The Annals of Applied Probability
2012, Vol. 22, No. 6, 2460–2504
DOI: 10.1214/11-AAP839

A DIFFUSION APPROXIMATION THEOREM FOR A NONLINEAR PDE WITH APPLICATION TO RANDOM BIREFRINGENT OPTICAL FIBERS[1]

BY A. DE BOUARD AND M. GAZEAU

CMAP, CNRS and École Polytechnique

In this article we propose a generalization of the theory of diffusion approximation for random ODE to a nonlinear system of random Schrödinger equations. This system arises in the study of pulse propagation in randomly birefringent optical fibers. We first show existence and uniqueness of solutions for the random PDE and the limiting equation. We follow the work of Garnier and Marty [*Wave Motion* **43** (2006) 544–560], Marty [Problèmes d'évolution en milieux aléatoires: Théorèmes limites, schémas numériques et applications en optique (2005) Univ. Paul Sabatier], where a linear electric field is considered, and we get an asymptotic dynamic for the nonlinear electric field.

1. Introduction. The Manakov PMD equation has been introduced by Wai and Menyuk in [31] to study light propagation over long distance in random birefringent optical fibers. Due to the various length scales present in this problem, a small parameter ε appears in the rescaled equation. Our aim in this paper is to prove a diffusion limit theorem for this equation for which we will have to generalize the perturbed test function method [5, 20, 24] to the case of infinite dimension. In [18, 22], a limit theorem is proved for the linear part of the Manakov PMD equation using the Fourier transform and the theory of diffusion approximation for random ODE. Obviously the method in [18, 22] does not work for a nonlinear PDE. In [12, 22], a limit theorem is proved for a nonlinear scalar PDE driven by a one-dimensional noise. The proof relies on the fact that the solution processes are continuous functions of the noise. These methods are no longer applicable to the limit equation that we will consider which is driven by a three-dimensional noise, because the solution cannot be written as a continuous function of the noise. Indeed, in a general setting a strong solution of a stochastic equation is only a measurable function of the initial data and the Brownian motion driving the equation. However, in the case of a one-dimensional noise, Doss [14] and Sussman [27] proved that the solution of such an equation can be written as a continuous

Received May 2011; revised November 2011.

[1]Supported in part by the French ANR research Project ANR-07-44 BLAN-0250 and by a grant from Région Ile-de-France.

MSC2010 subject classifications. 35Q55, 60H15.

Key words and phrases. Nonlinear Schrödinger equation, stochastic partial differential equations, white noise, diffusion limit.

function of the Brownian motion. This result has been extended by Yamato [32] to multidimensional Brownian motions when the Lie algebra generated by the vector fields of the equation is nilpopent of step p. He actually proves the equivalence between the nilpotent hypothesis and the fact that the solution can be written as a continuous function of iterated Stratonovich integrals. In our case the vector fields driving the Manakov PMD equation are functions of the Pauli matrices and the nilpotent hypothesis of Yamato is not satisfied. This motivates the use of the perturbed test function method. Note that the method has been used for a linear PDE in [13] and a PDE with bounded diffusion coefficients in [25].

We are also interested in the mathematical analysis of both the Manakov PMD and the limit equations. Using a unitary transformation, we are able to establish Strichartz estimates for the transformed equation, that are not available for the Manakov PMD equation. This result will then enable us to prove global existence of solutions. The limiting equation is also studied. We use a compactness method to study the existence and uniqueness of solutions of this latter equation, due to the lack of nilpotent hypothesis and to the absence of unitary transformation similar to the Manakov PMD case.

1.1. *Presentation of the model.* Optical fibers are thin, transparent and flexible fibers along which the light propagates to transmit information over long distances and so are of huge interest in modern communications. In a perfect fiber, the two transverse components of the electric field are degenerate in the sense that they propagate with the same characteristics: group velocity, chromatic dispersion, refractive indices ($n_1 = n_2$), etc. However, during the fabrication process the fiber may present defects like an ellipticity of the core or suffer from mechanical distortions like stress constraints or twisting [1, 2]. These phenomena induce modal birefringence ($n_1 \neq n_2$) characterized by an orientation angle θ and an amplitude b. If $n_1 > n_2$, we then define a slow axe and a rapid axe corresponding, respectively, to the mode indices n_1 and n_2. The orientation angle θ describes the rotation of the local polarization axes with respect to the initial axes. The birefringence strength (or degree of modal birefringence) is given by $b = |n_1 - n_2|k_0 = k_1 - k_2$, where k_1, k_2 are the components of the wave vector and k_0 the wavenumber of the incident light in vacuum. The beat length $L_B = \frac{2\pi}{k_1 - k_2}$ indicates the length required for the polarization to return to its initial state. There exist several types of birefringence that do not have the same effect on the electric field. Usually linear birefringence is studied (in the absence of Kerr effect, a linearly polarized light remains linearly polarized), although it has been shown that the birefringence could also be elliptic (occurring in case of twisting, see Menyuk [23]). In case of a uniform anisotropy along the fiber, the birefringence parameters (θ, b) are constant. However, in realistic configurations, the anisotropy is not uniform along the fiber. We assume, as in [28–31], that the birefringence is randomly varying, implying polarization mode dispersion (PMD). The difference of velocity of the two modes, due to random change of the birefringence (and so of the refractive indices), induces coupling between the two polarized modes and pulse spreading: PMD is one of the limiting factors of high bit rate transmission.

In [31], Wai and Menyuk assumed that there is no polarization-dependent loss and considered that communication fibers are nearly linearly birefringent. We here use one of the models introduced in [31] for which the local axes of birefringence are bended with an angle θ randomly varying along the propagation axe and that b and b' (the frequency derivative of b) are constant along this axe. Let us recall that the Pauli matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let us consider the coupled nonlinear Schrödinger equation transformed into the frame of the local axes of birefringence [21, 31]

$$(1.1)\qquad \begin{aligned} &i\frac{\partial \Psi}{\partial t} + \widetilde{\Sigma}(t)\Psi + ib'\sigma_3\frac{\partial \Psi}{\partial x} + \frac{d_0}{2}\frac{\partial^2 \Psi}{\partial x^2} \\ &\qquad + \frac{5}{6}|\Psi|^2\Psi + \frac{1}{6}(\Psi^*\sigma_3\Psi)\sigma_3\Psi + \frac{1}{3}N(\Psi) = 0, \end{aligned}$$

where d_0 is the group velocity dispersion parameter, $N(\Psi) = (\overline{\Psi}_1\Psi_2^2, \overline{\Psi}_2\Psi_1^2)^t$ and

$$\widetilde{\Sigma}(t) = \begin{pmatrix} b & -\dfrac{i}{2}\dfrac{d\theta(t)}{dt} \\ \dfrac{i}{2}\dfrac{d\theta(t)}{dt} & -b \end{pmatrix}.$$

We recall that in the context of fiber optics, x corresponds to the retarded time while t corresponds to the distance along the fiber. We introduce a new vector field $\widetilde{\Psi} = \exp(-ibt\sigma_3)\Psi$. The evolution of $\widetilde{\Psi}$ is given by the previous equation (1.1) replacing $\widetilde{\Sigma}$ and $N(\Psi)$, respectively, by

$$\widetilde{\widetilde{\Sigma}}(t) = \begin{pmatrix} 0 & -\dfrac{i}{2}\dfrac{d\theta(t)}{dt}e^{-2ibt} \\ \dfrac{i}{2}\dfrac{d\theta(t)}{dt}e^{2ibt} & 0 \end{pmatrix} \quad \text{and} \quad N(\widetilde{\Psi}) = \begin{pmatrix} \overline{\widetilde{\Psi}}_1\widetilde{\Psi}_2^2 e^{-4ibt} \\ \overline{\widetilde{\Psi}}_2\widetilde{\Psi}_1^2 e^{4ibt} \end{pmatrix}.$$

Following Wai and Menyuk [21, 29–31], we denote by l the fiber length. We also denote by l_d the dispersion length scale and l_{nl} the nonlinear length scale related to Kerr effect. The fiber autocorrelation length l_c is the length over which two polarization components remain correlated. We consider, as in [31], a typical configuration where $l \sim l_d \sim l_{nl} \gg l_c \gg L_B$, that is, we consider "relatively small" propagation distances. Under these assumptions and the assumptions on $d\theta/dt$ below, the term $N(\widetilde{\Psi})$ is rapidly oscillating and will be neglected [2, 21, 31], its effect being averaged out to zero. As in [21, 31], we introduce a unitary matrix

$$(1.2)\qquad T(t) = \begin{pmatrix} u_1(t) & \overline{u}_2(t) \\ -u_2(t) & \overline{u}_1(t) \end{pmatrix},$$

the solution of

$$i\,\frac{\partial T(t)}{\partial t}+\tilde{\tilde{\tilde{\Sigma}}}(t)T(t)=0. \tag{1.3}$$

We also consider, for $t\in\mathbb{R}_+$, the matrix

$$\begin{aligned}\boldsymbol{\sigma}(u(t))&=\begin{pmatrix}|u_1|^2-|u_2|^2 & 2\overline{u}_1\overline{u}_2\\ 2u_1u_2 & |u_2|^2-|u_1|^2\end{pmatrix}=\begin{pmatrix}m_3 & m_1-im_2\\ m_1+im_2 & -m_3\end{pmatrix}\\ &=\sigma_1m_1(t)+\sigma_2m_2(t)+\sigma_3m_3(t),\end{aligned}\tag{1.4}$$

which characterizes the linear birefringence and where m_1, m_2, m_3 are real-valued processes. Then we can remove the rapid variation of the state of polarization in the evolution of $\widetilde{\Psi}$ using the change of variable $\widetilde{\Psi}(t)=T(t)X(t)$. We obtain

$$\begin{aligned}&i\,\frac{\partial X}{\partial t}+ib'\boldsymbol{\sigma}(u(t))\,\frac{\partial X}{\partial x}+\frac{d_0}{2}\,\frac{\partial^2 X}{\partial x^2}\\ &\qquad+\frac{5}{6}|X|^2X+\frac{1}{6}(X^*\sigma_3X)\sigma_3X+\frac{1}{6}N_u(X)=0,\end{aligned}\tag{1.5}$$

where $N_u(X)=(N_{1,u}(X),N_{2,u}(X))^t$ satisfy

$$\begin{aligned}N_{1,u}(X)&=(m_1^2+m_2^2)(2|X_2|^2-|X_1|^2)X_1\\ &\quad+(m_1-im_2)m_3(2|X_1|^2-|X_2|^2)X_2\\ &\quad+(m_1-im_2)^2X_2^2\overline{X}_1+(m_1+im_2)m_3X_1^2\overline{X}_2,\end{aligned}\tag{1.6}$$

$$\begin{aligned}N_{2,u}(X)&=(m_1^2+m_2^2)(2|X_1|^2-|X_2|^2)X_2\\ &\quad-(m_1+im_2)m_3(2|X_2|^2-|X_1|^2)X_1\\ &\quad-(m_1-im_2)m_3X_2^2\overline{X}_1+(m_1+im_2)^2X_1^2\overline{X}_2.\end{aligned}\tag{1.7}$$

Assuming, as in [18, 22, 30], that the correlation length of $d\theta/dt$ is much shorter than the birefringence beat length and that $|d\theta/dt|\ll b$, we set $d\theta/dt=2\varepsilon_0\alpha(t)$, where ε_0 is a small dimensionless parameter and α a Markov process with good ergodic properties. Thus, we may replace the process u by ν, with [18, 22]

$$\begin{aligned}d\nu(t)&=i\sqrt{\gamma_c}\big(\sigma_1\nu(t)\circ dW_1(t)+\sigma_2\nu(t)\circ dW_2(t)\big)+i\gamma_s\sigma_3\nu(t)\,dt\\ &=i\sqrt{\gamma_c}\big(\sigma_1\nu(t)\,dW_1(t)+\sigma_2\nu(t)\,dW_2(t)\big)+i\gamma_s\sigma_3\nu(t)\,dt\\ &\quad-\gamma_c\nu(t)\,dt,\end{aligned}\tag{1.8}$$

where $|\nu_1(0)|^2+|\nu_2(0)|^2=1$, $W=(W_1,W_2)$ is a $2d$ real-valued Brownian motion and $\circ$ denotes the Stratonovich product. The second equation is the corresponding Itô equation. In addition, γ_c,γ_s are two constants determined by α and given by

$$\gamma_c=\int_0^\infty\cos(2bt)\mathbb{E}(\alpha(0)\alpha(t))\,dt\quad\text{and}\quad\gamma_s=\int_0^\infty\sin(2bt)\mathbb{E}(\alpha(0)\alpha(t))\,dt.$$

Then $\nu(t) \in \mathbb{S}^3$ a.s., the unit sphere in $\mathbb{C}^2 \sim \mathbb{R}^4$. We denote by Λ the unique invariant probability measure of ν (see Section 5) and by $\mathbb{E}_\Lambda(\cdot)$ the expectation with respect to Λ. Thus, replacing u by ν in (1.5), we obtain a new equation describing the evolution of the electric field envelope $X = (X_1, X_2)^t$:

$$
\begin{aligned}
& i\,\frac{\partial X}{\partial t} + \frac{d_0}{2}\,\frac{\partial^2 X}{\partial x^2} + \frac{8}{9}|X|^2 X \\
& \qquad = -ib'\boldsymbol{\sigma}(\nu(t))\,\frac{\partial X}{\partial x} - \frac{1}{6}\big(N_\nu(X) - \mathbb{E}_\Lambda(N_\nu(X))\big);
\end{aligned} \tag{1.9}
$$

indeed, the process $m = (m_1, m_2, m_3)$ is now defined as $m = (g_1(\nu), g_2(\nu), g_3(\nu))$ and it can be proved (see Section 5) that

$$
\begin{aligned}
\mathbb{E}_\Lambda(N_{1,\nu}(X)) &= \tfrac{2}{3}(2|X_2|^2 - |X_1|^2)X_1, \\
\mathbb{E}_\Lambda(N_{2,\nu}(X)) &= \tfrac{2}{3}(2|X_1|^2 - |X_2|^2)X_2.
\end{aligned}
$$

We set

$$
F_{\nu(t)}(X(t)) = \tfrac{8}{9}|X|^2 X - \tfrac{1}{6}\big(N_\nu(X) - \mathbb{E}_\Lambda(N_\nu(X))\big). \tag{1.10}
$$

Equation (1.9) is of great interest for the study of dispersion because the main effects leading to signal distortions (Kerr effect, chromatic dispersion, PMD) can be easily identified: on the left-hand side, the first term describes the evolution of the pulse along the fiber. The second one corresponds to the chromatic dispersion and the last term to the Kerr effect averaged on the Poincaré sphere. On the right-hand side of the equation, the first term describes the linear PMD effect and the second term describes nonlinear PMD.

The Manakov PMD equation (1.9) is written in dimensionless form. According to the length scales we consider, we set $X_\varepsilon(t,x) = \frac{1}{\varepsilon}X(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ and $\nu_\varepsilon(t) = \nu(\frac{t}{\varepsilon^2})$, where ν is the solution of (1.8); then the electric field X_ε has the following evolution:

$$
i\,\frac{\partial X_\varepsilon(t)}{\partial t} + \frac{ib'}{\varepsilon}\boldsymbol{\sigma}(\nu_\varepsilon(t))\,\frac{\partial X_\varepsilon(t)}{\partial x} + \frac{d_0}{2}\,\frac{\partial^2 X_\varepsilon(t)}{\partial x^2} + F_{\nu_\varepsilon(t)}(X_\varepsilon(t)) = 0, \tag{1.11}
$$

where the term $F_{\nu_\varepsilon(t)}(X_\varepsilon(t))$ is given by (1.10).

In various physical situations, the long time behavior of a phenomenon subject to random perturbations requires to take care of the different characteristic length scales of the problem. In this context Papanicolaou, Stroock and Varadhan [24] and Blankenship and Papanicolaou [5] introduced the approximation diffusion theory for random ordinary differential equations. This method has been used to study wave propagation in random media [17] and, in particular, in randomly birefringent fibers [18, 22], but only few results exist on limit theorems for random PDEs. In the latter, the authors studied the evolution, in an optical fiber, of the linear field envelope X_ε given by

$$
i\,\frac{\partial X_\varepsilon(t)}{\partial t} + \frac{ib'}{\varepsilon}\boldsymbol{\sigma}(\nu_\varepsilon(t))\,\frac{\partial X_\varepsilon(t)}{\partial x} + \frac{d_0}{2}\,\frac{\partial^2 X_\varepsilon(t)}{\partial x^2} = 0
$$

and proved that the asymptotic dynamics, when ε goes to zero, is given by

$$i\,dX(t)+\left(\frac{d_0}{2}\frac{\partial^2 X(t)}{\partial x^2}\right)dt+i\sqrt{\gamma}\sum_{k=1}^{3}\sigma_k\frac{\partial X(t)}{\partial x}\circ dW_k(t)=0,$$

where $W=(W_1,W_2,W_3)$ is a 3d Brownian motion, and $\gamma=(b')^2/6\gamma_c$. Note that the linear PMD effect reduces to one single parameter γ in front of the three Brownian motions. Generalizing the perturbed test function method, we will prove that the asymptotic dynamic of (1.11) is given by the stochastic nonlinear evolution:

$$\begin{aligned}&i\,dX(t)+\left(\frac{d_0}{2}\frac{\partial^2 X(t)}{\partial x^2}+F(X(t))\right)dt\\&\quad+i\sqrt{\gamma}\sum_{k=1}^{3}\sigma_k\frac{\partial X(t)}{\partial x}\circ dW_k(t)\\&\quad=0,\end{aligned}\tag{1.12}$$

where the nonlinear function F reduces to $F(X(t))=\frac{8}{9}|X(t)|^2X(t)$ that is simply the expectation, with respect to the invariant measure Λ, of $F_{\nu_\varepsilon(t)}(X_\varepsilon(t))$. We will also make use of the following equivalent Itô formulation:

$$\begin{aligned}&i\,dX(t)+\left(\left(\frac{d_0}{2}-\frac{3i\gamma}{2}\right)\frac{\partial^2 X(t)}{\partial x^2}+F(X)(t)\right)dt\\&\quad+i\sqrt{\gamma}\sum_{k=1}^{3}\sigma_k\frac{\partial X(t)}{\partial x}\,dW_k(t)\\&\quad=0.\end{aligned}\tag{1.13}$$

Note that a different regime concerned with long propagation distances and corresponding to $l\gg l_{nl}\sim l_d$ is of physical interest; however, this regime would lead to another asymptotic analysis which is beyond the scope of this paper.

This paper is organized as follows: in Section 1.2 we give notation that will be used along the paper and state the main results. Section 2 is devoted to the proof of well-posedness for the Manakov PMD equation. In Section 3 we study the local well-posedness of the limiting equation (1.12). Finally, in Section 4 we prove the convergence in law of X_ε to X as ε goes to zero. This paper ends with Section 5 where we recall some results obtained in [18, 22] about the driving process ν, and Section 6 where proofs of technical results used in Section 4 are gathered.

1.2. *Notation and main results.* Before stating the main results of this article, let us give some definitions and notation.

For all $p\geq 1$, we define $\mathbb{L}^p(\mathbb{R})=(L^p(\mathbb{R};\mathbb{C}))^2$ the Lebesgue spaces of functions with values in $\mathbb{C}^2$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we define a scalar product on

$\mathbb{L}^2(\mathbb{R})$ by

$$(u, v)_{\mathbb{L}^2} = \sum_{i=1}^{2} \mathcal{R}\mathrm{e}\left\{ \int_{\mathbb{R}} u_i \overline{v}_i \, dx \right\}.$$

We denote by $\mathbb{W}^{m,p}$, $m \in \mathbb{N}^*$, $p \in \mathbb{N}^*$ the space of functions in $\mathbb{L}^p$ such that their m first derivatives are in $\mathbb{L}^p$. If $p = 2$, then we denote $\mathbb{H}^m(\mathbb{R}) = \mathbb{W}^{m,2}(\mathbb{R})$, $m \in \mathbb{N}$. We will also use $\mathbb{H}^{-m}$ the topological dual space of $\mathbb{H}^m$ and denote $\langle \cdot, \cdot \rangle$ the paring between $\mathbb{H}^m$ and $\mathbb{H}^{-m}$. The Fourier transform of a tempered distribution $v \in \mathcal{S}'(\mathbb{R})$ is either denoted by $\widehat{v}$ or $\mathcal{F}v$. If $s \in \mathbb{R}$, then $\mathbb{H}^s$ is the fractional Sobolev space of tempered distributions $v \in \mathcal{S}'(\mathbb{R})$ such that $(1 + |\xi|^2)^{s/2} \widehat{v} \in \mathbb{L}^2$. Let $(E, \| \cdot \|_E)$ and $(F, \| \cdot \|_F)$ be two Banach spaces. We denote by $\mathcal{L}(E, F)$ the space of linear continuous functions from E into F, endowed with its natural norm. If I is an interval of $\mathbb{R}$ and $1 \le p \le +\infty$, then $L^p(I; E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \mapsto \|f(t)\|_E$ is in $L^p(I)$. The space $L^p(\Omega, E)$ is defined similarly where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. We denote by $L^p_w(I, E)$ the space $L^p(I, E)$ endowed with the weak (or weak star) topology. For a real number $0 < \alpha < 1$ and $p \ge 1$, we denote by $W^{\alpha,p}([0, T], E)$ the fractional Sobolev space of functions u in $L^p(0, T; E)$ satisfying

$$\int_0^T \int_0^T \frac{\|u(t) - u(s)\|_E^p}{|t - s|^{\alpha p + 1}} \, ds \, dt < +\infty.$$

The space $C^\beta([0, T]; E)$ is the space of Hölder continuous functions of order $\beta > 0$ with values in E and we denote by $\mathcal{M}(E)$ the set of probability measures on E, endowed with the topology of the weak convergence $\sigma(\mathcal{M}(E), C_b(E))$.

We will use the space

$$\mathcal{K} = \big(C([0, T], \mathbb{H}^1_{\mathrm{loc}}) \cap C_w([0, T], \mathbb{H}^1) \cap L^\infty_w(0, T; \mathbb{H}^2) \big) \times C([0, T], \mathbb{R}),$$

where $C_w([0, T], \mathbb{H}^m), m \in \mathbb{Z}$ is the space of functions f in $L^\infty(0, T; \mathbb{H}^m)$, weakly continuous from $[0, T]$ into $\mathbb{H}^m$. As the solution of our limit equation will not necessary be global in time, we need to introduce a space of exploding paths, as in [3], by adding a point Δ, which acts as a cemetery point, at infinity in $\mathbb{H}^1$; then

$$\begin{aligned} \mathcal{E}(\mathbb{H}^1) = \big\{ & f \in C([0, T], \mathbb{H}^1 \cup \{\Delta\}), \\ & f(t_0) = \Delta \text{ for } t_0 \in [0, T] \Rightarrow f(t) = \Delta \text{ for } t \in [t_0, T] \big\}. \end{aligned}$$

We define a topology on $\mathbb{H}^1 \cup \{\Delta\}$ such that the open sets of $\mathbb{H}^1 \cup \{\Delta\}$ are the open sets of $\mathbb{H}^1$ and the complementary in $\mathbb{H}^1 \cup \{\Delta\}$ of the closed bounded sets in $\mathbb{H}^1$. For any $f \in C([0, T], \mathbb{H}^1 \cup \{\Delta\})$ we denote the blowing-up time $\tau(f)$ by

$$\tau(f) = \inf\{t \in [0, T], f(t) = \Delta\}$$

with the convention $\tau(f) = +\infty$ if $f(t) \neq \Delta$ for all $t \in [0, T]$. We endow the space $\mathcal{E}(\mathbb{H}^1)$ with the topology induced by the uniform convergence in $\mathbb{H}^1$ on every compact set of $[0, \tau(f))$.

Let $(\mathcal{A}, \mathcal{G}, \mathbb{Q})$ be a probability space endowed with the complete filtration $(\mathcal{G}_t)_{t\geq 0}$ generated by a two-dimensional Brownian motion $W = (W_1, W_2)$ which is driving the diffusion process ν given by (1.8). We first state an existence and uniqueness result for (1.11).

THEOREM 1.1. *Let $\varepsilon > 0$ and suppose that $X_\varepsilon(0) = v \in \mathbb{L}^2(\mathbb{R})$, then there exists a unique global solution X_ε to* (1.11) *such that*, $\mathbb{Q}$*-almost surely*,

$$X_\varepsilon \in C(\mathbb{R}_+, \mathbb{L}^2) \cap C^1(\mathbb{R}_+, \mathbb{H}^{-2}) \cap \mathbb{L}^8_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{L}^4).$$

Moreover, equation (1.11) *preserves the $\mathbb{L}^2$ norm, that is, for all $t \in \mathbb{R}_+$*

$$\|X_\varepsilon(t)\|_{\mathbb{L}^2} = \|v\|_{\mathbb{L}^2}.$$

If, in addition, $X_\varepsilon(0) = v \in \mathbb{H}^1$ (*resp.*, $\mathbb{H}^2$, *resp.*, $\mathbb{H}^3$), *then the corresponding solution is in $C(\mathbb{R}_+, \mathbb{H}^1)$* [*resp.*, $C(\mathbb{R}_+, \mathbb{H}^2)$, *resp.*, $C(\mathbb{R}_+, \mathbb{H}^3)$].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a three-dimensional real-valued Brownian motion $W = (W_1, W_2, W_3)$. We denote by $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ the complete filtration generated by W. The next theorem gives existence and uniqueness of the local solution for (1.12)

THEOREM 1.2. *Let $X_0 = v \in \mathbb{H}^1(\mathbb{R})$, then there exists a maximal stopping time $\tau^*(v, \omega)$ and a unique strong solution X* (*in the probabilistic sense*) *to* (1.12), *such that $X \in C([0, \tau^*), \mathbb{H}^1(\mathbb{R}))$ $\mathbb{P}$-a.s. Furthermore, the $\mathbb{L}^2$ norm is almost surely preserved, that is*, $\forall t \in [0, \tau^*)$, $\|X(t)\|_{\mathbb{L}^2} = \|v\|_{\mathbb{L}^2}$ *and the following alternative holds for the maximal existence time of the solution*:

$$\tau^*(v, \omega) = +\infty \quad \textit{or} \quad \limsup_{t \nearrow \tau^*(v,\omega)} \|X(t)\|_{\mathbb{H}^1} = +\infty.$$

Moreover, if $v \in \mathbb{H}^2$, then $X \in C([0, \tau^*), \mathbb{H}^2(\mathbb{R}))$ and τ^* satisfies

$$\tau^*(v, \omega) = +\infty \quad \textit{or} \quad \lim_{t \nearrow \tau^*(v,\omega)} \|X(t)\|_{\mathbb{H}^1} = +\infty. \tag{1.14}$$

Note that we do not obtain global existence for (1.12), due to the lack of control of the evolution of the $\mathbb{H}^1$ norm (see Remark 3.1).

Using these existence theorems, we are able to prove a diffusion approximation result for the nonlinear system of PDEs (1.11).

THEOREM 1.3. *Let $X_\varepsilon(0) = X_0 = v$ be in $\mathbb{H}^3(\mathbb{R})$, then the solution X_ε of* (1.11) *given by Theorem* 1.1 *converges in law to the solution X of* (1.12) *in $\mathcal{E}(\mathbb{H}^1)$, that is, for all functions f in $C_b(\mathcal{E}(\mathbb{H}^1))$*,

$$\lim_{\varepsilon\to 0} \mathcal{L}(X_\varepsilon)(f) = \mathcal{L}(X)(f).$$

Note that we consider here the Manakov PMD equation (1.11), but the method may be carried out to other nonlinear Schrödinger equations. Let us first emphasize the key points that allow us to prove Theorem 1.3.

The first point is that the noise term is a linear function of the unknown X_ε. This particular structure leads to a stochastic partial differential equation for the limiting equation. The second point is the fact that the Pauli matrices are Hermitian. This is important to obtain the conservation of the $\mathbb{L}^2$ norm for both equations. Finally, we use that the driving process ν is a homogeneous Markov ergodic process defined on a compact state space such that $\mathbb{E}_\Lambda(\boldsymbol{\sigma}(y)) = 0$. The hypothesis on the driving noise may be weakened as in the case of a random ordinary differential equation assuming good mixing properties (e.g., exponential decay of the covariance function). The boundedness of $\boldsymbol{\sigma}(\nu_\varepsilon(t))$ seems to be necessary. It is used to prove uniform bounds in Lemma 4.5 for tightness. On the other hand, the lack of Strichartz estimates for the limiting equation (1.12) is a negative aspect. Thus, we use that $F(v)$ is locally Lipschitz in $\mathbb{H}^1(\mathbb{R})$ to prove existence and uniqueness of a local solution to (1.12). But if $\boldsymbol{\sigma}(\nu_\varepsilon(t))$ were a one-dimensional process, larger dimension and larger power in the nonlinear term could be considered.

Other types of nonlinear Schrödinger equations may be considered replacing, for example, $i\frac{\partial X_\varepsilon}{\partial x}$ by X_ε and assuming that the matrices σ_k are real valued and symmetric. This latter equation is simpler to handle using Strichartz estimates for the fundamental solution and because $\boldsymbol{\sigma}(\nu_\varepsilon(t))X_\varepsilon(t)$ can be treated as a perturbation as far as we are concerned with existence of solutions.

2. The Manakov PMD equation: Proof of Theorem 1.1. The point here is that no Strichartz estimates are available for (1.11) because of the lack of commutativity of the matrix $\boldsymbol{\sigma}$ at a different time: $\boldsymbol{\sigma}(\nu(t))\boldsymbol{\sigma}(\nu(s)) \neq \boldsymbol{\sigma}(\nu(s))\boldsymbol{\sigma}(\nu(t))$. Consequently, only local existence and uniqueness for initial data in $\mathbb{H}^1$ can be easily proved directly on (1.11). The idea of the proof is then to find a unitary transformation such that Strichartz estimates are available for the transformed equation. This change of unknown is given in the next result.

LEMMA 2.1. *Let us denote for $t \in \mathbb{R}_+$*

$$Z_\varepsilon(t) = \begin{pmatrix} \nu_{1,\varepsilon}(t) & \overline{\nu}_{2,\varepsilon}(t) \\ -\nu_{2,\varepsilon}(t) & \overline{\nu}_{1,\varepsilon}(t) \end{pmatrix},$$

where $\nu_\varepsilon = \nu(t/\varepsilon^2)$, ν given by (1.8). Assuming that $X_\varepsilon \in C([0,T],\mathbb{L}^2)$, we set $\Psi_\varepsilon(t) = Z_\varepsilon(t)X_\varepsilon(t)$; then the evolution of the electric field Ψ_ε is given by the stochastic Itô equation

$$\begin{aligned} i\,d\Psi_\varepsilon(t) &+ \left\{\frac{ib'}{\varepsilon}\sigma_3\frac{\partial\Psi_\varepsilon}{\partial x} + \frac{d_0}{2}\frac{\partial^2\Psi_\varepsilon}{\partial x^2} + \frac{5}{6}|\Psi_\varepsilon|^2\Psi_\varepsilon + \frac{1}{6}(\Psi_\varepsilon^*\sigma_3\Psi_\varepsilon)\sigma_3\Psi_\varepsilon\right\}dt \\ &+ \frac{\gamma_s}{\varepsilon^2}\sigma_3\Psi_\varepsilon\,dt + \frac{i\gamma_c}{\varepsilon^2}\Psi_\varepsilon\,dt - \frac{\sqrt{\gamma_c}}{\varepsilon}\big(\sigma_1\Psi_\varepsilon\,d\widetilde{W}_1(t) + \sigma_2\Psi_\varepsilon\,d\widetilde{W}_2(t)\big) = 0, \end{aligned} \tag{2.1}$$

where $\widetilde{W}_j(t) = \varepsilon W_j(t/\varepsilon^2)$, $j = 1, 2$, *and with initial conditions*

$$\Psi_\varepsilon(0) = \begin{pmatrix} \nu_{1,\varepsilon}(0)v_1 + \overline{\nu}_{2,\varepsilon}(0)v_2 \\ -\nu_{2,\varepsilon}(0)v_1 + \overline{\nu}_{1,\varepsilon}(0)v_2 \end{pmatrix} = \psi_0.$$

PROOF. Using the equation satisfied by ν_ε and because $|\nu_{1,\varepsilon}(t)|^2 + |\nu_{2,\varepsilon}(t)|^2 = 1$ for any $t \geq 0$, we obtain

$$i\, dZ_\varepsilon(t) Z_\varepsilon^{-1} \Psi_\varepsilon(t) = -\frac{\gamma_s}{\varepsilon^2}\sigma_3 \Psi_\varepsilon\, dt - \frac{i\gamma_c}{\varepsilon^2}\Psi_\varepsilon\, dt + \frac{\sqrt{\gamma_c}}{\varepsilon}\sigma_1 \Psi_\varepsilon\, d\widetilde{W}_1(t) + \frac{\sqrt{\gamma_c}}{\varepsilon}\sigma_2 \Psi_\varepsilon\, d\widetilde{W}_2(t).$$

The nonlinear part of (2.1) is obtained as in the derivation of (1.5). $\square$

We first investigate the behavior of the linear equation

$$i\frac{\partial \Psi_\varepsilon}{\partial t} + \frac{1}{\varepsilon} i b' \sigma_3 \frac{\partial \Psi_\varepsilon}{\partial x} + \frac{d_0}{2}\frac{\partial^2 \Psi_\varepsilon}{\partial x^2} = 0 \tag{2.2}$$

with initial condition $\Psi_\varepsilon(0) = \psi_0 \in \mathbb{L}^2$.

PROPOSITION 2.1. *The unbounded matrix operator* $H_\varepsilon = \frac{id_0}{2} I_2 \frac{\partial^2}{\partial x^2} - \frac{b'}{\varepsilon}\sigma_3 \frac{\partial}{\partial x}$ *defined on* $\mathscr{D}(H_\varepsilon) = \mathbb{H}^2$ *is the infinitesimal generator of a unique strongly continuous unitary group* $U_\varepsilon(t)$ *on* $\mathbb{L}^2$. *Moreover,* $U_\varepsilon(t)$ *may be expressed as a convolution kernel, that is, for* $\psi_0 \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned} U_\varepsilon(t)\psi_0 &= A_\varepsilon(t) \star \psi_0 \\ &= \frac{1}{\sqrt{2\pi i d_0 t}} \begin{pmatrix} \exp\left\{\dfrac{i}{2}\dfrac{(x - b't/\varepsilon)^2}{d_0 t}\right\} & 0 \\ 0 & \exp\left\{\dfrac{i}{2}\dfrac{(x + b't/\varepsilon)^2}{d_0 t}\right\} \end{pmatrix} \star \psi_0. \end{aligned}$$

PROOF. Assuming $\psi_0 \in \mathcal{S}(\mathbb{R})$ and taking the Fourier transform, in the space variable, of (2.2), we obtain readily

$$\frac{\partial \widehat{\Psi}_\varepsilon}{\partial t} = -\frac{1}{\varepsilon} i b' \sigma_3 \xi \widehat{\Psi}_\varepsilon - i\frac{d_0 \xi^2}{2}\widehat{\Psi}_\varepsilon.$$

Since σ_3 does not depend on time, we obtain

$$\widehat{\Psi}_\varepsilon(t) = R_\varepsilon(t)\widehat{\psi}_0 = \begin{pmatrix} \exp\left\{-\dfrac{id_0}{2}\xi^2 t - i\dfrac{b'}{\varepsilon}\xi t\right\} & 0 \\ 0 & \exp\left\{-\dfrac{id_0}{2}\xi^2 t + i\dfrac{b'}{\varepsilon}\xi t\right\} \end{pmatrix} \widehat{\psi}_0.$$

The statement of Proposition 2.1 follows then in a classical way, setting $A_\varepsilon(t) = \mathcal{F}^{-1}(R_\varepsilon(t))$. $\square$

The explicit formulation of the kernel given in Proposition 2.1 allows immediately to get the following dispersive estimates: if $p \geq 2$, $t \neq 0$, then $U_\varepsilon \in \mathcal{L}(\mathbb{L}^{p'}, \mathbb{L}^p)$ where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$ and for all $\psi_0 \in \mathbb{L}^{p'}$,

$$\|U_\varepsilon(t)\psi_0\|_{\mathbb{L}^p} \leq (2\pi|d_0||t|)^{-1/2+1/p}\|\psi_0\|_{\mathbb{L}^{p'}}. \tag{2.3}$$

Using then classical arguments (see [7, 19]), one may prove Strichartz inequalities for $U_\varepsilon(t)$.

PROPOSITION 2.2. *The following properties hold*:

(1) *For every* $\psi_0 \in \mathbb{L}^2(\mathbb{R})$, $U_\varepsilon(\cdot)\psi_0 \in L^8(\mathbb{R}; \mathbb{L}^4) \cap C(\mathbb{R}; \mathbb{L}^2)$. *Furthermore, there exists a constant* C *such that*

$$\|U_\varepsilon(\cdot)\psi_0\|_{L^8(\mathbb{R};\mathbb{L}^4)} \leq C\|\psi_0\|_{\mathbb{L}^2} \qquad \text{for every } \psi_0 \in \mathbb{L}^2.$$

(2) *Let* I *be an interval of* $\mathbb{R}$ *and* $t_0 \in I$. *Let* $f \in L^{8/7}(I, \mathbb{L}^{4/3})$, *then the function*

$$t \mapsto \int_{t_0}^t U_\varepsilon(t-s)f(s)\,ds$$

belongs to $L^8(I, \mathbb{L}^4) \cap C(I, \mathbb{L}^2)$. *Furthermore, there exists a constant* C *independent of* I *such that for every* $f \in L^{8/7}(I, \mathbb{L}^{4/3})$

$$\left\|\int_{t_0}^{\cdot} U_\varepsilon(\cdot - s)f(s)\,ds\right\|_{L^8(I,\mathbb{L}^4)\cap L^\infty(I,\mathbb{L}^2)} \leq C\|f\|_{L^{8/7}(I,\mathbb{L}^{4/3})}.$$

We now turn to the study of the nonlinear problem. We will use, as is classical, a cutoff argument on the nonlinear term which is not Lipschitz. The cutoff we consider here is of the same form as the one considered in [9]. We first prove an existence and uniqueness result for this truncated equation, then deduce from this result the existence of a unique solution for (2.1). We denote:

$$f(\Psi_\varepsilon) = \tfrac{5}{6}|\Psi_\varepsilon|^2\Psi_\varepsilon + \tfrac{1}{6}(\Psi_\varepsilon^*\sigma_3\Psi_\varepsilon)\sigma_3\Psi_\varepsilon.$$

Let $\Theta \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\Theta \subset [-2; 2]$ such that $\Theta(x) = 1$ for $|x| \leq 1$ and $0 \leq \Theta(x) \leq 1$ for $x \in \mathbb{R}$. Let $R > 0$ and $\Theta_R(x) = \Theta(x/R)$. We then consider the following equation:

$$\begin{aligned}
\Psi_\varepsilon^R(t) = {} & U_\varepsilon(t)\psi_0 + \frac{i\gamma_s}{\varepsilon^2}\int_0^t U_\varepsilon(t-s)\sigma_3\Psi_\varepsilon^R(s)\,ds \\
& - \frac{\gamma_c}{\varepsilon^2}\int_0^t U_\varepsilon(t-s)\Psi_\varepsilon^R(s)\,ds \\
& + i\int_0^t U_\varepsilon(t-s)\Theta_R\big(\|\Psi_\varepsilon^R\|_{L^8(0,s;\mathbb{L}^4)}\big)f(\Psi_\varepsilon^R(s))\,ds \\
& - \frac{i\sqrt{\gamma_c}}{\varepsilon}\int_0^t U_\varepsilon(t-s)\sigma_1\Psi_\varepsilon^R(s)\,d\widetilde{W}_1(s) \\
& - \frac{i\sqrt{\gamma_c}}{\varepsilon}\int_0^t U_\varepsilon(t-s)\sigma_2\Psi_\varepsilon^R(s)\,d\widetilde{W}_2(s),
\end{aligned} \tag{2.4}$$

which is the mild form of the Itô equation,

$$
\begin{aligned}
& i\,d\Psi_\varepsilon^R(t)+\left\{\frac{ib'}{\varepsilon}\sigma_3\frac{\partial\Psi_\varepsilon^R(t)}{\partial x}+\frac{d_0}{2}\frac{\partial^2\Psi_\varepsilon^R(t)}{\partial x^2}+\frac{\gamma_s}{\varepsilon^2}\sigma_3\Psi_\varepsilon^R(t)+\frac{i}{\varepsilon^2}\gamma_c\Psi_\varepsilon^R(t)\right\}dt \\
& \quad -\frac{\sqrt{\gamma_c}}{\varepsilon}\sigma_1\Psi_\varepsilon^R\,d\widetilde{W}_1(t)-\frac{\sqrt{\gamma_c}}{\varepsilon}\sigma_2\Psi_\varepsilon^R\,d\widetilde{W}_2(t) \\
& \quad +\Theta_R\big(\|\Psi_\varepsilon^R\|_{L^8(0,t;\mathbb{L}^4)}\big)f(\Psi_\varepsilon^R(t))\,dt=0
\end{aligned}
\tag{2.5}
$$

with initial condition $\Psi_\varepsilon^R(0)=\psi_0$.

PROPOSITION 2.3. *Let $\Psi_\varepsilon^R(0)=\psi_0\in\mathbb{L}^2(\mathbb{R})$. Let $T>0$ and $\mathcal{U}_c^T=C([0,T];\mathbb{L}^2)\cap L^8(0,T;\mathbb{L}^4)$; then (2.4) has a unique strong adapted solution $\Psi_\varepsilon^R\in L^8(\mathcal{A};\mathcal{U}_c^T)$, for any $T>0$.*

PROOF. We use a fixed point argument in the Banach space $L^8(\mathcal{A};\mathcal{U}_c^T)$ for sufficiently small time T depending on R. We first need to establish estimates on the stochastic integrals

$$
J_{j,\varepsilon}\Psi_\varepsilon(t)=\int_0^t U_\varepsilon(t-s)\sigma_j\Psi_\varepsilon(s)\,d\widetilde{W}_j(s),\qquad j=1,2.
$$

LEMMA 2.2. *Let $T>0$; then for each adapted process $\Psi_\varepsilon\in L^8(\mathcal{A};\mathcal{U}_c^T)$ and for $j=1,2$ the stochastic integral $J_{j,\varepsilon}\Psi_\varepsilon$ belongs to $L^8(\mathcal{A};\mathcal{U}_c^T)$. Moreover, for any $T>0$ and t in $[0,T]$ we have the estimates*

$$
\mathbb{E}\big(\|J_{j,\varepsilon}\Psi_\varepsilon\|^8_{L^8(0,T;\mathbb{L}^4)\cap L^\infty(0,T;\mathbb{L}^2)}\big)\le CT^4\mathbb{E}\big(\|\Psi_\varepsilon\|^8_{L^\infty(0,T;\mathbb{L}^2)}\big).
$$

PROOF. Since $\Psi_\varepsilon\in L^8(\mathcal{A};\mathcal{U}_c^T)$ and is adapted, we may apply the Burkholder–Davis–Gundy inequality in the Banach space $\mathbb{L}^4(\mathbb{R})$ (which is UMD space [6]):

$$
\begin{aligned}
\mathbb{E}\big(\|J_{j,\varepsilon}\Psi_\varepsilon\|^8_{L^8(0,T;\mathbb{L}^4)}\big) &= \mathbb{E}\bigg(\int_0^T\bigg\|\int_0^t U_\varepsilon(t-s)\sigma_j\Psi_\varepsilon(s)\,d\widetilde{W}_j(s)\bigg\|^8_{\mathbb{L}^4}dt\bigg) \\
&\le \int_0^T\mathbb{E}\bigg(\sup_{0\le u\le t}\bigg\|\int_0^u U_\varepsilon(t-s)\sigma_j\Psi_\varepsilon(s)\,d\widetilde{W}_j(s)\bigg\|^8_{\mathbb{L}^4}\bigg)dt \\
&\le C\mathbb{E}\bigg(\int_0^T\bigg(\int_0^t\|U_\varepsilon(t-s)\sigma_j\Psi_\varepsilon(s)\|^2_{\mathbb{L}^4}\,ds\bigg)^4dt\bigg).
\end{aligned}
$$

Using the Hölder inequality in time, Fubini and a change of variable,

$$
\begin{aligned}
&\mathbb{E}\bigg(\int_0^T\bigg(\int_0^t\|U_\varepsilon(t-s)\sigma_j\Psi_\varepsilon(s)\|^2_{\mathbb{L}^4}\,ds\bigg)^4dt\bigg) \\
&\qquad\le T^3\mathbb{E}\bigg(\int_0^T\|U_\varepsilon(\cdot)\sigma_j\Psi_\varepsilon(s)\|^8_{L^8(0,T;\mathbb{L}^4)}\,ds\bigg).
\end{aligned}
$$

On the other hand, by Proposition 2.2,

$$\mathbb{E}\left(\int_0^T \|U_\varepsilon(\cdot)\sigma_j\Psi_\varepsilon(s)\|^8_{L^8(0,T;\mathbb{L}^4)}\,ds\right) \le C\mathbb{E}\left(\int_0^T \|\Psi_\varepsilon(s)\|^8_{\mathbb{L}^2}\,ds\right)$$
$$\le CT\mathbb{E}\big(\|\Psi_\varepsilon\|^8_{L^\infty(0,T;\mathbb{L}^2)}\big).$$

Combining these inequalities leads to the estimate in $L^8(0,T;\mathbb{L}^4)$. The other estimate is proved using the Burkholder inequality in Hilbert space and the unitary property of the group U_ε. Finally, $U_\varepsilon(t)$ being a unitary semigroup in $\mathbb{L}^2$, Theorem 6.10 in [8] tells us that, provided $\Psi_\varepsilon \in L^8(\mathcal{A}, L^2(0,T;\mathbb{L}^2))$, then $J_{j,\varepsilon}\Psi_\varepsilon(\cdot)$ has continuous modification with values in $\mathbb{L}^2(\mathbb{R})$. $\square$

Given $\Psi^R_\varepsilon \in L^8(\mathcal{A};\mathcal{U}^T_c)$, we denote by $\mathcal{T}\Psi^R_\varepsilon(t)$ the right-hand side of (2.4). Since the group $U_\varepsilon(\cdot)$ maps $\mathbb{L}^2(\mathbb{R})$ into $C(\mathbb{R},\mathbb{L}^2(\mathbb{R}))$, Proposition 2.2 and Lemma 2.2 easily imply that the mapping $\mathcal{T}$ maps $L^8(\mathcal{A};\mathcal{U}^T_c)$ into itself. Let now Ψ^R_ε and Φ^R_ε being adapted processes with values in $L^8(\mathcal{A};\mathcal{U}^T_c)$, then using Proposition 2.2, the same arguments as in [9] for the cutoff and Lemma 2.2 applied to $J_{j,\varepsilon}(\Phi^R_\varepsilon(t) - \Psi^R_\varepsilon(t))$, we get

$$\mathbb{E}(\|\mathcal{T}\Psi^R_\varepsilon - \mathcal{T}\Phi^R_\varepsilon\|^8_{\mathcal{U}^T_c})^{1/8} \le \left(\frac{CT}{\varepsilon^2} + \frac{CT^{1/2}}{\varepsilon} + C(R)T^{1/2}\right)\mathbb{E}(\|\Psi^R_\varepsilon - \Phi^R_\varepsilon\|^8_{\mathcal{U}^T_c})^{1/8}.$$

We conclude that $\mathcal{T}$ is a contraction mapping if T is chosen such that $CT/\varepsilon^2 + CT^{1/2}/\varepsilon + C(R)T^{1/2} < 1$. As usual, iterating the procedure, we deduce the existence of a unique solution of (2.4) in $L^8(\mathcal{A};\mathcal{U}^T_c)$ for all $T > 0$. $\square$

Our aim is now to get global existence for the process Ψ_ε, the solution of (2.1) which may be constructed from the above results. Let us set

$$\kappa^R_\varepsilon(\psi_0,\omega) = \inf\{t \ge 0, \|\Psi^R_\varepsilon\|_{L^8(0,t;\mathbb{L}^4)} \ge R\},$$

which is a $\mathcal{G}_\varepsilon(t)$ stopping time. It can be proved using Strichartz estimates and the integral formulation (2.4) (see [9, 10]) that κ^R_ε is nondecreasing with R and that $\Psi^R_\varepsilon = \Psi^{R'}_\varepsilon$ on $[0,\kappa^R_\varepsilon]$ for $R < R'$. Thus, we are able to define a local solution Ψ_ε to (2.1) on the random interval $[0,\kappa^*_\varepsilon(\psi_0))$, where $\kappa^*_\varepsilon(\psi_0) = \lim_{R\to+\infty}\kappa^R_\varepsilon$, by setting $\Psi_\varepsilon(t) = \Psi^R_\varepsilon(t)$ on $[0,\kappa^R_\varepsilon]$. It remains to prove that $\kappa^*_\varepsilon = +\infty$ almost surely. From the construction of the stopping time κ^*_ε it is clear that a.s.,

$$\text{if } \kappa^*_\varepsilon(\psi_0) < +\infty \qquad \text{then } \lim_{t\nearrow\kappa^*_\varepsilon(\psi_0)} \|\Psi^R_\varepsilon\|_{L^8(0,t;\mathbb{L}^4)} = +\infty. \tag{2.6}$$

The arguments are adapted from [9]. We first prove the following lemma:

LEMMA 2.3. *Let $\Psi_\varepsilon(0) = \psi_0$ be as in Proposition 2.3 and Ψ^R_ε be the corresponding solution of (2.5); then for any $t < T$*

$$\|\Psi^R_\varepsilon(t)\|_{\mathbb{L}^2} = \|\psi_0\|_{\mathbb{L}^2} \qquad a.s.,$$

and there is a constant $M_\varepsilon > 0$, depending on T and $\|\psi_0\|_{\mathbb{L}^2}$, but independent of R, such that

$$\mathbb{E}\big(\|\Psi_\varepsilon^R\|_{L^8(0,T;\mathbb{L}^4)}\big) \le M_\varepsilon(T). \tag{2.7}$$

PROOF. To prove that the $\mathbb{L}^2$ norm of the solution Ψ_ε^R of (2.5) is constant in time, we apply formally the Itô formula to $\frac{1}{2}\|\Psi_\varepsilon^R(t)\|_{\mathbb{L}^2}^2$ and notice that by integration by parts

$$\left(b'\sigma_3 \frac{\partial \Psi_\varepsilon^R}{\partial x}, \Psi_\varepsilon^R\right)_{\mathbb{L}^2} = -\left(\Psi_\varepsilon^R, b'\sigma_3 \frac{\partial \Psi_\varepsilon^R}{\partial x}\right)_{\mathbb{L}^2} = 0.$$

Since $\sigma_j^* = \sigma_j$, $j = 1, 2, 3$, where $*$ stands for the conjuguate transpose, we get

$$(\Psi_\varepsilon^R(t), i\sigma_j \Psi_\varepsilon^R(t))_{\mathbb{L}^2} = 0 \qquad \text{for } j = 1, 2, 3.$$

Moreover, because the Itô corrections cancel with the damping term $-\frac{\gamma_c}{\varepsilon^2}\Psi_\varepsilon^R$ of (2.5), we get $\|\Psi_\varepsilon^R(t)\|_{\mathbb{L}^2} = \|\psi_0\|_{\mathbb{L}^2}, \forall t \le T$. The computations can be made rigorous by a regularization procedure.

In order to prove (2.7), we follow the procedure in [9, 10]. Using the integral formulation (2.4), the conservation of the $\mathbb{L}^2$-norm and Proposition 2.2, we obtain for a.e. $\omega \in \Omega$ and for all time T_1 such that $T \ge T_1 > 0$

$$\|\Psi_\varepsilon^R\|_{L^8(0,T_1;\mathbb{L}^4)} \le K_\varepsilon(\omega) + CT_1^{1/2}\|\Psi_\varepsilon^R\|_{L^8(0,T_1;\mathbb{L}^4)}^3, \tag{2.8}$$

where

$$K_\varepsilon(\omega) = C\left(1 + \frac{T}{\varepsilon^2}\right)\|\psi_0\|_{\mathbb{L}^2} + \frac{1}{\varepsilon}\sum_{j=1}^{2}\|J_{j,\varepsilon}\Psi_\varepsilon^R\|_{L^8(0,T;\mathbb{L}^4)}.$$

From inequality (2.8) it follows that $\|\Psi_\varepsilon\|_{L^8(0,T_1;\mathbb{L}^4)} \le 2K_\varepsilon(\omega)$ if T_1 is chosen, for example, such that $T_1(\omega) = \inf(T, 2^{-6}(C^{1/2}K_\varepsilon)^{-4})$. If $T_1 < T$ we can reiterate the process on small time intervals $[lT_1, (l+1)T_1] \subset [0, T]$ (keeping R fixed and varying l) to get $\|\Psi_\varepsilon\|_{L^8(lT_1,(l+1)T_1;\mathbb{L}^4)} \le 2K_\varepsilon(\omega)$. Summing these estimates, using $T_1 = 2^{-6}C^{-2}(K_\varepsilon)^{-4}$ and the Young inequality, we obtain

$$\|\Psi_\varepsilon^R\|_{L^8(0,T;\mathbb{L}^4)} \le C(T)(K_\varepsilon(\omega))^5.$$

Taking the expectation in the above inequality, using the Hölder inequality and Lemma 2.2, we get the following estimate:

$$\mathbb{E}\big(\|\Psi_\varepsilon^R\|_{L^8(0,T;\mathbb{L}^4)}\big) \le C(T)\left(\left(1 + \frac{T}{\varepsilon^2}\right)^5 \|\psi_0\|_{\mathbb{L}^2}^5 + \frac{CT^{5/2}}{\varepsilon^5}\|\psi_0\|_{\mathbb{L}^2}^5\right), \tag{2.9}$$

from which (2.7) follows. $\square$

We easily deduce from Lemma 2.3 and (2.6) that $\kappa_\varepsilon^* = +\infty$ a.s. and as in [9] the existence and uniqueness of a solution Ψ_ε of (2.1), a.s. in $\mathcal{U}_c^T$ for any $T > 0$.

To end the proof of Theorem 1.1, we have to extend those results to the process X_ε. For a.e. ω in $\mathcal{A}$ and for each $t \geq 0$ we set $X_\varepsilon(t) = Z_\varepsilon^{-1}(t)\Psi_\varepsilon(t)$. By definition of the process $Z_\varepsilon^{-1}(t)$ [which, in particular, is measurable with respect to $\mathcal{G}_\varepsilon(t)$] and properties of Ψ_ε, we easily deduce that $X_\varepsilon(t)$ is adapted and continuous with values in $\mathbb{L}^2$, and satisfy (1.11), hence is C^1 with values in $\mathbb{H}^{-2}$. By unitarity of Z_ε we also deduce that for all $t \geq 0$

$$\|\Psi_\varepsilon(t)\|_{\mathbb{L}^2}^2 = (X_\varepsilon(t), Z_\varepsilon^{-1}(t)Z_\varepsilon(t)X_\varepsilon(t))_{\mathbb{L}^2} = \|X_\varepsilon(t)\|_{\mathbb{L}^2}^2,$$

and since the coefficients of $Z_\varepsilon^{-1}(t)$ are a.s. uniformly bounded, $X_\varepsilon \in L^8_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{L}^4)$ a.s.; Theorem 1.1 is proved.

We now extend the previous global existence results to more regular initial data. T being fixed, we denote

$$\mathcal{V}^T = L^\infty(0, T; \mathbb{H}^1) \cap L^8(0, T; \mathbb{W}^{1,4})$$

and

$$\mathcal{V}_c^T = C(0, T; \mathbb{H}^1) \cap L^8(0, T; \mathbb{W}^{1,4}).$$

PROPOSITION 2.4. *Let $\Psi_\varepsilon(0) = \psi_0 \in \mathbb{H}^1$ and let $T > 0$; then equation (2.1) has a unique strong solution Ψ_ε with trajectories in $C(0, T; \mathbb{H}^1)$.*

PROOF. Let ψ_0 be in $\mathbb{H}^1$. Given $\Psi_\varepsilon^R \in L^8(\mathcal{A}; \mathcal{V}^T)$, we denote by $\mathcal{T}\Psi_\varepsilon^R(t)$ the right-hand side of (2.4) and $\mathcal{U}^T = L^\infty(0, T; \mathbb{L}^2) \cap L^8(0, T; \mathbb{L}^4)$. By Proposition 2.2, Lemma 2.2 applied to $\partial_x\Psi_\varepsilon^R$ and the Hölder inequality, we deduce that

$$\|\mathcal{T}\partial_x\Psi_\varepsilon^R\|_{L^8(\mathcal{U}^T)} \leq C\|\partial_x\psi_0\|_{\mathbb{L}^2} + \left(\frac{CT}{\varepsilon^2} + \frac{CT^{1/2}}{\varepsilon} + CT^{1/2}4R^2\right)\|\partial_x\Psi_\varepsilon^R\|_{L^8(\mathcal{U}^T)}.$$

Therefore, we conclude that choosing $R_0 = 2C\|\Psi_0\|_{\mathbb{H}^1}$, $\mathcal{T}$ maps the closed ball of $L^8(\mathcal{A}; \mathcal{V}^T)$ with radius R_0 into itself, provided T is small enough depending only on R and ε, but not on R_0. Combining with the fact that $\mathcal{T}$ is a contraction in $L^8(\mathcal{A}; \mathcal{U}^T)$ and that the balls of $L^8(\mathcal{A}; \mathcal{V}^T)$ are closed for the norm in $L^8(\mathcal{A}; \mathcal{U}^T)$, we conclude to the existence of a unique fixed point $\Psi_\varepsilon^R \in L^8(\mathcal{A}; \mathcal{V}^T)$. Using Proposition 2.2 and Lemma 2.2, we get continuity of the solution in $\mathbb{H}^1$. Since the cutoff only depends on the $L^8(0, T, \mathbb{L}^4(\mathbb{R}))$ norm, we deduce that there is a unique global solution Ψ_ε to (2.1) with paths in $C([0, T]; \mathbb{H}^1)$. Since the transformation Z_ε does not depend on x, we conclude that these results still hold true for X_ε. $\square$

PROPOSITION 2.5. *Let $\Psi_\varepsilon(0) = \psi_0 \in \mathbb{H}^m$, $m = 2, 3$. Let $T > 0$; then equation (2.1) has a unique strong solution Ψ_ε with paths in $C([0, T]; \mathbb{H}^m)$, $m = 2, 3$.*

PROOF. We consider equation (2.5) but with $\Theta_R(\|\Psi_\varepsilon^R\|_{L^8(0,t;\mathbb{L}^4)})$ replaced by $\Theta_R(\|\Psi_\varepsilon^R(t)\|_{\mathbb{H}^1}^2)$. Given Ψ_ε^R in $L^8(\mathcal{A}; L^\infty(0,T;\mathbb{H}^2(\mathbb{R})))$, we denote by $\mathcal{T}\Psi_\varepsilon^R(t)$ the right-hand side of the integral formulation of this equation. We easily prove that $\mathcal{T}$ maps the closed ball of $L^8(\mathcal{A}; L^\infty(0,T;\mathbb{H}^2(\mathbb{R})))$ with radius R_0 into itself, for $R_0 = 2C\|\Psi_0\|_{\mathbb{H}^2}$, provided that T is small enough, depending only on R and ε, but not on R_0. Using that this ball is closed for the norm in $L^8(\mathcal{A}; L^\infty(0,T;\mathbb{H}^1(\mathbb{R})))$ and that $\mathcal{T}$ is a contraction for the norm in $L^8(\mathcal{A}; L^\infty(0,T;\mathbb{H}^1(\mathbb{R})))$, we deduce that there exists a unique solution Ψ_ε with paths in $C(0,T;\mathbb{H}^2(\mathbb{R}))$ a.s., which is global since the solution is global in $\mathbb{H}^1$. Existence and uniqueness in $\mathbb{H}^3$ can be proved by the same arguments. Again those results are easily extended to X_ε and this concludes the proof of Theorem 1.1. $\square$

3. The limiting equation: Proof of Theorem 1.2. In order to prove a local existence and uniqueness result for the system (1.12), we use a compactness approach (see, e.g., [16]) motivated by the fact that we do not know if Strichartz estimates are available for (1.12). Indeed, no transformation similar to the Manakov PMD case seems to be available, as the equation $dX(t) = -\sqrt{\gamma}\sum_{k=1}^3 \sigma_k \frac{\partial X(t)}{\partial x}\, dW_k(t)$ cannot be solved in a simple way. We first prove existence of a unique solution in $\mathbb{H}^1$ for the linear part of the equation, defining then a random propagator, and then consider the nonlinear part as a perturbation. We will strongly use the fact that the nonlinearity is locally Lipschitz in $\mathbb{H}^1$. The regularity in $\mathbb{H}^2$ will follow with the same arguments as for (2.1). Let us consider the linear part of (1.12),

$$\begin{aligned} dX(t) &= \left(i\frac{d_0}{2}\frac{\partial^2 X}{\partial x^2}\right)dt - \sqrt{\gamma}\sum_{k=1}^3 \sigma_k \frac{\partial X(t)}{\partial x} \circ dW_k(t) \\ &= \left(i\frac{d_0}{2} + \frac{3\gamma}{2}\right)\frac{\partial^2 X}{\partial x^2}\,dt - \sqrt{\gamma}\sum_{k=1}^3 \sigma_k \frac{\partial X(t)}{\partial x}\, dW_k(t) \end{aligned} \tag{3.1}$$

with initial data $X(0) = v \in \mathbb{H}^2$. We introduce, for $\eta > 0$, the mollifier $J_\eta = (I - \eta\frac{\partial^2}{\partial x^2})^{-1}$. We denote by X_η the solution of the regularized Itô equation

$$dX_\eta(t) = \left(i\frac{d_0}{2} + \frac{3\gamma}{2}\right)\frac{\partial^2 J_\eta^2 X_\eta}{\partial x^2}\,dt - \sqrt{\gamma}\sum_{k=1}^3 \sigma_k \frac{\partial J_\eta X_\eta(t)}{\partial x}\, dW_k(t) \tag{3.2}$$

and $X_\eta(0) = v \in \mathbb{H}^2$. Since the operators $\partial_x^2 J_\eta^2$ and $\partial_x J_\eta$ are bounded from $\mathbb{H}^1$ into $\mathbb{H}^1$ (with constants depending on η), we easily get, thanks to the Doob inequality, the Fubini theorem, the Itô isometry and the independence of $(W_k)_{k=1,2,3}$, the existence and uniqueness of a solution X_η to (3.2) with paths in $C([0,T],\mathbb{H}^2)$ for any $T > 0$. Moreover, it is easy to see that the $\mathbb{H}^2$ norm of X_η is conserved since the Pauli matrices are Hermitian. Consequently, the process

$$M_\eta(t) = -X_\eta(t) + X_\eta(0) + \int_0^t \left(\frac{id_0}{2} + \frac{3\gamma}{2}\right) \frac{\partial^2 J_\eta^2 X_\eta}{\partial x^2}\, ds$$

is a $\mathcal{F}_t$ martingale with paths in $C([0,T],\mathbb{L}^2)$. Let us compute the quadratic variation. Let $a = (a_1, a_2)^t$ and $b = (b_1, b_2)^t$ be in $\mathbb{L}^2$ and $T \geq t \geq s \geq 0$; then

$$\begin{aligned}
&\mathbb{E}\big((a, M_\eta(t))_{\mathbb{L}^2}(b, M_\eta(t))_{\mathbb{L}^2} - (a, M_\eta(s))_{\mathbb{L}^2}(b, M_\eta(s))_{\mathbb{L}^2} | \mathcal{F}_s\big) \\
&\quad = \gamma \sum_{k=1}^3 \mathbb{E}\left(\int_s^t \left(a, \sigma_k \frac{\partial J_\eta X_\eta}{\partial x}\right)_{\mathbb{L}^2} \left(b, \sigma_k \frac{\partial J_\eta X_\eta}{\partial x}\right)_{\mathbb{L}^2} du \Big| \mathcal{F}_s\right).
\end{aligned}$$

We deduce that the quadratic variation of $M_\eta(t)$ is given by

$$(b, \langle\langle M_\eta(t)\rangle\rangle a)_{\mathbb{L}^2} = \gamma \sum_{k=1}^3 \int_0^t \left(a, \sigma_k \frac{\partial J_\eta X_\eta}{\partial x}\right)_{\mathbb{L}^2} \left(b, \sigma_k \frac{\partial J_\eta X_\eta}{\partial x}\right)_{\mathbb{L}^2} du. \tag{3.3}$$

Using the conservation of the $\mathbb{H}^2$ norm and equation (3.2), we get for all $0 \leq \alpha < \frac{1}{2}$

$$\mathbb{E}\big(\|X_\eta\|_{C^\alpha([0,T];\mathbb{L}^2)}\big) \leq C_\alpha(T), \tag{3.4}$$

where $C_\alpha(T)$ is a constant independent of η. Using the Ascoli–Arzela and Banach–Alaoglu theorems, the Markov inequality and inequality (3.4), we get that the sequence $(\mathcal{L}(X_\eta))_{\eta>0}$ is tight on $C_w([0,T],\mathbb{H}^1(\mathbb{R})) \cap L_w^\infty(0,T,\mathbb{H}^2)$. The Skorokhod theorem [4, 15] implies that on some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_t, \widetilde{\mathbb{P}})$, there exist a sequence of stochastic processes $(\widetilde{X}_\eta)_{\eta>0}$, and a process $\widetilde{X}$, such that

$$\mathcal{L}(\widetilde{X}_\eta) = \mathcal{L}(X_\eta), \qquad \mathcal{L}(\widetilde{X}) = \mathcal{L}(X)$$

and $\lim_{\eta\to 0} \widetilde{X}_\eta = \widetilde{X}$, $\widetilde{\mathbb{P}}$-a.s. in $C_w([0,T],\mathbb{H}^1) \cap L_w^\infty(0,T,\mathbb{H}^2)$. For all $\eta > 0$ and $t \in [0,T]$ we define the process

$$\widetilde{M}_\eta(t) = -\widetilde{X}_\eta(t) + \widetilde{X}_\eta(0) + \int_0^t \left(\frac{id_0}{2} + \frac{3\gamma}{2}\right) \frac{\partial^2 J_\eta^2 \widetilde{X}_\eta}{\partial x^2}(s)\, ds.$$

We deduce from the above laws equality that $\widetilde{M}_\eta(t)$ is a square integrable continuous martingale with values in $\mathbb{L}^2$ with respect to the filtration $\widetilde{\mathcal{F}}_t$ and that the quadratic variation $\langle\langle \widetilde{M}_\eta(t)\rangle\rangle$ is given by formula (3.3) replacing X_η by $\widetilde{X}_\eta$. Let $a \in \mathbb{H}^1$, then by the above martingale property we get for all $s \leq t$

$$\mathbb{E}\big((a, \widetilde{M}_\eta(t) - \widetilde{M}_\eta(s))_{\mathbb{L}^2} | \widetilde{\mathcal{F}}_s\big) = 0.$$

Using the almost sure convergence in $C_w([0,T],\mathbb{H}^1(\mathbb{R}))$ of X_η, the boundedness in $\mathbb{H}^{-1}$ of the operator J_η and the conservation of the $\mathbb{H}^1$ norm, we get the almost sure convergence in $C_w([0,T],\mathbb{H}^{-1}(\mathbb{R}))$ of $\widetilde{M}_\eta$ to $\widetilde{M}$, where

$$\widetilde{M}(t)=\widetilde{X}(t)-\widetilde{X}(0)-\int_0^t\left(\frac{id_0}{2}+\frac{3\gamma}{2}\right)\frac{\partial^2\widetilde{X}}{\partial x^2}(s)\,ds.$$

Hence, $\widetilde{M}$ is a weakly continuous martingale with values in $\mathbb{H}^{-1}$. Moreover, using the a.s. convergence in $C_w([0,T],\mathbb{H}^1(\mathbb{R}))$ and the dominated convergence theorem, we get for all $t,s\in[0,T]$, $t\geq s$ and for any $a,b\in\mathbb{H}^1$,

$$\begin{aligned}&\lim_{\eta\to 0}\mathbb{E}(\langle b,\langle\langle\widetilde{M}_\eta(t)\rangle\rangle a\rangle|\widetilde{\mathcal{F}}_s)\\&\qquad=\gamma\sum_{k=1}^{3}\mathbb{E}\left(\int_0^t\left\langle a,\sigma_k\frac{\partial\widetilde{X}}{\partial x}(u)\right\rangle\left\langle b,\sigma_k\frac{\partial\widetilde{X}}{\partial x}(u)\right\rangle du\,\middle|\,\widetilde{\mathcal{F}}_s\right).\end{aligned}$$

Thus, the quadratic variation $\langle b,\langle\langle\widetilde{M}(t)\rangle\rangle a\rangle$ is given, for all $t\in[0,T]$, by

$$\langle b,\langle\langle\widetilde{M}(t)\rangle\rangle a\rangle=\gamma\sum_{k=1}^{3}\int_0^t\left\langle a,\sigma_k\frac{\partial\widetilde{X}}{\partial x}(u)\right\rangle\left\langle b,\sigma_k\frac{\partial\widetilde{X}}{\partial x}(u)\right\rangle du. \tag{3.5}$$

Noticing that $\widetilde{M}(0)=0$ and using the representation theorem for continuous square integrable martingales, we obtain that, on a possibly enlarged space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{F}}_t,\widetilde{\mathbb{P}})$, one can find a Brownian motion $\widetilde{W}=(\widetilde{W}_1,\widetilde{W}_2,\widetilde{W}_3)$ such that

$$\langle a,\widetilde{M}(t)\rangle=\sqrt{\gamma}\int_0^t\sum_{k=1}^{3}\left\langle a,\sigma_k\frac{\partial\widetilde{X}}{\partial x}(s)\right\rangle d\widetilde{W}_k(s).$$

Thus, we deduce that $(\widetilde{X},\widetilde{W})$ is a weak solution of (3.1) on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{F}}_t,\widetilde{\mathbb{P}})$ with values in $C_w([0,T],\mathbb{H}^1(\mathbb{R}))\cap L^\infty(0,T,\mathbb{H}^2)$. To conclude the proof, we have to prove pathwise uniqueness of the solution and strong continuity in $\mathbb{H}^1$. Since $\widetilde{X}\in L^\infty(0,T,\mathbb{H}^2)$ is the solution of (3.1), we easily deduce that $\widetilde{X}\in C^\alpha([0,T],\mathbb{L}^2)$ for any $\alpha\in[0,1/2)$. By interpolation we obtain that $\widetilde{X}\in C([0,T],\mathbb{H}^1)$. It follows, using the Itô formula, that pathwise uniqueness holds for (3.1) in $C([0,T],\mathbb{H}^1)$. This implies, by the Yamada–Watanabe theorem, that the solution exists in the strong sense. Thus, we can define a random unitary propagator $U(t,s)$ which is strongly continuous from $\mathbb{H}^2$ into $\mathbb{H}^1$. This random propagator can be extended to a random propagator from $\mathbb{H}^1$ into $\mathbb{H}^1$ using the continuity of X in $\mathbb{H}^1$, the density of $\mathbb{H}^2$ into $\mathbb{H}^1$ and the isometry property of $U(t,s)$ in $\mathbb{H}^1$.

The local existence of the nonlinear problem (1.12) in $\mathbb{H}^1$ follows from the construction of the random propagator U: we consider a cutoff function $\Theta\in C_c^\infty(\mathbb{R})$, $\Theta\geq 0$ satisfying

$$\Theta_R(\|X(t)\|_{\mathbb{H}^1}^2)=\begin{cases}1, & \text{if } \|X(t)\|_{\mathbb{H}^1}^2\leq R,\\ 0, & \text{if } \|X(t)\|_{\mathbb{H}^1}^2\geq 2R,\end{cases}$$

and first construct a solution X^R of the cutoff equation,

$$\begin{aligned} & i\,dX^R(t) + \left(\frac{d_0}{2}\frac{\partial^2 X^R}{\partial x^2} + \Theta_R(\|X^R(t)\|_{\mathbb{H}^1}^2)F(X^R)(t)\right)dt \\ & \quad + i\sqrt{\gamma}\sum_{k=1}^{3}\sigma_k\frac{\partial X^R(t)}{\partial x}\circ dW_k(t) \\ & \quad = 0 \end{aligned} \tag{3.6}$$

with initial data $X^R(0) = v \in \mathbb{H}^1$ and whose integral formulation is given a.e. by

$$X^R(t) = U(t,0)v + i\int_0^t \Theta_R(\|X^R(s)\|_{\mathbb{H}^1}^2)U(t,s)F(X^R(s))\,ds. \tag{3.7}$$

The existence and uniqueness of $X^R \in L^\rho(\Omega; C(0,T;\mathbb{H}^1))$, the solution of (3.7), is easily obtained by a fixed point argument since the nonlinear term is globally Lipschitz. Introducing the nondecreasing stopping time

$$\tau^R = \inf\{t \geq 0, \|X^R(t)\|_{\mathbb{H}^1}^2 \geq R\},$$

we may then define a local solution X to (1.12) on a random interval $[0, \tau^*(v))$, where $\tau^*(v) = \lim_{R\to+\infty}\tau^R$ almost surely, by setting $X(t) = X^R(t)$ on $[0, \tau^R]$. Then for any stopping time $\tau < \tau^*$ we have constructed a unique local solution with paths a.s. in $C([0,\tau], \mathbb{H}^1)$. It follows from the construction of the stopping time τ^* that if $\tau^* < +\infty$, then $\limsup_{t\to\tau^*}\|X(t)\|_{\mathbb{H}^1} = +\infty$. Let us now prove that if $v \in \mathbb{H}^2$, then the maximal stopping time satisfies the following alternative:

$$\tau^* = +\infty \quad \text{or} \quad \lim_{t\to\tau^*}\|X(t)\|_{\mathbb{H}^1} = +\infty. \tag{3.8}$$

We note that the random propagator commutes with derivation. Hence, if $v \in \mathbb{H}^2$, then $U(\cdot, 0)v \in C([0,T], \mathbb{H}^2)$. We easily deduce, using (3.1) and interpolating $\mathbb{H}^1$ between $\mathbb{H}^2$ and $\mathbb{L}^2$, that $U(\cdot,0)v \in C^\beta([0,T], \mathbb{H}^1)$ for $\beta \in [0, 1/4)$. By a fixed point argument in $\mathbb{H}^2$ and equation (3.7), we conclude that $X \in C^\beta([0,\tau], \mathbb{H}^1)$ for any stopping time $\tau < \tau^*$ and for the same maximal time existence τ^*. Hence, using the condition on τ^* and uniform continuity of X in $\mathbb{H}^1$, we get that (3.8) holds.

REMARK 3.1. We were not able to prove the global well-posedness for (1.12). Due to the lack of Strichartz estimates, we cannot control the evolution of the $\mathbb{H}^1$ norm. Even though the deterministic energy provides a control on the $\mathbb{H}^1$ norm because we are in the subcritical case, its evolution for a solution of (1.12), which is given in the next lemma, involves terms which are not well controlled. However, we cannot really conclude to the real occurrence of blow up or not in this model. It is clear that on a physical point of view such a phenomenon should not occur.

LEMMA 3.1. *Let the functional H be defined for $u \in \mathbb{H}^1(\mathbb{R})$ by*

$$H(u) = \frac{d_0}{4} \int_{\mathbb{R}} \left| \frac{\partial u}{\partial x} \right|^2 dx - \frac{2}{9} \int_{\mathbb{R}} |u|^4 \, dx.$$

Then for any stopping time τ such that $\tau < \tau^*$, we have

$$\begin{aligned} H(X(\tau)) &= H(X_0) + \sqrt{\gamma} \frac{8}{9} \sum_{k=1}^{3} \int_0^\tau \left\langle |X|^2 X, \sigma_k \frac{\partial X}{\partial x} \right\rangle \circ dW_k(s) \\ &= H(X_0) + \sqrt{\gamma} \frac{8}{9} \sum_{k=1}^{3} \int_0^\tau \left\langle |X|^2 X, \sigma_k \frac{\partial X}{\partial x} \right\rangle dW_k(s) \\ &\quad + \frac{2\gamma}{9} \int_0^\tau \int_{\mathbb{R}} (\partial_x |X_1|^2 + \partial_x |X_2|^2)^2 \, dx \, ds \\ &\quad - \frac{4}{9} \gamma \int_0^\tau \int_{\mathbb{R}} \left| X_1 \frac{\partial X_2}{\partial x} - \frac{\partial X_1}{\partial x} X_2 \right|^2 dx \, ds \\ &\quad + \frac{12}{9} \gamma \int_0^\tau \int_{\mathbb{R}} \partial_x |X_1|^2 \, \partial_x |X_2|^2 \, dx \, ds. \end{aligned}$$

PROOF. The first equality follows by Stratonovich differential calculus applied to the functional H and because the process X is the solution of (1.12). The calculation can be made rigorous by localization (H is C^2 but not bounded) and regularization through convolution. The second equality is obtained writing the evolution of H in its Itô formulation, that is,

$$\begin{aligned} H(X(\tau)) = H(X_0) &+ \sqrt{\gamma} \frac{8}{9} \sum_{k=1}^{3} \int_0^\tau \left\langle |X|^2 X, \sigma_k \frac{\partial X}{\partial x} \right\rangle dW_k(s) \\ &+ \frac{24}{9} \gamma \int_0^\tau \langle X, \partial_x X \, \mathcal{R}\mathrm{e}(X . \partial_x \overline{X}) \rangle \, ds \\ &- \frac{8}{9} \gamma \sum_{k=1}^{3} \int_0^\tau \langle X, \sigma_k \partial_x X \, \mathcal{R}\mathrm{e}(X . \overline{\sigma}_k \partial_x \overline{X}) \rangle \, ds, \end{aligned}$$

where we used the unitary of the Pauli matrices and $\sigma_k = \sigma_k^*$, for $k = 1, 2, 3$. Easy calculations lead to the expression given above. □

4. Diffusion limit of the Manakov PMD equation: Proof of Theorem 1.3. The aim of this part is the proof of the convergence result given in Theorem 1.3. For this purpose we have to cutoff equation (1.11) in order to get uniform bounds, with respect to ε, of high order moments of the $\mathbb{H}^2$ norm of the solution. Let us denote by X_ε^R the solution of the cutoff equation

$$\begin{cases} i\,\dfrac{\partial X_\varepsilon^R(t)}{\partial t} + \dfrac{ib'}{\varepsilon}\boldsymbol{\sigma}(\nu_\varepsilon(t))\,\dfrac{\partial X_\varepsilon^R}{\partial x} + \dfrac{d_0}{2}\,\dfrac{\partial^2 X_\varepsilon^R}{\partial x^2} \\ \qquad + \Theta_R(\|X_\varepsilon^R(t)\|_{\mathbb{H}^1}^2)F_{\nu_\varepsilon(t)}(X_\varepsilon^R) = 0, \\ X_0 = v \in \mathbb{H}^3(\mathbb{R}). \end{cases} \tag{4.1}$$

The proof will consist of the following steps:

(1) We prove uniform bounds on the solution X_ε^R of (4.1). These bounds will enable us to prove tightness on $\mathcal{K}$.

(2) We use the perturbed test function method to get convergence of the generators in some sense [17, 20, 24]. This method formally gives a candidate for the limit process.

(3) Setting $Z_\varepsilon^R = (X_\varepsilon^R, \|X_\varepsilon^R(\cdot)\|_{\mathbb{H}^1}^2)$, we then prove that the family of laws $\mathcal{L}(Z_\varepsilon^R) = \mathbb{P}\circ(Z_\varepsilon^R)^{-1}$ is tight on $\mathcal{K}$ and we deduce that the process Z_ε^R converges in law, up to a subsequence.

(4) Combining the previous steps and using the martingale problem formulation, we identify the limit and conclude to the weak convergence of the whole sequence X_ε^R.

(5) Finally, we get rid of the cutoff and we conclude that the sequence $(X_\varepsilon)_{\varepsilon>0}$ converges in law to X in $\mathcal{E}(\mathbb{H}^1)$ using the Skorokhod theorem.

4.1. *Uniform bounds on X_ε^R.* Recall that a unique solution $\Psi_\varepsilon^R \in C(\mathbb{R}_+, \mathbb{H}^3)$ of the following equation exists (see Section 2):

$$\begin{aligned} &i\,d\Psi_\varepsilon^R(t) + \left\{\frac{ib'}{\varepsilon}\sigma_3\,\frac{\partial \Psi_\varepsilon^R(t)}{\partial x} + \frac{d_0}{2}\,\frac{\partial^2\Psi_\varepsilon^R(t)}{\partial x^2} + \frac{\gamma_s}{\varepsilon^2}\sigma_3\Psi_\varepsilon^R(t) + \frac{i}{\varepsilon^2}\gamma_c\Psi_\varepsilon^R(t)\right\}dt \\ &\qquad - \frac{\sqrt{\gamma_c}}{\varepsilon}\sigma_1\Psi_\varepsilon^R\,d\widetilde{W}_1(t) - \frac{\sqrt{\gamma_c}}{\varepsilon}\sigma_2\Psi_\varepsilon^R\,d\widetilde{W}_2(t) \\ &\qquad + \Theta_R(\|\Psi_\varepsilon^R(t)\|_{\mathbb{H}^1}^2)f(\Psi_\varepsilon^R(t))\,dt = 0. \end{aligned} \tag{4.2}$$

A solution X_ε^R to (4.1) is then easily deduced from $X_\varepsilon^R(t) = Z_\varepsilon^{-1}(t)\Psi_\varepsilon^R(t)$.

LEMMA 4.1. *Let $\psi_0 \in \mathbb{H}^3$ and Ψ_ε^R be the solution of* (4.2); *then for all $T > 0$ there exists a positive constant $C(R,T)$ independent of ε, such that, a.s. for every t in* $[0,T]$,

$$\|\Psi_\varepsilon^R(t)\|_{\mathbb{H}^3} \le C(R,T).$$

Similar bounds hold for $X_\varepsilon^R(t) = Z_\varepsilon^{-1}(t)\Psi_\varepsilon^R(t)$ for any $t \in [0,T]$ since Z_ε^{-1} is almost surely bounded.

PROOF. The bounds on the $\mathbb{H}^3$ norm are obtained using an energy method. Using a regularization procedure, the Itô formula applied to $\|\partial_x \Psi_\varepsilon^R(t)\|_{\mathbb{L}^2}^2$ and equation (4.2), we obtain for all $t \in [0, T]$

$$\|\partial_x \Psi_\varepsilon^R(t)\|_{\mathbb{L}^2}^2 = \|\partial_x \psi_0\|_{\mathbb{L}^2}^2 + 2\int_0^t \langle \partial_x \Psi_\varepsilon^R(s), d\,\partial_x \Psi_\varepsilon^R(s)\rangle + \frac{2\gamma_c}{\varepsilon^2}\int_0^t \|\partial_x \Psi_\varepsilon^R(s)\|_{\mathbb{L}^2}^2\,ds,$$

hence,

$$\begin{aligned}\|\partial_x \Psi_\varepsilon^R(t)\|_{\mathbb{L}^2}^2 &\le \|\partial_x \psi_0\|_{\mathbb{L}^2}^2 \\ &\quad + 2\int_0^t \Theta_R(\|\Psi_\varepsilon^R(s)\|_{\mathbb{H}^1}^2)\|\partial_x f(\Psi_\varepsilon^R(s))\|_{\mathbb{L}^2}\|\partial_x \Psi_\varepsilon^R(s)\|_{\mathbb{L}^2}\,ds \\ &\le \|\partial_x \psi_0\|_{\mathbb{L}^2}^2 + C(R)\int_0^t \|\partial_x \Psi_\varepsilon^R(s)\|_{\mathbb{L}^2}^2\,ds.\end{aligned}$$

By the Gronwall lemma we deduce that

$$\|\partial_x \Psi_\varepsilon^R(t)\|_{\mathbb{L}^2}^2 \le \|\partial_x \psi_0\|_{\mathbb{L}^2}^2 \exp(C(R)T).$$

Using the same procedure for $\|\partial_x^2 X_\varepsilon^R\|_{\mathbb{L}^2}^2$, the Gagliardo–Nirenberg and Young inequalities,

$$\begin{aligned}&\|\partial_x^2 \Psi_\varepsilon^R(t)\|_{\mathbb{L}^2}^2 - \|\partial_x^2 \psi_0\|_{\mathbb{L}^2}^2 \\ &\quad \le C\int_0^t \Theta_R(\|\Psi_\varepsilon^R(s)\|_{\mathbb{H}^1}^2)\big((\|\Psi_\varepsilon^R(s)\|_{\mathbb{L}^\infty}^2 + 1)\|\partial_x^2 \Psi_\varepsilon^R(s)\|_{\mathbb{L}^2}^2 \\ &\qquad + \|\Psi_\varepsilon^R(s)\|_{\mathbb{L}^\infty}^4 \|\partial_x \Psi_\varepsilon^R(s)\|_{\mathbb{L}^2}^6\big)\,ds.\end{aligned}$$

By Sobolev embeddings, properties of the cutoff function and again the Gronwall lemma, we conclude

$$\|\partial_x^2 \Psi_\varepsilon^R(t)\|_{\mathbb{L}^2}^2 \le \|\partial_x^2 \psi_0\|_{\mathbb{L}^2}^2 C(R, T).$$

A bound on $\|\partial_x^3 X_\varepsilon^R\|_{\mathbb{L}^2}^2$ may be obtained similarly using the previous estimates and the Gronwall lemma. $\square$

REMARK 4.1. To prove the convergence result, we need initial data in $\mathbb{H}^3(\mathbb{R})$. We will explain later where exactly we need this extra regularity, but this is mainly due to the fact that we prove tightness in $C([0, T], \mathbb{H}^1)$.

REMARK 4.2. Note that we first prove convergence in law for the couple of random variables $(X_\varepsilon^R, \|X_\varepsilon^R(\cdot)\|_{\mathbb{H}^1}^2)$. This is due to the fact that the cutoff is not continuous for the weak topology in $\mathbb{H}^1$ or for the strong topology in $\mathbb{H}^1_{\mathrm{loc}}$. These arguments have already been used in [11].

4.2. *The perturbed test function method.* Note that the process X_ε^R is not Markov due to the presence of ν_ε. However, $(X_\varepsilon^R, \nu_\varepsilon)$ is Markov, by construction of ν. We denote by $\mathscr{L}_\varepsilon^R$ its infinitesimal generator. Let us compute $\mathscr{L}_\varepsilon^R f$ for f sufficiently smooth such that f maps $\mathbb{H}^{-1} \times \mathbb{S}^3$ into $\mathbb{R}$ and is of class C_b^2. Let $\langle\cdot,\cdot\rangle$ be the duality product between $\mathbb{H}^1$ and $\mathbb{H}^{-1}$. Then, for $\varepsilon > 0$ and for X_ε^R, the solution of the Manakov PMD equation (4.1),

$$\begin{aligned}
&f(X_\varepsilon^R(t), \nu_\varepsilon(t)) - f(v, y)\\
&\quad = f(X_\varepsilon^R(t), \nu_\varepsilon(t)) - f(v, \nu_\varepsilon(t)) + f(v, \nu_\varepsilon(t)) - f(v, y)\\
&\quad = \langle D_v f(v, \nu_\varepsilon(t)), X_\varepsilon^R(t) - v\rangle + R(X_\varepsilon^R(t), v)\\
&\qquad + f(v, \nu_\varepsilon(t)) - f(v, y),
\end{aligned}$$

where

$$R(X_\varepsilon^R(t), v) = \int_0^1 (1-\theta)\big\langle D_v^2 f\big(v + \theta\big(X_\varepsilon^R(t) - v\big)\big)\big(X_\varepsilon^R(t) - v\big), X_\varepsilon^R(t) - v\big\rangle\, d\theta$$

and $D_v^2 f(v) \in \mathcal{L}(\mathbb{H}^{-1}, \mathbb{H}^1)$. Thus,

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}\big(f(X_\varepsilon^R(t), \nu_\varepsilon(t)) - f(v, y) | (X(0), \nu(0)) = (v, y)\big)\\
&\quad = \mathbb{E}\bigg(\bigg\langle D_v f(v, \nu_\varepsilon(t)), \frac{X_\varepsilon^R(t) - v}{t}\bigg\rangle\bigg|(X(0), \nu(0)) = (v, y)\bigg)\\
&\qquad + \mathbb{E}\bigg(\frac{R(X_\varepsilon^R(t), v)}{t}\bigg| X(0) = v\bigg) + \mathbb{E}\bigg(\frac{f(v, \nu_\varepsilon(t)) - f(v, y)}{t}\bigg| \nu(0) = y\bigg).
\end{aligned}$$

We know by Theorem 1.1 that if $v \in \mathbb{H}^3$, then $X_\varepsilon^R \in C^1([0, T], \mathbb{H}^1)$. Thus, by the mean value theorem, equation (4.1), the almost sure boundedness of ν, Lemma 4.1 and the conservation of the $\mathbb{L}^2$ norm,

$$\begin{aligned}
&\frac{1}{t}\|X_\varepsilon^R(t) - v\|_{\mathbb{L}^2}\\
&\quad \le \sup_{s\in[0,t]} \|\partial_s X_\varepsilon^R(s)\|_{\mathbb{L}^2}\\
&\quad \le \sup_{s\in[0,t]} \bigg(\bigg\|\frac{b'}{\varepsilon}\boldsymbol{\sigma}(\nu_\varepsilon(t))\,\partial_x X_\varepsilon^R(s)\bigg\|_{\mathbb{L}^2} + \bigg\|\frac{d_0}{2}\,\partial_x^2 X_\varepsilon^R(s)\bigg\|_{\mathbb{L}^2}\\
&\qquad\qquad + \|\Theta_R(\|X_\varepsilon^R(s)\|_{\mathbb{H}^1}^2) F_{\nu_\varepsilon(s)}(X_\varepsilon^R(s))\|_{\mathbb{L}^2}\bigg)\\
&\quad \le \bigg(\frac{b'}{\varepsilon} + \frac{d_0}{2}\bigg) C(R, T) + 2RC\|v\|_{\mathbb{L}^2}.
\end{aligned}$$

Thus, by the boundedness of $D_v^2 f$, the continuity of $t \mapsto X_\varepsilon^R(t)$ in $\mathbb{L}^2$ and the previous bounds, we conclude that

$$\frac{R(X_\varepsilon^R(t), v)}{t} \le C(R, T, \varepsilon) \sup_{w \in \mathbb{H}^1} \|D_v^2 f(w)\|_{\mathcal{L}(\mathbb{H}^{-1}, \mathbb{H}^1)} (1 + \|v\|_{\mathbb{L}^2}) \|X_\varepsilon(t) - v\|_{\mathbb{L}^2}$$

and the right-hand side above tends to zero as t goes to zero. Now, we perform the change of variables $t' = t/\varepsilon^2$ to get

$$\frac{1}{t}\mathbb{E}\big(f(v, \nu_\varepsilon(t)) - f(v, y) | \nu(0) = y\big) = \frac{1}{\varepsilon^2 t'}\mathbb{E}\big(f(v, \nu(t')) - f(v, y) | \nu(0) = y\big).$$

Thus, using the Markov property of the process ν, and using (4.1) again, we get an expression of the infinitesimal generator $\mathscr{L}_\varepsilon^R$ of the Markov process $(X_\varepsilon^R, \nu_\varepsilon)$:

$$\begin{aligned}
\mathscr{L}_\varepsilon^R f(v, y) &= \lim_{t \to 0} \frac{1}{t}\big(\mathbb{E}\big(f(X_\varepsilon^R(t), \nu_\varepsilon(t)) - f(v, y) | (X(0), \nu(0)) = (v, y)\big)\big) \\
&= \langle D_v f(v, y), \partial_t X_\varepsilon^R(t)|_{t=0}\rangle + \frac{1}{\varepsilon^2}\mathscr{L}_\nu f(v, y) \\
&= \left\langle D_v f(v, y), \frac{i d_0}{2} \frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2) F_y(v) \right\rangle \\
&\quad - \frac{1}{\varepsilon}\left\langle D_v f(v, y), b' \boldsymbol{\sigma}(y) \frac{\partial v}{\partial x} \right\rangle + \frac{1}{\varepsilon^2}\mathscr{L}_\nu f(v, y),
\end{aligned} \tag{4.3}$$

where $\mathscr{L}_\nu$ is the infinitesimal generator of ν and $\mathscr{D}_\nu$ its domain. The perturbed test function method gives (by identifying its infinitesimal generator) an idea of the limit law of the sequence $(X_\varepsilon^R)_{\varepsilon > 0}$. It provides in addition convergences that are useful to prove the weak convergence of the sequence of measures $(\mathcal{L}(X_\varepsilon^R))_{\varepsilon > 0}$.

PROPOSITION 4.1 (Perturbed test function method). *There exists a limiting infinitesimal generator $(\mathscr{L}^R, \mathscr{D}^R)$ such that for all sufficiently smooth and real-valued functions $f \in \mathscr{D}^R$ and for all positive ε, there exists a test function f_ε and positive constants $C_1(K)$ and $C_2(K)$ satisfying*

$$\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} |f_\varepsilon(v, y) - f(v)| \le \varepsilon C_1(K), \tag{4.4}$$

$$\sup_{\substack{v \in \mathcal{B}(K) \\ y \in \mathbb{S}^3}} |\mathscr{L}_\varepsilon^R f_\varepsilon(v, y) - \mathscr{L}^R f(v)| \le \varepsilon C_2(K), \tag{4.5}$$

where $\mathcal{B}(K)$ denotes the closed ball of $\mathbb{H}^3(\mathbb{R})$ with radius K.

PROOF. The idea is to prove that for all suitable test functions f, one can find a function f_ε of the form

$$f_\varepsilon(v, y) = f(v) + \varepsilon f^1(v, y) + \varepsilon^2 f^2(v, y), \tag{4.6}$$

such that Proposition 4.1 holds. We plug this expression of f_ε into (4.3) and formally compute the expression of $\mathscr{L}_\varepsilon^R f_\varepsilon$:

$$
\begin{aligned}
\mathscr{L}_\varepsilon^R f_\varepsilon(v,y) = {} & \Big\langle D_v f(v), \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2)F_y(v)\Big\rangle \\
& - \Big\langle D_v f^1(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\Big\rangle \\
& + \mathscr{L}_\nu f^2(v,y) + \frac{1}{\varepsilon}\mathscr{L}_\nu f^1(v,y) - \frac{1}{\varepsilon}\Big\langle D_v f(v), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\Big\rangle \\
& + \varepsilon\Big\langle D_v f^1(v,y), \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2)F_y(v)\Big\rangle \\
& - \varepsilon\Big\langle D_v f^2(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\Big\rangle \\
& + \varepsilon^2\Big\langle D_v f^2(v,y), \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2)F_y(v)\Big\rangle,
\end{aligned} \tag{4.7}
$$

and we notice that $\mathscr{L}_\nu f(v)$ is identically zero because f does not depend on $\nu = (\nu_1, \nu_2)$. The aim is to wisely choose the functions f^1 and f^2 and the regularity of f so that $\mathscr{L}_\varepsilon^R f_\varepsilon$ is well defined and that f_ε and $\mathscr{L}_\varepsilon^R f_\varepsilon$ converge in the sense of Proposition 4.1. In particular, we need to cancel the terms with a factor $1/\varepsilon$ and we need the terms with factors ε or ε^2 to be $\mathcal{O}(\varepsilon)$ on bounded sets. In order to cancel the $1/\varepsilon$ terms, we look for a function f^1 solution of the Poisson equation

$$
\mathscr{L}_\nu f^1(v,y) = \Big\langle D_v f(v), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\Big\rangle. \tag{4.8}
$$

By Corollary 5.1, we know that

$$
\mathbb{E}_\Lambda(g_j(\nu)) = 0 \qquad \forall j = 1,2,3.
$$

We deduce that $\langle D_v f(v), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\rangle$, which is a linear combination of $m_j = g_j(y)$ [see (1.4)], is of null mass with respect to the invariant measure Λ. Hence, $\langle D_v f(v), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\rangle$ is a function of $y \in \mathbb{S}^3$, which satisfies the assumptions of Proposition 5.1, provided that f is sufficiently smooth, that is, $f \in C^1(\mathbb{H}^{-1})$ and $v \in \mathbb{L}^2$. It follows that the solution f^1 of the Poisson equation (4.8) can be written as

$$
\begin{aligned}
f^1(v,y) &= \mathscr{L}_\nu^{-1}\Big(\Big\langle D_v f(v), b'\boldsymbol{\sigma}(\cdot)\frac{\partial v}{\partial x}\Big\rangle\Big)(y) \\
&= -\Big\langle D_v f(v), b'\widetilde{\boldsymbol{\sigma}}(y)\frac{\partial v}{\partial x}\Big\rangle,
\end{aligned} \tag{4.9}
$$

where

$$
\widetilde{\boldsymbol{\sigma}}(y) = \int_0^{+\infty} \mathbb{E}\big(\cdot\boldsymbol{\sigma}(\nu(t))|\nu(0) = y\big)\,dt. \tag{4.10}
$$

By Proposition 5.1, there is a positive constant M such that

$$\|\widetilde{\boldsymbol{\sigma}}(y)\|_\infty \le M \qquad \forall y \in \mathbb{S}^3, \tag{4.11}$$

and $f^1(v,y)$ is a continuous bounded function of y for $v \in \mathbb{L}^2$. We now have to choose the function f^2, but we cannot choose $\mathscr{L}_v f^2$ cancelling the terms

$$\left\langle D_v f(v), \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|^2_{\mathbb{H}^1})F_y(v)\right\rangle - \left\langle D_v f^1(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\right\rangle,$$

because they do not satisfy the null mass condition with respect to Λ. Hence, we look for a solution f^2 of the Poisson equation

$$\begin{aligned} \mathscr{L}_v f^2(v,y) = &-\langle D_v f(v), i\Theta_R(\|v\|^2_{\mathbb{H}^1})F_y(v)\rangle \\ &+ \langle D_v f(v), i\Theta_R(\|v\|^2_{\mathbb{H}^1})F(v)\rangle \\ &+ \left\langle D_v f^1(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\right\rangle \\ &- \mathbb{E}_\Lambda\left(\left\langle D_v f^1(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\right\rangle\right), \end{aligned} \tag{4.12}$$

where, due to (4.9),

$$\begin{aligned} &\left\langle D_v f^1(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\right\rangle \\ &\quad = -(b')^2\left\langle D_v^2 f(v)\widetilde{\boldsymbol{\sigma}}(y)\frac{\partial v}{\partial x}, \boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\right\rangle \\ &\qquad - (b')^2\left\langle D_v f(v), \widetilde{\boldsymbol{\sigma}}(y)\boldsymbol{\sigma}(y)\frac{\partial^2 v}{\partial x^2}\right\rangle. \end{aligned} \tag{4.13}$$

Moreover, thanks to expression (4.13), the Fubini theorem and Corollary 5.1,

$$\begin{aligned} &-\mathbb{E}_\Lambda\left(\left\langle D_v f^1(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\right\rangle\right) \\ &\quad = (b')^2\sum_{j,k=1}^3\left\langle D_v^2 f(v)\sigma_k\frac{\partial v}{\partial x}, \sigma_j\frac{\partial v}{\partial x}\right\rangle\int_0^{+\infty}\mathbb{E}_\Lambda(g_k(\nu(t))g_j(\nu(0)))\,dt \\ &\qquad + (b')^2\sum_{j,k=1}^3\left\langle D_v f(v), \sigma_k\sigma_j\frac{\partial^2 v}{\partial x^2}\right\rangle\int_0^{+\infty}\mathbb{E}_\Lambda(g_k(\nu(t))g_j(\nu(0)))\,dt \\ &\quad = \frac{\gamma}{2}\sum_{k=1}^3\left\langle D_v^2 f(v)\sigma_k\frac{\partial v}{\partial x}, \sigma_k\frac{\partial v}{\partial x}\right\rangle + \frac{3\gamma}{2}\left\langle D_v f(v), \frac{\partial^2 v}{\partial x^2}\right\rangle, \end{aligned} \tag{4.14}$$

where $\gamma = (b')^2/6\gamma_c$. Provided that f is of class $C^2(\mathbb{H}^{-1})$ and $v \in \mathbb{H}^1$ and because $f^1(v,\cdot)$ is of class $C_b^2(\mathbb{S}^3)$ for any $v \in \mathbb{H}^1$, we can now define, by Proposition 5.1, a unique solution, up to a constant, to the Poisson equation (4.12). This solution f^2 is expressed as

$$
\begin{aligned}
f^2(v,y) &= \mathscr{L}_\nu^{-1}\big(\langle D_v f(v), i\Theta_R(\|v\|_{\mathbb{H}^1}^2)\big(F_y(v)-F(v)\big)\rangle\big) \\
&\quad - \mathscr{L}_\nu^{-1}\bigg(\bigg\langle D_v f^1(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\bigg\rangle \\
&\qquad - \mathbb{E}_\Lambda\bigg(\bigg\langle D_v f^1(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\bigg\rangle\bigg)\bigg) \\
&= \langle D_v f(v), i\Theta_R(\|v\|_{\mathbb{H}^1}^2)\widetilde{F}(v,y)\rangle \\
&\quad - (b')^2\sum_{k,l=1}^{3}\bigg\langle D_v^2 f(v)\sigma_k\frac{\partial v}{\partial x}, \sigma_l\frac{\partial v}{\partial x}\bigg\rangle \widetilde{\widetilde{g}}_{k,l}(y) \\
&\quad - \bigg\langle D_v f(v), (b')^2\widetilde{\widetilde{\boldsymbol{\sigma}}}(y)\frac{\partial^2 v}{\partial x^2}\bigg\rangle,
\end{aligned}
\tag{4.15}
$$

where

$$
\widetilde{F}(v,y) = \int_0^{+\infty}\mathbb{E}\big(F_{\nu(t)}(v)-F(v)|\nu(0)=y\big)\,dt
$$

and

$$
\widetilde{\widetilde{g}}_{k,l}(y) = \int_0^{+\infty}\bigg(\int_t^{+\infty}\mathbb{E}\big(g_k(\nu(s))g_l(\nu(t))|\nu(0)=y\big)\,ds - \frac{\gamma}{2(b')^2}\delta_{kl}\bigg)\,dt
$$

and

$$
\widetilde{\widetilde{\boldsymbol{\sigma}}}(y) = \int_0^{+\infty}\bigg(\int_t^{+\infty}\mathbb{E}\big(\boldsymbol{\sigma}(\nu(s))\boldsymbol{\sigma}(\nu(t))|\nu(0)=y\big)\,ds - \frac{3\gamma}{2(b')^2}\bigg)\,dt.
$$

Replacing $\mathscr{L}_\nu f^1$ and $\mathscr{L}_\nu f^2$ in (4.7), respectively, by the right-hand side of (4.8) and (4.12), and using expression (4.14), we get

$$
\begin{aligned}
\mathscr{L}_\varepsilon^R f_\varepsilon(v,y) &= \bigg\langle D_v f(v), \bigg(\frac{id_0}{2}+\frac{3\gamma}{2}\bigg)\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2)F(v)\bigg\rangle \\
&\quad + \frac{\gamma}{2}\sum_{k=1}^{3}\bigg\langle D_v^2 f(v)\sigma_k\frac{\partial v}{\partial x}, \sigma_k\frac{\partial v}{\partial x}\bigg\rangle \\
&\quad + \varepsilon\bigg\langle D_v f^1(v,y), \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2)F_y(v)\bigg\rangle \\
&\quad - \varepsilon\bigg\langle D_v f^2(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\bigg\rangle \\
&\quad + \varepsilon^2\bigg\langle D_v f^2(v,y), \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2)F_y(v)\bigg\rangle,
\end{aligned}
\tag{4.16}
$$

and we define the limiting operator by

$$
\begin{aligned}
\mathscr{L}^R f(v) = &\Big\langle D_v f(v), \Big(\frac{id_0}{2} + \frac{3\gamma}{2}\Big)\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|^2_{\mathbb{H}^1})F(v)\Big\rangle \\
&+ \frac{\gamma}{2}\sum_{k=1}^{3}\Big\langle D_v^2 f(v)\sigma_k \frac{\partial v}{\partial x}, \sigma_k \frac{\partial v}{\partial x}\Big\rangle.
\end{aligned} \tag{4.17}
$$

Hence, if we define $\mathscr{D}^R$ as the space of functions which are the restriction to $\mathbb{H}^3$ of functions f from $\mathbb{H}^{-1}$ into $\mathbb{R}$ of class $C^3(\mathbb{H}^{-1})$ and such that f and its first three derivatives are bounded on bounded sets of $\mathbb{H}^{-1}$, then the functions f^1 and f^2 are well defined for $f \in \mathscr{D}^R$. Moreover, if $f \in \mathscr{D}^R$, then $\mathscr{L}^R_\varepsilon f_\varepsilon$ is well defined for $v \in \mathbb{H}^3$.

We now write that

$$
\sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} |f_\varepsilon(v,y) - f(v)| \le \varepsilon \sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} |f^1(v,y)| + \varepsilon^2 \sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} |f^2(v,y)|
$$

and use the following result, which is proved in Section 6.

LEMMA 4.2. *Let $f \in \mathscr{D}^R$ and f^1 and f^2 be, respectively, solutions of* (4.8) *and* (4.12). *Then*

$$
\sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} |f^1(v,y)| \le C_1(K) \quad \textit{and} \quad \sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} |f^2(v,y)| \le C_2(K).
$$

This proves the first convergence of Proposition 4.1. With $\mathscr{L}^R f(v)$ given by (4.17), the second convergence (4.5) in Proposition 4.1 follows from (4.16) and the next lemma, which is proved in Section 6. $\square$

LEMMA 4.3. *Let $f \in \mathscr{D}^R$ and f^1, f^2 be, respectively, solutions of* (4.8) *and* (4.12). *Then*

$$
\sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} \Big|\Big\langle D_v f^1(v,y), \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|^2_{\mathbb{H}^1})F_y(v)\Big\rangle\Big| \le C_1(K),
$$

$$
\sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} \Big|\Big\langle D_v f^2(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\Big\rangle\Big| \le C_2(K),
$$

$$
\sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} \Big|\Big\langle D_v f^2(v,y), \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|^2_{\mathbb{H}^1})F_y(v)\Big\rangle\Big| \le C_3(K).
$$

4.3. *Tightness of the family of probability measures* $(\mathcal{L}(Z_\varepsilon^R))_{\varepsilon>0}$. To prove tightness on $\mathcal{K}$ of the sequence of probability measure $\mathcal{L}(Z_\varepsilon^R) = \mathbb{P} \circ (Z_\varepsilon^R)^{-1}$, we need to obtain uniform bounds in ε on Z_ε^R in the space

$$\big(C([0,T],\mathbb{H}^2) \cap C^\alpha([0,T],\mathbb{H}^{-1})\big) \times C^\delta([0,T],\mathbb{R})$$

for suitable $\alpha, \delta > 0$. Note that uniform bounds of X_ε^R in $C([0,T],\mathbb{H}^2)$ are given by Lemma 4.1. The perturbed test function method will enable us to get the uniform bound in $C^\alpha([0,T],\mathbb{H}^{-1})$. Such bounds cannot be directly obtained using (4.1) because of the $1/\varepsilon$ term. In order to obtain such bounds, we use again the perturbed test function method for convenient test functions. Let $(\widetilde{e}_j)_{j\in\mathbb{N}^*}$ be a complete orthonormal system in $\mathbb{L}^2$. Recall that $\langle\cdot,\cdot\rangle$ is the duality product between $\mathbb{H}^1$–$\mathbb{H}^{-1}$ and $(\cdot,\cdot)_{\mathbb{L}^2}$ the inner product in $\mathbb{L}^2$. By definition of $\mathbb{H}^s$, $s \in \mathbb{R}$, we can define a complete orthonormal system $(e_j)_{j\in\mathbb{N}^*}$ on $\mathbb{H}^1$ from $(\widetilde{e}_j)_{j\in\mathbb{N}^*}$,

$$\begin{aligned}
\|v\|_{\mathbb{H}^{-1}}^2 &= \|(1+\xi^2)^{-1/2}\widehat{v}\|_{\mathbb{L}^2}^2 \\
&= \sum_{j=1}^{+\infty} \big((1+\xi^2)^{-1/2}\widehat{v}, \widehat{\widetilde{e}}_j\big)_{\mathbb{L}^2}^2 \\
&= \sum_{j=1}^{+\infty} \langle e_j, v\rangle^2,
\end{aligned}$$

where $e_j = \mathcal{F}^{-1}((1+\xi^2)^{-1/2}\widehat{\widetilde{e}}_j)$ for any $j \in \mathbb{N}^*$. We denote by $(f_j)_{j\in\mathbb{N}^*}$ the family of test functions in $\mathscr{D}^R$ defined by

$$\begin{aligned}
f_j : \mathbb{H}^{-1} &\to \mathbb{R}, \\
v &\mapsto f_j(v) = \langle e_j, v\rangle.
\end{aligned}$$

For $v \in \mathbb{H}^3$, we also consider particular perturbed test functions $f_{j,\varepsilon}$ of the form

$$f_{j,\varepsilon}(v,y) = f_j(v) + \varepsilon f_j^1(v,y), \tag{4.18}$$

where, for all j in $\mathbb{N}^*$, $f_j^1(v,y) = \langle e_j, \varphi^1(v,y)\rangle$ for a given function φ^1 with values in $\mathbb{H}^2$. We now choose φ^1 as a solution of the Poisson equation in y:

$$\mathscr{L}_v\varphi^1(v,y) - b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x} = 0, \tag{4.19}$$

whose explicit formulation is given by (see Proposition 5.1)

$$\varphi^1(v,y) = -b'\widetilde{\boldsymbol{\sigma}}(y)\frac{\partial v}{\partial x}, \tag{4.20}$$

where $\widetilde{\boldsymbol{\sigma}}(y)$ is given by (4.10). We point out that φ^1 behaves in its first variable like $\frac{\partial}{\partial x}$ and is linear in v. Consequently, for all j in $\mathbb{N}^*$,

$$\begin{aligned}
&\mathscr{L}_\varepsilon^R f_{j,\varepsilon}(X_\varepsilon^R(t), \nu_\varepsilon(t)) \\
&\quad = \left\langle e_j, \frac{id_0}{2} \frac{\partial^2 X_\varepsilon^R(t)}{\partial x^2} + i\Theta_R(\|X_\varepsilon^R(t)\|_{\mathbb{H}^1}^2) F_{\nu_\varepsilon(t)}(X_\varepsilon^R(t))\right\rangle \\
&\qquad + \left\langle e_j, (b')^2 \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(t)) \boldsymbol{\sigma}(\nu_\varepsilon(t)) \frac{\partial^2 X_\varepsilon^R(t)}{\partial x^2}\right\rangle \qquad (4.21)\\
&\qquad - \varepsilon \left\langle e_j, b' \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(t)) \frac{\partial}{\partial x}\left(\frac{id_0}{2} \frac{\partial^2 X_\varepsilon^R(t)}{\partial x^2}\right.\right. \\
&\qquad\qquad \left.\left. + i\Theta_R(\|X_\varepsilon^R(t)\|_{\mathbb{H}^1}^2) F_{\nu_\varepsilon(t)}(X_\varepsilon^R(t))\right)\right\rangle.
\end{aligned}$$

For all $t \in [0, T]$, we define the process M_ε^R with values in $\mathbb{H}^{-1}$ given for any j in $\mathbb{N}^*$ by

$$\begin{aligned}
&\langle e_j, M_\varepsilon^R(t)\rangle \\
&\quad = f_{j,\varepsilon}(X_\varepsilon^R, \nu_\varepsilon)(t) - f_{j,\varepsilon}(v, y) - \int_0^t \mathscr{L}_\varepsilon^R f_{j,\varepsilon}(X_\varepsilon^R(s), \nu_\varepsilon(s))\, ds \\
&\quad = \langle e_j, X_\varepsilon^R - v\rangle + \varepsilon \langle e_j, \varphi^1(X_\varepsilon^R, \nu_\varepsilon) - \varphi^1(v, y)\rangle \\
&\qquad - \int_0^t \mathscr{L}_\varepsilon^R f_{j,\varepsilon}(X_\varepsilon^R(s), \nu_\varepsilon(s))\, ds.
\end{aligned}$$

Given the fact that $\mathscr{L}_\varepsilon^R$ is the infinitesimal generator of the continuous Markov process $(X_\varepsilon^R, \nu_\varepsilon)$ and $\mathscr{L}_\varepsilon^R f_{j,\varepsilon}$ is well defined because $f_j \in \mathscr{D}^R$, then $\langle e_j, M_\varepsilon^R(t)\rangle$ is a real-valued continuous martingale. Moreover, it is a square integrable martingale, as follows from the bounds on the $\mathbb{H}^3$ norm of X_ε^R obtained in Lemma 4.1. To prove tightness of the family of probability measures $\mathcal{L}(Z_\varepsilon^R)$ on $\mathcal{K}$, we need estimates of moments on the processes X_ε^R and $\|X_\varepsilon^R(\cdot)\|_{\mathbb{H}^1}^2$. Before proving these estimates we introduce a process Y_ε^R close in probability to X_ε^R for which it will be easier to get those estimates, using, in particular, the Kolmogorov criterion. The idea is to use Lemma 4.4 below to get tightness of the family $\mathcal{L}(Z_\varepsilon^R)$ from convergence in law of a subsequence of Y_ε^R.

LEMMA 4.4. *Let us define the process Y_ε^R as*

$$X_\varepsilon^R(t) - Y_\varepsilon^R(t) = \varepsilon\big(\varphi^1(v, y) - \varphi^1(X_\varepsilon^R(t), \nu_\varepsilon(t))\big) \qquad \forall t \in [0, T]; \tag{4.22}$$

then for all $\delta > 0$,

$$\mathbb{P}\big(\|X_\varepsilon^R - Y_\varepsilon^R\|_{C([0,T],\mathbb{H}^1)} > \delta\big) \le \frac{\varepsilon}{\delta} C_1(T, R).$$

PROOF. Using the Markov inequality and Lemma 4.1, we get for all $\delta > 0$,

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,T]} \|X_\varepsilon^R(t) - Y_\varepsilon^R(t)\|_{\mathbb{H}^1} > \delta\Big) \\
&\quad \le \frac{\varepsilon}{\delta}\mathbb{E}\Big(\sup_{t\in[0,T]} \|\varphi^1(X_\varepsilon^R(t), \nu_\varepsilon(t)) - \varphi^1(v, y)\|_{\mathbb{H}^1}\Big) \\
&\quad \le \frac{\varepsilon}{\delta}\mathbb{E}\Big(\sup_{t\in[0,T]} \Big\| b'\widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(t)) \frac{\partial X_\varepsilon^R(t)}{\partial x} - b'\widetilde{\boldsymbol{\sigma}}(y) \frac{\partial v}{\partial x}\Big\|_{\mathbb{H}^1}\Big) \\
&\quad \le \varepsilon \frac{2M}{\delta} C(T, R),
\end{aligned}
$$

where M is given by (4.11). $\square$

Note that the process Y_ε^R is also defined by the identity, for all j in $\mathbb{N}^*$,

$$
\begin{aligned}
\langle e_j, Y_\varepsilon^R(t)\rangle &= \langle e_j, X_\varepsilon^R(t)\rangle - \varepsilon\langle e_j, \varphi^1(v, y) - \varphi^1(X_\varepsilon^R(t), \nu_\varepsilon(t))\rangle \\
&= \langle e_j, M_\varepsilon^R(t)\rangle + \langle e_j, v\rangle + \int_0^t \mathscr{L}_\varepsilon^R f_{j,\varepsilon}(X_\varepsilon^R(s), \nu_\varepsilon(s))\, ds \qquad (4.23)\\
&\qquad \forall t \in [0, T].
\end{aligned}
$$

LEMMA 4.5. *For all $1 \ge \varepsilon > 0$, there exist three positive constants $C_1(T, R)$, $C_2(T, R)$ and $C_3(T, R)$ depending on final time T and on the cutoff radius R, but independent of ε, such that*

$$\mathbb{E}\big(\|Y_\varepsilon^R\|^4_{C([0,T],\mathbb{H}^2)}\big) \le C_1(T, R), \tag{4.24}$$

$$\mathbb{E}\big(\|Y_\varepsilon^R\|_{C^\alpha([0,T],\mathbb{H}^{-1})}\big) \le C_2(T, R), \tag{4.25}$$

$$\mathbb{E}\big(\|\|Y_\varepsilon^R\|^2_{\mathbb{H}^1}\|_{C^\delta([0,T],\mathbb{R})}\big) \le C_3(T, R), \tag{4.26}$$

where $0 < \alpha < \frac{1}{2}$ and $\delta = \alpha/3 > 0$.

PROOF. Thanks to Lemma 4.1, we know that the solution X_ε^R of (4.1) is uniformly bounded, for all ε, in $\mathbb{H}^3$ by a constant C depending on R and T. We conclude, using the explicit formulation of φ_1 given by (4.20) and (4.22), that (4.24) holds.

To prove inequality (4.25), we first need an intermediate estimate that will be proved in Section 6.

LEMMA 4.6. *There exists a positive constant $C(R, T)$ such that for all $t, s \in [0, T]$*

$$\mathbb{E}\big(\|Y_\varepsilon^R(t) - Y_\varepsilon^R(s)\|^4_{\mathbb{H}^{-1}}\big) \le C(R, T)(t - s)^2.$$

Then we deduce from Lemma 4.6

$$\mathbb{E}\big(\|Y_\varepsilon^R\|^4_{\mathbb{W}^{\gamma,4}([0,T],\mathbb{H}^{-1})}\big) \le C(R,T)$$

for any $\gamma < 1/2$. We use the Sobolev embedding $\mathbb{W}^{\gamma,4}([0,T],\mathbb{H}^{-1}) \hookrightarrow C^\alpha([0,T],\mathbb{H}^{-1})$ for $\gamma - \alpha > 1/4$ and $\gamma < 1/2$, which implies $\alpha < 1/4$. Thus, we deduce the second inequality (4.25).

It remains to prove the last bound (4.26). Note that for $t, s \in [0,T]$

$$\begin{aligned}
&\big|\|Y_\varepsilon^R(t)\|^2_{\mathbb{H}^1} - \|Y_\varepsilon^R(s)\|^2_{\mathbb{H}^1}\big| \\
&\quad \le C \sup_{r\in[0,T]} \|Y_\varepsilon^R(r)\|_{\mathbb{H}^1} \|Y_\varepsilon^R(t) - Y_\varepsilon^R(s)\|_{\mathbb{H}^1} \\
&\quad \le C \sup_{r\in[0,T]} \|Y_\varepsilon^R(r)\|_{\mathbb{H}^1} \sup_{r\in[0,T]} \|Y_\varepsilon^R(r)\|^{2/3}_{\mathbb{H}^2} \|Y_\varepsilon^R(t) - Y_\varepsilon^R(s)\|^{1/3}_{\mathbb{H}^{-1}}.
\end{aligned}$$

It follows that if $\delta = \alpha/3$,

$$\big\|\|Y_\varepsilon^R(\cdot)\|^2_{\mathbb{H}^1}\big\|_{C^\delta([0,T],\mathbb{R})} \le C \sup_{r\in[0,T]} \|Y_\varepsilon^R(r)\|^{5/3}_{\mathbb{H}^2} \|Y_\varepsilon^R\|^{1/3}_{C^\alpha([0,T],\mathbb{H}^{-1})}.$$

Inequality (4.26) is then implied by the Hölder inequality, (4.24) and (4.25). $\square$

REMARK 4.3. The extra $\mathbb{H}^3$ regularity is needed precisely in the first step of the above proof in order to estimate the $\mathbb{H}^2$ norm of Y_ε^R, which involves the gradient of X_ε^R.

PROPOSITION 4.2. *The family of laws $(\mathcal{L}(Z_\varepsilon^R))_{\varepsilon>0}$ is tight on $\mathcal{K}$.*

PROOF. We set $\widetilde{Z}_\varepsilon^R = (Y_\varepsilon^R, \|Y_\varepsilon^R(\cdot)\|^2_{\mathbb{H}^1})$. Denoting by $\mathcal{B}(K)$ the closed ball of $(C([0,T];\mathbb{H}^2(\mathbb{R})) \cap C^\alpha([0,T];\mathbb{H}^{-1}(\mathbb{R}))) \times C^\delta([0,T];\mathbb{R})$ with radius K, for α and δ as in Lemma 4.5, we deduce using the Ascoli–Arzela and Banach–Alaoglu theorems that $\mathcal{B}(K)$ is compact in $\mathcal{K}$. Using the Markov inequality and Lemma 4.5, we get

$$\begin{aligned}
&\mathbb{P}\big(\widetilde{Z}_\varepsilon^R \notin \mathcal{B}(K)\big) \\
&\quad \le \frac{1}{K}\mathbb{E}\big(\max\big\{\|Y_\varepsilon^R\|_{C([0,T];\mathbb{H}^2)}, \|Y_\varepsilon^R\|_{C^\alpha([0,T];\mathbb{H}^{-1})}, \big\|\|Y_\varepsilon^R\|^2_{\mathbb{H}^1}\big\|_{C^\delta([0,T])}\big\}\big) \\
&\quad \le \frac{1}{K}\max(C_1^{1/4}(T,R), C_2(T,R), C_3(T,R)).
\end{aligned}$$

We conclude that the family of laws $(\mathcal{L}(\widetilde{Z}_\varepsilon^R))_{\varepsilon>0}$ is tight on $\mathcal{K}$ and by the Prokhorov theorem we obtain the relative compactness of the sequence of laws $(\mathcal{L}(\widetilde{Z}_\varepsilon^R))_{\varepsilon>0}$, that is, up to a subsequence, the sequence $\mathcal{L}(\widetilde{Z}_\varepsilon^R)$ weakly converges to a probability measure $\mathcal{L}(\widehat{Z}^R)$ where $\widehat{Z}^R = (\widehat{X}^R, \gamma^R)$. We may now use Lemma 4.4 to prove that the family of laws $\mathcal{L}(Z_\varepsilon^R)$ is tight. Indeed, it easily follows from Lemma 4.4 and the above convergence in law that for all $g \in C_b(\mathcal{K})$

$$\lim_{\varepsilon\to 0} \mathbb{E}(g(Z_\varepsilon^R)) = \mathbb{E}(g(\widehat{Z}^R)). \qquad \square$$

4.4. *Convergence in law of the process* X_ε^R. In order to get the convergence in law of the whole sequence $(X_\varepsilon^R)_{\varepsilon>0}$, it remains to characterize the limit, that is, to prove that $\widehat{X}^R = X^R$, the solution of (3.6), and that $\gamma^R(t) = \|X^R(t)\|_{\mathbb{H}^1}^2$ for any $t \in [0, T]$. The tool here will be the use of the martingale problem formulation introduced by Stroock and Varadhan in [26].

PROPOSITION 4.3. *The whole sequence* X_ε^R *converges in law to* X^R *in* $C([0, T], \mathbb{H}^1)$.

PROOF. In order to prove that any subsequence of X_ε^R converges to the same limit X^R, the solution of (3.6), we will prove the convergence of the martingale problem for suitable test functions $f \in \mathscr{D}^R$. To this purpose let us define, for $a \in \mathbb{H}^1$ with compact support, the particular test function $f_a(\cdot) = \langle a, \cdot\rangle$, so that $f_a \in \mathscr{D}^R$. From this particular choice, we construct a perturbed test function $f_{a,\varepsilon}$,

$$f_{a,\varepsilon}(v, y) = f_a(v) + \varepsilon f_a^1(v, y) + \varepsilon^2 f_a^2(v, y),$$

obtained thanks to Proposition 4.1. The correctors f_a^1 and f_a^2 are chosen to be the solution of the Poisson equations (4.8) and (4.12) for f_a. Let us denote by Z_ε^R a subsequence converging to $\widehat{Z}^R$ and define the $\mathbb{H}^{-1}$ valued process $\mathbf{N}_\varepsilon^R(Z_\varepsilon^R(t))$, associated to (4.1),

$$\begin{aligned}\langle a, \mathbf{N}_\varepsilon^R(Z_\varepsilon^R(t))\rangle &= f_{a,\varepsilon}(X_\varepsilon^R(t), \nu_\varepsilon(t)) - f_{a,\varepsilon}(v, y)\\ &\qquad - \int_0^t \mathscr{L}_\varepsilon^R f_{a,\varepsilon}(X_\varepsilon^R(s), \nu_\varepsilon(s))\,ds,\end{aligned}$$

where $\mathscr{L}_\varepsilon^R$ is given by (4.7). We also define the process $\mathbf{N}^R(Z_\varepsilon^R(t))$,

$$\langle a, \mathbf{N}^R(Z_\varepsilon^R(t))\rangle = f_a(X_\varepsilon^R(t)) - f_a(v) - \int_0^t \mathscr{L}^R f_a(X_\varepsilon^R(s))\,ds,$$

where $\mathscr{L}^R$ is given by expression (4.17). Moreover, we denote by $\mathscr{L}_{\gamma^R}^R$ the operator whose expression is given by (4.17) replacing $\|\widehat{X}^R(t)\|_{\mathbb{H}^1}$ by $\gamma^R(t)$ in the cutoff function. Let us now define $\langle a, \mathbf{N}^R(\widehat{Z}^R(t))\rangle$ by

$$\langle a, \mathbf{N}^R(\widehat{Z}^R(t))\rangle = f_a(\widehat{X}^R(t)) - f_a(v) - \int_0^t \mathscr{L}_{\gamma^R}^R f_a(\widehat{X}^R(s))\,ds. \tag{4.27}$$

The process $\langle a, \mathbf{N}_\varepsilon^R(Z_\varepsilon^R(t))\rangle$ is a real continuous martingale because $(X_\varepsilon^R, \nu_\varepsilon)$ is a Markov process and because $\mathscr{L}_\varepsilon^R f_{a,\varepsilon}$ is well defined since $X_\varepsilon^R(t) \in \mathbb{H}^3$. Moreover, it is a square integrable martingale, as follows from the bounds on the $\mathbb{H}^3$ norm of X_ε^R obtained in Lemma 4.1. The above martingale property implies that for all $t, s \in [0, T], t \geq s$,

$$\mathbb{E}[\langle a, \mathbf{N}_\varepsilon^R(Z_\varepsilon^R(t)) - \mathbf{N}_\varepsilon^R(Z_\varepsilon^R(s))\rangle | \sigma(Z_\varepsilon^R(u), \nu_\varepsilon(u)), u \leq s] = 0.$$

It follows, in particular, that for all test functions $h_1, \dots, h_m \in C_b(\mathbb{H}^1_{\mathrm{loc}} \times \mathbb{R})$ and $0 \leq t_1 < \cdots < t_m \leq s \leq t$,

$$\mathbb{E}\left[\langle a, \mathbf{N}_\varepsilon^R(Z_\varepsilon^R(t)) - \mathbf{N}_\varepsilon^R(Z_\varepsilon^R(s))\rangle \prod_{j=1}^m h_j(Z_\varepsilon^R(t_j))\right] = 0.$$

Using Proposition 4.1, Lemma 4.1 and the boundedness of the functions h_j, we get

$$\mathbb{E}\left(\langle a, \mathbf{N}_\varepsilon^R(Z_\varepsilon^R(t)) - \mathbf{N}^R(Z_\varepsilon^R(t)) - \mathbf{N}_\varepsilon^R(Z_\varepsilon^R(s)) + \mathbf{N}^R(Z_\varepsilon^R(s))\rangle \prod_{j=1}^m h_j(Z_\varepsilon^R(t_j))\right)$$
$$\leq \varepsilon C(R, T).$$

Let us consider a cutoff function $\chi_{R_0} \in C_c^\infty(\mathcal{K})$ satisfying

$$\chi_{R_0}(u) = \begin{cases} 1, & \text{if } u \in \mathcal{B}_\mathcal{K}(R_0), \\ 0, & \text{if } u \notin \mathcal{B}_\mathcal{K}(2R_0), \end{cases}$$

where $\mathcal{B}_\mathcal{K}(R_0)$ denotes the closed ball of radius R_0 of the space $\mathcal{K}$ and R_0 is chosen such that $X_\varepsilon^R \in \mathcal{B}_\mathcal{K}(R_0)$ a.s. (see Lemma 4.1). Note that by continuity of the functions χ_{R_0} and $\{h_j\}_{j\in\{1,\dots,m\}}$, respectively, in $\mathcal{K}$ and $\mathbb{H}^1_{\mathrm{loc}} \times \mathbb{R}$, by continuity of $f_a(\cdot)$ for the weak topology in $\mathbb{H}^1$, by continuity and boundedness of Θ_R in $C([0, T]; \mathbb{R})$, by continuity of F from $\mathbb{H}^1$ to $\mathbb{H}^{-1}$ and the bounds on $F(X_\varepsilon^R(t))$ obtained thanks to Lemma 4.1, the function

$$\langle a, \mathbf{N}^R(Z_\varepsilon^R(t))\rangle \chi_{R_0}(Z_\varepsilon^R) \prod_{j=1}^m h_j(Z_\varepsilon^R(t_j))$$

is a bounded and continuous function of Z_ε^R from $\mathcal{K}$ into $\mathbb{R}$. We deduce by convergence in law of Z_ε^R to $\widehat{Z}^R$ in $\mathcal{K}$, since the test function a is compactly supported, that for all $t, s \in [0, T], t \geq s$

$$\mathbb{E}\left(\langle a, \mathbf{N}^R(\widehat{Z}^R(t)) - \mathbf{N}^R(\widehat{Z}^R(s))\rangle \chi_{R_0}(\widehat{Z}^R) \prod_{j=1}^m h_j(\widehat{Z}^R(t_j))\right) = 0. \tag{4.28}$$

Since, almost surely, X_ε^R belongs to the closed ball $\mathcal{B}_\mathcal{K}(R_0)$, we deduce that almost surely $\widehat{X}^R \in \mathcal{B}_\mathcal{K}(R_0)$. Thus, we conclude from (4.28) that $\langle a, \mathbf{N}^R(\widehat{Z}^R(\cdot))\rangle$ is a continuous square integrable martingale with respect to the filtration $\mathcal{G}_t = \sigma(\widehat{Z}^R(s), s \leq t)$ and this holds for any $a \in \mathbb{H}^1$ with compact support.

In order to identify the equation satisfied by $\widehat{X}^R$, we consider, for $a, b \in \mathbb{H}^1$ with compact support, the function $g_{a,b}(v) = f_a(v) f_b(v) \in \mathscr{D}^R$ and the perturbed test function $g_{a,b,\varepsilon}$,

$$g_{a,b,\varepsilon}(v, y) = g_{a,b}(v) + \varepsilon g^1_{a,b}(v, y) + \varepsilon^2 g^2_{a,b}(v, y),$$

obtained thanks to Proposition 4.1. Thus, functions $g^1_{a,b}(v, y)$ and $g^2_{a,b}(v, y)$ are chosen to be solutions of the Poisson equations (4.8) and (4.12) for $g_{a,b}$. Let us now define the real-valued continuous martingale

$$\begin{aligned}\mathbf{H}^R_{a,b,\varepsilon}(Z^R_\varepsilon(t)) &= g_{a,b,\varepsilon}(X^R_\varepsilon(t), \nu_\varepsilon(t)) - g_{a,b,\varepsilon}(v, y)\\ &\quad - \int_0^t \mathscr{L}^R_\varepsilon g_{a,b,\varepsilon}(X^R_\varepsilon(s), \nu_\varepsilon(s))\, ds.\end{aligned}$$

Using the same arguments as before, we may prove that

$$\begin{aligned}&\lim_{\varepsilon \to 0} \mathbb{E}\Bigg(\big(\mathbf{H}^R_{a,b,\varepsilon}(Z^R_\varepsilon(t)) - \mathbf{H}^R_{a,b,\varepsilon}(Z^R_\varepsilon(s))\big)\chi_{R_0}(Z^R_\varepsilon) \prod_{j=1}^m h_j(Z^R_\varepsilon(t_j))\Bigg)\\ &\quad = \mathbb{E}\Bigg(\big(\mathbf{H}^R_{a,b}(\widehat{Z}^R(t)) - \mathbf{H}^R_{a,b}(\widehat{Z}^R(s))\big)\chi_{R_0}(\widehat{Z}^R) \prod_{j=1}^m h_j(\widehat{Z}^R(t_j))\Bigg),\end{aligned}$$

where

$$\begin{aligned}\mathbf{H}^R_{a,b}(\widehat{Z}(t)) &= g_{a,b}(\widehat{X}^R(t)) - g_{a,b}(v)\\ &\quad - \int_0^t \mathscr{L}^R_{\gamma^R} g_{a,b}(\widehat{X}^R(s))\, ds.\end{aligned}$$

From the above convergence and the martingale property of $\mathbf{H}^R_{a,b,\varepsilon}(Z^R_\varepsilon(t))$, we deduce that $\mathbf{H}^R_{a,b}(\widehat{Z}^R(\cdot))$ is a continuous real-valued martingale. A classical computation then shows that the quadratic variation of the martingale $\mathbf{N}^R(\widehat{Z}^R(t))$ defined in (4.27) is given by

$$\begin{aligned}&\langle b, \langle\!\langle \mathbf{N}^R(\widehat{Z}^R(t))\rangle\!\rangle a\rangle\\ &\quad = \int_0^t \mathscr{L}^R_{\gamma^R}(f_a(\widehat{Z}^R(s)) f_b(\widehat{Z}^R(s))) - f_a(\widehat{Z}^R(s))\mathscr{L}^R_{\gamma^R} f_b(\widehat{Z}^R(s))\\ &\qquad - f_b(\widehat{Z}^R(s))\mathscr{L}^R_{\gamma^R} f_a(\widehat{Z}^R(s))\, ds.\end{aligned}$$

Applying the operator $\mathscr{L}^R_{\gamma^R}$, respectively, to the test functions f_a and $g_{a,b}$, we obtain that

$$\mathscr{L}^R_{\gamma^R} f_a(\widehat{Z}^R(t)) = \left\langle a, \left(\frac{id_0}{2} + \frac{3\gamma}{2}\right)\frac{\partial^2 \widehat{X}^R}{\partial x^2} + i\Theta_R(\|\gamma^R(t)\|^2_{\mathbb{H}^1}) F(\widehat{X}^R)\right\rangle$$

and

$$
\begin{aligned}
&\mathscr{L}^R_{\gamma^R} g_{a,b}(\widehat{Z}^R(t)) \\
&\quad = f_b(\widehat{X}^R(t))\Big\langle a, \Big(\frac{id_0}{2}+\frac{3\gamma}{2}\Big)\frac{\partial^2 \widehat{X}^R(t)}{\partial x^2} + i\Theta_R(\gamma^R(t))F(\widehat{X}^R(t))\Big\rangle \\
&\qquad + f_a(\widehat{X}^R(t))\Big\langle b, \Big(\frac{id_0}{2}+\frac{3\gamma}{2}\Big)\frac{\partial^2 \widehat{X}^R(t)}{\partial x^2} + i\Theta_R(\gamma^R(t))F(\widehat{X}^R(t))\Big\rangle \\
&\qquad + \gamma\sum_{k=1}^{3}\Big\langle a, \sigma_k\frac{\partial \widehat{X}^R(t)}{\partial x}\Big\rangle\Big\langle b, \sigma_k\frac{\partial \widehat{X}^R(t)}{\partial x}\Big\rangle.
\end{aligned}
$$

We deduce that the quadratic variation is given by formula (3.5) with $\widetilde{X}$ replaced by $\widehat{X}^R$. Thus, using the martingale representation theorem, we can write the $\mathcal{G}_t$-martingale $\mathbf{N}^R(\widehat{Z}^R(t))$ as the stochastic integral

$$
\langle a, \mathbf{N}^R(\widehat{Z}^R(t))\rangle = \sqrt{\gamma}\int_0^t \sum_{k=1}^{3}\Big\langle a, \sigma_k\frac{\partial \widehat{X}^R(s)}{\partial x}\Big\rangle dW_k(s),
$$

where $W=(W_1,W_2,W_3)$ is a real-valued Brownian motion on a possibly enlarged space $(\Omega,\mathcal{G},\mathcal{G}_t,\mathbb{P})$. We deduce that $(\widehat{X}^R, W)$ is a weak solution in $C([0,T];\mathbb{H}^1_{\mathrm{loc}})\cap C_w([0,T];\mathbb{H}^1)\cap L^\infty_w(0,T;\mathbb{H}^2)$ of the equation

$$
\begin{cases}
id\widehat{X}^R(t) + \Big(\dfrac{d_0}{2}\partial_x^2\widehat{X}^R(t) + \Theta_R(\gamma^R(t))F(\widehat{X}^R(t))\Big)dt \\
\qquad + i\sqrt{\gamma}\displaystyle\sum_{k=1}^{3}\sigma_k\,\partial_x\widehat{X}^R(t)\circ dW_k(t) = 0, \\
X_0 = v \in \mathbb{H}^3.
\end{cases}
\tag{4.29}
$$

The next step consists in proving that almost surely $\gamma^R(t)=\|\widehat{X}^R(t)\|^2_{\mathbb{H}^1}$. Using the Skorokhod representation theorem, we can construct new random variables (that we still denote Z^R_ε, $\widehat{Z}^R$) on a new common probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ with, respectively, $\mathcal{L}(Z^R_\varepsilon)$ and $\mathcal{L}(\widehat{Z}^R)$ as probability measures and with values in $\mathcal{K}$ such that

$$
\lim_{\varepsilon\to 0} Z^R_\varepsilon = \widehat{Z}^R, \qquad \mathbb{P}\text{-a.s. in } \mathcal{K}.
$$

Since $\widehat{X}^R \in L^\infty(0,T;\mathbb{H}^2)$, we deduce using (4.29) that $\widehat{X}^R\in C([0,T];\mathbb{L}^2)$. Hence, applying the Itô formula, it is easy to see, since Θ_R is a real-valued function, that almost surely

$$
\|\widehat{X}^R(t)\|_{\mathbb{L}^2} = \|v\|_{\mathbb{L}^2} = \|X^R_\varepsilon(t)\|_{\mathbb{L}^2} \qquad \forall t\in[0,T], \forall\varepsilon>0.
$$

Thus, we deduce the strong convergence of $X^R_\varepsilon(t)$ to $\widehat{X}^R(t)$ in $\mathbb{L}^2$, a.s. for each $t\in[0,T]$. Since X^R_ε converges to $\widehat{X}^R$ in $L^\infty_w(0,T;\mathbb{H}^2)$, we get using Lemma 4.1

that

$$\|\widehat{X}^R\|_{L^\infty(0,T;\mathbb{H}^2)} \leq \liminf_{\varepsilon\to 0} \|X_\varepsilon^R\|_{L^\infty(0,T;\mathbb{H}^2)} \leq C(R,T), \qquad \mathbb{P}\text{-a.s.}$$

Interpolating $\mathbb{H}^1$ between $\mathbb{L}^2$ and $\mathbb{H}^2$, we conclude that

$$\lim_{\varepsilon\to 0} \|X_\varepsilon^R(t) - \widehat{X}^R(t)\|_{\mathbb{H}^1} = 0 \qquad \forall t \in [0,T], \qquad \mathbb{P}\text{-a.s.,} \tag{4.30}$$

and $\widehat{X}^R \in C([0,T];\mathbb{H}^1)$; it follows that, almost surely for all t in $[0,T]$, $\gamma^R(t) = \|\widehat{X}^R(t)\|_{\mathbb{H}^1}^2$ and $\widehat{X}^R$ is a solution of (3.6). Thus, the limit in law of X_ε^R is unique and is given by the solution X^R of (3.6).

The final step consists in recovering the convergence in law in $C([0,T],\mathbb{H}^1)$. Since Y_ε^R is uniformly bounded in ε in $C^\alpha([0,T],\mathbb{H}^{-1}) \cap C([0,T];\mathbb{H}^2)$ with $0 \leq \alpha < 1/2$, we deduce that it is a.s. uniformly bounded in ε in $C^\beta([0,T],\mathbb{H}^1)$ with $\beta = \alpha/3$. Moreover, using pointwise convergence (4.30), expression (4.22) and uniform bounds (4.1), we get pointwise convergence in $\mathbb{H}^1$ of Y_ε^R to X^R. We conclude that Y_ε^R converges in law to X^R in $C([0,T],\mathbb{H}^1(\mathbb{R}))$ and by Lemma 4.4, the convergence in law of X_ε^R to X^R in $C([0,T],\mathbb{H}^1(\mathbb{R}))$ follows. □

REMARK 4.4. Using the Arzela–Ascoli and Banach–Alaoglu theorems, Lemma 4.5 and the Tychonov theorem, we deduce that $(\mathcal{L}(X_\varepsilon^R))_{R\in\mathbb{N}}$ is tight on $\mathcal{K}^{\mathbb{N}}$. Thus, the same arguments as above lead to the convergence in law of $(X_\varepsilon^R)_{R\in\mathbb{N}}$ to $(X^R)_{R\in\mathbb{N}}$ (see [11]).

4.5. *Convergence of* $(X_\varepsilon)_{\varepsilon>0}$ *to* X. Using the Skorokhod theorem, we can construct new random variables $\widetilde{X}_\varepsilon^R$, $\widetilde{X}^R$ on a common probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_t, \widetilde{\mathbb{P}})$ and with values in $C([0,T],\mathbb{H}^1)$ such that for any $R > 0$,

$$\begin{cases} \widetilde{\mu}_\varepsilon^R = \mu_\varepsilon^R, \\ \widetilde{\mu}^R = \mu^R, \end{cases} \quad \text{and} \quad \widetilde{X}_\varepsilon^R \underset{\varepsilon\to 0}{\to} \widetilde{X}^R, \qquad \widetilde{\mathbb{P}}\text{-a.s. in } C([0,T],\mathbb{H}^1).$$

We define the escape times $\widetilde{\tau}^R$ and $\widetilde{\tau}_\varepsilon^R$ associated to the cutoff:

$$\widetilde{\tau}^R = \inf\{t \in [0,T], \|\widetilde{X}^R(t)\|_{\mathbb{H}^1} > R\}$$

and

$$\widetilde{\tau}_\varepsilon^R = \inf\{t \in [0,T], \|\widetilde{X}_\varepsilon^R(t)\|_{\mathbb{H}^1} > R\}.$$

Let $\widetilde{X}_\varepsilon$ and $\widetilde{X}$ be the processes, with values in $\mathcal{E}(\mathbb{H}^1)$, defined, respectively, by $\widetilde{X}_\varepsilon(t) = \widetilde{X}_\varepsilon^R(t)$ for $t < \widetilde{\tau}_\varepsilon^R$ and $\widetilde{X}(t) = \widetilde{X}^R(t)$ for $t < \widetilde{\tau}^R$, $\widetilde{X}(t) = \Delta$ for $t \geq \tau^* = \lim_{R\to+\infty} \widetilde{\tau}^R$. Then if $\tau < \tau^*$ a.s. is a stopping time, the process $\widetilde{X}_\varepsilon$ converges to $\widetilde{X}$ a.s. in $C([0,\tau],\mathbb{H}^1(\mathbb{R}))$. Hence, the convergence in law in $\mathcal{E}(\mathbb{H}^1)$ follows.

5. Study of the driving process $\boldsymbol{\nu}$. We recall in this appendix some results obtained in [18, 22] about the driving process ν.

PROPOSITION 5.1. *The process $\nu = (\nu_1, \nu_2)^t$ is a Feller process that evolves on the unit sphere $\mathbb{S}^3$ of $\mathbb{C}^2 \sim \mathbb{R}^4$. Furthermore, it admits a unique invariant measure Λ, which is the uniform measure on $\mathbb{S}^3$, under which it is ergodic. For all $f \in C_b^2(\mathbb{S}^3)$ satisfying the Fredholm alternative (or null mass condition) $\mathbb{E}_\Lambda(f(\nu)) = \int_{\mathbb{S}^3} f(y)\Lambda(dy) = 0$, the Poisson equation $\mathscr{L}_\nu u(y) + f(y) = 0$ admits a unique solution of class $C_b^2(\mathbb{S}^3)$, up to a constant, which can be written as $u(y) = \int_0^{+\infty} \mathbb{E}[f(\nu(t))|\nu_0 = y]\,dt$.*

Let us recall that $\boldsymbol{\sigma}(\nu(t)) = \sigma_1 m_1 + \sigma_2 m_2 + \sigma_3 m_3$ where $m_j(t) = g_j(\nu(t))$. We now state a result related to the effect of the random PMD on the pulse evolution.

COROLLARY 5.1. (1) *The process $m = (m_1, m_2, m_3) \in \mathbb{S}^3$ is a Feller process with a unique invariant measure $\Lambda \circ g^{-1}$ under which it is ergodic.*

(2) *For $j = 1, 2, 3$, $\mathbb{E}_\Lambda(g_j(\nu)) = \mathbb{E}_{\Lambda\circ g_j^{-1}}(m) = 0$ and $\mathbb{E}_\Lambda(g_j(\nu(t))g_k(\nu(t))) = \delta_{jk}/3$. As a consequence,*

$$\mathbb{E}_\Lambda(N_{1,\nu}(X)) = \tfrac{2}{3}(2|X_2|^2 - |X_1|^2)X_1,$$
$$\mathbb{E}_\Lambda(N_{2,\nu}(X)) = \tfrac{2}{3}(2|X_1|^2 - |X_2|^2)X_2.$$

(3) *For $j, k = 1, 2, 3$,*

$$\int_0^{+\infty} \mathbb{E}_\Lambda[g_j(\nu(0))g_k(\nu(t))]\,dt = \begin{cases} \dfrac{1}{12\gamma_c}, & \text{if } j = k,\\ 0, & \text{if } j \neq k, \end{cases}$$

where γ_c is the constant appearing in (1.8).

6. Proof of technical lemmas.

PROOF OF LEMMA 4.2. Let v be in $\mathbb{H}^3$. Using the explicit representation (4.9) of f^1, we obtain, since $D_v f(v) \in \mathbb{H}^1(\mathbb{R})$, that

$$\begin{aligned}|f^1(v, y)| &= \left|\left\langle D_v f(v), b'\tilde{\boldsymbol{\sigma}}(y)\frac{\partial v}{\partial x}\right\rangle\right|\\ &\leq b'\|D_v f(v)\|_{\mathbb{H}^1}\left\|\frac{\partial v}{\partial x}\right\|_{\mathbb{H}^{-1}}\sum_{j=1}^{3}\left|\int_0^{+\infty}\mathbb{E}\big(g_j(\nu(t))|\nu(0) = y\big)\,dt\right|.\end{aligned}$$

Moreover, by Proposition 5.1 the integral $\int_0^{+\infty} \mathbb{E}(g_j(\nu(t))|\nu(0) = y)\,dt$ converges because g_j is a bounded function of $\nu \in \mathbb{S}^3$. Since $v \mapsto D_v f(v)$ is a continuous function which is bounded on bounded sets of $\mathbb{H}^{-1}$, we deduce that

$$\sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} |f_\varepsilon^1(v,y)| \le b'C(K).$$

The function f^2 given by (4.15) may be bounded using the same arguments. Indeed,

$$\langle D_v f(v), i\Theta_R(\|v\|_{\mathbb{H}^1}^2)\widetilde{F}(v,y)\rangle \le \|D_v f(v)\|_{\mathbb{H}^1}\|\widetilde{F}(v,y)\|_{\mathbb{H}^{-1}}.$$

Since for all $v \in \mathbb{H}^3$, $y \mapsto F_y(v) - F(v)$ is a function of class C_b^2 on $\mathbb{S}^3$, with values in $\mathbb{H}^{-1}$, satisfying the null mass condition of Proposition 5.1, the term $\widetilde{F}(v,y)$ is bounded. Moreover, $v \mapsto F_y(v) - F(v)$ is bounded in $\mathbb{H}^{-1}$ on bounded sets of $\mathbb{H}^1$ by the continuous embeddings $\mathbb{H}^1(\mathbb{R}) \hookrightarrow \mathbb{L}^4(\mathbb{R})$ and $\mathbb{L}^{4/3}(\mathbb{R}) \hookrightarrow \mathbb{H}^{-1}(\mathbb{R})$. In addition,

$$\begin{aligned}
&\left|(b')^2 \sum_{k,l=1}^3 \left\langle D_v^2 f(v)\sigma_k \frac{\partial v}{\partial x}, \sigma_l \frac{\partial v}{\partial x}\right\rangle \widetilde{\widetilde{g}}_{k,l}(y) + \left\langle D_v f(v), (b')^2 \widetilde{\widetilde{\boldsymbol{\sigma}}}(y)\frac{\partial^2 v}{\partial x^2}\right\rangle\right| \\
&\quad \le C\sum_{k,l=1}^3 \left(\left|\left\langle D_v^2 f(v)\sigma_k \frac{\partial v}{\partial x}, \sigma_l \frac{\partial v}{\partial x}\right\rangle\right| + \left|\left\langle D_v f(v), \sigma_k\sigma_l \frac{\partial^2 v}{\partial x^2}\right\rangle\right|\right) \\
&\quad \le C\left(\|D_v^2 f(v)\|_{\mathscr{L}(\mathbb{H}^{-1},\mathbb{H}^1)} \left\|\frac{\partial v}{\partial x}\right\|_{\mathbb{H}^{-1}}^2 + \|D_v f(v)\|_{\mathbb{H}^1}\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{\mathbb{H}^{-1}}\right).
\end{aligned}$$

Since $v \mapsto D_v f(v)$ and $v \mapsto D_v^2 f(v)$ are bounded on bounded sets of $\mathbb{H}^{-1}(\mathbb{R})$, we conclude the proof of the lemma. $\square$

PROOF OF LEMMA 4.3. Replacing f^1 by its expression (4.9), we get

$$\begin{aligned}
&\left\langle D_v f^1(v,y), \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2)F_y(v)\right\rangle \\
&\quad = -\left\langle D_v^2 f(v) b'\widetilde{\boldsymbol{\sigma}}(y)\frac{\partial v}{\partial x}, \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2)F_y(v)\right\rangle \\
&\qquad - \left\langle D_v f(v), b'\widetilde{\boldsymbol{\sigma}}(y)\frac{id_0}{2}\frac{\partial^3 v}{\partial x^3} + ib'\widetilde{\boldsymbol{\sigma}}(y)\Theta_R(\|v\|_{\mathbb{H}^1}^2)\,\partial_x F_y(v)\right\rangle.
\end{aligned}$$

By the assumptions on f, $v \mapsto D_v f(v)$ and $v \mapsto D_v^2 f(v)$ are continuous bounded functions on bounded sets of $\mathbb{H}^{-1}(\mathbb{R})$. Moreover, $D_v^2 f(v) \in \mathcal{L}(\mathbb{H}^{-1},\mathbb{H}^1)$, $D_v f(v) \in \mathbb{H}^1$ and $\frac{\partial^3 v}{\partial x^3} \in \mathbb{L}^2$. Using the bound (4.11), we deduce that

$$\sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} \left|\left\langle D_v f^1(v,y), \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2)F_y(v)\right\rangle\right| \le C(K).$$

Let us now compute the first derivative of f^2 using expression (4.15); for all h in $\mathbb{H}^1$ and v in $\mathbb{H}^3$,

$$
\begin{aligned}
&\langle D_v f^2(v,y), h\rangle \\
&\quad = \langle D_v^2 f(v)h, i\Theta_R(\|v\|_{\mathbb{H}^1}^2)\widetilde{F}(v,y)\rangle \\
&\qquad + \langle D_v f(v), 2i\Theta_R'(\|v\|_{\mathbb{H}^1}^2)(v,h)_{\mathbb{H}^1}\widetilde{F}(v,y)\rangle \\
&\qquad + \langle D_v f(v), i\Theta_R(\|v\|_{\mathbb{H}^1}^2) D_v\widetilde{F}(v,y).h\rangle \\
&\qquad - (b')^2 \sum_{k,l=1}^{3} D_v^3 f(v).\left(\sigma_k \frac{\partial v}{\partial x}, \sigma_l \frac{\partial v}{\partial x}, h\right)\widetilde{\widetilde{g}}_{k,l}(y) \\
&\qquad - 2(b')^2 \sum_{k,l=1}^{3} \left\langle D_v^2 f(v)\sigma_k \frac{\partial h}{\partial x}, \sigma_l \frac{\partial v}{\partial x}\right\rangle \widetilde{\widetilde{g}}_{k,l}(y) \\
&\qquad - \left\langle D_v^2 f(v)h, (b')^2 \widetilde{\widetilde{\boldsymbol{\sigma}}}(y)\frac{\partial^2 v}{\partial x^2}\right\rangle \\
&\qquad - \left\langle D_v f(v), (b')^2 \widetilde{\widetilde{\boldsymbol{\sigma}}}(y)\frac{\partial^2 h}{\partial x^2}\right\rangle.
\end{aligned}
$$

Taking, respectively, $h = \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2)F_y(v)$ and $h = b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}$, we conclude

$$
\sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} \left|\left\langle D_v f^2(v,y), \frac{id_0}{2}\frac{\partial^2 v}{\partial x^2} + i\Theta_R(\|v\|_{\mathbb{H}^1}^2)F_y(v)\right\rangle\right| \le C(K)
$$

and

$$
\sup_{\substack{v\in\mathcal{B}(K)\\ y\in\mathbb{S}^3}} \left|\left\langle D_v f^2(v,y), b'\boldsymbol{\sigma}(y)\frac{\partial v}{\partial x}\right\rangle\right| \le C(K),
$$

since $v \mapsto D_v^3 f(v)$ is bounded on the bounded set of $\mathbb{H}^{-1}(\mathbb{R})$ with values in $\mathcal{L}_3(\mathbb{H}^{-1}, \mathbb{R})$ and $\frac{\partial^4 v}{\partial x^4} \in \mathbb{H}^{-1}$. $\square$

PROOF OF LEMMA 4.6. Let us recall that the family $\{e_i\}_{i\in\mathbb{N}^*}$ denotes a complete orthonormal system of $\mathbb{H}^1$ constructed from a complete orthonormal system $\{\widetilde{e}_i\}_{i\in\mathbb{N}^*}$ in $\mathbb{L}^2$ and $\langle\cdot,\cdot\rangle$ is the duality product between $\mathbb{H}^1$–$\mathbb{H}^{-1}$. Then

$$
\|Y_\varepsilon^R(t) - Y_\varepsilon^R(s)\|_{\mathbb{H}^{-1}}^4 = \left\{\sum_{i=1}^{+\infty}\langle e_i, Y_\varepsilon^R(t) - Y_\varepsilon^R(s)\rangle^2\right\}^2.
$$

Using twice the Young inequality and the expression of Y_ε^R given by (4.23) and (4.21), we obtain

$$
\begin{aligned}
&\|Y_\varepsilon^R(t) - Y_\varepsilon^R(s)\|_{\mathbb{H}^{-1}}^4 \\
&\quad \le C \left\| \frac{d_0}{2} \int_s^t \partial_x^2 X_\varepsilon^R(t')\,dt' \right\|_{\mathbb{H}^{-1}}^4 \\
&\qquad + C \left\| \int_s^t \Theta_R(\|X_\varepsilon^R(t')\|_{\mathbb{H}^1}^2) F_{\nu_\varepsilon(t')}(X_\varepsilon^R(t'))\,dt' \right\|_{\mathbb{H}^{-1}}^4 \\
&\qquad + C \left\| \int_s^t (b')^2 \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(t'))\boldsymbol{\sigma}(\nu_\varepsilon(t'))\,\partial_x^2 X_\varepsilon^R(t')\,dt' \right\|_{\mathbb{H}^{-1}}^4 \\
&\qquad + C \left[\sum_{i=1}^{+\infty} \langle e_i, M_\varepsilon^R(t) - M_\varepsilon^R(s)\rangle^2 \right]^2 \\
&\qquad + C\varepsilon^4 \left\| \int_s^t b'\widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(t'))\,\partial_x \left(\frac{d_0}{2}\,\partial_x^2 X_\varepsilon^R(t') \right.\right. \\
&\qquad\qquad\qquad \left.\left. + \Theta_R(\|X_\varepsilon(t')\|_{\mathbb{H}^1}^2) F_{\nu_\varepsilon(t')}(X_\varepsilon^R(t')) \right) dt' \right\|_{\mathbb{H}^{-1}}^4.
\end{aligned}
$$

We bound each term separately. Using Lemma 4.1,

$$
\left\| \int_s^t \frac{d_0}{2} \frac{\partial^2 X_\varepsilon^R(t')}{\partial x^2}\,dt' \right\|_{\mathbb{H}^{-1}}^4 \le C(R,T)(t-s)^4.
$$

Using that F is cubic and Lemma 4.1,

$$
\left\| \int_s^t \Theta_R(\|X_\varepsilon(t')\|_{\mathbb{H}^1}^2) F_{\nu_\varepsilon(t')}(X_\varepsilon^R(t'))\,dt' \right\|_{\mathbb{H}^{-1}}^4 \le C(R,T)(t-s)^4
$$

and using Lemma 4.1 and the bound (4.11),

$$
\left\| \int_s^t (b')^2 \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(t'))\boldsymbol{\sigma}(\nu_\varepsilon(t'))\,\frac{\partial^2 X_\varepsilon^R(t')}{\partial x^2}\,dt' \right\|_{\mathbb{H}^{-1}}^4 \le C(R,T)(t-s)^4.
$$

Finally, we bound the ε^4 term that is well defined because X_ε^R has values in $\mathbb{H}^3$. Using the Cauchy–Schwarz inequality, Lemma 4.1 and (4.11), we get for all $\varepsilon < 1$,

$$
\begin{aligned}
&\varepsilon^4 \left\| \int_s^t b'\widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(t'))\,\frac{\partial}{\partial x}\left(\frac{d_0}{2}\,\frac{\partial^2 X_\varepsilon^R(t')}{\partial x^2} + \Theta_R(\|X_\varepsilon\|_{\mathbb{H}^1}^2) F_{\nu_\varepsilon(t')}(X_\varepsilon^R(t')) \right) dt' \right\|_{\mathbb{H}^{-1}}^4 \\
&\quad \le \varepsilon^4 (b')^4 M^4 \left\| \int_s^t \frac{d_0}{2}\,\frac{\partial^3 X_\varepsilon^R(t')}{\partial x^3} + \frac{\partial}{\partial x}\big(\Theta_R(\|X_\varepsilon\|_{\mathbb{H}^1}^2) F_{\nu_\varepsilon(t')}(X_\varepsilon^R(t'))\big)\,dt' \right\|_{\mathbb{H}^{-1}}^4 \\
&\quad \le C(R,T)(t-s)^4.
\end{aligned}
$$

Taking the expectation and adding the previous estimates, we deduce that

$$\begin{aligned}
&\mathbb{E}\big(\|Y_\varepsilon^R(t) - Y_\varepsilon^R(s)\|_{\mathbb{H}^{-1}}^4\big)\\
&\qquad \le C(R,T)(t-s)^4 + C\mathbb{E}\bigg(\bigg[\sum_{j\in\mathbb{N}^*}\langle e_j, M_\varepsilon^R(t) - M_\varepsilon^R(s)\rangle^2\bigg]^2\bigg).
\end{aligned}$$

In order to prove a uniform bound, with respect to ε, of the second term, we will use the Burkholder–Davis–Gundy inequality and, consequently, we have to compute the quadratic variation $\langle\langle M_\varepsilon^R(t)\rangle\rangle$ of $M_\varepsilon^R(t)$ defined, for all $j \in \mathbb{N}^*$, by

$$\langle e_j, M_\varepsilon^R(t)\rangle = f_{j,\varepsilon}(X_\varepsilon^R(t), \nu_\varepsilon(t)) - f_{j,\varepsilon}(v, y) - \int_0^t \mathscr{L}_\varepsilon^R f_{j,\varepsilon}(X_\varepsilon^R(s), \nu_\varepsilon(s))\, ds,$$

where $\mathscr{L}_\varepsilon^R f_{j,\varepsilon}(X_\varepsilon^R(s), \nu_\varepsilon(s))$ is given by (4.21). The next lemma states that the process $\langle\langle M_\varepsilon^R(t)\rangle\rangle$ can be expressed only in terms of the infinitesimal generator $\mathscr{L}_\nu$ of the Markov process ν.

LEMMA 6.1. *For all j in $\mathbb{N}^*$*

$$\begin{aligned}
&\langle e_j, \langle\langle M_\varepsilon^R(t)\rangle\rangle e_j\rangle\\
&\quad = (b')^2 \int_0^t \mathscr{L}_\nu\bigg(\bigg\langle e_j, \tilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\, \frac{\partial X_\varepsilon^R}{\partial x}(s)\bigg\rangle^2\bigg)\, ds\\
&\qquad - 2(b')^2 \int_0^t \bigg\langle e_j, \tilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\, \frac{\partial X_\varepsilon^R}{\partial x}(s)\bigg\rangle\bigg\langle e_j, \mathscr{L}_\nu\tilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\, \frac{\partial X_\varepsilon^R}{\partial x}(s)\bigg\rangle\, ds.
\end{aligned}$$

Thus, using the Burkholder–Davis–Gundy inequality,

$$\mathbb{E}\Bigg(\bigg(\sum_{j=1}^{+\infty}\langle e_j, M_\varepsilon^R(t) - M_\varepsilon^R(s)\rangle^2\bigg)^2\Bigg) \le C(R,T)|t-s|^2,$$

thanks to Lemma 4.1 and Proposition 5.1. Adding the previous estimates,

$$\mathbb{E}\big(\|Y_\varepsilon^R(t) - Y_\varepsilon^R(s)\|_{\mathbb{H}^{-1}}^4\big) \le C(R,T)|t-s|^2,$$

and Lemma 4.6 is proved. $\square$

PROOF OF LEMMA 6.1. A classical computation shows that for all $j \in \mathbb{N}^*$,

$$\begin{aligned}
\langle e_j, \langle\langle M_\varepsilon^R(t)\rangle\rangle e_j\rangle &= \int_0^t \mathscr{L}_\varepsilon^R\big(f_{j,\varepsilon}(X_\varepsilon^R(s), \nu_\varepsilon(s))^2\big)\, ds\\
&\qquad - 2\int_0^t f_{j,\varepsilon}(X_\varepsilon^R(s), \nu_\varepsilon(s))\mathscr{L}_\varepsilon^R f_{j,\varepsilon}(X_\varepsilon^R(s), \nu_\varepsilon(s))\, ds.
\end{aligned}$$

Now, for all j in $\mathbb{N}^*$,

$$(f_{j,\varepsilon}(X_\varepsilon^R(s),\nu_\varepsilon(s)))^2 = \langle e_j, X_\varepsilon^R(s)\rangle^2 - 2b'\varepsilon\langle e_j, X_\varepsilon^R(s)\rangle\left\langle e_j, \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\,\frac{\partial X_\varepsilon^R}{\partial x}(s)\right\rangle + (b')^2\varepsilon^2\left\langle e_j, \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\,\frac{\partial X_\varepsilon^R}{\partial x}(s)\right\rangle^2.$$

Thus, we get

$$\begin{aligned}
&\mathscr{L}_\varepsilon^R(f_{j,\varepsilon}(X_\varepsilon^R(s),\nu_\varepsilon(s)))^2\\
&\quad= 2\langle e_j, X_\varepsilon^R(s)\rangle\left\langle e_j, \frac{id_0}{2}\,\frac{\partial^2 X_\varepsilon^R(s)}{\partial x^2} + i\Theta_R(\|X_\varepsilon^R(s)\|_{\mathbb{H}^1}^2)F_{\nu_\varepsilon(s)}(X_\varepsilon^R(s))\right\rangle\\
&\qquad - 2b'\varepsilon\left\langle e_j, \frac{id_0}{2}\,\frac{\partial^2 X_\varepsilon^R(s)}{\partial x^2} + i\Theta_R(\|X_\varepsilon^R(s)\|_{\mathbb{H}^1}^2)F_{\nu_\varepsilon(s)}(X_\varepsilon^R(s))\right\rangle\\
&\qquad\quad \times\left\langle e_j, \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\,\frac{\partial X_\varepsilon^R}{\partial x}(s)\right\rangle\\
&\qquad - 2b'\varepsilon\langle e_j, X_\varepsilon^R(s)\rangle\\
&\qquad\quad \times\left\langle e_j, \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\,\frac{\partial}{\partial x}\left(\frac{id_0}{2}\,\frac{\partial^2 X_\varepsilon^R(s)}{\partial x^2} + i\Theta_R(\|X_\varepsilon^R(s)\|_{\mathbb{H}^1}^2)F_{\nu_\varepsilon(s)}(X_\varepsilon^R(s))\right)\right\rangle\\
&\qquad - 2(b')^2\left\langle e_j, \mathscr{L}_\nu\widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\,\frac{\partial X_\varepsilon^R}{\partial x}(s)\right\rangle\left\langle e_j, \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\,\frac{\partial X_\varepsilon^R}{\partial x}(s)\right\rangle\\
&\qquad + (b')^2\mathscr{L}_\nu\left(\left\langle e_j, \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\,\frac{\partial X_\varepsilon^R}{\partial x}(s)\right\rangle^2\right)\\
&\qquad + 2b'\left\langle e_j, \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\,\frac{\partial}{\partial x}\left(b'\boldsymbol{\sigma}(\nu_\varepsilon(s))\,\frac{\partial X_\varepsilon^R(s)}{\partial x}\right)\right\rangle\langle e_j, X_\varepsilon^R(s)\rangle\\
&\qquad + 2(b')^2\varepsilon^2\left\langle e_j, \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\,\frac{\partial X_\varepsilon^R}{\partial x}(s)\right\rangle\left\langle e_j, \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\frac{id_0}{2}\,\frac{\partial^3 X_\varepsilon^R(s)}{\partial x^3}\right\rangle\\
&\qquad + 2(b')^2\varepsilon^2\left\langle e_j, \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\,\frac{\partial X_\varepsilon^R}{\partial x}(s)\right\rangle\\
&\qquad\quad \times\left\langle e_j, i\widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\Theta_R(\|X_\varepsilon^R(s)\|_{\mathbb{H}^1}^2)\,\frac{\partial}{\partial x}F_{\nu_\varepsilon(s)}(X_\varepsilon^R(s))\right\rangle\\
&\qquad - 2(b')^2\varepsilon\left\langle e_j, \widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\,\frac{\partial X_\varepsilon^R}{\partial x}(s)\right\rangle\left\langle e_j, b'\widetilde{\boldsymbol{\sigma}}(\nu_\varepsilon(s))\boldsymbol{\sigma}(\nu_\varepsilon(s))\,\frac{\partial^2 X_\varepsilon^R(s)}{\partial x^2}\right\rangle.
\end{aligned}$$

The same kinds of computations for the term $2f_{j,\varepsilon}\mathscr{L}_\varepsilon^R f_{j,\varepsilon}$ lead to the result. $\square$

Acknowledgments. The authors wish to thank the referees for fruitful comments.

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CMAP, CNRS UMR 7641
ÉCOLE POLYTECHNIQUE
91128 PALAISEAU CEDEX
FRANCE
E-MAIL: debouard@cmap.polytechnique.fr
gazeau@cmap.polytechnique.fr