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A SCALING ANALYSIS OF A CAT AND MOUSE MARKOV CHAIN

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If (C_n) is a Markov chain on a discrete state space $\mathcal{S}$, a Markov chain (C_n, M_n) on the product space $\mathcal{S} \times \mathcal{S}$, the cat and mouse Markov chain, is constructed. The first coordinate of this Markov chain behaves like the original Markov chain and the second component changes only when both coordinates are equal. The asymptotic properties of this Markov chain are investigated. A representation of its invariant measure is, in particular, obtained. When the state space is infinite it is shown that this Markov chain is in fact null recurrent if the initial Markov chain (C_n) is positive recurrent and reversible. In this context, the scaling properties of the location of the second component, the mouse, are investigated in various situations: simple random walks in $\mathbb{Z}$ and $\mathbb{Z}^2$ reflected a simple random walk in $\mathbb{N}$ and also in a continuous time setting. For several of these processes, a time scaling with rapid growth gives an interesting asymptotic behavior related to limiting results for occupation times and rare events of Markov processes.

1. Introduction. The PageRank algorithm of Google, as designed by Brin and Page [10] in 1998, describes the web as an oriented graph $\mathcal{S}$ whose nodes are the web pages and the html links between these web pages, the links of the graph. In this representation, the importance of a page is defined as its weight for the stationary distribution of the associated random walk on the graph. Several off-line algorithms can be used to estimate this equilibrium distribution on such a huge state space, they basically use numerical procedures (matrix-vector multiplications). See Berkhin [4], for example. Several on-line algorithms that update the ranking scores while exploring the graph have been recently proposed to avoid some of the shortcomings of off-line algorithms, in particular, in terms of computational complexity.

The starting point of this paper is an algorithm designed by Abiteboul et al. [1] to compute the stationary distribution of a finite recurrent Markov chain. In this setting, to each node of the graph is associated a number, the "cash" of the node. The algorithm works as follows: at a given time, the node x with the largest value V_x of cash is visited, V_x is set to 0 and the value of the cash of each of its d_x neighbors is incremented by V_x/d_x. Another possible strategy to update cash

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variables is as follows: a random walker updates the values of the cash at the nodes of its random path in the graph. This policy is referred to as the Markovian variant. Both strategies have the advantage of simplifying the data structures necessary to manage the algorithm. It turns out that the asymptotic distribution, in terms of the number of steps of the algorithm, of the vector of the cash variables gives an accurate estimation of the equilibrium distribution; see Abiteboul et al. [1] for the complete description of the procedure to get the invariant distribution. See also Litvak and Robert [23]. The present paper does not address the problem of estimating the accuracy of these algorithms, it analyzes the asymptotic properties of a simple Markov chain which appears naturally in this context.

Cat and mouse Markov chain. It has been shown in Litvak and Robert [23] that, for the Markovian variant of the algorithm, the distribution of the vector of the cash variables can be represented with the conditional distributions of a Markov chain (C_n, M_n) on the discrete state space $\mathcal{S}\times\mathcal{S}$. The sequence (C_n), representing the location of the cat, is a Markov chain with transition matrix $P = (p(x, y))$ associated to the random walk on the graph $\mathcal{S}$. The second coordinate, the location of the mouse, (M_n) has the following dynamic:

– If $M_n \neq C_n$, then $M_{n+1} = M_n$,
– If $M_n = C_n$, then, conditionally on M_n, the random variable M_{n+1} has distribution $(p(M_n, y), y \in \mathcal{S})$ and is independent of C_{n+1}.

This can be summarized as follows: the cat moves according to the transition matrix $P = (p(x, y))$ and the mouse stays idle unless the cat is at the same site, in which case the mouse also moves independently according to $P = (p(x, y))$.

The terminology "cat and mouse problem" is also used in a somewhat different way in game theory, the cat playing the role of the "adversary." See Coppersmith et al. [11] and references therein.

The asymptotic properties of this interesting Markov chain (C_n, M_n) for a number of transition matrices P are the subject of this paper. In particular, the asymptotic behavior of the location mouse (M_n) is investigated. The distribution of (M_n) plays an important role in the algorithm designed by Abiteboul et al. [1]; see Litvak and Robert [23] for further details. It should be noted that (M_n) is not, in general, a Markov chain.

Outline of the paper. Section 2 analyzes the recurrence properties of the Markov chain (C_n, M_n) when the Markov chain (C_n) is recurrent. A representation of the invariant measure of (C_n, M_n) in terms of the reversed process of (C_n) is given.

Since the mouse moves only when the cat arrives at its location, it may seem quite likely that the mouse will spend most of the time at nodes which are unlikely for the cat. It is shown that this is indeed the case when the state space is finite and if the Markov chain (C_n) is reversible but not in general.

When the state space is infinite and if the Markov chain (C_n) is reversible, it turns out that the Markov chain (C_n, M_n) is in fact null recurrent. A precise description of the asymptotic behavior of the sequence (M_n) is done via a scaling in time and space for several classes of simple models. Interestingly, the scalings used are quite diverse, as it will be seen. They are either related to asymptotics of rare events of ergodic Markov chains or to limiting results for occupation times of recurrent random walks:

(1) *Symmetric simple random walks*. The cases of symmetric simple random walks in $\mathbb{Z}^d$ with $d = 1$ and 2 are analyzed in Section 3. Note that for $d \geq 3$ the Markov chain (C_n) is transient so that in this case the location of the mouse does not change with probability 1 after some random time:
   - In the one-dimensional case, $d = 1$, if $M_0 = C_0 = 0$, on the linear time scale $t \to nt$, as n gets large, it is shown that the location of the mouse is of the order of $\sqrt[4]{n}$. More precisely, the limit in distribution of the process $(M_{\lfloor nt \rfloor}/\sqrt[4]{n}, t \geq 0)$ is a Brownian motion $(B_1(t))$ taken at the local time at 0 of another independent Brownian motion $(B_2(t))$. See Theorem 2 below.

     This result can be (roughly) described as follows. Under this linear time scale the location of the cat, a simple symmetrical random walk, is of the order of $\sqrt{n}$ by Donsker's theorem. It turns out that it will encounter $\sim\sqrt{n}$ times the mouse. Since the mouse moves only when it encounters the cat and that it also follows the sample path of a simple random walk, after $\sqrt{n}$ steps its order of magnitude will be therefore of the order of $\sqrt[4]{n}$.
   - When $d = 2$, on the linear time scale $t \to nt$, the location of the mouse is of the order of $\sqrt{\log n}$. More precisely, the finite marginals of the rescaled processes $(M_{\lfloor \exp(nt) \rfloor}/\sqrt{n}, t \geq 0)$ converge to the corresponding finite marginals of a Brownian motion in $\mathbb{R}^2$ on a time scale which is an independent *discontinuous* stochastic process with independent and nonhomogeneous increments.

(2) *Reflected simple random walk*. Section 4 investigates the reflected simple random walk on the integers. A jump of size $+1$ (resp., -1) occurs with probability p [resp., $(1 - p)$] and the quantity $\rho = p/(1 - p)$ is assumed to be strictly less than 1 so that the Markov chain (C_n) is ergodic.

   If the location of the mouse is far away from the origin, that is, $M_0 = n$ with n large and the cat is at equilibrium, a standard result shows that it takes a duration of time of the order of ρ^{-n} for the cat to hit the mouse. This suggests an exponential time scale $t \to \rho^{-n}t$ to study the evolution of the successive locations of the mouse. For this time scale it is shown that the location of the mouse is still of the order of n as long as $t < W$ where W is some nonintegrable random variable. At time $t = W$ on the exponential time scale, the mouse has hit 0 and after that time the process $(M_{\lfloor t\rho^{-n} \rfloor}/n)$ oscillates between 0 and above $1/2$ on every nonempty time interval.

(3) *Continuous time random walks*. Section 5 introduces the cat and mouse process for continuous time Markov processes. In particular, a discrete Ornstein–Uhlenbeck process, the $M/M/\infty$ queue, is analyzed. This is a birth and death process whose birth rates are constant and the death rate at $n \in \mathbb{N}$ is proportional to n. When $M_0 = n$, contrary to the case of the reflected random walk, there does not seem to exist a time scale for which a nontrivial functional theorem holds for the corresponding rescaled process. Instead, it is possible to describe the asymptotic behavior of the location of the mouse after the pth visit of the cat. It has a multiplicative representation of the form $nF_1F_2 \cdots F_p$ where (F_p) are i.i.d. random variables on [0, 1].

The examples analyzed are quite specific. They are, however, sufficiently representative of the different situations for the dynamic of the mouse:

(1) One considers the case when an integer valued Markov chain (C_n) is ergodic and the initial location of the mouse is far away from 0. The correct time scale to investigate the evolution of the location of the mouse is given by the duration of time for the occurrence of a rare event for the original Markov chain. When the cat hits the mouse at this level, before returning to the neighborhood of 0, it changes the location of the mouse by an additive (resp., multiplicative) step in the case of the reflected random walk (resp., $M/M/\infty$ queue).

(2) For null recurrent homogeneous random walks, the distribution of the duration of times between two visits of the cat to the mouse do not depend on the location of the mouse but it is nonintegrable. The main problem is therefore to get a functional renewal theorem associated to an i.i.d. sequence (T_n) of nonnegative random variables such that $\mathbb{E}(T_1) = +\infty$. More precisely, if

$$N(t) = \sum_{i \geq 1} \mathbb{1}_{\{T_1 + \cdots + T_i \leq t\}},$$

one has to find $\phi(n)$ such that the sequence of processes $(N(nt)/\phi(n), t \geq 0)$ converges as n goes to infinity. When the tail distribution of T_1 has a polynomial decay, several technical results are available. See Garsia and Lamperti [12], for example. This assumption is nevertheless not valid for the two-dimensional case. In any case, it turns out that the best way (especially for $d = 2$) to get such results is to formulate the problem in terms of occupation times of Markov processes for which several limit theorems are available. This is the key of the results in Section 3.

The fact that for all the examples considered jumps occur on the nearest neighbors does not change this qualitative behavior. Under more general conditions analogous results should hold. Additionally, this simple setting has the advantage of providing explicit expressions for most of the constants involved.

2. The cat and mouse Markov chain. In this section we consider a general transition matrix $P = (p(x, y), x, y \in \mathcal{S})$ on a discrete state space $\mathcal{S}$. Throughout the paper, it is assumed that P is aperiodic, irreducible without loops, that is, $p(x, x) = 0$ for all $x \in \mathcal{S}$ and with an invariant measure π. Note that it is not assumed that π has a finite mass. The sequence (C_n) will denote a Markov chain with transition matrix $P = (p(x, y))$. It will represent the sequence of nodes which are sequentially updated by the random walker.

The transition matrix of the reversed Markov chain (C_n^*) is denoted by

$$p^*(x, y) = \frac{\pi(y)}{\pi(x)} p(y, x)$$

and, for $y \in \mathcal{S}$, one defines

$$H_y^* = \inf\{n > 0 : C_n^* = y\} \quad \text{and} \quad H_y = \inf\{n > 0 : C_n = y\}.$$

The Markov chain (C_n, M_n) on $\mathcal{S} \times \mathcal{S}$ referred to as the "cat and mouse Markov chain" is introduced. Its transition matrix $Q = (q(\cdot, \cdot))$ is defined as follows: for x, $y, z \in \mathcal{S}$,

$$\begin{cases} q[(x, y), (z, y)] = p(x, z), & \text{if } x \neq y; \\ q[(y, y), (z, w)] = p(y, z)p(y, w). \end{cases} \tag{1}$$

The process (C_n) [resp., (M_n)] will be defined as the position of the cat (resp., the mouse). Note that the position (C_n) of the cat is indeed a Markov chain with transition matrix $P = (p(\cdot, \cdot))$. The position of the mouse (M_n) changes only when the cat is at the same position. In this case, starting from $x \in \mathcal{S}$ they both move independently according to the stochastic vector $(p(x, \cdot))$.

Since the transition matrix of (C_n) is assumed to be irreducible and aperiodic, it is not difficult to check that the Markov chain (C_n, M_n) is aperiodic and visits with probability 1 all the elements of the diagonal of $\mathcal{S} \times \mathcal{S}$. In particular, there is only one irreducible component. Note that (C_n, M_n) itself is not necessarily irreducible on $\mathcal{S} \times \mathcal{S}$, as the following example shows: take $\mathcal{S} = \{0, 1, 2, 3\}$ and the transition matrix $p(0, 1) = p(2, 3) = p(3, 1) = 1$ and $p(1, 2) = 1/2 = p(1, 0)$; in this case the element $(0, 3)$ cannot be reached starting from $(1, 1)$.

THEOREM 1 (Recurrence). *The Markov chain (C_n, M_n) on $\mathcal{S} \times \mathcal{S}$ with transition matrix Q defined by relation* (1) *is recurrent*: *the measure ν defined as*

$$\nu(x, y) = \pi(x) \mathbb{E}_x \left(\sum_{n=1}^{H_y^*} p(C_n^*, y) \right), \qquad x, y \in \mathcal{S}, \tag{2}$$

is invariant. Its marginal on the second coordinate is given by, for $y \in \mathcal{S}$,

$$\nu_2(y) \stackrel{\text{def.}}{=} \sum_{x \in \mathcal{S}} \nu(x, y) = \mathbb{E}_\pi(p(C_0, y) H_y),$$

and it is equal to π on the diagonal, $\nu(x, x) = \pi(x)$ *for $x \in \mathcal{S}$.*

In particular, with probability 1, the elements of $\mathcal{S} \times \mathcal{S}$ for which ν is nonzero are visited infinitely often and ν is, up to a multiplicative coefficient, the unique invariant measure. The recurrence property is not surprising: the positive recurrence property of the Markov chain (C_n) shows that cat and mouse meet infinitely often with probability one. The common location at these instants is a Markov chain with transition matrix P and therefore recurrent. Note that the total mass of ν,

$$\nu(\mathcal{S} \times \mathcal{S}) = \sum_{y \in \mathcal{S}} \mathbb{E}_\pi(p(C_0, y) H_y)$$

can be infinite when $\mathcal{S}$ is countable. See Kemeny et al. [20] for an introduction on recurrence properties of discrete countable Markov chains.

The measure ν_2 on $\mathcal{S}$ is related to the location of the mouse under the invariant measure ν.

PROOF OF THEOREM 1. From the ergodicity of (C_n) it is clear that $\nu(x, y)$ is finite for $x, y \in \mathcal{S}$. One has first to check that ν satisfies the equations of invariant measure for the Markov chain (C_n, M_n),

$$\nu(x, y) = \sum_{z \neq y} \nu(z, y) p(z, x) + \sum_{z} \nu(z, z) p(z, x) p(z, y), \qquad x, y \in \mathcal{S}. \tag{3}$$

For $x, y \in \mathcal{S}$,

$$\begin{aligned} \sum_{z \neq y} \nu(z, y) p(z, x) &= \sum_{z \neq y} \pi(x) p^*(x, z) \mathbb{E}_z\left(\sum_{n=1}^{H_y^*} p(C_n^*, y)\right) \\ &= \pi(x) \mathbb{E}_x\left(\sum_{n=2}^{H_y^*} p(C_n^*, y)\right) \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\sum_{z \in \mathcal{S}} \nu(z, z) p(z, x) p(z, y) \\ &\qquad = \sum_{z \in \mathcal{S}} \pi(x) p^*(x, z) p(z, y) \mathbb{E}_z\left(\sum_{n=0}^{H_z^*-1} p(C_n^*, z)\right). \end{aligned} \tag{5}$$

The classical renewal argument for the invariant distribution π of the Markov chain (C_n^*), and any bounded function f on $\mathcal{S}$, gives that

$$\mathbb{E}_\pi(f) = \frac{1}{\mathbb{E}_z(H_z^*)} \mathbb{E}_z\left(\sum_{n=0}^{H_z^*-1} f(C_n^*)\right);$$

see Theorem 3.2, page 12, of Asmussen [3], for example. In particular, we have $\pi(z) = 1/\mathbb{E}_z(H_z^*)$, and

$$\begin{aligned}\mathbb{E}_z\left(\sum_{n=0}^{H_z^*-1} p(C_n^*, z)\right) &= \mathbb{E}_z(H_z^*)\mathbb{E}_\pi(p(C_0^*, z)) = \frac{\sum_{x\in S}\pi(x)p(x,z)}{\pi(z)} \\ &= \frac{\pi(z)}{\pi(z)} = 1.\end{aligned} \tag{6}$$

Substituting the last identity into (5), we obtain

$$\begin{aligned}\sum_{z\in\mathcal{S}} \nu(z,z)p(z,x)p(z,y) &= \sum_{z\in\mathcal{S}} \pi(x)p^*(x,z)p(z,y) \\ &= \pi(x)\mathbb{E}_x(p(C_1^*, y)).\end{aligned} \tag{7}$$

Relations (3)–(5) and (7) show that ν is indeed an invariant distribution. At the same time, from (6) one gets the identity $\nu(x,x) = \pi(x)$ for $x \in \mathcal{S}$.

The second marginal is given by, for $y \in \mathcal{S}$,

$$\begin{aligned}\sum_{x\in\mathcal{S}} \nu(x,y) &= \sum_{t\geq 1}\sum_{x\in\mathcal{S}} \pi(x)\mathbb{E}_x\big(p(C_t^*, y)\mathbb{1}_{\{H_y^*\geq t\}}\big) \\ &= \sum_{t\geq 1} \mathbb{E}_\pi\big(p(C_t^*, y)\mathbb{1}_{\{H_y^*\geq t\}}\big) \\ &= \sum_{x\in S}\sum_{z_1,\dots,z_{t-1}\neq y}\sum_{z_t\in S} \pi(x)p^*(x,z_1)p^*(z_1,z_2)\cdots p^*(z_{t-1},z_t)p(z_t,y) \\ &= \sum_{x\in S}\sum_{z_1,\dots,z_{t-1}\neq y}\sum_{z_t\in S} p(z_1,x)p(z_2,z_1)\cdots p(z_t,z_{t-1})\pi(z_t)p(z_t,y) \\ &= \sum_{t\geq 1} \mathbb{E}_\pi\big(p(C_0, y)\mathbb{1}_{\{H_y\geq t\}}\big) = \mathbb{E}_\pi(p(C_0,y)H_y),\end{aligned}$$

and the theorem is proved. □

The representation (2) of the invariant measure can be obtained (formally) through an iteration of the equilibrium equations (3). Since the first coordinate of (C_n, M_n) is a Markov chain with transition matrix P and ν is the invariant measure for (C_n, M_n), the first marginal of ν is thus equal to $\alpha\pi$ for some $\alpha > 0$, that is,

$$\sum_y \nu(x,y) = \alpha\pi(x), \qquad x \in \mathcal{S}.$$

The constant α is in fact the total mass of ν. In particular, from (2), one gets that the quantity

$$h(x) \stackrel{\text{def.}}{=} \sum_{y\in\mathcal{S}} \mathbb{E}_x\left(\sum_{n=1}^{H_y^*} p(C_n^*, y)\right), \qquad x \in \mathcal{S},$$

is independent of $x \in \mathcal{S}$ and equal to α. Note that the parameter α can be infinite.

PROPOSITION 1 (Location of the mouse in the reversible case). *If (C_n) is a reversible Markov chain, with the definitions of the above theorem, for $y \in \mathcal{S}$, the relation*

$$\nu_2(y) = 1 - \pi(y)$$

holds. If the state space $\mathcal{S}$ is countable, the Markov chain (C_n, M_n) is then null recurrent.

PROOF. For $y \in \mathcal{S}$, by reversibility,

$$\begin{aligned}\nu_2(y) &= \mathbb{E}_\pi(p(C_0, y)H_y) = \sum_x \pi(x)p(x, y)\mathbb{E}_x(H_y) \\ &= \sum_x \pi(y)p(y, x)\mathbb{E}_x(H_y) = \pi(y)\mathbb{E}_y(H_y - 1) \\ &= 1 - \pi(y).\end{aligned}$$

The proposition is proved. □

COROLLARY 1 (Finite state space). *If the state space $\mathcal{S}$ is finite with cardinality N, then (C_n, M_n) converges in distribution to (C_∞, M_∞) such that*

$$\mathbb{P}(C_\infty = x, M_\infty = y) = \alpha^{-1}\pi(x)\mathbb{E}_x\left(\sum_{n=1}^{H_y^*} p(C_n^*, y)\right), \qquad x, y \in \mathcal{S}, \tag{8}$$

with

$$\alpha = \sum_{y \in \mathcal{S}} \mathbb{E}_\pi(p(C_0, y)H_y)$$

in particular, $\mathbb{P}(C_\infty = M_\infty = x) = \alpha^{-1}\pi(x)$. If the Markov chain (C_n) is reversible, then

$$\mathbb{P}(M_\infty = y) = \frac{1 - \pi(y)}{N - 1}.$$

Tetali [29] showed, via linear algebra, that if (C_n) is a general recurrent Markov chain, then

$$\sum_{y \in \mathcal{S}} \mathbb{E}_\pi(p(C_0, y)H_y) \le N - 1. \tag{9}$$

See also Aldous and Fill [2]. It follows that the value $\alpha = N - 1$ obtained for reversible chains is the maximal possible value of α. The constant α^{-1} is the probability that the cat and mouse are at the same location.

In the reversible case, Corollary 1 implies the intuitive fact that the less likely a site is for the cat, the more likely it is for the mouse. This is, however, false in general. Consider a Markov chain whose state space $\mathcal{S}$ consists of r cycles with respective sizes $m_1, \dots, m_r$ with one common node 0,

$$\mathcal{S} = \{0\} \cup \bigcup_{k=1}^{r} \{(k,i) : 1 \le i \le m_k\},$$

and with the following transitions: for $1 \le k \le r$ and $2 \le i \le m_k$,

$$p\big((k,i),(k,i-1)\big) = 1, \qquad p((k,1),0) = 1 \quad \text{and} \quad p(0,(k,m_k)) = \frac{1}{r}.$$

Define $m = m_1 + m_2 + \cdots + m_r$. It is easy to see that

$$\pi(0) = \frac{r}{m+r} \quad \text{and} \quad \pi(y) = \frac{1}{m+r}, \qquad y \in \mathcal{S} - \{0\}.$$

One gets that for the location of the mouse, for $y \in \mathcal{S}$,

$$\nu_2(y) = E_\pi(p(C_0, y)H_y) = \begin{cases} \pi(y)(m - m_k + 1), & \text{if } y = (k, m_k), 1 \le k \le r, \\ \pi(y), & \text{otherwise.} \end{cases}$$

Observe that for any y distinct from 0 and (k, m_k), we have $\pi(0) > \pi(y)$ and $\nu_2(0) > \nu_2(y)$; the probability to find a mouse in 0 is larger than in y. Note that in this example one easily obtains $c = 1/r$.

3. Random walks in $\mathbb{Z}$ and $\mathbb{Z}^2$.

In this section the asymptotic behavior of the mouse when the cat follows a recurrent random walk in $\mathbb{Z}$ and $\mathbb{Z}^2$ is analyzed. The jumps of the cat are uniformly distributed on the neighbors of the current location.

3.1. *One-dimensional random walk.*

The transition matrix P of this random walk is given by

$$p(x, x+1) = \tfrac{1}{2} = p(x, x-1), \qquad x \in \mathbb{Z}.$$

Decomposition into cycles. If the cat and the mouse start at the same location, they stay together a random duration of time G which is geometrically distributed with parameter $1/2$. Once they are at different locations for the first time, they are at distance 2 so that the duration of time T_2 until they meet again has the same distribution as the hitting time of 0 by the random walk which starts at 2. The process

$$\big((C_n, M_n), 0 \le n \le G + T_2\big)$$

is defined as a *cycle*. The sample path of the Markov chain (C_n, M_n) can thus be decomposed into a sequence of cycles. It should be noted that, during a cycle, the mouse moves only during the period with duration G.

Since one investigates the asymptotic properties of the sample paths of (M_n) on the linear time scale $t \to nt$ for n large, to get limit theorems one should thus estimate the number of cycles that occur in a time interval $[0, nt]$. For this purpose, we compare the cycles of the cat and mouse process to the cycles of a simple symmetric random walk, which are the time intervals between two successive visits to zero by the process (C_n). Observe that a cycle of (C_n) is equal to $1 + T_1$, where T_1 is the time needed to reach zero starting from 1. Further, T_2 is the sum of two independent random variables distributed as T_1. Hence, one guesses that on the linear time scale $t \to nt$ the number of cycles on $[0, nt]$ for (C_n, M_n) is asymptotically equivalent to 1/2 of the number of cycles on $[0, nt]$ for (C_n), as $n \to \infty$. It is well known that the latter number is of the order $\sqrt{n}$. Then the mouse makes order of $\sqrt{n}$ steps of a simple symmetric random walk, and thus its location must be of the order $\sqrt[4]{n}$.

To make this argument precise, we first prove technical Lemma 1, which says that only $o(\sqrt{n})$ of (C_n)-cycles can be fitted into the time interval of the order $\sqrt{n}$. Next, Lemma 2 proves that the number of cycles of length $T_2 + 2$ on $[0, nt]$, scaled by $\sqrt{n}$, converges to 1/2 of the local time of a Brownian motion, analogously to the corresponding result for the number of cycles of a simple symmetric random walk [22]. Finally, the main limiting result for the location of the mouse is given by Theorem 2.

LEMMA 1. *For any* $x, \varepsilon > 0$ *and* $K > 0$,

$$\lim_{n\to+\infty} \mathbb{P}\left(\inf_{0\le k\le\lfloor x\sqrt{n}\rfloor} \frac{1}{\sqrt{n}} \sum_{i=k}^{k+\lfloor\varepsilon\sqrt{n}\rfloor} (1 + T_{1,i}) \le K\right) = 0,$$

where $(T_{1,i})$ *are i.i.d. random variables with the same distribution as the first hitting time of* 0 *of* (C_n), $T_1 = \inf\{n > 0 : C_n = 0$ *with* $C_0 = 1\}$.

PROOF. If E is an exponential random variable with parameter 1 independent of the sequence $(T_{1,i})$, by using the fact that, for $u \in (0, 1)$, $\mathbb{E}(u^{T_1}) = (1 - \sqrt{1 - u^2})/u$, then for $n \ge 2$,

$$\log \mathbb{P}\left(\frac{1}{\sqrt{n}} \sum_{i=0}^{\lfloor\varepsilon\sqrt{n}\rfloor} (1 + T_{1,i}) \le E\right) = \lfloor\varepsilon\sqrt{n}\rfloor \log\bigl(1 - \sqrt{1 - e^{-2/\sqrt{n}}}\bigr) \le -\varepsilon\sqrt[4]{n}.$$

Denote by

$$m_n = \inf_{0\le k\le\lfloor x\sqrt{n}\rfloor} \frac{1}{\sqrt{n}} \sum_{i=k}^{k+\lfloor\varepsilon\sqrt{n}\rfloor} (1 + T_{1,i})$$

the above relation gives

$$\mathbb{P}(m_n \le E) \le \sum_{k=0}^{\lfloor x\sqrt{n}\rfloor} \mathbb{P}\left(\frac{1}{\sqrt{n}} \sum_{i=k}^{k+\lfloor\varepsilon\sqrt{n}\rfloor} (1 + T_{1,i}) \le E\right) \le \bigl(\lfloor x\sqrt{n}\rfloor + 1\bigr)e^{-\varepsilon\sqrt[4]{n}},$$

hence,

$$\sum_{n=2}^{+\infty} \mathbb{P}(m_n \leq E) < +\infty$$

and, consequently, with probability 1, there exists N_0 such that, for any $n \geq N_0$, we have $m_n > E$. Since $\mathbb{P}(E \geq K) > 0$, the lemma is proved. □

LEMMA 2. *Let, for $n \geq 1$, $(T_{2,i})$ i.i.d. random variables with the same distribution as $T_2 = \inf\{k > 0 : C_k = 0$ with $C_0 = 2\}$ and*

$$u_n = \sum_{\ell=1}^{+\infty} \mathbb{1}_{\{\sum_{k=1}^{\ell}(2+T_{2,k})<n\}},$$

then the process $(u_{\lfloor tn \rfloor}/\sqrt{n})$ converges in distribution to $(L_B(t)/2)$, where $L_B(t)$ is the local time process at time $t \geq 0$ of a standard Brownian motion.

PROOF. The variable T_2 can be written as a sum $T_1 + T_1'$ of independent random variables T_1 and T_1' having the same distribution as T_1 defined in the above lemma. For $k \geq 1$, the variable $T_{2,k}$ can be written as $T_{1,2k-1} + T_{1,2k}$. Clearly,

$$\frac{1}{2}\sum_{\ell=1}^{+\infty} \mathbb{1}_{\{\sum_{k=1}^{\ell}(1+T_{1,k})<n\}} - \frac{1}{2} \leq u_n \leq \frac{1}{2}\sum_{\ell=1}^{+\infty} \mathbb{1}_{\{\sum_{k=1}^{\ell}(1+T_{1,k})<n\}}.$$

Furthermore,

$$\left(\sum_{\ell=1}^{+\infty} \mathbb{1}_{\{\sum_{k=1}^{\ell}(1+T_{1,k})<n\}}, n \geq 1\right) \stackrel{\text{dist.}}{=} (r_n) \stackrel{\text{def.}}{=} \left(\sum_{\ell=1}^{n-1} \mathbb{1}_{\{C_\ell=0\}}, n \geq 1\right),$$

where (C_n) is the symmetric simple random walk.

A classical result by Knight [22] (see also Borodin [8] and Perkins [25]) gives that the process $(r_{\lfloor nt \rfloor}/\sqrt{n})$ converges in distribution to $(L_B(t))$ as n gets large. The lemma is proved. □

The main result of this section can now be stated.

THEOREM 2 (Scaling of the location of the mouse). *If $(C_0, M_0) \in \mathbb{N}^2$, the convergence in distribution*

$$\lim_{n\to+\infty}\left(\frac{1}{\sqrt[4]{n}} M_{\lfloor nt \rfloor}, t \geq 0\right) \stackrel{\text{dist.}}{=} \big(B_1(L_{B_2}(t)), t \geq 0\big)$$

holds, where $(B_1(t))$ and $(B_2(t))$ are independent standard Brownian motions on $\mathbb{R}$ and $(L_{B_2}(t))$ is the local time process of $(B_2(t))$ at 0.

The location of the mouse at time T is therefore of the order of $\sqrt[4]{T}$ as T gets large. The limiting process can be expressed as a Brownian motion slowed down by the process of the local time at 0 of an independent Brownian motion. The quantity $L_{B_2}(T)$ can be interpreted as the scaled duration of time the cat and the mouse spend together.

PROOF OF THEOREM 2. Without loss of generality, one can assume that $C_0 = M_0$. A coupling argument is used. Take:

– i.i.d. geometric random variables (G_i) such that $\mathbb{P}(G_1 \geq p) = 1/2^{p-1}$ for $p \geq 1$;
– (C_k^a) and $(C_{j,k}^b)$, $j \geq 1$, i.i.d. independent symmetric random walks starting from 0;

and assume that all these random variables are independent. One denotes, for $m = 1, 2$ and $j \geq 1$,

$$T_{m,j}^b = \inf\{k \geq 0 : C_{j,k}^b = m\}.$$

Define

$$(C_k, M_k) = \begin{cases} (C_k^a, C_k^a), & 0 \leq k < G_1, \\ (C_{G_1}^a - 2I_1 + I_1 C_{1,k-G_1}^b, C_{G_1}^a), & G_1 \leq k \leq \tau_1, \end{cases}$$

with $I_1 = C_{G_1}^a - C_{G_1-1}^a$, $\tau_1 = G_1 + T_{2,1}^b$. It is not difficult to check that

$$[(C_k, M_k), 0 \leq k \leq \tau_1]$$

has the same distribution as the cat and mouse Markov chain during a cycle as defined above.

Define $t_0 = 0$ and $t_i = t_{i-1} + \tau_i$, $s_0 = 0$ and $s_i = s_{i-1} + G_i$. The $(i+1)$th cycle is defined as

$$(C_k, M_k) = \begin{cases} (C_{k-t_i+s_i}^a, C_{k-t_i+s_i}^a), & t_i \leq k < t_i + G_{i+1}, \\ (C_{s_{i+1}}^a - 2I_{i+1} + I_{i+1} C_{i+1,k-t_i-G_{i+1}}^b, C_{s_{i+1}}^a), & \\ & t_i + G_{i+1} \leq k \leq t_{i+1}, \end{cases}$$

with $I_{i+1} = C_{s_{i+1}}^a - C_{s_{i+1}-1}^a$ and $\tau_{i+1} = G_{i+1} + T_{2,i+1}^b$. The sequence (C_n, M_n) has the same distribution as the Markov chain with transition matrix Q defined by relation (1).

With this representation, the location M_n of the mouse at time n is given by $C_{\kappa_n}^a$, where κ_n is the number of steps the mouse has made up to time n, formally defined as

$$\kappa_n \stackrel{\text{def.}}{=} \sum_{i=1}^{+\infty} \left[\sum_{\ell=1}^{i-1} G_\ell + (n - t_{i-1}) \right] \mathbb{1}_{\{t_{i-1} \leq n \leq t_{i-1}+G_i\}} + \sum_{i=1}^{+\infty} \left[\sum_{\ell=1}^{i} G_\ell \right] \mathbb{1}_{\{t_{i-1}+G_i < n < t_i\}},$$

in particular,

$$\sum_{\ell=1}^{\nu_n} G_\ell \le \kappa_n \le \sum_{\ell=1}^{\nu_n+1} G_\ell \tag{10}$$

with ν_n defined as the number of cycles of the cat and mouse process up to time n:

$$\nu_n = \inf\{\ell : t_{\ell+1} > n\} = \inf\left\{\ell : \sum_{k=1}^{\ell+1}(G_k + T_{2,k}^b) > n\right\}.$$

Define

$$\overline{\nu}_n = \inf\left\{\ell : \sum_{k=1}^{\ell+1}(2 + T_{2,k}^b) > n\right\},$$

then, for $\delta > 0$, on the event $\{\overline{\nu}_n > \nu_n + \delta\sqrt{n}\}$,

$$\begin{aligned} n &\ge \sum_{k=1}^{\nu_n+\delta\sqrt{n}} (2 + T_{2,k}^b) \ge \sum_{k=1}^{\nu_n+1} [G_k + T_{2,k}^b] + \sum_{k=\nu_n+2}^{\nu_n+\delta\sqrt{n}} (2 + T_{2,k}^b) - \sum_{k=1}^{\nu_n+1} (G_k - 2) \\ &\ge n + \sum_{k=\nu_n+2}^{\nu_n+\delta\sqrt{n}} (2 + T_{2,k}^b) - \sum_{k=1}^{\nu_n+1} (G_k - 2). \end{aligned}$$

Hence,

$$\sum_{k=\nu_n+2}^{\nu_n+\delta\sqrt{n}} (2 + T_{2,k}^b) \le \sum_{k=1}^{\nu_n+1} (G_k - 2);$$

since $T_{1,k}^b \le 2 + T_{2,k}^b$, the relation

$$\begin{aligned} \{\overline{\nu}_n - \nu_n > \delta\sqrt{n}\} &\subset \left\{\inf_{1\le\ell\le\nu_n} \sum_{k=\ell}^{\ell+\lfloor\delta\sqrt{n}\rfloor} T_{1,k}^b \le \sum_{k=1}^{\nu_n+1} (G_k - 2), \overline{\nu}_n > \nu_n\right\} \\ &\subset \left\{\inf_{1\le\ell\le\overline{\nu}_n} \sum_{k=\ell}^{\ell+\lfloor\delta\sqrt{n}\rfloor} T_{1,k}^b \le \sup_{1\le\ell\le\overline{\nu}_n} \sum_{k=1}^{\ell+1}(G_k - 2)\right\} \end{aligned} \tag{11}$$

holds. Since $\mathbb{E}(G_1) = 2$, Donsker's theorem gives the following convergence in distribution:

$$\lim_{K\to+\infty}\left(\frac{1}{\sqrt{K}} \sum_{k=1}^{\lfloor tK\rfloor+1} (G_k - 2), 0 \le t \le 1\right) \stackrel{\text{dist.}}{=} \left(\text{var}(G_1)W(t), 0 \le t \le 1\right),$$

where $(W(t))$ is a standard Brownian motion, and, therefore,

$$\lim_{K\to+\infty} \frac{1}{\sqrt{K}} \sup_{1\le\ell\le K} \sum_{k=1}^{\ell+1}(G_k - 2) \stackrel{\text{dist.}}{=} \text{var}(G_1) \sup_{0\le t\le 1} W(t). \tag{12}$$

For $t > 0$, define

$$\big(\Delta_n(s), 0 \le s \le t\big) \stackrel{\text{def.}}{=} \left(\frac{1}{\sqrt{n}}(\overline{\nu}_{\lfloor ns\rfloor} - \nu_{\lfloor ns\rfloor}), 0 \le s \le t\right).$$

By relation (11) one gets that, for $0 \le s \le t$,

$$\begin{aligned} \{\Delta_n(s) > \delta\} &\subset \left\{\inf_{1\le\ell\le\overline{\nu}_{\lfloor ns\rfloor}} \sum_{k=\ell}^{\ell+\lfloor\delta\sqrt{n}\rfloor} T_{1,k}^b \le \sup_{1\le\ell\le\overline{\nu}_{\lfloor ns\rfloor}} \sum_{k=1}^{\ell+1}(G_k - 2)\right\} \\ &\subset \left\{\inf_{1\le\ell\le\overline{\nu}_{\lfloor nt\rfloor}} \sum_{k=\ell}^{\ell+\lfloor\delta\sqrt{n}\rfloor} T_{1,k}^b \le \sup_{1\le\ell\le\overline{\nu}_{\lfloor nt\rfloor}} \sum_{k=1}^{\ell+1}(G_k - 2)\right\}. \end{aligned} \tag{13}$$

Letting $\varepsilon > 0$, by Lemma 2 and relation (12), there exist some $x_0 > 0$ and n_0 such that if $n \ge n_0$, then, respectively,

$$\mathbb{P}\big(\overline{\nu}_{\lfloor nt\rfloor} \ge x_0\sqrt{n}\big) \le \varepsilon \quad \text{and} \quad \mathbb{P}\left(\sup_{1\le\ell\le x_0\sqrt{n}} \sum_{k=1}^{\ell+1}(G_k - 2) \ge x_0\sqrt{n}\right) \le \varepsilon. \tag{14}$$

By using relation (13),

$$\begin{aligned} &\left\{\sup_{0\le s\le t} \Delta_n(s) > \delta\right\} \\ &\qquad \subset \big\{\overline{\nu}_{\lfloor nt\rfloor} \ge x_0\sqrt{n}\big\} \\ &\qquad\quad \cup \left\{\inf_{1\le\ell\le\overline{\nu}_{\lfloor nt\rfloor}} \sum_{k=\ell}^{\ell+\lfloor\delta\sqrt{n}\rfloor} T_{1,k}^b \le \sup_{1\le\ell\le\overline{\nu}_{\lfloor nt\rfloor}} \sum_{k=1}^{\ell+1}(G_k - 2), \overline{\nu}_{\lfloor nt\rfloor} < x_0\sqrt{n}\right\} \\ &\qquad \subset \big\{\overline{\nu}_{\lfloor nt\rfloor} \ge x_0\sqrt{n}\big\} \cup \left\{\inf_{1\le\ell\le x_0\sqrt{n}} \sum_{k=\ell}^{\ell+\lfloor\delta\sqrt{n}\rfloor} T_{1,k}^b \le \sup_{1\le\ell\le x_0\sqrt{n}} \sum_{k=1}^{\ell+1}(G_k - 2)\right\}. \end{aligned}$$

With a similar decomposition with the partial sums of $(G_k - 2)$, relations (14) give the inequality, for $n \ge n_0$,

$$\mathbb{P}\Big(\sup_{0\le s\le t} \Delta_n(s) > \delta\Big) \le 2\varepsilon + \mathbb{P}\left(\inf_{1\le k\le x_0\sqrt{n}} \frac{1}{\sqrt{n}} \sum_{i=k}^{k+\lfloor\delta\sqrt{n}\rfloor} T_{1,i}^b \le x_0\right).$$

By Lemma 1, the left-hand side is thus arbitrarily small if n is sufficiently large. In a similar way the same results holds for the variable $\sup(-\Delta_n(s) : 0 \le s \le t)$. The variable $\sup(|\Delta_n(s)| : 0 \le s \le t)$ converges therefore in distribution to 0. Consequently, by using relation (10) and the law of large numbers, the same property holds for

$$\sup_{0\le s\le t} \frac{1}{\sqrt{n}}\big(\kappa_{\lfloor ns\rfloor} - 2\overline{\nu}_{\lfloor ns\rfloor}\big).$$

Donsker's theorem gives that the sequence of processes $(C^a_{\lfloor\sqrt{n}s\rfloor}/\sqrt[4]{n}, 0 \le s \le t)$ converges in distribution to $(B_1(s), 0 \le s \le t)$. In particular, for ε and $\delta > 0$, there exists some n_0 such that if $n \ge n_0$, then

$$\mathbb{P}\left(\sup_{0\le u,v\le t, |u-v|\le\delta} \frac{1}{\sqrt[4]{n}}\left|C^a_{\lfloor\sqrt{n}u\rfloor} - C^a_{\lfloor\sqrt{n}v\rfloor}\right| \ge \delta\right) \le \varepsilon;$$

see Billingsley [6], for example. Since $M_n = C^a_{\kappa_n}$ for any $n \ge 1$, the processes

$$\left(\frac{1}{\sqrt[4]{n}} M_{\lfloor ns\rfloor}, 0 \le s \le t\right) \quad \text{and} \quad \left(\frac{1}{\sqrt[4]{n}} C^a_{2\overline{\nu}_{\lfloor ns\rfloor}}, 0 \le s \le t\right)$$

have therefore the same asymptotic behavior for the convergence in distribution. Since, by construction (C^a_k) and $(\overline{\nu}_n)$ are independent, with Skorohod's representation theorem, one can assume that, on an appropriate probability space with two independent Brownian motions $(B_1(s))$ and $(B_2(s))$, the convergences

$$\lim_{n\to+\infty} \left(C^a_{\lfloor\sqrt{n}s\rfloor}/\sqrt[4]{n}, 0 \le s \le t\right) = \left(B_1(s), 0 \le s \le t\right),$$

$$\lim_{n\to+\infty} \left(\overline{\nu}_{\lfloor ns\rfloor}/\sqrt{n}\right) = \left(L_{B_2}(s)/2, 0 \le s \le t\right)$$

hold almost surely for the norm of the supremum. This concludes the proof of the theorem. □

3.2. *Random walk in the plane*. The transition matrix P of this random walk is given by, for $x \in \mathbb{Z}^2$,

$$p\big(x, x + (1, 0)\big) = p\big(x, x - (1, 0)\big) = p\big(x, x + (0, 1)\big) = p\big(x, x - (0, 1)\big) = \tfrac{1}{4}.$$

Decomposition into cycles. In the one-dimensional case, when the cat and the mouse start at the same location, when they are separated for the first time, they are at distance 2, so that the next meeting time has the same distribution as the hitting time of 0 for the simple random walk when it starts at 2. For $d = 2$, because of the geometry, the situation is more complicated. When the cat and the mouse are separated for the first time, there are several possibilities for the patterns of their respective locations and not only one as for $d = 1$. A finite Markov chain has to be introduced that describes the relative position of the mouse with respect to the cat.

Let $e_1 = (1, 0)$, $e_{-1} = -e_1$, $e_2 = (0, 1)$, $e_{-2} = -e_2$ and the set of unit vectors of $\mathbb{Z}^2$ is denoted by $\mathcal{E} = \{e_1, e_{-1}, e_2, e_{-2}\}$. Clearly, when the cat and the mouse are at the same site, they stay together a geometric number of steps whose mean is $4/3$. When they are just separated, up to a translation, a symmetry or a rotation, if the mouse is at e_1, the cat will be at e_2, e_{-2} or $-e_1$ with probability $1/3$. The next time the cat will meet the mouse corresponds to one of the instants of visit to $\mathcal{E}$ by the sequence (C_n). If one considers only these visits, then, up to a translation, it is not difficult to see that the position of the cat and of the mouse is a Markov chain with transition matrix Q_R defined below.

DEFINITION 1. Let $e_1 = (1,0)$, $e_{-1} = -e_1$, $e_2 = (0,1)$, $e_{-2} = -e_2$ and the set of unit vectors of $\mathbb{Z}^2$ is denoted by $\mathcal{E} = \{e_1, e_{-1}, e_2, e_{-2}\}$.

If (C_n) is a random walk in the plane, (R_n) denotes the sequence in $\mathcal{E}$ such that (R_n) is the sequence of unit vectors visited by (C_n) and

$$r_{ef} \stackrel{\text{def.}}{=} \mathbb{P}(R_1 = f \mid R_0 = e), \qquad e, f \in \mathcal{E}. \tag{15}$$

A transition matrix Q_R on $\mathcal{E}^2$ is defined as follows: for $e, f, g \in \mathcal{E}$,

$$\begin{cases} Q_R((e,g),(f,g)) = r_{ef}, & e \neq g, \\ Q_R((e,e),(e,-e)) = 1/3, \\ Q_R((e,e),(e,\overline{e})) = Q_R\big((e,e),(e,-\overline{e})\big) = 1/3, \end{cases} \tag{16}$$

with the convention that $\overline{e}$, $-\overline{e}$ are the unit vectors orthogonal to e, μ_R denotes the invariant probability distribution associated to Q_R and $D_{\mathcal{E}}$ is the diagonal of $\mathcal{E}^2$.

A characterization of the matrix R is as follows. Let

$$\tau^+ = \inf(n > 0 : C_n \in \mathcal{E}) \quad \text{and} \quad \tau = \inf(n \geq 0 : C_n \in \mathcal{E}),$$

then clearly $r_{ef} = \mathbb{P}(C_{\tau^+} = f \mid C_0 = e)$. For $x \in \mathbb{Z}^2$, define

$$\phi(x) = \mathbb{P}(C_\tau = e_1 \mid C_0 = x).$$

By symmetry, it is easily seen that the coefficients of R can be determined by ϕ. For $x \notin \mathcal{E}$, by looking at the state of the Markov chain at time 1, one gets the relation

$$\Delta\phi(x) \stackrel{\text{def.}}{=} \phi(x+e_1) + \phi(x+e_{-1}) + \phi(x+e_2) + \phi(x+e_{-2}) - 4\phi(x) = 0$$

and $\phi(e_i) = 0$ if $i \in \{-1, 2, -2\}$ and $\phi(e_1) = 1$. In other words, ϕ is the solution of a *discrete Dirichlet problem*: it is a harmonic function (for the discrete Laplacian) on $\mathbb{Z}^2$ with fixed values on $\mathcal{E}$. Classically, there is a unique solution to the Dirichlet problem; see Norris [24], for example. An explicit expression of ϕ is, apparently, not available.

THEOREM 3. *If $(C_0, M_0) \in \mathbb{N}^2$, the convergence in distribution of finite marginals*

$$\lim_{n \to +\infty} \left(\frac{1}{\sqrt{n}} M_{\lfloor e^{nt} \rfloor}, t \geq 0 \right) \stackrel{\text{dist.}}{=} (W(Z(t)))$$

holds, with

$$(Z(t)) = \left(\frac{16 \mu_R(D_{\mathcal{E}})}{3\pi} L_B(T_t) \right),$$

where μ_R is the probability distribution on $\mathcal{E}^2$ introduced in Definition 1, the process $(W(t)) = (W_1(t), W_2(t))$ is a two-dimensional Brownian motion and:

– $(L_B(t))$ *the local time at* 0 *of a standard Brownian motion* $(B(t))$ *on* $\mathbb{R}$ *independent of* $(W(t))$.
– *For* $t \geq 0$, $T_t = \inf\{s \geq 0 : B(s) = t\}$.

PROOF. The proof follows the same lines as before: a convenient construction of the process to decouple the time scale of the visits of the cat and the motion of the mouse. The arguments which are similar to the ones used in the proof of the one-dimensional case are not repeated.

Let (R_n, S_n) be the Markov chain with transition matrix Q_R that describes the relative positions of the cat and the mouse at the instances of visits of (C_n, M_n) to $\mathcal{E} \times \mathcal{E}$ up to rotation, symmetry and translation. For N visits to the set $\mathcal{E} \times \mathcal{E}$, the proportion of time the cat and the mouse will have met is given by

$$\frac{1}{N} \sum_{\ell=1}^{N} \mathbb{1}_{\{R_\ell = S_\ell\}};$$

this quantity converges almost surely to $\mu_R(D_\mathcal{E})$.

Now one has to estimate the number of visits of the cat to the set $\mathcal{E}$. Kasahara [18] (see also Bingham [7] and Kasahara [17]) gives that, for the convergence in distribution of the finite marginals, the following convergence holds:

$$\lim_{n \to +\infty} \left(\frac{1}{n} \sum_{i=0}^{\lfloor e^{nt} \rfloor} \mathbb{1}_{\{C_i \in \mathcal{E}\}} \right) \stackrel{\text{dist.}}{=} \left(\frac{4}{\pi} L_B(T_t) \right).$$

The rest of the proof follows the same lines as in the proof of Theorem 2. $\square$

REMARK. Tanaka's Formula (see Rogers and Williams [27]) gives the relation

$$L(T_t) = t - \int_0^{T_t} \operatorname{sgn}(B(s))\, dB(s),$$

where $\operatorname{sgn}(x) = -1$ if $x < 0$ and $+1$ otherwise. Since the process (T_t) has independent increments and that the T_t's are stopping times, one gets that $(L(T_t))$ has also independent increments. With the function $t \to T_t$ being discontinuous, the limiting process $(W(Z(t)))$ is also discontinuous. This is related to the fact that the convergence of processes in the theorem is minimal: it is only for the convergence in distribution of finite marginals. For $t \geq 0$, the distribution of $L(T_t)$ is an exponential distribution with mean $2t$; see Borodin and Salminen [9], for example. The characteristic function of

$$W_1\left(\frac{16 \mu_R(D_\mathcal{E}) L(T_t)}{3\pi} \right)$$

at $\xi \in \mathbb{C}$ such that $\operatorname{Re}(\xi) = 0$ can be easily obtained as

$$\mathbb{E}\left(e^{i\xi W_1[Z(t)]}\right) = \frac{\alpha_0^2}{\alpha_0^2 + \xi^2 t} \qquad \text{with } \alpha_0 = \frac{\sqrt{3\pi}}{4\sqrt{\mu_R(D_\mathcal{E})}}.$$

With a simple inversion, one gets that the density of this random variable is a bilateral exponential distribution given by

$$\frac{\alpha_0}{2\sqrt{t}}\exp\left(-\frac{\alpha_0}{\sqrt{t}}|y|\right), \qquad y \in \mathbb{R}.$$

The characteristic function can be also represented as

$$\mathbb{E}\big(e^{i\xi W_1[Z(t)]}\big) = \frac{\alpha_0^2}{\alpha_0^2 + \xi t} = \exp\left(\int_{-\infty}^{+\infty} (e^{i\xi u} - 1)\Pi(t,u)\,du\right)$$

with

$$\Pi(t,u) = \frac{e^{-\alpha_0|u|/\sqrt{t}}}{|u|}, \qquad u \in \mathbb{R}.$$

$\Pi(t,u)\,du$ is in fact the associated Lévy measure of the nonhomogeneous process with independent increments $(W_1(Z(t)))$. See Chapter 5 of Gikhman and Skorohod [13].

4. The reflected random walk. In this section the cat follows a simple ergodic random walk on the integers with a reflection at 0; an asymptotic analysis of the evolution of the sample paths of the mouse is carried out. Despite being a quite simple example, it exhibits already an interesting scaling behavior.

Let P denote the transition matrix of the simple reflected random walk on $\mathbb{N}$,

$$\begin{cases} p(x,x+1) = p, & x \geq 0, \\ p(x,x-1) = 1-p, & x \neq 0, \\ p(0,0) = 1-p. \end{cases} \tag{17}$$

It is assumed that $p \in (0, 1/2)$ so that the corresponding Markov chain is positive recurrent and reversible and its invariant probability distribution is a geometric random variable with parameter $\rho \stackrel{\text{def.}}{=} p/(1-p)$. In this case, one can check that the measure ν on $\mathbb{N}^2$ defined in Theorem 1 is given by

$$\begin{cases} \nu(x,y) = \rho^x(1-\rho), & 0 \leq x < y-1, \\ \nu(y-1,y) = \rho^{y-1}(1-\rho)(1-p), \\ \nu(y,y) = \rho^y(1-\rho), \\ \nu(y+1,y) = \rho^{y+1}(1-\rho)p, \\ \nu(x,y) = \rho^x(1-\rho), & x > y+1. \end{cases}$$

The following proposition describes the scaling for the dynamics of the cat.

PROPOSITION 2. *If, for $n \geq 1$, $T_n = \inf\{k > 0 : C_k = n\}$, then, as n goes to infinity, the random variable $T_n/\mathbb{E}_0(T_n)$ converges in distribution to an exponentially distributed random variable with parameter* 1 *and*

$$\lim_{n\to+\infty} \mathbb{E}_0(T_n)\rho^n = \frac{1+\rho}{(1-\rho)^2}$$

with $\rho = p/(1-p)$.

If $C_0 = n$, *then* T_0/n *converges almost surely to* $(1+\rho)/(1-\rho)$.

PROOF. The first convergence result is standard; see Keilson [19] for closely related results. Note that the Markov chain (C_n) has the same distribution as the embedded Markov chain of the $M/M/1$ queue with arrival rate p and service rate q. The first part of the proposition is therefore a discrete analogue of the convergence result of Proposition 5.11 of Robert [26].

If $C_0 = n$ and define by induction $\tau_n = 0$ and, for $0 \le i \le n$,

$$\tau_i = \inf\{k \ge 0 : C_{k+\tau_{i+1}} = i\},$$

hence, $\tau_n + \cdots + \tau_i$ is the first time when the cat crosses level i. The strong Markov property gives that the $(\tau_i, 0 \le i \le n-1)$ are i.i.d. A standard calculation (see Grimmett and Stirzaker [15], e.g.) gives that

$$\mathbb{E}(u^{\tau_1}) = \frac{1 - \sqrt{1 - 4pqu}}{2pu}, \qquad 0 \le u \le 1,$$

hence, $\mathbb{E}(\tau_0) = (1+\rho)/(1-\rho)$. Since $T_0 = \tau_{n-1} + \cdots + \tau_0$, the last part of the proposition is therefore a consequence of the law of large numbers. □

Additive jumps. An intuitive picture of the main phenomenon is as follows. It is assumed that the mouse is at level n for some n large. If the cat starts at 0, according to the above proposition, it will take of the order of ρ^{-n} steps to reach the mouse. The cat and the mouse will then interact for a short amount of time until the cat returns in the neighborhood of 0, leaving the mouse at some new location M. Note that, because n is large, the reflection condition does not play a role for the dynamics of the mouse at this level and by spatial homogeneity outside 0, one has that $M = n + M'$ where M' is some random variable whose distribution is independent of n. Hence, when the cat has returned to the mouse k times after hitting 0 and then went back to 0 again, the location of the mouse can be represented as $n + M'_1 + \cdots + M'_k$, where (M'_i) are i.i.d. with the same distribution as M'. Roughly speaking, on the exponential time scale $t \to \rho^{-n}t$, it will be seen that the successive locations of the mouse can be represented with the random walk associated to M' with a negative drift, that is, $\mathbb{E}(M') < 0$.

The section is organized as follows: one investigates the properties of the random variable M' and the rest of the section is devoted to the proof of the functional limit theorem. The main ingredient is also a decomposition of the sample path of (C_n, M_n) into cycles. A cycle starts and ends with the cat at 0 and the mouse is visited at least once by the cat during the cycle.

Free process. Let (C'_n, M'_n) be the cat and mouse Markov chain associated to the simple random walk on $\mathbb{Z}$ without reflection (the free process):

$$p'(x, x+1) = p = 1 - p'(x, x-1) \qquad \forall x \in \mathbb{Z}.$$

PROPOSITION 3. *If $(C'_0, M'_0) = (0,0)$, then the asymptotic location of the mouse for the free process $M'_\infty = \lim_{n\to\infty} M'_n$ is such that, for $u \in \mathbb{C}$ such that $|u| = 1$,*

$$\mathbb{E}(u^{M'_\infty}) = \frac{\rho(1-\rho)u^2}{-\rho^2 u^2 + (1+\rho)u - 1}, \tag{18}$$

in particular,

$$\mathbb{E}(M'_\infty) = -\frac{1}{\rho} \quad \textit{and} \quad \mathbb{E}\left(\frac{1}{\rho^{M'_\infty}}\right) = 1.$$

Furthermore, the relation

$$\mathbb{E}\left(\sup_{n\geq 0} \frac{1}{\sqrt{\rho}^{M'_n}}\right) < +\infty \tag{19}$$

holds. If (S_k) is the random walk associated to a sequence of i.i.d. random variables with the same distribution as M'_∞ and (E_i) are i.i.d. exponential random variables with parameter $(1+\rho)/(1-\rho)^2$, then the random variable W defined by

$$W = \sum_{k=0}^{+\infty} \rho^{-S_k} E_k \tag{20}$$

is almost surely finite with infinite expectation.

PROOF. Let $\tau = \inf\{n \geq 1 : C'_n < M'_n\}$, then, by looking at the different cases, one has

$$M'_\tau = \begin{cases} 1, & \text{if } M_1 = 1, C_1 = -1, \\ 1 + M''_\tau, & \text{if } M_1 = 1, C_1 = 1, \\ -1 + M''_\tau, & \text{if } M_1 = -1, \end{cases}$$

where M''_τ is an independent r.v. with the same distribution as M'_τ. Hence, for $u \in \mathbb{C}$ such that $|u| = 1$, one gets that

$$\mathbb{E}(u^{M'_\tau}) = \left((1-p)\frac{1}{u} + p^2 u\right)\mathbb{E}(u^{M'_\tau}) + p(1-p)u$$

holds. Since $M'_\tau - C'_\tau = 2$, after time τ, the cat and the mouse meet again with probability ρ^2. Consequently,

$$M'_\infty \stackrel{\text{dist.}}{=} \sum_{i=1}^{1+G} M'_{\tau,i},$$

where $(M'_{\tau,i})$ are i.i.d. random variables with the same distribution as M'_τ and G is an independent geometrically distributed random variable with parameter ρ^2. This identity gives directly the expression (18) for the characteristic function of M'_∞ and also the relation $\mathbb{E}(M'_\infty) = -1/\rho$.

Recall that the mouse can move one step up only when it is at the same location as the cat, hence, one gets the upper bound

$$\sup_{n\geq 0} M'_n \leq U \stackrel{\text{def.}}{=} 1 + \sup_{n\geq 0} C'_n$$

and the fact that $U - 1$ has the same distribution as the invariant distribution of the reflected random walk (C_n), that is, a geometric distribution with parameter ρ gives directly inequality (19).

Let $N = (N_t)$ be a Poisson process with rate $(1-\rho)^2/(1+\rho)$, then one can check the following identity for the distributions:

$$W \stackrel{\text{dist.}}{=} \int_0^{+\infty} \rho^{-S_{N_t}}\, dt. \tag{21}$$

By the law of large numbers, (S_{N_t}/t) converges almost surely to $-(1+\rho)/[(1-\rho)^2\rho]$. One gets therefore that W is almost surely finite. From (18), one gets $u \mapsto \mathbb{E}(u^{M'_\infty})$ can be analytically extended to the interval

$$\frac{1+\rho-\sqrt{(1-\rho)(1+3\rho)}}{2\rho^2} < u < \frac{1+\rho+\sqrt{(1-\rho)(1+3\rho)}}{2\rho^2}$$

in particular, for $u = 1/\rho$ and its value is $\mathbb{E}(\rho^{-M'_\infty}) = 1$. This gives by (20) and Fubini's theorem that $\mathbb{E}(W) = +\infty$. $\square$

Note that $\mathbb{E}(\rho^{-M'_\infty}) = 1$ implies that the exponential moment $\mathbb{E}(u^{M'_\infty})$ of the random variable M'_∞ is finite for u in the interval $[1, 1/\rho]$.

Exponential functionals. The representation (21) shows that the variable W is an exponential functional of a compound Poisson process. See Yor [30]. It can be seen as the invariant distribution of the auto-regressive process (X_n) defined as

$$X_{n+1} \stackrel{\text{def.}}{=} \rho^{-A_n} X_n + E_n, \qquad n \geq 0.$$

The distributions of these random variables are investigated in Guillemin et al. [16] when (A_n) are nonnegative. See also Bertoin and Yor [5]. The above proposition shows that W has a heavy-tailed distribution. As it will be seen in the scaling result below, this has a qualitative impact on the asymptotic behavior of the location of the mouse. See Goldie [14] for an analysis of the asymptotic behavior of tail distributions of these random variables.

A scaling for the location of the mouse. The rest of the section is devoted to the analysis of the location of the mouse when it is initially far away from the location of the cat. Define

$$s_1 = \inf\{\ell \geq 0 : C_\ell = M_\ell\} \quad \text{and} \quad t_1 = \inf\{\ell \geq s_1 : C_\ell = 0\}$$

and, for $k \geq 1$,

$$s_{k+1} = \inf\{\ell \geq t_k : C_\ell = M_\ell\} \quad \text{and} \quad t_{k+1} = \inf\{\ell \geq s_{k+1} : C_\ell = 0\}. \tag{22}$$

Proposition 2 suggests an exponential time scale for a convenient scaling of the location of the mouse. When the mouse is initially at n and the cat at the origin, it takes the duration s_1 of the order of ρ^{-n} so that the cat reaches this level. Just after that time, the two processes behave like the free process on $\mathbb{Z}$ analyzed above, hence, when the cat returns to the origin (at time t_1), the mouse is at position $n + M'_\infty$. Note that on the extremely fast exponential time scale $t \to \rho^{-n} t$, the (finite) time that the cat and mouse spend together is vanishing, and so is the time needed for the cat to reach zero from $n + M'_\infty$ (linear in n by the second statement of Proposition 2). Hence, on the exponential time scale, s_1 is a finite exponential random variable, and s_2 is distributed as a sum of two i.i.d. copies of s_1. The following proposition presents a precise formulation of this description, in particular, a proof of the corresponding scaling results. For the sake of simplicity, and because of the topological intricacies of convergence in distribution, in a first step the convergence result is restricted on the time interval $[0, s_2]$, that is, on the two first "cycles." Theorem 4 below gives the full statement of the scaling result.

PROPOSITION 4. *If $M_0 = n \geq 1$ and $C_0 = 0$, then, as n goes to infinity, the random variable $(M_{t_1} - n, \rho^n t_1)$ converges in distribution to (M'_∞, E_1) and the process*

$$\left(\frac{M_{\lfloor t\rho^{-n}\rfloor}}{n} \mathbb{1}_{\{0 \leq t < \rho^n s_2\}}\right)$$

converges in distribution for the Skorohod topology to the process

$$\left(\mathbb{1}_{\{t < E_1 + \rho^{-M'_\infty} E_2\}}\right),$$

where the distribution of M'_∞ is as defined in Proposition 3, and it is independent of E_1 and E_2, two independent exponential random variables with parameter $(1+\rho)/(1-\rho)^2$.

PROOF. For $T > 0$, $\mathcal{D}([0, T], \mathbb{R})$ denotes the space of cadlag functions, that is, of right continuous functions with left limits, and d^0 is the metric on this space defined by, for $x, y \in \mathcal{D}([0, T], \mathbb{R})$,

$$d^0(x, y) = \inf_{\varphi \in \mathcal{H}} \left[\sup_{0 \leq s < t < T} \left| \log \frac{\varphi(t) - \varphi(s)}{t - s} \right| + \sup_{0 \leq s < T} |x(\varphi(s)) - y(s)| \right],$$

where $\mathcal{H}$ is the set of nondecreasing functions φ such that $\varphi(0)=0$ and $\varphi(T)=T$. See Billingsley [6].

An upper index n is added on the variables s_1, s_2, t_1 to stress the dependence on n. Take three independent Markov chains (C_k^a), (C_k^b) and (C_k^c) with transition matrix P such that $C_0^a=C_0^c=0$, $C_0^b=n$ and, for $i=a,b,c$, T_p^i denotes the hitting time of $p\geq 0$ for (C_k^i). Since $((C_k,M_k),s_1^n\leq k\leq t_1^n)$ has the same distribution as $((n+C'_k,n+M'_k),0\leq k<T_0^b)$, by the strong Markov property, the sequence $(M_k,k\leq s_2^n)$ has the same distribution as $(N_k,0\leq k\leq T_n^a+T_0^b+T_n^c)$, where

$$N_k=\begin{cases} n, & k\leq T_n^a,\\ n+M'_{k-T_n^a}, & T_n^a\leq k\leq T_n^a+T_0^b,\\ n+M'_{T_0^b}, & T_n^a+T_0^b\leq k\leq T_n^a+T_0^b+T^c_{n+M'_{T_0^b}}.\end{cases}\tag{23}$$

Here $((C_k^b-n,M'_k),0\leq k\leq T_0^b)$ is a sequence with the same distribution as the free process with initial starting point $(0,0)$ and killed at the hitting time of $-n$ by the first coordinate. Additionally, it is independent of the Markov chains (C_k^a) and (C_k^c). In particular, the random variable $M_{t_1}-n$, the jump of the mouse from its initial position when the cat hits 0, has the same distribution as $M'_{T_0^b}$. Since T_0^b converges almost surely to infinity, $M'_{T_0^b}$ is converging in distribution to M'_∞.

Proposition 2 and the independence of (C_k^a) and (C_k^c) show that the sequences $(\rho^n T_n^a)$ and $(\rho^n T_n^c)$ converge in distribution to two independent exponential random variables E_1 and E_2 with parameter $(1+\rho)/(1-\rho)^2$. By using Skorohod's Representation theorem, (see Billingsley [6]) up to a change of probability space, it can be assumed that these convergences hold for the almost sure convergence.

By representation (23), the rescaled process $((M_{\lfloor t\rho^{-n}\rfloor}/n)\mathbb{1}_{\{0\leq t<\rho^n s_2\}},t\leq T)$ has the same distribution as

$$x_n(t)\stackrel{\text{def.}}{=}\begin{cases} 1, & t<\rho^n T_n^a,\\ 1+\dfrac{1}{n}M'_{\lfloor\rho^{-n}t-T_n^a\rfloor}, & \rho^n T_n^a\leq t<\rho^n(T_n^a+T_0^b),\\ 1+\dfrac{1}{n}M'_{T_0^b}, & \rho^n(T_n^a+T_0^b)\leq t<\rho^n(T_n^a+T_0^b+T^c_{n+M'_{T_0^b}}),\\ 0, & t\geq\rho^n(T_n^a+T_0^b+T^c_{n+M'_{T_0^b}}),\end{cases}$$

for $t\leq T$. Proposition 2 shows that T_0^b/n converges almost surely to $(1-\rho)/(1+\rho)$ so that $(\rho^n(T_n^a+T_0^b))$ converges to E_1 and, for $n\geq 1$,

$$\rho^n T^c_{n+M'_{T_0^b}}=\rho^{-M'_{T_0^b}}\rho^{n+M'_{T_0^b}}T^c_{n+M'_{T_0^b}}\longrightarrow\rho^{-M'_\infty}E_2,$$

almost surely as n goes to infinity. Additionally, one has also

$$\lim_{n\to+\infty}\frac{1}{n}\sup_{k\geq 0}|M'_k|=0,$$

almost surely. Define

$$x_\infty = \left(\mathbb{1}_{\{t<T\wedge(E_1+\rho^{-M'_\infty}E_2)\}}\right),$$

where $a \wedge b = \min(a, b)$ for $a, b \in \mathbb{R}$.

Time change. For $n \geq 1$ and $t > 0$, define u_n (resp., v_n) as the minimum (resp., maximum) of $t \wedge \rho^n \lceil T_n^a + T_0^b + T^c_{n+M'_{T_0^b}} \rceil$ and $t \wedge (E_1 + \rho^{-M'_\infty} E_2)$, and

$$\varphi_n(s) = \begin{cases} \dfrac{v_n}{u_n} s, & 0 \leq s \leq u_n, \\ v_n + (s - u_n)\dfrac{T - v_n}{T - u_n}, & u_n < s \leq T. \end{cases}$$

Noting that $\varphi_n \in \mathcal{H}$, by using this function in the definition of the distance d^0 on $\mathcal{D}([0, T], \mathbb{R})$ to have an upper bound of $(d(x_n, x_\infty))$ and with the above convergence results, one gets that, almost surely, the sequence $(d(x_n, x_\infty))$ converges to 0. The proposition is proved. $\square$

THEOREM 4 (Scaling for the location of the mouse). *If $M_0 = n$, $C_0 = 0$, then the process*

$$\left(\frac{M_{\lfloor t\rho^{-n}\rfloor}}{n} \mathbb{1}_{\{t<\rho^n t_n\}}\right)$$

converges in distribution for the Skorohod topology to the process $(\mathbb{1}_{\{t<W\}})$, where W is the random variable defined by (20).

If H_0 is the hitting time of 0 *by (M_n),*

$$H_0 = \inf\{s \geq 0 : M_s = 0\},$$

then, as n goes to infinity, $\rho^n H_0$ converges in distribution to W.

PROOF. In the same way as in the proof of Proposition 4, it can be proved that for $p \geq 1$, the random vector $[(M_{t_k} - n, \rho^n t_k), 1 \leq k \leq p]$ converges in distribution to the vector

$$\left(S_k, \sum_{i=0}^{k-1} \rho^{-S_i} E_i\right)$$

and, for $k \geq 0$, the convergence in distribution

$$\lim_{n\to+\infty} \left(\frac{M_{\lfloor t\rho^{-n}\rfloor}}{n} \mathbb{1}_{\{0\leq t<\rho^n t_k\}}\right) = \left(\mathbb{1}_{\{t<E_1+\rho^{-S_1}E_2+\cdots+\rho^{-S_{k-1}}E_k\}}\right) \tag{24}$$

holds for the Skorohod topology.

Let $\phi : [0, 1] \to \mathbb{R}_+$ be defined by $\phi(s) = \mathbb{E}(\rho^{-sM'_\infty})$, then $\phi(0) = \phi(1) = 1$ and $\phi'(0) < 0$, since ϕ is strictly convex then for all $s < 1$, $\phi(s) < 1$.

If $C_0 = M_0 = n$, and the sample path of $(M_k - n, k \geq 0)$ follows the sample path of a reflected random walk starting at 0, we have, in particular, that the supremum of its successive values is integrable. By Proposition 3, as n goes to infinity, $M_{t_1} - n$ is converging in distribution to M'_∞. Lebesgue's theorem gives therefore that the averages are also converging, hence, since $\mathbb{E}(M'_\infty)$ is negative, there exists N_0 such that if $n \geq N_0$,

$$\mathbb{E}_{(n,n)}(M_{t_1}) \stackrel{\text{def.}}{=} \mathbb{E}(M_{t_1} \mid M_0 = C_0 = n) \leq n + \frac{1}{2}\mathbb{E}(M'_\infty) = n - \frac{1}{2\rho}. \tag{25}$$

Note that t_1 has the same distribution as T_0 in Proposition 2 when $C_0 = n$. Proposition 2 now implies that there exists $K_0 \geq 0$ so that, for $n \geq N_0$,

$$\rho^{n/2}\mathbb{E}_{(0,n)}\big(\sqrt{t_1}\big) \leq K_0. \tag{26}$$

The identity $\mathbb{E}(1/\rho^{M'_\infty}) = 1$ implies that $\mathbb{E}(\rho^{-M'_\infty/2}) < 1$, and inequality (19) and Lebesgue's theorem imply that one can choose $0 < \delta < 1$ and N_0, so that

$$\mathbb{E}\big(\rho^{(n-M_{t_1})/2}\big) \leq \delta \tag{27}$$

holds for $n \geq N_0$. Let $\nu = \inf\{k \geq 1 : M_{t_k} \leq N_0\}$ and, for $k \geq 1$, $\mathcal{G}_k$ the σ-field generated by the random variables (C_j, M_j) for $j \leq t_k$. Because of inequality (25), one can check that the sequence

$$\left(M_{t_{k\wedge\nu}} + \frac{1}{2\rho}(k \wedge \nu), k \geq 0\right)$$

is a super-martingale with respect to the filtration $(\mathcal{G}_k)$, hence,

$$\mathbb{E}(M_{t_{k\wedge\nu}}) + \frac{1}{2\rho}\mathbb{E}(k \wedge \nu) \leq \mathbb{E}(M_0) = n.$$

Since the location of the mouse is nonnegative, by letting k go to infinity, one gets that $\mathbb{E}(\nu) \leq 2\rho n$. In particular, ν is almost surely a finite random variable.

Intuitively, t_ν is the time when the mouse reaches the area below a finite boundary N_0. Our goal now is to prove that the sequence $(\rho^n t_\nu)$ converges in distribution to W. For $p \geq 1$ and on the event $\{\nu \geq p\}$,

$$\big(\rho^n(t_\nu - t_p)\big)^{1/2} = \left(\sum_{k=p}^{\nu-1} \rho^n(t_{k+1} - t_k)\right)^{1/2} \leq \sum_{k=p}^{\nu-1} \sqrt{\rho^n(t_{k+1} - t_k)}. \tag{28}$$

For $k \geq p$, inequality (26) and the strong Markov property give that the relation

$$\rho^{M_{t_k}/2}\mathbb{E}\big[\sqrt{t_{k+1} - t_k} \mid \mathcal{G}_k\big] = \rho^{M_{t_k}/2}\mathbb{E}_{(0,M_{t_k})}\big[\sqrt{t_1}\big] \leq K_0$$

holds on the event $\{\nu > k\} \subset \{M_{t_k} > N_0\}$. One gets therefore that

$$\begin{aligned}\mathbb{E}\big(\sqrt{\rho^n(t_{k+1} - t_k)}\mathbb{1}_{\{k<\nu\}}\big) &= \mathbb{E}\big(\rho^{(n-M_{t_k})/2}\mathbb{1}_{\{k<\nu\}}\rho^{M_{t_k}/2}\mathbb{E}\big[\sqrt{t_{k+1} - t_k} \mid \mathcal{G}_k\big]\big) \\ &\leq K_0\mathbb{E}\big(\rho^{(n-M_{t_k})/2}\mathbb{1}_{\{k<\nu\}}\big)\end{aligned}$$

holds, and, with inequality (27) and again the strong Markov property,

$$\mathbb{E}\big(\rho^{(n-M_{t_k})/2}\mathbb{1}_{\{k<\nu\}}\big)=\mathbb{E}\big(\rho^{-\sum_{j=0}^{k-1}(M_{t_{j+1}}-M_{t_j})/2}\mathbb{1}_{\{k<\nu\}}\big)$$
$$\le\delta\mathbb{E}\big(\rho^{-\sum_{j=0}^{k-2}(M_{t_{j+1}}-M_{t_j})/2}\mathbb{1}_{\{k-1<\nu\}}\big)\le\delta^k.$$

Relation (28) gives therefore that

$$\mathbb{E}\big(\sqrt{\rho^n(t_\nu-t_p)}\big)\le\frac{K_0\delta^p}{1-\delta}.$$

For $\xi\ge 0$,

$$\begin{aligned}|\mathbb{E}(e^{-\xi\rho^n t_\nu})-\mathbb{E}(e^{-\xi\rho^n t_p})|&\le\big|\mathbb{E}\big(1-e^{-\xi\rho^n(t_\nu-t_p)^+}\big)\big|+\mathbb{P}(\nu<p)\\&=\int_0^{+\infty}\xi e^{-\xi u}\mathbb{P}\big(\rho^n(t_\nu-t_p)\ge u\big)\,du+\mathbb{P}(\nu<p)\\&\le\frac{K_0\delta^p}{1-\delta}\int_0^{+\infty}\frac{\xi}{\sqrt{u}}e^{-\xi u}\,du+\mathbb{P}(\nu<p)\end{aligned}\tag{29}$$

by using Markov's inequality for the random variable $\sqrt{\rho^n(t_\nu-t_p)}$. Since $\rho^n t_p$ converges in distribution to $E_0+\rho^{-S_1}E_1+\cdots+\rho^{-S_p}E_p$, one can prove that, for $\varepsilon>0$, by choosing a fixed p sufficiently large and that if n is large enough, then the Laplace transforms at $\xi\ge 0$ of the random variables $\rho^n t_\nu$ and W are at a distance less than ε.

At time t_ν the location M_{t_ν} of the mouse is $x\le N_0$ and the cat is at 0. Since the sites visited by M_n are a Markov chain with transition matrix $(p(x,y))$, with probability 1, the number R of jumps for the mouse to reach 0 is finite. By recurrence of (C_n), almost surely, the cat will meet the mouse R times in a finite time. Consequently, if H_0 is the time when the mouse hits 0 for the first time, then by the strong Markov property, the difference H_0-t_ν is almost surely a finite random variable. The convergence in distribution of $(\rho^n H_0)$ to W is therefore proved. □

Nonconvergence of scaled process after W. Theorem 4 could suggest that the convergence holds for a whole time axis, that is,

$$\lim_{n\to+\infty}\left(\frac{M_{\lfloor t\rho^{-n}\rfloor}}{n},t\ge 0\right)=\big(\mathbb{1}_{\{t<W\}},t\ge 0\big)$$

for the Skorohod topology. That is, after time W the rescaled process stays at 0 like for fluid limits of stable stochastic systems. However, it turns out that this convergence does not hold at all for the following intuitive (and nonrigorous) reason. Each time the cat meets the mouse at x large, the location of the mouse is at $x+M'_\infty$ when the cat returns to 0, where M'_∞ is the random variable defined in Proposition 3. In this way, after the kth visit of the cat, the mouse is at the kth position of a random walk associated to M'_∞ starting at x. Since $\mathbb{E}(1/\rho^{M'_\infty})=1$, Kingman's result (see Kingman [21]) implies that the hitting time of δn, with $0<\delta<1$, by this random walk started at 0 is of the order of $\rho^{-\delta n}$. For each of the steps of the random walk, the cat needs also of the order of $\rho^{-\delta n}$ units of time. Hence, the mouse reaches the level δn in order of $\rho^{-2\delta n}$ steps, and this happens on any finite interval $[s,t]$ on the time scale $t \to \rho^{-n}t$ only if $\delta \le 1/2$. Thus, it is very likely that the next relation holds:

$$\lim_{n\to+\infty} \mathbb{P}\left(\sup_{s\le u\le t} \frac{M_{\lfloor u\rho^{-n}\rfloor}}{n} = \frac{1}{2}\right) = 1.$$

Note that this implies that for $\delta \le 1/2$ on the time scale $t \to \rho^{-n}t$ the mouse will cross the level δn infinitely often on any finite interval! The difficulty in proving this statement is that the mouse is not at $x + M'_\infty$ when the cat returns at 0 at time τ_x but at $x + M'_{\tau_x}$, so that the associated random walk is not space-homogeneous but only asymptotically close to the one described above. Since an exponentially large number of steps of the random walks are considered, controlling the accuracy of the approximation turns out to be a problem. Nevertheless, a partial result is established in the next proposition.

PROPOSITION 5. *If $M_0 = C_0 = 0$, then for any $s, t > 0$ with $s < t$, the relation*

$$\lim_{n\to+\infty} \mathbb{P}\left(\sup_{s\le u\le t} \frac{M_{\lfloor u\rho^{-n}\rfloor}}{n} \ge \frac{1}{2}\right) = 1 \tag{30}$$

holds.

It should be kept in mind that, since (C_n, M_n) is recurrent, the process (M_n) returns infinitely often to 0 so that relation (30) implies that the scaled process exhibits oscillations for the norm of the supremum on compact intervals.

PROOF OF PROPOSITION 5. First it is assumed that $s = 0$. If $C_0 = 0$ and $T_0 = \inf\{k > 0 : C_k = 0\}$, then, in particular, $\mathbb{E}(T_0) = 1/(1-\rho)$. The set $\mathcal{C} = \{C_0, \dots, C_{T_0-1}\}$ is a cycle of the Markov chain, and denote by B its maximal value. The Markov chain can be decomposed into independent cycles $(\mathcal{C}_n, n \ge 1)$ with the corresponding values (T_0^n) and (B_n) for T_0 and B. Kingman's result (see Theorem 3.7 of Robert [26], e.g.) shows that there exists some constant K_0 such that $\mathbb{P}(B \ge n) \sim K_0\rho^n$. Taking $0 < \delta < 1/2$, for $\alpha > 0$,

$$U_n \stackrel{\text{def.}}{=} \rho^{(1-\delta)n} \sum_{k=1}^{\lfloor \alpha\rho^{-n}\rfloor} \left[\mathbb{1}_{\{B_k \ge \delta n\}} - \mathbb{P}(B \ge \delta n)\right],$$

then, by Chebyshev's inequality, for $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}(|U_n| \ge \varepsilon) &\le \rho^{(2-2\delta)n}\alpha\rho^{-n}\frac{\operatorname{Var}(\mathbb{1}_{\{B\ge\delta n\}})}{\varepsilon^2} \le \frac{\alpha}{\varepsilon^2}\rho^{(1-2\delta)n}\mathbb{P}(B \ge \delta n)\\ &\le \frac{\alpha K_0}{\varepsilon^2}\rho^{(1-\delta)n}.\end{aligned}$$

By using Borel–Cantelli's lemma, one gets that the sequence (U_n) converges almost surely to 0, hence, almost surely,

$$\lim_{n\to+\infty}\rho^{(1-\delta)n}\sum_{k=1}^{\lfloor\alpha\rho^{-n}\rfloor}\mathbb{1}_{\{B_k\geq\delta n\}}=\alpha K_0. \tag{31}$$

For $x\in\mathbb{N}$, let ν_x be the number of cycles up to time x, and the strong law of large numbers gives that, almost surely,

$$\lim_{x\to+\infty}\frac{\nu_x}{x}=\lim_{x\to+\infty}\frac{1}{x}\sum_{k=1}^{x}\mathbb{1}_{\{C_k=0\}}=1-\rho. \tag{32}$$

Denote by $x_n\stackrel{\text{def.}}{=}\lfloor\rho^{-n}t\rfloor$. For $\alpha_0>0$, the probability that the location of the mouse is never above level δn on the time interval $(0,x_n]$ is

$$\begin{aligned}\mathbb{P}&\Big(\sup_{1\leq k\leq\lfloor\rho^{-n}t\rfloor}M_k\leq\delta n\Big)\\ &\leq\mathbb{P}\left(\sup_{1\leq k\leq\lfloor\rho^{-n}t\rfloor}M_k\leq\delta n,\,\rho^{(1-\delta)n}\sum_{i=0}^{\nu_{x_n}-1}\mathbb{1}_{\{B_i\geq\delta n\}}\geq\frac{\alpha_0K_0}{2}\right)\\ &\quad+\mathbb{P}\left(\rho^{(1-\delta)n}\sum_{i=0}^{\nu_{x_n}-1}\mathbb{1}_{\{B_i\geq\delta n\}}<\frac{\alpha_0K_0}{2}\right).\end{aligned} \tag{33}$$

By the definition of x_n and (32), $\nu_{x_n}-1$ is asymptotically equivalent to $(1-\rho)\lfloor\rho^{-n}t\rfloor$, hence, if α_0 is taken to be $(1-\rho)t$, by (31), one gets that the last expression converges to 0 as n gets large. In the second term, the mouse stays below level δn, so a visit of the cat to δn on a cycle is necessarily at least one meeting of the cat and the mouse on this cycle. Further, it is clear that $\nu_{x_n}-1$ is not larger than $x_n=\lfloor\rho^{-n}t\rfloor$. Finally, recall that the mouse moves only when met by the cat and the sequence of successive sites visited by the mouse is a also a simple reflected random walk. Hence, if $\alpha_1=\alpha_0K_0/2$,

$$\begin{aligned}\mathbb{P}&\left(\sup_{1\leq k\leq\lfloor\rho^{-n}t\rfloor}M_k\leq\delta n,\,\rho^{(1-\delta)n}\sum_{i=0}^{\nu_{x_n}-1}\mathbb{1}_{\{B_i\geq\delta n\}}\geq\alpha_1\right)\\ &\leq\mathbb{P}\left(\sup_{1\leq k\leq\lfloor\rho^{-n}t\rfloor}M_k\leq\delta n,\,\rho^{(1-\delta)n}\sum_{i=0}^{\lfloor\rho^{-n}t\rfloor}\mathbb{1}_{\{C_i=M_i\}}\geq\alpha_1\right)\\ &\leq\mathbb{P}\Big(\sup_{1\leq k\leq\lfloor\alpha_1\rho^{-(1-\delta)n}\rfloor}C_k\leq\delta n\Big)=\mathbb{P}\big(T_{\lfloor\delta n\rfloor+1}\geq\lfloor\alpha_1\rho^{-(1-\delta)n}\rfloor\big)\end{aligned}$$

with the notation of Proposition 2, but this proposition shows that the random variable $\rho^{\lfloor\delta n\rfloor}T_{\lfloor\delta n\rfloor+1}$ converges in distribution as n gets large. Consequently, since

$\delta < 1/2$, the expression

$$\mathbb{P}\big(T_{\lfloor \delta n\rfloor+1} \geq \lfloor \alpha_1 \rho^{-(1-\delta)n}\rfloor\big) = \mathbb{P}\big(\rho^{\lfloor \delta n\rfloor} T_{\lfloor \delta n\rfloor+1} \geq \alpha_1 \rho^{-(1-2\delta)n}\big)$$

converges to 0. The relation

$$\lim_{n\to+\infty} \mathbb{P}\bigg(\sup_{0\leq u\leq t} \frac{M_{\lfloor u\rho^{-n}\rfloor}}{n} \geq \frac{1}{2}\bigg) = 1$$

has been proved.

The proof of the same result on the interval $[s,t]$ uses a coupling argument. Define the cat and mouse Markov chain $(\widetilde{C}_k, \widetilde{M}_k)$ as follows:

$$(\widetilde{C}_k, k \geq 0) = \big(C_{\lfloor s\rho^{-n}\rfloor+k}, k \geq 0\big)$$

and the respective jumps of the sequences $(M_{\lfloor s\rho^{-n}\rfloor+k})$ and $(\widetilde{M}_k)$ are independent except when $M_{\lfloor s\rho^{-n}\rfloor+k} = \widetilde{M}_k$, in which case they are the same. In this way, one checks that $(\widetilde{C}_k, \widetilde{M}_k)$ is a cat and mouse Markov chain with the initial condition

$$(\widetilde{C}_0, \widetilde{M}_0) = \big(C_{\lfloor s\rho^{-n}\rfloor}, 0\big).$$

By induction on k, one gets that $M_{\lfloor s\rho^{-n}\rfloor+k} \geq \widetilde{M}_k$ for all $k \geq 0$. Because of the ergodicity of (C_k), the variable $C_{\lfloor s\rho^{-n}\rfloor}$ converges in distribution as n get large. Thus, $\widetilde{C}_0$ is on a finite distance from 0 with probability one, and in the same way as before, one gets that

$$\lim_{n\to+\infty} \mathbb{P}\bigg(\sup_{0\leq u\leq t-s} \frac{\widetilde{M}_{\lfloor u\rho^{-n}\rfloor}}{n} \geq \frac{1}{2}\bigg) = 1,$$

therefore,

$$\liminf_{n\to+\infty} \mathbb{P}\bigg(\sup_{s\leq u\leq t} \frac{M_{\lfloor u\rho^{-n}\rfloor}}{n} \geq \frac{1}{2}\bigg) \geq \liminf_{n\to+\infty} \mathbb{P}\bigg(\sup_{0\leq u\leq t-s} \frac{\widetilde{M}_{\lfloor u\rho^{-n}\rfloor}}{n} \geq \frac{1}{2}\bigg) = 1.$$

This completes the proof of relation (30). □

5. Continuous time Markov chains. Let $Q = (q(x,y), x, y \in \mathcal{S})$ be the Q-matrix of a continuous time Markov chain on $\mathcal{S}$ such that, for any $x \in \mathcal{S}$,

$$q_x \overset{\text{def.}}{=} \sum_{y:\, y\neq x} q(x,y)$$

is finite and that the Markov chain is positive recurrent and π is its invariant probability distribution. The transition matrix of the underlying discrete time Markov chain is denoted as $p(x,y) = q(x,y)/q_x$; for $x \neq y$, note that $p(\cdot,\cdot)$ vanishes on the diagonal. See Norris [24] for an introduction on Markov chains and Rogers and Williams [28] for a more advanced presentation.

The analogue of the Markov chain (C_n, M_n) in this setting is the Markov chain $(C(t), M(t))$ on $\mathcal{S}^2$ whose infinitesimal generator Ω is defined by, for $x, y \in \mathcal{S}$,

$$\begin{aligned}\Omega(f)(x,y) &= \sum_{z\in\mathcal{S}} q(x,z)[f(z,y)-f(x,y)]\mathbb{1}_{\{x\neq y\}} \\ &\quad + \sum_{z,z'\in\mathcal{S}} q_x p(x,z)p(x,z')[f(z,z')-f(x,x)]\mathbb{1}_{\{x=y\}}\end{aligned} \tag{34}$$

for any function f on $\mathcal{S}^2$ vanishing outside a finite set. The first coordinate is indeed a Markov chain with Q-matrix Q and when the cat and the mouse are at the same site x, after an exponential random time with parameter q_x, they jump independently according to the transition matrix P. Note that if one looks at the sequence of sites visited by $(C(t), M(t))$, then it has the same distribution as the cat and mouse Markov chain associated to the matrix P. For this reason, the results obtained in Section 2 can be proved easily in this setting. In particular, $(C(t), M(t))$ is null recurrent when $(C(t))$ is reversible.

PROPOSITION 6. *If, for $t \geq 0$,*

$$U(t) = \int_0^t \mathbb{1}_{\{M(s)=C(s)\}}\, ds$$

and $S(t) = \inf\{s > 0 : U(s) \geq t\}$, then the process $(M(S(t)))$ has the same distribution as $(C(t))$, that is, it is a Markov process with Q-matrix Q.

This proposition simply states that, up to a time change, the mouse moves like the cat. In discrete time this is fairly obvious; the proof is in this case a little more technical.

PROOF OF PROPOSITION 6. If f is a function on $\mathcal{S}$, then by characterization of Markov processes, one has that the process

$$(H(t)) \stackrel{\text{def.}}{=} \left(f(M(t)) - f(M(0)) - \int_0^t \Omega(\bar{f})(C(s), M(s))\, ds\right)$$

is a local martingale with respect to the natural filtration $(\mathcal{F}_t)$ of $(C(t), M(t))$, where $\bar{f} : \mathcal{S}^2 \to \mathbb{R}$ such that $\bar{f}(x,y) = f(y)$ for $x, y \in \mathcal{S}$. The fact that, for $t \geq 0$, $S(t)$ is a stopping time and that $s \to S(s)$ is nondecreasing, and Doob's optional stopping theorem imply that $(H(S(t)))$ is a local martingale with respect to the filtration $(\mathcal{F}_{S(t)})$. Since

$$\begin{aligned}&\int_0^{S(t)} \Omega(\bar{f})(C(s), M(s))\, ds \\ &\qquad = \sum_{y\in\mathcal{S}} \int_0^{S(t)} q(M(s), y)\mathbb{1}_{\{C(s)=M(s)\}}\big(f(y) - f(M(s))\big)\, ds\end{aligned}$$

$$= \int_0^{S(t)} \mathbb{1}_{\{C(s)=M(s)\}} Q(f)(M(s))\, ds$$
$$= \int_0^t Q(f)(M(S(s)))\, ds,$$

the infinitesimal generator Q is defined for $x \in \mathcal{S}$ in a standard way as

$$Q(f)(x) = \sum_{y \in \mathcal{S}} q(x, y)[f(y) - f(x)].$$

One therefore gets that

$$\left(f(M(S(t))) - f(M(0)) - \int_0^t Q(f)(M(S(s)))\, ds \right)$$

is a local martingale for any function f on $\mathcal{S}$. This implies that $(M(S(t)))$ is a Markov process with Q-matrix Q, that is, that $(M(S(t)))$ has the same distribution as $(C(t))$. See Rogers and Williams [27]. $\square$

The example of the $M/M/\infty$ process. The example of the $M/M/\infty$ queue is investigated in the rest of this section. The associated Markov process can be seen as an example of a discrete Ornstein–Uhlenbeck process. As it will be shown, there is a significant qualitative difference with the example of Section 4 which is a discrete time version of the $M/M/1$ queue. The Q-matrix is given by

$$\begin{cases} q(x, x+1) = \rho, \\ q(x, x-1) = x. \end{cases} \tag{35}$$

The corresponding Markov chain is positive recurrent and reversible and its invariant probability distribution is Poisson with parameter ρ.

PROPOSITION 7. *If $C(0) = x \le n - 1$ and*

$$T_n = \inf\{s > 0 : C(s) = n\},$$

then, as n tends to infinity, the variable $T_n / E_x(T_n)$ converges in distribution to an exponentially distributed random variable with parameter 1 *and*

$$\lim_{n \to +\infty} \mathbb{E}_x(T_n) \rho^n / (n-1)! = e^{-\rho}.$$

If $C(0) = n$, then $T_0 / \log n$ converges in distribution to 1.

See Chapter 6 of Robert [26]. It should be remarked that the duration of time it takes to reach n starting from 0 is essentially the time it takes to go to n starting from $n - 1$.

Multiplicative jumps. The above proposition gives the order of magnitude for the duration of time for the cat to hit the mouse. As before, the cat returns "quickly" to the neighborhood of 0, but, contrary to the reflected random walk, it turns out that the cat will take the mouse down for some time before leaving the mouse. The next proposition shows that if the mouse is at n, its next location after the visit of the cat is of the order of nF for a certain random variable F.

PROPOSITION 8. *If* $C(0) = M(0) = n$ *and*

$$T_0 = \inf\{s > 0 : C(s) = 0\},$$

then, as n goes to infinity, the random variable $M(T_0)/n$ *converges in distribution to a random variable* F *on* $[0, 1]$ *such that* $\mathbb{P}(F \leq x) = x^{\rho}$.

PROOF. Let $\tau = \inf\{s > 0 : M(s) = M(s-) + 1\}$ be the instant of the first upward jump of $(M(s))$. Since $(M(S(s)))$ has the same distribution as $(C(s))$, one gets that $U(\tau)$, with $U(t)$ defined as in Proposition 6, has the same distribution as the time till a first upward jump of $(C(s))$, which is an exponential random variable with parameter ρ by definition (35). Now, think of $(M(S(s)))$ as the process describing the number of customers in an $M/M/\infty$ queue, which contains n customers at time 0. Let (E_i) be i.i.d. exponential random variables with parameter 1. For $1 \leq i \leq n$, E_i is the service time of the ith initial customer. At time τ, the process of the mouse will have run only for $U(\tau)$, so the ith customer is still there if $E_i > U(\tau)$. Note that there is no arrival up to time τ, and, hence,

$$M(\tau) \stackrel{\text{dist.}}{=} 1 + \sum_{i=1}^{n} \mathbb{1}_{\{E_i > U(\tau)\}}.$$

Consequently, by conditioning on the value of $U(\tau)$, by the law of large numbers one obtains that the sequence $(M(\tau)/n)$ converges in distribution to the random variable $F \stackrel{\text{def.}}{=} \exp(-U(\tau))$, which implies directly that $\mathbb{P}(F \leq x) = x^{\rho}$.

It remains to show that $(M(T_0)/n)$ converges in distribution to the same limit as $(M(\tau)/n)$. The fact that the mouse moves only when it meets the cat gives the following:

– On the event $\tau \geq T_0$, necessarily $M(\tau-) = C(\tau-) = 0$ because if the mouse did not move upward before time T_0, then it has reached 0 together with the cat. In this case, at time τ, the mouse makes its first jump upward from 0 to 1. Thus, the quantity

$$\mathbb{P}(\tau \geq T_0) \leq \mathbb{P}\big(M(\tau) = 1\big) = \mathbb{P}\big(U(\tau) > \max\{E_1, \dots, E_n\}\big)$$

converges to 0 as $n \to \infty$.

– Just before time τ, the mouse and the cat are at the same location and

$$\mathbb{P}\big(C(\tau) = M(\tau-) - 1\big) = \mathbb{E}\left[\frac{M(\tau-)}{\rho + M(\tau-)}\right]$$

converges to 1 as n gets large.

The above statements imply that with probability converging to one, the cat will find itself below the mouse for the first time strictly above level zero and before time T_0. We now show that after this event the cat will hit zero before returning back to the mouse. If $\varepsilon > 0$, then

$$\begin{aligned}\mathbb{E}\big(\mathbb{P}_{C(\tau)}\big(T_0 \geq T_{M(\tau)}\big)\big) &\leq \mathbb{E}\big(\mathbb{P}_{M(\tau)-1}\big(T_0 \geq T_{M(\tau)}\big)\big)\\ &\leq \mathbb{P}\left(\frac{M(\tau)}{n} \leq \varepsilon\right) + \sup_{k \geq \lfloor \varepsilon n \rfloor} \mathbb{P}_k(T_0 \geq T_{k+1}),\end{aligned}$$

hence, by Proposition 7, for ε (resp., n) sufficiently small (resp., large), the above quantity is arbitrarily small. This result implies that the probability of the event $\{M(\tau) = M(T_0)\}$ converges to 1. The proposition is proved. □

An underlying random walk. If $C(0) = 0$ and $M(0) = n$, the next time the cat returns to 0, Proposition 8 shows that the mouse will be at a location of the order of nF_1, where $F_1 = \exp(-E_1/\rho)$ and E_1 is an exponential random variable with parameter 1. After the pth round, the location of the mouse is of the order of

$$n \prod_{k=1}^{p} F_k = n \exp\left(-\frac{1}{\rho} \sum_{k=1}^{p} E_k\right), \tag{36}$$

where (E_k) are i.i.d. with the same distribution as E_1. A precise statement of this nonrigorous statement can be formulated easily. From (36), one gets that after the order of $\rho \log n$ rounds, the location of the mouse is within a finite interval.

The corresponding result for the reflected random walk exhibits an additive behavior. Theorem 4 gives that the location of the mouse is of the order of

$$n + \sum_{i=1}^{p} A_i \tag{37}$$

after p rounds, where (A_k) are i.i.d. copies of M'_∞, distribution of which is given by the generating function of relation (18). In this case the number of rounds after which the location of the mouse is located within a finite interval is of the order of n.

As Theorem 4 shows, for the reflected random walk, $t \to \rho^{-n} t$ is a convenient time scaling to describe the location of the mouse until it reaches a finite interval. This is not the case for the $M/M/\infty$ queue, since the duration of the first round of the cat, of the order of $(n-1)!/\rho^n$ by Proposition 7, dominates by far the duration of the subsequent rounds, that is, when the location of the mouse is at xn with $x < 1$.

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