+ corresponding to the positive eigenvalues of A_λ. We have

$$d_\lambda(\theta) \begin{cases} > 0, & \text{if } \theta \in (\theta_-, \theta_+), \\ < 0, & \text{if } \theta \in (\theta_+, \theta_- + \pi). \end{cases} \tag{9}$$

Let us first notice that, under Assumption 1.1, the critical points d_0, d_1 and d_λ are different.

LEMMA 2.2. *Under Assumption 1.1 if θ is a critical point of d_λ, then $d_0(\theta)\, d_1(\theta) < 0$. In particular, θ is not a critical point of d_i, $i \in \{0, 1\}$.*

PROOF. Assume that there exists θ such that $d_\lambda(\theta) = 0 = d_0(\lambda)$. Then $d_1(\theta) = 0$. As a consequence, u_θ is an eigenvector for A_0, A_1 and A_λ associated to the respective eigenvalues η_0, η_1 and $\eta_\lambda = (1-\lambda)\eta_0 + \lambda\eta_1$. This implies that the second eigenvalue of A_λ is also a convex combination of two complex numbers with negative real part [consider the relation $\mathrm{Tr}(A_\lambda) = (1-\lambda)\,\mathrm{Tr}(A_0) + \lambda\,\mathrm{Tr}(A_0)$]. This cannot hold under Assumption 1.1. As a consequence, $d_0(\theta)\, d_1(\theta) \neq 0$. Since $d_\lambda(\theta) = 0$, we get that $d_0(\theta)$ and $d_1(\theta)$ have opposite signs. □

Without loss of generality we can assume that $d_0(\theta_+) < 0$ and $d_1(\theta_+) > 0$. Because of the equality $d_\lambda = (1-\lambda)\, d_0(\theta) + \lambda\, d_1(\theta)$ we have constraints on the signs of the d_i. Let us list all the possibilities:

(a) d_0 vanishes 0, 1 or 2 times on (θ_-, θ_+), and d_1 does not vanish at all;
(b) d_1 vanishes 0, 1 or 2 times on $(\theta_+, \theta_- + \pi)$, and d_0 does not vanish at all;
(c) d_1 vanishes 2 times on $(\theta_+, \theta_- + \pi)$ at points $\theta_{1m} < \theta_{1M}$, and d_0 vanishes 1 or 2 times on $(\theta_{1m}, \theta_{1M})$;
(d) d_0 vanishes 2 times on (θ_-, θ_+) at points $\theta_{0m} < \theta_{0M}$, and d_1 vanishes 1 or 2 times on $(\theta_{0m}, \theta_{0M})$;

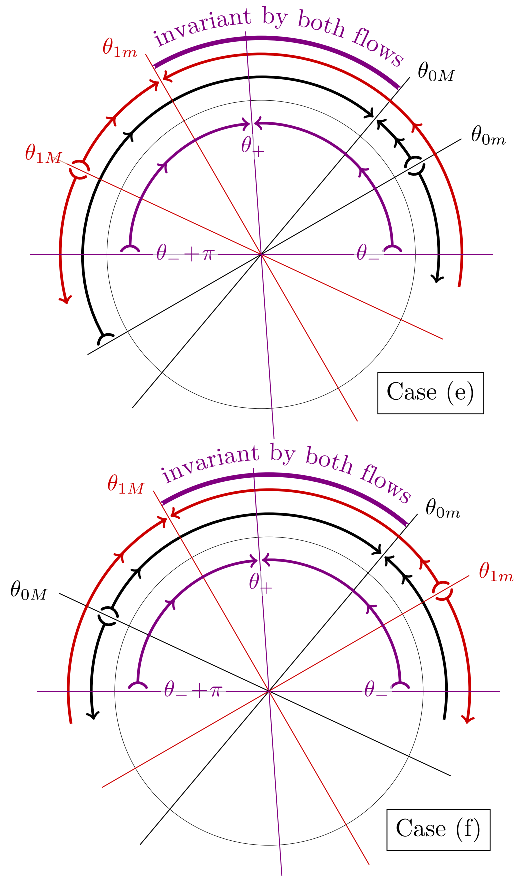

FIG. 1. *The three flows in cases* (e) *and* (f). *The outer arrows, in red, represent the flow of* d_1. *The middle ones, in blue, represent* d_0 *and the inner ones the averaged flow* d_λ. *In the two cases, there is a region around* θ_+, *that is, left invariant by both flows. The regions on each side are unstable and lead back to the invariant region.*

(e) d_1 vanishes 1 or 2 times on $(\theta_+, \theta_- + \pi)$ at points $\theta_{1m} \leq \theta_{1M}$, and d_0 vanishes 1 or 2 times on (θ_-, θ_+) at points $\theta_{0m} \leq \theta_{0M}$;

(f) d_0 vanishes 2 times at points $\theta_{0m} < \theta_{0M}$ and d_1 vanishes 2 times at points $\theta_{1m} < \theta_{1M}$ such that $\theta_{1m} < \theta_{0m} < \theta_+ < \theta_{1M} < \theta_{0M}$.

In the last two cases we have a subinterval of $(\theta_-, \theta_- + \pi)$, that is, invariant for both of the systems $\dot{\theta}_t = d_i(\theta_t) : (\theta_{0M}, \theta_{1m})$ in case (e), $(\theta_{0m}, \theta_{1M})$ in case (f); see Figure 1.

2.4. *Ergodic properties of the angular process.* Since the asymptotic behavior of $R_t = \|X_t\|$ depends on the long time behavior of the process $(U, I) = (e_\Theta, I)$, let us briefly study its ergodicity (recurrent and transient points, number of invariant measures...).

First, we remark that when Assumption 1.1 is satisfied, there exists $\varepsilon > 0$ such that:

- the points $\{(\theta, i) : \theta \in (\theta_- - \varepsilon, \theta_- + \varepsilon), i = 0, 1\}$ lead with positive probability to (θ_+, j) and $(\theta_+ - \pi, j)$, $j = 0, 1$;
- the points $\{(\theta, i) : \theta \in (\theta_- + \pi - \varepsilon, \theta_- + \pi + \varepsilon), i = 0, 1\}$ lead with positive probability to (θ_+, j) and $(\theta_+ + \pi, j)$, $j = 0, 1$.

Thus if one of the sets $(\theta_- - \varepsilon, \theta_- + \varepsilon) \times \{0, 1\}$ or $(\theta_- + \pi - \varepsilon, \theta_- + \pi + \varepsilon) \times \{0, 1\}$ is attained with positive probability starting from $(\theta_+, 0)$, then the Markov process (U_t, I_t) on the circle is recurrent. This is the case in situations (a), (b), (c), (d) described above. In these situations the process (U_t, I_t) is irreducible and has a unique invariant measure.

In cases (e) and (f), (U_t, I_t) has exactly two distinct recurrent classes and two invariant measures supported by two intervals on the circles corresponding to the invariant interval defined above and its symmetric. Let μ_β and $\tilde{\mu}_\beta$ be these two ergodic invariant measures. For any initial measure μ on $\mathbb{T} \times \{0, 1\}$,

$$\frac{1}{t} \int_0^t f(U_s, I_s)\, ds \xrightarrow[t\to\infty]{\mathbb{P}_\mu\text{-a.s.}} P \int f(u, i)\, d\mu_\beta(u, i) + (1 - P) \int f(u, i)\, d\tilde{\mu}_\beta(u, i),$$

where $P \in \{0, 1\}$ is a random variable such that $\mathbb{P}(P = 1)$ is the probability that (U, I) reaches the class of $(e_{\theta_+}, 0)$ when the law of (U_0, I_0) is μ. Now by symmetry we have

$$\int f(u, i)\, d\tilde{\mu}_\beta(u, i) = \int f(-u, i)\, d\mu_\beta(u, i),$$

so that, if $f(-u, i) = f(u, i)$, in all cases, we have

$$\frac{1}{t} \int_0^t f(U_s, I_s)\, ds \xrightarrow[t\to\infty]{\mathbb{P}_\mu\text{-a.s.}} \int f(u, i)\, d\mu_\beta(u, i).$$

Finally notice that the invariant measures are always absolutely continuous with respect to $\lambda_{\mathbb{T}} \otimes (\delta_0 + \delta_1)$, where $\lambda_{\mathbb{T}}$ is the Lebesgue measure on $\mathbb{T}$.

2.5. *Many jumps*: *Blow up*. In the sequel, μ_β stands for any invariant measure of (U, I), and we identify $u = e_\theta$ with θ. As $\mathcal{A}(\theta, i) = \langle A_i e_\theta, e_\theta \rangle = \mathcal{A}(\theta + \pi, i)$ we get [see the expression (6)]

$$\frac{1}{t} \log(R_t / R_0) \xrightarrow[t\to\infty]{\text{a.s.}} \int \mathcal{A}(\theta, i)\, d\mu_\beta(\theta, i).$$

As a consequence, for any probability measure ν on $\mathbb{R}^2 \times \{0, 1\}$ such that $\nu(\{0\} \times \{0, 1\}) = 0$, the convergence (2) in Theorem 1.6 holds with

$$\chi(\beta) = \int \mathcal{A}(\theta, i)\, d\mu_\beta(\theta, i).$$

In order to prove that $\chi(\beta)$ is positive when β is large we use the following lemma, which will be proved in Section 2.6.

LEMMA 2.3. *When β is large, the invariant measures are concentrated around the stable points θ_+ and $\tilde{\theta}_+ = \theta_+ + \pi$ of the averaged dynamical system. More precisely, for any $\epsilon > 0$, and any neighborhood $K \subset \mathbb{T}$ of the set $\{\theta_+, \tilde{\theta}_+\}$, there exists a $\beta(K, \epsilon)$ such that, for any $\beta \geq \beta(K, \epsilon)$,*

$$\mu_\beta\big(K \times \{0, 1\}\big) \geq 1 - \epsilon.$$

Thanks to this result, we can now prove

$$\int \mathcal{A}(\theta, i)\, d\mu_\beta(\theta, i) > 0$$

for β large enough. For $\theta = \theta_+$ or $\theta = \tilde{\theta}_+$, we know that

$$\int \mathcal{A}(\theta_+, i)\, d\mu_\beta(\theta, i) = \langle A_\lambda e_{\theta_+}, e_{\theta_+}\rangle = \lambda_+ > 0.$$

Moreover $\mathcal{A}(\cdot, i)$ is continuous for $i = 0, 1$. Choose K, a neighborhood of $\theta_+, \tilde{\theta}_+$, such that

$$\forall (\theta, i) \in K \times \{0, 1\} \qquad \mathcal{A}(\theta, i) \geq \frac{2\lambda_+}{3}.$$

Thanks to Lemma 2.3, for β large enough,

$$\mu_\beta\big(K \times \{0, 1\}\big) \geq 1 - \frac{\lambda_+}{6\|\mathcal{A}\|_\infty}.$$

Therefore,

$$\begin{aligned}\left|\int \mathcal{A}(\theta, i)\, d\mu_\beta(\theta, i) - \lambda_+\right| &\leq \int \big|\mathcal{A}(\theta, i) - \mathcal{A}(\theta_+, i)\big| \mathbb{1}_{\theta \in K}\, d\mu_\beta \\ &\quad + \int \big|\mathcal{A}(\theta, i) - \mathcal{A}(\theta_+, i)\big| \mathbb{1}_{\theta \notin K}\, d\mu_\beta \\ &\leq \frac{\lambda_+}{3} + 2\|\mathcal{A}\|_\infty \mu_\beta\big(\bar{K} \times \{0, 1\}\big) \\ &\leq \frac{2\lambda_+}{3}.\end{aligned}$$

This shows that $\chi(\beta) \geq \frac{\lambda_+}{3} > 0$. Hence R_t converges a.s. to infinity; this completes the proof of Theorem 1.6.

2.6. *The invariant measures concentrate near the attractive points.* This section is devoted to the proof of Lemma 2.3. The idea is that the averaged system gets back quickly to the stable points, so most of the mass of the invariant measure μ_β should be located near these stable points. To quantify this attraction to the stable points, we find a Lyapunov function, in the following sense.

LEMMA 2.4. *Suppose that there exists a function $(\theta, i) \mapsto f_\beta(\theta, i)$ that satisfies*

$$\begin{aligned} & f_\beta(\theta, i) \geq a > 0, \\ & L_\beta f_\beta(\theta, i) \leq -\rho f_\beta(\theta, i) + C\mathbb{1}_{\{\theta \in K\}}. \end{aligned} \tag{10}$$

Then $\mu_\beta(K) \geq a\rho/C$.

PROOF. Integrating (10) with respect to the invariant measure μ_β, we get

$$0 = \int L_\beta f_\beta \, d\mu_\beta \leq -\rho \int f_\beta \, d\mu_\beta + C\mu_\beta(K),$$

which proves the result. □

The Lyapunov function f_β will be constructed by the classical "perturbation" method; for details, see, for example, [7]. We start from a test function f (depending only on θ) adapted to the averaged dynamical system driven by d_λ, and build a perturbation $f_\beta = f - \beta^{-1} g$ of this function such that $L_\beta f_\beta \approx L_A f$; this perturbed function will satisfy the hypotheses of Lemma 2.4 with appropriate constants.

Let K be a small neighborhood of the stable points θ_+, $\tilde{\theta}_+$ and $\epsilon > 0$. There exists a 2π-periodic function f that satisfies the following properties:

(1) f is $\mathcal{C}^2(\mathbb{R})$;
(2) $f(\theta_-) = f(\tilde{\theta}_-) = 2$, $f(\theta_+) = f(\tilde{\theta}_+) = 1$;
(3) $f'(\theta_-) = f'(\theta_+) = f'(\tilde{\theta}_+) = f'(\tilde{\theta}_-) = 0$;
(4) $f''(\theta_-) = -1$, $f''(\theta_+) = \epsilon$;
(5) f is monotonous between its critical points.

Notice that, by design, f decreases along the trajectories of the averaged system,

$$\forall \theta \in [0, 2\pi] \qquad L_A f(\theta) = d_\lambda(\theta) f'(\theta) \leq 0. \tag{11}$$

In the sequel, we still denote by f the function $(\theta, i) \in \mathbb{T} \times \{0, 1\} \mapsto f(\theta)$. Let us define g and f_β on $\mathbb{T} \times \{0, 1\}$ by

$$\begin{aligned} g(\theta, i) &= L_A f(\theta) - L_C f(\theta, i), \\ f_\beta(\theta, i) &= f(\theta) - \frac{1}{\beta} g(\theta, i), \end{aligned}$$

where L_C is the continuous part of L_β defined in (7), and L_A is given by (8). A straightforward computation ensures that, for any $\theta \in \mathbb{T}$, $i \mapsto g(\theta, i)$ is solution of

$$L_J g(\theta, \cdot) = L_C f(\theta, \cdot) - L_A f(\theta) = -g(\theta, \cdot). \tag{12}$$

Indeed, keeping in mind that $L_A f$ does not depend on i, we have for any $(\theta, i) \in \mathbb{T} \times \{0, 1\}$,

$$\begin{aligned} L_J g(\theta, i) &= \lambda_i \big(g(\theta, 1-i) - g(\theta, i)\big) \\ &= \lambda_i \big(-L_C f(\theta, 1-i) + L_C f(\theta, i)\big) \\ &= L_C f(\theta, i) - \big(\lambda_i L_C f(\theta, 1-i) + (1-\lambda_i) L_C f(\theta, i)\big) \\ &= L_C f(\theta, i) - \big(\lambda_i d_{1-i}(\theta) f'(\theta) + \lambda_{1-i} d_i(\theta) f'(\theta)\big) \\ &= L_C f(\theta, i) - L_A f(\theta). \end{aligned}$$

Thus, we get from equation (12) that

$$\begin{aligned} L_\beta f_\beta(\theta, i) &= L_C f(\theta, i) - \beta^{-1} L_C g(\theta, i) + \beta L_J f(\theta, i) - L_J g(\theta, i) \\ &= L_A f(\theta) - \beta^{-1} L_C g(\theta, i) \end{aligned}$$

since $L_J f(\theta, i) = 0$ according to the fact that f does not depend on $i \in \{0, 1\}$. The definition of g ensures that

$$L_\beta f_\beta(\theta, i) = L_A f(\theta) + \beta^{-1} R f(\theta, i), \tag{13}$$

where

$$\begin{aligned} R f(\theta, i) &= L_C L_C f(\theta, i) - L_C L_A f(\theta, i), \\ L_C L_C f(\theta, i) &= d_i(\theta)^2 f''(\theta) + d_i(\theta)\, d_i'(\theta) f'(\theta), \\ L_C L_A f(\theta, i) &= d_i(\theta)\, d_\lambda(\theta) f''(\theta) + d_i(\theta)\, d_\lambda'(\theta) f'(\theta). \end{aligned}$$

Thus there exists $\bar{R}_\epsilon$ such that for any $(\theta, i) \in \mathbb{R} \times \{0, 1\}$, $|Rf(\theta, i)| \le \bar{R}_\epsilon$. In particular, if β is sufficiently large, one can assume that

$$\tfrac{1}{2} \le 1 - \epsilon \le f_\beta(\theta, i) \le 3. \tag{14}$$

Let us prove (10) between two critical points $\theta_- < \theta_+$, splitting the interval $[\theta_-, \theta_+]$ in three regions,

$$[\theta_-, \theta_- + l_-], \qquad [\theta_- + l_-, \theta_+ - l_+] \quad \text{and} \quad [\theta_+ - l_+, \theta_+],$$

where l_- and l_+ depend on f, ε and K (but not on β).

First region. Since θ_- is a critical point of d_λ, one has $L_A f(\theta_-) = 0$. Moreover $f'(\theta_-)$ is equal to 0 since f reaches its minimum at θ_-. From (13), the expressions of $L_C L_C f$ and $L_C L_A f$, we get that

$$L_\beta f_\beta(\theta_-, i) = \beta^{-1} Rf(\theta_-, i) = \beta^{-1} d_i(\theta_-)^2 f''(\theta_-) \le -\beta^{-1} c_u,$$

where

$$c_u = \min\bigl(d_0(\theta_-)^2, d_1(\theta_-)^2\bigr) > 0. \tag{15}$$

By continuity, we can find $l_- > 0$ (that does not depend on β) such that $Rf(\theta, i) \le -c_u/2$ for $\theta \in [\theta_-, \theta_- + l_-]$. Remembering (11), we obtain

$$\begin{aligned} L_\beta f_\beta(\theta, i) &\le \beta^{-1} Rf(\theta, i) \\ &\le -\frac{c_u}{2}\beta^{-1} \\ &\le -\frac{c_u}{6}\beta^{-1} f_\beta(\theta, i), \end{aligned} \tag{16}$$

where the last line follows from (14).

Second region. For $\theta \in [\theta_- + l_-, \theta_+ - l_+]$, $|d_\lambda(\theta)|$ and $|f'(\theta)|$ are bounded below, so $L_A f(\theta) \le -\rho$ for some $\rho > 0$ that does not depend on β. Since Rf is bounded,

$$L_\beta f_\beta \le -\frac{\rho}{2}$$

for β large enough. Then (16) also holds when β is large.

Third region. Since θ_+ is a critical point of d_λ and an extremum of f, $L_A f(\theta_+) = 0$ and from (13),

$$L_\beta f_\beta(\theta_+, i) = \beta^{-1} Rf(\theta_+, i) = \beta^{-1} d_i(\theta_+)^2 f''(\theta_+) \le \beta^{-1} c_d \epsilon,$$

where

$$c_d = \max\bigl(d_0(\theta_+)^2, d_1(\theta_+)^2\bigr). \tag{17}$$

By continuity, we can find $l_+ > 0$ such that, for any $\theta \in [\theta_+ - l_+, \theta_+]$,

$$0 \le Rf(\theta, i) \le 2c_d\epsilon \quad \text{and} \quad 1 \le f(\theta, i) \le 1 + \epsilon.$$

Notice that l_+ does not depend on β. Without loss of generality, one can assume that K contains $[\theta_+ - l_+, \theta_+]$. We use (11) once more to get, for $\theta \in [\theta_+ - l_+, \theta_+]$,

$$\begin{aligned} L_\beta f_\beta(\theta, i) &\le 2c_d\epsilon\beta^{-1} \\ &\le -\frac{c_u}{6}\beta^{-1} f_\beta(\theta, i) + \frac{c_u}{6}\beta^{-1} f_\beta(\theta, i) + 2c_d\epsilon\beta^{-1} \\ &\le -\frac{c_u}{6}\beta^{-1} f_\beta(\theta, i) + \beta^{-1}\biggl((1+\epsilon)\frac{c_u}{6} + 2c_d\epsilon\biggr). \end{aligned}$$

Conclusion. Gathering the three estimates provides (10) with

$$a = \min_{\theta,i} f_\beta(\theta, i), \qquad \rho = \frac{c_u}{6}\beta^{-1} \quad \text{and} \quad C = \beta^{-1}\left((1+\epsilon)\frac{c_u}{6} + 2c_d\epsilon\right).$$

By (11), $a \geq 1-\epsilon$ when β is large. By Lemma 2.4,

$$\mu(K) \geq \frac{(1-\epsilon)\rho}{C} = \frac{1-\epsilon}{1+\epsilon+12(c_d/c_u)\epsilon}.$$

This can be arbitrarily close to 1 if we choose ϵ small enough.

3. Two explicit examples with a phase transition. In this section we perform a detail study of Examples 1.3 and 1.4. It has been pointed out in Section 2.4 that the angular processes associated to these two examples are of different types. The first one has two recurrent classes whereas the second one is ergodic. Nevertheless, we are able to get a perfect picture of the asymptotic of $\|X_t\|$ as a function of β for these two examples. As the studies are similar we present precisely the analysis of Example 1.4, and we provide more briefly the key expressions for Example 1.3.

3.1. *Example* 1.4. Let a and b be two positive real numbers, $\lambda = 1/2$, and set

$$A_0 = \begin{pmatrix} -1 & ab \\ -a/b & -1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -1 & -a/b \\ ab & -1 \end{pmatrix}$$

and

$$A_{1/2} = \frac{A_1 + A_0}{2} = \begin{pmatrix} -1 & a(b-1/b)/2 \\ a(b-1/b)/2 & -1 \end{pmatrix}.$$

The eigenvalues of A_0 and A_1 are equal to $-1 \pm ia$, whereas the eigenvalues of $A_{1/2}$ are $-1 \pm a(b-1/b)/2$. If $a(b-1/b) > 2$, that is, $b > 1+\sqrt{1+a^2}$, the matrix $A_{1/2}$ admits a positive and a negative eigenvalue. The associated eigenvectors are $(1, 1)$ and $(1, -1)$. The generator of the process (Θ_t, I_t) is given by

$$L_\beta f(\theta, i) = d_i(\theta)\partial_\theta f(\theta, i) + \frac{\beta}{2}\big(f(\theta, 1-i) - f(\theta, i)\big),$$

where

$$d_0(\theta) = -a/b\cos^2(\theta) - ab\sin^2(\theta) < 0,$$
$$d_1(\theta) = ab\cos^2(\theta) + a/b\sin^2(\theta) > 0.$$

LEMMA 3.1. *The invariant measure μ_β of the angular process is given by*

$$\mu_\beta(d\theta, i) = \frac{1}{C(\beta)}\frac{1}{|d_i(\theta)|}e^{\beta v(\theta)}\mathbb{1}_{[0,2\pi]}(\theta)\,d\theta,$$

where

$$v(\theta) = \begin{cases} \dfrac{1}{2a}\big(\arctan\big(b\tan(\theta)\big) - \arctan\big(b^{-1}\tan(\theta)\big)\big), & \text{if } \theta \neq \pm\dfrac{\pi}{2}, \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

and

$$C(\beta) = \int_0^{2\pi} \left[\frac{1}{d_1(\theta)} - \frac{1}{d_0(\theta)}\right] e^{\beta v(\theta)}\, d\theta.$$

REMARK 3.2. Notice that v belongs to $\mathcal{C}^\infty(\mathbb{T})$ and is π-periodic. Moreover, $v'(\theta) = 0$ if and only if $\theta = \pm\pi/4 + k\pi$. Finally, the function v reaches its maximum at $\pi/4 + k\pi$ and its minimum at $-\pi/4 + k\pi$.

PROOF OF LEMMA 3.1. If μ_β is an invariant measure for (Θ, I), then, for any smooth function f on $\mathbb{T} \times \{0, 1\}$, one has

$$\int_{\mathbb{T}\times\{0,1\}} L_\beta f(\theta, i)\, d\mu_\beta(\theta, i) = 0.$$

Let us look for an invariant measure μ_β on $\mathbb{T} \times \{0, 1\}$ that can be written as

$$\mu_\beta(d\theta, i) = \rho_0(\theta)\mathbb{1}_0(i)\, d\theta + \rho_1(\theta)\mathbb{1}_1(i)\, d\theta,$$

where ρ_0 and ρ_1 are two smooth and 2π-periodic functions. If f does not depend on the discrete variable $i \in \{0, 1\}$, that is, $f(\theta, i) = f(\theta)$, then

$$\begin{aligned} &\int_{\mathbb{T}\times\{0,1\}} L_\beta f(\theta)\, d\mu_\beta(\theta, i) \\ &\quad = \int_{\mathbb{T}} \partial_\theta f(\theta)(d_0\rho_0)(\theta)\, d\theta + \int_{\mathbb{T}} \partial_\theta f(\theta)(d_1\rho_1)(\theta)\, d\theta \end{aligned}$$

and an integration by parts leads to

$$\int_{\mathbb{T}\times\{0,1\}} L_\beta f(\theta)\, d\mu_\beta(\theta, i) = -\int_{\mathbb{T}} f(\theta)[d_0\rho_0 + d_1\rho_1]'(\theta)\, d\theta.$$

This ensures that $d_0\rho_0 + d_1\rho_1$ must be constant. Let us assume that one can find ρ_0 and ρ_1 such that $d_0\rho_0 + d_1\rho_1 = 0$. Now, if f is such that $f(\theta, 0) = f(\theta)$ et $f(\theta, 1) = 0$, we get

$$\begin{aligned} &\int_{\mathbb{T}\times\{0,1\}} L_\beta f(\theta, i)\, d\mu_\beta(\theta, i) \\ &\quad = \int_{\mathbb{T}} \left[d_0(\theta)\partial_\theta f(\theta) - \frac{\beta}{2} f(\theta)\right]\rho_0(\theta)\, d\theta + \int_{\mathbb{T}} \frac{\beta}{2} f(\theta)\rho_1(\theta)\, d\theta \end{aligned}$$

and after an integration by parts,

$$\begin{aligned} &\int_{\mathbb{T}\times\{0,1\}} L_\beta f(\theta, i)\, d\mu_\beta(\theta, i) \\ &\quad = \int_{\mathbb{T}} f(\theta)\left[-(d_0\rho_0)'(\theta) + \frac{\beta}{2}\big(\rho_1(\theta) - \rho_0(\theta)\big)\right] d\theta. \end{aligned}$$

Let us define $\phi = d_0\rho_0$. Then $\rho_0 = \frac{\phi}{d_0}$ and $\rho_1 = -\frac{\phi}{d_1}$. The function ϕ is solution of the following ordinary differential equation

$$\phi' = -\frac{\beta}{2}\left(\frac{1}{d_1} + \frac{1}{d_0}\right)\phi. \tag{19}$$

This equation admits a solution on $\mathbb{T}$ (i.e., 2π-periodic) since the integral of $\frac{1}{d_1} + \frac{1}{d_0}$ on $[-\pi, \pi]$ is equal to 0. In fact this is already true on $[-\pi/2, \pi/2]$. Since d_0 and d_1 are explicit trigonometric functions, one can find an explicit expression for ϕ. Notice that

$$\left[\arctan\left(b^{-1}\tan(\theta)\right)\right]' = \frac{1}{b}\cdot\frac{1+\tan^2(\theta)}{1+\tan^2(\theta)/b^2} = \frac{1}{b\cos^2(\theta) + 1/b\sin^2(\theta)} = \frac{a}{d_1(\theta)},$$

$$\left[\arctan\left(b\tan(\theta)\right)\right]' = -\frac{a}{d_0(\theta)}.$$

The differential equation (19) becomes $\phi' = \beta v'\phi$, where v is given by (18) and its solutions are given by

$$\phi = K\exp(\beta v).$$

This relation provides the expression of ρ_0 and ρ_1 up to the multiplicative constant K. Since we are looking for probability measures, K is such that

$$K\int_{\mathbb{T}}\left(\frac{1}{d_0(\theta)} - \frac{1}{d_1(\theta)}\right)\phi(\theta)\,d\theta = 1.$$

Conversely, it is easy to check that the measure given in Lemma 3.1 is invariant for L_β. $\square$

Let us now consider the function χ given by

$$\chi(\beta) = \int \mathcal{A}(\theta, i)\,d\mu_\beta(\theta, i).$$

LEMMA 3.3. *The function $\beta \mapsto \chi(\beta)$ is a $\mathcal{C}^1$ and monotonous map on $[0, +\infty)$ such that χ' has the sign of $b^2 - 1$ and*

$$\chi(0) = -1, \qquad \lim_{\beta\to\infty}\chi(\beta) = \frac{a(b^2-1)}{2b} - 1.$$

PROOF. From the definition of A_i and $\mathcal{A}$, we get that, for $i \in \{0, 1\}$,

$$\mathcal{A}(\theta, i) = \langle A_i e_\theta, e_\theta\rangle = \frac{a(b^2-1)}{2b}\sin(2\theta) - 1.$$

For sake of simplicity, $\mathcal{A}(\theta)$ stands for $\mathcal{A}(\theta, 0) = \mathcal{A}(\theta, 1)$. Thus, $\chi(\beta)$ is given by

$$\chi(\beta) = \int_0^{2\pi}\mathcal{A}(\theta)\tilde{\mu}_\beta(d\theta),$$

where

$$\tilde{\mu}_\beta(d\theta) = \frac{1}{C(\beta)}\left(\frac{1}{d_1(\theta)} - \frac{1}{d_0(\theta)}\right)e^{\beta v(\theta)}\mathbb{1}_{[0,2\pi]}\,d\theta.$$

Its derivative is given by

$$\begin{aligned}\chi'(\beta) &= \int_0^{2\pi} \mathcal{A}(\theta)v(\theta)\tilde{\mu}_\beta(d\theta) - \frac{C'(\beta)}{C(\beta)}\int_0^{2\pi}\mathcal{A}(\theta)\tilde{\mu}_\beta(d\theta)\\ &= \int_0^{2\pi} \mathcal{A}(\theta)v(\theta)\tilde{\mu}_\beta(d\theta) - \int_0^{2\pi} v(\theta)\tilde{\mu}_\beta(d\theta)\int_0^{2\pi}\mathcal{A}(\theta)\tilde{\mu}_\beta(d\theta).\end{aligned}$$

In other words, one has

$$\begin{aligned}\chi'(\beta) &= \operatorname{Cov}_{\tilde{\mu}_\beta}\big(\mathcal{A}(\cdot), v(\cdot)\big)\\ &= \frac{a(b^2-1)}{2b}\operatorname{Cov}_{\tilde{\mu}_\beta}\big(\sin(2\cdot), v(\cdot)\big).\end{aligned}$$

The mean of $\sin(2\cdot)$ with respect to $\tilde{\mu}_\beta$ is equal to 0. Besides, $\theta \mapsto v(\theta)\sin(2\theta)$ is nonnegative (and nonconstant) on $\mathbb{T}$. Thus, χ' has the sign of b^2-1.

If $\beta = 0$, one has

$$\begin{aligned}\chi(0) &= \frac{1}{C(0)}\int_0^{2\pi}\left(\frac{a(b^2-1)}{2b}\sin(2\theta) - 1\right)\left(\frac{1}{d_1(\theta)} - \frac{1}{d_0(\theta)}\right)d\theta\\ &= -\frac{1}{C(0)}\int_0^{2\pi}\left(\frac{1}{d_1(\theta)} - \frac{1}{d_0(\theta)}\right)d\theta = -1 < 0.\end{aligned}$$

Finally, as β goes to ∞, the probability measure ν_β converges to a probability measure concentrated on the points $\{\pi/4, 5\pi/4\}$, where v reaches its maximum. We get

$$\lim_{\beta\to+\infty}\chi(\beta) = \frac{a(b^2-1)}{2b} - 1.$$

This completes the proof. $\square$

COROLLARY 3.4. *If* $b > 1 + \sqrt{1+a^2}$, *then there exists* $\beta_c \in (0, +\infty)$ *such that* χ *is negative on* $(0, \beta_c)$ *and positive on* $(\beta_c, +\infty)$.

3.2. *Example* 1.3. Let us define A_0 and A_1 by

$$A_0 = \begin{pmatrix} -1 & 2b \\ 0 & -1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -1 & 0 \\ 2b & -1 \end{pmatrix}$$

with $b > 0$. Then A_0 and A_1 are two Jordan matrices, and the eigenvalues of $A_{1/2}$ are given by $-1 \pm b$. In this case,

$$d_0(\theta) = -2b\sin^2(\theta) \le 0 \quad \text{and} \quad d_1(\theta) = 2b\cos^2(\theta) \ge 0$$

and (Θ, I) has two recurrent classes

$$C_1 = \{(\theta, i) : \theta \in (0, \pi/2), i = 0, 1\},$$
$$C_2 = \{(\theta, i) : \theta \in (\pi, 3\pi/2), i = 0, 1\}.$$

It can be shown, following the lines of the previous section, that the ergodic invariant measure μ_β of the angular process on C_1 is given by

$$\mu_\beta(d\theta, i) = \frac{1}{C(\beta)} \cdot \frac{1}{|d_i(\theta)|} e^{\beta v(\theta)} \mathbb{1}_{(0,\pi/2)}(\theta)\, d\theta,$$

where

$$v(\theta) = -\frac{1}{2b\sin(2\theta)} \quad \text{and} \quad C(\beta) = \frac{2}{b}\int_0^{\pi/2} \frac{1}{\sin^2(2\theta)} e^{\beta v(\theta)}\, d\theta.$$

Moreover, for any $\beta > 0$,

$$\chi(\beta) = -1 + \frac{1}{C(\beta)} \int_0^{\pi/2} \frac{2}{\sin(2\theta)} e^{\beta v(\theta)}\, d\theta.$$

In particular, the map $\beta \mapsto \chi(\beta)$ is a $\mathcal{C}^1$ increasing function on $[0, +\infty)$ such that

$$\chi(0) = -1, \qquad \lim_{\beta\to\infty} \chi(\beta) = -1 + b.$$

COROLLARY 3.5. *If $b > 1$, then there exists $\beta_c \in (0, +\infty)$ such that χ is negative on $(0, \beta_c)$ and positive on $(\beta_c, +\infty)$.*

4. Application to matrix products. The process studied in the preceding sections is linked to some products of random matrices. Let us consider the embedded chain of our process defined by the sequence of the positions of the process X at the times when the second coordinate I changes, that is, the positions at the times when one changes the flow. The jump times are given by sums of independent random variables with exponential law of parameters $\lambda_0\beta$ and $\lambda_1\beta$. To study this embedded chain is to study the linear images of vectors by products of independent random matrices which distributions are the image laws of exponential law of parameter 1 by the two mappings

$$s \mapsto \exp\big((s/\beta\lambda_0)A_0\big) \quad \text{and} \quad s \mapsto \exp\big((s/\beta\lambda_1)A_1\big).$$

Let us denote $(T_k)_{k\geq 0}$ the sequence of the jump times of the second coordinate (with the convention $T_0 = 0$) and $(Z_k)_{k\geq 0}$ the sequence of the positions of X at these times,

$$Z_k = X_{T_k}.$$

The embedded chain and the process $(X_t)t \geq 0$ are linked as follows. For $t \in]T_k, T_{k+1}]$ one has

$$X_t = \exp\left(\frac{t - T_k}{\beta\lambda_{i_k}} A_{i_k}\right) Z_k,$$

where i_k is 0 or 1 depending on the evenness of k. Thus

$$Z_k = U_k U_{k-1} \cdots U_1 X_0 \qquad \text{where } U_l = \exp\left(\frac{T_l - T_{l-1}}{\beta\lambda_{i_{l-1}}} A_{i_{l-1}}\right).$$

For example, we can fix that $i_0 = 0$, which means that at time 0, X is driven by the vector field $x \mapsto A_0 x$.

Let $e^{(1)}$ and $e^{(2)}$ be the element of the canonical basis of $\mathbb{R}^2$, $X_t^{(1)}$ and $X_t^{(2)}$ the processes starting from $e^{(1)}$ and $e^{(2)}$, respectively. From the equality

$$X_t^{(1)} = \exp\left(\frac{t - T_k}{\beta\lambda_{i_k}} A_{i_k}\right) U_k U_{k-1} \cdots U_1 e^{(1)},$$

we get

$$\begin{aligned}\|U_k U_{k-1} \cdots U_1\| &\geq \|U_k U_{k-1} \cdots U_1 e^{(1)}\| \\ &\geq \|\exp(-((t - T_k)/\beta\lambda_{i_k}) A_{i_k}) X_t^{(1)}\| \\ &\geq \|\exp(((t - T_k)/\beta\lambda_{i_k}) A_{i_k})\|^{-1} \|X_t^{(1)}\|.\end{aligned}$$

On the other hand, for $t \in]T_k, T_{k+1}]$, we have

$$\begin{aligned}\|U_k U_{k-1} \cdots U_1\| &\leq \|U_k U_{k-1} \cdots U_1 e^{(1)}\| + \|U_k U_{k-1} \cdots U_1 e^{(2)}\| \\ &= \sum_{j=1}^{2} \|\exp(-((t - T_k)/\beta\lambda_{i_k}) A_{i_k})\| \|X_t^{(j)}\| \\ &\leq 2\|\exp(-((t - T_k)/\beta\lambda_{i_k}) A_{i_k})\| \max(\|X_t^{(1)}\|, \|X_t^{(2)}\|).\end{aligned}$$

According to Theorem 1.6 almost surely both limits

$$\lim_{t\to\infty} \frac{1}{t} \log\|X_t^{(1)}\| \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t} \log\|X_t^{(2)}\|$$

exist and are equal to $\chi(\beta)$. Moreover, almost surely, the ratio $(t - T_k)/t$ tends to 0 and, as T_k is the sum of independent random variables of parameter $\lambda\beta$ and $(1 - \lambda)\beta$, the strong law of large numbers gives

$$\frac{T_{2k}}{2k} \xrightarrow[k\to\infty]{} \frac{1}{2\lambda\beta} + \frac{1}{2(1 - \lambda)\beta} = \frac{1}{2\lambda(1 - \lambda)\beta},$$

so that T_k/k almost surely tends toward $(2\lambda(1 - \lambda)\beta)^{-1}$. Putting things together we get that, almost surely,

$$\lim_{k\to\infty} \frac{1}{k} \log\|U_k U_{k-1} \cdots U_1\| = \frac{\chi(\beta)}{2\lambda(1 - \lambda)\beta}.$$

In particular this limit has the same sign as $\chi(\beta)$, it is negative for small β and positive for large β.

This does give an example of a product of independent matrices, the eigenvalues of which are of modulus less than one, with a positive Lyapunov exponent, but in this case the matrices $(U_k)_k$ do not have the same distribution; it depends on the evenness of k. If we group the U_k by 2 we get a product of independent matrices with the same distribution, but their eigenvalues are not always of modulus less than one. Some matrices in the image of

$$(s,t) \mapsto \exp\left(\frac{t}{\beta\lambda_1} A_1\right) \exp\left(\frac{s}{\beta\lambda_0} A_0\right)$$

are hyperbolic.

So let us slightly modify the process we began with. When the second coordinate is $i \in \{0, 1\}$, at each date given by the sum of independent random variables with exponential law of parameter $\lambda_i \beta$, one chooses independently with probability $1/2$ to keep the flow i or with probability $1/2$ to flip to the flow $1 - i$. As an independent geometric random sum of exponential independent random variables is still an exponential random variable, in continuous time, this modification is simply a change of parameter β (replaced par $\beta/2$). The embedded chain defined by the position at times given by (not the changes of flow but) the sums of exponential random variables, also corresponds to a products of independent random matrices, and this time, all matrices considered have eigenvalues of modulus less than one.

Let (D_k) denote the sequence of dates considered in this case. It is a sum of k independent exponential variables of parameters $\beta\lambda_0$ and $\beta\lambda_1$ and, almost surely, asymptotically, half of them are of parameter $\beta\lambda_0$, half of them of parameter $\beta\lambda_1$. So that, as before, D_k/k almost surely tends to $(2\lambda(1-\lambda)\beta)^{-1}$. These remarks and the preceding computation give the following proposition.

PROPOSITION 4.1. *Let A_0 and A_1 two matrices such that Assumption 1.1 is satisfied. Let $(V_k)_{k\geq 1}$ be a sequence of independent matrices with distribution given by the half sum of the image measures of the exponential law of parameter* 1 *by the two mappings*

$$s \mapsto \exp\left(\frac{s}{\beta\lambda_0} A_0\right) \quad \textit{and} \quad t \mapsto \exp\left(\frac{t}{\beta\lambda_1} A_1\right).$$

Then almost surely, one has

$$\lim_{k\to\infty} \frac{1}{k} \log \|V_k V_{k-1} \cdots V_1\| = \frac{\chi(\beta/2)}{2\lambda(1-\lambda)\beta}$$

and if β is sufficiently large, this limit is positive.

Thus we have obtained examples of product of random independent identically distributed matrices, the eigenvalues of which have modulus less than one, with a positive Lyapounov exponent.

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