

ASYMPTOTIC PROPERTIES OF EIGENMATRICES OF A LARGE SAMPLE COVARIANCE MATRIX

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Let $S_n = \frac{1}{n} X_n X_n^*$ where $X_n = \{X_{ij}\}$ is a $p \times n$ matrix with i.i.d. complex standardized entries having finite fourth moments. Let $Y_n(\mathbf{t}_1, \mathbf{t}_2, \sigma) = \sqrt{p}(\mathbf{x}_n(\mathbf{t}_1)^*(S_n + \sigma I)^{-1} \mathbf{x}_n(\mathbf{t}_2) - \mathbf{x}_n(\mathbf{t}_1)^* \mathbf{x}_n(\mathbf{t}_2) m_n(\sigma))$ in which $\sigma > 0$ and $m_n(\sigma) = \int \frac{dF_{y_n}(x)}{x + \sigma}$ where $F_{y_n}(x)$ is the Marčenko–Pastur law with parameter $y_n = p/n$; which converges to a positive constant as $n \rightarrow \infty$, and $\mathbf{x}_n(\mathbf{t}_1)$ and $\mathbf{x}_n(\mathbf{t}_2)$ are unit vectors in \mathbb{C}^p , having indices \mathbf{t}_1 and \mathbf{t}_2 , ranging in a compact subset of a finite-dimensional Euclidean space. In this paper, we prove that the sequence $Y_n(\mathbf{t}_1, \mathbf{t}_2, \sigma)$ converges weakly to a $(2m + 1)$ -dimensional Gaussian process. This result provides further evidence in support of the conjecture that the distribution of the eigenmatrix of S_n is asymptotically close to that of a Haar-distributed unitary matrix.

1. Introduction. Suppose that $\{x_{jk}, j, k = 1, 2, \dots\}$ is a double array of complex random variables that are independent and identically distributed (i.i.d.) with mean zero and variance 1. Let $\mathbf{x}_j = (x_{1j}, \dots, x_{pj})'$ and $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$, we define

$$(1.1) \quad \mathbf{S}_n = \frac{1}{n} \sum_{k=1}^n \mathbf{x}_k \mathbf{x}_k^* = \frac{1}{n} \mathbf{X} \mathbf{X}^*,$$

where \mathbf{x}_k^* and \mathbf{X}^* are the transposes of the complex conjugates of \mathbf{x}_k and \mathbf{X} , respectively. The matrix \mathbf{S}_n defined in (1.1) can be viewed as the sample covariance matrix of a p -dimensional random sample with size n . When the dimension p is fixed and the sample size n is large, the spectral behavior of \mathbf{S}_n has been extensively investigated in the literature due to its importance in multivariate statistical inference [see, e.g., Anderson (1951, 1989)]. However, when the dimension p is proportional to the sample size n in the limit; that is, $\frac{p}{n} \rightarrow y > 0$ as $n \rightarrow \infty$, the classical asymptotic theory will induce serious inaccuracy. This phenomenon can be easily explained from the viewpoint of random matrix theory (RMT).

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Before introducing our advancement of the theory, we will first give a brief review of some well-known properties of \mathbf{S}_n in RMT. We define the empirical spectral distribution (ESD) of \mathbf{S}_n by

$$F^{\mathbf{S}_n}(x) = \frac{1}{p} \sum_{j=1}^p I(\lambda_j \leq x),$$

where λ_j 's are eigenvalues of \mathbf{S}_n . First, it has long been known that $F^{\mathbf{S}_n}(x)$ converges almost surely to the standard Marčenko–Pastur law [MPL; see, e.g., Marčenko and Pastur (1967), Wachter (1978) and Yin (1986)] $F_y(x)$, which has a density function $(2\pi xy)^{-1} \sqrt{(b-x)(x-a)}$, supported on $[a, b] = [(1 - \sqrt{y})^2, (1 + \sqrt{y})^2]$. For the case $y > 1$, $F_y(x)$ has a point mass $1 - 1/y$ at 0. If its fourth moment is finite, as $n \rightarrow \infty$, the largest eigenvalue of \mathbf{S}_n converges to b while the smallest eigenvalue (when $y \leq 1$) or the $(p - n + 1)$ st smallest eigenvalue (when $y > 1$) converges to a [see Bai (1999) for a review]. The central limit theorem (CLT) for linear spectral statistics (LSS) of \mathbf{S}_n has been established in Bai and Silverstein (2004).

While results on the eigenvalues of \mathbf{S}_n are abundant in the literature, not much work has been done on the behavior of the eigenvectors of \mathbf{S}_n . It has been conjectured that the eigenmatrix; that is, the matrix of orthonormal eigenvectors of \mathbf{S}_n , is asymptotically Haar-distributed. This conjecture has yet to be formally proven due to the difficulty of describing the “asymptotically Haar-distributed” properties when the dimension p increases to infinity. Silverstein (1981) was the first one to create an approach to characterize the eigenvector properties. We describe his approach as follows: denoting the spectral decomposition of \mathbf{S}_n by $\mathbf{U}_n^* \mathbf{\Lambda} \mathbf{U}_n$, if \mathbf{x}_{ij} is normally distributed, \mathbf{U}_n has a Haar measure on the orthogonal matrices and is independent of the eigenvalues in $\mathbf{\Lambda}$. For any unit vector $\mathbf{x}_n \in \mathbb{C}^p$, the vector $\mathbf{y}_n = (y_1, \dots, y_p) = \mathbf{U}_n \mathbf{x}_n$ performs like a uniform distribution over the unit sphere in \mathbb{C}^p . As such, for $t \in [0, 1]$, a stochastic process

$$X_n(t) = \sqrt{p/2} \left(\sum_{i=1}^{\lfloor pt \rfloor} (y_i^2 - 1/p) \right)$$

is defined. If $\mathbf{z} = (z_1, \dots, z_p)' \sim N(0, I_p)$, then \mathbf{y}_n has the same distribution as $\mathbf{z}/\|\mathbf{z}\|$ and $X_n(t)$ is identically distributed with

$$\tilde{X}_n(t) = \sqrt{p/2} \|\mathbf{z}\|^{-2} \left(\sum_{i=1}^{\lfloor pt \rfloor} (z_i^2 - \|\mathbf{z}\|^2/p) \right).$$

Applying Donsker’s theorem [Donsker (1951)], $X_n(t)$ tends to a standard Brownian bridge.

For any general large sample covariance, it is important to examine the behavior of the $X_n(t)$ process. Silverstein (1981, 1984, 1989) prove that the integral of polynomial functions with respect to $X_n(t)$ will tend to a normal distribution. To overcome the difficulty of tightness, Silverstein (1990) takes $\mathbf{x}_n = (\pm 1, \dots, \pm 1)/\sqrt{p}$

so that the process $X_n(t)$ will tend to the standard Brownian bridge instead. In addition, Bai, Miao and Pan (2007) investigate the process $X_n(t)$, defined for $\mathbf{T}_n^{1/2} \mathbf{S}_n \mathbf{T}_n^{1/2}$ with $(\mathbf{T}_n^{1/2})^2 = \mathbf{T}_n$, a nonnegative positive definite matrix.

However, so far, the process $X_n(t)$ is assumed to be generated only by one unit vector \mathbf{x}_n in \mathbb{C}^p . This imposes restrictions on many practical situations. For example, in the derivation of the limiting properties of the bootstrap corrected Markowitz portfolio estimates, we need to consider two unparallel vectors simultaneously [see Bai, Liu and Wong (2009) and Markowitz (1952, 1959, 1991)]. In this paper, we will go beyond the boundaries of their studies to investigate the asymptotics of the eigenmatrix for any general large sample covariance matrix \mathbf{S}_n when \mathbf{x}_n runs over a subset of the p -dimensional unit sphere in which $\mathbb{C}_1^p = \{\mathbf{x}_n : \|\mathbf{x}_n\| = 1, \mathbf{x}_n \in \mathbb{C}^p\}$.

We describe the approach we introduced in this paper as follows: if \mathbf{V}_n is Haar-distributed, for any pair of p -vectors \mathbf{x} and \mathbf{y} satisfying $\mathbf{x} \perp \mathbf{y}$, $(\mathbf{V}_n \mathbf{x}, \mathbf{V}_n \mathbf{y})$ possesses the same joint distribution as

$$(1.2) \quad (\mathbf{z}_1 \quad \mathbf{z}_2) \begin{pmatrix} \mathbf{z}_1^* \mathbf{z}_1 & \mathbf{z}_1^* \mathbf{z}_2 \\ \mathbf{z}_2^* \mathbf{z}_1 & \mathbf{z}_2^* \mathbf{z}_2 \end{pmatrix}^{-1/2},$$

where \mathbf{z}_1 and \mathbf{z}_2 are two independent p -vectors whose components are i.i.d. standard normal variables. As n tends to infinity, we have

$$(1.3) \quad \frac{1}{p} \begin{pmatrix} \mathbf{z}_1^* \mathbf{z}_1 & \mathbf{z}_1^* \mathbf{z}_2 \\ \mathbf{z}_2^* \mathbf{z}_1 & \mathbf{z}_2^* \mathbf{z}_2 \end{pmatrix} \longrightarrow I_2.$$

Therefore, any group of functionals defined by these two random vectors should be asymptotically independent of each other. We shall adopt this setup to explore the conjecture that \mathbf{U}_n is asymptotically Haar-distributed.

We consider \mathbf{x} and \mathbf{y} to be two p -vectors with an angle θ . Thereafter, we find two orthonormal vectors $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ such that

$$\mathbf{x} = \|\mathbf{x}\| \boldsymbol{\alpha}_1 \quad \text{and} \quad \mathbf{y} = \|\mathbf{y}\| (\boldsymbol{\alpha}_1 \cos \theta + \boldsymbol{\alpha}_2 \sin \theta).$$

By (1.2) and (1.3), we have

$$(1.4) \quad \mathbf{V}_n \mathbf{x} \sim p^{-1/2} \|\mathbf{x}\| \mathbf{z}_1 \quad \text{and} \quad \mathbf{V}_n \mathbf{y} \sim p^{-1/2} \|\mathbf{y}\| (\mathbf{z}_1 \cos \theta + \mathbf{z}_2 \sin \theta).$$

Let $\sigma > 0$ be a positive constant, we now consider the following three quantities:

$$(1.5) \quad \mathbf{x}^* (\mathbf{S}_n + \sigma I)^{-1} \mathbf{x}, \quad \mathbf{x}^* (\mathbf{S}_n + \sigma I)^{-1} \mathbf{y} \quad \text{and} \quad \mathbf{y}^* (\mathbf{S}_n + \sigma I)^{-1} \mathbf{y}.$$

We hypothesize that if \mathbf{U}_n is asymptotically Haar-distributed and is asymptotically independent of $\boldsymbol{\Lambda}$, then the above three quantities should be asymptotically equivalent to

$$(1.6) \quad p^{-1} \|\mathbf{x}\|^2 \mathbf{z}_1^* (\boldsymbol{\Lambda} + \sigma I)^{-1} \mathbf{z}_1, \\ p^{-1} \|\mathbf{x}\| \|\mathbf{y}\| \mathbf{z}_1^* (\boldsymbol{\Lambda} + \sigma I)^{-1} (\mathbf{z}_1 \cos \theta + \mathbf{z}_2 \sin \theta) \quad \text{and} \\ p^{-1} \|\mathbf{y}\|^2 (\cos \theta \mathbf{z}_1 + \sin \theta \mathbf{z}_2)^* (\boldsymbol{\Lambda} + \sigma I)^{-1} (\mathbf{z}_1 \cos \theta + \mathbf{z}_2 \sin \theta),$$

respectively. We then proceed to investigate the stochastic processes related to these functionals. By using the Stieltjes transform of the sample covariance matrix, we have

$$p^{-1} \mathbf{z}_1^* (\mathbf{\Lambda} + \sigma I)^{-1} \mathbf{z}_1 \rightarrow m(\sigma) = -\frac{1 + \sigma - y - \sqrt{(1 + y + \sigma)^2 - 4y}}{2y\sigma} \quad \text{a.s.},$$

where $m(\sigma)$ is a solution to the quadratic equation

$$(1.7) \quad m(1 + \sigma - y + y\sigma m) - 1 = 0.$$

Here, the selection of $m(\sigma)$ is due to the fact that $m(\sigma) \rightarrow 0$ as $\sigma \rightarrow \infty$. By using the same argument, we conclude that

$$p^{-1} (\cos \theta \mathbf{z}_1 + \sin \theta \mathbf{z}_2)^* (\mathbf{\Lambda} + \sigma I)^{-1} (\mathbf{z}_1 \cos \theta + \mathbf{z}_2 \sin \theta) \rightarrow m(\sigma) \quad \text{a.s.}$$

Applying the results in Bai, Miao and Pan (2007), it can be easily shown that, for the complex case,

$$(1.8) \quad p^{-1/2} [\mathbf{z}_1^* (\mathbf{\Lambda} + \sigma I)^{-1} \mathbf{z}_1 - pm_n(\sigma)] \rightarrow N(0, W),$$

and for the real case, the limiting variance is $2W$, where $W = W(\sigma, \sigma)$, $m_n(\sigma)$ is $m(\sigma)$ with y replaced by y_n such that

$$m_n(\sigma) = -\frac{1 + \sigma - y_n - \sqrt{(1 + y_n + \sigma)^2 - 4y_n}}{2y_n\sigma},$$

$$y_n = p/n$$

and

$$W(\sigma_1, \sigma_2) = \frac{m(\sigma_1)m(\sigma_2)}{1 - y(1 - \sigma_1 y m(\sigma_1))(1 - \sigma_2 m(\sigma_2))}.$$

Here, the definitions of “real case” and “complex case” are given in Theorem 1 as stated in the next section. By the same argument, one could obtain a similar result such that

$$(1.9) \quad p^{-1/2} [(\mathbf{z}_1 \cos \theta + \mathbf{z}_2 \sin \theta)^* (\mathbf{\Lambda} + \sigma I)^{-1} (\mathbf{z}_1 \cos \theta + \mathbf{z}_2 \sin \theta) - pm_n(\sigma)].$$

We normalize the second term in (1.6) and, thereafter, derive the CLT for the joint distribution of all three terms stated in (1.6) after normalization. More notably, we establish some limiting behaviors of the processes defined by these normalized quantities.

2. Main results. Let $\mathbb{S} = \mathbb{S}_p$ be a subset of the unit p -sphere \mathbb{C}_1^p indexed by an m -dimensional hyper-cube $T = [0, 2\pi]^m$. For any m arbitrarily chosen orthogonal unit p -vectors $\mathbf{x}_1, \dots, \mathbf{x}_{m+1} \in \mathbb{C}_1^p$, we define

$$(2.1) \quad \mathbb{S} = \{ \mathbf{x}_n(\mathbf{t}) = \mathbf{x}_1 \cos t_1 + \mathbf{x}_2 \sin t_1 \cos t_2 + \dots + \mathbf{x}_m \sin t_1 \dots \sin t_{m-1} \cos t_m + \mathbf{x}_{m+1} \sin t_1 \dots \sin t_{m-1} \sin t_m, \mathbf{t} \in T \}.$$

If \mathbb{S} is chosen in the form of (2.1), then the inner product $\mathbf{x}_n(\mathbf{t}_1)^* \mathbf{x}_n(\mathbf{t}_2)$ is a function of \mathbf{t}_1 and \mathbf{t}_2 only (i.e., independent of n). Also, the norm of the difference (we call it norm difference in this paper) $\| \mathbf{x}_n(\mathbf{t}_1) - \mathbf{x}_n(\mathbf{t}_2) \|$ satisfies the Lipschitz condition. If the time index set is chosen arbitrarily, we could assume that the angle, $\vartheta_n(\mathbf{t}_1, \mathbf{t}_2)$, between $\mathbf{x}_n(\mathbf{t}_1)$ and $\mathbf{x}_n(\mathbf{t}_2)$ tends to a function of \mathbf{t}_1 and \mathbf{t}_2 whose norm difference satisfies the Lipschitz condition.

Thereafter, we define a stochastic process $\mathbf{Y}_n(\mathbf{u}, \sigma)$ mapping from the time index set $T \times T \times I$ to \mathbb{S} with $I = [\sigma_{10}, \sigma_{20}]$ ($0 < \sigma_{10} < \sigma_{20}$) such that

$$Y_n(\mathbf{u}, \sigma) = \sqrt{p}(\mathbf{x}_n(\mathbf{t}_1)^* (\mathbf{S}_n + \sigma I)^{-1} \mathbf{x}_n(\mathbf{t}_2) - \mathbf{x}_n(\mathbf{t}_1)^* \mathbf{x}_n(\mathbf{t}_2)) m_n(\sigma),$$

where $(\mathbf{u}, \sigma) = (\mathbf{t}_1, \mathbf{t}_2, \sigma) \in T \times T \times I$.

REMARK 1. If the sample covariance matrix \mathbf{S}_n is real, the vectors \mathbf{x}_n and \mathbf{y}_n will be real, and thus, the set \mathbb{S} has to be defined as a subset of unit sphere $\mathbb{R}_1^p = \{ \mathbf{x} \in \mathbb{R}^p, \|\mathbf{x}\| = 1 \}$. The time index can be similarly described for the complex case. In what follows, we shall implicitly use the convention for the real case.

We have the following theorem.

THEOREM 1. Assume that the entries of \mathbf{X} are i.i.d. with mean 0, variance 1, and finite fourth moments. If the variables are complex, we further assume $EX_{11}^2 = 0$ and $E|X_{11}|^4 = 2$, and refer to this case as the complex case. If the variables are real, we assume $EX_{11}^4 = 3$ and refer to it as the real case. Then, as $n \rightarrow \infty$, the process $Y_n(\mathbf{t}_1, \mathbf{t}_2, \sigma)$ converges weakly to a multivariate Gaussian process $Y(\mathbf{t}_1, \mathbf{t}_2, \sigma)$ with mean zero and variance-covariance function $EY(\mathbf{t}_1, \mathbf{t}_2, \sigma_1)Y(\mathbf{t}_3, \mathbf{t}_4, \sigma_2)$ satisfying

$$EY(\mathbf{t}_1, \mathbf{t}_2, \sigma_1)Y(\mathbf{t}_3, \mathbf{t}_4, \sigma_2) = \vartheta(\mathbf{t}_1, \mathbf{t}_4)\vartheta(\mathbf{t}_3, \mathbf{t}_2)W(\sigma_1, \sigma_2)$$

for the complex case and satisfying

$$EY(\mathbf{t}_1, \mathbf{t}_2, \sigma_1)Y(\mathbf{t}_3, \mathbf{t}_4, \sigma_2) = (\vartheta(\mathbf{t}_1, \mathbf{t}_4)\vartheta(\mathbf{t}_3, \mathbf{t}_2) + \vartheta(\mathbf{t}_1, \mathbf{t}_3)\vartheta(\mathbf{t}_4, \mathbf{t}_2))W(\sigma_1, \sigma_2)$$

for the real case where

$$W(\sigma_1, \sigma_2) = \frac{ym(\sigma_1)m(\sigma_2)}{1 - y(1 - \sigma_1m(\sigma_1))(1 - \sigma_2m(\sigma_2))}$$

and

$$\vartheta(\mathbf{t}, \mathbf{s}) = \lim \mathbf{x}_n^*(\mathbf{t})\mathbf{x}_n(\mathbf{s}).$$

We will provide the proof of this theorem in the next section. We note that Bai, Miao and Pan (2007) have proved that

$$\sqrt{p}[\mathbf{x}_n(\mathbf{t}_1)^*(\mathbf{S}_n + \sigma I)^{-1}\mathbf{x}_n(\mathbf{t}_1) - m_n(\sigma)] \longrightarrow N(0, W)$$

for the complex case and proved that the asymptotic variance is $2W$ for the real case.

More generally, if \mathbf{x} and \mathbf{y} are two orthonormal vectors, applying Theorem 1, we obtain the limiting distribution of the three quantities stated in (1.5) with normalization such that

$$(2.2) \quad \sqrt{p} \begin{pmatrix} \mathbf{x}^*(\mathbf{S}_n + \sigma I)^{-1}\mathbf{x} - m_n(\sigma) \\ \mathbf{x}^*(\mathbf{S}_n + \sigma I)^{-1}\mathbf{y} \\ \mathbf{y}^*(\mathbf{S}_n + \sigma I)^{-1}\mathbf{y} - m_n(\sigma) \end{pmatrix} \longrightarrow N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} W & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & W \end{pmatrix} \right)$$

for the complex case while the asymptotic covariance matrix is

$$\begin{pmatrix} 2W & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & 2W \end{pmatrix}$$

for the real case.

REMARK 2. This theorem shows that the three quantities stated in (1.5) are asymptotically independent of one another. It provides a stronger support to the conjecture that \mathbf{U}_n is asymptotically Haar-distributed than those established in the previous literature.

In many practical applications, such as wireless communications and electrical engineering [see, e.g., Evans and Tse (2000)], we are interested in extending the process $Y_n(\mathbf{u}, \sigma)$ defined on a region $T \times T \times D$ where D is a compact subset of the complex plane and is disjoint with the interval $[a, b]$, the support of the MPL. We can define a complex measure by putting complex mass $\mathbf{x}^*(\mathbf{t}_1)\mathbf{U}_n^*\mathbf{e}_j\mathbf{e}_j'\mathbf{U}_n\mathbf{y}(\mathbf{t}_2)$ at λ_j , the j th eigenvalue of \mathbf{S}_n , where \mathbf{e}_j is the p -vector with 1 in its j th entry and 0 otherwise. In this situation, the Stieltjes transform of this complex measure is

$$s_n(z) = \mathbf{x}^*(\mathbf{S}_n - z\mathbf{I})^{-1}\mathbf{y},$$

where $z = \mu + iv$ with $v \neq 0$. When considering the CLT of LSS associated with the complex measure defined above, we need to examine the limiting properties of the Stieltjes transforms, which lead to the extension of the process $Y_n(\mathbf{u}, \sigma)$ to $Y_n(\mathbf{u}, -z)$, where z is an index number in D .

If $\mathbf{x}^*\mathbf{y}$ is a constant (or has a limit, we still denote it as $\mathbf{x}^*\mathbf{y}$ for simplicity), it follows from Lemma 6 that

$$\mathbf{x}^*(\mathbf{S}_n - z\mathbf{I})^{-1}\mathbf{y} \longrightarrow \mathbf{x}^*\mathbf{y}s(z),$$

where

$$s(z) = \begin{cases} \frac{1 - z - y + \sqrt{(1 - z + y)^2 - 4y}}{2yz}, & \text{when } \Im(z) > 0, \\ \bar{s}(\bar{z}), & \text{when } \Im(z) < 0, \end{cases}$$

$$= \frac{1 - z - y + \operatorname{sgn}(\Im(z))\sqrt{(1 - z + y)^2 - 4y}}{2yz} \quad \text{if } \Im(z) \neq 0,$$

is the Stieltjes transform of MPL, in which, by convention, the square root \sqrt{z} takes the one with the positive imaginary part. When $z \neq 0$ is real, $s(z)$ is defined as the limit from the upper complex plane. By definition, $m(\sigma) = s(-\sigma + i0) = \lim_{v \downarrow 0} s(-\sigma + iv)$. In calculating the limit, we follow the conventional sign of the square root of a complex number that the real part of $\sqrt{(-\sigma + iv - 1 - y)^2 - 4y}$ should have the opposite sign of v , and thus

$$m(\sigma) = -\frac{1 + \sigma - y - \sqrt{(1 + y + \sigma)^2 - 4y}}{2y\sigma}.$$

Now, we are ready to extend the process $Y_n(\mathbf{u}, \sigma)$ to

$$Y_n(\mathbf{u}, z) = \sqrt{p}[\mathbf{x}_n^*(\mathbf{t}_1)(\mathbf{S}_n - z\mathbf{I})^{-1}\mathbf{x}_n(\mathbf{t}_2) - \mathbf{x}_n^*(\mathbf{t}_1)\mathbf{x}_n(\mathbf{t}_2)s(z, y_n)],$$

where $s(z, y_n)$ is the Stieltjes transform of the LSD of \mathbf{S}_n in which y is replaced by y_n . Here, $z = u + iv$ with $v > 0$ or $v < 0$. Thereby, we obtain the following theorem.

THEOREM 2. *Under the conditions of Theorem 1, the process $Y_n(\mathbf{u}, z)$ tends to a multivariate Gaussian process $Y(\mathbf{u}, z)$ with mean 0 and covariance function $E(Y(\mathbf{u}, z_1)Y(\mathbf{u}, z_2))$ satisfying*

$$(2.3) \quad E(Y(\mathbf{u}, z_1)Y(\mathbf{u}, z_2)) = \vartheta(\mathbf{t}_1, \mathbf{t}_4)\vartheta(\mathbf{t}_3, \mathbf{t}_2)W(z_1, z_2)$$

for the complex case and satisfying

$$(2.4) \quad E(Y(\mathbf{u}, z_1)Y(\mathbf{u}, z_2)) = (\vartheta(\mathbf{t}_1, \mathbf{t}_4)\vartheta(\mathbf{t}_3, \mathbf{t}_2) + \vartheta(\mathbf{t}_1, \mathbf{t}_3)\vartheta(\mathbf{t}_4, \mathbf{t}_2))W(z_1, z_2)$$

for the real case where

$$W(z_1, z_2) = \frac{ys(z_1)s(z_2)}{1 - y(1 + z_1s(z_1))(1 + z_2s(z_2))}.$$

Theorem 2 follows from Theorem 1 and Vitali lemma [see Lemma 2.3 of Bai and Silverstein (2004)] since both $Y(\mathbf{u}, z)$ and $Y_n(\mathbf{u}, z)$ are analytic functions when z is away from $[a, b]$, the support of MPL.

Suppose that $f(x)$ is analytic on an open region containing the interval $[a, b]$. We construct an LSS with respect to the complex measure as defined earlier; that is,

$$\sum_{j=1}^p f(\lambda_j) \mathbf{x}^*(\mathbf{t}_1) \mathbf{U}_n^* \mathbf{e}_j \mathbf{e}_j' \mathbf{U}_n \mathbf{y}(\mathbf{t}_2).$$

We then consider the normalized quantity

$$X_n(f) = \sqrt{p} \left(\sum_{j=1}^p f(\lambda_j) \mathbf{x}^*(\mathbf{t}_1) \mathbf{U}_n^* \mathbf{e}_j \mathbf{e}_j' \mathbf{U}_n \mathbf{y}(\mathbf{t}_2) - \mathbf{x}^*(\mathbf{t}_1) \mathbf{y}(\mathbf{t}_2) \int f(x) dF_{y_n}(x) \right), \tag{2.5}$$

where F_y is the standardized MPL. By applying the Cauchy formula

$$f(x) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z - x} dz,$$

where \mathcal{C} is a contour enclosing x , we obtain

$$X_n(f, \mathbf{u}) = -\frac{\sqrt{p}}{2\pi i} \left(\oint_{\mathcal{C}} [\mathbf{x}_n^*(\mathbf{t}_1) (\mathbf{S}_n - z\mathbf{I})^{-1} \mathbf{y}(\mathbf{t}_2) - \mathbf{x}_n^*(\mathbf{t}_1) \mathbf{y}(\mathbf{t}_2) s_n(z)] f(z) dz \right), \tag{2.6}$$

where \mathcal{C} is a contour enclosing the interval $[a, b]$, $\mathbf{u} = (\mathbf{t}_1, \mathbf{t}_2)$, and

$$s_n(z) = \frac{1 - z - y_n + \text{sgn}(\Im(z)) \sqrt{(1 - z + y_n)^2 - 4y_n}}{2y_n z}.$$

Thereafter, we obtain the following two corollaries.

COROLLARY 1. *Under the conditions of Theorem 1, for any k functions f_1, \dots, f_k analytic on an open region containing the interval $[a, b]$, the k -dimensional process*

$$(X_n(f_1, \mathbf{u}_1), \dots, X_n(f_k, \mathbf{u}_k))$$

tends to the k -dimensional stochastic multivariate Gaussian process with mean zero and covariance function satisfying

$$E(X(f, \mathbf{u}) X(g, \mathbf{v})) = -\frac{\theta}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} W(z_1, z_2) f(z_1) g(z_2) dz_1 dz_2,$$

where $\theta = \vartheta(\mathbf{t}_1, \mathbf{t}_4) \vartheta(\mathbf{t}_3, \mathbf{t}_2)$ for the complex case and $= \vartheta(\mathbf{t}_1, \mathbf{t}_4) \vartheta(\mathbf{t}_3, \mathbf{t}_2) + \vartheta'(\mathbf{t}_1, \mathbf{t}_3) \overline{\vartheta'}(\mathbf{t}_4, \mathbf{t}_2)$ for the real case. Here, \mathcal{C}_1 and \mathcal{C}_2 are two disjoint contours that enclose the interval $[a, b]$ such that the functions f_1, \dots, f_k are analytic inside and on them.

COROLLARY 2. *The covariance function in Corollary 1 can also be written as*

$$E(X(f, \mathbf{u})X(g, \mathbf{v})) = \theta \left(\int_a^b f(x)g(x) dF_y(x) - \int_a^b f(x) dF_y(x) \int_a^b g(x) dF_y(x) \right),$$

where θ has been defined in Corollary 1.

3. The proof of Theorem 1. To prove Theorem 1, by Lemma 7, it is sufficient to show that $Y_n(\mathbf{u}, \sigma) - EY_n(\mathbf{u}, \sigma)$ tends to the limit process $Y(\mathbf{u}, \sigma)$. We will first prove the property of the finite-dimensional convergence in Section 3.1 before proving the tightness property in Section 3.3. Throughout the paper, the limit is taken as $n \rightarrow \infty$.

3.1. *Finite-dimensional convergence.* Under the assumption of a finite fourth moment, we follow Bai, Miao and Pan (2007) to truncate the random variables X_{ij} at $\varepsilon_n \sqrt[4]{n}$ for all i and j in which $\varepsilon_n \rightarrow 0$ before renormalizing the random variables to have mean 0 and variance 1. Therefore, it is reasonable to impose an additional assumption that $|X_{ij}| \leq \varepsilon_n \sqrt[4]{n}$ for all i and j .

Suppose \mathbf{s}_j denotes the j th column of $\frac{1}{\sqrt{n}}X_n$. Let $\mathbf{A}(\sigma) = \mathbf{S}_n + \sigma \mathbf{I}$ and $\mathbf{A}_j(\sigma) = \mathbf{A}(\sigma) - \mathbf{s}_j \mathbf{s}_j^*$. Let \mathbf{x}_n and \mathbf{y}_n be any two vectors in \mathbb{C}_1^p . We define

$$\begin{aligned} \xi_j(\sigma) &= \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j - \frac{1}{n} \text{tr} \mathbf{A}_j^{-1}(\sigma), \\ \gamma_j &= \mathbf{s}_j^* \mathbf{A}_j^{-1} \mathbf{y}_n \mathbf{x}_n^* \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j - \frac{1}{n} \mathbf{x}_n^* \mathbf{A}_j^{-1}(\sigma) \mathbf{A}_j^{-1}(\sigma) \mathbf{y}_n, \\ \beta_j(\sigma) &= \frac{1}{1 + \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j}, \\ b_j(\sigma) &= \frac{1}{1 + n^{-1} \text{tr} \mathbf{A}_j^{-1}(\sigma)} \end{aligned}$$

and

$$\bar{b} = \frac{1}{1 + n^{-1} E \text{tr} \mathbf{A}^{-1}(\sigma)}.$$

We also define the σ -field $\mathcal{F}_j = \sigma(\mathbf{s}_1, \dots, \mathbf{s}_j)$. We denote by $E_j(\cdot)$ the conditional expectation when \mathcal{F}_j is given. By convention, E_0 denotes the unconditional expectation.

Using the martingale decomposition, we have

$$\begin{aligned}
 \mathbf{A}^{-1}(\sigma) - \mathbf{E}\mathbf{A}^{-1}(\sigma) &= \sum_{j=1}^n (\mathbf{E}_j - \mathbf{E}_{j-1})[\mathbf{A}^{-1}(\sigma) - \mathbf{A}_k^{-1}(\sigma)] \\
 (3.1) \qquad \qquad \qquad &= \sum_{j=1}^n (\mathbf{E}_j - \mathbf{E}_{j-1})\beta_j \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 Y_n(\mathbf{u}, \sigma) &= \sqrt{p} \sum_{j=1}^n (\mathbf{E}_j - \mathbf{E}_{j-1}) \mathbf{x}(\mathbf{t}_1)^* [\mathbf{A}^{-1}(\sigma) - \mathbf{A}_k^{-1}(\sigma)] \mathbf{x}(\mathbf{t}_2) \\
 &= \sqrt{p} \sum_{j=1}^n (\mathbf{E}_j - \mathbf{E}_{j-1}) \beta_j \mathbf{x}(\mathbf{t}_1)^* \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma) \mathbf{x}(\mathbf{t}_2).
 \end{aligned}$$

Consider the K -dimensional distribution of $\{Y_n(\mathbf{u}_1, \sigma_1), \dots, Y_n(\mathbf{u}_K, \sigma_K)\}$ where $(\mathbf{u}_i, \sigma_i) = (\mathbf{t}_{i1}, \mathbf{t}_{i2}, \sigma_i) \in T \times T \times I$. Invoking Lemma 3, we will have

$$\sum_{i=1}^K a_i (Y_n(\mathbf{u}_i, \sigma_i) - \mathbf{E}Y_n(\mathbf{u}_i, \sigma_i)) \Rightarrow N(0, \boldsymbol{\alpha}' \Sigma \boldsymbol{\alpha})$$

for any constants $a_i, i = 1, \dots, K$, where

$$\boldsymbol{\alpha} = (a_1, \dots, a_K)'$$

and

$$\Sigma_{ij} = \mathbf{E}Y(\mathbf{t}_{i1}, \mathbf{t}_{i2}, \sigma_i)Y(\mathbf{t}_{j1}, \mathbf{t}_{j2}, \sigma_j) = \vartheta(\mathbf{t}_{i1}, \mathbf{t}_{j2})\vartheta(\mathbf{t}_{j1}, \mathbf{t}_{i2})W(\sigma_i, \sigma_j)$$

for the complex case and

$$\Sigma_{ij} = (\vartheta(\mathbf{t}_{i1}, \mathbf{t}_{j2})\vartheta(\mathbf{t}_{j1}, \mathbf{t}_{i2}) + \vartheta(\mathbf{t}_{i1}, \mathbf{t}_{j1})\vartheta(\mathbf{t}_{j2}, \mathbf{t}_{i2}))W(\sigma_i, \sigma_j)$$

for the real case.

To this end, we will verify the Liapounov condition and calculate the asymptotic covariance matrix Σ (see Lemma 3) in the next subsections.

3.1.1. *Verification of Liapounov's condition.* By (3.1), we have

$$\begin{aligned}
 (3.2) \qquad \sum_{i=1}^K a_i (Y_n(\sigma_i) - \mathbf{E}Y_n(\sigma_i)) \\
 &= \sqrt{p} \sum_{j=1}^n (\mathbf{E}_j - \mathbf{E}_{j-1}) \sum_{i=1}^K (a_i \beta_j (\mathbf{x}^*(\mathbf{t}_{i1}) \mathbf{A}_j^{-1}(\sigma_i) \mathbf{s}_j \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma_i) \mathbf{x}(\mathbf{t}_{i2}))).
 \end{aligned}$$

The Liapounov condition with power index 4 follows by verifying that

$$(3.3) \quad p^2 \sum_{j=1}^n \mathbb{E} \left| \sum_{i=1}^K a_i \beta_j \mathbf{x}^*(\mathbf{t}_{i1}) \mathbf{A}_j^{-1}(\sigma_i) \mathbf{s}_j \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma_i) \mathbf{x}(\mathbf{t}_{i2}) \right|^4 \longrightarrow 0.$$

The limit (3.3) holds if one can prove that, for any $\mathbf{x}_n, \mathbf{y}_n \in \mathbb{C}_1^p$,

$$(3.4) \quad p^2 \sum_{j=1}^n \mathbb{E} |\beta_j \mathbf{x}_n^* \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma) \mathbf{y}_n|^4 \longrightarrow 0.$$

To do this, applying Lemma 2.7 of Bai and Silverstein (1998), for any $q \geq 2$, we get

$$(3.5) \quad \begin{cases} \max_j \mathbb{E} |\mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma) \mathbf{y}_n \mathbf{x}_n^* \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j|^q = O(n^{-1-q/2}), \\ \max_j \mathbb{E} |\gamma_j(\sigma)|^q = O(n^{-1-q/2}) \quad \text{and} \\ \max_j \mathbb{E} |\xi_j(\sigma)|^q = O(n^{-q/2}). \end{cases}$$

When $q > 2$, the $O(\cdot)$ can be replaced by $o(\cdot)$ in the first two inequalities. The assertion in (3.4) will then easily follow from the estimations in (3.5) and the observation that $|\beta_j(\sigma)| < 1$.

3.1.2. *Simplification of $Y_n(\mathbf{u}) - \mathbb{E}Y_n(\mathbf{u})$.* For any $\mathbf{x}_n, \mathbf{y}_n \in \mathbb{C}_1^p$, from (3.1), we have

$$(3.6) \quad \begin{aligned} & \mathbf{x}_n^* \mathbf{A}^{-1}(\sigma) \mathbf{y}_n - \mathbb{E} \mathbf{x}_n^* \mathbf{A}^{-1}(\sigma) \mathbf{y}_n \\ &= \sum_{j=1}^n (\bar{b} \mathbb{E}_j \gamma_j + \mathbb{E}_j (b_j - \bar{b}) \gamma_j \\ & \quad + (\mathbb{E}_j - \mathbb{E}_{j-1}) b_j(\sigma) \beta_j(\sigma) \xi_j(\sigma) \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma) \mathbf{y}_n \mathbf{x}_n^* \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j). \end{aligned}$$

For the third term on the right-hand side of (3.6), applying (3.5), we have

$$\begin{aligned} & \mathbb{E} \left| \sqrt{p} \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) b_j(\sigma) \beta_j(\sigma) \xi_j(\sigma) \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma) \mathbf{y}_n \mathbf{x}_n^* \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j \right|^2 \\ &= p \sum_{j=1}^n \mathbb{E} |(\mathbb{E}_j - \mathbb{E}_{j-1}) b_j(\sigma) \beta_j(\sigma) \xi_j(\sigma) \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma) \mathbf{y}_n \mathbf{x}_n^* \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j|^2 \\ &\leq p \sum_{j=1}^n (\mathbb{E} |\xi_j(\sigma)|^4 \mathbb{E} |\mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma) \mathbf{y}_n \mathbf{x}_n^* \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j|^4)^{1/2} = o(n^{-1/2}). \end{aligned}$$

For the second term on the right-hand side of (3.6), we have

$$\begin{aligned} & \mathbb{E} \left| \sqrt{p} \sum_{j=1}^n \mathbb{E}_j (b_j(\sigma) - \bar{b}(\sigma)) \gamma_j(\sigma) \right|^2 \\ & \leq p \sum_{j=1}^n (\mathbb{E} |b_j(\sigma) - \bar{b}(\sigma)|^4 \mathbb{E} |\gamma_j(\sigma)|^4)^{1/2} \\ & = o(n^{-3/2}) \cdot \left(\max_j \mathbb{E} |\text{tr} \mathbf{A}_j^{-1}(\sigma) - \text{tr} \mathbf{A}^{-1}(\sigma)|^4 \right)^{1/2} \\ & = o(n^{-1/2}), \end{aligned}$$

where the last step follows from applying the martingale decomposition and the Burkholder inequality and using the fact that

$$|\text{tr} \mathbf{A}_j^{-1}(\sigma) - \text{tr} \mathbf{A}^{-1}(\sigma)| \leq 1/\sigma$$

and

$$\mathbb{E} |\text{tr} \mathbf{A}^{-1}(\sigma) - \text{tr} \mathbf{A}^{-1}(\sigma)|^4 = O(n^2).$$

Thus, we conclude that

$$(3.7) \quad \sqrt{p}(\mathbf{x}_n^* \mathbf{A}^{-1}(\sigma) \mathbf{y}_n - \mathbb{E} \mathbf{x}_n^* \mathbf{A}^{-1}(\sigma) \mathbf{y}_n) = \sqrt{p} \sum_{j=1}^n \bar{b} \mathbb{E}_j \gamma_j + o_p(1).$$

3.2. *Asymptotic covariances.* To compute Σ , by the limiting property in (3.7), we only need to compute the limit

$$v_{i,j} = \lim p \sum_{k=1}^n \bar{b}(\sigma_i) \bar{b}(\sigma_j) \mathbb{E}_{k-1} \mathbb{E}_k \gamma_k(\mathbf{t}_{i1}, \mathbf{t}_{i2}, \sigma_i) \mathbb{E}_k \gamma_k(\mathbf{t}_{j1}, \mathbf{t}_{j2}, \sigma_j),$$

in which, for any $i, k = 1, \dots, K$, we have

$$\begin{aligned} \gamma_k(\mathbf{t}_{i1}, \mathbf{t}_{i2}, \sigma_i) &= \mathbf{s}_k^* \mathbf{A}_k^{-1}(\sigma_i) \mathbf{x}(\mathbf{t}_{i2}) \mathbf{x}^*(\mathbf{t}_{i1}) \mathbf{A}_k^{-1}(\sigma_i) \mathbf{s}_k \\ &\quad - \frac{1}{n} \mathbf{x}^*(\mathbf{t}_{i1}) \mathbf{A}_k^{-1}(\sigma_i) \mathbf{A}_k^{-1}(\sigma_i) \mathbf{x}(\mathbf{t}_{i2}). \end{aligned}$$

By Lemma 4, we obtain $\bar{b}(\sigma) \rightarrow b(\sigma) = 1/(1 + ym(\sigma))$. Thus, we only need to calculate

$$(3.8) \quad v_{i,j} = \lim p \sum_{k=1}^n b(\sigma_i) b(\sigma_j) \mathbb{E}_{k-1} \mathbb{E}_k \gamma_k(\mathbf{t}_{i1}, \mathbf{t}_{i2}, \sigma_i) \mathbb{E}_k \gamma_k(\mathbf{t}_{j1}, \mathbf{t}_{j2}, \sigma_j).$$

For simplicity, we will use $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, \sigma_1$ and σ_2 to denote $\mathbf{x}(\mathbf{t}_{i1}), \mathbf{x}(\mathbf{t}_{i2}), \mathbf{x}(\mathbf{t}_{j1}), \mathbf{x}(\mathbf{t}_{j2}), \sigma_i$ and σ_j . For $\mathbf{X} = (X_1, \dots, X_p)'$ of i.i.d. entries with mean 0 and variance 1, and $\mathbf{A} = (\mathbf{A}_{ij})$ and $\mathbf{B} = (\mathbf{B}_{ij})$ to be Hermitian matrices, the following

equality holds:

$$\begin{aligned} & \mathbf{E}(\mathbf{X}^* \mathbf{A} \mathbf{X} - \text{tr} \mathbf{A})(\mathbf{X}^* \mathbf{B} \mathbf{X} - \text{tr} \mathbf{B}) \\ &= \text{tr} \mathbf{A} \mathbf{B} + |\mathbf{E}X_1^2|^2 \text{tr} \mathbf{A} \mathbf{B}^T + \sum \mathbf{A}_{ii} \mathbf{B}_{ii} (\mathbf{E}|X_1|^4 - 2 - |\mathbf{E}X_1^2|^2). \end{aligned}$$

Using this equality, we get

$$(3.9) \quad \begin{aligned} \nu &= \lim \frac{pb(\sigma_1)b(\sigma_2)}{n^2} \sum_{k=1}^n \text{tr}(\mathbf{E}_k \mathbf{A}_k^{-1}(\sigma_1) \mathbf{y} \mathbf{x}^* \mathbf{A}_k^{-1}(\sigma_1)) \\ & \quad \times \mathbf{E}_k \mathbf{A}_k^{-1}(\sigma_2) \mathbf{v} \mathbf{u}^* \mathbf{A}_k^{-1}(\sigma_2) \end{aligned}$$

for the complex case and obtain

$$(3.10) \quad \begin{aligned} \nu &= \lim \frac{pb(\sigma_1)b(\sigma_2)}{n^2} \sum_{k=1}^n \text{tr}(\mathbf{E}_k \mathbf{A}_k^{-1}(\sigma_1) \mathbf{y} \mathbf{x}^* \mathbf{A}_k^{-1}(\sigma_1)) \\ & \quad \times \mathbf{E}_k \mathbf{A}_k^{-1}(\sigma_2) (\mathbf{v} \mathbf{u}^* + \mathbf{u} \mathbf{v}^*) \mathbf{A}_k^{-1}(\sigma_2) \end{aligned}$$

for the real case.

One could easily calculate the limit in (3.9) by applying the method used in [Bai, Miao and Pan \(2007\)](#) and by using the proof of their equation (4.7). Therefore, we only need to calculate the limit of

$$(3.11) \quad \begin{aligned} & \frac{yb(\sigma_1)b(\sigma_2)}{n} \sum_{k=1}^n \text{tr} \mathbf{E}_k (\mathbf{A}_k^{-1}(\sigma_1) \mathbf{y} \mathbf{x}^* \mathbf{A}_k^{-1}(\sigma_1)) \mathbf{E}_k (\mathbf{A}_k^{-1}(\sigma_2) \mathbf{v} \mathbf{u}^* \mathbf{A}_k^{-1}(\sigma_2)) \\ &= \frac{yb(\sigma_1)b(\sigma_2)}{n} \sum_{k=1}^n \mathbf{E}_k (\mathbf{x}^* \mathbf{A}_k^{-1}(\sigma_1) \check{\mathbf{A}}_k^{-1}(\sigma_2) \mathbf{v}) (\mathbf{u}^* \check{\mathbf{A}}_k^{-1}(\sigma_2) \mathbf{A}_k^{-1}(\sigma_1) \mathbf{y}), \end{aligned}$$

where $\check{\mathbf{A}}_k^{-1}(z_2)$ is similarly defined as $\mathbf{A}_k^{-1}(\sigma_2)$ by using $(\mathbf{s}_1, \dots, \mathbf{s}_{k-1}, \check{\mathbf{s}}_{k+1}, \dots, \check{\mathbf{s}}_n)$ and by using the fact that $\check{\mathbf{s}}_{k+1}, \dots, \check{\mathbf{s}}_n$ are i.i.d. copies of $\mathbf{s}_{k+1}, \dots, \mathbf{s}_n$.

Following the arguments in [Bai, Miao and Pan \(2007\)](#), we only have to replace their vectors \mathbf{x}_n and \mathbf{x}_n^* connected with $\mathbf{A}_k^{-1}(\sigma_1)$ by \mathbf{y} and \mathbf{x}^* and replace those connected with $\mathbf{A}_k^{-1}(\sigma_2)$ by \mathbf{v} and \mathbf{u}^* , respectively. Going along with the same lines from their (4.7) to (4.23), we obtain

$$(3.12) \quad \begin{aligned} & \mathbf{E}_k \mathbf{x}^* \mathbf{A}_k^{-1}(\sigma_1) \check{\mathbf{A}}_k^{-1}(\sigma_2) \mathbf{v} \mathbf{u}^* \check{\mathbf{A}}_k^{-1}(\sigma_2) \mathbf{A}_k^{-1}(\sigma_1) \mathbf{y} \\ & \quad \times \left[1 - \frac{k-1}{n} \bar{b}(\sigma_1) \bar{b}(\sigma_2) \frac{1}{n} \text{tr} T^{-1}(\sigma_2) T^{-1}(\sigma_1) \right] \\ &= \mathbf{x}^* T^{-1}(\sigma_1) T^{-1}(\sigma_2) \mathbf{v} \mathbf{u}^* T^{-1}(\sigma_2) T^{-1}(\sigma_1) \mathbf{y} \\ & \quad \times \left(1 + \frac{k-1}{n} \bar{b}(\sigma_1) \bar{b}(\sigma_2) \frac{1}{n} \mathbf{E}_{k-1} \text{tr}(\mathbf{A}_k^{-1}(\sigma_1) \check{\mathbf{A}}_k^{-1}(\sigma_2)) \right) + o_p(1) \end{aligned}$$

and

$$E_k \operatorname{tr}(\mathbf{A}_k^{-1}(\sigma_1)\check{\mathbf{A}}_k^{-1}(\sigma_2)) = \frac{\operatorname{tr}(T^{-1}(\sigma_1)T^{-1}(\sigma_2)) + o_p(1)}{1 - ((k-1)/n^2)b(\sigma_1)b(\sigma_2)\operatorname{tr}(T^{-1}(\sigma_1)T^{-1}(\sigma_2))},$$

where

$$T(\sigma) = \left(\sigma + \frac{n-1}{n}b(\sigma) \right) \mathbf{I}.$$

We then obtain

$$\begin{aligned} (3.13) \quad d(\sigma_1, \sigma_2) &:= \lim \bar{b}(\sigma_1)\bar{b}(\sigma_2) \frac{1}{n} \operatorname{tr}(T^{-1}(\sigma_1)T^{-1}(\sigma_2)) \\ &= \frac{yb(\sigma_1)b(\sigma_2)}{(\sigma_1 + b(\sigma_1))(\sigma_2 + b(\sigma_2))} \end{aligned}$$

and

$$\begin{aligned} (3.14) \quad h(\sigma_1, \sigma_2) &:= b(\sigma_1)b(\sigma_2)\mathbf{x}^*T^{-1}(\sigma_1)T^{-1}(\sigma_2)\mathbf{v}\mathbf{u}^*T^{-1}(\sigma_2)T^{-1}(\sigma_1)\mathbf{y} \\ &= \frac{\mathbf{x}^*\mathbf{v}\mathbf{u}^*\mathbf{y}b(\sigma_1)b(\sigma_2)}{(\sigma_1 + b(\sigma_1))^2(\sigma_2 + b(\sigma_2))^2}. \end{aligned}$$

From (3.13) and (3.14), we get

The right-hand side of (3.11)

$$\begin{aligned} &\xrightarrow{\text{a.s.}} yh(\sigma_1, \sigma_2) \left(\int_0^1 \frac{1}{(1 - td(\sigma_1, z_2))} dt + \int_0^1 \frac{td(\sigma_1, \sigma_2)}{(1 - td(\sigma_1, \sigma_2))^2} dt \right) \\ &= \frac{yh(\sigma_1, \sigma_2)}{1 - d(\sigma_1, \sigma_2)} \\ &= \frac{y\mathbf{x}^*\mathbf{v}\mathbf{u}^*\mathbf{y}b(\sigma_1)b(\sigma_2)}{(\sigma_1 + b(\sigma_1))(\sigma_2 + b(\sigma_2))[(\sigma_1 + b(\sigma_1))(\sigma_2 + b(\sigma_2)) - yb(\sigma_1)b(\sigma_2)]}. \end{aligned}$$

In addition, from (1.7), we establish

$$(3.15) \quad \frac{1}{\sigma + b(\sigma)} = m(\sigma) \quad \text{and} \quad \frac{b(\sigma)}{\sigma + b(\sigma)} = 1 - \sigma m(\sigma).$$

Applying these identities, the limit of (3.11) can be simplified to

$$\mathbf{x}^*\mathbf{v}\mathbf{u}^*\mathbf{y}W(\sigma_1, \sigma_2),$$

where

$$W(\sigma_1, \sigma_2) = \frac{ym(\sigma_1)m(\sigma_2)}{1 - y(1 - \sigma_1m(\sigma_1))(1 - \sigma_2m(\sigma_2))}.$$

By symmetry, the limit of (3.10) for the real case can also be simplified to

$$(\mathbf{x}^* \mathbf{v} \mathbf{u}^* \mathbf{y} + \mathbf{x}^* \mathbf{u} \mathbf{v}^* \mathbf{y}) W(\sigma_1, \sigma_2).$$

Therefore, for the complex case, the covariance function of the process $Y(\mathbf{t}_{i1}, \mathbf{t}_{i2}, \sigma)$ is

$$EY(\mathbf{t}_{i1}, \mathbf{t}_{i2}, \sigma_1)Y(\mathbf{t}_{j1}, \mathbf{t}_{j2}, \sigma_2) = \vartheta(\mathbf{t}_{i1}, \mathbf{t}_{j2})\vartheta(\mathbf{t}_{i2}, \mathbf{t}_{j1})W(\sigma_1, \sigma_2),$$

while, for the real case, it is

$$\begin{aligned} EY(\mathbf{t}_{i1}, \mathbf{t}_{i2}, \sigma_1)Y(\mathbf{t}_{j1}, \mathbf{t}_{j2}, \sigma_2) \\ = (\vartheta(\mathbf{t}_{i1}, \mathbf{t}_{j2})\vartheta(\mathbf{t}_{j1}, \mathbf{t}_{i2}) + \vartheta(\mathbf{t}_{i1}, \mathbf{t}_{j1})\vartheta(\mathbf{t}_{j2}, \mathbf{t}_{i2}))W(\sigma_1, \sigma_2). \end{aligned}$$

3.3. Tightness.

THEOREM 3. *Under the conditions in Theorem 1, the sequence of $Y_n(\mathbf{u}, \sigma) - E(Y_n(\mathbf{u}, \sigma))$ is tight.*

For ease reference on the tightness, we quote a proposition from page 267 of Loève (1978) as follows.

PROPOSITION 1 (Tightness criterion). *The sequence $\{P_n\}$ of probability measure is tight if and only if:*

$$(i) \quad \sup_n P_n(x : |x(0)| > c) \longrightarrow 0 \quad \text{as } c \rightarrow \infty$$

and, for every $\varepsilon > 0$, as $\delta \rightarrow 0$, we have

$$(ii) \quad P_n(\omega_x(\delta) > \varepsilon) \longrightarrow 0,$$

where δ -oscillation is defined by

$$\omega_x(\delta) = \sup_{|\mathbf{t}-\mathbf{s}|<\delta} |x(\mathbf{t}) - x(\mathbf{s})|.$$

To complete the proof of the tightness for Theorem 3, we note that condition (i) in Proposition 1 is a consequence of finite-dimensional convergence which has been proved in the previous section. To demonstrate condition (ii) in Proposition 1, we will use the two lemmas given below. Therefore, to complete the proof of Theorem 3, by Proposition 1 and Lemma 1, it is sufficient to verify that

$$(3.16) \quad \sup_{\mathbf{u}_1, \mathbf{u}_2 \in T \times T} E \left| \frac{Y_n(\mathbf{u}_1) - Y_n(\mathbf{u}_2)}{\|(\mathbf{u}_1, \sigma_1) - (\mathbf{u}_2, \sigma_2)\|} \right|^{4m+2} < \infty.$$

This inequality will be proved in Lemma 2 stated below.

LEMMA 1. Suppose that $X_n(\mathbf{t})$ is a sequence of stochastic processes, defined on an m -dimensional time domain T , whose paths are continuous and Lipschitz; that is, there is a random variable $R = R_n$ such that

$$|X_n(\mathbf{t}) - X_n(\mathbf{s})| \leq R \|\mathbf{t} - \mathbf{s}\|.$$

If there is an $\alpha > m$ such that

$$(3.17) \quad \mathbb{E}|R|^\alpha < \infty,$$

then, for any fixed $\varepsilon > 0$, we have

$$(3.18) \quad \lim_{\delta \downarrow 0} \mathbb{P}_n(\omega_x(\delta) > \varepsilon) = 0.$$

PROOF. Without loss of generality, we assume that $T = [0, M]^m$. First, for any given $\varepsilon > 0$ and $\delta > 0$, we choose an integer K such that $MK^{-1} < \delta$ and $2^\alpha K^{m-\alpha} < 1/2$. For each $\ell = 1, 2, \dots$, we define

$$t_i(j, \ell) = \frac{jM}{K^\ell}, \quad j = 1, \dots, K^\ell.$$

Denoting by $\mathbf{t}(\mathbf{j}, \ell)$, $\mathbf{j} = (j_1, \dots, j_m)$, the vector whose i th entry is $t_i(j_i, \ell)$. Then, we have

$$\begin{aligned} & \mathbb{P}_n(\omega_x(\delta) \geq \varepsilon) \\ & \leq 2\mathbb{P}\left(\sup_{\mathbf{j}, 1} \sup_{|\mathbf{t}-\mathbf{t}(\mathbf{j}, 1)| \leq 2M\sqrt{m}/K} |X_n(\mathbf{t}) - X_n(\mathbf{t}(\mathbf{j}, 1))| \geq \varepsilon/2\right) \\ & \leq \sum_{\ell=1}^L \sum_{(\mathbf{j}, \ell+1)} 2\mathbb{P}(|X_n(\mathbf{t}(\mathbf{j}^*, \ell)) - X_n(\mathbf{t}(\mathbf{j}, \ell+1))| \geq 2^{-\ell-1}\varepsilon) \\ & \quad + 2\mathbb{P}\left(\sup_{\mathbf{t}(\mathbf{j}, L+1)} \sup_{\|\mathbf{t}-\mathbf{t}(\mathbf{j}, L+1)\| \leq 2\sqrt{m}M/K^L} |X_n(\mathbf{t}) - X_n(\mathbf{t}(\mathbf{j}, L+1))| \geq 2^{-L-2}\varepsilon\right) \\ & \leq \sum_{\ell=1}^{\infty} 2(K^\ell/M)^m \left(\frac{2\sqrt{m}M}{\varepsilon 2^{-\ell-1}K^\ell}\right)^\alpha \mathbb{E}|R|^\alpha \\ & = 2^{2+3\alpha} \varepsilon^{-\alpha} m^{\alpha/2} (M/K)^{\alpha-m} \mathbb{E}|R|^\alpha \\ & = 2^{2+3\alpha} \varepsilon^{-\alpha} m^{\alpha/2} \delta^{\alpha-m} \mathbb{E}|R|^\alpha \rightarrow 0 \quad \text{as } \delta \rightarrow 0, \end{aligned}$$

where the summation $\sum_{(\mathbf{j}, \ell+1)}$ runs over all possibilities of $j_i \leq K^{\ell+1}$, and $\mathbf{t}(\mathbf{j}^*, \ell)$ is the $\mathbf{t}(\mathbf{j}, \ell)$ vector closest to $\mathbf{t}(\mathbf{j}, \ell+1)$. Here, to prove the first inequality, one only needs to choose $\mathbf{t}(\mathbf{j}, 1)$ as the center of the first layer hypercube in which $\frac{1}{2}(\mathbf{t} + \mathbf{s})$ lies. The proof of the second inequality could be easily obtained by applying a simple induction. In the proof of the third inequality, the first term follows by the Chebyshev inequality and the fact that

$$|X_n(\mathbf{t}(\mathbf{j}^*, \ell)) - X_n(\mathbf{t}(\mathbf{j}, \ell+1))| \leq R \|\mathbf{t}(\mathbf{j}^*, \ell) - \mathbf{t}(\mathbf{j}, \ell+1)\| \leq R\sqrt{m}M/K^\ell.$$

At the same time, the second term tends to 0 for all fixed n when $L \rightarrow \infty$ because

$$\begin{aligned} & \mathbb{P}\left(\sup_{\mathbf{t}(\mathbf{j}, L+1)} \sup_{\|\mathbf{t}-\mathbf{t}(\mathbf{j}, L+1)\| \leq 2M\sqrt{m}K^{-L-2}} |X_n(\mathbf{t}) - X_n(\mathbf{t}(\mathbf{j}, L+1))| \geq 2^{-L-2}\varepsilon\right) \\ & \leq \mathbb{P}(|R| \geq (K/2)^{L+2}\varepsilon/2M\sqrt{m}) \rightarrow 0. \end{aligned}$$

Thus, the proof of the lemma is complete. \square

LEMMA 2. *Under the conditions of Theorem 1, the property in (3.16) holds for any m .*

PROOF. For simplicity, we only prove the lemma for a general m instead of $4m + 2$. For a constant L , we have

$$\begin{aligned} & \mathbb{E} \left| \frac{Y_n(\mathbf{u}_1, \sigma_1) - Y_n(\mathbf{u}_2, \sigma_2) - \mathbb{E}(Y_n(\mathbf{u}_1, \sigma_1) - Y_n(\mathbf{u}_2, \sigma_2))}{\|\mathbf{u}_1 - \mathbf{u}_2\| + |\sigma_1 - \sigma_2|} \right|^m \\ & \asymp p^{m/2} \mathbb{E} \left| \frac{\mathbf{x}_n(\mathbf{t}_1)^* \mathbf{A}^{-1}(\sigma_1) \mathbf{x}_n(\mathbf{t}_2) - \mathbb{E} \mathbf{x}_n(\mathbf{t}_1)^* \mathbf{A}^{-1}(\sigma_1) \mathbf{x}_n(\mathbf{t}_2)}{\|\mathbf{t}_1 - \mathbf{t}_3\| + \|\mathbf{t}_2 - \mathbf{t}_4\| + |\sigma_1 - \sigma_2|} \right. \\ & \quad \left. - \frac{(\mathbf{x}_n(\mathbf{t}_3)^* \mathbf{A}^{-1}(\sigma_2) \mathbf{x}_n(\mathbf{t}_4) - \mathbb{E} \mathbf{x}_n(\mathbf{t}_3)^* \mathbf{A}^{-1}(\sigma_2) \mathbf{x}_n(\mathbf{t}_4))}{\|\mathbf{t}_1 - \mathbf{t}_3\| + \|\mathbf{t}_2 - \mathbf{t}_4\| + |\sigma_1 - \sigma_2|} \right|^m \\ & \leq Ln^{m/2} \{ \mathbb{E} |(\mathbf{x}_n(\mathbf{t}_1) - \mathbf{x}_n(\mathbf{t}_3))^* \mathbf{A}^{-1}(\sigma_1) \mathbf{x}_n(\mathbf{t}_2) \\ & \quad - \mathbb{E}(\mathbf{x}_n(\mathbf{t}_1) - \mathbf{x}_n(\mathbf{t}_3))^* \mathbf{A}^{-1}(\sigma_1) \mathbf{x}_n(\mathbf{t}_2)| (\|\mathbf{t}_1 - \mathbf{t}_3\|)^{-1} \|^m \\ & \quad + \mathbb{E} |(\mathbf{x}_n^*(\mathbf{t}_3) \mathbf{A}^{-1}(\sigma_1) (\mathbf{x}_n(\mathbf{t}_2) - \mathbf{x}_n(\mathbf{t}_4)) \\ & \quad - \mathbb{E} \mathbf{x}_n^*(\mathbf{t}_3) \mathbf{A}^{-1}(\sigma_1) (\mathbf{x}_n(\mathbf{t}_2) - \mathbf{x}_n(\mathbf{t}_4)))| (\|\mathbf{t}_2 - \mathbf{t}_4\|)^{-1} \|^m \\ & \quad + \mathbb{E} |\mathbf{x}_n^*(\mathbf{t}_3) \mathbf{A}^{-1}(\sigma_1) \mathbf{A}^{-1}(\sigma_2) \mathbf{x}_n(\mathbf{t}_4) \\ & \quad - \mathbb{E} \mathbf{x}_n^*(\mathbf{t}_3) \mathbf{A}^{-1}(\sigma_1) \mathbf{A}^{-1}(\sigma_2) \mathbf{x}_n(\mathbf{t}_4)|^m \}, \end{aligned}$$

where $a \asymp b$ means a and b have the same order, that is, there exists a positive constant K such that $K^{-1}b < a < Kb$.

We note that $\|\mathbf{x}_n(\mathbf{t}_1) - \mathbf{x}_n(\mathbf{t}_3)\|/\|\mathbf{t}_1 - \mathbf{t}_3\| \leq 1$ or bounded for the general case. By applying the martingale decomposition in (3.1), the Burkholder inequality and the estimates in (3.5), we have

$$\begin{aligned} & n^{m/2} \mathbb{E} \left| \frac{(\mathbf{x}_n(\mathbf{t}_1) - \mathbf{x}_n(\mathbf{t}_3))^* \mathbf{A}^{-1}(\sigma_1) \mathbf{x}_n(\mathbf{t}_2) - \mathbb{E}(\mathbf{x}_n(\mathbf{t}_1) - \mathbf{x}_n(\mathbf{t}_3))^* \mathbf{A}^{-1}(\sigma_1) \mathbf{x}_n(\mathbf{t}_2)}{\|\mathbf{t}_1 - \mathbf{t}_3\|} \right|^m \\ & = O(1). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} & n^{m/2} \mathbb{E} \left| \frac{\mathbf{x}_n^*(\mathbf{t}_3) \mathbf{A}^{-1}(\sigma_1) (\mathbf{x}_n(\mathbf{t}_2) - \mathbf{x}_n(\mathbf{t}_4)) - \mathbb{E} \mathbf{x}_n^*(\mathbf{t}_3) \mathbf{A}^{-1}(\sigma_1) (\mathbf{x}_n(\mathbf{t}_2) - \mathbf{x}_n(\mathbf{t}_4))}{\|\mathbf{t}_2 - \mathbf{t}_4\|} \right|^m \\ & = O(1). \end{aligned}$$

Using the martingale decomposition and the Burkholder inequality, we get

$$\begin{aligned}
 & n^{m/2} \mathbb{E} |\mathbf{x}_n^*(\mathbf{t}_3) \mathbf{A}^{-1}(\sigma_1) \mathbf{A}^{-1}(\sigma_2) \mathbf{x}_n(\mathbf{t}_4) - \mathbb{E} \mathbf{x}_n^*(\mathbf{t}_2) \mathbf{A}^{-1}(\sigma_1) \mathbf{A}^{-1}(\sigma_2) \mathbf{x}_n(\mathbf{t}_4)|^m \\
 & \leq L n^{m/2} \left[\sum_{k=1}^n \mathbb{E} |\mathbf{x}_n^*(\mathbf{t}_3) [\mathbf{A}^{-1}(\sigma_1) \mathbf{A}^{-1}(\sigma_2) - \mathbf{A}_k^{-1}(\sigma_1) \mathbf{A}_k^{-1}(\sigma_2)] \mathbf{x}_n(\mathbf{t}_4)|^m \right. \\
 & \quad \left. + \mathbb{E} \left(\sum_{k=1}^n \mathbb{E}_{k-1} |\mathbf{x}_n^*(\mathbf{t}_3) [\mathbf{A}^{-1}(\sigma_1) \mathbf{A}^{-1}(\sigma_2) \right. \right. \\
 & \quad \quad \left. \left. - \mathbf{A}_k^{-1}(\sigma_1) \mathbf{A}_k^{-1}(\sigma_2)] \mathbf{x}_n(\mathbf{t}_4)|^2 \right)^{m/2} \right] \\
 & = O(1),
 \end{aligned}$$

which follows from applying the following decomposition:

$$\begin{aligned}
 & \mathbf{A}^{-1}(\sigma_1) \mathbf{A}^{-1}(\sigma_2) - \mathbf{A}_k^{-1}(\sigma_1) \mathbf{A}_k^{-1}(\sigma_2) \\
 & = \beta(\sigma_1) \mathbf{A}_k^{-1}(\sigma_1) \mathbf{s}_k \mathbf{s}_k^* \mathbf{A}_k^{-1}(\sigma_1) \mathbf{A}_k^{-1}(\sigma_2) + \beta(\sigma_2) \mathbf{A}_k^{-1}(\sigma_1) \mathbf{A}_k^{-1}(\sigma_2) \mathbf{s}_k \mathbf{s}_k^* \mathbf{A}_k^{-1}(\sigma_2) \\
 & \quad + \beta(\sigma_k) \beta(\sigma_2) \mathbf{A}_k^{-1}(\sigma_1) \mathbf{s}_k \mathbf{s}_k^* \mathbf{A}_k^{-1}(\sigma_1) \mathbf{A}_k^{-1}(\sigma_2) \mathbf{s}_k \mathbf{s}_k^* \mathbf{A}_k^{-1}(\sigma_2)
 \end{aligned}$$

and thereafter employing the results in (3.5). Thus, condition (3.16) is verified. \square

4. Proof of Corollary 2. Applying the quadratic equation (1.7), we have

$$(4.1) \quad \sigma = \frac{1}{m} - \frac{1}{1 + ym}.$$

Making a difference of σ_1 and σ_2 , we obtain

$$\sigma_1 - \sigma_2 = \frac{m(\sigma_2) - m(\sigma_1)}{m(\sigma_1)m(\sigma_2)} - \frac{y(m(\sigma_2) - m(\sigma_1))}{(1 + ym(\sigma_1))(1 + ym(\sigma_2))}.$$

We also establish

$$(4.2) \quad \frac{m(\sigma_2) - m(\sigma_1)}{\sigma_1 - \sigma_2} = \frac{m(\sigma_1)m(\sigma_2)(1 + ym(\sigma_1))(1 + ym(\sigma_2))}{(1 + ym(\sigma_1))(1 + ym(\sigma_2)) - ym(\sigma_1)m(\sigma_2)}.$$

Finally, we conclude that

$$\begin{aligned}
 & \frac{m(\sigma_2) - m(\sigma_1)}{\sigma_1 - \sigma_2} - m(\sigma_1)m(\sigma_2) \\
 & = \frac{ym^2(\sigma_1)m^2(\sigma_2)}{(1 + ym(\sigma_1))(1 + ym(\sigma_2)) - ym(\sigma_1)m(\sigma_2)} = W(\sigma_1, \sigma_2)
 \end{aligned}$$

by noticing that $1 + ym(\sigma) = m(\sigma)/(1 - \sigma m(\sigma))$, an easy consequence of (4.1).

Furthermore, one could easily show that the left-hand side of the above equation is

$$\int_a^b \frac{dF_y(x)}{(x + \sigma_1)(x + \sigma_2)} - \int_a^b \frac{dF_y(x)}{x + \sigma_1} \int_a^b \frac{dF_y(x)}{x + \sigma_2}.$$

By using the unique extension of analytic functions, we have

$$W(z_1, z_2) = \int_a^b \frac{dF_y(x)}{(x - z_1)(x - z_2)} - \int_a^b \frac{dF_y(x)}{x - z_1} \int_a^b \frac{dF_y(x)}{x - z_2}.$$

Substituting this into Corollary 1, we complete the proof of Corollary 2.

APPENDIX

LEMMA 3 [Theorem 35.12 of Billingsley (1995)]. *Suppose that, for each n , $X_{n,1}, X_{n,2}, \dots, X_{n,r_n}$ is a real martingale difference sequence with respect to the increasing σ -field $\{\mathcal{F}_{n,j}\}$ having second moments. If, as $n \rightarrow \infty$,*

- (i) $\sum_{j=1}^{r_n} E(X_{n,j}^2 | \mathcal{F}_{n,j-1}) \xrightarrow{i.p.} \sigma^2$ and
- (ii) $\sum_{j=1}^{r_n} E(X_{n,j}^2 I_{(|Y_{n,j}| \geq \varepsilon)}) \rightarrow 0,$

where σ^2 is a positive constant and ε is an arbitrary positive number, then

$$\sum_{j=1}^{r_n} X_{n,j} \xrightarrow{\mathcal{D}} N(0, \sigma^2).$$

In what follows, \mathbf{s}_j , \mathbf{A}^{-1} and \mathbf{A}_j^{-1} are defined in Section 3 and \mathbf{M}_j and \mathbf{M} refer to any pair of matrices which are independent of \mathbf{s}_j .

LEMMA 4. *Under the conditions of Theorem 1, for any matrix M_j bounded in norm and independent of \mathbf{s}_j , we have*

$$(A.1) \quad \max_j \left| \frac{1}{n} (\mathbf{s}_j^* M_j \mathbf{s}_j - \text{tr } M_j) \right| \xrightarrow{a.s.} 0.$$

The proof of this lemma could be easily obtained by applying the truncation technique and invoking Lemma 2.7 of Bai and Silverstein (1998).

LEMMA 5. *Under the conditions of Theorem 1, for any $\mathbf{x}_n, \mathbf{y}_n \in \mathbb{C}_1^p$,*

$$(A.2) \quad \sup_j |\mathbf{x}_n^* \mathbf{A}^{-1} M \mathbf{y}_n - \mathbf{x}_n^* \mathbf{A}_j^{-1} M \mathbf{y}_n| \xrightarrow{a.s.} 0.$$

Similarly, for any matrix M with bounded norm and independent of \mathbf{s}_i , we have

$$(A.3) \quad \max_{i,j} |\mathbb{E}_j \mathbf{x}_n^* \mathbf{A}_j^{-1}(\sigma) M \mathbf{y}_n - \mathbb{E}_j \mathbf{x}_n^* \mathbf{A}_{ij}^{-1}(\sigma) M \mathbf{y}_n| \xrightarrow{a.s.} 0.$$

PROOF. Using

$$(A.4) \quad \mathbf{A}^{-1}(\sigma) = \mathbf{A}_j^{-1}(\sigma) - \mathbf{A}_j^{-1}(\sigma) \mathbf{s}_j \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma) \beta_j(\sigma),$$

we obtain

$$\begin{aligned} \sup_j |\mathbf{x}_n^* \mathbf{A}^{-1} M \mathbf{y}_n - \mathbf{x}_n^* \mathbf{A}_j^{-1} M \mathbf{y}_n| &\leq \sup_j \frac{1}{n} |\mathbf{x}_n^* \mathbf{A}_j^{-1} \mathbf{s}_j \mathbf{s}_j^* \mathbf{A}_j^{-1} M \mathbf{y}_n| \\ &= \sup_j \frac{1}{n} |\mathbf{x}_n^* \mathbf{A}_j^{-1} \mathbf{A}_j^{-1} M \mathbf{y}_n| + o(1), \end{aligned}$$

which, in turn, implies (A.2). Here, we adopt (A.1) in the last step above. The conclusion (A.3) can be proved in a similar way. \square

LEMMA 6. Under the conditions of Theorem 1, for any $\mathbf{x}_n, \mathbf{y}_n \in \mathbb{C}_1^p$, we have

$$(A.5) \quad \mathbf{x}_n^* \mathbf{A}^{-1}(\sigma) \mathbf{y}_n - \mathbf{x}_n^* \mathbf{y}_n m(\sigma) \xrightarrow{a.s.} 0$$

and

$$(A.6) \quad \max_j |\mathbb{E}_j \mathbf{x}_n^* \mathbf{A}_j^{-1}(\sigma) \mathbf{y}_n - \mathbf{x}_n^* \mathbf{y}_n m(\sigma)| \xrightarrow{a.s.} 0.$$

PROOF. By using the formula $\mathbf{A} = \sum_{j=1}^n \mathbf{s}_j \mathbf{s}_j^* + \sigma \mathbf{I}$ and multiplying \mathbf{x}_n^* from the left- and multiplying $\mathbf{A}^{-1} \mathbf{y}_n$ from the right-hand side of the equation, we obtain

$$\mathbf{x}_n^* \mathbf{A}^{-1}(\sigma) \mathbf{y}_n = \sigma^{-1} \mathbf{x}_n^* \mathbf{y}_n - \frac{1}{n\sigma} \sum_{j=1}^n \mathbf{x}_n^* \mathbf{s}_j \mathbf{s}_j^* \mathbf{A}_j^{-1}(\sigma) \beta_j(\sigma) \mathbf{y}_n.$$

As $\beta_j(\sigma) \xrightarrow{a.s.} b = \frac{1}{1+ym(\sigma)}$ uniformly in j , we apply Lemmas 4 and 5 and obtain

$$\mathbf{x}_n^* \mathbf{A}^{-1}(\sigma) \mathbf{y}_n = \sigma^{-1} \mathbf{x}_n^* \mathbf{y}_n - \sigma^{-1} \mathbf{x}_n^* \mathbf{A}^{-1}(\sigma) \mathbf{y}_n b(\sigma) + o(1).$$

This, in turn, implies that

$$\mathbf{x}_n^* \mathbf{A}^{-1}(\sigma) \mathbf{y}_n = \frac{\mathbf{x}_n^* \mathbf{y}_n + o(1)}{\sigma + b(\sigma)}.$$

The conclusion in (A.5) could then follow from the fact that

$$m(\sigma) = \frac{1}{\sigma + b(\sigma)},$$

whereas the conclusion in (A.6) can be proved by employing the same method. \square

LEMMA 7. Under the conditions of Theorem 1, for any $\mathbf{x}_n, \mathbf{y}_n \in \mathbb{C}_1^p$, we have

$$\sqrt{n}(\mathbf{x}_n^* \mathbf{E}(\mathbf{S}_n + \sigma I)^{-1} \mathbf{y}_n - \mathbf{x}_n^* \mathbf{y}_n m_n(\sigma)) \longrightarrow 0.$$

PROOF. When $\mathbf{y}_n = \mathbf{x}_n$, Lemma 7 in our paper reduces to the conclusion (5.5) $\rightarrow 0$ as shown in Bai, Miao and Pan (2007). To complete the proof, one could simply keep \mathbf{x}_n^* unchanged and substitute \mathbf{x}_n by $\mathbf{y}_n = (\mathbf{x}_n^* \mathbf{y}_n) \mathbf{x}_n + \mathbf{z}_n$ in the proof of the above conclusion. Thereafter, the proof of this lemma follows. \square

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