The Annals of Applied Probability
2011, Vol. 21, No. 3, 951–986
DOI: 10.1214/10-AAP713

A POSITIVE RECURRENT REFLECTING BROWNIAN MOTION WITH DIVERGENT FLUID PATH

BY MAURY BRAMSON[1]

University of Minnesota, Twin Cities

Semimartingale reflecting Brownian motions (SRBMs) are diffusion processes with state space the d-dimensional nonnegative orthant, in the interior of which the processes evolve according to a Brownian motion, and that reflect against the boundary in a specified manner. The data for such a process are a drift vector θ, a nonsingular $d \times d$ covariance matrix Σ, and a $d \times d$ reflection matrix R. A standard problem is to determine under what conditions the process is positive recurrent. Necessary and sufficient conditions for positive recurrence are easy to formulate for $d = 2$, but not for $d > 2$.

Associated with the pair (θ, R) are fluid paths, which are solutions of deterministic equations corresponding to the random equations of the SRBM. A standard result of Dupuis and Williams [*Ann. Probab.* **22** (1994) 680–702] states that when every fluid path associated with the SRBM is attracted to the origin, the SRBM is positive recurrent. Employing this result, El Kharroubi, Ben Tahar and Yaacoubi [*Stochastics Stochastics Rep.* **68** (2000) 229–253, *Math. Methods Oper. Res.* **56** (2002) 243–258] gave sufficient conditions on (θ, Σ, R) for positive recurrence for $d = 3$; Bramson, Dai and Harrison [*Ann. Appl. Probab.* **20** (2009) 753–783] showed that these conditions are, in fact, necessary.

Relatively little is known about the recurrence behavior of SRBMs for $d > 3$. This pertains, in particular, to necessary conditions for positive recurrence. Here, we provide a family of examples, in $d = 6$, with $\theta = (-1, -1, \dots, -1)^T$, $\Sigma = I$ and appropriate R, that are positive recurrent, but for which a linear fluid path diverges to infinity. These examples show in particular that, for $d \geq 6$, the converse of the Dupuis–Williams result does not hold.

1. Introduction. This paper is concerned with the class of d-dimensional diffusion processes known as semimartingale reflecting Brownian motions (SRBMs). Such processes arise as approximations for open d-station queueing networks (see, e.g., Harrison and Nguyen [10] and Williams [17, 18]). The state space for a process $Z = \{Z(t), t \geq 0\}$ in this class is $S = \mathbb{R}_+^d$, the nonnegative orthant. The data of the process consists of a drift vector θ, a nonsingular covariance matrix Σ, and a $d \times d$ reflection matrix R that specifies the boundary behavior. In the interior

Received December 2009; revised May 2010.

[1]Supported in part by NSF Grant CCF-0729537.

MSC2010 subject classifications. 60K25, 68M20, 90B15.

Key words and phrases. Reflecting Brownian motion, fluid model, queueing networks, heavy traffic limits.

of the orthant, $Z(\cdot)$ behaves as an ordinary Brownian motion with parameters θ and Σ and, roughly speaking, $Z(\cdot)$ is pushed in direction R^k whenever the boundary $\{z \in S : z_k = 0\}$ is hit, for $k = 1, \dots, d$, where R^k is the kth column of R. The process is Feller [16] and so is strong Markov.

A precise description for $Z(\cdot)$ is given by

$$Z(t) = Z(0) + B(t) + \theta t + RY(t), \qquad t \geq 0, \tag{1.1}$$

where $B(\cdot)$ is an unconstrained Brownian motion with covariance vector Σ and no drift, with $B(0) = 0$, and $Y(\cdot)$ is a d-dimensional process with components $Y_1(\cdot), \dots, Y_d(\cdot)$ such that

(1.2) $Y(\cdot)$ is continuous and nondecreasing, with $Y(0) = 0$,

(1.3) $Y_k(\cdot)$ only increases at times t at which $Z_k(t) = 0$, $\quad k = 1, \dots, d$,

(1.4) $Z(t) \in S$ for all $t \geq 0$.

(Display (1.3) means that $Y_k(t_2) > Y_k(t_1)$, for $t_2 > t_1$, implies $Z_k(t) = 0$ at some $t \in [t_1, t_2]$.) For an SRBM with data (θ, Σ, R) to exist, it is necessary and sufficient that R be *completely-S*. Completely-S means that each principal submatrix R' is an *S-matrix*, that is, for some $w \geq 0$, $R'w > 0$ holds. The complete definition and basic properties of $Z(\cdot)$ are reviewed in Appendix A of Bramson, Dai and Harrison [2].

An SRBM is said to be *positive recurrent* if the expected time to hit an arbitrary open neighborhood of the origin is finite for every starting state. A necessary and sufficient condition for positive recurrence, for $d = 2$, is that

$$R \text{ is nonsingular with } R^{-1}\theta < 0 \tag{1.5}$$

and that R is a P-matrix (El Kharroubi, Ben Tahar and Yaacoubi [7]). (That is, each principal submatrix of R has a positive determinant.) Necessary and sufficient conditions, for $d = 3$, are known, but are more complicated. El Kharroubi, Ben Tahar and Yaacoubi [8] gave sufficient conditions; Bramson, Dai and Harrison [2] showed these conditions are necessary. Another proof of the sufficiency of these conditions was recently given in Dai and Harrison [4]. In the special case where R is an M-matrix, (1.5) is necessary and sufficient for positive recurrence in all d (Harrison and Williams [11]); (1.5) is always necessary for positive recurrence [7].

Associated with the parameters θ and R are *fluid paths*, which are solutions of deterministic equations corresponding to (1.1)–(1.4). More precisely, a fluid path is a pair of continuous functions $y, z : [0, \infty) \to \mathbb{R}^d$ that satisfy

(1.6) $z(t) = z(0) + \theta t + Ry(t)$ for all $t \geq 0$,

(1.7) $y(\cdot)$ is continuous and nondecreasing, with $y(0) = 0$,

(1.8) $y_k(\cdot)$ only increases at times t at which $z_k(t) = 0$, $\quad k = 1, \dots, d$,

(1.9) $z(t) \in S$ for all $t \geq 0$.

A fluid path (y, z) is *attracted to the origin* if $z(t) \to 0$ as $t \to \infty$; it is *divergent* if $|z(t)| \to \infty$ as $t \to \infty$ [where $|u| \stackrel{\text{def}}{=} \Sigma_i |u_i|$, for $u = (u_i) \in \mathbb{R}^d$].

The following result, from Dupuis and Williams [6], gives a sufficient condition for positive recurrence of an SRBM in terms of the associated fluid paths.

THEOREM 1.1 (Dupuis–Williams). *Let $Z(\cdot)$ be a d-dimensional SRBM with data (θ, Σ, R). If every fluid path associated with (θ, R) is attracted to the origin, then $Z(\cdot)$ is positive recurrent.*

Theorem 1.1 provides an important ingredient for demonstrating the sufficiency of the conditions in [8] for positive recurrence of an SRBM, for $d = 3$, that were alluded to above. An open question is whether a converse of Theorem 1.1 holds for $d > 3$, that is, whether $Z(\cdot)$ positive recurrent implies that every fluid path is attracted to the origin.

A fluid path (y, z) is *linear* if $y(t) = ut$ and $z(t) = vt$ for given vectors $u, v \geq 0$. ($u \geq 0$ means $u_i \geq 0$ for $i = 1, \ldots, d$.) When $y(\cdot)$ and $z(\cdot)$ are linear, the fluid path properties (1.6)–(1.9) can be expressed as solutions of the linear complementarity problem

$$u, v \geq 0, \qquad v = \theta + Ru, \qquad u \cdot v = 0, \tag{1.10}$$

where $u \cdot v \stackrel{\text{def}}{=} \sum_i u_i v_i$. A solution (u, v) of (1.10) is *stable* if $v = 0$ and *divergent* otherwise. It is *nondegenerate* if u and v together have exactly d positive components, and it is *degenerate* otherwise. It is easy to see that, for a converse to Theorem 1.1 to hold, all linear fluid paths associated with a positive recurrent SRBM must be stable.

In this article, we provide a family of examples, in $d = 6$, for which the SRBM is positive recurrent, yet possesses a divergent linear fluid path. We set

$$\theta = (-1, -1, \ldots, -1)^T, \qquad \Sigma = I \tag{1.11}$$

(where "^{T}" denotes the transpose), and denote by R the 6×6 matrix with

$$R = J_1 + J_2, \tag{1.12}$$

where J_1 satisfies $(J_1)_{i,j} = 1$, for $i, j = 1, \ldots, 6$, and

$$J_2 = \begin{bmatrix} 0 & \delta_2 & \delta_2 & \delta_2 & \delta_2 & -\delta_4 \\ 0 & 0 & -\delta_3 & -\delta_3 & -\delta_3 & -\delta_3 \\ 0 & -\delta_3 & 0 & -\delta_3 & -\delta_3 & -\delta_3 \\ 0 & -\delta_3 & -\delta_3 & 0 & -\delta_3 & -\delta_3 \\ 0 & -\delta_3 & -\delta_3 & -\delta_3 & 0 & -\delta_3 \\ \delta_1 & -\delta_3 & -\delta_3 & -\delta_3 & -\delta_3 & 0 \end{bmatrix}. \tag{1.13}$$

Here, we assume that $\delta_i > 0$, $i = 1, \ldots, 4$, with

$$\delta_2 + \delta_3 \leq \tfrac{1}{6}\delta_4 \tag{1.14}$$

and

$$\delta_1 \le \delta_3 \le 0.1, \qquad \delta_4 < 1. \tag{1.15}$$

One can, for example, choose

$$\delta_1 = \delta_2 = \delta_3 = 0.05, \qquad \delta_4 = 0.6. \tag{1.16}$$

The matrix R has been chosen so that $R_{i,i} = 1$ for $i = 1, \dots, 6$. The roles of the coordinates $i = 2, \dots, 5$ with respect to R are indistinguishable, and the role of $i = 6$ differs from those of $i = 2, \dots, 5$ only in its interaction with the coordinate $i = 1$ through $R_{1,6}$ and $R_{6,1}$. Since all entries of R are positive, it is immediate that R is completely-S. The role of the relations in (1.14) and (1.15) will be explained in the next subsection.

The main result in this article is the following theorem.

THEOREM 1.2. *Let $Z(\cdot)$ denote the SRBM with $\theta = (-1, -1, \dots, -1)^T$, $\Sigma = I$ and R satisfying* (1.12)–(1.15). *Then $Z(\cdot)$ is positive recurrent*, *but possesses a divergent linear fluid path.*

One can check that (u, v), with $u = e_1$ and $v = \delta_1 e_6$, defines a divergent linear fluid path (e_i denotes the ith unit vector). Since u and v together have a total of two positive components, the fluid path is degenerate. [Related divergent fluid paths are easy to construct, e.g., (y, z) with $y(t) = e_1 t$ and $z(t) = \sum_{k=2}^{5} e_k + \delta_1 e_6 t$.] In order to demonstrate Theorem 1.2, it suffices to show $Z(\cdot)$ is positive recurrent.

Similar examples exist that satisfy the analog of Theorem 1.2, but with $d > 6$. One can construct such examples by inserting additional coordinates $Z_i(\cdot)$ that are independent of $Z_1(\cdot), \dots, Z_6(\cdot)$, with $\theta_i = -1$ and $R_{i,i} = 1$.

In the remainder of the section, we summarize how the matrix R affects the evolution of $Z(\cdot)$ and leads to its positive recurrence. We also outline the rest of the paper.

Sketch of positive recurrence. The reflection matrix R that we have chosen has the following properties, which we will use in the next three paragraphs. For θ given by (1.11), all of the coordinates $Z_k(\cdot)$, $k = 1, \dots, 6$, have drift -1, which is compensated for by R, which pushes a coordinate away from 0 whenever any of the coordinates is being reflected there. (Although the motion induced by R is not absolutely continuous, we will also refer to it as "drift" here.) Because of the choice of θ, for $k, k' = 2, \dots, 5$,

$$Z_{k'}(\cdot) - Z_k(\cdot) \text{ has no drift}$$

except when one of the coordinates is being reflected; when the coordinate k is reflected, the difference has negative drift because of the term δ_3 in J_2. Also, for $k = 2, \dots, 5$, $Z_6(\cdot) - Z_k(\cdot)$ has no drift except when $Z_k(\cdot)$ is reflected, in which case the difference has negative drift, or when $Z_6(\cdot)$ or $Z_1(\cdot)$ is reflected, in which case it has positive drift, the last case occurring because of the term δ_1. On the other hand, when the first coordinate is being reflected, for $k = 2, \dots, 5$,

$$Z_1(\cdot) - Z_k(\cdot) \text{ has no drift}$$

and, when one of the other four coordinates $k = 2, \dots, 5$ is being reflected, the difference has positive drift because of the term δ_2 in J_2. But, when $Z_6(\cdot)$ is reflected, the difference acquires a negative drift because of the term δ_4 in J_2 and (1.14).

The process $Z(\cdot)$ is positive recurrent, although its deterministic analog $z(\cdot)$ possesses a divergent linear fluid path in the direction e_6 when $u = e_1$. This difference in behavior occurs due to the following interaction between the different coordinates of $Z(\cdot)$. When $Z_1(\cdot)$ is close to 0 [for instance, when $Z_k(\cdot)$, $k = 2, \dots, 5$, are larger], it may remain small for an extended period of time, with the other coordinates perhaps increasing. Nonetheless, as we will see, after a finite expected time, one of the coordinates k, $k = 2, \dots, 5$, will hit 0. Because of the reflections against 0 by this coordinate and perhaps by the other three coordinates, the coordinate $k = 1$ will acquire, on the average, a positive drift and therefore increase linearly. When this occurs, each of the coordinates $k = 2, \dots, 5$ will drift toward 0 and afterward remain close to 0.

The sixth coordinate increases linearly in time when the first coordinate undergoes repeated reflection. However, when the first coordinate is instead increasing, the sixth coordinate will drift back to 0 on account of the terms $(J_2)_{6,j} = -\delta_3$, $j = 2, \dots, 5$. Moreover, on account of (1.14), the term $(J_2)_{1,6} = -\delta_4$ is sufficiently smaller than $-\delta_2$ so that, when the sixth coordinate starts reflecting at 0, the negative drift induced in the first coordinate more than compensates for the positive drift induced in the first coordinate by the reflection of the other four coordinates. As a consequence, the first coordinate acquires a negative net drift. After this occurs, the coordinates $k = 2, \dots, 6$ will all remain close to 0 until the first coordinate hits 0, in which case the behavior outlined above can repeat. This behavior prevents any of the coordinates from typically moving too far from 0, and will ensure that the system is positive recurrent.

The proof of Theorem 1.2 is organized as follows. In Section 2, we give a number of bounds on $Y(\cdot)$ and $Z(\cdot)$ that are derived by applying elementary Brownian motion estimates to (1.1). These bounds are employed in the rest of the paper. In Section 3, we demonstrate a version of Foster's Criterion that will be used here. We also recall and then employ the main result in Ratzkin and Treibergs [15], which states that for a Brownian pursuit problem, the presence of four "predators" is enough for them to capture the "prey" in finite expected time. In our context, $Z_k(\cdot)$, $k = 2, \dots, 5$, will play the role of the predators and $Z_1(\cdot)$ will play the role of the prey. This behavior will justify the claim in the above discussion that one of the coordinates with $k = 2, \dots, 5$ will hit 0 after a finite expected time.

In Section 4, we state the main steps in the proof of Theorem 1.2 in the form of a series of five propositions, and show how the theorem follows from them.

Depending on whether or not $Y_1(\cdot)$ is initially growing quickly, Proposition 4.1 states that, during this time, either the coordinates $Z_2(\cdot), \dots, Z_6(\cdot)$ decrease by an appropriate factor or $Z_6(\cdot)$ increases linearly. In the first case, it follows from Proposition 4.2 that $Z_1(\cdot)$ will also remain small and so, as desired, the norm of the SRBM decreases by a factor over the time interval. In the second case, the argument proceeds along the lines sketched above in the comparison of $Z(\cdot)$ with the divergent fluid path, and employs Propositions 4.3, 4.4 and 4.5.

In Section 5, we demonstrate Propositions 4.1 and 4.2 and, in Section 6, we demonstrate Propositions 4.3, 4.4 and 4.5. The reasoning employs the interaction of the different components $Z_k(\cdot)$, $k = 1, \dots, 6$, and draws from the different bounds in Sections 2 and 3.

2. Basic estimates.

In this section, we give a number of elementary bounds that will be used in the remainder of the article. In Lemma 2.1, we give bounds on standard one-dimensional Brownian motion $B(\cdot)$. (All of the bounds in the lemma hold in greater generality; see, e.g., [12], page 59, and [13].) These bounds will then be applied in the rest of the section to obtain bounds on the quantities $Y(\cdot)$ and $Z(\cdot)$ in (1.1), the equation describing the evolution of SRBM. Here and elsewhere in the paper, the notation $C_1, C_2, \dots$ will be employed for positive constants whose precise value is not of interest to us, with the same symbol often being reused.

LEMMA 2.1. *Let $B(\cdot)$ denote a standard Brownian motion. Then, for each $t \geq 0$,*

$$E\Big[\max_{0 \leq s \leq s' \leq t} \big(B(s') - B(s)\big)^2\Big] \leq 8t. \tag{2.1}$$

For given $\varepsilon > 0$, there exist $C_1, \varepsilon' > 0$ such that, for each $t \geq 0$,

$$P\Big(\max_{0 \leq s \leq t} |B(s)| \geq \varepsilon t\Big) \leq C_1 e^{-\varepsilon' t}. \tag{2.2}$$

For given $\varepsilon > 0$, there exist $C_1, \varepsilon' > 0$ such that, for each $u \geq 0$,

$$P\Big(\inf_{t \geq 0} \big(\varepsilon t + u - |B(t)|\big) \leq 0\Big) \leq C_1 e^{-\varepsilon' u}, \tag{2.3}$$

and, for each $u > 0$ and $t \geq 0$,

$$P\Big(\min_{0 \leq s \leq s' \leq t} \big(\varepsilon(s' - s) + u - |B(s') - B(s)|\big) \leq 0\Big) \leq C_1(t+1)e^{-\varepsilon' u}. \tag{2.4}$$

PROOF. Since $(B(s') - B(s))^2 \leq 2(B(s')^2 + B(s)^2)$, it follows from the Reflection Principle that the left-hand side of (2.1) is at most

$$4E\Big[\max_{0 \leq s \leq t} B(s)^2\Big] \leq 8E[B(t)^2] = 8t.$$

The bound (2.2) follows by applying the Reflection Principle to both $B(\cdot)$ and $-B(\cdot)$.

Again applying the Reflection Principle to $B(\cdot)$ and $-B(\cdot)$, it follows that, for given $\varepsilon > 0$,

$$\begin{aligned}P\left(\frac{1}{2}(\varepsilon t' + u) - \max_{0\le s\le t'} |B(s)| \le 0\right) &\le 4P\left(\frac{1}{2}(\varepsilon t' + u) - |B(t')| \le 0\right)\\ &\le C_2 \exp\bigl(-(\varepsilon t' + u)^2/8t'\bigr) \le C_2 e^{-\frac{1}{8}\varepsilon u},\end{aligned}$$

where C_2 does not depend on t' or u. Setting $t' = 2^i$, $i = 0, 1, 2, \ldots,$ one obtains bounds whose exceptional probabilities sum to at most $C_1 e^{-\varepsilon' u}$, for $\varepsilon' = \frac{1}{8}\varepsilon$ and appropriate C_1. The bound in (2.3) follows quickly from this.

It follows from (2.3) that, for each $i = 0, 1, 2, \ldots,$

$$P\left(\inf_{s'\ge i}\left(\varepsilon(s' - i) + \frac{1}{2}u - |B(s') - B(i)|\right) \le 0\right) \le C_1 e^{-\frac{1}{2}\varepsilon' u}. \tag{2.5}$$

Using the Reflection Principle, it is easy to check that, for appropriate C_3, $\varepsilon'' > 0$ and all $u \ge 0$,

$$P\left(\max_{0\le s\le 1} |B(i + s) - B(i)| \ge \frac{1}{2}u - \varepsilon\right) \le C_3 e^{-\varepsilon'' u}.$$

Together with (2.5), this implies

$$P\Bigl(\inf_{s\in[i,i+1), s'\ge s} \bigl(\varepsilon(s' - s) + u - |B(s') - B(s)|\bigr) \le 0\Bigr) \le C_1 e^{-\varepsilon' u}$$

for new choices of C_1 and ε'. Summing over $i < t$ gives the bounds in (2.4). □

The next lemma provides elementary upper and lower bounds on $Y_k(\cdot)$.

LEMMA 2.2. *For each $t \ge 0$ and $\ell = 2, \ldots, 5$,*

$$\sum_{k=1}^{6} Y_k(t) \ge t - Z_\ell(0) - B_\ell(t) \tag{2.6}$$

and, for each $\ell = 1, \ldots, 6$,

$$\sum_{k=1}^{6} Y_k(t) \ge \frac{1}{2}\bigl(t - Z_\ell(0) - B_\ell(t)\bigr). \tag{2.7}$$

For each $t \ge 0$ and $k = 1, \ldots, 6$,

$$Y_k(t) \le t + \max_{0\le s\le t} (-B_k(s)) \tag{2.8}$$

and, for a given $\varepsilon \geq 0$, *there exist* C_1 *and* $\varepsilon' > 0$ *so that*

$$P\big(Y_k(t) \geq (1+\varepsilon)t\big) \leq C_1 e^{-\varepsilon' t}. \tag{2.9}$$

PROOF. Since $\delta_3 \geq 0$, it follows from (1.1) that, for $\ell = 2, \dots, 5$,

$$Z_\ell(t) \leq Z_\ell(0) + B_\ell(t) - t + \sum_{k=1}^{6} Y_k(t), \tag{2.10}$$

from which (2.6) immediately follows. Since each of the entries of J_2 in (1.13) is less than 1, the analog of (2.10) holds for $\ell = 1, \dots, 6$, but with the term $2\sum_{k=1}^{6} Y_k(t)$. This implies (2.7).

Let τ denote the time in $[0,t]$ at which $Y_k(t)$ is first attained, for given k. It follows from (1.1) that

$$Y_k(t) \leq \tau - B_k(\tau) \leq t + \max_{0\leq s\leq t}(-B_k(s)),$$

which imples (2.8). The bound (2.9) follows from (2.8) and (2.2). □

We next obtain a number of upper bounds on $Z_k(\cdot)$. The following lemma is elementary.

LEMMA 2.3. *Let* $B(\cdot)$ *denote a standard Brownian motion. For each* k, t *and* x,

$$P\Big(\max_{0\leq s\leq t} Z_k(s) - Z_k(0) \geq 7t + x\Big) \leq 16P\big(B(t) \geq x\big). \tag{2.11}$$

Consequently, for all t, *and appropriate* C_1 *and* $\varepsilon' > 0$,

$$P\Big(\max_{0\leq s\leq t} Z_k(s) - Z_k(0) \geq 8t\Big) \leq C_1 e^{-\varepsilon' t}. \tag{2.12}$$

PROOF. It follows from (1.1) that, since all entries for J_2 in (1.13) are at most $\frac{1}{6}$,

$$\max_{0\leq s\leq t} Z_k(s) - Z_k(0) \leq \max_{0\leq s\leq t} B_k(s) + \frac{7}{6}\sum_{\ell=1}^{6} Y_\ell(t).$$

By (2.8) of Lemma 2.2, this is at most

$$7t + \frac{7}{6}\sum_{\ell=1}^{6} \max_{0\leq s\leq t}(-B_\ell(s)) + \max_{0\leq s\leq t} B_k(s).$$

The inequality in (2.11) follows from this and the Reflection Principle. The inequality in (2.12) is an immediate consequence of (2.11). □

The following lemma requires a bit more work. Here, we employ the notation $N_k(t)$, $k = 1, \dots, 6$, with $N_6(t) = Y_1(t)$ and $N_k(t) = 0$ for $k \neq 6$; x_+ denotes the positive part of $x \in \mathbb{R}$.

LEMMA 2.4. *For each k, $k = 2, \dots, 6$, $t \geq 0$ and x,*

$$P\Big(\max_{0\leq s\leq t} Z_k(s) - Z_k(0) - \delta_1 N_k(t) \geq x\Big) \leq 48 P\bigg(B(t) \geq \frac{1}{4}x\bigg), \tag{2.13}$$

where $B(\cdot)$ is standard Brownian motion. Consequently, for given $\varepsilon > 0$, there exist $C_1, \varepsilon' > 0$ such that, for each $t \geq 0$,

$$P\Big(\max_{0\leq s\leq t} Z_k(s) - Z_k(0) - \delta_1 N_k(t) \geq \varepsilon t\Big) \leq C_1 e^{-\varepsilon' t}. \tag{2.14}$$

Also, for $k = 2, \dots, 5$,

$$E\Big[\Big(\Big(\max_{0\leq s\leq t} Z_k(t) - Z_k(0)\Big)_+\Big)^2\Big] \leq 24 \cdot 16t. \tag{2.15}$$

PROOF. Let τ_k denote the last time r, $r \leq s$, at which $Z_k(r) = 0$; if the set is empty, let $\tau_k = 0$. Let τ denote the last time r, $r \leq s$, at which $Z_\ell(r) = 0$ for any $\ell = 2, \dots, 6$; denote this coordinate by L. If the set is empty, set $\tau = 0$. We also abbreviate by setting $B_k(r_1, r_2) = B_k(r_2) - B_k(r_1)$ and $N_k(r_1, r_2) = N_k(r_2) - N_k(r_1)$.

We claim that for given k, $k = 2, \dots, 6$,

$$Z_k(\tau) - Z_k(0) \leq B_k(\tau_k, \tau) - B_L(\tau_k, \tau) + \delta_1 N_k(\tau). \tag{2.16}$$

To see this, note that subtraction of the equations for the kth and Lth coordinates of (1.1) implies

$$\begin{aligned} Z_k(\tau) - Z_k(\tau_k) &= Z_L(\tau) - Z_L(\tau_k) + B_k(\tau_k, \tau) - B_L(\tau_k, \tau) \\ &\quad + \delta_1 N_k(\tau_k, \tau) - \delta_1 N_L(\tau_k, \tau) - \delta_3\big(Y_L(\tau) - Y_L(\tau_k)\big) \\ &\leq B_k(\tau_k, \tau) - B_L(\tau_k, \tau) + \delta_1 N_k(\tau). \end{aligned}$$

When $\tau_k > 0$, $Z_k(\tau_k) = 0$ holds, and so (2.16) follows.

Let $\tau' = \tau \vee \tau_1$. Note that, when $\tau \neq \tau' > 0$,

$$Z_1(\tau') - Z_1(\tau) = -Z_1(\tau) \leq 0.$$

Also, since $Z_6(r) > 0$ for $r \in (\tau, \tau']$, it follows from the definition of J_2 that

$$\big(R\big(Y(\tau') - Y(\tau)\big)\big)_k \leq \big(R\big(Y(\tau') - Y(\tau)\big)\big)_1.$$

Subtraction of the kth and 1st coordinates of (1.1), together with these two inequalities, implies that

$$Z_k(\tau') - Z_k(\tau) \leq B_k(\tau, \tau') - B_1(\tau, \tau') + \delta_1 N_k(\tau, \tau'). \tag{2.17}$$

It is easy to see that

$$Z_k(s) - Z_k(\tau') \leq B_k(\tau', s). \tag{2.18}$$

Combining (2.16), (2.17) and (2.18) implies

$$Z_k(s) - Z_k(0) \le B_k(\tau_k, s) - B_L(\tau_k, \tau) - B_1(\tau, \tau') + \delta_1 N_k(t). \tag{2.19}$$

By employing the upper bounds $\frac{1}{4}x$ on $-B_1(s,s')$ and $B_k(s,s')$, and $\frac{1}{2}x$ on $-B_\ell(s,s')$, $\ell \ne 1, k$, for all $0 \le s \le s' \le t$, one therefore obtains that, for all x,

$$\begin{aligned} &P\Big(\max_{0\le s\le t} Z_k(s) - Z_k(0) - \delta_1 N_k(t) \ge x\Big) \\ &\quad \le 12P\left(\max_{0\le s\le s'\le t} \big(B(s') - B(s)\big) \ge \frac{1}{2}x\right). \end{aligned} \tag{2.20}$$

It follows from the Reflection Principle that the right-hand side of (2.20) is at most $48P(B(t) \ge \frac{1}{4}x)$, which implies (2.13).

The inequalities in (2.14) and (2.15) follow directly from (2.13). □

We will employ (2.13) to show (2.21) of the following lemma. On account of the thin tail of $\max_{0\le s\le t} Z_k(s)$, restricting its expectation to a set F decreases the expectation proportionally to $P(F)$, except for a logarithmic factor; a similar statement holds for the second moment. The lemma will be important for our calculations later in the article.

LEMMA 2.5. *For an appropriate constant C_1, all $t \ge 0$ and all measurable sets F with $P(F) > 0$,*

$$E\Big[\max_{0\le s\le t} Z_k(s)^2; F\Big] \le C_1 P(F)\big(t \log\big(e/P(F)\big) + M\big) \tag{2.21}$$

for $k = 2, \dots, 5$, when $Z_k(0) \le \sqrt{M}$, and

$$E\Big[\max_{0\le s\le t} Z_k(s); F\Big] \le C_1 P(F)\big(\sqrt{t}\log\big(e/P(F)\big) + t + M\big) \tag{2.22}$$

for $k = 1, 2, \dots, 6$, when $Z_k(0) \le M$.

PROOF. On account of (2.13), we can construct a standard normal random variable W on the probability space so that, for $k = 2, \dots, 5$,

$$E\Big[\max_{0\le s\le t} \big(\big(Z_k(s) - Z_k(0)\big)_+\big)^2; F\Big] \le C_2 t E[W^2; F], \tag{2.23}$$

where $C_2 = 48 \cdot 16$. (The inequality follows by integrating by parts and employing $E[(4B(t))^2] = 16t$.) Choosing a so that $P(W^2 \ge a) = P(F)$, the right-hand side of (2.23) is at most

$$C_2 t\left(aP(F) + \int_a^\infty P(W^2 \ge x)\,dx\right). \tag{2.24}$$

The random variable W^2 has an exponentially tight tail in the sense that, for appropriate $C_3, C_4 > 0$ and all y, x with $0 \le y \le x$,

$$P(W^2 \ge x) \le C_3 e^{-C_4(x-y)} P(W^2 \ge y). \tag{2.25}$$

Setting $y = 0$ and $x = a$, this implies $a \le \frac{1}{C_4}\log(C_3/P(F))$. Application of (2.25) with $y = a$ therefore implies (2.24) is at most

$$\begin{aligned} &C_2 t\big(aP(F) + (C_3/C_4)P(W^2 \ge a)\big) \\ &\quad \le (C_2/C_4)tP(F)\big(\log\big(1/P(F)\big) + C_3 + \log C_3\big). \end{aligned}$$

So, for appropriate C_5,

$$E\Big[\max_{0\le s\le t}\big((Z_k(s) - Z_k(0))_+\big)^2; F\Big] \le C_5 t P(F)\log\big(e/P(F)\big). \tag{2.26}$$

For $Z_k(0) \le \sqrt{M}$, (2.21) follows from this by considering the complementary events $\{\max_{s\le t} Z_k(s) > 2\sqrt{M}\}$ and $\{\max_{s\le t} Z_k(s) \le 2\sqrt{M}\}$, and noting that, on the former,

$$\max_{0\le s\le t} Z_k(s)^2 \le 4 \max_{0\le s\le t}\big(Z_k(s) - Z_k(0)\big)^2$$

and, on the latter, $\max_{s\le t} Z_k(s)^2 \le 4M$.

In order to show (2.22), we note that, for $k = 1, \dots, 6$, it follows from (1.1) and (2.8) that

$$\begin{aligned} Z_k(s) - Z_k(0) &\le B_k(s) - s + \frac{7}{6}\sum_{\ell=1}^{6} Y_\ell(s) \\ &\le 6s + B_k(s) + \frac{7}{6}\sum_{\ell=1}^{6}\max_{0\le r\le s}(-B_\ell(r)). \end{aligned} \tag{2.27}$$

This, together with the Reflection Principle, implies that

$$\begin{aligned} P\Big(\max_{0\le s\le t} Z_k(s) - Z_k(0) - 6t \ge x\Big) &\le 7P\Big(\max_{0\le s\le t} B(s) \ge \frac{1}{8}x\Big) \\ &\le 14P\Big(B(t) \ge \frac{1}{8}x\Big), \end{aligned} \tag{2.28}$$

where $B(\cdot)$ is standard Brownian motion.

Reasoning as in the first part of the proof, we can construct a standard normal random variable W so that

$$E\Big[\max_{0\le s\le t} Z_k(s) - Z_k(0) - 6t; F\Big] \le C_2\sqrt{t}\,E[W; F], \tag{2.29}$$

where $C_2 = 14 \cdot 8$. Since W has an exponentially tight tail, we can reason as through (2.26) to show that

$$E\Big[\max_{0\le s\le t} Z_k(s) - Z_k(0) - 6t; F\Big] \le C_5\sqrt{t}\,P(F)\log\big(e/P(F)\big) \tag{2.30}$$

for appropriate C_5. This implies (2.22) for $Z_k(0) \le M$ and appropriate C_1. $\square$

We now apply Lemma 2.4 to obtain sharper bounds on $Y_k(\cdot)$, with $k = 2, \dots, 6$, than those in Lemma 2.2, provided bounds on $Y_1(\cdot)$ are given.

LEMMA 2.6. *For given* $\varepsilon > 0$, *there exist* C_1 *and* $\varepsilon' > 0$ *such that, for all* $t \ge 0$ *and* $k = 2, \dots, 6$,

$$P\left(Y_k(t) + (1-\delta_3)\sum_{\ell=2,\ell\neq k}^{6} Y_\ell(t) \ge (1+\varepsilon)t + \delta_1 N_k(t)\right) \le C_1 e^{-\varepsilon' t}. \tag{2.31}$$

There exist C_1 *and* $\varepsilon' > 0$ *such that, for all* $t \ge 0$ *and* $k = 2, \dots, 6$,

$$P\left(Y_k(t) \le \frac{1}{5}t - \frac{1}{\delta_3}\big(Z_k(0) + 2Y_1(t)\big)\right) \le C_1 e^{-\varepsilon' t}. \tag{2.32}$$

PROOF. It follows from (1.1) that

$$Z_k(t) - Z_k(0) \ge B_k(t) - t + Y_k(t) + (1-\delta_3)\sum_{\ell=2,\ell\neq k}^{6} Y_\ell(t) \tag{2.33}$$

for $k = 2, \dots, 6$. Together with (2.14) of Lemma 2.4, (2.33) implies (2.31).

Summing the arguments inside $P(\cdot)$ in (2.31), over $\ell = 2, \dots, 6$, gives

$$P\left(\left(1 - \frac{4}{5}\delta_3\right)\sum_{\ell=2}^{6} Y_\ell(t) \ge (1+\varepsilon)t + \delta_1 Y_1(t)\right) \le 5C_1 e^{-\varepsilon' t}.$$

Since $\delta_3 \le \frac{1}{10}$,

$$\frac{1-\delta_3}{1-4/5\delta_3} \le 1 - \frac{1}{5}\delta_3 - \frac{1}{10}\delta_3^2, \tag{2.34}$$

which implies that, for small enough ε,

$$P\left((1-\delta_3)\sum_{\ell=2}^{6} Y_\ell(t) - t \ge -\frac{1}{5}\left(1 + \frac{1}{2}\delta_3\right)\delta_3 t + \delta_1 Y_1(t)\right) \le 5C_1 e^{-\varepsilon' t}. \tag{2.35}$$

By (1.1), one has, for $k = 2, \dots, 6$,

$$Z_k(t) - Z_k(0) \le B_k(t) - t + \delta_3 Y_k(t) + (1-\delta_3)\sum_{\ell=2}^{6} Y_\ell(t) + (1+\delta_1)Y_1(t).$$

Off of the exceptional set in (2.35), this is at most

$$-\tfrac{1}{5}(1 + \tfrac{1}{2}\delta_3)\delta_3 t + B_k(t) + \delta_3 Y_k(t) + 2Y_1(t).$$

Solving for $Y_k(t)$, together with the obvious exponential bound on $B_k(t)$, produces (2.32) for a new choice of C_1 and ε'. $\square$

In Lemma 2.4, we gave upper bounds on $Z_k(\cdot)$ for $k = 2, \dots, 6$. Here, we employ (2.31) and (2.32) of Lemma 2.6 to obtain an upper bound on $Z_1(\cdot)$. The bound implies in particular that, for large t, $Y_1(t) > 0$ and hence $Z_1(s) = 0$ at some $s \le t$.

LEMMA 2.7. *For given $\varepsilon > 0$, there exist C_1 and $\varepsilon' > 0$ such that, for each $t \ge 0$,*

$$P\left(Z_1(t) - Z_1(0) - \frac{3}{\delta_3}(Y_1(t) + Z_6(0)) \ge -\frac{1}{30}\delta_4 t\right) \le C_1 e^{-\varepsilon' t}. \tag{2.36}$$

PROOF. On account of (1.1),

$$\begin{aligned} Z_1(t) - Z_1(0) \le B_1(t) - t + Y_1(t) + (1 + \delta_2) \sum_{k=2}^{6} Y_k(t) \\ - (\delta_2 + \delta_4) Y_6(t). \end{aligned} \tag{2.37}$$

One bounds $(1 + \delta_2) \sum_{k=2}^{6} Y_k(t)$ by employing (2.31) after summing over $\ell = 2, \dots, 6$, and one bounds $(\delta_2 + \delta_4) Y_6(t)$ by employing (2.32). It then follows with a little algebra that the right-hand side of (2.37) is at most

$$\begin{aligned} &\left(\frac{9}{10}\delta_2 + \delta_3 - \frac{1}{5}\delta_4\right) t + \frac{3}{\delta_3}(Y_1(t) + Z_6(0)) \\ &\le -\frac{1}{30}\delta_4 t + \frac{3}{\delta_3}(Y_1(t) + Z_6(0)) \end{aligned} \tag{2.38}$$

off of a set of probability $C_1 e^{-\varepsilon' t}$, for appropriate C_1 and $\varepsilon' > 0$. For the bound on the left-hand side of (2.38), one employs the bounds on δ_i in (1.14) and (1.15), together with (2.31), (2.32) and an analog of (2.34). For the inequality in (2.38), one uses $\delta_2 + \delta_3 \le \frac{1}{6}\delta_4$. It follows from (2.37) and (2.38) that, off of the exceptional set,

$$Z_1(t) - Z_1(0) \le -\frac{1}{30}\delta_4 t + \frac{3}{\delta_3}(Y_1(t) + Z_6(0)),$$

which implies (2.36). □

3. A Brownian pursuit model and Foster's Criterion. In this section, we first discuss a Brownian pursuit model, which was mentioned briefly at the end of Section 1. Using a result of Ratzkin and Treibergs [15], it is employed to show that the expected time for at least one of the coordinates $Z_k(\cdot)$, $k = 2, \dots, 5$, to hit 0 is finite. In Proposition 3.2, we apply this result to obtain a lower bound on $\sum_{k=2}^{5} Y_k(\cdot)$ that will be used later in the paper. We then show an appropriate version of Foster's Criterion. Foster's Criterion is a tool for showing the positive recurrence of a Markov process. Since the stopping times we will employ are random, we need a variant of the standard version.

A Brownian pursuit model. The pursuit model consists of n standard one-dimensional Brownian motions, $X_k(\cdot)$, $k = 2, \dots, n+1$, that "pursue" another Brownian motion $X_1(\cdot)$. The n Brownian motions are referred to as *predators* and the other Brownian motion as the *prey*. The prey will be said to be *captured* at time t if t is the first time at which $X_1(t) = X_k(t)$ for some $k = 2, \dots, n+1$. All Brownian motions are assumed to move independently.

One wishes to know whether the expected time for capture is finite or infinite. When there are initially predators on each side of the prey, one can show that the expected capture time is finite. When all of the predators are on one side of the prey, the expected capture time is infinite for $n \le 3$ and finite for $n \ge 4$. This and a number of related problems were considered in Bramson and Griffeath [3] in the context of simple symmetric random walk. There, the behavior for $n \le 3$ was demonstrated and simulations were given that suggested the behavior for $n \ge 4$. Li and Shao [14] showed finite expected capture time for Brownian motion for $n \ge 5$ and Ratzkin and Treibergs [15] more recently showed this for $n = 4$.

Ratzkin and Treibergs [15] showed finite expected capture time by bounding the tail of the capture time T. Their result can be formulated as follows.

THEOREM 3.1. *For any initial state where all four of the predators are within distance* 1 *and to the right of the prey*,

$$P(T > t) \le C_1 t^{-(1+\eta)} \tag{3.1}$$

for appropriate C_1 *and all* $t \ge 0$, *where* $\eta = 0.000073$. *Consequently*,

$$E[T] < \infty. \tag{3.2}$$

The analogous result for $n = 5$ is less delicate, which [15] showed with $\eta = 0.0634$. The reasoning in both [14] and [15] relies on rephrasing the pursuit model in terms of an eigenvalue problem for the departure time of an n-dimensional Brownian motion from an appropriate generalized cone. This type of problem was also studied in DeBlassie [5]. (See [14] for additional references.)

We will employ both (3.1) and (3.2) for Proposition 3.2. (The first inequality is not needed, but applying it makes one of the steps more explicit.) We note that, by (1.1), for $k = 2, \dots, 5$ and all $t \ge 0$,

$$\begin{aligned} Z_k(t) - Z_1(t) \le {} & \big(Z_k(0) - Z_1(0)\big) + \big(B_k(t) - B_1(t)\big) \\ & - \delta_2 \sum_{k=2}^{5} Y_k(t) + (\delta_4 - \delta_3) Y_6(t). \end{aligned} \tag{3.3}$$

When $Y_6(t) = 0$, this implies

$$Z_k(t) - Z_1(t) \le \big(Z_k(0) - Z_1(0)\big) + \big(B_k(t) - B_1(t)\big). \tag{3.4}$$

Set

$$T_1(x) = \min\{t : Z_1(t) - Z_1(0) \geq x\}, \tag{3.5}$$

$$T_6 = \min\{t : Z_6(t) = 0\}. \tag{3.6}$$

By employing Theorem 3.1 and (3.4), it is easy to show the following proposition.

PROPOSITION 3.1. *Suppose that for a given* $x \geq 0$, $max_{k=2,\dots,5} Z_k(0) \leq x$. *Then, for* $\eta = 0.000073$ *and an appropriate constant* C_1 *not depending on* x,

$$P\big(T_1(x) \wedge T_6 \geq x^2 t\big) \leq C_1 t^{-(1+\eta)} \tag{3.7}$$

for all $t \geq 0$. *Consequently, for appropriate* C_2 *not depending on* x,

$$E[T_1(x) \wedge T_6] < C_2 x^2. \tag{3.8}$$

PROOF. By scaling space and time by $2x$ and $4x^2$, respectively, it follows from (3.1) of Theorem 3.1 that

$$P\Big(B_1(s) - \min_{2\leq k\leq 5} B_k(s) < 2x \text{ for all } s \leq x^2 t\Big) \leq C_1 t^{-(1+\eta)}$$

for a new choice of C_1. On account of (3.4) and the bounds on $Z_k(0)$, $k = 2, \dots, 5$, this implies that

$$\begin{aligned} &P\big(Z_1(s) - Z_1(0) < x \text{ for all } s \leq x^2 t;\, T_6 \geq x^2 t\big) \\ &\quad \leq P\Big(Z_1(s) - Z_1(0) - \min_{2\leq k\leq 5} Z_k(s) < x \text{ for all } s \leq x^2 t;\, T_6 \geq x^2 t\Big) \\ &\quad \leq C_1 t^{-(1+\eta)}. \end{aligned}$$

The inequality in (3.7) follows immediately. □

Application of Proposition 3.1. We define the stopping times

$$T_2(x) = \min\left\{t : \sum_{k=2}^{5} Y_k(t) = \frac{1}{2}(t + x^2)\right\} \wedge T_6 \wedge 5x^{5/\eta}, \tag{3.9}$$

where η is as in Theorem 3.1. In Sections 4–6, we will require upper bounds on $E[T_2(x)]$ in order to ensure the linear growth of $Z_1(\cdot)$ mentioned at the end of Section 1. Here, we employ Proposition 3.1 to obtain the following bounds.

PROPOSITION 3.2. *Suppose that* $\max_{k=2,\dots,5} Z_k(0) \leq x$, *with* $x \geq 2$. *Then, for appropriate* C_1 *not depending on* x,

$$E[T_2(x)] \leq C_1 x^2. \tag{3.10}$$

In Sections 4–6, we will also require upper bounds on $P(A)$, where

$$A = \{\omega : T_2(x) = 5x^{5/\eta}\}. \tag{3.11}$$

These bounds are obtained in Proposition 3.3, which we state shortly.

In order to demonstrate Propositions 3.2 and 3.3, we need to rule out certain behavior of $Z(\cdot)$ except on sets of small probability. For this, we introduce the following notation. Let $S_1(x)$ denote the last time t before $T_1(x)$ at which $Z_1(t) - Z_1(0) = \frac{1}{2}x$, for given x. Set

$$\tau = \min\Big\{t : \min_{2 \le k \le 5} Z_k(t) = 0 \text{ for } t \ge S_1(x)\Big\}. \tag{3.12}$$

(If τ does not occur, set $\tau = \infty$.) Neither $S_1(x)$ nor τ is a stopping time. We also set

$$t_e = x^{5/\eta}, \qquad t_f = 5x^{5/\eta} \quad \text{and} \quad T_1'(x) = 4\big(T_1(x) + x^2\big). \tag{3.13}$$

Using this notation, we define:

$$A_1 = \{\omega : T_1(x) \wedge T_6 > t_e\}, \tag{3.14}$$

$$A_2 = \{\omega : T_1(x) \le T_6 \wedge \tau \wedge t_e\}, \tag{3.15}$$

$$A_3 = \{\omega : \tau < T_1(x) \le t_e, T_6 > T_1'(x)\}, \tag{3.16}$$

$$A_4 = \left\{\omega : \sum_{k=2}^{5} Y_k(T_1'(x)) < \frac{1}{2}\big(T_1'(x) + x^2\big)\right\}, \tag{3.17}$$

$$A_5 = \{\omega : T_6 > T_1'(x)\}. \tag{3.18}$$

One can check that

$$A_5 \subseteq A_1 \cup A_2 \cup A_3. \tag{3.19}$$

Also, note that

$$T_2(x) \le T_1'(x) \wedge T_6 \qquad \text{on } A_4^c. \tag{3.20}$$

Using this notation, it is not difficult to show the following lemma.

LEMMA 3.1. *For A as in* (3.11),

$$A \subseteq A' \stackrel{\text{def}}{=} A_1 \cup A_2 \cup (A_3 \cap A_4). \tag{3.21}$$

PROOF. Set

$$T_2'(x) = \min\left\{t : \sum_{k=2}^{5} Y_k(t) = \frac{1}{2}(t + x^2)\right\} \wedge T_6$$

and $A_6 = \{\omega : T_2'(x) \ge t_f\}$. It suffices to show that $A_6 \subseteq A'$.

Since $T_1'(x) < t_f$ on A_3,

$$A_3 \cap A_6 \subseteq A_3 \cap A_4.$$

Consequently, by (3.19) and the definition of A',

$$A_5 \cap A_6 \subseteq A_1 \cup A_2 \cup (A_3 \cap A_6) \subseteq A'. \tag{3.22}$$

On the other hand,

$$A_5^c \cap A_6 \subseteq A_1 \subseteq A'.$$

Together with (3.22), this implies $A_6 \subseteq A'$, as desired. □

The bounds on $P(A')$ in Proposition 3.3 will be applied in the proof of Proposition 3.2 and the bounds on $P(A)$ will be applied in the proof of Proposition 4.4. Proposition 3.1, Lemma 2.1 and (1.1) are the main tools in the proof of Proposition 3.3.

PROPOSITION 3.3. *Suppose that* $\max_{k=2,\dots,5} Z_k(0) \le x$, *with* $x \ge 1$. *Then, for an appropriate* C_1 *not depending on* x,

$$P(A) \le P(A') \le C_1 x^{-5/\eta - 2}. \tag{3.23}$$

PROOF. In addition to A_1, A_2, A_3 and A_4, we employ the set

$$A_7 = \{\omega : Z_1(s) = 0 \text{ for some } s \in [\tau, T_1'(x)]\}. \tag{3.24}$$

We proceed to obtain upper bounds on each of $P(A_1)$, $P(A_2)$, $P(A_3 \cap A_7)$ and $P(A_3 \cap A_4 \cap A_7^c)$. We first note that, by applying (3.7) of Proposition 3.1, with $t = x^{5/\eta - 2}$,

$$P(A_1) \le C_2 x^{-5/\eta - 2} \tag{3.25}$$

for appropriate C_2.

In order to bound $P(A_2)$, we need to show that, over $[S_1(x), t_e]$, $T_1(x)$ typically will not occur before $T_6 \wedge \tau$ occurs; on this set, $Z_1(\cdot)$ will drift toward 0 and away from x. First note, by (1.1), that when $T_1(x) \le T_6 \wedge \tau$,

$$Z_1(T_1(x)) - Z_1(S_1(x)) = \big(B_1(T_1(x)) - B_1(S_1(x))\big) - \big(T_1(x) - S_1(x)\big).$$

It then follows from the definitions of $T_1(x)$ and $S_1(x)$ that

$$B_1(T_1(x)) - B_1(S_1(x)) = T_1(x) - S_1(x) + \tfrac{1}{2}x. \tag{3.26}$$

But, by (2.4) of Lemma 2.1, the probability of (3.26) occurring when $T_1(x) \le t_e$ is at most

$$C_2(t_e + 1)e^{-\varepsilon' x} \le C_3 e^{-\frac{1}{2}\varepsilon' x}$$

for appropriate C_2, C_3 and $\varepsilon' > 0$. Consequently,

$$P(A_2) \le C_3 e^{-\frac{1}{2}\varepsilon' x}. \tag{3.27}$$

We next show that $P(A_3 \cap A_7)$ is small. This event will typically not occur because the coordinates $k = 2, \dots, 5$ that are reflecting at 0 after τ will impart a positive drift to $Z_1(\cdot)$. Restricting our attention to the event A_3, let K be the index at which $Z_K(\tau) = 0$. Also, let τ' be any random time with

$$\tau \le \tau' \le T_1'(x) \wedge \min\{s > \tau : Z_1(s) = 0\}. \tag{3.28}$$

Since $\tau' \le T_1'(x) < T_6$, it follows from (1.1) that

$$\sum_{k=2}^{5} \bigl(Y_k(\tau') - Y_k(\tau)\bigr) \ge (\tau' - \tau) - \bigl(B_K(\tau') - B_K(\tau)\bigr). \tag{3.29}$$

Applying (1.1) for the first coordinate and then substituting in (3.29), one obtains

$$Z_1(\tau') - Z_1(\tau) \ge \tilde{B}(\tau') - \tilde{B}(\tau) + \delta_2(\tau' - \tau), \tag{3.30}$$

where $\tilde{B}(t) \stackrel{\text{def}}{=} B_1(t) - (1 + \delta_2) B_K(t)$.

Again applying (2.4) of Lemma 2.1, one has

$$\begin{aligned} P\bigl(\tilde{B}(\tau') - \tilde{B}(\tau) + \delta_2(\tau' - \tau) \le -\tfrac{1}{4}x\bigr) &\le C_2\bigl(T_1'(x) + 1\bigr)e^{-\varepsilon' x} \\ &\le C_3 e^{-\frac{1}{2}\varepsilon' x}, \end{aligned} \tag{3.31}$$

with $T_1(x) \le t_e$ and the definitions of $T_1'(x)$ and t_e being used in the latter inequality. Applying this to (3.30), one obtains that, since $Z_1(\tau) \ge \frac{1}{2}x$,

$$P\bigl(Z_1(\tau') \le \tfrac{1}{4}x; A_3\bigr) \le C_3 e^{-\frac{1}{2}\varepsilon' x}$$

for τ' as in (3.28). This implies that

$$P(A_3 \cap A_7) \le C_3 e^{-\frac{1}{2}\varepsilon' x}. \tag{3.32}$$

We now show that

$$P(A_3 \cap A_4 \cap A_7^c) \le C_2 e^{-\frac{1}{4}\varepsilon' x} \tag{3.33}$$

for an appropriate choice of C_2. On the set $A_3 \cap A_7^c$, it follows from (1.1) that

$$\sum_{k=2}^{5} Y_k(T_1'(x)) \ge \bigl(T_1'(x) - \tau\bigr) - \bigl(B_K(T_1'(x)) - B_K(\tau)\bigr), \tag{3.34}$$

where K is the index at which $Z_K(\tau) = 0$. Since $\tau < T_1(x)$, it follows from the definition of $T_1'(x)$ that the right-hand side of (3.34) is at least

$$\tfrac{1}{2}\bigl(T_1'(x) + x^2\bigr) + \bigl[\tfrac{1}{2}x^2 + \tfrac{1}{4}\bigl(T_1'(x) - \tau\bigr) - \bigl(B_K(T_1'(x)) - B_K(\tau)\bigr)\bigr].$$

Again applying (2.4), this is greater than $\frac{1}{2}(T_1'(x)+x^2)$ off of a set of probability $C_2e^{-\varepsilon' x^2}$, for appropriate C_2 and $\varepsilon'>0$. Consequently,

$$\begin{aligned} &P(A_3\cap A_4\cap A_7^c)\\ &\quad = P\left(\sum_{k=2}^{5} Y_k(T_1'(x)) < \frac{1}{2}\big(T_1'(x)+x^2\big);\, A_3\cap A_7^c\right)\\ &\quad \le C_2e^{-\varepsilon' x^2}. \end{aligned} \tag{3.35}$$

One has

$$A\subseteq A' = A_1\cup A_2\cup(A_3\cap A_4)\subseteq A_1\cup A_2\cup(A_3\cap A_7)\cup(A_3\cap A_4\cap A_7^c).$$

Combining (3.25), (3.27), (3.32) and (3.35) therefore implies (3.23) for an appropriate choice of C_2. □

Using Proposition 3.3, the demonstration of Proposition 3.2 is quick.

PROOF OF PROPOSITION 3.2. It follows from (3.19) that

$$A_4^c\cup A_5^c\supseteq \big(A_1\cup A_2\cup(A_3\cap A_4)\big)^c = (A')^c.$$

Because of (3.20),

$$T_2(x)\le T_1'(x)\wedge T_6 \tag{3.36}$$

on A_4^c. On the other hand, (3.36) holds trivially on A_5^c. Along with (3.13), this implies that

$$T_2(x)\le 4\big(T_1(x)+x^2\big)\wedge T_6\le 4\big(T_1(x)\wedge T_6\big)+4x^2 \tag{3.37}$$

on $(A')^c$, and so, by (3.8) of Proposition 3.1,

$$E[T_2(x);\,(A')^c]\le C_3x^2 \tag{3.38}$$

for appropriate C_3.

The bound $T_2(x)\le 5x^{5/\eta}$ always holds and so, by Proposition 3.3,

$$E[T_2(x);\,A']\le C_4/x^2 \tag{3.39}$$

for appropriate C_4. Inequality (3.10) follows immediately from (3.38) and (3.39). □

Foster's Criterion. Foster's Criterion is a standard tool for showing positive recurrence of a Markov process when the process has a "uniformly negative drift" off of a bounded set in the state space (see, e.g., Bramson [1] or Foss and Konstantopoulos [9]). Versions of Foster's Criterion typically employ deterministic stopping times whose length depends only on the initial state. Here, we require a version of Foster's Criterion with random times, which is given below.

We state the proposition for SRBM defined on the induced Z-path space, consisting of continuous paths on $\mathbb{R}_+^6$ with the natural filtration, in order to facilitate the definition of the sequence of stopping times employed in its proof. The SRBM can always be projected onto this space. The proof of the proposition employs an elementary martingale argument that extends to more general Feller processes.

Here and later on in the article, we employ the norm

$$\|z\| = z_1 + \sum_{k=2}^{5} z_k^2 + z_6 \qquad \text{for } z = (z_1, \ldots, z_6), z_k \geq 0. \tag{3.40}$$

We set, for $\delta > 0$,

$$\tau_A(\delta) = \inf\{t \geq \delta : Z(t) \in A\};$$

$E_z[\cdot]$ denotes the expectation for the process with $Z(0) = z$, and $\mathcal{F}(t), t \geq 0$, denotes the filtration of σ-algebras associated with the SRBM.

PROPOSITION 3.4. *Suppose that, for some $\delta, \varepsilon, \kappa > 0$ and a family of stopping times $\sigma(z), z \in \mathbb{R}_+^6$, with $\sigma(z) \geq \delta$, $E_z[\sigma(z)]$ is measurable in z and the SRBM $Z(\cdot)$ satisfies*

$$E_z[\|Z(\sigma(z))\|] \leq (\|z\| \vee \kappa) - \varepsilon E_z[\sigma(z)] \tag{3.41}$$

for all z. Then

$$E_z[\tau_A(\delta)] \leq \frac{1}{\varepsilon}(\|z\| \vee \kappa) \qquad \textit{for all } z, \tag{3.42}$$

where $A = \{z : \|z\| \leq \kappa\}$. Hence, $Z(\cdot)$ is positive recurrent.

PROOF. The argument is a slight modification of that for the generalized Foster's Criterion given on page 94 of [1]. Set $\sigma_0 = 0$, and let $\sigma_1 < \sigma_2 < \cdots$ denote the stopping times defined inductively, with $\sigma_n - \sigma_{n-1}$, conditioned on $Z(\sigma_{n-1}) = z$, having the same law as $\sigma(z)$ given $Z(0) = z$. By (3.41) and the Strong Markov Property, for all z,

$$E_z[\|Z(\sigma_n)\| | \mathcal{F}(\sigma_{n-1})] \leq \big(\|Z(\sigma_{n-1})\| \vee \kappa\big) - \varepsilon E_{Z(\sigma_{n-1})}[\sigma(Z(\sigma_{n-1}))] \tag{3.43}$$

for almost all ω.

Set $M(0) = \|z\| \vee \kappa$ and

$$M(n) = \|Z(\sigma_n)\| + \varepsilon\sigma_n \qquad \text{for } n \geq 1. \tag{3.44}$$

Also, set $\mathcal{G}(n) = \mathcal{F}(\sigma_n)$. On account of (3.43),

$$E_z[M(n)|\mathcal{G}(n-1)] \le M(n-1) \qquad \text{for } n \le \rho, \tag{3.45}$$

where ρ is the first time $n > 0$ at which $M(n) \in A$. So, $M(n \wedge \rho)$ is a nonnegative supermartingale on $\mathcal{G}(n)$.

It follows from the Optional Sampling Theorem that

$$E_z[M(\rho)] \le \|z\| \vee \kappa. \tag{3.46}$$

Note that $\tau_A(\delta) \le \sigma_\rho$. Therefore, by (3.44) and (3.46),

$$\varepsilon E_z[\tau_A(\delta)] \le E_z[M(\rho)] \le \|z\| \vee \kappa, \tag{3.47}$$

which implies (3.42) as desired. □

4. Main steps of the proof of Theorem 1.2. Here, we present the main steps of the proof of Theorem 1.2, postponing their proofs until Sections 5 and 6. Our goal is to show that (3.41) of Proposition 3.4 is satisfied for each SRBM satisfying the conditions of Theorem 1.2. It then follows from the proposition that the SRBM is positive recurrent.

We employ the notation $D_1, D_2, \dots$ and $\varepsilon_1, \varepsilon_2, \dots,$ as well as the previous notation $C_1, C_2, \dots,$ to denote positive constants. As earlier, C_i denote terms whose precise value is not of interest to us, with the same symbol sometimes being reused. The terms D_i and ε_i will sometimes take general values in the statements of the propositions, in which case specific values will be employed at the end of the section to demonstrate (3.41). We state the values of D_i and ε_i we will apply, in most cases, when they are first introduced.

Proposition 4.1 is the first result. It states in essence that, after an appropriate time, either the norm of the initial state of the process decreases by a large factor or the sixth coordinate is bounded away from 0. In the first case, (3.41) will be demonstrated by using Proposition 4.2. In the second case, this will be done by using Propositions 4.3–4.5. In the statement of Proposition 4.1, one can choose $D_1 = 24 \cdot 16 \cdot 4 + 4$ and $D_2 = 24 \cdot 16 \cdot 40/\delta_1\delta_3$. At the end of the section, we will set $\varepsilon_1 = \varepsilon_2^2$; the term $\varepsilon_2 \in (0, \delta_1\delta_2\delta_3/1200]$, with the exact value being specified then.

PROPOSITION 4.1. *Suppose that* $Z(0) = z$ *with* $z_1 \le M$, $z_k^2 \le M$, *for* $k = 2, \dots, 5$, *and* $z_6 \le M$. (a) *For given* $\varepsilon_1 > 0$, *there exist* C_1, $D_1 \ge 1$ *and* $\varepsilon' > 0$ *such that, for all* M,

$$P\big(Z_k(M) > D_1 M\big) \le C_1 e^{-\varepsilon' M} \qquad \textit{for } k = 1, 6, \tag{4.1}$$

$$P\big(Z_k(M) > \varepsilon_1 M\big) \le C_1 e^{-\varepsilon' M} \qquad \textit{for } k = 2, \dots, 5, \tag{4.2}$$

$$E[Z_k(M)^2] \le D_1 M \qquad \textit{for } k = 2, \dots, 5. \tag{4.3}$$

(b) *For appropriate* $D_2 > 0$ *and each* $\varepsilon_2 \in (0, \frac{1}{40}\delta_1\delta_3]$, *there exist sets* $F_1 \in \mathcal{F}(M)$, $F_2 \in \mathcal{F}(M)$ *and* $\varepsilon' > 0$ *such that, for large enough* M,

$$P\big((F_1 \cup F_2)^c\big) \le e^{-\varepsilon' M} \qquad \textit{for } k = 1, 6, \tag{4.4}$$

$$Z_6(M) \le \varepsilon_2 M \textit{ on } F_1, \tag{4.5}$$

$$E[Z_k(M)^2; F_1] \le \varepsilon_2 D_2 M \qquad \textit{for } k = 2, \dots, 5, \tag{4.6}$$

$$Z_6(M) \ge \varepsilon_2 M \textit{ on } F_2. \tag{4.7}$$

Depending on whether F_1 or F_2 holds, we proceed in different ways. Under F_1, we consider the evolution of the SRBM for an additional time $D_3 M$. For this, we employ Proposition 4.2, which is given below.

We introduce the following terminology for Proposition 4.2. Set

$$\varepsilon_3 = 6\varepsilon_2/\delta_3 \quad \text{and} \quad D_3 = 250 D_1/\delta_3\delta_4, \tag{4.8}$$

for given $\varepsilon_2 > 0$ and $D_1 \ge 1$. Let U_1 be the first time t on the interval $[0, (D_3 - \varepsilon_3)M]$ at which $Z_1(t) = \frac{12\varepsilon_2}{\delta_3} M$, with $U_1 = (D_3 - \varepsilon_3)M$ if this does not occur. Set $U_2 = U_1 + \varepsilon_3 M \le D_3 M$. The proposition states that $Z_1(U_2)$, $Z_6(U_2)$ and $Z_k(U_2)$, $k = 2, \dots, 5$, are all small in an appropriate sense. The argument requires $Z_1(t) > 0$ for $t \le U_2$, which enables all other coordinates to drift toward 0.

PROPOSITION 4.2. *Suppose* $Z(0) = z$ *satisfies*

$$(12\varepsilon_2/\delta_3)M \le z_1 \le D_1 M, \tag{4.9}$$

$$z_k \le \varepsilon_2 M \qquad \textit{for } k = 2, \dots, 6, \tag{4.10}$$

for given $D_1 \ge 1$ *and* $\varepsilon_2 \in (0, 1]$. *Then, for* U_2 *as given above and large enough* M,

$$E[Z_k(U_2)] \le (70\varepsilon_2/\delta_3)M \qquad \textit{for } k = 1, 6, \tag{4.11}$$

$$E[Z_k(U_2)^2] \le (24 \cdot 97\varepsilon_2/\delta_3)M \qquad \textit{for } k = 2, \dots, 5. \tag{4.12}$$

When F_2 occurs, we follow the sketch given near the end of Section 1. In this case, we restart the SRBM at time M and apply Proposition 4.3. In the proposition, we employ the stopping times $T_3(\cdot)$ and $T_4(\cdot)$. We define

$$T_3(M) = \min\left\{t : \sum_{k=2}^{5} Y_k(t) = \frac{1}{6}(\delta_1 t + \varepsilon_2 M)\right\} \tag{4.13}$$

for given $\varepsilon_2 \in (0, 1]$. We then set $T_4(M) = T_3(M)$ off of a set G_M that will be specified in the proof of the proposition, with $T_4(M) \le T_3(M) \wedge T_6$ holding on G_M, where T_6 is the stopping time that was defined in (3.6). [$T_3(M) < T_6$ will hold off of G_M.] The set G_M will be negligible in the sense of (4.23) and (4.24).

In addition to the bounds on G_M in (4.23) and (4.24), Proposition 4.3 gives upper and lower bounds on $T_3(M)$ and $Z_k(T_3(M))$, for $k = 1, 6$, and upper bounds on $Z_k(T_3(M))$, for $k = 2, \dots, 5$. We will set the constant ε_4 in the proposition equal to $\frac{1}{10}$ at the end of the section.

PROPOSITION 4.3. *Suppose* $Z(0) = z$ *satisfies*

$$z_k \le D_1 M \qquad \text{for } k = 1, 6, \tag{4.14}$$

$$z_k \le \varepsilon_2^2 M \qquad \text{for } k = 2, \dots, 5, \tag{4.15}$$

$$z_6 \ge \varepsilon_2 M, \tag{4.16}$$

for given M, D_1 *and* $\varepsilon_2 \in (0, \frac{1}{20}]$. *Then, on* G_M^c *and* $T_3(M) < \infty$,

$$T_3(M) \ge \tfrac{1}{30}\varepsilon_2 M, \tag{4.17}$$

$$T_3(M) \le T_6, \tag{4.18}$$

$$Z_k(T_3(M)) \le \frac{31 D_1}{\varepsilon_2} T_3(M) \qquad \text{for } k = 1, 6, \tag{4.19}$$

$$Z_k(T_3(M)) \le 31\varepsilon_2 T_3(M) \qquad \text{for } k = 2, \dots, 5, \tag{4.20}$$

$$Z_1(T_3(M)) \ge \tfrac{1}{12}\delta_1\delta_2 T_3(M), \tag{4.21}$$

$$Z_6(T_3(M)) \ge \tfrac{1}{2}\delta_1 T_3(M). \tag{4.22}$$

For given $\varepsilon_4 > 0$ *and large enough* M,

$$E[Z_k(T_4(M)); G_M] \le \varepsilon_4 \qquad \text{for } k = 1, 6, \tag{4.23}$$

$$E[Z_k(T_4(M))^2; G_M] \le \varepsilon_4 \qquad \text{for } k = 2, \dots, 5. \tag{4.24}$$

We define stopping times $T_3'(M)$ as follows. For given $M > 0$ and $z = (z_1, \dots, z_6)$, set

$$T_3'(M) = T_3(M) \wedge T_6 \wedge 5N_M(z)^{5/2\eta}, \tag{4.25}$$

where

$$N_M(z) = \Bigl(\max_{k=2,\dots,5} z_k^2\Bigr) \vee M \tag{4.26}$$

and $\eta = 0.000073$ as in Section 3. Assuming random initial conditions that satisfy the analog of (4.3) in Proposition 4.1, we give, in Proposition 4.4, bounds on $E[T_3'(M)]$. Moreover, the truncation event

$$A_M = \{\omega : T_3'(M) = 5N_M(z)^{5/2\eta}\} \tag{4.27}$$

is small in the sense of (4.30) and, under further initial conditions, is small as in (4.31). Propositions 3.2 and 3.3 are the key ingredients in the proof.

PROPOSITION 4.4. *Suppose that* $Z(0)$ *satisfies*

$$E[Z_k(0)^2] \le D_1 M \qquad \text{for } k = 2, \dots, 5, \tag{4.28}$$

and given $M \ge 4$ *and* D_1. *Then, for appropriate* C_2 *not depending on* M,

$$E[T_3'(M)] \le C_2 M \tag{4.29}$$

and

$$E[Z_k(T_3'(M))^2; A_M] \le C_2/\sqrt{M}. \tag{4.30}$$

If, in addition, $Z_k(0) \le D_1 M$ *for* $k = 1, 6$, *then*

$$E[Z_k(T_3'(M)); A_M] \le C_3/M \tag{4.31}$$

for appropriate C_3 *not depending on* M.

On the set $G_M^c \cap A_M^c$, we continue to follow the evolution of $Z(\cdot)$ after the elapsed time $M + T_3(M)$. [Note that, on $G_M^c \cap A_M^c$, $T_3(M) = T_3'(M)$.] We wish to show that, provided $Z_k(\cdot)$, $k = 1, 6$, are initially "large" but $Z_k(\cdot)$, $k = 2, \dots, 5$, are initially "small," then all coordinates will typically be small at an appropriate random time. This is done in Proposition 4.5. The bounds (4.39) and (4.40) will allow us to demonstrate (3.41) under the event F_2 in Proposition 4.1.

In order to state Proposition 4.5, we define

$$T_5(M_1) = \min\{t : Z_1(t) = \varepsilon_5 M_1\} \wedge D_4 M_1, \tag{4.32}$$

$$T_5'(M_1) = T_5(M_1) + \tfrac{1}{2}\varepsilon_5 M_1, \tag{4.33}$$

for given $M_1 > 0$, D_4 and $\varepsilon_5 > 0$. Note that

$$T_5'(M_1) \le \big(D_4 + \tfrac{1}{2}\varepsilon_5\big) M_1 \tag{4.34}$$

always holds. We employ the constants $\varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8$ and D_4, D_5 in the proposition. A specific value of $\varepsilon_5 \in (0, \frac{1}{72}\delta_1\delta_2]$ will be assigned at the end of the section; there, we will also employ $\varepsilon_6 = 31\varepsilon_2$, $\varepsilon_7 = \frac{1}{12}\delta_1\delta_2$, $\varepsilon_8 = \frac{1}{20}\delta_3\varepsilon_5$ and $D_5 = 31 D_1/\varepsilon_2$; D_4 is specified in the proposition.

PROPOSITION 4.5. *Let* $T_5(\cdot)$ *and* $T_5'(\cdot)$ *be as in* (4.32) *and* (4.33) *for given* $\varepsilon_5 > 0$. *Suppose* $Z(0) = z$ *satisfies*

$$z_k \le \varepsilon_6 M_1 \qquad \text{for } k = 2, \dots, 5, \tag{4.35}$$

$$\varepsilon_7 M_1 \le z_k \le D_5 M_1 \qquad \text{for } k = 1, 6, \tag{4.36}$$

for given $M_1 > 0$, $\varepsilon_6 > 0$, $\varepsilon_7 \ge 6\varepsilon_5 \vee 3\varepsilon_6$ *and* $D_5 > 0$. *Then, for given* $\varepsilon_8 > 0$ *and* $D_4 = 10 D_5 \delta_4/\delta_2\delta_3$,

$$P\big(T_5(M_1) = D_4 M_1\big) \le C_1 e^{-\varepsilon' M_1}, \tag{4.37}$$

$$P\big(Z_k(T_5(M_1)) \ge \varepsilon_8 M_1\big) \le C_1 e^{-\varepsilon' M_1} \qquad \text{for } k = 2, \dots, 6, \tag{4.38}$$

for appropriate C_1 *and* $\varepsilon' > 0$ *not depending on* M_1. *Moreover,*

$$E[Z_k(T_5'(M_1))] \le 6\varepsilon_5 M_1 + C_4 \qquad \text{for } k = 1, 6, \tag{4.39}$$

$$E[Z_k(T_5'(M_1))^2] \le 24 \cdot 8\varepsilon_5 M_1 + C_4 \qquad \text{for } k = 2, \dots, 5, \tag{4.40}$$

for appropriate C_4 *not depending on* M_1.

Demonstration of Theorem 1.2. It suffices to consider the SRBM $Z(\cdot)$ on the induced Z-path space. We will show that, for $z \in \mathbb{R}_+^6$ and an appropriate stopping time $\sigma(z)$, the assumption (3.41) of Proposition 3.4 is satisfied. The proposition will then imply $Z(\cdot)$ is positive recurrent. We abbreviate by setting $\sigma(z) = \sigma$ and dropping the subscript z from $E_z[\cdot]$.

We will express σ in terms of a related stopping time σ', which we construct piecemeal by using the sets appearing in the previous propositions. Assume that $\|z\| = M$. Then $z_1 \le M$, $z_k^2 \le M$, for $k = 2, \dots, 5$, and $z_6 \le M$, and so the assumptions of Proposition 4.1 are satisfied. It follows from the proposition that (4.1)–(4.3) hold for given M and (4.4)–(4.7) hold for large enough M. Let H_1 denote the union of the set where $(F_1 \cup F_2)^c$ occurs and where either the event in (4.1) or the event in (4.2) occurs. On H_1, we set $\sigma' = M$. It follows from Lemma 2.5, (4.1), (4.2) and (4.4) that, for large enough M,

$$E[\|Z(\sigma')\|;\, H_1] \le 1. \tag{4.41}$$

Suppose next that the event $F_1 \cap H_1^c$ holds. Then either (a) $Z_1(M) < (12\varepsilon_2/\delta_3)M$ or (b) $Z_1(M) \ge (12\varepsilon_2/\delta_3)M$; denote the former of these events by H_2 and the latter by H_3. Under H_2, we set $\sigma' = M$. Then, on account of (4.5) and (4.6) of Proposition 4.1, with $D_2 = 24 \cdot 16 \cdot 40/\delta_1\delta_3$,

$$\begin{aligned} E[\|Z(\sigma')\|;\, H_2] &\le \left(\frac{12\varepsilon_2}{\delta_3} + 4\varepsilon_2 D_2 + \varepsilon_2\right)M \\ &\le (97 \cdot 16 \cdot 40\varepsilon_2/\delta_1\delta_3)M. \end{aligned} \tag{4.42}$$

When H_3 occurs, we set $\sigma' = M + U_2$, where U_2 is defined below (4.8). (Here and later on, stopping times such as U_2 refer to the restarted process.) The process restarted at time M satisfies conditions (4.9) and (4.10) of Proposition 4.2. It follows from (4.11) and (4.12) of the proposition that

$$E[\|Z(\sigma')\|;\, H_3] \le (140 + 96 \cdot 97)\frac{\varepsilon_2}{\delta_3}M \le 97^2\frac{\varepsilon_2}{\delta_3}M. \tag{4.43}$$

The bounds (4.41)–(4.43) consider the behavior of $Z(\sigma)$ off of $F_2 \cap H_1^c$. We now consider the behavior on $F_2 \cap H_1^c$, for which there are two cases. Denote by H_4 the subset of $F_2 \cap H_1^c$ corresponding to the union of the events G_M and A_M for the restarted process, which appear in the proof of Proposition 4.3 and in (4.27). Let

$$\sigma' \stackrel{\text{def}}{=} M + \big(T_4(M) \wedge 5N_M(Z(M))^{5/\eta}\big) \le M + T_3'(M),$$

that is, σ' is the earlier of the times at which either the event G_M or A_M occurs. The restarted process satisfies both (4.14)–(4.16) of Proposition 4.3 and (4.28) of Proposition 4.4. It therefore follows from (4.23) and (4.24), with $\varepsilon_4 = \frac{1}{10}$, and (4.30) and (4.31) that

$$E[\|Z(\sigma')\|;\, H_4] \le 1 \tag{4.44}$$

for large enough M.

We also consider the behavior of $Z(\sigma')$ on $H_5 \stackrel{\text{def}}{=} F_2 \cap H_1^c \cap H_4^c$. On account of (4.19)–(4.22) of Proposition 4.3, the conditions (4.35) and (4.36) of Proposition 4.5 are satisfied for the process restarted at time $M + T_3(M) = M + T_3'(M)$, for $M_1 = T_3(M)$ and D_5, ε_6 and ε_7 as specified before Proposition 4.5. Also, $\varepsilon_7 \ge 6\varepsilon_5 \vee 3\varepsilon_6$ holds for $\varepsilon_2 \le \delta_1\delta_2\delta_3/1200$ and ε_5 as specified before the proposition. Inequalities (4.39) and (4.40) therefore hold for $T_5'(M_1)$ chosen as in (4.33). Setting $\sigma' = M + T_3(M) + T_5'(T_3(M))$, it follows from these inequalities that

$$\begin{aligned} E[\|Z(\sigma')\|; H_5] &\le 97 \cdot 8\varepsilon_5 E[T_3(M); H_5] + C_4 \\ &\le 97 \cdot 8\varepsilon_5 E[T_3'(M)] + C_4 \end{aligned} \tag{4.45}$$

for appropriate C_4. On account of (4.3) of Proposition 4.1, one can apply Proposition 4.4 to $Z(\cdot)$ restarted at time M, which gives the upper bound in (4.29) on $E[T_3'(M)]$. Applying this to (4.45), one obtains

$$E[\|Z(\sigma')\|; H_5] \le 98 \cdot 8\varepsilon_5 C_2 M \tag{4.46}$$

for large enough M and appropriate C_2.

Adding the bounds in (4.41)–(4.46) for $E[Z(\sigma'); H_i]$, $i = 1, \dots, 5$, one obtains

$$E[\|Z(\sigma')\|] \le C_5(\varepsilon_2 + \varepsilon_5)M$$

for large enough M, with C_5 depending on δ_1 and δ_3. So far, we have not specified the values of ε_2 and ε_5; we now set

$$\varepsilon_2 = \varepsilon_5 = (1/4C_5) \wedge (\delta_1\delta_2\delta_3/1200).$$

It follows that

$$E[\|Z(\sigma')\|] \le \tfrac{1}{2}M \tag{4.47}$$

for $\|z\| = M$ and $M \ge M_0$, for appropriate $M_0 \ge 1$.

We define σ in terms of σ', by setting $\sigma = \sigma'$ when $\|z\| = M$ for $M \ge M_0$, and $\sigma = M \vee 1$ for $M \le M_0$. When $\|z\| = M$ and $M \ge M_0$, this implies

$$E[\|Z(\sigma)\|] \le \tfrac{1}{2}M. \tag{4.48}$$

On the other hand, by applying (2.22) of Lemma 2.5 to (4.1), it follows for all M that

$$E[Z_k(M \vee 1)] \le C_1(M \vee 1) \qquad \text{for } k = 1, 6$$

and appropriate $C_1 \ge D_1 \vee 1$. Together with (4.3), this implies

$$E[\|Z(M \vee 1)\|] \le 6C_1(M \vee 1) \tag{4.49}$$

for all M. Setting $\kappa = 12C_1(M_0 \vee 1)$, it follows from (4.48) and (4.49) that

$$E[\|Z(\sigma)\|] \le (\|z\| \vee \kappa) - \tfrac{1}{2}(M \vee 1) \tag{4.50}$$

for $\|z\| = M$ and all M.

We also wish to show that, for $\|z\| = M$,

$$E[\sigma] \le C_3(M \vee 1) \tag{4.51}$$

for some C_3. This is a quick consequence of the definition of σ on $H_1, \dots, H_5$ for $\|z\| \ge M_0$. On $H_1 \cup H_2$, $\sigma = M$; on H_3, $\sigma \le D_3 M$; on H_4, $\sigma \le M + T_3'(M)$; and on H_5, $\sigma = M + T_3'(M) + T_5'(T_3'(M))$. It therefore follows from (4.29) of Proposition 4.4 and (4.34) that

$$\begin{aligned} E[\sigma] &\le M + D_3 M + E[T_3'(M)] + E[T_5'(T_3'(M))] \\ &\le \big(1 + D_3 + C_2 + C_2\big(D_4 + \tfrac{1}{2}\varepsilon_5\big)\big)M \le C_3 M \end{aligned} \tag{4.52}$$

for $\|z\| \ge M_0$ and appropriate C_2 and C_3. Together with $\sigma = M \vee 1$ for $\|z\| < M_0$, this implies (4.51).

Combining (4.50) and (4.51), one obtains

$$E[\|Z(\sigma)\|] \le (\|z\| \vee \kappa) - (1/2C_3)E[\sigma]. \tag{4.53}$$

This implies (3.41) of Proposition 3.4, with $\varepsilon = 1/2C_3$. Since $Z(\cdot)$ is Feller and σ is defined in terms of hitting times of closed sets, one can check that $E_z[\sigma(z)] = E[\sigma]$ is measurable in z. By applying the proposition, (3.42) follows and hence $Z(\cdot)$ is positive recurrent. This demonstrates Theorem 1.2.

5. Demonstration of Propositions 4.1 and 4.2. Proposition 4.1 constitutes the first step of the proof of Theorem 1.2. It provides elementary upper bounds (4.1)–(4.3) on $Z_k(M)$, $k = 1, \dots, 6$, and on $E[Z_k(M)^2]$, $k = 2, \dots, 5$, that are valid over all M. It states that, off of the exceptional set in (4.4), either $Z_k(M)$ will be small for all $k = 2, \dots, 6$ or $Z_6(M)$ will be large, in the sense of (4.5)–(4.7). This dichotomy depends on the rate of growth of $Y_1(\cdot)$ as given by the set F_3 in (5.1) of the proof, although the actual correspondence is a bit more complicated. The proof of Proposition 4.1 relies on the application of lemmas from Section 2 to the equation (1.1) of the SRBM.

PROOF OF PROPOSITION 4.1. Both inequalities in (4.1) follow directly from (2.12) of Lemma 2.3, with $D_1 \ge 9$. Inequality (4.2) follows from (2.14) of Lemma 2.4, with a new choice of C_1. For (4.3), one can restrict the expectation to the set $\{Z_k(M) > 2\sqrt{M}\}$ and its complement. One then applies (2.15) to the first part and a trivial bound to the second part to obtain (4.3), with $D_1 \ge 24 \cdot 16 \cdot 4 + 4$.

For the inequalities (4.4)–(4.7), we first set

$$F_3 = \{\omega : Y_1(M) - Y_1(\tau_k) > \varepsilon_9 M \text{ for some } k = 2, \dots, 5\}. \tag{5.1}$$

Here, $\varepsilon_9 \stackrel{\text{def}}{=} 2\varepsilon_2/\delta_1$ and τ_k is the last time before M at which $Z_k(t) = 0$ for any t; if the set is empty, let $\tau_k = 0$. On F_3, we denote by K one of the indices k satisfying (5.1).

We consider the behavior on F_3 and F_3^c separately, first considering the behavior on F_3. One has, by applying (1.1) to the Kth and 6th coordinates,

$$\begin{aligned} Z_6(M) - Z_K(M) = \big(Z_6(\tau_K) - Z_K(\tau_K)\big) + \big(B_6(M) - B_6(\tau_K)\big) \\ - \big(B_K(M) - B_K(\tau_K)\big) + \delta_1\big(Y_1(M) - Y_1(\tau_K)\big) \\ + \delta_3\big(Y_6(M) - Y_6(\tau_K)\big). \end{aligned} \tag{5.2}$$

On F_3, it follows from (2.3) of Lemma 2.1 that, except on a set $F_4 \in \mathcal{F}(M)$ of exponentially small probability in M,

$$Z_6(M) \geq \delta_1 \varepsilon_9 M - \varepsilon M \geq \tfrac{1}{2}\delta_1\varepsilon_9 M = \varepsilon_2 M \tag{5.3}$$

for $\varepsilon = \frac{1}{2}\delta_1\varepsilon_9$ and large enough M. This gives the inequality in (4.7) on the set $F_3 \cap F_4^c$.

We now consider the behavior of $Z(\cdot)$ on F_3^c. Set $t_1 = (1 - 20\varepsilon_9/\delta_3)M$; since $\varepsilon_2 \leq \frac{1}{40}\delta_1\delta_3$, $t_1 \geq 0$ holds. It follows from (2.14) of Lemma 2.4 that, except on a set $F_5 \in \mathcal{F}(M)$ of exponentially small probability in M,

$$Z_k(t_1) \leq Z_k(0) + \varepsilon M \leq \varepsilon_9 M \tag{5.4}$$

for $k = 2, \dots, 5$, $\varepsilon = \varepsilon_9/2$ and large enough M. Restarting $Z(\cdot)$ at time t_1, it follows from (2.32) of Lemma 2.6 and (5.4) that, except on a set $F_6 \in \mathcal{F}(M)$ of exponentially small probability,

$$Y_k(M) - Y_k(t_1) \geq 4\frac{\varepsilon_9}{\delta_3}M - \frac{\varepsilon_9}{\delta_3}M - \frac{2}{\delta_3}\big(Y_1(M) - Y_1(t_1)\big). \tag{5.5}$$

On F_3^c, when $\tau_k < t_1$, the last term on the right-hand side of (5.5) is at most $2\varepsilon_9 M/\delta_3$, which implies

$$Y_k(M) - Y_k(t_1) > 0, \tag{5.6}$$

and hence $Z_k(\tau_k') = 0$ for some $\tau_k' \in [t_1, M]$. This contradicts the definition of τ_k, and so $\tau_k \geq t_1$.

Let τ_k' be the smallest such time. Since τ_k' is a stopping time, we may restart $Z(\cdot)$ at τ_k'. Applying (2.15) of Lemma 2.4, it follows that

$$E[Z_k(M)^2; F_3^c \cap F_5^c \cap F_6^c] \leq 24 \cdot 16 \cdot 20\frac{\varepsilon_9}{\delta_3}M = \varepsilon_2 D_2 M \tag{5.7}$$

for $k = 2, \dots, 5$ and $D_2 = 24 \cdot 16 \cdot 40/\delta_1\delta_3$.

We now conclude the demonstration of (4.4)–(4.7). Denoting the set on which the inequality in (4.7) holds by F_2, one has by (5.3) that $F_2 \supseteq F_3 \cap F_4^c$. Setting $F_1 = F_2^c \cap F_3^c \cap F_5^c \cap F_6^c$, then (4.5) is automatically satisfied and (4.6) holds because of (5.7). Since $(F_1 \cup F_2)^c \subseteq F_4 \cup F_5 \cup F_6$, (4.4) follows, for appropriate $\varepsilon' > 0$, from the upper bounds on the probabilities of F_4, F_5 and F_6. It follows from the definition of F_2 that $F_2 \in \mathcal{F}(M)$; since $F_i \in \mathcal{F}(M)$, $i = 2, \dots, 6$, one also has $F_1 \in \mathcal{F}(M)$. $\square$

Proposition 4.2 states that, if z_k, $k = 2, \dots, 6$, are all small and z_1 is bounded below, but is not too large, then $Z_k(U_2)$, $k = 1, \dots, 6$, are all small in the sense of (4.11) and (4.12). The proof considers the behavior of $Z(t)$ over $[U_1, U_2]$. The stopping time U_1 was defined so that $Z_1(U_1)$ is relatively small, but large enough so that, over $[0, U_2]$ with $U_2 = U_1 + \varepsilon_3 M$, $Z_1(t) > 0$ holds. The interval $[U_1, U_2]$ is both large enough to obtain the desired behavior of $Z_k(U_2)$, $k = 2, \dots, 5$, in (4.12) and short enough so (4.11) holds for $Z_k(U_2)$, $k = 1, 6$. As with Proposition 4.1, the proof applies the lemmas of Section 2 to (1.1).

PROOF OF PROPOSITION 4.2. We first show (4.11) for $k = 1$. It follows from Lemma 2.7, (4.9) and (4.10) that, on the set where $Z_1(t) > 0$ for $t \in [0, \frac{1}{2} D_3 M]$,

$$Z_1\left(\frac{1}{2} D_3 M\right) \le D_1 M + \frac{3\varepsilon_2}{\delta_3} M - \frac{1}{60} \delta_4 D_3 M \tag{5.8}$$

except for a set F_7 of exponentially small probability in M. Since the right-hand side of (5.8) is negative for D_3 satisfying (4.8) and $Z_1(0) \ge \frac{12\varepsilon_2}{\delta_3} M$, $Z_1(t) = \frac{12\varepsilon_2}{\delta_3} M$ must occur at some $t \le \frac{1}{2} D_3 M$; hence $Z_1(U_1) = \frac{12\varepsilon_2}{\delta_3} M$ on F_7^c. By (2.12) of Lemma 2.3 and (4.8), this in turn implies that $Z_1(U_2) \le \frac{60\varepsilon_2}{\delta_3} M$ off of an additional set of exponentially small probability. Together with (2.22) of Lemma 2.5, this implies (4.11) for $k = 1$ and large M.

Restarting $Z(\cdot)$ at U_1, it follows from (1.1) and (4.8) that, except on a set F_8 of exponentially small probability in M,

$$Z_1(U_1 + s) \ge \frac{12\varepsilon_2}{\delta_3} M + B_1(s) - s > 0$$

for $s \le \varepsilon_3 M$. Consequently, on F_8^c,

$$Z_1(t) > 0 \qquad \text{for } t \le U_2. \tag{5.9}$$

Since $Z_6(0) \le \varepsilon_2 M$, one can therefore employ (2.14) of Lemma 2.4, with small enough $\varepsilon > 0$, together with (2.22) of Lemma 2.5, to obtain (4.11) for $k = 6$.

We still need to show (4.12). For this, one can employ the conditions (4.8), (4.10) and (5.9) and argue similarly to (5.4) through (5.6), in the proof of Proposition 4.1, to conclude that, for $k = 2, \dots, 5$,

$$Z_k(\tau'_k) = 0 \qquad \text{for some } \tau'_k \in [U_1, U_2],$$

off of a set F_9 of exponentially small probability in M. Letting τ'_k denote the first such time, we restart $Z(\cdot)$ at τ'_k. Applying (2.15) and (2.21), it follows that

$$\begin{aligned} E[Z_k(U_2)^2] &\le E[Z_k(U_2)^2; F_9^c] + E[Z_k(U_2)^2; F_9] \\ &\le (24 \cdot 16 + 1)\varepsilon_3 M \le (24 \cdot 97\varepsilon_2/\delta_3) M \end{aligned}$$

for large enough M. This implies (4.12). □

6. Demonstration of Propositions 4.3, 4.4 and 4.5. The proofs of Propositions 4.3, 4.4 and 4.5 rely on the application of the lemmas in Section 2 to the equation (1.1) of the SRBM $Z(\cdot)$. Proposition 4.4 also relies on Propositions 3.2 and 3.3. The reasoning behind the proofs follows in spirit the sketch given near the end of Section 1 and in Section 4.

We first demonstrate Proposition 4.3. The proposition states that, off of the exceptional set G_M defined in the proof, the inequalities (4.17)–(4.22) all hold. In particular, $Z_k(T_3(M))$, $k = 2, \dots, 5$, will be small and $Z_k(T_3(M))$, $k = 1, 6$, will be bounded below, but not too large. These inequalities, except for (4.21), will follow from their analogs (6.1)–(6.4) that hold over $[\frac{1}{30}\varepsilon_2 M, T_3(M)]$ and $[0, T_3(M)]$. The exceptional set G_M will be shown to be small in the sense of (4.23) and (4.24).

The lower bounds on $Z_k(T_3(M))$, $k = 1, 6$, constitute the more delicate part of the argument and depend on the condition $z_6 \geq \varepsilon_2 M$ in (4.16). Arguing as in (6.17)–(6.20), we will show that the growth of $Y_1(\cdot)$ causes $Z_6(\cdot)$ to increase linearly. On the other hand, as shown below (6.6), the growth of $Y_k(\cdot)$, $k = 2, \dots, 5$, together with $Y_6(T_3(M)) = 0$, causes $Z_1(\cdot)$ to eventually increase linearly. The stopping time $T_3(M)$ has been chosen so that both features are present.

PROOF OF PROPOSITION 4.3. We first specify the set G_M used in the definition of $T_4(M)$. We abbreviate by setting $M' = \frac{1}{30}\varepsilon_2 M$. Writing $G_M = \bigcup_{i=1}^5 G_i$, the sets G_i are defined as follows:

$$G_1 = \left\{\omega : \sum_{k=2}^{5} Y_k(M') \geq 5M'\right\}, \tag{6.1}$$

$$G_2 = \left\{\omega : Z_k(s) \geq \frac{31}{\varepsilon_2} D_1 s \text{ for some } s \in [M', T_3(M)], k = 1, 6\right\}, \tag{6.2}$$

$$G_3 = \{\omega : Z_k(s) \geq 31\varepsilon_2 s \text{ for some } s \in [M', T_3(M)], k = 2, \dots, 5\}, \tag{6.3}$$

$$G_4 = \left\{\omega : Z_6(s) \leq \tfrac{1}{2}\delta_1 s \text{ for some } s \in [0, T_3(M)]\right\}, \tag{6.4}$$

$$G_5 = \left\{\omega : B_2(s) - B_1(s) \geq \tfrac{1}{12}\delta_1\delta_2 s \text{ for some } s \in [M', T_3(M)]\right\}. \tag{6.5}$$

Inequality (4.17) follows from (6.1) and the definition of $T_3(M)$. Inequalities (4.19) and (4.20) follow by setting $s = T_3(M)$ in (6.2) and (6.3); both (4.18) and (4.22) follow from (6.4). The demonstration of (4.21) requires a little work. First note that, on G_4^c, (1.1), (2.6) of Lemma 2.2 and (4.15) imply that

$$\begin{aligned} Z_1(T_3(M)) &\geq B_1(T_3(M)) - T_3(M) + \sum_{k=1}^{6} Y_k(T_3(M)) + \delta_2 \sum_{k=2}^{5} Y_k(T_3(M)) \\ &\geq \frac{1}{6}\delta_1\delta_2 T_3(M) + \left(\frac{1}{6}\delta_2\varepsilon_2 - \varepsilon_2^2\right)M + B_1(T_3(M)) - B_2(T_3(M)) \\ &\geq \frac{1}{6}\delta_1\delta_2 T_3(M) + B_1(T_3(M)) - B_2(T_3(M)), \end{aligned} \tag{6.6}$$

which, on G_5^c, is at least $\frac{1}{12}\delta_1\delta_2 T_3(M)$. So,

$$Z_1(T_3(M)) \geq \tfrac{1}{12}\delta_1\delta_2 T_3(M) \qquad \text{on } G_4^c \cap G_5^c,$$

which demonstrates (4.21).

We need to show (4.23) and (4.24). For this, we define V_i, $i = 2, 3, 4, 5$, to be the first time at which the event in G_i occurs, with

$$V_1 = \inf\left\{s : \sum_{k=2}^{5} Y_k(s) \geq 5M'\right\},$$

if G_1 occurs; off of these sets, define $V_i = T_3(M)$ for $i = 1, \dots, 5$. We complete our definition of $T_4(M)$ in (4.13) by setting

$$T_4(M) = V_1 \wedge V_2 \wedge V_3 \wedge V_4 \wedge V_5. \tag{6.7}$$

Note that $V_4 \leq T_6$. It follows from this and (6.7) that $T_4(M) \leq T_3(M) \wedge T_6$; moreover, $T_4(M)$ is a stopping time.

We note that, by (2.9) of Lemma 2.2,

$$P(G_1) \leq C_1 e^{-\varepsilon' M} \tag{6.8}$$

for appropriate C_1 and $\varepsilon' > 0$. Using (2.21) and (2.22) of Lemma 2.5, it therefore follows that, for given $\varepsilon_{10} > 0$ and large enough M,

$$E[Z_k(T_4(M)); G_1] \leq \varepsilon_{10} \qquad \text{for } k = 1, 6, \tag{6.9}$$

$$E[Z_k(T_4(M))^2; G_1] \leq \varepsilon_{10} \qquad \text{for } k = 2, \dots, 5. \tag{6.10}$$

We require more detailed estimates for $G_2, \dots, G_5$. For each $i = 2, \dots, 5$ and $j = 1, 2, \dots,$ we denote by $G_i(j)$ the event for which G_i first occurs on $[j, j+1]$. We first consider the behavior on G_3. We recall that, by (2.14) of Lemma 2.4, for $k = 2, \dots, 5$ and given $\varepsilon > 0$,

$$P\bigl(Z_k(s) - Z_k(0) \geq \varepsilon j \text{ for some } s \leq j\bigr) \leq C_1 e^{-\varepsilon' j} \tag{6.11}$$

for each $j = 1, 2, \dots,$ and appropriate C_1 and $\varepsilon' > 0$. On account of (4.15), it follows for small enough ε that

$$P\bigl(Z_k(s) \geq 31\varepsilon_2 s \text{ for some } s \in [j-1, j]\bigr) \leq C_1 e^{-\varepsilon' j} \tag{6.12}$$

for $j \geq M'$. It therefore follows from (2.21) and (2.22) that

$$E[Z_k(V_3); G_3(j)] \leq e^{-\frac{1}{2}\varepsilon' j} \qquad \text{for } k = 1, 6, \tag{6.13}$$

$$E[Z_k(V_3)^2; G_3(j)] \leq e^{-\frac{1}{2}\varepsilon' j} \qquad \text{for } k = 2, \dots, 5, \tag{6.14}$$

for $j \geq M'$ and large enough M. Summing over j gives

$$E[Z_k(V_i); G_i] \leq \varepsilon_{10} \qquad \text{for } k = 1, 6, \tag{6.15}$$

$$E[Z_k(V_i)^2; G_i] \leq \varepsilon_{10} \qquad \text{for } k = 2, \dots, 5, \tag{6.16}$$

for $i = 3$, given $\varepsilon_{10} > 0$ and large enough M.

The inequalities (6.15) and (6.16) hold for $i = 2$ and $i = 5$ for the same reasons, except that one applies (2.12) of Lemma 2.3 in place of (2.14) and (4.14) in place of (4.15) for $i = 2$, and one applies (2.3) of Lemma 2.1 for $i = 5$. The inequalities (6.15) and (6.16) also hold for $i = 4$, although this requires more work; we now do this.

We note that, by (2.6) of Lemma 2.2, (4.13) and (4.15),

$$Y_1(s) \geq s - \big(\varepsilon_2^2 M + B_2(s)\big) - \tfrac{1}{6}(\delta_1 s + \varepsilon_2 M) \tag{6.17}$$

for $s \leq T_3(M) \wedge T_6$. Together with (1.1), (4.16) and $\varepsilon_2 \leq \frac{1}{20}$, this implies

$$\begin{aligned} Z_6(s) &\geq \varepsilon_2 M + B_6(s) - s + (1+\delta_1) Y_1(s) \\ &\geq \tfrac{3}{4}(\delta_1 s + \varepsilon_2 M) + \big(B_6(s) - (1+\delta_1) B_2(s)\big). \end{aligned} \tag{6.18}$$

It follows from (2.2) of Lemma 2.1 that

$$P\big(Z_6(s) \leq \tfrac{1}{2}\delta_1 s \text{ for some } s \in [j-1, j]\big) \leq C_1 e^{-\varepsilon' j} \tag{6.19}$$

for $j \geq M'$, and appropriate C_1 and $\varepsilon' > 0$. Also, by (2.3),

$$P\big(Z_6(s) \leq \tfrac{1}{2}\delta_1 s \text{ for some } s \leq M'\big) \leq C_1 e^{-\varepsilon' M} \tag{6.20}$$

for appropriate C_1 and $\varepsilon' > 0$. Proceeding similarly to (6.12)–(6.16), the inequalities (6.15) and (6.16) with $i = 4$ also hold.

On account of (6.9), (6.10), (6.15) and (6.16), for $i = 2, \ldots, 5$, it follows for large enough M that

$$E[Z_k(T_4(M)); G_M] \leq 5\varepsilon_{10} \qquad \text{for } k = 1, 6, \tag{6.21}$$

$$E[Z_k(T_4(M))^2; G_M] \leq 5\varepsilon_{10} \qquad \text{for } k = 2, \ldots, 5, \tag{6.22}$$

where ε_{10} is as in (6.9) and (6.10). This implies (4.23) and (4.24) for $\varepsilon_4 = 5\varepsilon_{10}$, and completes the proof of the proposition. □

The demonstration of Proposition 4.4 is based on the comparison between $T_3'(M, z)$ and $T_2(\sqrt{N_M(z)})$ in (6.23). This enables one to employ the upper bounds on $E[T_2(x)]$ and $P(A)$ from Propositions 3.2 and 3.3.

PROOF OF PROPOSITION 4.4. Let $T_3'(M, z)$ and $A_M(z)$ denote the analogs of $T_3'(M)$ and A_M, with $Z(0) = z$ being specified. Comparing $T_3'(M, z)$ with $T_2(x)$ in (3.9), for $x = \sqrt{N_M(z)}$, it is easy to see that

$$T_3'(M, z) \leq T_2\big(\sqrt{N_M(z)}\big). \tag{6.23}$$

It therefore follows from Proposition 3.2 that, for appropriate C_1,

$$E[T_3'(M, z)] \leq E\big[T_2\big(\sqrt{N_M(z)}\big)\big] \leq C_1 N_M(z). \tag{6.24}$$

Integrating (6.24) over z and applying (4.28), one obtains

$$\begin{aligned} E[T_3'(M)] &= E\big[E[T_3'(M)|Z(0)=z]\big] \le C_1 E[N_M(Z(0))] \\ &\le C_1\Big(E\Big[\max_{k=2,\dots,5} Z_k(0)^2\Big] + M\Big) \le C_1(5D_1+1)M. \end{aligned} \tag{6.25}$$

This implies (4.29) with $C_2 = C_1(5D_1+1)$.

Since the truncated values $T_6 \wedge 5N_M(z)^{5/\eta}$ and $T_6 \wedge 5x^{5/\eta}$ in (4.25) and (3.9) are equal, it is easy to check that

$$A_M(z) \subseteq A \tag{6.26}$$

for given z, where A is the event in (3.11) with $x = \sqrt{N_M(z)}$. It therefore follows from Proposition 3.3 that, for appropriate C_1,

$$P(A_M(z)) \le P(A) \le C_1 N_M(z)^{-\frac{5}{2\eta}-1}. \tag{6.27}$$

Together with (2.21), (4.27) and (6.27) imply that, for given z,

$$E[Z_k(T_3'(M))^2; A_M(Z(0))|Z(0)=z] \le C_2/\sqrt{N_M(z)} \le C_2/\sqrt{M} \tag{6.28}$$

for $k = 2,\dots,5$ and appropriate C_2. Similarly, by (2.22), (4.27) and (6.27),

$$E[Z_k(T_3'(M)); A_M(Z(0))|Z(0)=z] \le C_3/N_M(z) \le C_3/M \tag{6.29}$$

for $k = 1, 6$ and appropriate C_3. Integrating (6.28) and (6.29) over z produces (4.30) and (4.31). □

We now demonstrate Proposition 4.5. We first show (4.37) and (4.38), which are then used to show (4.39) and (4.40). On account of the upper bounds on z_1 and z_6 in (4.36), $Z_1(\cdot)$ will drift toward 0 so that $Z_1(t) = \varepsilon_5 M_1$ typically occurs before time $D_4 M_1$. This will imply (4.37). Since $Y_1(T_5(M_1)) = 0$, the coordinates $k = 2,\dots,6$ drift toward 0 over $[0, T_5(M_1)]$, implying (4.38). The elapsed time between $T_5(M_1)$ and $T_5'(M_1)$ is short enough so $Z_k(T_5'(M_1))$, $k = 1, 6$, will typically still be small, and so (4.39) will hold. It is also short enough so $Y_1(T_5'(M_1)) = 0$ and long enough for $Z_k(t) = 0$, $k = 2,\dots,5$, to typically occur, from which (4.40) will follow.

PROOF OF PROPOSITION 4.5. We first demonstrate (4.37). To do so, we analyze $Z(t)$ when $Y_1(t) = 0$, for given $t \ge 0$. By (1.1),

$$Z_1(t) = Z_1(0) + B_1(t) - t + (1+\delta_2)\sum_{k=1}^{6} Y_k(t) - (\delta_2+\delta_4)Y_6(t). \tag{6.30}$$

Since $Y_1(t) = 0$, it follows from (2.32) of Lemma 2.6 and (4.36) that

$$Y_6(t) \ge \frac{1}{5}t - \frac{D_5}{\delta_3}M_1 \tag{6.31}$$

off of a set of exponentially small probability in M_1. Together, (1.14), (4.36), (6.30) and (6.31) imply that

$$Z_1(t) \le \frac{2\delta_4 D_5}{\delta_3} M_1 + B_1(t) - \left(1 + \frac{\delta_2 + \delta_4}{5}\right)t + (1+\delta_2)\sum_{k=2}^{6} Y_k(t). \tag{6.32}$$

Now, by (2.31) of Lemma 2.6, one has that, for given $\varepsilon > 0$,

$$\begin{aligned} -\left(1 + \frac{\delta_2 + \delta_4}{5}\right)t + (1+\delta_2)\sum_{k=2}^{6} Y_k(t) \\ \le \left[\frac{(1+\varepsilon)(1+\delta_2)}{1-\delta_3} - 1 - \frac{\delta_2+\delta_4}{5}\right]t \end{aligned} \tag{6.33}$$

off of a set of exponentially small probability in M_1. One can check that, because of (1.14) and $\delta_3 \le \frac{1}{10}$, the right-hand side of (6.33) is less than $-(\frac{1}{5}\delta_2 + \varepsilon)t$ for ε chosen small enough. Combining (6.32), (6.33) and applying (2.2) of Lemma 2.1, one therefore obtains

$$Z_1(t) < \frac{2\delta_4 D_5}{\delta_3} M_1 - \frac{1}{5}\delta_2 t \tag{6.34}$$

off of a set of exponentially small probability in M_1, provided $Y_1(t) = 0$. But, since $Z_1(t) \ge 0$, it follows from (6.34) that, off this set, $Y_1(D_4 M_1) > 0$ for $D_4 = (10 D_5 \delta_4/\delta_2\delta_3)M_1$. So, $Z_1(t) = 0$ for some $t \le D_4 M_1$, which implies (4.37).

Set $\tau_k = \min\{t : Z_k(t) = 0\}$ for $k = 2, \dots, 6$. In order to demonstrate (4.38), we show that

$$\tau_k \le T_5(M_1) \qquad \text{for } k = 2, \dots, 6, \tag{6.35}$$

off of a set of exponentially small probability in M_1. Inequality (4.38) then follows from (4.37) and (2.14) of Lemma 2.4 for a small enough choice of $\varepsilon > 0$.

We claim that, off of a set of exponentially small probability in M_1,

$$\tau_6 < T_5(M_1). \tag{6.36}$$

To see this, note that, when $Y_6(t) = 0$, it follows from (6.30), (2.6) of Lemma 2.2, (4.35) and (4.36) and $\varepsilon_7 \ge 3\varepsilon_6$ that

$$Z_1(t) \ge \tfrac{1}{3}\varepsilon_7 M_1 + \delta_2 t + \big(B_1(t) - (1+\delta_2)B_2(t)\big). \tag{6.37}$$

Since $\varepsilon_7 \ge 6\varepsilon_5$, on account of (2.3) of Lemma 2.1, this is greater than $\varepsilon_5 M_1$ off of a set of exponentially small probability in M_1. Together with (4.37), this implies the claim.

Also note that, for each $k = 2, \dots, 5$, it follows from (1.1) and (4.35) and (4.36) that

$$Z_k(t) - Z_6(t) \le B_k(t) - B_6(t) - \tfrac{2}{3}\varepsilon_7 M_1$$

on $Y_k(t)=0$. Together with (2.2) of Lemma 2.1, this implies

$$\tau_6 \ge \tau_k \wedge D_4 M_1 \ge \tau_k \wedge T_5(M_1) \qquad \text{for } k=2,\dots,5, \tag{6.38}$$

off of a set of exponentially small probability in M_1. Together with (6.36), this implies (6.35).

In order to demonstrate (4.39) for $k=1$ and $k=6$, we restart $Z(\cdot)$ at time $T_5(M_1)$ and then apply (2.12) of Lemma 2.3. Using the definition of $T_5(M_1)$, for $k=1$, and (4.38), with $\varepsilon_8 \stackrel{\text{def}}{=} \frac{1}{20}\delta_3\varepsilon_5 \le \varepsilon_5$, for $k=6$, one obtains

$$P\big(Z_k(T_5'(M_1)) \ge 5\varepsilon_5 M_1\big) \le C_1 e^{-\varepsilon' M_1} \qquad \text{for } k=1,6, \tag{6.39}$$

for appropriate C_1 and $\varepsilon' > 0$. Inequality (4.39) then follows from (2.22) of Lemma 2.5.

In order to demonstrate (4.40), we restart $Z(\cdot)$ at time $T_5(M_1)$, denoting the new process by $\tilde{Z}(\cdot)$ and the corresponding Brownian motion by $\tilde{B}(\cdot)$. By (1.1),

$$\tilde{Z}_1(t) \ge \varepsilon_5 M_1 - t + \tilde{B}_1(t),$$

and so $\tilde{Z}_1(t) > 0$ on $[0, \frac{1}{2}\varepsilon_5 M_1]$ off of a set of exponentially small probability in M_1. Since $\varepsilon_8 < \frac{1}{10}\delta_3\varepsilon_5$, it follows, by (2.32) of Lemma 2.6 and (4.38), that

$$\tilde{Y}_k\left(\frac{1}{2}\varepsilon_5 M_1\right) \ge \left(\frac{1}{10}\varepsilon_5 - \frac{\varepsilon_8}{\delta_3}\right) M_1 > 0 \qquad \text{for } k=2,\dots,5, \tag{6.40}$$

off of a set F_{10} of exponentially small probability in M_1.

We denote by $\tilde{\tau}_k$ the first time at which $\tilde{Z}_k(t)=0$. Restarting the process at $\tilde{\tau}_k$, it follows from (2.15) of Lemma 2.4 that

$$E\big[\tilde{Z}_k\big(\tfrac{1}{2}\varepsilon_5 M_1\big)^2; F_{10}^c\big] \le 24 \cdot 8\varepsilon_5 M_1 \qquad \text{for } k=2,\dots,5.$$

On account of the upper bounds on $P(F_{10})$ and (2.21) of Lemma 2.5, one obtains

$$E\big[\tilde{Z}_k\big(\tfrac{1}{2}\varepsilon_5 M_1\big)^2\big] \le 24 \cdot 8\varepsilon_5 M_1 + C_4 \qquad \text{for } k=2,\dots,5 \tag{6.41}$$

for appropriate C_4, which depends on ε_5 but not on M_1. This implies (4.40). □

REFERENCES

[1] BRAMSON, M. (2008). *Stability of Queueing Networks*. *Lecture Notes in Math*. **1950**. Springer, Berlin. MR2445100

[2] BRAMSON, M., DAI, J. G. and HARRISON, J. M. (2010). Positive recurrence of reflecting Brownian motion in three dimensions. *Ann. Appl. Probab.* **20** 753–783. MR2650048

[3] BRAMSON, M. and GRIFFEATH, D. (1991). Capture problems for coupled random walks. In *Random Walks*, *Brownian Motion*, *and Interacting Particle Systems*. *Progress in Probability* **28** 153–188. Birkhäuser, Boston, MA. MR1146445

[4] DAI, J. G. and HARRISON, J. M. (2010). Reflecting Brownian motion in three dimensions: A new proof of sufficient conditions for positive recurrence. *Math. Meth. Oper. Res.* To appear.

[5] DEBLASSIE, R. D. (1987). Exit times from cones in $\mathbb{R}^n$ of Brownian motion. *Probab. Theory Related Fields* **74** 1–29. MR0863716

[6] DUPUIS, P. and WILLIAMS, R. J. (1994). Lyapunov functions for semimartingale reflecting Brownian motions. *Ann. Probab.* **22** 680–702. MR1288127

[7] EL KHARROUBI, A., BEN TAHAR, A. and YAACOUBI, A. (2000). Sur la récurrence positive du mouvement brownien réflechi dans l'orthant positif de $\mathbb{R}^n$. *Stochastics Stochastics Rep.* **68** 229–253. MR1746181

[8] EL KHARROUBI, A., BEN TAHAR, A. and YAACOUBI, A. (2002). On the stability of the linear Skorohod problem in an orthant. *Math. Methods Oper. Res.* **56** 243–258. MR1938213

[9] FOSS, S. S. and KONSTANTOPOULOS, T. (2004). An overview of some stochastic stability methods. *J. Oper. Res. Soc. Japan* **47** 275–303. MR2174067

[10] HARRISON, J. M. and NGUYEN, V. (1993). Brownian models of multiclass queueing networks: Current status and open problems. *Queueing Syst.* **13** 5–40. MR1218842

[11] HARRISON, J. M. and WILLIAMS, R. J. (1987). Brownian models of open queueing networks with homogeneous customer populations. *Stochastics* **22** 77–115. MR0912049

[12] LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Spaces: Isoperimetry and Processes. Ergebnisse der Mathematik und Ihrer Grenzgebiete* (3) [*Results in Mathematics and Related Areas* (3)] **23**. Springer, Berlin. MR1102015

[13] LI, W. V. (1999). A Gaussian correlation inequality and its applications to small ball probabilities. *Electron. Comm. Probab.* **4** 111–118 (electronic). MR1741737

[14] LI, W. V. and SHAO, Q.-M. (2001). Capture time of Brownian pursuits. *Probab. Theory Related Fields* **121** 30–48. MR1857107

[15] RATZKIN, J. and TREIBERGS, A. (2009). A capture problem in Brownian motion and eigenvalues of spherical domains. *Trans. Amer. Math. Soc.* **361** 391–405. MR2439411

[16] TAYLOR, L. M. and WILLIAMS, R. J. (1993). Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant. *Probab. Theory Related Fields* **96** 283–317. MR1231926

[17] WILLIAMS, R. J. (1995). Semimartingale reflecting Brownian motions in the orthant. In *Stochastic Networks. The IMA Volumes in Mathematics and its Applications* **71** 125–137. Springer, New York. MR1381009

[18] WILLIAMS, R. J. (1996). On the approximations of queueing networks in heavy traffic. In *Stochastic Networks: Theory and Applications* (F. P. Kelly, S. Zachary and I. Ziedins, eds.). Oxford Univ. Press.

UNIVERSITY OF MINNESOTA
TWIN CITIES CAMPUS
SCHOOL OF MATHEMATICS
INSTITUTE OF TECHNOLOGY
127 VINCENT HALL
206 CHURCH STREET S.E.
MINNEAPOLIS, MINNESOTA 55455
USA
E-MAIL: bramson@math.umn.edu