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OPTIMAL MULTIPLE STOPPING TIME PROBLEM

BY MAGDALENA KOBYLANSKI, MARIE-CLAIRE QUENEZ AND ELISABETH ROUY-MIRONESCU

Université Paris Est, Université Paris-Diderot and Ecole Centrale de Lyon

We study the optimal multiple stopping time problem defined for each stopping time S by $v(S) = \operatorname{ess\,sup}_{\tau_1,\dots,\tau_d \geq S} E[\psi(\tau_1,\dots,\tau_d)|\mathcal{F}_S]$.

The key point is the construction of a *new reward* ϕ such that the value function $v(S)$ also satisfies $v(S) = \operatorname{ess\,sup}_{\theta \geq S} E[\phi(\theta)|\mathcal{F}_S]$. This new reward ϕ is not a right-continuous adapted process as in the classical case, but a family of random variables. For such a reward, we prove a new existence result for optimal stopping times under weaker assumptions than in the classical case. This result is used to prove the existence of optimal multiple stopping times for $v(S)$ by a constructive method. Moreover, under strong regularity assumptions on ψ, we show that the new reward ϕ can be aggregated by a progressive process. This leads to new applications, particularly in finance (applications to American options with multiple exercise times).

Introduction. The present work on the optimal multiple stopping time problem, following the optimal single stopping time problem, involves proving the existence of the maximal reward, finding necessary or sufficient conditions for the existence of optimal stopping times and providing a method to compute these optimal stopping times.

The results are well known in the case of the optimal single stopping time problem. Consider a *reward* given by a right-continuous left-limited (RCLL) positive adapted process $(\phi_t)_{0\leq t\leq T}$ on $\mathbb{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$, $\mathbb{F}$ satisfying the usual conditions, and look for the maximal reward

$$v(0) = \sup\{E[\phi_\tau], \tau \in T_0\},$$

where $T \in]0, \infty[$ is the fixed time horizon and T_0 is the set of stopping times θ smaller than T. From now on, the process $(\phi_t)_{0\leq t\leq T}$ will be denoted by (ϕ_t). In order to compute $v(0)$, we introduce for each $S \in T_0$ the *value function* $v(S) = \operatorname{ess\,sup}\{E[\phi_\tau|\mathcal{F}_S], \tau \in T_S\}$, where T_S is the set of stopping times in T_0 greater than S. The value function is given by a family of random variables $\{v(S), S \in T_0\}$. By using the right continuity of the reward (ϕ_t), it can be shown that there exists an adapted process (v_t) which *aggregates* the family of random variables $\{v(S), S \in T_0\}$ that is such that $v_S = v(S)$ a.s. for each $S \in T_0$. This

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process is the *Snell envelope* of (ϕ_t), that is, the smallest supermartingale process that dominates ϕ. Moreover, when the reward (ϕ_t) is continuous, the stopping time defined trajectorially by

$$\overline{\theta}(S) = \inf\{t \geq S, v_t = \phi_t\}$$

is optimal. For details, see El Karoui (1981), Karatzas and Shreve (1998) or Peskir and Shiryaev (2006).

In the present work, we show that computing the value function for the optimal multiple stopping time problem

$$v(S) = \operatorname{ess\,sup}\{E[\psi(\tau_1, \dots, \tau_d)|\mathcal{F}_S], \tau_1, \dots, \tau_d \in T_S\},$$

reduces to computing the value function for an optimal single stopping time problem

$$u(S) = \operatorname{ess\,sup}\{E[\phi(\theta)|\mathcal{F}_S], \theta \in T_S\},$$

where the *new reward* ϕ is no longer an RCLL process, but a family $\{\phi(\theta), \theta \in T_0\}$ of positive random variables which satisfies some compatibility properties. For this new optimal single stopping time problem with a reward $\{\phi(\theta), \theta \in T_0\}$, we show that the minimal optimal stopping time for the value function $u(S)$ is no longer given by a hitting time of processes, but by the essential infimum

$$\theta^*(S) := \operatorname{ess\,inf}\{\theta \in T_S, u(\theta) = \phi(\theta) \text{ a.s.}\}.$$

This method also has the advantage that it no longer requires any aggregation results that need stronger hypotheses and whose proofs are rather technical.

By using the reduction property $v(S) = u(S)$ a.s., we give a method to construct by induction optimal stopping times $(\tau_1^*, \dots, \tau_d^*)$ for $v(S)$, which are also defined as essential infima, in terms of *nested* optimal single stopping time problems.

Some examples of optimal multiple stopping time problems have been studied in different mathematical fields. In finance, this type of problem appears in, for instance, the study of *swing options* [e.g., Carmona and Touzi (2008), Carmona and Dayanik (2008)] in the case of ordered stopping times. In the nonordered case, some optimal multiple stopping time problems appear as useful mathematical tools to establish some large deviations estimations [see Kobylanski and Rouy (1998)]. Further applications can be imagined in, for example, finance and insurance [see Kobylanski, Quenez and Rouy-Mironescu (2010)]. In a work in preparation [see Kobylanski and Quenez (2010)], the Markovian case will be studied in detail and some applications will be presented.

The paper is organized as follows. In Section 1 we revisit the optimal single stopping time problem for admissible families. We prove the existence of optimal stopping times when the family ϕ is right- and left-continuous in expectation along stopping times. We also characterize the minimal optimal stopping times. In Section 2 we solve the optimal double stopping time problem. Under quite weak assumptions, we show the existence of a pair of optimal stopping times and give a construction of those optimal stopping times. In Section 3 we generalize the results obtained in Section 2 to the optimal d-stopping-times problem. Also, we study the simpler case of a symmetric reward. In this case, the problem clearly reduces to ordered stopping times, and our general characterization of the optimal multiple stopping time problem in terms of nested optimal single stopping time problems straightforwardly reduces to a sequence of optimal single stopping time problems defined by backward induction. We apply these results to *swing options* and, in this particular case, our results correspond to those of Carmona and Dayanik (2008). In the last section, we prove some aggregation results and characterize the optimal stopping times in terms of hitting times of processes.

Let $\mathbb{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a probability space, where $T \in]0, \infty[$ is the fixed time horizon and $(\mathcal{F}_t)_{0\le t\le T}$ is a filtration satisfying the usual conditions of right continuity and augmentation by the null sets of $\mathcal{F} = \mathcal{F}_T$. We suppose that $\mathcal{F}_0$ contains only sets of probability 0 or 1. We denote by T_0 the collection of stopping times of $\mathbb{F}$ with values in $[0, T]$. More generally, for any stopping time S, we denote by T_S the class of stopping times $\theta \in T_0$ with $S \le \theta$ a.s.

We use the following notation: for real-valued random variables X and X_n, $n \in \mathbb{N}$, the notation "$X_n \uparrow X$" means "the sequence (X_n) is nondecreasing and converges to X a.s."

1. The optimal single stopping time problem revisited.

We first recall some classical results on the optimal single stopping time problem.

1.1. *Classical results*. The following classical results, namely the supermartingale property of the value function, the optimality criterium and the right continuity in expectation of the value function are well known [see El Karoui (1981) or Karatzas and Shreve (1998) or Peskir and Shiryaev (2006)]. They are very important tools in optimal stopping theory and will often be used in this paper in the (unusual) case of a reward given by an admissible family of random variables defined as follows.

DEFINITION 1.1. A family of random variables $\{\phi(\theta), \theta \in T_0\}$ is said to be *admissible* if it satisfies the following conditions:

1. for all $\theta \in T_0$, $\phi(\theta)$ is an $\mathcal{F}_\theta$-measurable $\overline{\mathbb{R}}^+$-valued random variable;
2. for all $\theta, \theta' \in T_0$, $\phi(\theta) = \phi(\theta')$ a.s. on $\{\theta = \theta'\}$.

REMARK 1.1. Let (ϕ_t) be a positive progressive process. The family defined by $\phi(\theta) = \phi_\theta$ is admissible.

Note also that the definition of admissible families corresponds to the notion of T_0-systems introduced by El Karoui (1981).

For the convenience of the reader, we recall the definition of the essential supremum and its main properties in Appendix A.

Suppose the reward is given by an admissible family $\{\phi(\theta), \theta \in T_0\}$. The *value function at time* S, where $S \in T_0$, is given by

$$v(S) = \operatorname*{ess\,sup}_{\theta \in T_S} E[\phi(\theta)|\mathcal{F}_S]. \tag{1.1}$$

PROPOSITION 1.1 (Admissibility of the value function). *The value function that is the family of random variables* $\{v(S), S \in T_0\}$ *defined by* (1.1) *is an admissible family.*

PROOF. Property 1 of admissibility for $\{v(S), S \in T_0\}$ follows from the existence of the essential supremum (see Theorem A.1 in Appendix A).

Take $S, S' \in T_0$ and let $A = \{S = S'\}$. For each $\theta \in T_S$, put $\theta_A = \theta \mathbf{1}_A + T\mathbf{1}_{A^c}$. As $A \in \mathcal{F}_S \cap \mathcal{F}_{S'}$, we have a.s. on A, $E[\phi(\theta)|\mathcal{F}_S] = E[\phi(\theta_A)|\mathcal{F}_S] = E[\phi(\theta_A)|\mathcal{F}_{S'}] \le v(S')$, hence taking the essential supremum over $\theta \in T_S$, we have $v(S) \le v(S')$ a.s., and by symmetry of S and S', we have shown property 2 of admissibility. $\square$

PROPOSITION 1.2. *There exists a sequence of stopping times* $(\theta^n)_{n\in\mathbb{N}}$ *with* θ^n *in* T_S *such that*

$$E[\phi(\theta^n)|\mathcal{F}_S] \uparrow v(S) \qquad a.s.$$

PROOF. For each $S \in T_0$, one can show that the set $\{E[\phi(\theta)|\mathcal{F}_S], \theta \in T_S\}$ is closed under pairwise maximization. Indeed, let $\theta, \theta' \in T_0$ and $A = \{E[\phi(\theta')|\mathcal{F}_S] \le E[\phi(\theta)|\mathcal{F}_S]\}$. One has $A \in \mathcal{F}_S$. Let $\tau = \theta\mathbf{1}_A + \theta'\mathbf{1}_{A^c}$, a stopping time. It is easy to check that $E[\phi(\tau)|\mathcal{F}_S] = E[\phi(\theta)|\mathcal{F}_S] \vee E[\phi(\theta')|\mathcal{F}_S]$. The result follows by a classical result (see Theorem A.1 in Appendix A). $\square$

Recall that for each fixed $S \in T_0$, an admissible family $\{h(\theta), \theta \in T_S\}$ is said to be a *supermartingale system* (resp., a *martingale system*) if, for any $\theta, \theta' \in T_0$ such that $\theta \ge \theta'$ a.s.,

$$E[h(\theta)|\mathcal{F}_{\theta'}] \le h(\theta') \qquad \text{a.s.} \qquad (\text{resp., } E[h(\theta)|\mathcal{F}_{\theta'}] = h(\theta') \text{ a.s.}).$$

PROPOSITION 1.3.

- *The value function* $\{v(S), S \in T_0\}$ *is a supermartingale system.*
- *Furthermore, it is characterized as the* Snell envelope system *associated with* $\{\phi(S), S \in T_0\}$, *that is, the smallest supermartingale system which is greater (a.s.) than* $\{\phi(S), S \in T_0\}$.

PROOF. Let us prove the first part. Fix $S \geq S'$ a.s. By Proposition 1.2, there exists an optimizing sequence (θ^n) for $v(S)$. By the monotone convergence theorem, $E[v(S)|\mathcal{F}_{S'}] = \lim_{n\to\infty} E[\phi(\theta^n)|\mathcal{F}_{S'}]$ a.s. Now, for each n, since $\theta^n \geq S'$ a.s., we have $E[\phi(\theta^n)|\mathcal{F}_{S'}] \leq v(S')$ a.s. Hence, $E[v(S)|\mathcal{F}_{S'}] \leq v(S')$ a.s., which gives the supermartingale property of the value function.

Let us prove the second part. Let $\{v'(S), S \in T_0\}$ be a supermartingale system such that for each $\theta \in T_0$, $v'(\theta) \geq \phi(\theta)$ a.s. Fix $S \in T_0$. By the properties of v', for all $\theta \in T_S$, $v'(S) \geq E[v'(\theta)|\mathcal{F}_S] \geq E[\phi(\theta)|\mathcal{F}_S]$ a.s. Taking the supremum over $\theta \in T_S$, we have $v'(S) \geq v(S)$ a.s. □

Now, recall the following Bellman optimality criterium [see, e.g., El Karoui (1981)].

PROPOSITION 1.4 (Optimality criterium). *Fix $S \in T_0$ and let $\theta^* \in T_S$ be such that $E[\phi(\theta^*)] < \infty$. The three following assertions are equivalent*:

1. *θ^* is S-optimal for $v(S)$, that is,*

$$v(S) = E[\phi(\theta^*)|\mathcal{F}_S] \qquad \textit{a.s.}; \tag{1.2}$$

2. *$v(\theta^*) = \phi(\theta^*)$ a.s. and $E[v(S)] = E[v(\theta^*)]$*;
3. *$E[v(S)] = E[\phi(\theta^*)]$.*

REMARK 1.2. Note that since the value function is a supermartingale system, equality $E[v(S)] = E[v(\theta^*)]$ is equivalent to the fact that the family $\{v(\theta), \theta \in T_{S,\theta^*}\}$ is a martingale system.

PROOF OF PROPOSITION 1.4. Let us show that assertion 1 implies assertion 2. Suppose assertion 1 is satisfied. Since the value function v is a supermartingale system greater that ϕ, we clearly have

$$v(S) \geq E[v(\theta^*)|\mathcal{F}_S] \geq E[\phi(\theta^*)|\mathcal{F}_S] \qquad \text{a.s.}$$

Since equality (1.2) holds, this implies that the previous inequalities are actually equalities.

In particular, $E[v(\theta^*)|\mathcal{F}_S] = E[\phi(\theta^*)|\mathcal{F}_S]$ a.s., but as inequality $v(\theta^*) \geq \phi(\theta^*)$ holds a.s., and as $E[\phi(\theta^*)] < \infty$, we have $v(\theta^*) = \phi(\theta^*)$ a.s.

Moreover, $v(S) = E[v(\theta^*)|\mathcal{F}_S]$ a.s., which gives $E[v(S)] = E[v(\theta^*)]$. Hence, assertion 2 is satisfied.

Clearly, assertion 2 implies assertion 3. It remains to show that 3 implies 1.

Suppose that 3 is satisfied. Since $v(S) \geq E[\phi(\theta^*)|\mathcal{F}_S]$ a.s., this gives $v(S) = E[\phi(\theta^*)|\mathcal{F}_S]$ a.s. Hence, 1 is satisfied. □

REMARK 1.3. It is clear that

$$E[v(S)] = \sup_{\theta \in T_S} E[\phi(\theta)]. \tag{1.3}$$

By assertion 3 of Proposition 1.4, a stopping time $\theta^* \in T_S$ such that $E[\phi(\theta^*)] < \infty$ is S-optimal for $v(S)$ if and only if it is optimal for the optimal stopping time problem (1.3), that is,

$$\sup_{\theta \in T_S} E[\phi(\theta)] = E[\phi(\theta^*)].$$

We now give a regularity result on v [see Lemma 2.13 in El Karoui (1981)]. Let us first introduce the following definition.

DEFINITION 1.2. An admissible family $\{\phi(\theta), \theta \in T_0\}$ is said to be *right- (resp., left-) continuous along stopping times in expectation [RCE (resp., LCE)]* if for any $\theta \in T_0$ and any sequence $(\theta_n)_{n\in\mathbb{N}}$ of stopping times such that $\theta_n \downarrow \theta$ a.s. (resp., $\theta_n \uparrow \theta$ a.s.), one has $E[\phi(\theta)] = \lim_{n\to\infty} E[\phi(\theta_n)]$.

REMARK 1.4. If (ϕ_t) is a continuous adapted process such that $E[\sup_{t\in[0,T]} \phi_t] < \infty$, then the family defined by $\phi(\theta) = \phi_\theta$ is clearly RCE and LCE. Also, if (ϕ_t) is an RCLL adapted process such that its jumps are totally inaccessible, then the family defined by $\phi(\theta) = \phi_\theta$ is clearly RCE and even LCE.

PROPOSITION 1.5. *Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family which is RCE. The family $\{v(S), S \in T_0\}$ is then RCE.*

PROOF. Since $\{v(S), S \in T_0\}$ is a supermartingale system, the function $S \mapsto E[v(S)]$ is a nonincreasing function of stopping times. Suppose it is not RCE at $S \in T_0$. If $E[v(S)] < \infty$, then there exists a constant $\alpha > 0$ and a sequence of stopping times $(S_n)_{n\in\mathbb{N}}$ such that $S_n \downarrow S$ a.s. and such that

$$\lim_{n\to\infty} \uparrow E[v(S_n)] + \alpha \le E[v(S)]. \tag{1.4}$$

Now, recall that $E[v(S)] = \sup_{\theta\in T_S} E[\phi(\theta)]$ [see (1.3)]. Hence, there exists $\theta' \in T_S$ such that

$$\sup_{n\in\mathbb{N}} \sup_{\theta\in T_{S_n}} E[\phi(\theta)] + \frac{\alpha}{2} \le E[\phi(\theta')].$$

Hence, for all $n \in \mathbb{N}$, $E[\phi(\theta' \vee S_n)] + \frac{\alpha}{2} \le E[\phi(\theta')]$. As $\theta' \vee S_n \downarrow \theta'$ a.s., we obtain, by taking the limit when $n \to \infty$ and using the RCE property of ϕ, that

$$E[\phi(\theta')] + \frac{\alpha}{2} \le E[\phi(\theta')],$$

which gives the expected contradiction in the case $E[v(S)] < \infty$.

Otherwise, instead of (1.4), we have $\lim_{n\to\infty} \uparrow E[v(S_n)] \le C$ for some constant $C > 0$, and similar arguments as in the finite case lead to a contradiction as well. □

1.2. *New results*. We will now give a new result which generalizes the classical existence result of an optimal stopping time stated in the case of a reward process to the case of a reward family of random variables.

THEOREM 1.1 (Existence of optimal stopping times). *Let* $\{\phi(\theta), \theta \in T_0\}$ *be an admissible family that satisfies the integrability condition*

$$v(0) = \sup_{\theta \in T_0} E[\phi(\theta)] < \infty$$

and which is RCE and LCE along stopping times. Then, for each $S \in T_0$, *there exists an optimal stopping time for* $v(S)$. *Moreover, the random variable defined by*

$$\theta^*(S) := \operatorname{ess\,inf}\{\theta \in T_S, v(\theta) = \phi(\theta) \text{ a.s.}\} \tag{1.5}$$

is the minimal optimal stopping time for $v(S)$.

Let us emphasize that in this theorem, the optimal stopping time $\theta^*(S)$ is not defined trajectorially, but as an essential infimum of random variables. In the classical case, that is, when the reward is given by an adapted RCLL process, recall that the minimal optimal stopping time is given by the random variable $\overline{\theta}(S)$ defined trajectorially by

$$\overline{\theta}(S) = \inf\{t \geq S, v_t = \phi_t\}.$$

The definition of $\theta^*(S)$ as an essential infimum allows the assumption on the regularity of the reward to be relaxed. More precisely, whereas in the previous works (mentioned in the Introduction), the reward was given by an RCLL and LCE process, in our setting, the reward is given by an RCE and LCE family of random variables. The idea of the proof is classical: we use an approximation method introduced by Maingueneau (1978), but our setting allows us to simplify and shorten the proof.

PROOF OF THEOREM 1.1. The proof will be divided into two parts.

Part I: In this part, we will prove the existence of an optimal stopping time.

Fix $S \in T_0$. We begin by constructing a family of stopping times [see Maingueneau (1978) or El Karoui (1981)]. For $\lambda \in]0, 1[$, define the $\mathcal{F}_S$-measurable random variable $\theta^\lambda(S)$ by

$$\theta^\lambda(S) := \operatorname{ess\,inf}\{\theta \in T_S, \lambda v(\theta) \leq \phi(\theta) \text{ a.s.}\}. \tag{1.6}$$

The following lemma holds.

LEMMA 1.1. *The stopping time* $\theta^\lambda(S)$ *is a* $(1-\lambda)$-*optimal stopping time for*

$$E[v(S)] = \sup_{\theta \in T_S} E[\phi(\theta)], \tag{1.7}$$

that is,

$$\lambda E[v(S)] \leq E[\phi(\theta^\lambda(S))]. \tag{1.8}$$

Suppose now that we have proven Lemma 1.1.

Since $\lambda \mapsto \theta^\lambda(S)$ is nondecreasing, for $S \in T_0$, the stopping time

$$\hat{\theta}(S) := \lim_{\lambda \uparrow 1} \uparrow \theta^\lambda(S) \tag{1.9}$$

is well defined. Let us show that $\hat{\theta}(S)$ is optimal for $v(S)$.

By letting $\lambda \uparrow 1$ in inequality (1.8), and since ϕ is LCE, we easily derive that $E[v(S)] = E[\phi(\hat{\theta}(S))]$. Consequently, by the optimality criterium 3 of Proposition 1.4, $\hat{\theta}(S)$ is S-optimal for $v(S)$. This completes part I.

Part II: Let us now prove that $\theta^*(S) = \hat{\theta}(S)$ a.s., where $\theta^*(S)$ is defined by (1.5), and that it is the minimal optimal stopping time for $v(S)$.

For each $S \in T_0$, the set $\mathbb{T}_S = \{\theta \in T_S, v(\theta) = \phi(\theta) \text{ a.s.}\}$ is not empty (since T belongs to $\mathbb{T}_S$) and is closed under pairwise minimization. Hence, there exists a sequence $(\theta_n)_{n\in\mathbb{N}}$ of stopping times in $\mathbb{T}_S$ such that $\theta_n \downarrow \theta^*(S)$ a.s. Consequently, $\theta^*(S)$ is a stopping time.

Let θ be an optimal stopping time for $v(S)$. By the optimality criterium (Proposition 1.4), and since, by assumption, $E[\phi(\theta)] < \infty$, we have $v(\theta) = \phi(\theta)$ a.s. and hence

$$\theta^*(S) \le \operatorname{ess\,inf}\{\theta \in T_0, \theta \text{ optimal for } v(S)\} \qquad \text{a.s.}$$

Now, for each $\lambda < 1$, the stopping time $\theta^\lambda(S)$ defined by (1.6) clearly satisfies $\theta^\lambda(S) \le \theta^*(S)$ a.s. Passing to the limit when $\lambda \uparrow 1$, we obtain $\hat{\theta}(S) \le \theta^*(S)$. As $\hat{\theta}(S)$ is optimal for $v(S)$, this implies that $\hat{\theta}(S) \ge \operatorname{ess\,inf}\{\theta \in T_0, \theta \text{ optimal for } v(S)\}$ a.s. Hence,

$$\theta^*(S) = \hat{\theta}(S) = \operatorname{ess\,inf}\{\theta \in T_0, \theta \text{ optimal for } v(S)\} \qquad \text{a.s.,}$$

which gives the desired result. The proof of Theorem 1.1 is thus complete. □

It now remains to prove Lemma 1.1.

PROOF OF LEMMA 1.1. We have to prove inequality (1.8). This will be done by means of the following steps.

Step 1: Fix $\lambda \in]0,1[$. It is easy to check that the set $\mathbb{T}_S^\lambda = \{\theta \in T_S, \lambda v(\theta) \le \phi(\theta) \text{ a.s.}\}$ is nonempty (since $T \in \mathbb{T}_S^\lambda$) and closed by pairwise minimization. By Theorem A.1 in the Appendix, there exists a sequence (θ^n) in $\mathbb{T}_S$ such that $\theta^n \downarrow \theta^\lambda(S)$ a.s. Therefore, $\theta^\lambda(S)$ is a stopping time and $\theta^\lambda(S) \ge S$ a.s. Moreover, we have $\lambda v(\theta^n) \le \phi(\theta^n)$ a.s. for all n. Taking expectation and using the RCE properties of v and ϕ, we obtain

$$\lambda E[v(\theta^\lambda(S)]) \le E[\phi(\theta^\lambda(S))]. \tag{1.10}$$

Step 2: Let us show that for each $\lambda \in]0,1[$ and each $S \in T_0$,

$$v(S) = E[v(\theta^\lambda(S))|\mathcal{F}_S] \qquad \text{a.s.} \tag{1.11}$$

For each $S \in T_0$, let us define the random variable $J(S) = E[v(\theta^\lambda(S))|\mathcal{F}_S]$. Step 2 amounts to showing that $J(S) = v(S)$ a.s.

Since $\{v(S), S \in T_0\}$ is a supermartingale system and since $\theta^\lambda(S) \geq S$ a.s., we have that

$$J(S) = E[v(\theta^\lambda(S))|\mathcal{F}_S] \leq v(S) \qquad \text{a.s.}$$

It remains to show the reverse inequality.

Step 2a: Let us show that the family $\{J(S), S \in T_0\}$ is a supermartingale system.

Let $S, S' \in T_0$ be such that $S' \geq S$ a.s. As $\theta^\lambda(S') \geq \theta^\lambda(S) \geq S$ a.s., we have

$$E[J(S')|\mathcal{F}_S] = E[v(\theta^\lambda(S'))|\mathcal{F}_S] = E\big[E\big[v(\theta^\lambda(S'))|\mathcal{F}_{\theta^\lambda(S)}\big]|\mathcal{F}_S\big] \qquad \text{a.s.}$$

Now, since $\{v(S), S \in T_0\}$ is a supermartingale system, $E[v(\theta^\lambda(S'))|\mathcal{F}_{\theta^\lambda(S)}] \leq v(\theta^\lambda(S))$ a.s. Consequently,

$$E[J(S')|\mathcal{F}_S] \leq E[v(\theta^\lambda(S))|\mathcal{F}_S] = J(S) \qquad \text{a.s.}$$

Step 2b: Let us show that for each $S \in T_0$ and each $\lambda \in]0, 1[$,

$$\lambda v(S) + (1 - \lambda)J(S) \geq \phi(S) \qquad \text{a.s.}$$

Fix $S \in T_0$ and $\lambda \in]0, 1[$.

On $\{\lambda v(S) \leq \phi(S)\}$, we have $\theta^\lambda(S) = S$ a.s. Hence, on $\{\lambda v(S) \leq \phi(S)\}$, $J(S) = E[v(\theta^\lambda(S))|\mathcal{F}_S] = E[v(S)|\mathcal{F}_S] = v(S)$ and therefore

$$\lambda v(S) + (1 - \lambda)J(S) = v(S) \geq \phi(S) \qquad \text{a.s.}$$

Furthermore, on $\{\lambda v(S) > \phi(S)\}$, as $J(S)$ is nonnegative, we have

$$\lambda v(S) + (1 - \lambda)J(S) \geq \lambda v(S) \geq \phi(S) \qquad \text{a.s.,}$$

and the proof of Step 2b is complete.

Now, the family $\{\lambda v(S) + (1 - \lambda)J(S), S \in T_0\}$ is a supermartingale system by convex combination of two supermartingale systems. Hence, as the value function $\{v(S), S \in T_0\}$ is characterized as the smallest supermartingale system which dominates $\{\phi(S), S \in T_0\}$, we derive that for each $S \in T_0$,

$$\lambda v(S) + (1 - \lambda)J(S) \geq v(S) \qquad \text{a.s.}$$

Now, by the integrability assumption made on ϕ, we have $v(S) < \infty$ a.s. Hence, we have $J(S) \geq v(S)$ a.s. Consequently, for each $S \in T_0$, $J(S) = v(S)$ a.s., which completes Step 2.

Finally, Step 1 [inequality (1.10)] and Step 2 [equality (1.11)] give

$$\lambda E[v(S)] = \lambda E[v(\theta^\lambda(S))] \leq E[\phi(\theta^\lambda(S))].$$

In other words, $\theta^\lambda(S)$ is a $(1 - \lambda)$-optimal stopping time for (1.7), which completes the proof of Lemma 1.1. □

REMARK 1.5. Recall that in the previous works [see, e.g., Karatzas and Shreve (1998), Proposition D.10 and Theorem D.12], the proof of the existence of optimal stopping times requires the value function to be aggregated and thus the use of some fine aggregation results such as Proposition 4.1. In our work, since we only work with families of random variables, we do not need any aggregation techniques, which simplifies and shortens the proof.

Under some regularity assumptions on the reward, we can show that the value function family is left-continuous along stopping times in expectation. More precisely, we have the following.

PROPOSITION 1.6. *Suppose that the admissible family* $\{\phi(\theta), \theta \in T_0\}$ *is LCE and RCE, and satisfies the integrability condition* $v(0) = \sup_{\theta \in T_0} E[\phi(\theta)] < \infty$. *The value function* $\{v(S), S \in T_0\}$ *defined by* (1.1) *is then LCE.*

PROOF. Let $S \in T_0$ and let (S_n) be a sequence of stopping times such that $S_n \uparrow S$ a.s. Note that by the supermartingale property of v, we have

$$E[v(S_n)] \geq E[v(S)]. \tag{1.12}$$

Now, by Theorem 1.1, the stopping time $\theta^*(S_n)$ defined by (1.5) is optimal for $v(S_n)$. Moreover, it is clear that $(\theta^*(S_n))_n$ is a nondecreasing sequence of stopping times dominated by $\theta^*(S)$.

Let us define $\overline{\theta} = \lim_{n\to\infty} \uparrow \theta^*(S_n)$. Note that $\overline{\theta}$ is a stopping time. Also, as for each n, $\theta^*(S_n) \geq S_n$ a.s., it follows that $\overline{\theta} \geq S$ a.s. Therefore, since ϕ is LCE,

$$E[v(S)] \geq E[\phi(\overline{\theta})] = \lim_{n\to\infty} E[\phi(\theta^*(S_n))] = \lim_{n\to\infty} E[v(S_n)].$$

This, together with (1.12), gives $E[v(S)] = \lim_{n\to\infty} E[v(S_n)]$. □

REMARK 1.6. In this proof, we have also proven that $\overline{\theta}$ is optimal for $v(S)$. Hence, by the optimality criterium, $v(\overline{\theta}) = \phi(\overline{\theta})$ a.s., which implies that $\overline{\theta} \geq \theta^*(S)$ a.s. Moreover, since for each n, $\theta^*(S_n) \leq \theta^*(S)$ a.s., by letting n tend to ∞, we clearly have that $\overline{\theta} \leq \theta^*(S)$ a.s. Hence, $\overline{\theta} = \lim_{n\to\infty} \uparrow \theta^*(S_n) = \theta^*(S)$ a.s. Thus, we have also shown that the map $S \mapsto \theta^*(S)$ is left-continuous along stopping times.

2. The optimal double stopping time problem.

2.1. *Definition and first properties of the value function.* We now consider the optimal double stopping time problem. We introduce the following definitions.

DEFINITION 2.1. The family $\{\psi(\theta, S), \theta, S \in T_0\}$ is *biadmissible* if it satisfies:

1. for all $\theta, S \in T_0$, $\psi(\theta, S)$ is an $\mathcal{F}_{\theta \vee S}$-measurable $\overline{\mathbb{R}}^+$-valued r.v.;
2. for all $\theta, \theta', S, S' \in T_0$, $\psi(\theta, S) = \psi(\theta', S')$ a.s. on $\{\theta = \theta'\} \cap \{S = S'\}$.

REMARK 2.1. Let Ψ be a biprocess, that is, a function

$$\Psi : [0, T]^2 \times \Omega \to \mathbb{R}^+; (t, s, \omega) \mapsto \Psi_{t,s}(\omega)$$

such that for almost all ω, the map $(t, s) \mapsto \Psi_{t,s}(\omega)$ is *right-continuous* (i.e., $\Psi_{t,s} = \lim_{(t',s') \to (t^+, s^+)} \Psi_{t',s'}$), and for each $(t, s) \in [0, T]^2$, $\Psi_{t,s}$ is $\mathcal{F}_{t \vee s}$-measurable. In this case, the family $\{\psi(\theta, S), \theta, S \in T_0\}$ defined by

$$\psi(\theta, S)(\omega) := \Psi_{\theta(\omega), S(\omega)}(\omega)$$

is clearly biadmissible.

For a biadmissible family $\{\psi(\theta, S), \theta, S \in T_0\}$, let us consider the value function associated with the reward family $\{\psi(\theta, S), \theta, S \in T_0\}$:

$$v(S) = \operatorname{ess} \sup_{\tau_1, \tau_2 \in T_S} E[\psi(\tau_1, \tau_2) | \mathcal{F}_S]. \tag{2.1}$$

As in the case of the single stopping time problem, we have the following properties.

PROPOSITION 2.1. *Let $\{\psi(\theta, S), \theta, S \in T_0\}$ be a biadmissible family of random variables. The following properties then hold*:

(1) *the family $\{v(S), S \in T_0\}$ is an admissible family of random variables*;
(2) *for each $S \in T_0$, there exists a sequence of pairs of stopping times $((\tau_1^n, \tau_2^n))_{n \in \mathbb{N}}$ in $T_S \times T_S$ such that $\{E[\psi(\tau_1^n, \tau_2^n) | \mathcal{F}_S]\}_{n \in \mathbb{N}}$ is nondecreasing and a.s.*

$$E[\psi(\tau_1^n, \tau_2^n) | \mathcal{F}_S] \uparrow v(S);$$

(3) *the family of random variables $\{v(S), S \in T_0\}$ is a supermartingale system, that is, it satisfies the dynamic programming principle.*

PROOF. (1) As in the case of single stopping time, property 1 of admissibility for $\{v(S), S \in T_0\}$ follows from the existence of the essential supremum.

Take $S, S' \in T_0$ and put $A = \{S = S'\}$, and for each $\tau_1, \tau_2 \in T_S$, put $\tau_1^A = \tau_1 \mathbf{1}_A + T \mathbf{1}_{A^c}$ and $\tau_2^A = \tau_2 \mathbf{1}_A + T \mathbf{1}_{A^c}$. As $A \in \mathcal{F}_S \cap \mathcal{F}_{S'}$, one has, a.s. on A, $E[\psi(\tau_1, \tau_2) | \mathcal{F}_S] = E[\psi(\tau_1^A, \tau_2^A) | \mathcal{F}_S] = E[\psi(\tau_1^A, \tau_2^A) | \mathcal{F}_{S'}] \le v(S')$. Hence, taking the essential supremum over $\tau_1, \tau_2 \in T_S$, we have $v(S) \le v(S')$ a.s., and, by symmetry, we have shown property 2 of admissibility. Hence, the family $\{v(S), S \in T_0\}$ is an admissible family of random variables.

The proofs of (2) and (3) can be easily adapted from the proofs of Proposition 1.2 and Proposition 1.3. □

Following the case of single stopping time, we now give some regularity results on the value function.

DEFINITION 2.2. A biadmissible family $\{\psi(\theta, S), \theta, S \in T_0\}$ is said to be *right-continuous along stopping times in expectation (RCE)* if, for any $\theta, S \in T_0$ and any sequences $(\theta_n)_{n\in\mathbb{N}} \in T_0$ and $(S_n)_{n\in\mathbb{N}} \in T_0$ such that $\theta_n \downarrow \theta$ and $S_n \downarrow S$ a.s., one has $E[\psi(\theta, S)] = \lim_{n\to\infty} E[\psi(\theta_n, S_n)]$.

PROPOSITION 2.2. *Suppose that the biadmissible family* $\{\psi(\theta, S), \theta, S \in T_0\}$ *is RCE. The family* $\{v(S), S \in T_0\}$ *defined by* (2.1) *is then RCE.*

PROOF. The proof follows the proof of Proposition 1.5. □

2.2. *Reduction to an optimal single stopping time problem.* In this section, we will show that the optimal double stopping time problem (2.1) can be reduced to an optimal single stopping time problem associated with a *new reward family*.

More precisely, for each stopping time $\theta \in T_S$ let us introduce the two $\mathcal{F}_\theta$-measurable random variables

$$u_1(\theta) = \operatorname*{ess\,sup}_{\tau_1\in T_\theta} E[\psi(\tau_1, \theta)|\mathcal{F}_\theta], \qquad u_2(\theta) = \operatorname*{ess\,sup}_{\tau_2\in T_\theta} E[\psi(\theta, \tau_2)|\mathcal{F}_\theta]. \tag{2.2}$$

Note that since $\{\psi(\theta, S), \theta, S \in T_0\}$ is biadmissible, for each fixed $\theta \in T_0$, the families $\{\psi(\tau_1, \theta), \tau_1 \in T_0\}$ and $\{\psi(\theta, \tau_2), \tau_2 \in T_0\}$ are admissible. Hence, by Proposition 1.1 the families $\{u_1(\theta), \theta \in T_S\}$ and $\{u_2(\theta), \theta \in T_S\}$ are admissible. Put

$$\phi(\theta) = \max[u_1(\theta), u_2(\theta)]. \tag{2.3}$$

The family $\{\phi(\theta), \theta \in T_S\}$, which is called the *new reward family*, is also clearly admissible. Consider the value function associated with the new reward

$$u(S) = \operatorname*{ess\,sup}_{\theta\in T_S} E[\phi(\theta)|\mathcal{F}_S] \qquad \text{a.s.} \tag{2.4}$$

THEOREM 2.1 (Reduction). *Suppose that* $\{\psi(\theta, S), \theta, S \in T_0\}$ *is a biadmissible family. For each stopping time* S, *consider* $v(S)$ *defined by* (2.1) *and* $u(S)$ *defined by* (2.2), (2.3), (2.4). *Then*,

$$v(S) = u(S) \qquad a.s.$$

PROOF. Let S be a stopping time.

Step 1: First, let us show that $v(S) \le u(S)$ a.s.

Let $\tau_1, \tau_2 \in T_S$. Put $A = \{\tau_1 \le \tau_2\}$. As A is in $\mathcal{F}_{\tau_1} \cap \mathcal{F}_{\tau_2}$, we have

$$E[\psi(\tau_1, \tau_2)|\mathcal{F}_S] = E[\mathbf{1}_A E[\psi(\tau_1, \tau_2)|\mathcal{F}_{\tau_1}]|\mathcal{F}_S] + E[\mathbf{1}_{A^c} E[\psi(\tau_1, \tau_2)|\mathcal{F}_{\tau_2}]|\mathcal{F}_S].$$

By noticing that on A we have $E[\psi(\tau_1, \tau_2)|\mathcal{F}_{\tau_1}] \le u_2(\tau_1) \le \phi(\tau_1 \wedge \tau_2)$ a.s. and, similarly, on A^c we have $E[\psi(\tau_1, \tau_2)|\mathcal{F}_{\tau_2}] \le u_1(\tau_2) \le \phi(\tau_1 \wedge \tau_2)$ a.s., we get

$$E[\psi(\tau_1, \tau_2)|\mathcal{F}_S] \le E[\phi(\tau_1 \wedge \tau_2)|\mathcal{F}_S] \le u(S) \qquad \text{a.s.}$$

By taking the supremum over τ_1 and τ_2 in T_S, we complete Step 1.

Step 2: Let us now show that $v(S) \ge u(S)$ a.s.

We clearly have $v(S) \ge \operatorname{ess\,sup}_{\tau_2 \in T_S} E[\psi(S, \tau_2)|\mathcal{F}_S] = u_2(S)$ a.s. By similar arguments, $v(S) \ge u_1(S)$ a.s. and, consequently,

$$v(S) \ge \max[u_1(S), u_2(S)] = \phi(S) \qquad \text{a.s.}$$

Thus, $\{v(S), S \in T_0\}$ is a supermartingale system which is greater than $\{\phi(S), S \in T_0\}$. Now, by Proposition 1.3, $\{u(S), S \in T_0\}$ is the smallest supermartingale system which is greater than $\{\phi(S), S \in T_0\}$. Consequently, Step 2 follows, which completes the proof. □

Note that the reduction to an optimal single stopping time problem associated with a new reward will be the key property used to construct optimal multiple stopping times and to establish an existence result for them (see Sections 2.3–2.5).

2.3. *Properties of optimal stopping times*. In this section, we are given a biadmissible family $\{\psi(\theta, S), \theta, S \in T_0\}$ such that $E[\operatorname{ess\,sup}_{\theta, S \in T_0} \psi(\theta, S)] < \infty$.

PROPOSITION 2.3 (A necessary condition of optimality). *Let S be a stopping time and consider the value function $v(S)$ defined by* (2.1) *for all $\theta \in T_S$, $u_1(\theta), u_2(\theta)$ defined by* (2.2), *$\phi(\theta)$ defined by* (2.3) *and $u(S)$ defined by* (2.4).

Suppose that the pair (τ_1^*, τ_2^*) is optimal for $v(S)$ and put $A = \{\tau_1^* \le \tau_2^*\}$. Then:

(1) *$\tau_1^* \wedge \tau_2^*$ is optimal for $u(S)$;*
(2) *τ_2^* is optimal for $u_2(\tau_1^*)$ a.s. on A;*
(3) *τ_1^* is optimal for $u_1(\tau_2^*)$ a.s. on A^c.*

Moreover $A = \{\tau_1^* \le \tau_2^*\} \subset B = \{u_1(\tau_1^* \wedge \tau_2^*) \le u_2(\tau_1^* \wedge \tau_2^*)\}$.

PROOF. Let $S \in T_0$ and suppose that the pair of stopping times (τ_1^*, τ_2^*) is optimal for $v(S)$. As $u(S) = v(S)$ a.s., we obtain equality in Step 1 of the proof of Theorem 2.1. More precisely,

$$v(S) = E[\psi(\tau_1^*, \tau_2^*)|\mathcal{F}_S] = E[\phi(\tau_1^* \wedge \tau_2^*)|\mathcal{F}_S] = u(S) \qquad \text{a.s.},$$

$$E[\psi(\tau_1^*, \tau_2^*)|\mathcal{F}_{\tau_1^*}] = u_2(\tau_1^*) = u_2(\tau_1^* \wedge \tau_2^*) = \phi(\tau_1^* \wedge \tau_2^*) \qquad \text{a.s. on } A,$$

$$E[\psi(\tau_1^*, \tau_2^*)|\mathcal{F}_{\tau_2^*}] = u_1(\tau_2^*) = u_1(\tau_1^* \wedge \tau_2^*) = \phi(\tau_1^* \wedge \tau_2^*) \qquad \text{a.s. on } A^c,$$

which easily leads to (1), (2), (3) and $A \subset B$. □

REMARK 2.2. Note that, in general, for a pair (τ_1^*, τ_2^*) of optimal stopping times for $v(S)$, the inclusion $A \subset B$ is strict. Indeed if $\psi \equiv 0$, then $v = u = u_1 = u_2 = \phi = 0$, and all pairs of stopping times are optimal. Consider $\tau_1^* = T$, $\tau_2^* = 0$. In this case, $A = \varnothing$ and $B = \Omega$.

We now give a sufficient condition for optimality.

PROPOSITION 2.4 (Construction of optimal stopping times). *Using the notation of Proposition 2.3, suppose that*:

1. θ^* *is optimal for* $u(S)$;
2. θ_2^* *is optimal for* $u_2(\theta^*)$;
3. θ_1^* *is optimal for* $u_1(\theta^*)$

and put $B = \{u_1(\theta^*) \le u_2(\theta^*)\}$. *The pair of stopping times* (τ_1^*, τ_2^*) *defined by*

$$\tau_1^* = \theta^* \mathbf{1}_B + \theta_1^* \mathbf{1}_{B^c}, \qquad \tau_2^* = \theta_2^* \mathbf{1}_B + \theta^* \mathbf{1}_{B^c} \tag{2.5}$$

is then optimal for $v(S)$.

Moreover, $\tau_1^* \wedge \tau_2^* = \theta^*$ *and* $B = \{\tau_1^* \le \tau_2^*\}$.

PROOF. Let θ^* be an optimal stopping time for $u(S)$, that is, $u(S) = E[\phi(\theta^*)|\mathcal{F}_S]$ a.s. Let θ_1^* be an optimal stopping time for $u_1(\theta^*)$ (i.e., $u_1(\theta^*) = E[\psi(\theta_1^*, \theta^*)|\mathcal{F}_{\theta^*}]$ a.s.) and let θ_2^* be an optimal stopping time for $u_2(\theta^*)$ (i.e., $u_2(\theta^*) = E[\psi(\theta^*, \theta_2^*)|\mathcal{F}_{\theta^*}]$ a.s.). We introduce the set $B = \{u_1(\theta^*) \le u_2(\theta^*)\}$. Note that B is in $\mathcal{F}_{\theta^*}$.

Let τ_1^*, τ_2^* be the stopping times defined by (2.5). We clearly have the inclusion

$$B \subset \{\tau_1^* \le \tau_2^*\}. \tag{2.6}$$

Since $u(S) = E[\phi(\theta^*)|\mathcal{F}_S]$ and $\phi(\theta^*) = \max[u_1(\theta^*), u_2(\theta^*)]$, we have

$$u(S) = E[\mathbf{1}_B u_2(\theta^*) + \mathbf{1}_{B^c} u_1(\theta^*)|\mathcal{F}_S].$$

The optimality of θ_1^* and θ_2^* gives that a.s.

$$\begin{aligned} u(S) &= E[\mathbf{1}_B \psi(\theta^*, \theta_2^*) + \mathbf{1}_{B^c} \psi(\theta_1^*, \theta^*)|\mathcal{F}_S] \\ &= E[\mathbf{1}_B \psi(\tau_1^*, \tau_2^*) + \mathbf{1}_{B^c} \psi(\tau_1^*, \tau_2^*)|\mathcal{F}_S] = E[\psi(\tau_1^*, \tau_2^*)|\mathcal{F}_S]. \end{aligned}$$

As $u(S) = v(S)$ a.s., the pair of stopping times (τ_1^*, τ_2^*) is S-optimal for $v(S)$. By Proposition 2.3, we have $\{\tau_1^* \le \tau_2^*\} \subset B$. Hence, by (2.6), $B = \{\tau_1^* \le \tau_2^*\}$. □

REMARK 2.3. Proposition 2.4 still holds true if condition 2 holds true on the set B and condition 3 holds true on the set B^c.

Note that by Remark 2.2, we do not have a characterization of optimal pairs of stopping times. However, it is possible to give a characterization of *minimal optimal* stopping times in a particular sense (see Appendix B).

2.4. *Regularity of the new reward.* Before studying the problem of the existence of optimal stopping times, we have to state some regularity properties of the new reward family $\{\phi(\theta), \theta \in T_0\}$.

Let us introduce the following definition.

DEFINITION 2.3. A biadmissible family $\{\psi(\theta, S), \theta, S \in T_0\}$ is said to be *uniformly right- (resp., left-) continuous in expectation along stopping times [URCE (resp., ULCE)]* if $v(0) = \sup_{\theta, S \in T_0} E[\psi(\theta, S)] < \infty$ and if, for each $\theta, S \in T_0$ and each sequence of stopping times $(S_n)_{n\in\mathbb{N}}$ such that $S_n \downarrow S$ a.s. (resp., $S_n \uparrow S$ a.s.),

$$\lim_{n\to\infty} \sup_{\theta\in T_0} |E[\psi(\theta, S)] - E[\psi(\theta, S_n)]| = 0 \quad \text{and}$$

$$\lim_{n\to\infty} \sup_{\theta\in T_0} |E[\psi(S, \theta)] - E[\psi(S_n, \theta)]| = 0.$$

The following right continuity property holds true for the new reward family.

THEOREM 2.2. *Suppose that the biadmissible family $\{\psi(\theta, S), \theta, S \in T_0\}$ is URCE (resp., both URCE and ULCE). The family $\{\phi(S), S \in T_0\}$ defined by (2.3) is then RCE (resp., both RCE and LCE).*

PROOF. As $\phi(\theta) = \max[u_1(\theta), u_2(\theta)]$, it is sufficient to show the RCE (resp., both RCE and LCE) properties for the family $\{u_1(\theta), \theta \in T_0\}$.

Let us introduce the following value function for each $S, \theta \in T_0$:

$$U_1(\theta, S) = \operatorname{ess\,sup}_{\tau_1 \in T_\theta} E[\psi(\tau_1, S)|\mathcal{F}_\theta] \qquad \text{a.s.} \tag{2.7}$$

As for all $\theta \in T_0$,

$$u_1(\theta) = U_1(\theta, \theta) \qquad \text{a.s.},$$

it is sufficient to prove that $\{U_1(\theta, S), \theta, S \in T_0\}$ is RCE (resp., both RCE and LCE), that is, if $\theta, S \in T_0$ and $(\theta_n)_n$, $(S_n)_n$ in T_0 are such that $\theta_n \downarrow \theta$ and $S_n \downarrow S$ a.s. (resp., $\theta_n \uparrow \theta$ and $S_n \uparrow S$ a.s.), then $\lim_{n\to\infty} E[U_1(\theta_n, S_n)] = E[U_1(\theta, S)]$. Now, we have

$$\begin{aligned} &|E[U_1(\theta, S)] - E[U_1(\theta_n, S_n)]| \\ &\qquad \le \underbrace{|E[U_1(\theta, S)] - E[U_1(\theta_n, S)]|}_{\text{(I)}} + \underbrace{|E[U_1(\theta_n, S)] - E[U_1(\theta_n, S_n)]|}_{\text{(II)}}. \end{aligned}$$

Let us show that (I) tends to 0 as $n \to \infty$. For each $S \in T_0$, $\{\psi(\theta, S), \theta \in T_0\}$ is an admissible family of positive random variables which is RCE (resp., both RCE and LCE). By Proposition 1.5 (resp., Proposition 1.6), the value function $\{U_1(\theta, S), \theta \in T_0\}$ is RCE (resp., both RCE and LCE). It follows that (I) converges to 0 as n tends to ∞.

Let us show that (II) tends to 0 as $n \to \infty$. By definition of the value function $U_1(\cdot,\cdot)$ (2.7), it follows that

$$|E[U_1(\theta_n, S)] - E[U_1(\theta_n, S_n)]| \le \sup_{\tau \in T_0} |E[\psi(\tau, S)] - E[\psi(\tau, S_n)]|.$$

which converges to 0 since $\{\psi(\theta, S), \theta, S \in T_0\}$ is URCE (resp., both URCE and ULCE). The proof of Theorem 2.2 is thus complete. □

COROLLARY 2.1. *Suppose that* $v(0) = \sup_{\theta, S \in T_0} E[\psi(\theta, S)] < \infty$. *Under the same hypothesis as Theorem* 2.2, *the family* $\{v(S), S \in T_0\}$ *defined by* (2.1) *is RCE* (*resp*., *both RCE and LCE*).

PROOF. This follows from the fact that $v(S) = u(S)$ a.s. (Theorem 2.1), where $\{u(S), S \in T_0\}$ is the value function family associated with the new reward $\{\phi(S), S \in T_0\}$. Applying Propositions 1.5 and 1.6, we obtain the required properties. □

We will now turn to the problem of the existence of optimal stopping times.

2.5. *Existence of optimal stopping times*. Let $\{\psi(\theta, S), \theta, S \in T_0\}$ be a biadmissible family which is URCE and ULCE. Suppose that $v(0) < \infty$.

By Theorem 2.2, the admissible family of positive random variables $\{\phi(\theta), \theta \in T_0\}$ defined by (2.3) is RCE and LCE. By Theorem 1.1, the stopping time

$$\theta^* = \operatorname{ess\,inf}\{\theta \in T_S, u(\theta) = \phi(\theta) \text{ a.s.}\}$$

is optimal for $u(S)\,[= v(S)]$, that is,

$$u(S) = \operatorname*{ess\,sup}_{\theta \in T_S} E[\phi(\theta)|\mathcal{F}_S] = E[\phi(\theta^*)|\mathcal{F}_S] \qquad \text{a.s.}$$

Moreover, the families $\{\psi(\theta, \theta^*), \theta \in T_{\theta^*}\}$ and $\{\psi(\theta^*, \theta), \theta \in T_{\theta^*}\}$ are admissible and are RCE and LCE. Consider the following optimal stopping time problems defined for each $S \in T_{\theta^*}$:

$$v_1(S) = \operatorname*{ess\,sup}_{\theta \in T_S} E[\psi(\theta, \theta^*)|\mathcal{F}_S] \quad \text{and} \quad v_2(S) = \operatorname*{ess\,sup}_{\theta \in T_S} E[\psi(\theta^*, \theta)|\mathcal{F}_S].$$

By Theorem 1.1 the stopping times θ_1^* and θ_2^* defined by $\theta_1^* = \operatorname{ess\,inf}\{\theta \in T_{\theta^*}, v_1(\theta) = \psi(\theta, \theta^*) \text{ a.s.}\}$ and $\theta_2^* = \operatorname{ess\,inf}\{\theta \in T_{\theta^*}, v_2(\theta) = \psi(\theta^*, \theta) \text{ a.s.}\}$ are optimal stopping times for $v_1(\theta^*)$ and $v_2(\theta^*)$, respectively. Note that $v_1(\theta^*) = u_1(\theta^*)$ and $v_2(\theta^*) = u_2(\theta^*)$ a.s.

Let τ_1^* and τ_2^* be the stopping times defined by

$$\tau_1^* = \theta^* \mathbf{1}_B + \theta_1^* \mathbf{1}_{B^c}, \qquad \tau_2^* = \theta^* \mathbf{1}_{B^c} + \theta_2^* \mathbf{1}_B, \tag{2.8}$$

where $B = \{u^1(\theta^*) \le u^2(\theta^*)\}$. By Proposition 2.4, the pair (τ_1^*, τ_2^*) is optimal for $v(S)$. Consequently, we have proven the following theorem.

THEOREM 2.3 (Existence of an optimal pair of stopping times). *Let* $\{\psi(\theta, S), \theta, S \in T_0\}$ *be a biadmissible family which is URCE and ULCE. Suppose that* $v(0) < \infty$.

The pair of stopping times (τ_1^*, τ_2^*) *defined by* (2.8) *is then optimal for* $v(S)$ *defined by* (2.1).

REMARK 2.4. Note that since θ^*, θ_1^*, θ_2^* are minimal optimal, by results in Appendix B, (τ_1^*, τ_2^*) is minimal optimal for $v(S)$ (in the sense defined in Appendix B).

3. The optimal d-stopping time problem. Let $d \in \mathbb{N}$, $d \geq 2$. In this section, we show that computing the value function for the optimal d-stopping time problem

$$v(S) = \operatorname{ess\,sup}\{E[\psi(\tau_1, \ldots, \tau_d)|\mathcal{F}_S], \tau_1, \ldots, \tau_d \in T_S\}$$

reduces to computing the value function for an optimal single stopping time problem, that is,

$$v(S) = \operatorname{ess\,sup}\{E[\phi(\theta)|\mathcal{F}_S], \theta \in T_S\} \qquad \text{a.s.,}$$

for a *new reward* ϕ. This new reward is expressed in terms of optimal $(d-1)$-stopping time problems. Hence, by induction, the initial optimal d-stopping time problem can be reduced to *nested* optimal single stopping time problems.

3.1. *Definition and initial properties of the value function.*

DEFINITION 3.1. We say that the family of random variables $\{\psi(\theta), \theta \in T_0^d\}$ is a *d-admissible family* if it satisfies the following conditions:

1. for all $\theta = (\theta_1, \ldots, \theta_d) \in T_0^d$, $\psi(\theta)$ is an $\mathcal{F}_{\theta_1 \vee \cdots \vee \theta_d}$ measurable $\overline{\mathbb{R}}^+$-valued random variable;
2. for all $\theta, \theta' \in T_0^d$, $\psi(\theta) = \psi(\theta')$ a.s. on $\{\theta = \theta'\}$.

For each stopping time $S \in T_0$, we consider the value function associated with the reward $\{\psi(\theta), \theta \in T_0^d\}$:

$$v(S) = \operatorname*{ess\,sup}_{\tau \in T_S^d} E[\psi(\tau)|\mathcal{F}_S]. \tag{3.1}$$

As in the optimal double stopping time problem, the value function satisfies the following properties.

PROPOSITION 3.1. *Let* $\{\psi(\theta), \theta \in T_0^d\}$ *be a d-admissible family of random variables. The following properties then hold*:

1. $\{v(S), S \in T_0\}$ *is an admissible family of random variables*;

2. *For each $S \in T_0$, there exists a sequence of stopping times $(\theta^n)_{n\in\mathbb{N}}$ in T_S^d such that the sequence $\{E[\psi(\theta^n)|\mathcal{F}_S]\}_{n\in\mathbb{N}}$ is nondecreasing and such that $v(S) = \lim_{n\to\infty} \uparrow E[\psi(\theta^n)|\mathcal{F}_S]$ a.s.*;
3. *The family of random variables $\{v(S),\ S \in T_0\}$ defined by (3.1) is a supermartingale system.*

The proof is an easy generalization of the optimal double stopping time problem (Proposition 2.1).

Following the case with single or double stopping time, we now state the following result on the regularity of the value function.

PROPOSITION 3.2. *Suppose that the d-admissible family $\{\psi(\theta), \theta \in T_0^d\}$ is RCE and that $v(0) < \infty$. The family $\{v(S), S \in T_0\}$ defined by (3.1) is then RCE.*

The definition of RCE and the proof of this property are easily derived from the single or double stopping time case (see Definition 2.2 and Proposition 2.2).

3.2. *Reduction to an optimal single stopping time problem.* The optimal d-stopping time problem (3.1) can be expressed in terms of an optimal single stopping time problem as follows.

For $i = 1, \dots, d$ and $\theta \in T_0$, consider the random variable

$$u^{(i)}(\theta) = \operatorname{ess}\sup_{\tau_1,\dots,\tau_{i-1},\tau_{i+1},\dots,\tau_d \in T_\theta^{d-1}} E[\psi(\tau_1, \dots, \tau_{i-1}, \theta, \tau_{i+1}, \dots, \tau_d)|\mathcal{F}_\theta]. \tag{3.2}$$

Note that this notation is adapted to the d-dimensional case.

In the two-dimensional case ($d = 2$), we have

$$u^{(1)}(\theta) = \operatorname{ess}\sup_{\tau_2\in T_\theta} E[\psi(\theta, \tau_2)|\mathcal{F}_\theta] = u_2(\theta) \qquad \text{a.s.}$$

and

$$u^{(2)}(\theta) = \operatorname{ess}\sup_{\tau_1\in T_\theta} E[\psi(\tau_1, \theta)|\mathcal{F}_\theta] = u_1(\theta) \qquad \text{a.s.},$$

by definition of $u_1(\theta)$ and $u_2(\theta)$ [see (2.2)]. Thus, the notation in the two-dimensional case was different, but more adapted to that simpler case.

For each $\theta \in T_0$, define the $\mathcal{F}_\theta$-measurable random variable called the *new reward*,

$$\phi(\theta) = \max[u^{(1)}(\theta), \dots, u^{(d)}(\theta)], \tag{3.3}$$

and for each stopping time S, define the $\mathcal{F}_S$-measurable variable

$$u(S) = \operatorname{ess}\sup_{\theta\in T_S} E[\phi(\theta)|\mathcal{F}_S]. \tag{3.4}$$

THEOREM 3.1 (Reduction). *Let* $\{\psi(\theta), \theta \in T_0^d\}$ *be a* d*-admissible family of random variables and for each stopping time* S, *consider* $v(S)$ *defined by* (3.1) *and* $u(S)$ *defined by* (3.2), (3.3) *and* (3.4). *Then*,

$$v(S) = u(S) \qquad a.s.$$

PROOF. *Step 1*: Let us prove that for all $S \in T_0$, $v(S) \leq u(S)$ a.s.

Let S be a stopping time and $\tau = (\tau_1, \dots, \tau_d) \in T_S^d$. There exists $(A_i)_{i=1,\dots,d}$ with $\Omega = \bigcup_i A_i$, where $A_i \cap A_j =$ for $i \neq j$, $\tau_1 \wedge \cdots \wedge \tau_d = \tau_i$ a.s. on A_i and A_i are in $\mathcal{F}_{\tau_1 \wedge \cdots \wedge \tau_d}$ for $i = 1, \dots, d$ (for $d = 2$, one can take $A_1 = \{\tau_1 \leq \tau_2\}$ and $A_2 = A_1^c$). We have

$$E[\psi(\tau)|\mathcal{F}_S] = \sum_{i=1}^{d} E[\mathbf{1}_{A_i} E[\psi(\tau)|\mathcal{F}_{\tau_i}]|\mathcal{F}_S].$$

By noticing that on A_i one has a.s. $E[\psi(\tau)|\mathcal{F}_{\tau_i}] \leq u^{(i)}(\tau_i) \leq \phi(\tau_i) = \phi(\tau_1 \wedge \cdots \wedge \tau_d)$, we get $E[\psi(\tau)|\mathcal{F}_S] \leq E[\phi(\tau_1 \wedge \cdots \wedge \tau_d)|\mathcal{F}_S] \leq u(S)$ a.s. By taking the supremum over $\tau = (\tau_1, \dots, \tau_d)$, we complete Step 1.

Step 2: Let us show that for all $S \in T_0$, $v(S) \geq u(S)$ a.s.

This follows from the fact that $\{v(S), S \in T_0\}$ is a supermartingale system greater than $\{\phi(S), S \in T_0\}$ and that $\{u(S), S \in T_0\}$ is the smallest supermartingale system of this class. $\square$

Note that the new reward is expressed in terms of optimal $(d-1)$-stopping time problems. Hence, by induction, the initial optimal d-stopping time problem can be reduced to nested optimal single stopping time problems. In the case of a symmetric reward, the problem reduces to ordered stopping times and the nested optimal single stopping time problems simply reduce to a sequence of optimal single stopping time problems defined by backward induction (see Section 3.6 and the application to *swing options*).

3.3. *Properties of optimal stopping times in the* d*-stopping time problem.* Let $\{\psi(\theta), \theta \in T_0^d\}$ be a d-admissible family. Let us introduce the following notation: for $i = 1, \dots, d$, $\theta \in T_0$ and $\tau_1, \dots, \tau_{d-1}$ in T_0, consider the random variable

$$\psi^{(i)}(\tau_1, \dots, \tau_{d-1}, \theta) = \psi(\tau_1, \dots, \tau_{i-1}, \theta, \tau_i, \dots, \tau_{d-1}). \tag{3.5}$$

Using this notation, note that for each $i = 1, \dots, d$, the value function $u^{(i)}$ defined at (3.2) can be written

$$u^{(i)}(\theta) = \operatorname*{ess\,sup}_{\tau \in T_\theta^{d-1}} E\big[\psi^{(i)}(\tau, \theta)|\mathcal{F}_\theta\big]. \tag{3.6}$$

PROPOSITION 3.3 (Construction of optimal stopping times). *Suppose that*:

1. *there exists an optimal stopping time* θ^* *for* $u(S)$;

2. *for* $i = 1, \dots, d$, *there exist* $(\theta_1^{(i)*}, \dots, \theta_{i-1}^{(i)*}, \theta_{i+1}^{(i)*}, \dots, \theta_d^{(i)*}) = \theta^{(i)*}$ *in* T_θ^{d-1} *such that* $u^{(i)}(\theta^*) = E[\psi^{(i)}(\theta^{(i)*}, \theta^*)|\mathcal{F}_{\theta^*}]$.

Let $(B_i)_{i=1,\dots,d}$ *with* $\Omega = \bigcup_i B_i$ *be such that* $B_i \cap B_j = \varnothing$ *for* $i \neq j$, $\phi(\theta^*) = u^{(i)}(\theta^*)$ *a.s. on* B_i *and* B_i *is* $\mathcal{F}_{\theta^*}$*-measurable for* $i = 1, \dots, d$. *Put*

$$\tau_j^* = \theta^* \mathbf{1}_{B_j} + \sum_{i \neq j, i=1}^{d} \theta_j^{(i)*} \mathbf{1}_{B_i}. \tag{3.7}$$

Then, $(\tau_1^*, \dots, \tau_d^*)$ *is optimal for* $v(S)$, *and* $\tau_1^* \wedge \cdots \wedge \tau_d^* = \theta^*$.

PROOF. It is clear that $\tau_1^* \wedge \cdots \wedge \tau_d^* = \theta^*$, and a.s.

$$\begin{aligned}
u(S) = E[\phi(\theta^*)|\mathcal{F}_S] &= \sum_{i=1}^{d} E[\mathbf{1}_{B_i} u^{(i)}(\theta^*)|\mathcal{F}_S] \\
&= \sum_{i=1}^{d} E[\mathbf{1}_{B_i} E[\psi^{(i)}(\theta^{(i)*}, \theta^*)|\mathcal{F}_{\theta^*}]|\mathcal{F}_S] \\
&= \sum_{i=1}^{d} E[\mathbf{1}_{B_i} E[\psi(\theta_1^{(i)*}, \dots, \theta_{i-1}^{(i)*}, \theta^*, \theta_{i+1}^{(i)*}, \dots, \theta_d^{(i)*})|\mathcal{F}_{\theta^*}]|\mathcal{F}_S] \\
&= E[\psi(\tau_1^*, \dots, \tau_{i-1}^*, \tau_i^*, \tau_{i+1}^*, \dots, \tau_d^*)|\mathcal{F}_S] \leq v(S) = u(S). \qquad \square
\end{aligned}$$

REMARK 3.1. As in the bidimensional case, one can easily derive a necessary condition for obtaining optimal stopping times. Moreover, for an adapted partial order relation on $\mathbb{R}^d$, one can also derive a characterization of minimal optimal d-stopping times. This result is given in Appendix B.2.

Before studying the existence of an optimal d-stopping time for $v(S)$, we will study the regularity properties of the new reward $\{\phi(\theta), \theta \in T_0\}$ defined by (3.3).

3.4. *Regularity of the new reward.* Let us introduce the following definition of uniform continuity.

DEFINITION 3.2. A d-admissible family $\{\psi(\theta), \theta \in T_0^d\}$ is said to be *uniformly right- (resp., left-) continuous along stopping times in expectation [URCE (resp., ULCE)]* if $v(0) < \infty$, and for each $i = 1, \dots, d$, $S \in T_0$ and sequence of stopping times $(S_n)_{n \in \mathbb{N}}$ such that $S_n \downarrow S$ a.s. (resp., $S_n \uparrow S$ a.s.), we have

$$\lim_{n \to \infty} \sup_{\theta \in T_0^{d-1}} \left| E[\psi^{(i)}(\theta, S_n)] - E[\psi^{(i)}(\theta, S)] \right| = 0 \qquad \text{a.s.}$$

PROPOSITION 3.4. *Let* $\{\psi(\theta), \theta \in T_0^d\}$ *be a* d*-admissible family which is URCE* (*resp*., *both URCE and ULCE*). *The family of positive random variables* $\{\phi(S), S \in T_0\}$ *defined by* (3.3) *is then RCE* (*resp*., *both RCE and LCE*).

PROOF. The proof uses an induction argument. For $d = 1$ and $d = 2$, the result has already been shown. Fix $d \geq 1$ and suppose by induction that the property holds for any d-admissible family which is URCE (resp., both URCE and ULCE). Let $\{\psi(\theta), \theta \in T_S^{d+1}\}$ be a $(d+1)$-admissible family which is URCE (resp., both URCE and ULCE). As $\phi(\theta) = \max[u^{(1)}(\theta), \dots, u^{(d+1)}(\theta)]$, it is sufficient to show the RCE (resp., both RCE and LCE) properties for the family $\{u^{(i)}(\theta), \theta \in T_0\}$ for all $i = 1, \dots, d+1$.

Let us introduce the following value function for each $S, \theta \in T_0$:

$$U^{(i)}(\theta, S) = \operatorname*{ess\,sup}_{\tau \in T_\theta^d} E[\psi^{(i)}(\tau, S) | \mathcal{F}_\theta] \qquad \text{a.s.} \tag{3.8}$$

As for all $\theta \in T_0$,

$$u^{(i)}(\theta) = U^{(i)}(\theta, \theta) \qquad \text{a.s.,}$$

it is sufficient to prove that the biadmissible family $\{U^{(i)}(\theta, S), \theta, S \in T_0\}$ is RCE (resp., both RCE and LCE) as in the bidimensional case.

Let $\theta, S \in T_0$ and $(\theta_n)_n$, $(S_n)_n$ be monotonic sequences of stopping times that converge, respectively, to θ and S a.s. We have

$$\begin{aligned} &E[|U^{(i)}(\theta, S) - U^{(i)}(\theta_n, S_n)|] \\ &\quad \leq \underbrace{E[|U^{(i)}(\theta, S) - U^{(i)}(\theta_n, S)|]}_{\text{(I)}} + \underbrace{E[|U^{(i)}(\theta_n, S) - U^{(i)}(\theta_n, S_n)|]}_{\text{(II)}}. \end{aligned}$$

Let us show that (I) tends to 0 as $n \to \infty$. Note that for each $S \in T_0$, $\{\psi^{(i)}(\tau, S), \tau \in T_0^d\}$ is a d-admissible family of positive random variables which is URCE (resp., both URCE and ULCE) and $\{U^{(i)}(\theta, S), \theta \in T_0\}$ is the corresponding value function family. By the induction assumption, this family is RCE (resp., both RCE and LCE). Hence, (I) converges a.s. to 0 as n tends to ∞ when (θ_n) is monotonic.

Let us now show that (II) tends to 0 as $n \to \infty$. By definition of the value function $U^{(i)}(\cdot, \cdot)$ (3.8), it follows that

$$E[|U^{(i)}(\theta_n, S) - U^{(i)}(\theta_n, S_n)|] \leq \sup_{\theta \in T_0^d} |E[\psi^{(i)}(\theta, S)] - E[\psi^{(i)}(\theta, S_n)]|,$$

and the right-hand side tends to 0 by the URCE (resp., both URCE and ULCE) properties of ψ. $\square$

3.5. *Existence of optimal stopping times*. By Theorem 1.1, the regularity properties of the new reward will ensure the existence of an optimal stopping time $\theta^* \in T_0$ for $u(S)$. By Proposition 3.3, this will allow us to show by induction the existence of an optimal stopping time for $v(S)$.

THEOREM 3.2 (Existence of optimal stopping times). *Let $\{\psi(\theta), \theta \in T_0^d\}$ be a d-admissible family of positive random variables which is URCE and ULCE. There then exists a $\tau^* \in T_S^d$ optimal for $v(S)$, that is, such that*

$$v(S) = \operatorname{ess\,sup}_{\tau \in T_S^d} E[\psi(\tau)|\mathcal{F}_S] = E[\psi(\tau*)|\mathcal{F}_S].$$

PROOF. The result is proved by induction on d. For $d = 1$ the result is just Theorem 1.1. Suppose now that $d \geq 1$ and suppose by induction that for all d-admissible families which are URCE and ULCE, optimal d-stopping times do exist. Let $\{\psi(\theta), \theta \in T_S^{d+1}\}$ be a $(d+1)$-admissible family which is URCE and ULCE. The existence of an optimal $(d+1)$-stopping time for the associated value function $v(S)$ will be derived by applying Proposition 3.3. Now, by Proposition 3.4, the new reward family $\{\phi(\theta), \theta \in T_0\}$ is LCE and RCE. By Theorem 1.1, there exists an optimal stopping time θ^* for $u(S)$. Thus, we have proven that condition 1 of Proposition 3.3 is satisfied.

Note now that for $i = 1, \dots, d+1$, the d-admissible families $\{\psi^{(i)}(\theta, \theta^*), \theta \in T_0^d\}$ are URCE and ULCE. Thus, by the induction hypothesis, for each $\theta \in T_0$, there exists an optimal $\theta^{*(i)} \in T_{\theta^*}^d$ for the value function $U^{(i)}(\theta^*, \theta^*)$ defined by (3.8). Noting that $U^{(i)}(\theta^*, \theta^*) = u^{(i)}(\theta^*)$, we have proven that condition 2 of Proposition 3.3 is satisfied. Now applying Proposition 3.3, the result follows. □

3.6. *Symmetric case*. Suppose that $\psi(\tau_1, \dots, \tau_d)$ is symmetric with respect to $(\tau_1, \dots, \tau_d)$, that is,

$$\psi(\tau_1, \dots, \tau_d) = \psi(\tau_{\sigma(1)}, \dots, \tau_{\sigma(d)})$$

for each permutation σ of $\{1, \dots, d\}$. By symmetry we can suppose that $\tau_1 \leq \tau_2 \leq \dots \leq \tau_d$, that is, that the value function $v(S)$ coincides with

$$v_d(S) = \operatorname{ess\,sup}_{(\tau_1, \dots, \tau_d) \in \mathcal{S}_S^d} E[\psi(\tau_1, \dots, \tau_d)|\mathcal{F}_S],$$

where $\mathcal{S}_S^d = \{\tau_1, \dots, \tau_d \in T_S \text{ s.t. } \tau_1 \leq \tau_2 \leq \dots \leq \tau_d\}$. It follows that the value functions $u^{(i)}(\theta)$ and the new reward $\phi(\theta)$ coincide and are simply given for each $\theta \in T_0$ by the following random variable:

$$\phi_1(\theta) = \operatorname{ess\,sup}_{(\tau_2, \tau_3, \dots, \tau_d) \in \mathcal{S}_\theta^{d-1}} E[\psi(\theta, \tau_2, \dots, \tau_d)|\mathcal{F}_\theta].$$

The reduction property can be written as follows:

$$v(S) = \text{ess} \sup_{\theta \in T_S} E[\phi_1(\theta)|\mathcal{F}_S].$$

We then consider the value function $\phi_1(\theta_1)$. The associated new reward is given for θ_1, θ_2 such that $S \le \theta_1 \le \theta_2$ by

$$\phi_2(\theta_1, \theta_2) = \text{ess} \sup_{(\tau_3, \dots, \tau_d) \in \mathcal{S}_{\theta_2}^{d-2}} E[\psi(\theta_1, \theta_2, \tau_3, \dots, \tau_d)|\mathcal{F}_{\theta_2}].$$

Again, the reduction property gives

$$\phi_1(\theta_1) = \text{ess} \sup_{\theta \in T_{\theta_1}} E[\phi_2(\theta_1, \theta_2)|\mathcal{F}_{\theta_1}]. \tag{3.9}$$

We then consider the value function $\phi_2(\theta_1, \theta_2)$, and so on. Thus, by forward induction, we define the new rewards ϕ_i for $i = 1, 2, \dots, d-1$ by

$$\phi_i(\theta_1, \dots, \theta_i) = \text{ess} \sup_{(\tau_{i+1}, \dots, \tau_d) \in \mathcal{S}_{\theta_i}^{d-i}} E[\psi(\theta_1, \dots, \theta_i, \tau_{i+1}, \dots, \tau_d)|\mathcal{F}_{\theta_i}]$$

for each $(\theta_1, \dots, \theta_i) \in \mathcal{S}_S^i$. The reduction property gives

$$\phi_i(\theta_1, \dots, \theta_i) = \text{ess} \sup_{\theta_{i+1} \in T_{\theta_i}} E[\phi_{i+1}(\theta_1, \dots, \theta_i, \theta_{i+1})|\mathcal{F}_{\theta_i}]. \tag{3.10}$$

Note that for $i = d-1$,

$$\phi_{d-1}(\theta_1, \dots, \theta_{d-1}) = \text{ess} \sup_{\theta_d \in T_{\theta_{d-1}}} E[\Psi(\theta_1, \dots, \theta_{d-1}, \theta_d)|\mathcal{F}_{\theta_{d-1}}] \tag{3.11}$$

for each $(\theta_1, \dots, \theta_{d-1}) \in \mathcal{S}_S^{d-1}$.

Hence, using backward induction we can now define $\phi_{d-1}(\theta_1, \dots, \theta_{d-1})$ by (3.11) and then $\phi_{d-2}(\theta_1, \dots, \theta_{d-2}), \dots, \phi_2(\theta_1, \theta_2), \phi_1(\theta_1)$ by the induction formula (3.10). Consequently, we have the following characterization of the value function and construction of a multiple optimal stopping time (which are rather intuitive).

PROPOSITION 3.5.

- *Let $\{\psi(\theta), \theta \in T_0^d\}$ be a symmetric d-admissible family of random variables, and for each stopping time S, consider the associated value function $v(S)$.*

  Let ϕ_i, $i = d-1, d-2, \dots, 2, 1$, be defined by backward induction as follows: $\phi_{d-1}(\theta_1, \dots, \theta_{d-1})$ is given by (3.11) for each $(\theta_1, \dots, \theta_{d-1}) \in \mathcal{S}_S^{d-1}$. Also, for $i = d-2, \dots, 2, 1$ and each $(\theta_1, \dots, \theta_i) \in \mathcal{S}_S^i$, $\phi_i(\theta_1, \dots, \theta_i)$ is given in terms of the function ϕ_{i+1} by backward induction formula (3.10).

  The value function then satisfies

$$v(S) = \text{ess} \sup_{\theta \in T_S} E[\phi_1(\theta)|\mathcal{F}_S]. \tag{3.12}$$

- *Suppose that $\{\psi(\theta), \theta \in T_0^d\}$ is URCE and ULCE. Let θ_1^* be an optimal stopping time for $v(S)$ given by (3.12), let θ_2^* be an optimal stopping time for $\phi_1(\theta_1^*)$ given by (3.9) and for $i = 2, 3, \dots, d-1$, let θ_{i+1}^* be an optimal stopping time for $\phi_i(\theta_1^*, \dots, \theta_i^*)$ given by (3.10).*

  Then, $(\theta_1^*, \dots, \theta_d^*)$ is a multiple optimal stopping time for $v(S)$.

Some simple examples. First, consider the very simple additive case: suppose that the reward is given by

$$\psi(\tau_1, \dots, \tau_d) = Y(\tau_1) + Y(\tau_2) + \cdots + Y(\tau_d), \tag{3.13}$$

where Y is an admissible family of random variables such that $\sup_{\tau \in T_0} E[Y(\tau)] < \infty$. We then obviously have that $v(S) = dv^1(S)$, where $v^1(S)$ is the value function of the single optimal stopping time problem associated with reward Y. Also, if θ_1^* is an optimal stopping time for $v_1(S)$, then $(\theta_1^*, \dots, \theta_1^*)$ is optimal for $v(S)$.

Application to swing options. Let us now consider the more interesting additive case of *swing options*: suppose that $T = +\infty$ and that the reward is still given by (3.13), but the stopping times are separated by a fixed amount of time $\delta > 0$ (sometimes called "refracting time"). In this case, the value function is given by

$$v(S) = \operatorname{ess\,sup}\{E[\psi(\tau_1, \dots, \tau_d)|\mathcal{F}_S], (\tau_1, \dots, \tau_d) \in \mathcal{S}_S^d\},$$

where $\mathcal{S}_S^d = \{\tau_1, \dots, \tau_d \in T_S \text{ s.t. } \tau_i \in T_{\tau_{i-1}+\delta}, 2 \le i \le d-1\}$. All the previous properties then still hold. Again, the ϕ_i satisfy the following induction equality:

$$\phi_i(\theta_1, \dots, \theta_i) = \operatorname{ess} \sup_{\theta_{i+1} \in T_{\theta_i+\delta}} E[\phi_{i+1}(\theta_1, \dots, \theta_i, \theta_{i+1})|\mathcal{F}_{\theta_i}].$$

One can then easily derive that $\phi_{d-1}(\theta_1, \theta_2, \dots, \theta_{d-1}) = Y(\theta_1) + \cdots + Y(\theta_{d-1}) + Z_{d-1}(\theta_{d-1})$, where

$$Z_{d-1}(\theta_{d-1}) = \operatorname{ess} \sup_{\tau \in T_{\theta_{d-1}+\delta}} E[Y(\tau)|\mathcal{F}_{\theta_{d-1}}].$$

$$\phi_{d-2}(\theta_1, \dots, \theta_{d-2}) = Y(\theta_1) + \cdots + Y(\theta_{d-2}) + Z_{d-2}(\theta_{d-2}), \qquad \text{where}$$

$$Z_{d-2}(\theta_{d-2}) = \operatorname{ess} \sup_{\tau \in T_{\theta_{d-2}+\delta}} E[Y(\tau) + Z_{d-1}(\tau)|\mathcal{F}_{\theta_{d-2}}],$$

and so on. Hence, for $i = 1, 2, \dots, d-2$, $\phi_i(\theta_1, \dots, \theta_i) = Y(\theta_1) + \cdots + Y(\theta_i) + Z_i(\theta_i)$, where

$$Z_i(\theta_i) = \operatorname{ess} \sup_{\tau \in T_{\theta_i+\delta}} E[Y(\tau) + Z_{i+1}(\tau)|\mathcal{F}_{\theta_i}].$$

The value function satisfies

$$v(S) = \operatorname{ess} \sup_{\theta \in T_S} E[Y(\theta) + Z_1(\theta)|\mathcal{F}_S]. \tag{3.14}$$

This corresponds to Proposition 3.2 of Carmona and Dayanik (2008).

Suppose that Y is RCE and LCE. Let θ_1^* be the minimal optimal stopping time for $v(S)$ given by (3.14) and for $i = 1, 2, \dots, d-1$, let θ_{i+1}^* be the minimal optimal stopping time for $Z_i(\theta_i^*)$. The d-stopping time $(\theta_1^*, \dots, \theta_d^*)$ is then the minimal optimal stopping time for $v(S)$. This corresponds to Proposition 5.4 of Carmona and Dayanik (2008).

Note that the multiplicative case can be solved similarly. Further applications to American options with multiple exercise times are studied in Kobylanski and Quenez (2010).

4. Aggregation and multiple optimal stopping times. As explained in the Introduction, in previous works on the optimal single stopping time problem, the reward is given by an RCLL positive adapted process (ϕ_t). Moreover, when the reward (ϕ_t) is continuous, an optimal S-stopping time is given by

$$\overline{\theta}(S) = \inf\{t \geq S, v_t = \phi_t\}, \tag{4.1}$$

which corresponds to the first hitting time after S of 0 by the RCLL adapted process $(v_t - \phi_t)$. This formulation is very important since it gives a simple and efficient method to compute an optimal stopping time.

In the two-dimensional case, instead of considering a reward process, it is quite natural to suppose that the reward is given by a biprocess $(\Psi_{t,s})_{(t,s)\in[0,T]^2}$ such that a.s., the map $(t,s) \mapsto \Psi_{t,s}$ is continuous and for each $(t,s) \in [0,T]^2$, $\Psi_{t,s}$ is $\mathcal{F}_{t\vee s}$-measurable (see Remark 2.1).

We would like to construct some optimal stopping times by using hitting times of processes. By the existence and construction properties of optimal stopping times given in Theorem 2.3, we are led to construct θ^*, θ_1^* and θ_2^* as hitting times of processes. Since Ψ is a continuous biprocess, there is no problem for θ_1^*, θ_2^*. However, for θ^* we need to aggregate the new reward $\{\phi(\theta), \theta \in T_0\}$, which requires new aggregation results. These results hold under stronger assumptions on the reward than those made in the previous existence theorem (Theorem 2.3).

4.1. *Some general aggregation results.*

4.1.1. *Aggregation of a supermartingale system.* Recall the classical result of aggregation of a supermartingale system [El Karoui (1981)].

PROPOSITION 4.1. *Let $\{h(S), S \in T_0\}$ be a supermartingale system which is RCE and such that $h(0) < \infty$. There then exists an RCLL adapted process (h_t) which aggregates the family $\{h(S), S \in T_0\}$, that is, for each $S \in T_0$, $h_S = h(S)$ a.s.*

This lemma relies on a well-known result [see, e.g., El Karoui (1981) or Theorem 3.13 in Karatzas and Shreve (1994); for details, see the proof in Section 4.4].

Classically, the above Proposition 4.1 is used to aggregate the value function of the single stopping time problem. However, it cannot be applied to the new reward since it is no longer a supermartingale system. Thus, we will now state a new result on aggregation.

4.1.2. *A new result on aggregation of an admissible family.* Let us introduce the following right-continuous property for admissible families.

DEFINITION 4.1. An admissible family $\{\phi(\theta), \theta \in T_0\}$ is said to be *right-continuous along stopping times (RC)* if for any $\theta \in T_0$ and any sequence $(\theta_n)_{n\in\mathbb{N}}$ of stopping times such that $\theta_n \downarrow \theta$ a.s., we have $\phi(\theta) = \lim_{n\to\infty} \phi(\theta_n)$ a.s.

We state the following result.

THEOREM 4.1. *Suppose that the admissible family of positive random variables $\{\phi(\theta), \theta \in T_0\}$ is right-continuous along stopping times. There then exists a progressive process (ϕ_t) such that for each $\theta \in T_0$, $\phi_\theta = \phi(\theta)$ a.s. and such that there exists a nonincreasing sequence of right-continuous processes $(\phi_t^n)_{n\in\mathbb{N}}$ such that for each $(\omega, t) \in \Omega \times [0, T]$, $\lim_{n\to\infty} \phi_t^n(\omega) = \phi_t(\omega)$.*

PROOF. See Section 4.4. □

4.2. *The optimal stopping problem.* First, recall the following classical result [El Karoui (1981)].

PROPOSITION 4.2 (Aggregation of the value function). *Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family of random variables which is RCE. Suppose that $E[\operatorname{ess\,sup}_{\theta\in T_0} \phi(\theta)] < \infty$.*

There then exists an RCLL supermartingale (v_t) which aggregates the family $\{v(S), S \in T_0\}$ defined by (1.1), that is, for each stopping time S, $v(S) = v_S$ a.s.

PROOF. The family $\{v(S), S \in T_0\}$ is a supermartingale system (Proposition 1.3) and has the RCE property (Proposition 1.5). The result clearly follows by applying the aggregation property of supermartingale systems (Proposition 4.1). □

THEOREM 4.2. *Suppose the reward is given by an RC and LCE admissible family $\{\phi(\theta), \theta \in T_0\}$ such that $E[\operatorname{ess\,sup}_{\theta\in T_0} \phi(\theta)] < \infty$.*

Let (ϕ_t) be the progressive process given by Theorem 4.1 that aggregates this family. Let $\{v(S), S \in T_0\}$ be the family of value functions defined by (1.1), and let (v_t) be an RCLL adapted process that aggregates the family $\{v(S), S \in T_0\}$.

The random variable defined by

$$\overline{\theta}(S) = \inf\{t \geq S, v_t = \phi_t\} \tag{4.2}$$

is the minimal optimal stopping time for $v(S)$, that is, $\overline{\theta}(S) = \theta^*(S)$ a.s.

As for Theorem 1.1, the proof relies on the construction of a family of stopping times that are approximatively optimal. The details, which require some fine techniques of the general theory of processes, are given in Section 4.4.

REMARK 4.1. In the case of an RCLL reward process supposed to be LCE, the above theorem corresponds to the classical existence result [see El Karoui (1981) and Karatzas and Shreve (1998)].

4.3. *The optimal multiple stopping time problem.* For simplicity, we study only the case when $d = 2$. We will now prove that the minimal optimal pair of stopping times (τ_1^*, τ_2^*) defined by (2.8) can also be given in terms of *hitting times*. In order to do this, we first need to aggregate the value function and the new reward.

4.3.1. *Aggregation of the value function.*

PROPOSITION 4.3. *Suppose the reward is given by an RCE biadmissible family $\{\psi(\theta, S), \theta, S \in T_0\}$ such that $E[\operatorname{ess\,sup}_{\theta, S \in T_0} \psi(\theta, S)] < \infty$.*

There then exists a supermartingale (v_t) with RCLL paths that aggregates the family $\{v(S), S \in T_0\}$ defined by (2.1), that is, such for each $S \in T_0$, $v(S) = v_S$ a.s.

PROOF. The RCE property of $\{v(S), S \in T_0\}$ shown in Proposition 2.2, together with the supermartingale property [Proposition 2.1(3)] gives, by Proposition 4.1, the desired result. □

4.3.2. *Aggregation of the new reward.* We will now study the aggregation problem of the new reward family $\{\phi(\theta), \theta \in T_0\}$. Let us introduce the following definition.

DEFINITION 4.2. A biadmissible family $\{\psi(\theta, S), \theta, S \in T_0\}$ is said to be *uniformly right-continuous along stopping times (URC)* if $E[\operatorname{ess\,sup}_{\theta, S \in T_0} \psi(\theta, S)] < \infty$ and if for each nonincreasing sequence of stopping times $(S_n)_{n \in \mathbb{N}}$ in T_S which converges a.s. to a stopping time $S \in T_0$,

$$\lim_{n \to \infty} \Big[\operatorname*{ess\,sup}_{\theta \in T_S} \{|\psi(\theta, S_n) - \psi(\theta, S)|\}\Big] = 0 \qquad \text{a.s.}$$

and

$$\lim_{n \to \infty} \Big[\operatorname*{ess\,sup}_{\theta \in T_S} \{|\psi(S_n, \theta) - \psi(S, \theta)|\}\Big] = 0 \qquad \text{a.s.}$$

The following right continuity property holds true for the new reward family.

THEOREM 4.3. *Suppose that the admissible family of positive random variables* $\{\psi(\theta, S), \theta, S \in T_0\}$ *is URC. The family of positive random variables* $\{\phi(S), S \in T_0\}$ *defined by* (2.3) *is then RC.*

PROOF. As $\phi(\theta) = \max[u_1(\theta), u_2(\theta)]$, it is sufficient to show the RC property for the family $\{u_1(\theta), \theta \in T_0\}$.

Now, for all $\theta \in T_0$, $u_1(\theta) = U_1(\theta, \theta)$ a.s., where

$$U_1(\theta, S) = \operatorname*{ess\,sup}_{\tau_1 \in T_\theta} E[\psi(\tau_1, S)|\mathcal{F}_\theta] \qquad \text{a.s.} \tag{4.3}$$

Hence, it is sufficient to prove that $\{U_1(\theta, S), \theta, S \in T_0\}$ is RC.

Let $\theta, S \in T_0$ and $(\theta_n)_n$, $(S_n)_n$ be nonincreasing sequences of stopping times in T_0 that converge to θ and S a.s. We have

$$|U_1(\theta, S) - U_1(\theta_n, S_n)| \le \underbrace{|U_1(\theta, S) - U_1(\theta_n, S)|}_{\text{(I)}} + \underbrace{|U_1(\theta_n, S) - U_1(\theta_n, S_n)|}_{\text{(II)}}.$$

(I) *tends to* 0 *as* $n \to \infty$.

For each $S \in T_0$, as $\{\psi(\theta, S), \theta \in T_0\}$ is an admissible family of positive random variables which is RC, Proposition 4.3 gives the existence of an RCLL adapted process $(U_t^{1,S})$ such that for each stopping time $\theta \in T_0$,

$$U_\theta^{1,S} = U_1(\theta, S) \qquad \text{a.s.} \tag{4.4}$$

(I) can be rewritten as $|U_1(\theta, S) - U_1(\theta_n, S)| = |U_\theta^{1,S} - U_{\theta_n}^{1,S}|$ a.s., which converges a.s. to 0 as n tends to ∞ by the right continuity of the process $(U_t^{1,\theta})$.

(II) *tends to* 0 *as* $n \to \infty$.

By definition of the value function $U_1(\cdot, \cdot)$ (4.3), it follows that

$$\begin{aligned} |U_1(\theta_n, S) - U_1(\theta_n, S_n)| &\le E\Big(\operatorname*{ess\,sup}_{\tau_1 \in T_{\theta_n}} |\psi(\tau_1, S) - \psi(\tau_1, S_n)| \big| \mathcal{F}_{\theta_n}\Big) \\ &\le E(Z_m|\mathcal{F}_{\theta_n}) \qquad \text{a.s.} \end{aligned}$$

for any $n \ge m$, where $Z_m := \sup_{r \ge m}\{\operatorname{ess\,sup}_{\tau \in T_0} |\psi(\tau, S_r) - \psi(\tau, S)|\}$ and $(E(Z_m|\mathcal{F}_t))_{t \ge 0}$ is an RCLL version of the conditional expectation. Hence, by the right continuity of this process, for each fixed $m \in \mathbb{N}$, the sequence of random variables $(E(Z_m|\mathcal{F}_{\theta_n}))_{n \in \mathbb{N}}$ converges a.s. to $E(Z_m|\mathcal{F}_\theta)$ as n tends to ∞. It follows that for each $m \in \mathbb{N}$,

$$\limsup_{n \to \infty} |U_1(\theta_n, S) - U_1(\theta_n, S_n)| \le E(Z_m|\mathcal{F}_\theta) \qquad \text{a.s.} \tag{4.5}$$

Now, the sequence $(Z_m)_{m \in \mathbb{N}}$ converges a.s. to 0 and

$$|Z_m| \le 2 \operatorname*{ess\,sup}_{\theta, S \in T_0} \psi(\theta, S) \qquad \text{a.s.}$$

Note that the second member of this inequality is integrable. By the Lebesgue theorem for the conditional expectation, $E(Z_m|\mathcal{F}_\theta)$ converges to 0 in L^1 as m tends to ∞. The sequence $(Z_m)_{m\in\mathbb{N}}$ is decreasing. It follows that the sequence $\{E(Z_m|\mathcal{F}_\theta)\}_{m\in\mathbb{N}}$ is also decreasing and hence converges a.s. Since this sequence converges to 0 in L^1, its limit is also 0 almost surely. By letting m tend to ∞ in (4.5), we obtain

$$\limsup_{n\to\infty} |U_1(\theta_n, S) - U_1(\theta_n, S_n)| \le 0 \qquad \text{a.s.}$$

The proof of Theorem 4.3 is thus complete. □

COROLLARY 4.1 (Aggregation of the new reward). *Under the same hypothesis as Theorem 4.3, there exists some progressive right-continuous adapted process* (ϕ_t) *which aggregates the family* $\{\phi(\theta),\ \theta \in T_0\}$, *that is,* $\phi_\theta = \phi(\theta)$ *a.s. for each* $\theta \in T_0$, *and such that there exists a decreasing sequence of right-continuous processes* $(\phi_t^n)_{n\in\mathbb{N}}$ *that converges to* (ϕ_t).

PROOF. This follows from the right continuity of the new reward (Theorem 4.3) which we can aggregate (Theorem 4.1). □

REMARK 4.2. For the optimal d-stopping time problem, the same result holds for URC d-admissible families $\{\psi(\theta), \theta \in T_0^d\}$, that is, families that satisfy $E[\operatorname{ess\,sup}_{\theta\in T_0} \psi(\theta)] < \infty$ and

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{\theta\in T_0} \left|\psi^{(i)}(\theta, S) - \psi^{(i)}(\theta, S_n)\right| = 0$$

for $i = 1, \dots, d, \theta, S \in T_0$ and sequences (S_n) in T_0 such that $S_n \downarrow S$ a.s.

The proof is strictly the same, with $U_1(\theta, S)$ replaced by $U^{(i)}(\theta, S)$ for $\theta, S \in T_0$ and $\psi(\tau, S)$ with $\tau, S \in T_0$ replaced by $\psi^{(i)}(\tau, S)$, with $\tau \in T_0^{d-1}$ and $S \in T_0$.

4.3.3. *Optimal multiple stopping times as hitting times of processes.* As before, for the sake of simplicity, we suppose that $d = 2$. Suppose that $\{\psi(\theta, S), \theta, S \in T_0\}$ is a URC and ULCE biadmissible family. Let $\{\phi(\theta), \theta \in T_0\}$ be the new reward family. By Theorem 2.2, this family is LCE. Furthermore, by Theorem 4.3, this family is RC. Let (ϕ_t) be the progressive process that aggregates this family, given by Theorem 4.1. Let (u_t) be an RCLL process that aggregates the value function associated with (ϕ_t). By Theorem 4.2, the stopping time

$$\theta^* = \inf\{t \ge S, u_t = \phi_t\}$$

is optimal for $u(S)$.

The family $\{\psi(\theta, \theta^*),\ \theta \in T_{\theta^*}\}$ is admissible, RC and LCE. Let (ψ_t^1) be the progressive process that aggregates this family given by Theorem 4.1. Let (v_t^1) be an RCLL process that aggregates the value function associated with (ψ_t^1). By Theorem 4.2 the stopping time $\theta_1^* = \inf\{t \ge \theta^*, v_t^1 = \psi_t^1\}$ is optimal for $v_{\theta^*}^1$ and $v_{\theta^*}^1 = u^1(\theta^*)$.

The family $\{\psi(\theta^*, \theta),\ \theta \in T_{\theta^*}\}$ is admissible, RC and LCE. Let (ψ_t^2) be the progressive process that aggregates this family given by Theorem 4.1. Let (v_t^2) be an RCLL process that aggregates the value function associated with (ψ_t^2). By Theorem 4.2, the stopping time $\theta_2^* = \inf\{t \geq \theta^*, v_t^2 = \psi_t^2\}$ is optimal for $v_{\theta^*}^2$, and $v_{\theta^*}^2 = u_2(\theta^*)$.

By Proposition 2.4, the pair of stopping times (τ_1^*, τ_2^*) defined by

$$\tau_1^* = \theta^* \mathbf{1}_B + \theta_1^* \mathbf{1}_{B^c}, \qquad \tau_2^* = \theta_2^* \mathbf{1}_B + \theta^* \mathbf{1}_{B^c}, \tag{4.6}$$

where $B = \{u_1(\theta^*) \leq u_2(\theta^*)\} = \{v_{\theta^*}^1 \leq v_{\theta^*}^2\}$, is optimal for $v(S)$.

THEOREM 4.4. *Let $\{\psi(\theta, S), \theta, S \in T_0\}$ be a biadmissible family which is URC and ULCE. The pair of stopping times (τ_1^*, τ_2^*) defined by (4.6) is then optimal for $v(S)$.*

Note that the above construction of (τ_1^*, τ_2^*) as hitting times of processes requires stronger assumptions on the reward than those made in Theorem 2.3. Furthermore, let us emphasize that it also requires some new aggregation results (Theorems 4.1 and 4.2).

4.4. *Proofs of Proposition 4.1 and Theorems 4.1 and 4.2.* We now give the proofs of Proposition 4.1 and Theorems 4.1 and 4.2.

First, we give the short proof of the classical Proposition 4.1 which we recall here (for the reader's convenience).

PROPOSITION 4.1. *Let $\{h(S), S \in T_0\}$ be a supermartingale system which satisfies $h(0) < \infty$ and which is right-continuous along stopping times in expectation. There then exists an RCLL adapted process (h_t) which aggregates the family $\{h(S), S \in T_0\}$, that is, $h_S = h(S)$ a.s.*

PROOF. Let us consider the process $(h(t))_{0 \leq t \leq T}$. It is a supermartingale and the function $t \mapsto E(h(t))$ is right-continuous. By classical results [see Theorem 3.13 in Karatzas and Shreve (1994)], there exists an RCLL supermartingale $(h_t)_{0 \leq t \leq T}$ such that for each $t \in [0, T]$, $h_t = h(t)$ a.s. It is then clear that for each dyadic stopping time $S \in T_0$, $h_S = h(S)$ a.s. (for details, see Part 2 of the proof of Theorem 1.1). This implies that

$$E[h_S] = E[h(S)]. \tag{4.7}$$

Since the process $(h_t)_{0 \leq t \leq T}$ is RCLL and since the family $\{h(S), S \in T_0\}$ is right-continuous in expectation, equality (4.7) still holds for any stopping time $S \in T_0$. It then remains to show that $h_S = h(S)$ a.s., but this is classical. Let $A \in \mathcal{F}_S$ and define $S_A = S\mathbf{1}_A + T\mathbf{1}_{A^c}$. Since S_A is a stopping time, $E[h_{S_A}] = E[h(S_A)]$. Since $h_T = h(T)$ a.s., it gives that $E[h_S \mathbf{1}_A] = E[h(S)\mathbf{1}_A]$, from which the desired result follows. □

We now give the proof of Theorem 4.1.

THEOREM 4.1. *Suppose that the admissible family of positive random variables $\{\phi(\theta), \theta \in T_0\}$ is right-continuous along stopping times. There then exists a progressive process (ϕ_t) such that for each $\theta \in T_0$, $\phi_\theta = \phi(\theta)$ a.s. and such that there exists a nonincreasing sequence of right-continuous processes $(\phi_t^n)_{n\in\mathbb{N}}$ such that for each $(\omega, t) \in \Omega \times [0, T]$, $\lim_{n\to\infty} \phi_t^n(\omega) = \phi_t(\omega)$.*

PROOF. For each $n \in \mathbb{N}^*$, let us define a process $(\phi_t^n)_{t\geq 0}$ that is a function of (ω, t) by

$$\phi_t^n(\omega) = \sup_{s\in\mathbb{D}\cap]t,([2^n t]+1)/2^n[} \phi(s \wedge T) \tag{4.8}$$

for each $(\omega, t) \in \Omega \times [0, T]$, where $\mathbb{D}$ is the set of dyadic rationals.

For each $t \in [0, T]$ and each $\varepsilon > \frac{1}{2^n}$, the process (ϕ_t^n) is $(\mathcal{F}_{t+\varepsilon})$-adapted and, for each $\omega \in \Omega$, the function $t \mapsto \phi_t^n(\omega)$ is right-continuous. Hence, the process (ϕ_t^n) is also $(\mathcal{F}_{t+\varepsilon})$-progressive. Moreover, the sequence $(\phi_t^n)_{n\in\mathbb{N}^*}$ is decreasing. Let ϕ_t be its limit, that is, for each $(\omega, t) \in \Omega \times [0, T]$,

$$\phi_t(\omega) = \lim_{n\to\infty} \phi_t^n(\omega).$$

It follows that for each $\varepsilon > 0$, the process (ϕ_t) is $(\mathcal{F}_{t+\varepsilon})$-progressive. Thus, (ϕ_t) is $(\mathcal{F}_{t^+})$-progressive and consequently $(\mathcal{F}_t)$-progressive since $\mathcal{F}_{t^+} = \mathcal{F}_t$.

Step 1: Fix $\theta \in T_0$. Let us show that $\phi_\theta \leq \phi(\theta)$ a.s.

Let us suppose, by contradiction, that the above inequality does not hold. There then exists $\varepsilon > 0$ such that the set $A = \{\phi(\theta) \leq \phi_\theta - \varepsilon\}$ satisfies $P(A) > 0$.

Fix $n \in N$. For all $\omega \in A$, we have that $\phi(\theta)(\omega) \leq \phi_{\theta(\omega)}^n(\omega) - \varepsilon$, where $\phi_{\theta(\omega)}^n(\omega)$ is defined by (4.8) with t replaced by $\theta(\omega)$.

By definition of ϕ^n there exists $t \in]\theta(\omega), \frac{[2^n\theta(\omega)]+1}{2^n}[\cap \mathbb{D}$ such that

$$\phi(\theta)(\omega) \leq \phi(t)(\omega) - \frac{\varepsilon}{2}.$$

We introduce the following subset of $[0, T] \times \Omega$:

$$\overline{A}_n = \left\{(t, \omega), t \in \left]\theta(\omega), \frac{[2^n\theta(\omega)] + 1}{2^n}\right[\cap \mathbb{D} \text{ and } \phi(\theta)(\omega) \leq \phi(t)(\omega) - \frac{\varepsilon}{2}\right\}.$$

First, note that $\overline{A}_n$ is optional. Indeed, we have $\overline{A}_n = \bigcup_{t\in\mathbb{D}}\{t\} \times B_{n,t}$, where

$$B_{n,t} = \left\{\theta < t < \frac{[2^n\theta] + 1}{2^n}\right\} \cap \left\{\phi(\theta) \leq \phi(t) - \frac{\varepsilon}{2}\right\},$$

and the process $(\omega, t) \mapsto \mathbf{1}_{B_{n,t}}(\omega)$ is optional since θ and $\frac{[2^n\theta]+1}{2^n}$ are stopping times and $\{\phi(\theta), \theta \in T_0\}$ is admissible. Also, A is included in $\pi(\overline{A}_n)$, the projection of $\overline{A}_n$ onto Ω, that is,

$$A \subset \pi(\overline{A}_n) = \{\omega \in \Omega,\ \exists t \in [0, T] \text{ s.t. } (t, \omega) \in \overline{A}_n\}.$$

Hence, by a section theorem [see Dellacherie and Meyer (1975), Chapter IV], there exists a dyadic stopping time T_n such that for each ω in $\{T_n < \infty\}$, $(T_n(\omega), \omega) \in \overline{A}_n$ and

$$P(T_n < \infty) \geq P(\pi(\overline{A}_n)) - \frac{P(A)}{2^{n+1}} \geq P(A) - \frac{P(A)}{2^{n+1}}.$$

Hence, for all ω in $\{T_n < \infty\}$

$$\phi(\theta)(\omega) \leq \phi(T_n(\omega)) - \frac{\varepsilon}{2} \quad \text{and} \quad T_n(\omega) \in \left]\theta(\omega), \frac{[2^n\theta(\omega)] + 1}{2^n}\right[\cap \mathbb{D}.$$

Note that

$$P\left(\bigcap_{n\geq 1}\{T_n < \infty\}\right) \geq P(A) - \left(\sum_{n\geq 1} \frac{P(A)}{2^{n+1}}\right) \geq \frac{P(A)}{2} > 0.$$

Put $\overline{T}_n = T_1 \wedge \cdots \wedge T_n$. We have $\overline{T}_n \downarrow \theta$ and $\phi(\theta) \leq \phi(\overline{T}_n) - \frac{\varepsilon}{2}$ for each n on $\bigcap_{n\geq 1}\{T_n < \infty\}$. By letting n tend to ∞ in this inequality, since $\{\phi(\theta), \theta \in T_0\}$ is right-continuous along stopping times, we derive that $\phi(\theta) \leq \phi(\theta) - \frac{\varepsilon}{2}$ a.s. on $\bigcap_{n\geq 1}\{T_n < \infty\}$, which gives the desired contradiction.

Step 2: Fix $\theta \in T_0$. Let us show that $\phi(\theta) \leq \phi_\theta$ a.s.

Put $T^n = \frac{[2^n\theta]+1}{2^n}$. The sequence (T^n) is a nonincreasing sequence of stopping times such that $T^n \downarrow \theta$. Moreover, note that since the family $\{\phi(\theta), \theta \in T_0\}$ is admissible, for each $d \in \mathbb{D}$, for almost every $\omega \in \{T^{n+1} = d\}$, $\phi(T^{n+1})(\omega) = \phi(d)(\omega)$. Now, we have $T^{n+1} \in]\theta, T^n[\cap \mathbb{D}$. Also, for each $\omega \in \Omega$ and each $d \in]\theta(\omega), T^n(\omega)[\cap \mathbb{D}$,

$$\phi(d)(\omega) \leq \sup_{s\in]\theta(\omega), T^n(\omega)[\cap\mathbb{D}} \phi(s)(\omega) = \phi^n_{\theta(\omega)}(\omega),$$

where the last equality follows by the definition of $\phi^n_{\theta(\omega)}(\omega)$ [see (4.8), with t replaced by $\theta(\omega)$]. Hence,

$$\phi(T^{n+1}) \leq \phi^n_\theta \qquad \text{a.s.}$$

Letting n tend to ∞, by using the right-continuous property of $\{\phi(\theta), \theta \in T_0\}$ along stopping times and the convergence of $\phi^n_{\theta(\omega)}(\omega)$ to $\phi_{\theta(\omega)}(\omega)$ for each ω, we derive that $\phi(\theta) \leq \phi_\theta$ a.s. □

We now give the proof of Theorem 4.2.

THEOREM 4.2. $\overline{\theta}(S) = \inf\{t \geq S, v_t = \phi_t\}$ *is an optimal stopping time for* v_S.

PROOF. We begin by constructing a family of stopping times that are approximatively optimal. For $\lambda \in]0, 1[$, define the stopping time

$$\overline{\theta}^\lambda(S) := \inf\{t \geq S, \lambda v_t \leq \phi_t\} \wedge T. \tag{4.9}$$

The proof follows the proof of Theorem 1.1 exactly, except for Step 1, which corresponds to the following lemma.

LEMMA 4.1. *For each $S \in T_0$ and $\lambda \in]0,1[$,*

$$\lambda v_{\overline{\theta}^\lambda(S)} \le \phi_{\overline{\theta}^\lambda(S)} \qquad a.s. \tag{4.10}$$

By the same arguments as in the proof of Theorem 1.1, $\overline{\theta}^\lambda(S)$ is nondecreasing with respect to λ and converges as $\lambda \uparrow 1$ to an optimal stopping time which coincides with $\overline{\theta}(S)$ a.s. □

PROOF OF LEMMA 4.1. To simplify notation, $\overline{\theta}^\lambda(S)$ will be written as $\overline{\theta}^\lambda$. For the sake of simplicity, without loss of generality, we suppose that $t \mapsto v_t(\omega)$ is RCLL for each $\omega \in \Omega$.

Fix $\omega \in \Omega$. In the following, we use only simple analytic arguments.

By definition of $\overline{\theta}^\lambda(\omega)$ (1.6), for each $n \in \mathbb{N}^*$, there exists $t \in [\overline{\theta}^\lambda(\omega), \overline{\theta}^\lambda(\omega) + \frac{1}{n}[$ such that $\lambda v_t(\omega) \le \phi_t(\omega)$.

Also, note that for each $m \in \mathbb{N}^*$, $\phi_t(\omega) \le \phi_t^m(\omega)$.

Now, fix $m \in \mathbb{N}^*$ and $\alpha > 0$.

By the right continuity of $t \mapsto v_t(\omega)$ and $t \mapsto \phi_t^m(\omega)$, there exists $t_n^m(\omega) \in \mathbb{D} \cap [\overline{\theta}^\lambda(\omega), \overline{\theta}^\lambda(\omega) + \frac{1}{n}[$ such that

$$\lambda v_{t_n^m(\omega)}(\omega) \le \phi^m_{t_n^m(\omega)}(\omega) + \alpha. \tag{4.11}$$

Note that $\lim_{n\to\infty} t_n^m(\omega) = \overline{\theta}^\lambda(\omega)$ and $t_n^m(\omega) \ge \overline{\theta}^\lambda(\omega)$ for any n. Again, by using the right continuity of $t \mapsto v_t(\omega)$ and $t \mapsto \phi_t^m(\omega)$, and by letting n tend to ∞ in (4.11), we derive that

$$\lambda v_{\overline{\theta}^\lambda(\omega)}(\omega) \le \phi^m_{\overline{\theta}^\lambda(\omega)}(\omega) + \alpha,$$

and this inequality holds for each $\alpha > 0$, $m \in \mathbb{N}^*$ and $\omega \in \Omega$. By letting m tend to ∞ and α tend to 0, we derive that for each $\omega \in \Omega$, $\lambda v_{\overline{\theta}^\lambda(\omega)}(\omega) \le \phi_{\overline{\theta}^\lambda(\omega)}(\omega)$, which completes the proof of the lemma. □

APPENDIX A

We recall the following classical theorem [see, e.g., Karatzas and Shreve (1998), Neveu (1975)].

THEOREM A.1 (Essential supremum). *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{X}$ be a nonempty family of positive random variables defined on $(\Omega, \mathcal{F}, P)$. There exists a random variable X^* satisfying*:

1. *for all $X \in \mathcal{X}$, $X \le X^*$ a.s.*;
2. *if Y is a random variable satisfying $X \le Y$ a.s. for all $X \in \mathcal{X}$, then $X^* \le Y$ a.s.*

This random variable, which is unique a.s., is called the essential supremum of $\mathcal{X}$ *and is denoted* $\operatorname{ess\,sup} \mathcal{X}$.

Furthermore, if $\mathcal{X}$ *is closed under pairwise maximization* (i.e., $X, Y \in \mathcal{X}$ *implies* $X \vee Y \in \mathcal{X}$), *then there is a nondecreasing sequence* $\{Z_n\}_{n\in\mathbb{N}}$ *of random variables in* $\mathcal{X}$ *satisfying* $X^* = \lim_{n\to\infty} Z_n$ *a.s.*

APPENDIX B

B.1. Characterization of minimal optimal double stopping time. In order to give a characterization of *minimal optimal* stopping times, we introduce the following partial order relation on $\mathbb{R}^2$: $(a, b) \prec (a', b')$ if and only if

$$[(a \wedge b < a' \wedge b') \text{ or } (a \wedge b = a' \wedge b' \text{ and } a \leq a' \text{ and } b \leq b')].$$

Note that although the minimum of two elements of $\mathbb{R}^2$ is not defined, the infimum, that is, the greatest minorant of the couple, does exist and $\inf[(a, b), (a', b')] = \mathbf{1}_{\{a\wedge b<a'\wedge b'\}}(a, b) + \mathbf{1}_{\{a'\wedge b'<a\wedge b\}}(a', b') + \mathbf{1}_{\{a\wedge b=a'\wedge b'\}}(a \wedge a', b \wedge b')$.

Note also that if (τ_1^*, τ_2^*), $(\tau_1', \tau_2') \in T_0 \times T_0$ are optimal for $v(S)$, then the infimum of the couple $\inf[(\tau_1^*, \tau_2^*), (\tau_1', \tau_2')]$, in the sense of the relation $\prec$ a.s., is optimal for $v(S)$.

The two following assertions can be shown to be equivalent:

1. a pair $(\tau_1^*, \tau_2^*) \in T_0 \times T_0$ is *minimal optimal* for $v(S)$ (i.e, is the minimum for the order $\prec$ a.s. of the set $\{(\tau_1^*, \tau_2^*) \in T_S^2, v(S) = E[\psi(\tau_1^*, \tau_2^*)|\mathcal{F}_S]\}$), $\theta^* = \tau_1^* \wedge \tau_2^*$ and $\theta_1^*, \theta_2^* \in T_0$ are such that $\theta_2^* = \tau_2^*$ on $\{\tau_1^* < \tau_2^*\}$ and $\theta_1^* = \tau_1^*$ on $\{\tau_1^* > \tau_2^*\}$;
2. (a) $\theta^* \in T_0$ is minimal optimal for $u(S)$;
   (b) $\theta_2^* \in T_0$ is minimal optimal for $u_2(\theta^*)$ on $\{u_1(\theta^*) < u_2(\theta^*)\}$;
   (c) $\theta_1^* \in T_0$ is minimal optimal for $u_1(\theta^*)$ on $\{u_2(\theta^*) < u_1(\theta^*)\}$, and $\tau_1^* = \theta^*\mathbf{1}_{\{u_1(\theta^*)\leq u_2(\theta^*)\}} + \theta_1^*\mathbf{1}_{\{u_1(\theta^*)>u_2(\theta^*)\}}$, $\tau_2^* = \theta^*\mathbf{1}_{\{u_2(\theta^*)\leq u_1(\theta^*)\}} + \theta_2^* \times \mathbf{1}_{\{u_2(\theta^*)>u_1(\theta^*)\}}$.

B.2. Characterization of minimal optimal *d*-stopping times. Consider the following partial order relation $\prec_d$ on $\mathbb{R}^d$ defined by induction in the following way: for $d = 1$, $\forall a, a' \in \mathbb{R}$, $a \prec_1 a'$ if and only if $a \leq a'$, and for $d > 1$, $\forall (a_1, \dots, a_d), (a_1', \dots, a_d') \in \mathbb{R}^d$, $(a_1, \dots, a_d) \prec_d (a_1', \dots, a_d')$ if and only if either $a_1 \wedge \cdots \wedge a_d < a_1' \wedge \cdots \wedge a_d'$ or

$$\begin{cases} a_1 \wedge \cdots \wedge a_d = a_1' \wedge \cdots \wedge a_d', & \text{and, for } i = 1, \dots, d, \\ a_i = a_1 \wedge \cdots \wedge a_d \quad \Longrightarrow & \begin{cases} a_i' = a_1' \wedge \cdots \wedge a_d' \quad \text{and} \\ (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d) \\ \qquad \prec_{d-1} (a_1', \dots, a_{i-1}', a_{i+1}', \dots, a_d'). \end{cases} \end{cases}$$

Note that for $d = 2$ the order relation $\prec_2$ is the order relation $\prec$ defined above.

One can show that a d-stopping time $(\tau_1, \dots, \tau_d)$ is the d-minimal optimal stopping time for $v(S)$, that is, it is minimal for the order $\prec_d$ in the set $\{\tau \in T_S^d, v(S) = E[\psi(\tau)|\mathcal{F}_S]\}$ if and only if:

1. $\theta^* = \tau_1 \wedge \cdots \wedge \tau_d$ is minimal optimal for $u(S)$;
2. for $i = 1, \dots, d$, $\theta^{*(i)} = \tau_i \in T_S^{d-1}$ is the $(d-1)$-minimal optimal stopping time for $u^{(i)}(\theta^*)$ on the set $\{u^{(i)}(\theta^*) \geq \bigvee_{k \neq i} u^{(k)}(\theta^*)\}$.

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M. KOBYLANSKI
CNRS—UMR 8050 (LAMA)
UNIVERSITÉ DE MARNE-LA-VALLÉE
5, BOULEVARD DESCARTES
CITÉ DESCARTES—CHAMPS-SUR-MARNE
77454 MARNE-LA-VALLÉE CEDEX 2
FRANCE
E-MAIL: magdalena.kobylanski@univ-mlv.fr

M.-C. QUENEZ
CNRS—UMR 7599 (LPMA)
UNIVERSITÉ DENIS DIDEROT (P7)
175 RUE DU CHEVALERET
75251 PARIS CEDEX 05
FRANCE
E-MAIL: quenez@math.jussieu.fr

E. ROUY-MIRONESCU
CNRS
ECOLE CENTRALE DE LYON
INSTITUT CAMILLE JORDAN
UNIVERSITÉ DE LYON
43, BOULEVARD DU 11 NOVEMBRE 1918
69622 VILLEURBANNE CEDEX
FRANCE
E-MAIL: Elisabeth.Mironescu@ec-lyon.fr