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ON APPROXIMATIVE SOLUTIONS OF MULTISTOPPING PROBLEMS

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In this paper, we consider multistopping problems for finite discrete time sequences $X_1, \dots, X_n$. m-stops are allowed and the aim is to maximize the expected value of the best of these m stops. The random variables are neither assumed to be independent not to be identically distributed. The basic assumption is convergence of a related imbedded point process to a continuous time Poisson process in the plane, which serves as a limiting model for the stopping problem. The optimal m-stopping curves for this limiting model are determined by differential equations of first order. A general approximation result is established which ensures convergence of the finite discrete time m-stopping problem to that in the limit model. This allows the construction of approximative solutions of the discrete time m-stopping problem. In detail, the case of i.i.d. sequences with discount and observation costs is discussed and explicit results are obtained.

1. Introduction. In this paper, we consider multistopping problems for discrete time sequences $X_1, \dots, X_n$. In comparison to the usual stopping problem, there are m stops $1 \le T_1 < \cdots < T_m \le n$ allowed. The aim is to determine these stopping times in such a way that

$$E\Big[\max_{1 \le i \le m} X_{T_i}\Big] = E[X_{T_1} \vee \cdots \vee X_{T_m}] = \sup. \tag{1.1}$$

Thus, the gain of a stopping sequence $(T_i)_{i \le m}$ is the expected maximal value of the m choices X_{T_i}. In the case $m = 1$, this stopping problem reduces to the classical Moser problem [Moser (1956)]. We will see that optimal m-stopping times exist and are determined by a recursive description.

Our aim is to obtain *explicit* approximative solutions of the m-stopping problem in (1.1) under general distributional conditions. In particular, we do not assume that the random variables X_i are independent or identically distributed or are even of specific i.i.d. form with $X_i \sim U(0, 1)$ as assumed in several papers in the literature. Our basic assumption is convergence of the imbedded planar point process (1.2) of rescaled observations to some Poisson point process N in the plane,

$$N_n = \sum_{i=1}^{n} \delta_{(i/n, X_i^n)} \xrightarrow{d} N. \tag{1.2}$$

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Here $X_i^n = \frac{X_i - b_n}{a_n}$ is a nomalization of the X_i induced typically from the central limit theorem for maxima respectively related point process convergence results. Our aim is to prove that under some regularity conditions the optimal m-stopping problem of $X_1, \dots, X_n$ can be approximated by a suitable formulated m-stopping problem for the continuous time Poisson process N which serves as a limiting model for the discrete time model. Furthermore, we want to show that the stopping problem in the limit model can be solved in explicit form. The solution is described by an increasing sequence of stopping curves with their related threshold stopping times. These curves solve usual one-stopping problems for transformed Poisson processes and are characterized by differential equations of first order, which can be solved either in exact form or numerically. The solution for the limit model also allows us to construct approximative optimal stopping times for the discrete time model. We apply this approach in detail to the m-stopping of sequences $X_i = c_i Z_i + d_i$ *with discount* and observation costs and i.i.d. sequences Z_i.

It has been observed in several papers in the literature that optimal stopping may have an easier solution in a related form for a Poisson number of points or for imbedded homogeneous Poisson processes as for instance in the classical house selling problem or in best choice problems. For $m = 1$ [see, e.g., Chow, Robbins and Siegmund (1971), Sakaguchi (1976), Bruss and Rogers (1991), Gnedin and Sakaguchi (1992), Gnedin (1996), Baryshnikov and Gnedin (2000)]. For general reference, we refer to Ferguson (2007), Chapter 2. For $m \geq 1$, multistopping problems were introduced in Haggstrom (1967) who derived some structural results corresponding roughly to Theorem 2.3; compare also some extensions in Nikolaev (1999). The two stopping problem has been considered in the case of Poissonian streams in Saario and Sakaguchi (1992). In this paper, differential equations were derived corresponding to those for the one-stopping problems in Karlin (1962), Siegmund (1967) and Sakaguchi (1976). Multiple buying—selling problems were studied in Bruss and Ferguson (1997) based on a vector valued formulation with pay-off given by the sum of the m-choices instead of the max as in (1.1); see also the extension in Bruss (2010). In Kühne and Rüschendorf (2002) the case of 2-stopping problems for i.i.d. sequences was treated based on the approximative approach in Kühne and Rüschendorf (2000a). The results in this paper were rederived in Assaf, Goldstein and Samuel-Cahn (2004, 2006) and in Goldstein and Samuel-Cahn (2006). In case $m = 1$ based on this approximation for several classes of independent and dependent sequences optimal solutions have been found in explicit form [see Kühne and Rüschendorf (2000b, 2004) and Faller and Rüschendorf (2009)]. The present paper establishes an extension of the approximative approach as described above to m-stopping problems as in (1.1). It is based on the dissertation of Faller (2009) to which we refer for some technical details in the proofs.

The program to establish this approximation approach in general is based on the following steps. In Section 2, we formulate the necessing recursive characterization of the optimal solutions of the m-stopping problem corresponding to

Bellman's optimality equation. Section 3 is devoted to solve the m-stopping problem for the limit model of an inhomogeneous Poisson process. A particular difficulty arises from the fact that in the limit model the intensity function is typically infinite along a lower boundary curve, In consequence, known stationary Markovian techniques as for homogenous Poisson processes do not apply. The main result, Theorem 3.3, shows that the optimal m-stopping problem can be reduced to m 1-stopping problems for transformed Poission processes. The optimal stopping curves are characterized by a sequence of differential equations of first order.

In Section 4, we are able to derive explicit solutions for some classes of differential equations, as appearing in the description of the optimal stopping curves. This part is based on developments in Faller and Rüschendorf (2009) for the case $m = 1$. Section 5 gives the basic approximation theorem (Theorem 5.2) allowing to approximate the finite discrete problems by m-stopping in the limit model. The proof of this result needs to develop a new technique. It also uses essentially the extension of the convergence of multiple stopping times in Proposition 5.1 in Faller and Rüschendorf (2009) for $m = 1$ to $m \geq 1$. We restrict our presentation to the essential new part of this proof. Finally in Section 6 we obtain as application solutions in *explicit* form for optimal m-stopping problems for sequences $X_i = c_i Z_i + d_i$ with Z_i i.i.d. and with discount and observation costs c_i, d_i. It is remarkable that we get detailed results including the asymptotic constants as well as approximative optimal stopping sequences in explicit form. Our aim is to extend these results in subsequent papers to further classes of stopping problems as to selection problems, to the sum cost case as well as to some classes of dependent sequences. It seems also possible as done in the case $m = 1$, to extend this approach to the case where cluster processes arise in the limit.

2. *m*-stopping problems for finite sequences.

In this section, we give a formulation of the optimality principle for the m-stopping of discrete recursive sequences. Given a discrete time sequence $(X_i, \mathcal{F}_i)_{1 \leq i \leq n}$ in a probability space $(\Omega, \mathcal{A}, P)$ with filtration $\mathcal{F} = (\mathcal{F}_i)_{0 \leq i \leq n}$ the m-stopping problem $(1 \leq m \leq n)$ is to find stopping times $1 \leq T_1 < T_2 < \cdots < T_m \leq n$ w.r.t. the filtration $(\mathcal{F}_i)_{1 \leq i \leq n}$ such that

$$E\Big[\max_{1 \leq i \leq m} X_{T_i}\Big] = E[X_{T_1} \vee \cdots \vee X_{T_m}] = \sup. \tag{2.1}$$

In case $m = 1$, (2.1) is identical to the usual (one-)stopping problem. A well-known recursive solution of this problem [see Chow, Robbins and Siegmund (1971), Theorem 3.2] is based on the threshold curves $W_i = W_F(X_{i+1}, \ldots, X_n)$ of the optimal stopping time defined by

$$\begin{aligned} &W_n := -\infty, \\ &W_i := E[X_{i+1} \vee W_{i+1} | \mathcal{F}_i] \qquad \text{for } i = n-1, \ldots, 0. \end{aligned} \tag{2.2}$$

We need a version of this classical result for stopping times larger than a given stopping time S.

PROPOSITION 2.1 (Recursive solution of one-stopping problems).

(a) *For any time point* $0 \le k \le n-1$, *the* $\mathcal{F}$*-stopping time*

$$T(k) := \min\{k < i \le n : X_i > W_i\}$$

is optimal in the sense that for any $\mathcal{F}$*-stopping time* $T > k$ *we have*

$$E\big[X_{T(k)} | \mathcal{F}_k\big] = W_k \ge E[X_T | \mathcal{F}_k] \qquad P\text{-}a.s. \tag{2.3}$$

(b) *For any* $\mathcal{F}$*-stopping time* S, *the* $\mathcal{F}$*-stopping time*

$$T(S) = \min\{S < i \le n : X_i > W_i\}$$

is optimal in the sense that for any $\mathcal{F}$*-stopping time* T *with* $S < T$ *on* $\{S < n\}$ *and* $S = T$ *on* $\{S = n\}$ *we have*

$$E\big[X_{T(S)} | \mathcal{F}_S\big] = W_S \ge E[X_T | \mathcal{F}_S] \qquad P\text{-}a.s. \tag{2.4}$$

REMARK 2.2. For m stopping problems, the following variant of Proposition 2.1 will also be needed [for details of the proof, see Faller (2009)].

Let $Y_1, \ldots, Y_n : (\Omega, \mathcal{A}, P) \to E$ be random variables taking values in a measurable space E and $\mathcal{F} := (\mathcal{F}_i)_{0 \le i \le n}$ a filtration in $\mathcal{A}$ such that $\sigma(Y_i) \subset \mathcal{F}_i$ for all $1 \le i \le n$. Let S be an $\mathcal{F}$-stopping time, let $Z : (\Omega, \mathcal{A}, P) \to \overline{\mathbb{R}}$ be $\mathcal{F}_S$-measurable and $h : E \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be measurable with $Eh(Y_i, Z)^+ < \infty$. Also define recursively for $z \in \overline{\mathbb{R}}$

$$\begin{aligned} &W_n(z) := h(Y_n, z), \\ &W_i(z) := E[h(Y_{i+1}, z) \vee W_{i+1}(z) | \mathcal{F}_i] \qquad \text{for } i = n-1, \ldots, 0. \end{aligned} \tag{2.5}$$

Then the $\mathcal{F}$ stopping time

$$T(S, Z) := \min\{S < i \le n : h(Y_i, Z) > W_i(Z_i)\}, \tag{2.6}$$

where $Z_i := Z 1_{\{S \le i\}}$ is optimal in the sense that for any further $\mathcal{F}$-stopping time T with $S < T$ on $\{S < n\}$ and $S = T$ on $\{S = n\}$ we have

$$E\big[h\big(Y_{T(S,Z)}, Z\big) | \mathcal{F}_S\big] = W_S(Z_S) \ge E[h(Y_T, Z) | \mathcal{F}_S] \qquad P\text{-a.s.} \tag{2.7}$$

Similar as for the one-stopping problems the idea of solving (2.1) is simple. The ℓth stopping time T_ℓ should be i if the $(m-\ell)$-stopping value past i with guarantee value X_i is in expectation larger than the $(m-\ell+1)$-stopping value past i and with guarantee value reached before time i. This idea leads to the following construction. Define $W_i^0(x) := x$ for $x \in \overline{\mathbb{R}}$ and inductively for $1 \le m \le n$, $x \in \mathbb{R}$ define thresholds $W_k^m(x)$ by

$$\begin{aligned} &W_{n-m+1}^m(x) := x, \\ &W_i^m(x) := E[W_{i+1}^{m-1}(X_{i+1}) \vee W_{i+1}^m(x) | \mathcal{F}_i] \qquad \text{for } i = n-m, \ldots, 0. \end{aligned} \tag{2.8}$$

The related threshold stopping times are defined recursively for $k \le n - m$ by

$$\begin{aligned} T_1^m(k,x) &:= \min\{k < i \le n-m+1 : W_i^{m-1}(X_i) > W_i^m(x)\}, \\ T_\ell^m(k,x) &:= \min\{T_{\ell-1}^m(k,x) < i \le n-m+\ell : \\ &\qquad W_i^{m-l}(X_i) > W_i^{m-l+1}(x \vee M_{\ell-1,i})\} \end{aligned} \tag{2.9}$$

for $2 \le \ell \le m$ and $M_{j,i} := X_{T_j^m(k,x)} 1_{\{T_j^m(k,x) \le i\}}$.

Equation (2.9) corresponds to a sequence of m one-stopping problems for (more complicated) transformed sequences of random variables. The following result extends the classical recursive characterization of optimal stopping times for one-stopping problems in Proposition 2.1 to the case $m \ge 1$. Related structural results can be found in the papers of Haggstrom (1967), Saario and Sakaguchi (1992), Bruss and Ferguson (1997), Nikolaev (1999), Bruss and Delbaen (2001) and Kühne and Rüschendorf (2002).

THEOREM 2.3 (Recursive characterization of m-stopping problems). *The $\mathcal{F}$-stopping times $(T_\ell^m(k,x))_{1 \le \ell \le m}$ are optimal in the sense that for all $\mathcal{F}$-stopping times $(T_\ell)_{1 \le \ell \le m}$ with $k < T_1 < \cdots < T_m \le n$ we have*

$$\begin{aligned} &E[x \vee X_{T_1^m(k,x)} \vee \cdots \vee X_{T_m^m(k,x)} | \mathcal{F}_k] \\ &\quad = E[W_{T_1^m(k,x)}^{m-1}(x \vee X_{T_1^m(k,x)}) | \mathcal{F}_k] = W_k^m(x) \\ &\quad \ge E[x \vee X_{T_1} \vee \cdots \vee X_{T_m} | \mathcal{F}_k] \qquad P\text{-a.s.} \end{aligned}$$

The proof of Theorem 2.3 follows by induction in m based on Proposition 2.1 and Remark 2.2 similarly as in the case $m = 1$. For details, see Faller (2009), Satz 2.1 or Kühne and Rüschendorf (2002), Proposition 2.1. In general, the recursive characterization of optimal m-stopping times and values is difficult to evaluate. Our aim is to prove that one can construct optimal m-stopping times and values approximatively by considering related limiting m-stopping problems for Poisson processes in continuous time.

3. m-stopping of Poisson processes.

In this section, we deal with the optimal m-stopping problem for the limit model given by a Poisson point process N. We consider a Poisson process $N = \sum_k \delta_{(\tau_k, Y_k)}$ in the plane restricted to some set

$$M_f = \{(t,x) \in [0,1] \times \overline{\mathbb{R}}; x > f(t)\},$$

where $f : [0,1] \to \mathbb{R} \cup \{-\infty\}$ is a continuous lower boundary function of N. The intensity of N may be (and in typical cases is) infinite along the lower boundary f. As in Kühne and Rüschendorf (2000a), respectively, Faller and Rüschendorf (2009) who consider the case $m = 1$, we assume that the intensity measure μ of N is a Radon measure on M_f with the topology on M_f induced by the usual topology on $[0,1] \times \overline{\mathbb{R}}$. Thus any compact set $A \subset M_f$ has only finitely many points. By convergence in distribution "$N_n \xrightarrow{d} N$ on M_f," we mean convergence in distribution of the restricted point processes. This is the basic assumption made in this paper.

We generally assume the boundedness condition

$$E\Big[\Big(\sup_k Y_k\Big)^+\Big] < \infty. \tag{B}$$

Let $\mathcal{A}_t = \sigma(N(\cdot \cap [0,t] \times \overline{\mathbb{R}} \cap M_f))$, $t \in [0,1]$, denote the relevant filtration of the point process N. A stopping time for N or N-stopping time is a mapping $T : \Omega \to [0,1]$ with $\{T \le t\} \in \mathcal{A}_t$ for each $t \in [0,1]$. Denote by

$$\overline{Y}_T := \sup\{Y_k : 1 \le k \le N(M_f), T = \tau_k\}, \qquad \sup \varnothing := -\infty,$$

the reward w.r.t. stopping time T.

Let $v : \overline{M}_f \to \overline{\mathbb{R}}$ be a continuous transformation of the points of N such that

$$\left.\begin{array}{l} v(t,x) \le ax^+ + b \ \forall (t,x) \in M_f, \text{ with real constants } a, b \ge 0, \\ v(t,\cdot) \text{ is for each } t \text{ a monotonically nondecreasing function,} \\ v(\cdot,x) \text{ is for each } x \text{ a monotonically nonincreasing function.} \end{array}\right\} \tag{3.1}$$

Define $c := f(1)$ and for any guarantee value $x \in [c, \infty)$ and $t \in [0,1)$ the *optimal stopping curve* $\hat{u}$ of the transformed Poisson process by

$$\begin{aligned} &\hat{u}(t,x) := \sup\{E[v(T, \overline{Y}_T \vee x)] : T > t \text{ is an } N\text{-stopping time}\}, \\ &\hat{u}(1,x) := v(1,x). \end{aligned} \tag{3.2}$$

It will be shown in the following proposition that the treshold stopping time corresponding to $\hat{u}$ is an optimal stopping time for the Poisson process. For the basic notions of stopping of point processes; see Kühne and Rüschendorf (2000a), respectively, Faller and Rüschendorf (2009). The following proposition is the analogue of Proposition 2.1 for continuous time Poisson processes. It is essential for the solution of the m-stopping problem of N.

PROPOSITION 3.1 (Optimal stopping times larger than S). *Let N satisfy the boundedness condition* (B), *let v satisfy condition* (3.1) *and assume the following separation condition for the optimal stopping boundary $\hat{u}$*:

$$\hat{u}(t,c) > \hat{f}(t) := v(t, f(t)) \qquad \forall t \in [0,1). \tag{$\hat{\mathrm{S}}$}$$

Then:

(a) *$\hat{u}$ is continuous on $[0,1] \times [c,\infty]$ and for all $(t,x) \in [0,1] \times [c,\infty]$ holds*

$$\begin{aligned} \hat{u}(t,x) &= E\big[v\big(T(t,x), \overline{Y}_{T(t,x)} \vee x\big)\big] \\ &= E\big[v\big(T(t,x), \overline{Y}_{T(t,x)} \vee c\big) \vee v(1,x)\big] \end{aligned} \tag{3.3}$$

with the optimal stopping time

$$T(t,x) := \inf\{\tau_k > t : v(\tau_k, Y_k) > \hat{u}(\tau_k, x)\}, \qquad \inf \varnothing := 1.$$

$\hat{u}(\cdot, x)$ *is for* $x \in [c, \infty]$ *the optimal stopping curve of the transformed Poisson process* $\hat{N} := \sum_k \delta_{(\tau_k, v(\tau_k, Y_k))}$ *in* $M_{\hat{f}}$ *for the guarantee value* $v(1,x)$.

(b) *Let* S *be an* N*-stopping time, let* $Z \geq c$ *be real* $\mathcal{A}_S$*-measurable with* $EZ^+ < \infty$ *and* $\mathcal{T}(S)$ *the set of all* N*-stopping times* T *with* $T > S$ *on* $\{S < 1\}$ *and* $T = 1$ *on* $\{S = 1\}$. *Then* $T(S, Z) \in \mathcal{T}(S)$ *is optimal in the sense that*

$$\begin{aligned} E\big[v\big(T(S,Z), \overline{Y}_{T(S,Z)} \vee Z\big) | \mathcal{A}_S\big] &= \hat{u}(S, Z) \\ &\geq E[v(T, \overline{Y}_T \vee Z) | \mathcal{A}_S] \qquad P\text{-a.s.} \end{aligned} \tag{3.4}$$

for all $T \in \mathcal{T}(S)$.

PROOF. (a) The statement in (a) is proved by discretization. Since $\hat{f}$ is continuous and $\hat{u}(\cdot, c)$ is right continuous there exists a monotonically nonincreasing, continuous function $\hat{f}_2 : [0,1] \to [\hat{c}, \infty)$, $\hat{c} := \hat{f}(1) = v(1, c)$ such that $\hat{f} < \hat{f}_2 < \hat{u}(\cdot, c)$ on $[0, 1)$. Thus, for $t < 1$, the sets $[0,t] \times \overline{\mathbb{R}} \cap M_{\hat{f}_2}$ are compact in $M_{\hat{f}}$.

For $x \in [c, \infty)$, $n \in \mathbb{N}$ and $1 \leq i \leq 2$ define

$$M^n_{i/2^n}(x) := \sup_{\tau_k \in ((i-1)/2, i/2]} v(\tau_k, Y_k \vee x).$$

Consider the filtration $\mathcal{A}^n = (\mathcal{A}_{i/2^n})_{1 \leq i \leq 2^n}$. Then $M^n_{i/2^n}(x)$ is $\mathcal{A}_{i/2^n}$ measurable and $\mathcal{A}_{i/2^n}$, $\sigma(M^n_{(i+1)/2^n}(x))$ are independent. We define $w_n : [0,1] \times [c, \infty) \to \overline{\mathbb{R}}$ by

$$\begin{aligned} &w_n(t,x) := \sup\{E[M^n_T(x)] : T > t \text{ an } \mathcal{A}^n\text{-stopping time}\} \qquad \text{for } t \in [0,1), \\ &w_n(1,x) := v(1,x). \end{aligned} \tag{3.5}$$

Then for $t \in [0,1)$ by Proposition 2.1, we have

$$w_n(t,x) = E\big[M^n_{T_n(t,x)}(x)\big] = V^n_{\lfloor 2^n t \rfloor}(x)$$

with the optimal $\mathcal{A}^n$-stopping time

$$T_n(t,x) := \min\left\{t < \frac{i}{2^n} \leq 1 : M^n_{i/2^n}(x) > w_n\left(\frac{i}{2^n}, x\right)\right\}, \qquad \min \varnothing := 1,$$

and

$$\begin{aligned} &V^n_{2^n}(x) := v(1,x), \\ &V^n_i(x) := E\big[M^n_{(i+1)/2^n}(x) \vee V^n_{i+1}(x)\big], \qquad i = 2^n - 1, \dots, 0. \end{aligned} \tag{3.6}$$

The function $w_n(\cdot, x)$ is monotonically nonincreasing and constant on the intervals $[0, \frac{1}{2^n}), [\frac{1}{2^n}, \frac{2}{2^n}), \dots, [\frac{2^n-1}{2^n}, 1)$. We also have

$$(1) \quad w_n(t,x) \geq \hat{u}(t,x) \qquad \forall t \in [0,1],$$

$$(2) \quad w_n(t,x) \geq w_{n+1}(t,x) \qquad \forall t \in [0,1].$$

For the proof of (1) note that for any stopping time $T > t$, $T_n := \frac{\lceil T 2^n \rceil}{2^n}$ is an $\mathcal{A}^n$-stopping time with $T_n > t$ and $T_n - \frac{1}{2^n} < T \leq T_n$. Therefore,

$$M^n_{T_n}(x) = \sup_{\tau_k \in (T_n - 1/2^n, T_n]} v(\tau_k, Y_k \vee x) \geq v(T, \overline{Y}_T \vee x). \tag{3.7}$$

This implies $w_n(t,x) \geq \sup\{E[v(T, \overline{Y}_T \vee x)] : T > t \ N\text{-stopping time}\} = \hat{u}(t,x)$.

The proof of (2) is similar. If $T > t$ is an $\mathcal{A}^{n+1}$-stopping time, then $T' := \frac{\lceil T 2^n \rceil}{2^n}$ is an $\mathcal{A}^n$-stopping time with $T' > t$ and $T' - \frac{1}{2^n} < T \leq T'$. Thus, as above, we obtain $w_n(t,x) \geq w_{n+1}(t,x)$.

Relations (1) and (2) imply the existence of a monotonically nonincreasing function $w(\cdot, x) : [0,1] \to \mathbb{R} \cup \{-\infty\}$ with $w(\cdot, x) \geq \hat{u}(\cdot, x)$ and $w_n(\cdot, x) \downarrow w(\cdot, x)$ pointwise. It can be shown by our assumptions on v and N that w is continuous [see Faller (2009)].

For $\omega \in \Omega$ with $\hat{N}(\omega, K) < \infty$ for all compact $K \subset M_f$ and for $(t,x) \in [0,1] \times [c, \infty]$ and $t_n \downarrow t$, we have the convergence

$$M^n_{T_n(t_n,x)}(x) \to v\big(T(t,x), \overline{Y}_{T(t,x)} \vee x\big) \tag{3.8}$$

with the stopping time

$$\begin{aligned} T(t,x) &:= \inf\{\tau_k > t : v(\tau_k, Y_k \vee x) > w(\tau_k, x)\} \\ &\overset{(*)}{=} \inf\{\tau_k > t : v(\tau_k, Y_k) > w(\tau_k, x)\}, \qquad \inf \varnothing := 1. \end{aligned} \tag{3.9}$$

For the proof, note that monotone convergence of $w_n(\cdot, x)$ and continuity of the limit ω implies uniform convergence from above. Thus, for $x \in [c, \infty)$ points of N on the graph of $w(\cdot, x)$ are ignored by all stopping times $T_n(t,x)$ and $T(t,x)$. The second equality $(*)$ holds since $w(t,x) \geq \hat{u}(t,x) \geq v(t,x)$ and since by assumption $v(t, \cdot)$ is strictly monotonically increasing. This implies by Fatou's lemma the following sequence of inequalities:

$$\begin{aligned} \hat{u}(t,x) \leq w(t,x) &= \lim_{n\to\infty} w_n(t,x) = \lim_{n\to\infty} E\big[M^n_{T_n(t,x)}(x)\big] \\ &\leq E\big[v\big(T(t,x), \overline{Y}_{T(t,x)} \vee x\big)\big] \leq \hat{u}(t,x). \end{aligned}$$

Thus, $\hat{u}(\cdot, x) = w(\cdot, x)$ is continuous and $\hat{u}(t,x) = E[v(T(t,x), \overline{Y}_{T(t,x)} \vee x)]$. As $w(t,x) \geq v(t,x)$ implies that $\overline{Y}_{T(t,x)} > x$ for $T(t,x) < 1$, we have $\hat{u}(t,x) = E[v(T(t,x), \overline{Y}_{T(t,x)} \vee c) \vee v(1,x)]$, which means that $\hat{u}(\cdot, x)$ is the optimal stopping curve of the Poisson process $\hat{N}$ with guarantee value $v(1,x)$.

(b) To prove optimality of the stopping time $T(S,Z)$, set $S_n := \frac{\lceil S2^n \rceil}{2^n}$. Then S_n is an $\mathcal{A}^n$-stopping time and by (3.8) holds

$$M^n_{T_n(S_n,Z)}(Z) \to v\big(T(S,Z), \overline{Y}_{T(S,Z)} \vee Z\big) \qquad P\text{-a.s.} \tag{3.10}$$

Let $\mathcal{T}(S_n)$ be the set of all $\mathcal{A}^n$-stopping times T_n with $T_n > S_n$ on $\{S_n < 1\}$ and $T_n = S_n$ on $\{S_n = 1\}$. Let $T \in \mathcal{T}(S)$. By discretization $T > S$ in general does not imply $\frac{\lceil T2^n \rceil}{2^n} > \frac{\lceil S2^n \rceil}{2^n}$. Thus, we modify the discretization and define $T_n := \frac{\lceil T2^n \rceil}{2^n}\chi_{\{\lceil T2^n \rceil/2^n > S_n\}} + 1\chi_{\{\lceil T2^n \rceil/2^n = S_n\}} \in \mathcal{T}(S_n)$. Then analogously to (3.7)

$$v(T, \overline{Y}_T \vee Z) \le M^n_{T_n}(Z)\chi_{\{\lceil T2^n \rceil/2^n > S_n\}} + v(T, \overline{Y}_T \vee Z)\chi_{\{\lceil T2^n \rceil/2^n = S_n\}}.$$

This implies the inequalitites

$$\begin{aligned}
&E[v(T, \overline{Y}_T \vee Z)|\mathcal{A}_{S_n}] \\
&\quad \le E[M^n_{T_n}(Z)|\mathcal{A}_{S_n}]\chi_{\{\lceil T2^n \rceil/2^n > S_n\}} \\
&\qquad + E[v(T, \overline{Y}_T \vee Z)|\mathcal{A}_{S_n}]\chi_{\{\lceil T2^n \rceil/2^n = S_n\}} \\
&\quad \overset{(*)}{\le} \underbrace{E\big[M^n_{T_n(S_n,Z)}(Z)|\mathcal{A}_{S_n}\big]}_{=w_n(S_n,Z)}\chi_{\{\lceil T2^n \rceil/2^n > S_n\}} \\
&\qquad + E[v(T, \overline{Y}_T \vee Z)|\mathcal{A}_{S_n}]\chi_{\{\lceil T2^n \rceil/2^n = S_n\}}.
\end{aligned}$$

$(*)$ holds by Remark 2.2. Since we have $M^n_{i/2^n}(Z) = h(Y_i, Z)$, where $Y_i := N(\cdot \cap (\frac{i-1}{2^n}, \frac{i}{2^n}] \times \overline{\mathbb{R}} \cap M_f)$ and with $h : \mathcal{N}_R(M_f) \times [c,\infty) \to \overline{\mathbb{R}}$, $h(\sum_k \delta_{(t_k,y_k)}, x) := \sup_k v(t_k, y_k \vee x)$.

As $\mathcal{A}_S \subset \mathcal{A}_{S_n}$ we conclude

$$\begin{aligned}
&E[v(T, \overline{Y}_T \vee Z)|\mathcal{A}_S] \\
&\quad \le E\big[M^n_{T_n(S_n,Z)}(Z)\chi_{\{\lceil T2^n \rceil/2^n > S_n\}}|\mathcal{A}_S\big] \\
&\qquad + E\big[v(T, \overline{Y}_T \vee Z)\chi_{\{\lceil T2^n \rceil/2^n = S_n\}}|\mathcal{A}_S\big] \\
&\quad = w_n(S_n, Z)E\big[\chi_{\{\lceil T2^n \rceil/2^n > S_n\}}|\mathcal{A}_S\big] + E\big[v(T, \overline{Y}_T \vee Z)\chi_{\{\lceil T2^n \rceil/2^n = S_n\}}|\mathcal{A}_S\big],
\end{aligned}$$

and by the Lemma of Fatou we have by (3.10)

$$E[v(T, \overline{Y}_T \vee Z)|\mathcal{A}_S] \le E\big[v\big(T(S,Z), \overline{Y}_{T(S,Z)} \vee Z\big)|\mathcal{A}_S\big] = \hat{u}(S,Z).$$

As $T > S$ was chosen arbitrary this implies (b). □

In the sequel, we need the following *differentiability condition* to be fulfilled.

(D) Assume that there is a version of the density g of μ on M_f such that the intensity function

$$G(t,y) = \int_y^\infty g(t,z)\,dz$$

is continuous on $M_f \cap [0,1] \times \mathbb{R}$. Furthermore, we assume that $\lim_{y\to\infty} yG(t, y) = 0$ for all $t \in [0,1]$.

The following proposition determines the intensity function of transformed Poisson processes.

PROPOSITION 3.2 (Intensity function of transformed Poisson processes). *Let $N = \sum \delta_{(\tau_k, Y_k)}$ be a Poisson process with intensity function G satisfying the boundedness condition* (B). *Let $v : \overline{M}_f \to \overline{\mathbb{R}}$, $v = v(t,x)$ be a C^1-function monotonically nonincreasing in t and monotonically nondecreasing in x with $v(t,\infty) = \infty$ for all $t \in [0,1]$. Define $R(t,x) := (t, v(t,x))$ and $f_v(t) := v(t, f(t))$. Then $R(M_f) = M_{f_v}$, $R^{-1}(t,y) = (t, \xi(t,y))$ with a C^1-function $\xi : M_{f_v} \to \overline{\mathbb{R}}$.*

$\widehat{N} := \sum_k \delta_{(\tau_k, v(\tau_k, Y_k))}$ is a Poisson process on M_{f_v} with intensity measure $\widehat{\mu} = \mu \circ R^{-1}$ and intensity fuction $\widehat{G}(t,y) := G(t, \xi(t,y))$, $(t,y) \in M_{f_v}$.

PROOF. By Resnick [(1987), Proposition 3.7], $\widehat{N}$ is a Poisson process with intensity measure $\widehat{\mu} = \mu \circ R^{-1}$. The transformation formula implies that the density $\widehat{g}$ of $\widehat{\mu}$ is given by

$$\begin{aligned}\hat{g}(t,y) &= g(R^{-1}(t,y)) |\det J(R^{-1})(t,y)| \\ &= g(t, \xi(t,y)) \frac{\partial}{\partial y} \xi(t,y) = -\frac{\partial}{\partial y} G(t, \xi(t,y)).\end{aligned}$$ □

After this preparation, we now consider the m-stopping problem for Poisson processes. The aim is to solve

$$E[\overline{Y}_{T_1} \vee \cdots \vee \overline{Y}_{T_m}] = \sup, \tag{3.11}$$

where the supremum is over all N-stopping times[1] $0 \le T_1 < \cdots < T_m \le 1$.

This problem has been considered for Poisson processes on $[0,1] \times (c, \infty)$ already in Saario and Sakaguchi (1992) in the special case of intensity functions of the form

$$G(t,y) = \lambda\big(1 - F(y)\big) \tag{3.12}$$

with $\lambda > 0$ and F a continuous distribution function with $F(c) = 0$. Equation (3.12) models the case of i.i.d. random variables arriving at Poisson distributed arrival times. Sakaguchi and Saario (1995) derive for this case differential equations for the optimal stopping curves. Explicit solutions are however not given in any case. In the following, we extend these results to the case of general intensities. We subsequently also identify classes of examples of intensity functions which allow essentially explicit solutions.

In order to guarantee the existence of optimal m-stopping times, we restrict ourselves in the following to the case where the lower boundary is constant, $f \equiv c$.

[1] $T_1 < \cdots < T_m \le 1$ signifies that $T_{i-1} < T_i$ for each i on $\{T_{i-1} < 1\}$ and $T_i = 1$ on $\{T_{i-1} = 1\}$.

Define optimal m-stopping curves for guarantee value $x \in [c, \infty)$, $m \in \mathbb{N}$, and $t \in [0, 1)$ by[2]

$$
\begin{aligned}
u^m(t, x) &:= \sup\{E[\overline{Y}_{T_1} \vee \cdots \vee \overline{Y}_{T_m} \vee x] : t < T_1 < \cdots < T_m \le 1 \\
&\qquad\qquad N\text{-stopping times}\}, \qquad (3.13)\\
u^m(1, x) &:= x.
\end{aligned}
$$

Further let $u^0(t, x) := x$ for $(t, x) \in [0, 1] \times [c, \infty]$ and $u^m(t) := u^m(t, c)$ for $t \in [0, 1]$.

$u^m(\cdot, x)$ is called *optimal m-stopping curve* of N for guarantee value x. Define the inverse function $\xi^m : \overline{M}_{u^m} \to \overline{R}$ by

$$\xi^m(t, u^m(t, x)) = x \qquad \text{for } (t, x) \in [0, 1] \times [c, \infty]. \tag{3.14}$$

Further define $\gamma^m : [0, 1] \times [c, \infty] \to \overline{\mathbb{R}}$ by

$$\gamma^m(t, x) := \xi^{m-1}(t, u^m(t, x)) \tag{3.15}$$

as well as

$$\gamma^m(t) := \gamma^m(t, c) = \xi^{m-1}(t, u^m(t)). \tag{3.16}$$

Then $\gamma^m(t, x) > x$ iff $u^m(t, x) > u^{m-1}(t, x)$ and further

$$y > \gamma^m(t, x) \quad \Leftrightarrow \quad u^{m-1}(t, y) > u^m(t, x).$$

The optimal m-stopping for Poisson processes can be reduced by the previous structural results to m 1-stopping problem for transformed Poisson processes. The transformations are given by the optimal stopping curves u^m or equivalently by the inverses γ^m—both sequences of curves are defined recursively. Thus, we consider the transformed Poisson processes

$$N^m := \sum_k \delta_{(\tau_k, u^{m-1}(\tau_k, Y_k))} \qquad \text{on } M_{u^{m-1}}. \tag{3.17}$$

Define the (optimal) stopping times $T_\ell^m(t, k)$ with guarantee value x by

$$
\begin{aligned}
T_1^m(t, x) &:= \inf\{\tau_k > t : Y_k > \gamma^m(\tau_k, x)\},\\
T_\ell^m(t, x) &:= \inf\bigl\{\tau_k > T_{\ell-1}^m(t, x) : Y_k > \gamma^{m-\ell+1}\bigl(\tau_k, \overline{Y}_{T_{\ell-1}^m(t,x)} \vee x\bigr)\bigr\}.
\end{aligned}
\tag{3.18}
$$

The following theorem characterizes the optimal stopping time as threshold stopping time based on the optimal stopping curves. These are given by a system of m differential equations of first order.

THEOREM 3.3 (Optimal m-stopping of Poisson processes). *Let $f \equiv c$ and N satisfy the boundedness condition* (B) *and the separation condition* (S), *that is, $u^1(t) > c$ for $t \in [0, 1)$. Let $t_0(x) := \inf\{t \in [0, 1] : \mu((t, 1] \times (x, \infty]) = 0\}$.*

(a) *Then for* $m \in \mathbb{N}$, $(t,x) \in [0,1) \times [c,\infty)$ *holds*

$$u^m(t,x) = E\big[\overline{Y}_{T_1^m(t,x)} \vee \cdots \vee \overline{Y}_{T_m^m(t,x)} \vee x\big] = E\big[u^{m-1}\big(T_1^m(t,x), \overline{Y}_{T_1^m(t,x)} \vee x\big)\big]$$

with optimal stopping times $(T_\ell^m(t,x))_{1\le\ell\le m}$ *defined in* (3.18).

(b) *For* $(t,x) \in A := \{(t,x) \in (0,1] \times [c,\infty) : t < t_0(x)\}$ *holds* $u^m(t,x) > u^{m-1}(t,x)$ *while* $u^m(t,x) = u^{m-1}(t,x) = x$ *else. In particular,* $u^m(t) > u^{m-1}(t)$ *for* $t \in [0,1)$ *and* $u^m(\cdot,x)$ *is the optimal stopping curve of the transformed Poisson process* N^m.

(c) *Under the differentiability condition,* (D) $u^m(\cdot,x)$ *solves the differential equation*

$$\frac{\partial}{\partial t}u^m(t,x) = -\int_{u^m(t,x)}^{\infty} G(t,\xi^{m-1}(t,y))\,dy, \qquad t \in [0,1),$$
$$u^m(1,x) = x. \tag{3.19}$$

(d) *For* $x > -\infty$, (3.19) *has a unique solution. If* $c = -\infty$ *and if*

$$\liminf_{s\uparrow 1}\frac{u(s)}{b(s)} < \infty, \tag{3.20}$$

where $b(s) := E[\sup_{\tau_k > s} Y_k]$, *then also in this case* $u^m = u^m(\cdot,-\infty)$ *for* $m \ge 2$ *is uniquely determined by* (3.19).

PROOF. The proof is by induction in m. Our induction hypothesis is that the statement of Theorem 3.3 holds and moreover that for any n-stopping time S and any $\mathcal{A}_S$-measurable $Z \ge c$ with $EZ^+ < \infty$ we have P-a.s.

$$E\big[Z \vee \overline{Y}_{T_1^m(S,Z)} \vee \cdots \vee \overline{Y}_{T_m^m(S,Z)} | \mathcal{A}_S\big] = u^m(S,Z) \ge E[Z \vee \overline{Y}_{T_1} \vee \cdots \vee \overline{Y}_{T_m} | \mathcal{A}_S]$$

for all N-stopping times $S < T_1 < \cdots < T_m \le 1$. Further,

$$A = \{(t,x) \in [0,1] \times [c,\infty) : u^m(t,x) > u^{m-1}(t,x)\}. \tag{3.21}$$

For the one-stopping problem $m = 1$ the statement of Theorem 3.3 is contained in Faller and Rüschendorf (2009). Proposition 3.1 with $v(t,x) := x$ implies the first part of the induction hypothesis while the second part follows from Faller and Rüschendorf (2009), Lemma 2.1(c).

For the induction step $m \to m+1$, we obtain for all stopping times $S < T_1 < T_2 < \cdots < T_{m+1} \le 1$ and $Z \ge c$ $\mathcal{A}_S$-measurable by the induction hypothesis (note that $\mathcal{A}_S \subset \mathcal{A}_{T_1}$):

$$\begin{aligned} &E[(Z \vee \overline{Y}_{T_1}) \vee \overline{Y}_{T_2} \vee \cdots \vee \overline{Y}_{T_{m+1}} | \mathcal{A}_S] \\ &\quad\le E\big[(Z \vee \overline{Y}_{T_1}) \vee \overline{Y}_{T_1^m(T_1, Z\vee\overline{Y}_{T_1})} \vee \cdots \vee \overline{Y}_{T_m^m(T_1, Z\vee\overline{Y}_{T_1})} | \mathcal{A}_S\big] \\ &\quad= E[u^m(T_1, Z \vee \overline{Y}_{T_1}) | \mathcal{A}_S]. \end{aligned} \tag{3.22}$$

This expression is maximized by Proposition 3.1 by $T_1 = T_1^{m+1}(S, Z)$ where

$$T_1^{m+1}(t,x) := \inf\{\tau_k > t : u^m(\tau_k, Y_k) > \hat{u}(\tau_k, x)\}, \qquad \inf\varnothing := 1.$$

The maximizing value is given by $\hat{u}(S, Z)$.

For the proof, we need to show that $\hat{u}(t,c) > u^m(t)$ for $t \in [0,1)$. We next establish this and at the same time show (3.21) for $m+1$.

Note that for $x \in [c, \infty)$

$$\begin{aligned}\hat{u}(t,x) &= \sup\{E[u^m(T, \overline{Y}_T \vee x)] : T > t\ N\text{-stopping time}\}\\ &\geq E\big[u^m\big(T_1^m(t,x), \overline{Y}_{T_1^m(t,x)} \vee x\big)\big]\\ &\overset{(*)}{\geq} E\big[u^{m-1}\big(T_1^m(t,x), \overline{Y}_{T_1^m(t,x)} \vee x\big)\big]\\ &= u^m(t,x) \qquad \text{by induction hypothesis.}\end{aligned}$$

By (3.21), we have strict inequality in $(*)$ if and only if $P((T_1^m(t,x), \overline{Y}_{T_1^m(t,x)}) \in A) > 0$. Using Lemma 2.4 in Faller and Rüschendorf (2009), we see that this is equivalent to $\mu(A \cap M_{\gamma^m(\cdot,x)} \cap (t,1] \times \mathbb{R}) > 0$. This in turn is equivalent to

$$A \cap M_{\gamma^m(\cdot,x)} \cap (t,1] \times \mathbb{R} \neq \varnothing \tag{3.23}$$

[since $\gamma^m(\cdot, x)$ is monotonically nonincreasing and by definition of A]. We are going to show that this is fulfilled for all points $(t,x) \in A$.

So let $(t,x) \in A$ and thus by induction hypothesis $u^m(t,x) > u^{m-1}(t,x)$ or equivalently $\gamma^m(t,x) > x$. Under the assumption that $M_{\gamma^m(\cdot,x)} \cap (t,1] \times \mathbb{R} \subset A^c$, we obtain that also $(t, \gamma^m(t,x)) \in A^c$ since A^c is closed. This implies that

$$u^m(t, \gamma^m(t,x)) = u^{m-1}(t, \gamma^m(t,x)) = u^m(t,x).$$

Since $u^m(t, \cdot)$ is strictly increasing, it follows that $\gamma^m(t,x) = x$, which is a contradiction. Thus, (3.23) holds true.

With the choice $S := t$, $Z := x$ further, we obtain

$$\hat{u}(t,x) = E\big[u^m\big(T_1^{m+1}(t,x), \overline{Y}_{T_1^{m+1}(t,x)} \vee x\big)\big] = u^{m+1}(t,x).$$

Finally, in (3.22) holds

$$T_l^m\big(T_1^{m+1}(S,Z), Z \vee \overline{Y}_{T_1^{m+1}(S,Z)}\big) = T_{l+1}^{m+1}(S,Z).$$

By Proposition 3.1 $u^{m+1}(\cdot, x)$ is the optimal stopping curve of the Poisson process $N^{m+1} = \sum_k \delta_{(\tau_k, u^m(\tau_k, Y_k))}$ on M_{u^m} at the guarantee value x. We already proved that the separation condition is fulfilled for the stopping of N^{m+1} and by Proposition 3.2 N^{m+1} has the intensity function $G^{m+1}(t,y) := G(t, \xi^m(t,y))$. The existence and uniqueness results for the differential equation (3.19) therefore follow with our assumption from the corresponding result in Faller and Rüschendorf (2009) for the case $m = 1$. □

4. Explicit calculation of optimal *m*-stopping curves. For the case of one-stopping problems, some classes of intensity functions $G(t,y)$ have been introduced in Faller and Rüschendorf (2009) which allow to determine optimal stopping curves in explicit form. Solving the optimality equations in (3.19) for the sequence of optimal stopping curves for the m-stopping problem is in general much more demanding. However, for some of the classes considered in Faller and Rüschendorf (2009) explicit solutions can be given also in the m-stopping case.

We consider intensity functions $G(t,y)$ of the form

$$G(t,y) = H\left(\frac{y}{v(t)}\right)\frac{|v'(t)|}{v(t)} \tag{4.1}$$

or

$$G(t,y) = H\big(y - v(t)\big)|v'(t)| \tag{4.2}$$

as in Faller and Rüschendorf (2009) with $v(1) = 0$ or $v(1) = \infty$ in case (4.1) and $v(1) = -\infty$ in case (4.2). For the general motivation of these classes and these conditions, we refer to Faller and Rüschendorf (2009). In particular, we will see that the main application considered in this paper to m-stopping of i.i.d. sequences with discount and observation costs is covered by these classes.

We first state the results in the three cases mentioned and then give the proof.

Case 1: G satisfies (4.1) with v monotonically nonincreasing, $v(1) = 0$. Here $c = 0$. $H : (0,\infty] \to [0,\infty)$ is monotonically nonincreasing continuous, $\int_0^\infty H(x)\,dx > 0$ and we assume that $v : [0,1] \to [0,\infty)$ is a C^1-function with $v > 0$ on $[0,1)$.

We define

$$R^1(x) := x - \int_x^\infty H(y)\,dy, \qquad x \in (0,\infty), \tag{4.3}$$

and assume that there exists some $r > 0$ with $R^1(r) = 0$. Define $r_0 := 0$, $\Phi^0(x) := x$. Then for $m \geq 1$ by induction holds:

The function $R^m : (r_{m-1},\infty) \to \mathbb{R}$ given by

$$R^m(x) := x - \int_x^\infty H(\Phi^{m-1}(y))\,dy \tag{4.4}$$

has exactly one zero $r_m \in (r_{m-1},\infty)$ and the optimal m-stopping curves are given for $(t,x) \in [0,1) \times [0,\infty]$ by

$$u^m(t,x) = \phi^m\left(\frac{x}{v(t)}\right)v(t), \tag{4.5}$$

where $\phi^m : [0,\infty] \to [r_m,\infty]$ is the inverse function of $\Phi^m : [r_m,\infty] \to [0,\infty]$,

$$\Phi^m(x) := x \exp\left(-\int_x^\infty \left(\frac{1}{R^m(y)} - \frac{1}{y}\right)dy\right).$$

The system of functions (R^m, Φ^m), respectively, (u^m, ϕ^m) is by (4.5) recursively defined. In particular, it holds that

$$u^m(t) = r_m v(t) \tag{4.6}$$

and thus determination of the optimal stopping curves is reduced to finding a zero point of $\mathbb{R}^m$.

Case 2: G satisfies (4.1) *with v monotonically nondecreasing,* $v(1) = \infty$. Here $c = -\infty$. $H : (-\infty, \infty] \to [0, \infty)$ is monotonically nonincreasing continuous, $\int_{-\infty}^0 H(x)\,dx > 0$, $\int_0^\infty H(x)\,dx = 0$ and $\int_y^0 \frac{H(x)}{-x}\,dx < \infty$ for $y < 0$. Further, we assume that $v : [0, 1] \to [0, \infty]$ is a C^1-function with $v < \infty$ on $[0, 1)$.

We define

$$R^1(x) := x + \int_x^\infty H(y)\,dy, \qquad x \in (-\infty, \infty),$$

and assume that there exists some $r < 0$ with $R^1(r) = 0$. Define $r_0 := -\infty$, $\Phi^0(x) := x$. Then for $m \ge 1$ by induction holds:

The function $R^m : (r_{m-1}, 0) \to \mathbb{R}$ defined by

$$R^m(x) := x + \int_x^0 H(\Phi^{m-1}(y))\,dy$$

has exactly one zero $r_m \in (r_{m-1}, 0)$ and the optimal m-stopping curves are given for $(t, x) \in [0, 1) \times \overline{\mathbb{R}}$ by

$$u^m(t, x) = \begin{cases} x, & \text{if } x \ge 0, \\ \phi^m\left(\dfrac{x}{v(t)}\right) v(t), & \text{if } x < 0, \end{cases} \tag{4.7}$$

where $\phi^m : [-\infty, 0] \to [r_m, 0]$ is the inverse of $\Phi^m : [r_m, 0] \to [-\infty, 0]$,

$$\Phi^m(x) := x \exp\left(\int_x^0 \left(\frac{1}{y} - \frac{1}{R^m(y)} \right) dy \right).$$

In particular, $u^m(t) = r_m v(t)$.

Case 3: G satisfies (4.2) *with v monotonically nonincreasing* $v(1) = -\infty$. Then $c = -\infty$. $H : (-\infty, \infty] \to [0, \infty)$ is monotonically nonincreasing continuous, $\int_{-\infty}^\infty H(x)\,dx > 0$ and $\int_z^\infty \int_y^\infty H(x)\,dx\,dy < \infty$ for $z \in \mathbb{R}$. Further, we assume that $v : [0, 1] \to [-\infty, \infty)$ is a C^1-function with $v > -\infty$ on $[0, 1)$.

We define

$$R^1(x) := 1 - \int_x^\infty H(y)\,dy, \qquad x \in \mathbb{R},$$

and assume that there exists some $r \in \mathbb{R}$ such that $R^1(r) = 0$. Define $r_0 := -\infty$, $\Phi^0(x) := x$. Then for $m \ge 1$ by induction holds:

The function $R^m : (r_{m-1}, \infty) \to \mathbb{R}$ defined by

$$R^m(x) := 1 - \int_x^\infty H(\Phi^{m-1}(y))\,dy$$

has exactly one zero $r_m \in (r_{m-1}, \infty)$. The optimal m-stopping curves are given for $(t, x) \in [0, 1) \times \overline{\mathbb{R}}$ by

$$u^m(t, x) = \phi^m\big(x - v(t)\big) + v(t), \tag{4.8}$$

where $\phi^m : \overline{\mathbb{R}} \to [r_m, \infty]$ is the inverse of $\Phi^m : [r_m, \infty] \to \overline{\mathbb{R}}$,

$$\Phi^m(x) := x - \int_x^\infty \left(\frac{1}{R^m(y)} - 1\right) dy.$$

We have $u^m(t) = r_m + v(t)$.

PROOF. We only give the proof of Case 2. The proof of both other cases is similar. The proof is by induction in m where we additionally include that $R^m \geq R^{m-1}$ and thus $\Phi^m \geq \Phi^{m-1}$.

In the case $m = 1$, the statement has been shown in Faller and Rüschendorf (2009) [with $r_0 := -\infty$, $\Phi^0(x) := x$, $R^0(x) := x$].

Induction step $m \to m + 1 : u^{m+1}(\cdot, x)$ is the optimal stopping curve of N^{m+1} at the guarantee value x. N^{m+1} has the intensity function

$$G^{m+1}(t, y) = H\left(\Phi^m\left(\frac{y}{v(t)}\right)\right)\frac{v'(t)}{v(t)} \qquad \text{for } (t, y) \in M_{u^m}.$$

Thus, G^{m+1} again is of type (4.1) and we have to check the conditions of Case 2 in Faller and Rüschendorf (2009), who deal with optimal one-stopping w.r.t. this type of intensity functions. First, we note that R^{m+1} has a zero in $(r_m, 0)$ since $\Phi^m(x) \geq \Phi^{m-1}(x)$ and thus $R^{m+1} \geq R^m$. Further by substitution, we have

$$\int_y^0 \frac{H(\Phi^m(x))}{-x}\, dx \overset{\text{Subst.}}{=} \int_{\Phi^m(y)}^0 \frac{H(z)}{-z} \frac{-z}{\phi^m(z)} (\phi^m)'(z)\, dz < \infty,$$

as $\lim_{z\to 0} \frac{-z}{\phi^m(z)} = 1$ and $\lim_{z\to 0}(\phi^m)'(z) = 1$. Thus, the conditions hold true and the result follows. $\square$

For intensity functions G not of the form as in (4.1), (4.2) the optimality differential equations in Theorem 3.3 typically can only be solved numerically. In some cases, however, one can derive bounds for the optimal stopping curves $u^m(t, x)$ which can be used to derive necessary uniform integrability and separation conditions [see Faller (2009), pages 60–62] for the following approximation result.

5. Approximation of *m*-stopping problems. In this section, an extension of the approximation results in Kühne and Rüschendorf [(2004, Theorem 2.1] and Faller and Rüschendorf [(2009), Theorem 4.1], for optimal one-stopping problems for dependent sequences is given to the class of m-stopping problems. For the special case of i.i.d. sequences with distribution function F in the domain of the

Gumbel extreme value distribution Λ a corresponding approximation result was given in the case $m = 2$ in Kühne and Rüschendorf (2002). The following result concerns the dependent case and needs a new technique of proof which is based on discretization. The main result of this section states that under some conditions convergence of the finite imbedded point processes N_n to a Poisson process N implies approximation of the stopping behavior.

We use the same general assumptions as in Section 4 of Faller and Rüschendorf (2009) as well as the notation in Section 2 for the Poisson process N. In particular, $\gamma^1, \dots, \gamma^m$ are the functions defined in (3.15). Further, the lower boundary curve f of N is given by $f \equiv c$, N is a Poisson process on $[0, 1] \times (\overline{\mathbb{R}} \setminus \{c\})$ and $\mathcal{F}^n$ are the canonical filtrations induced by the imbedded point process N_n and we assume the convergence condition $N_n \xrightarrow{d} N$ on M_f as throughout this paper [see (1.2) and the introduction of Section 3].

The first result is an extension of Proposition 2.4 in Kühne and Rüschendorf (2000a) on the convergence of threshold stopping times to the case $m \geq 1$. For the technically involved proof, we refer to Faller (2009), Lemma 2.6.

PROPOSITION 5.1 (Convergence of multiple threshold stopping times). *Let $(t, x) \in [0, 1] \times [c, \infty)$ be fixed and let $v_n^m : [0, 1] \to \overline{\mathbb{R}}$ be functions such that $v_n^m \to \gamma^m(\cdot, x)$ uniformly on any interval $[0, s]$ with $s < 1$. Define the corresponding threshold stopping times*

$$\hat{T}_1^{n,m}(t, x) := \min\left\{tn < i \leq n - m + 1 : X_i^n > v_n^m\left(\frac{i}{n}\right)\right\},$$

$$\hat{T}_\ell^{n,m}(t, x) := \min\Bigg\{\hat{T}_{\ell-1}^{n,m}(t, x) < i \leq n - m + \ell : \\ X_i^n > \gamma^{m-\ell+1}\left(\frac{i}{n}, X^n_{\hat{T}_{\ell-1}^{n,m}(t,x)} \vee x\right)\Bigg\}$$

for $2 \leq \ell \leq m$. If $N_n \xrightarrow{d} N$ on M_c, we obtain convergence

$$\left(\frac{\hat{T}_\ell^{n,m}(t, x)}{n}, X^n_{\hat{T}_\ell^{n,m}(t,x)} \vee x\right)_{1 \leq \ell \leq m} \xrightarrow{d} \left(T_\ell^m(t, x), \overline{Y}_{T_\ell^m(t,x)} \vee x\right)_{1 \leq \ell \leq m}. \tag{5.1}$$

Let now $W_k^{n,m}(x)$ be the stopping thresholds for the m stopping of $X_1^n, \dots, X_n^n$ and the filtration $\mathcal{F}^n$ (see Section 2). The optimal *m-stopping curves* w.r.t. $\mathcal{F}^n$ are defined as follows. For $t \in [0, \frac{n-m+1}{n})$ and $x \in \overline{\mathbb{R}}$ let

$$u_n^m(t, x) := W_{\lfloor tn \rfloor}^{n,m}(x)$$

and $u_n^m(t, x) := W_{n-m+1}^{n,m}(x)$ for $t \in [\frac{n-m+1}{n}, 1]$.

More explicitly, we have for $t \in [0, \frac{n-m+1}{n})$ (see Theorem 2.3)

$$u_n^m(t,x) = \operatorname{ess\,sup}\{E[X_{T_1}^n \vee \cdots \vee X_{T_m}^n \vee x | \mathcal{F}_{\lfloor tn \rfloor}^n] : tn < T_1 < \cdots < T_m \le n \quad \mathcal{F}^n\text{-stopping times}\}$$
$$= E[X^n_{T_1^{n,m}(t,x)} \vee \cdots \vee X^n_{T_m^{n,m}(t,x)} \vee x | \mathcal{F}_{\lfloor tn \rfloor}^n] \qquad P\text{-a.s.} \tag{5.2}$$

The corresponding optimal m-stopping times are given by

$$T_1^{n,m}(t,x) := \min\left\{tn < i \le n-m+1 : u_n^{m-1}\left(\frac{i}{n}, X_i^n\right) > u_n^m\left(\frac{i}{n}, x\right)\right\},$$
$$T_\ell^{n,m}(t,x) := \min\left\{T_{\ell-1}^{n,m}(t,x) < i \le n-m+\ell : u_n^{m-\ell}\left(\frac{i}{n}, X_i^n\right) > u_n^{m-\ell+1}\left(\frac{i}{n}, M_{\ell-1,i}^{n,m} \vee x\right)\right\} \tag{5.3}$$

for $2 \le \ell \le m$, where $M_{j,i}^{n,m} := X^n_{T_j^{n,m}(t,x)} \chi_{\{T_j^{n,m}(t,x) \le i\}}$.

$u_n^m(\cdot, x)$ is right continuous and a piecewise constant curve in the space of random variables. We have the iterative representation (see Theorem 2.3)

$$u_n^m(t,x) = E\left[u_n^{m-1}\left(\frac{T_1^{n,m}(t,x)}{n}, X^n_{T_1^{n,m}(t,x)} \vee x\right) \Big| \mathcal{F}_{\lfloor tn \rfloor}^n\right] \qquad P\text{-a.s.}$$

Further, u_n^m are monotone in the sense that for $0 \le s \le t \le 1$

$$u_n^m(s,x) \ge E[u_n^m(t,x) | \mathcal{F}_{\lfloor sn \rfloor}^n] \qquad P\text{-a.s.}$$

In the opposite direction, we obtain for $0 \le s \le t \le 1$

$$u_n^m(s,x) \le E\left[\max_{s < i/n \le t} u_n^{m-1}\left(\frac{i}{n}, X_i^n\right) \vee u_n^m(t,x) | \mathcal{F}_{\lfloor sn \rfloor}^n\right] \qquad P\text{-a.s.} \tag{5.4}$$

This follows inductively from the recursive definition of the thresholds $W_\ell^m(x)$. We also need the following further conditions [for motivation, see Faller and Rüschendorf (2009)]:

(A) *Asymptotic independence condition.* For $0 \le s < t \le 1$

$$P\left(\max_{s < i/n \le t} X_i^n \le x | \mathcal{F}_{\lfloor sn \rfloor}^n\right) \xrightarrow{P} P\left(\sup_{s < \tau_k \le t} Y_k \le x\right) \qquad \forall x \in (c, \infty).$$

(U) *Uniform integrability condition.* M_n^+, with $M_n := \max_{1 \le i \le n} X_i^n$, is uniformly integrable and $E[\limsup_{n\to\infty} M_n^+] < \infty$.

(L) *Uniform integrability from below.* For some sequence $(v_n)_{n\in\mathbb{N}}$ of monotonically nonincreasing functions $v_n : [0,1] \to \mathbb{R} \cup \{-\infty\}$ with $v_n \to u$ pointwise, for all $t \in [0, 1)$ and the corresponding threshold stopping times

$$\hat{T}_n(t) := \min\left\{tn < i \le n : X_i^n > v_n\left(\frac{i}{n}\right)\right\}$$

holds

$$\lim_{s\uparrow 1}\limsup_{n\to\infty} E\big[X^n_{\hat{T}_n(t)}\chi_{\{\hat{T}_n(t)>sn\}}\big]=0. \tag{5.5}$$

A modified version of (L) is the condition (L^m):

(L^m) For $m\in\mathbb{N}$, there exists some sequence of monotonically nonincreasing functions $v^m_n:[0,1]\to\overline{\mathbb{R}}$ such that $v^m_n\to\gamma^m(\cdot,-\infty)$ pointwise and further the corresponding threshold stopping times

$$\hat{T}^{n,m}_1(t):=\min\biggl\{tn<i\le n-m+1:X^n_i>v^m_n\biggl(\frac{i}{n}\biggr)\biggr\}$$

satisfy

$$\lim_{s\uparrow 1}\limsup_{n\to\infty} E\big[X^n_{\hat{T}^{n,m}_1(t)}\chi_{\{\hat{T}^{n,m}_1(t)>sn\}}\big]=0.$$

Condition (L^m) in combination with (U) implies uniform integrability of $(X^n_{\hat{T}^{n,m}_1(t)})_{n\in\mathbb{N}}$. Denote

$$T^{n,m}_\ell:=T^{n,m}_\ell(0,c)\quad\text{and}\quad T^m_\ell:=T^m_\ell(0,c).$$

THEOREM 5.2 (Approximation of m-stopping problems). *Assume that $N_n\xrightarrow{d}N$ on $[0,1]\times(\overline{\mathbb{R}}\setminus\{c\})$ and also assume conditions* (A) *and* (U). *In case $c=-\infty$ also assume the modified uniform integrability condition* (L^m).

(a) *For all $(t,x)\in[0,1]\times[c,\infty)$ holds*

$$u^m_n(t,x)\xrightarrow{P}u^m(t,x).$$

If $c\in\mathbb{R}$ assume $X^n_n\xrightarrow{L^1}c$. Then we have in particluar

$$E[X^n_{T^{n,m}_1}\vee\cdots\vee X^n_{T^{n,m}_m}]\to u^m(0). \tag{5.6}$$

(b) *In case $(X^n_i)_{1\le i\le n}$ are independent random variables and if for $c\in\mathbb{R}$ we assume that $\mu(M_{\gamma^m})=\infty$ or $X^n_{n-i}\xrightarrow{P}c$ for $i=0,\dots,m-1$, then we obtain*

$$\biggl(\frac{T^{n,m}_\ell}{n},X^n_{T^{n,m}_\ell}\biggr)_{1\le\ell\le m}\xrightarrow{d}(T^m_\ell,\overline{Y}_{T^m_\ell}\vee c)_{1\le\ell\le m}.$$

(c) *If $c\in\mathbb{R}$ and $X^n_n\xrightarrow{L^1}c$, then*

$$\hat{T}^{n,m}_1:=\min\biggl\{1\le i\le n-m+1:X^n_i>\gamma^m\biggl(\frac{i}{n},c\biggr)\biggr\},$$

$$\hat{T}^{n,m}_\ell:=\min\biggl\{\hat{T}^{n,m}_{\ell-1}<i\le n-m+\ell:X^n_i>\gamma^{m-\ell+1}\biggl(\frac{i}{n},X^n_{\hat{T}^{n,m}_{\ell-1}}\vee c\biggr)\biggr\},$$
$$2\le\ell\le m,$$

defines an asymptotically optimal sequence of m-stopping times, that is, convergence as in (5.6) holds for these stopping times. In case $c = -\infty$,

$$\hat{T}_1^{n,m} := \min\left\{1 \le i \le n-m+1 : X_i^n > v_n^m\left(\frac{i}{n}\right)\right\},$$

$$\hat{T}_\ell^{n,m} := \min\left\{\hat{T}_{\ell-1}^{n,m} < i \le n-m+\ell : X_i^n > \gamma^{m-\ell+1}\left(\frac{i}{n}, X^n_{\hat{T}_{\ell-1}^{n,m}}\right)\right\},$$
$$2 \le \ell \le m,$$

are asymptotically optimal stopping times, where v_n^m are the threshold functions from condition (L^m).

PROOF. Since we use point process convergence on $[0,1] \times (\overline{\mathbb{R}} \setminus \{c\})$ and canonical filtrations, we can apply the Skorohod theorem and hence we assume w.l.o.g. P-a.s. convergence of the point processes.

(a) Consider at first the case $c \in \mathbb{R}$. Let $t \in [0,1)$ be a fixed element. We introduce at first discrete majorizing stopping problems. For $m \ge 1$ and $k > m$, define the discrete time points

$$a_i^k := \left(1 - \frac{i}{k}\right)t + \frac{i}{k}1, \qquad 0 \le i \le k,$$

and discrete time random variables

$$X_i^{n,k} := \max_{j/n \in (a_{i-1}^k, a_i^k]} X_j^n \vee c \qquad \text{for } 1 \le i \le k,$$

and consider the filtration $\mathcal{F}^{n,k} := (\mathcal{F}_i^{n,k})_{0 \le i \le k}$ with $\mathcal{F}_i^{n,k} := \mathcal{F}^n_{\lfloor a_i^k n \rfloor}$. The corresponding m-stopping curves are given inductively for $m \ge 1$ by backward induction for $i = k, \dots, 0$ by

$$^m W_{k-m+1}^{n,k}(x) := x,$$
$$^m W_i^{n,k}(x) := E[^{m-1}W_{i+1}^{n,k}(X_{i+1}^{n,k}) \vee {}^m W_{i+1}^{n,k}(x) | \mathcal{F}_i^{n,k}]$$
$$\text{for } i = k-m, \dots, 0.$$

These stopping problems majorize the original m-stopping problem,

$$\begin{aligned} ^m W_0^{n,k}(x) &= \operatorname{ess\,sup}\{E[X_{T_1'}^{n,k} \vee \cdots \vee X_{T_m'}^{n,k} \vee x : \mathcal{F}_0^{n,k}]: \\ &\qquad 0 < T_1' < \cdots < T_m' \le k \ \mathcal{F}^{n,k}\text{-stopping times}\} \\ &\overset{(*)}{=} \operatorname{ess\,sup}\{E[X_{T_1'}^{n,k} \vee \cdots \vee X_{T_m'}^{n,k} \vee x : \mathcal{F}_0^{n,k}]: \\ &\qquad 0 < T_1' \le \cdots \le T_m' \le k \ \mathcal{F}^{n,k}\text{-stopping times}\} \end{aligned}$$

$$\geq \operatorname{ess\,sup}\big\{E\big[X^n_{T_1} \vee \cdots \vee X^n_{T_m} \vee x|\mathcal{F}^n_{\lfloor tn\rfloor}\big]: \\ tn < T_1 < \cdots < T_m \leq n \ \mathcal{F}^n\text{-stopping times}\big\} \\ = u^m_n(t,x) \qquad P\text{-a.s.},$$

since for all $\mathcal{F}^n$-stopping times $tn < T_1 < \cdots < T_m \leq n$ it holds that $T_i' := \lceil \frac{1}{1-t}(\frac{T_i}{n} - t)k\rceil > 0$ are $\mathcal{F}^{n,k}$-stopping times with $a^k_{T_i'-1} < \frac{T_i}{n} \leq a^k_{T_i'}$, thus $X^{n,k}_{T_i'} \geq X^n_{T_i}$. For the proof of $(*)$ define for $\mathcal{F}^{n,k}$-stopping times $0 < T_1' \leq \cdots \leq T_m' \leq k$ the $\mathcal{F}^{n,k}$-stopping times $0 < T_1^* < \cdots < T_m^* \leq k$ by

$$T_1^* := T_1' \wedge (k-m+1),$$
$$T_\ell^* := \big((T_\ell'+1)\chi_{\{T^*_{\ell-1}=T'_\ell\}} + T'_\ell\chi_{\{T^*_{\ell-1}<T'_\ell\}}\big) \wedge (k-m+\ell), \qquad \ell = 2,\dots,m.$$

We will prove convergence as $n \to \infty$ to the stopping problem of

$$Y^k_i := \sup_{\tau_l \in (a^k_{i-1}, a^k_i]} Y_l \vee c \qquad \text{for } 1 \leq i \leq k,$$

with filtrations $\mathcal{A}^k := (\mathcal{A}^k_i)_{1\leq i\leq k}$, $\mathcal{A}^k_i := \mathcal{A}_{a^k_i}$ and optimal thresholds

$${}^m u^k_{k-m+1}(x) := x,$$
$${}^m u^k_i(x) := E[{}^{m-1}u^k_{i+1}(Y^k_{i+1}) \vee {}^m u^k_{i+1}(x)] \qquad \text{for } i = k-m,\dots,0.$$

By definition for $i \leq k-m$ holds

$$\begin{aligned}{}^m u^k_i(x) &= V\big({}^{m-1}u^k_{i+1}(Y^k_{i+1}) \vee x, \dots, {}^{m-1}u^k_{k-m+1}(Y^k_{k-m+1}) \vee x\big) \\ &= \sup\{E[{}^{m-1}u^k_T(Y^k_T) \vee x] : i < T \leq k-m+1 \ \mathcal{A}^k\text{-stopping times}\} \\ &= {}^m u^k(a^k_i, x),\end{aligned}$$

where ${}^m u^k(\cdot, x)$ are the optimal stopping curves of the processes

$${}^m N^k := \sum_{i=1}^{k-m+1} \delta_{(a^k_i, {}^{m-1}u^k_i(Y^k_i))} = \sum_{i=1}^{k-m+1} \delta_{(a^k_i, {}^{m-1}u^k(a^k_i, Y^k_i))}$$

at guarantee value x.

At first we establish that for any i the random variable Y^k_{i+1} is independent of the σ-algebra $\mathcal{F}^k_i := \sigma(\bigcup_{n\in\mathbb{N}} \mathcal{F}^{n,k}_i)$.

For the proof, note that by condition (A)

$$P(X^{n,k}_{i+1} \in \cdot|\mathcal{F}^{n,k}_i) \xrightarrow{P} P(Y^k_{i+1} \in \cdot).$$

Thus, we obtain by the continuous mapping theorem that for any continuous $f : \overline{\mathbb{R}} \to [0,1]$ we have

$$P\big(f(X^{n,k}_{i+1}) \in \cdot|\mathcal{F}^{n,k}_i\big) \xrightarrow{P} P\big(f(Y^k_{i+1}) \in \cdot\big).$$

This implies using uniform integrability that

$$E[f(X_{i+1}^{n,k})|\mathcal{F}_i^{n,k}] \xrightarrow{L^1} E[f(Y_{i+1}^k)].$$

On the other hand, by point process convergence it holds that $X_{i+1}^{n,k} \to Y_{i+1}^k$ P-a.s. and thus also $f(X_{i+1}^{n,k}) \xrightarrow{L^1} f(Y_{i+1}^k)$. This implies L^1-convergence of conditional expectations:

$$E[f(X_{i+1}^{n,k})|\mathcal{F}_i^{n,k}] \xrightarrow{L^1} E[f(Y_{i+1}^k)|\mathcal{F}_i^k].$$

In consequence, we obtain $E[f(Y_{i+1}^k)] = E[f(Y_{i+1}^k)|\mathcal{F}_i^k]$ P-a.s. for all continuous functions $f : \overline{\mathbb{R}} \to [0, 1]$, and thus independence of $\mathcal{F}_i^k$ and $\sigma(Y_{i+1}^k)$.

The next point to establish is proved by induction in m. The induction hypothesis is:

(1) For all $k > m$, $x \in [c, \infty)$ and $i = k - m + 1, \ldots, 0$

$${}^m W_i^{n,k}(x) \xrightarrow{P} {}^m u_i^k(x), \qquad n \to \infty.$$

(2) For all $s \in [t, 1]$ and all $x \in [c, \infty)$, we further have

$${}^m u^k(s, x) \to u^m(s, x), \qquad k \to \infty.$$

We do the induction step for $m - 1 \to m$: Assertion (1) we shall prove by backward induction on i: For $i = k - m + 1$ the assertion is trivial. We now consider the induction step from $i + 1$ to i: From the induction hypothesis, we know that

$${}^{m-1} W_{i+1}^{n,k}(x) \xrightarrow{P} {}^{m-1} u_{i+1}^k(x), \qquad n \to \infty,$$

for all $x \in [c, \infty)$. From this, the monotonicity of ${}^{m-1} W_{i+1}^{n,k}(x)$ in x and the continuity of ${}^{m-1} u_{i+1}^k(x)$ in x we can conclude that

$${}^{m-1} W_{i+1}^{n,k}(X_{i+1}^{n,k}) \xrightarrow{P} {}^{m-1} u_{i+1}^k(Y_{i+1}^k), \qquad n \to \infty.$$

For details, see Faller (2009). By the induction hypothesis for i, we also know that

$${}^m W_{i+1}^{n,k}(x) \xrightarrow{P} {}^m u_{i+1}^k(x), \qquad n \to \infty,$$

for $x \in [c, \infty)$, implying

$${}^{m-1} W_{i+1}^{n,k}(X_{i+1}^{n,k}) \vee {}^m W_{i+1}^{n,k}(x) \xrightarrow{L^1} {}^{m-1} u_{i+1}^k(Y_{i+1}^k) \vee {}^m u_{i+1}^k(x), \qquad n \to \infty.$$

From this, we get

$$\begin{aligned} &E[{}^{m-1} W_{i+1}^{n,k}(X_{i+1}^{n,k}) \vee {}^m W_{i+1}^{n,k}(x)|\mathcal{F}_i^{n,k}] \\ &\qquad \xrightarrow{L^1} E[{}^{m-1} u_{i+1}^k(Y_{i+1}^k) \vee {}^m u_{i+1}^k(x)|\mathcal{F}_i^k] \end{aligned}$$

as $n \to \infty$. The expression on the left-hand side equals ${}^m W_i^{n,k}(x)$, and since $\sigma(Y_{i+1}^k)$ and $\mathcal{F}_i^k$ are independent as shown above, the right-hand side equals ${}^m u_i^k(x)$. This completes the induction on i and the proof of assertion (1).

For the proof of assertion (2), observe that the process $\sum_{i=1}^k \delta_{(a_i^k, Y_i^k)}$ converges on $[t,1] \times (c,\infty]$ to $N = \sum_j \delta_{(\tau_j, Y_j)}$. Further, by induction hypothesis we have uniform convergence of ${}^{m-1}u^k(s,x)$ to $u^{m-1}(s,x)$ as $k \to \infty$. From this, we obtain convergence of the transformed point processes

$$ {}^m N^k = \sum_{i=1}^k \delta_{(a_i^k, {}^{m-1}u^k(a_i^k, Y_i^k))} \xrightarrow{d} N^m = \sum_j \delta_{(\tau_j, u^{m-1}(\tau_j, Y_j))}, \qquad k \to \infty, $$

on $M_{u^{m-1}} \cap [t,1] \times \overline{\mathbb{R}}$ and thus convergence of the optimal stopping curves of these processes, which proves (2).

Based on (1) and (2), we obtain the estimate

$$ \begin{aligned} &P\big(u_n^m(t,x) \ge u^m(t,x) + \varepsilon\big) \\ &\quad \le P\bigg({}^m W_0^{n,k}(x) \ge \underbrace{{}^m u^k(t,x)}_{{}^m u_0^k(x)} + \frac{\varepsilon}{2}\bigg) + P\bigg(u^m(t,x) \le {}^m u^k(t,x) - \frac{\varepsilon}{2}\bigg). \end{aligned} $$

The right-hand side converges for $n \to \infty$ and $k \to \infty$ to 0. Thus, we have shown

$$ \lim_{n\to\infty} P\big(u_n^m(t,x) \ge u^m(t,x) + \varepsilon\big) = 0. $$

To obtain convergence in probability, we next establish that $\liminf_{n\to\infty} E u_n^m(t, x) \ge u^m(t,x)$. This however is implied by the inequality

$$ E u_n^m(t,x) \ge E[X_{T_1}^n \vee \cdots \vee X_{T_m}^n \vee x] $$

holding true for all $\mathcal{F}^n$-stopping times $tn < T_1 < \cdots < T_m \le n$, and in particular for

$$ \begin{aligned} \hat{T}_1^{n,m}(t,x) &:= \min\bigg\{tn < i \le n - m + 1 : X_i^n > \gamma^m\bigg(\frac{i}{n}, x\bigg)\bigg\}, \\ \hat{T}_\ell^{n,m}(t,x) &:= \min\bigg\{\hat{T}_{\ell-1}^{n,m}(t,x) < i \le n - m + \ell : \\ &\qquad\qquad X_i^n > \gamma^{m-\ell+1}\bigg(\frac{i}{n}, X^n_{\hat{T}_{\ell-1}^{n,m}(t,x)} \vee x\bigg)\bigg\} \end{aligned} $$

for $2 \le \ell \le m$. Proposition 5.1 then implies the above statement.

For $c = -\infty$, we obtain similarly the convergence $u_n^m(t,x) \xrightarrow{P} u^m(t,x)$ for $x > -\infty$. Then the convergence of $u_n^m(t,-\infty) \xrightarrow{P} u^m(t)$ results as follows:

$$ u_n^m(t,-\infty) \le u_n^m(t,x) \xrightarrow{P} u^m(t,x) \downarrow u^m(t) \qquad \text{as } x \downarrow -\infty. $$

This implies that $\lim_{n\to\infty} P(u_n^m(t,-\infty) \geq u^m(t) + \varepsilon) = 0$ for all $\varepsilon > 0$. Let $\hat{T}_1^{n,m}(t)$ be the stopping times from condition (L^m) and let

$$\hat{T}_\ell^{n,m}(t) := \min\left\{\hat{T}_{\ell-1}^{n,m}(t) < i \leq n-m+\ell : X_i^n > \gamma^{m-\ell+1}\left(\frac{i}{n}, X^n_{\hat{T}_{\ell-1}^{n,m}(t)}\right)\right\}$$

for $2 \leq \ell \leq m$. Then we obtain by Proposition 5.1 and uniform integrability of $(X^n_{\hat{T}_1^{n,m}(t)})_{n\in\mathbb{N}}$ that

$$\begin{aligned} Eu_n^m(t,-\infty) &\geq E\bigl[X^n_{\hat{T}_1^{n,m}(t)} \vee \cdots \vee X^n_{\hat{T}_m^{n,m}(t)}\bigr] \\ &\xrightarrow{n\to\infty} E\bigl[\overline{Y}_{T_1^m(t,-\infty)} \vee \cdots \vee \overline{Y}_{T_m^m(t,-\infty)}\bigr] = u^m(t). \end{aligned}$$

Thus, $\liminf_{n\to\infty} Eu_n^m(t,-\infty) \geq u^m(t)$. As consequence, we obtain $u_n^m(t,-\infty) \xrightarrow{P} u^m(t)$ which was to be shown.

(b) For the proof of (b), see Faller (2009).

(c) For $c = -\infty$, we obtain the statement using uniform integrability and Proposition 5.1. For $c \in \mathbb{R}$ holds

$$\begin{aligned} &E[X^n_{\hat{T}_1^{n,m}} \vee \cdots \vee X^n_{\hat{T}_m^{n,m}}] \\ &\quad = E[X^n_{\hat{T}_1^{n,m}} \vee \cdots \vee X^n_{\hat{T}_m^{n,m}} \vee c] \\ &\qquad - \int_{\{X^n_{\hat{T}_1^{n,m}} \vee\cdots\vee X^n_{\hat{T}_m^{n,m}} < c\}} (c - X^n_{\hat{T}_1^{n,m}} \vee \cdots \vee X^n_{\hat{T}_m^{n,m}})\, dP. \end{aligned}$$

The first term converges by Proposition 5.1 to the stated limit. The modulus of the second term can be estimated from above by

$$\int_{\{X^n_{\hat{T}_m^{n,m}} < c\}} (c - X^n_{\hat{T}_m^{n,m}})\, dP \leq \int_{\{X_n^n < c\}} (c - X_n^n)\, dP \leq E|X_n^n - c| \to 0. \qquad \square$$

REMARK 5.3. The reason for restricting in (b) to independent sequences is the necessity to give estimates of $u_n(t,x)$ from above [cf. the case $m = 1$ in Faller (2009)]. In the dependent case, this amounts to (5.4). For $m \geq 2$ in contrast to the case $m = 1$ one has to consider terms of the form $\max_{s<i/n\leq t} u_n^{m-1}(\frac{i}{n}, X_i^n)$. It seems however difficult to establish the necessary point process convergence of $\sum_{i=1}^n \delta_{(i/n, u_n^{m-1}(i/n, X_i))}$ in the general dependent case.

6. Optimal m-stopping of i.i.d. sequences with discount and observation costs. As application, we study in this section the optimal m-stopping of i.i.d. sequences with discount and observation costs. In the case $m = 1$, this problem has been considered in various degree of generality in Kennedy and Kertz (1990), Kennedy and Kertz (1991), Kühne and Rüschendorf (2000b) and Faller and Rüschendorf (2009).

Let $(Z_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence with d.f. F in the domain of attraction of an extreme value distribution G, thus for some constants $a_n > 0$, $b_n \in \mathbb{R}$

$$n\big(1 - F(a_n x + b_n)\big) \to -\log G(x), \qquad x \in \mathbb{R}. \tag{6.1}$$

Consider $X_i = c_i Z_i + d_i$ the sequence with discount and observation factors, $c_i > 0$, $d_i \in \mathbb{R}$ and both sequences monotonically nondecreasing or nonincreasing. For convergence of the corresponding imbedded point processes

$$\hat{N}_n = \sum_{i=1}^{n} \delta_{(i/n,(X_i-\hat{b}_n)/\hat{a}_n)} \tag{6.2}$$

the following choices of $\hat{a}_n$, $\hat{b}_n$ turn out to be appropriate:

$$\begin{aligned} &\hat{a}_n := c_n a_n, \hat{b}_n := 0 \qquad \text{for } F \in D(\Phi_\alpha) \text{ or } F \in D(\Psi_\alpha),\\ &\hat{a}_n := c_n a_n, \hat{b}_n := c_n b_n + d_n \qquad \text{for } F \in D(\Lambda), \end{aligned} \tag{6.3}$$

where Φ_α, Ψ_α, Λ are the Fréchet, Weibull, and Gumbel distributions and a_n, b_n are the corresponding normalizations in (6.1). We give further conditions on c_i, d_i to establish point process convergence in (6.2). Related conditions are given in de Haan and Verkade (1987) in the treatment of i.i.d. sequences with trends, respectively, in Kühne and Rüschendorf (2000b).

Unlike before, c denotes here a general constant and not the guarantee value. The guarantee value of N is in case Φ_α given by 0 and in cases Ψ_α, Λ given generally by $-\infty$. This application shows in particular the importance of treating the case with lower boundary $-\infty$ as in Sections 2 and 3 of this paper, respectively, in Faller and Rüschendorf (2009). We state the optimality results for all three cases. It turns out that in all of the following examples the intensity functions of the transformed Poisson processes are of the form studied in Section 4. Hence, we obtain an explicit form of the solutions and optimal stopping curves.

We first consider the case of Fréchet limits.

THEOREM 6.1. *Let $F \in D(\Phi_\alpha)$ with $\alpha > 1$ and $F(0) = 0$ (i.e., $Z_i > 0$ P-a.s.). We assume that $b_n = 0$ and also convergence*

$$\frac{d_n}{c_n a_n} \to d, \qquad \frac{c_{\lfloor tn \rfloor}}{c_n} \to t^c \qquad \forall t \in [0,1]$$

with constants $c, d \in \mathbb{R}$, as well that c_n does not converge to 0. Assume that $c > -\frac{1}{\alpha}$ and that the function $R : (d,\infty) \to \mathbb{R}$,

$$R(x) := x + \frac{\alpha}{\alpha - 1}\frac{1}{1 + c\alpha}(x - d)^{-\alpha+1}, \qquad x \in (d,\infty), \tag{6.4}$$

has no zero point. Then it holds:

(a)

$$E[X_{T_1^{n,m}} \vee \cdots \vee X_{T_m^{n,m}}] / \hat{a}_n \to u^m(0) > 0, \tag{6.5}$$

where $u^m(t)$ is the m-stopping curve of the Poisson process $\hat{N}$ with intensity function

$$\hat{G}(t, y) = t^{c\alpha}(y - dt^{c+1/\alpha})^{-\alpha} = H\left(\frac{y}{v(t)}\right)\frac{v'(t)}{v(t)} \qquad \text{on } M_{\hat{f}}.$$

Here $v(t) := t^{c+1/\alpha}$, $H(x) := \frac{\alpha}{\alpha c+1}(x-d)^{-\alpha}$ *and* $\hat{f}(t) := dt^{c+1/\alpha}$.

(b) *Let* $\gamma^1, \dots, \gamma^m$ *be the functions defined in* (3.15) *for* $\hat{N}$. *Then*

$$\hat{T}_1^{n,m} := \min\left\{1 \le i \le n-m+1 : X_i > \hat{a}_n \gamma^m\left(\frac{i}{n}, d\right)\right\},$$

$$\hat{T}_\ell^{n,m} := \min\left\{\hat{T}_{\ell-1}^{n,m} < i \le n-m+\ell : X_i > \hat{a}_n \gamma^{m-\ell+1}\left(\frac{i}{n}, \left(\frac{1}{\hat{a}_n} X_{\hat{T}_{\ell-1}^{n,m}}\right) \vee d\right)\right\}$$

for $2 \le \ell \le m$ *are asmptotically optimal sequences of* m*-stopping times, that is, the limit in* (6.5) *is attained also for these sequences.*

The next result concerns the Weibull limit case.

THEOREM 6.2. *Let* $F \in D(\Psi_\alpha)$ *with* $\alpha > 0$ *and* $F(0) = 1$ (*i.e.*, $Z_i \le 0$ P*-a.s.*). *Further let* $a_n \downarrow 0$ *and* $b_n = 0$, *and*

$$\frac{d_n}{c_n a_n} \to d, \qquad \frac{c_{\lfloor tn \rfloor}}{c_n} \to t^c \qquad \forall t \in [0, 1]$$

for constants $c, d \in \mathbb{R}$. *If* $d_n > 0$, *then assume that either* $(d_n)_{n\in\mathbb{N}}$ *is monotonically nondecreasing or* $c_n a_n$ *does not converge to* 0.

(a) *If* $c < \frac{1}{\alpha}$ *and* $d \le 0$, *then it holds*

$$\frac{E[X_{T_1^{n,m}} \vee \cdots \vee X_{T_m^{n,m}}]}{\hat{a}_n} \to u_{c,d}^m(0) < 0. \tag{6.6}$$

(b) *If* $c > \frac{1}{\alpha}$ *and the function* $R : \mathbb{R} \to \mathbb{R}$,

$$R(x) := \begin{cases} x, & \text{if } x \ge d, \\ x - \dfrac{\alpha}{\alpha+1}\dfrac{1}{1-c\alpha}(-x+d)^{\alpha+1}, & \text{if } x < d, \end{cases} \tag{6.7}$$

has no zero point then (6.6) *holds with* $u_{c,d}^m(0) > 0$. *Here* $u_{c,d}^m(t)$ *is the* m*-stopping curve of the Poisson process* $\hat{N} = \hat{N}_{c,d}$. $\gamma_{c,d}^m$ *are the corresponding inverse functions defined in* (3.15) *and* (3.16).

(c) *Let* (w_n) *be an increasing sequence* $w_n < 0$ *such that* $n(1-F(w_n)) \to \frac{\alpha+1}{\alpha}$ [*e.g.*, $w_n = -(\frac{\alpha+1}{\alpha})^{1/\alpha} a_n$]. *Define functions* v_n^m *by*

$$v_n^m(t) := \frac{\gamma_{c,0}^m(t)}{u_{0,0}(t)} \frac{w_{\lfloor (1-t)n \rfloor}}{a_n} + \gamma_{c,d}^m(t) - \gamma_{c,0}^m(t),$$

where $\gamma_{c,0}^m(t) = -\Phi^{m-1}(r_m) u_{c,0}(t)$. *Then the* m-*stopping times defined by*

$$\hat{T}_1^{n,m} := \min\left\{1 \le i \le n-m+1 : X_i > \hat{a}_n v_n^m\left(\frac{i}{n}\right)\right\},$$

$$\hat{T}_\ell^{n,m} := \min\left\{\hat{T}_{\ell-1}^{n,m} < i \le n-m+\ell : X_i > \hat{a}_n \gamma_{c,d}^{m-\ell+1}\left(\frac{i}{n}, \frac{1}{\hat{a}_n} X_{\hat{T}_{\ell-1}^{n,m}}\right)\right\}$$

for $2 \le \ell \le m$, *are asymptotically optimal*, *that is*, *convergence as in* (6.6) *does also hold for them.*

The final result concerns the Gumbel case.

THEOREM 6.3. *Let* $F \in D(\Lambda)$ *and assume*

$$\frac{b_n}{a_n}\left(1 - \frac{c_{\lfloor tn \rfloor}}{c_n}\right) \to c \log(t), \qquad \frac{d_n - d_{\lfloor tn \rfloor}}{c_n a_n} \to d \log(t) \qquad \forall t \in [0,1]$$

for some constants $c, d \in \mathbb{R}$. *Assume also that* $(c_n)_{n\in\mathbb{N}}$ *and* $(d_n)_{n\in\mathbb{N}}$ *monotonically nondecreasing.*

(a) *If* $c + d < 1$, *then*

$$\frac{E[X_{T_1^{n,m}} \vee \cdots \vee X_{T_m^{n,m}}] - \hat{b}_n}{\hat{a}_n} \to u^m(0), \tag{6.8}$$

where $u^m(t)$ *is the* m-*stopping curve of the Poisson process* $\hat{N}$ *with intensity function*

$$\hat{G}(t,y) = e^{-y} t^{-(c+d)} \qquad \text{on } [0,1] \times \mathbb{R}.$$

(b) *Let* $\gamma^1, \dots, \gamma^m$ *be the inverse functions defined in* (3.15) *and* (3.16), *let* $(w_n)_{n\in\mathbb{N}}$ *be an increasing sequence with* $\lim_{n\to\infty} n(1-F(w_n)) = 1$ (*e.g.*, $w_n := b_n$). *Let* v_n^m *be defined as*

$$v_n^m(t) := \frac{w_{\lfloor (1-t)n \rfloor} - b_n}{a_n} + \gamma^m(t) - \log(1-t).$$

Then

$$\hat{T}_1^{n,m} := \min\left\{1 \le i \le n-m+1 : X_i > \hat{a}_n v_n^m\left(\frac{i}{n}\right) + \hat{b}_n\right\},$$

$$\hat{T}_\ell^{n,m} := \min\left\{\hat{T}_{\ell-1}^{n,m} < i \le n-m+\ell : \right.$$

$$\left. X_i > \hat{a}_n \gamma^{m-\ell+1}\left(\frac{i}{n}, \frac{X_{\hat{T}_{\ell-1}^{n,m}} - \hat{b}_n}{\hat{a}_n}\right) + \hat{b}_n\right\}$$

define an asymptotic optimal sequence of m-stopping times, that is, convergence as in (6.8) holds for them.

For details of the proof, we refer readers to Faller (2009), pages 75–77.

REFERENCES

ASSAF, D., GOLDSTEIN, L. and SAMUEL-CAHN, E. (2004). Two-choice optimal stopping. *Adv. in Appl. Probab.* **36** 1116–1147. MR2119857

ASSAF, D., GOLDSTEIN, L. and SAMUEL-CAHN, E. (2006). Maximizing expected value with two stage stopping rules. In *Random Walk, Sequential Analysis and Related Topics* 3–27. World Sci. Publ., Hackensack, NJ. MR2367696

BARYSHNIKOV, Y. M. and GNEDIN, A. V. (2000). Sequential selection of an increasing sequence from a multidimensional random sample. *Ann. Appl. Probab.* **10** 258–267. MR1765211

BRUSS, F. T. (2010). On a class of optimal stopping problems with mixed constraints. *Discrete Math. Theor. Comput. Sci.* **12** 363–380. MR2676679

BRUSS, F. T. and DELBAEN, F. (2001). Optimal rules for the sequential selection of monotone subsequences of maximum expected length. *Stochastic Process. Appl.* **96** 313–342. MR1865761

BRUSS, F. T. and FERGUSON, T. S. (1997). Multiple buying or selling with vector offers. *J. Appl. Probab.* **34** 959–973. MR1484028

BRUSS, F. T. and ROGERS, L. C. G. (1991). Embedding optimal selection problems in a Poisson process. *Stochastic Process. Appl.* **38** 267–278. MR1119984

CHOW, Y. S., ROBBINS, H. and SIEGMUND, D. (1971). *Great Expectations: The Theory of Optimal Stopping*. Houghton Mifflin, Boston, MA. MR0331675

DE HAAN, L. and VERKADE, E. (1987). On extreme-value theory in the presence of a trend. *J. Appl. Probab.* **24** 62–76. MR0876169

FALLER, A. (2009). Approximative Lösungen von Mehrfachstoppproblemen. Dissertation, Univ. Freiburg.

FALLER, A. and RÜSCHENDORF, L. (2009). On approximative solutions of optimal stopping problems. Preprint, Univ. Freiburg.

FERGUSON, T. S. (2007). Optimal stopping and applications. Electronic texts on homepage. Available at http://www.math.ucla.edu/~tom/Stopping/Contents.html.

GNEDIN, A. V. (1996). On the full information best-choice problem. *J. Appl. Probab.* **33** 678–687. MR1401465

GNEDIN, A. V. and SAKAGUCHI, M. (1992). On a best choice problem related to the Poisson process. In *Strategies for Sequential Search and Selection in Real Time (Amherst, MA*, 1990). *Contemp. Math.* **125** 59–64. Amer. Math. Soc., Providence, RI. MR1160609

GOLDSTEIN, L. and SAMUEL-CAHN, E. (2006). Optimal two-choice stopping on an exponential sequence. *Sequential Anal.* **25** 351–363. MR2271920

HAGGSTROM, G. W. (1967). Optimal sequential procedures when more than one stop is required. *Ann. Math. Statist.* **38** 1618–1626. MR0217946

KARLIN, S. (1962). Stochastic models and optimal policy for selling an asset. In *Studies in Applied Probability and Management Science* 148–158. Stanford Univ. Press, Stanford, CA. MR0137594

KENNEDY, D. P. and KERTZ, R. P. (1990). Limit theorems for threshold-stopped random variables with applications to optimal stopping. *Adv. in Appl. Probab.* **22** 396–411. MR1053237

KENNEDY, D. P. and KERTZ, R. P. (1991). The asymptotic behavior of the reward sequence in the optimal stopping of i.i.d. random variables. *Ann. Probab.* **19** 329–341. MR1085339

KÜHNE, R. and RÜSCHENDORF, L. (2000a). Approximation of optimal stopping problems. *Stochastic Process. Appl.* **90** 301–325. MR1794541

KÜHNE, R. and RÜSCHENDORF, L. (2000b). Optimal stopping with discount and observation costs. *J. Appl. Probab.* **37** 64–72. MR1761661

KÜHNE, R. and RÜSCHENDORF, L. (2002). On optimal two-stopping problems. In *Limit Theorems in Probability and Statistics*, *Vol. II* (*Balatonlelle*, 1999) 261–271. János Bolyai Math. Soc., Budapest. MR1979997

KÜHNE, R. and RÜSCHENDORF, L. (2004). Approximate optimal stopping of dependent sequences. *Theory Probab. Appl.* **48** 465–480.

MOSER, L. (1956). On a problem of Cayley. *Scripta Mathematica* **22** 289–292.

NIKOLAEV, M. L. (1999). On optimal multiple stopping of Markov sequences. *Theory Probab. Appl.* **43** 298–306.

RESNICK, S. I. (1987). *Extreme Values*, *Regular Variation*, *and Point Processes*. *Applied Probability. A Series of the Applied Probability Trust* **4**. Springer, New York. MR0900810

SAARIO, V. and SAKAGUCHI, M. (1992). Multistop best choice games related to the Poisson process. *Math. Japon.* **37** 41–51. MR1148515

SAKAGUCHI, M. (1976). Optimal stopping problems for randomly arriving offers. *Math. Japon.* **21** 201–217. MR0426860

SAKAGUCHI, M. and SAARIO, V. (1995). A class of best-choice problems with full information. *Math. Japon.* **41** 389–398. MR1326972

SIEGMUND, D. O. (1967). Some problems in the theory of optimal stopping rules. *Ann. Math. Statist.* **38** 1627–1640. MR0221666

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