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GENERALIZED INTEGRANDS AND BOND PORTFOLIOS: PITFALLS AND COUNTER EXAMPLES

BY ERIK TAFLIN[1]

EISTI

We construct Zero-Coupon Bond markets driven by a cylindrical Brownian motion in which the notion of generalized portfolio has important flaws: There exist bounded smooth random variables with generalized hedging portfolios for which the price of their risky part is $+\infty$ at each time. For these generalized portfolios, sequences of the prices of the risky part of approximating portfolios can be made to converges to any given extended real number in $[-\infty, \infty]$.

1. Introduction. In this article, we consider continuous time bond markets for which there exists a unique equivalent martingale measure (e.m.m.). It is well-known that the uniqueness of the e.m.m. does not in general imply that such a market is complete. We have here adopted the standard definition of complete market, which we shall call L^∞*-completeness* and which reads, omitting details:

Every random variable X in L^∞ is replicable by an admissible H'-valued self-financed portfolio process θ, where H' is the dual of H, the state space of the price process.

To our knowledge, such *noncompleteness results* were first established in [1] and [2] (see Proposition 4.7 of [1] and Proposition 6.9 of [2]). The considered price model was a jump-diffusion model with a finite dimensional Brownian motion (B.m.) and an infinite mark space, and H was the sup normed Banach space of continuous functions on $[0, \infty[$ with vanishing limit at ∞. It was also proved that this market is *approximately complete,* that is, the subspace of replicable random variables is dense in L^∞, if and only if the e.m.m. is unique (see Proposition 6.10 and Theorem 6.11 of [2]).

Similar results were proved in [11] (see Theorem 4.1, Theorem 4.2 and Remark 4.6 of [11]) for the case of price models introduced in [7], where the price is a H-valued process driven by a standard cylindrical B.m. (cf. [4]) and where H is a Sobolev space of continuous functions. Also various topological vector spaces A for which these markets are A-complete (change L^∞ to A in the above definition)

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[1]Chair in Mathematical Finance EISTI, and Laboratory AGM (UMR 8088-CNRS), 95000 Cergy, France.

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were specified in Theorem 4.3 of [11]. Hedging in the case of a Markovian price model was considered in [3].

The notion of admissible portfolio was weakened in [6] to that of *generalized self-financed bond portfolio*, which reestablished, for very general price processes having a unique e.m.m., the L^∞-*completeness* of the market, but with H' in the above definition replaced by the larger set $\mathcal{U}$ of bounded and unbounded linear forms in H (see the discussion in Section 4 of [6] and see also [5]).

The aim of the present paper is to study and establish properties of generalized self-financed bond portfolios. In particular, we are interested in the price of the risky part (or equivalently, in the price of the risk-free part) of generalized bond portfolios, for which the separation into risk-free and risky part makes sense. To this end, simple price models driven by a standard cylindrical B.m., of the kind introduced in [7] and [8] and with constant volatility operator, are considered. It is proved that the price model can be chosen such that some generalized self-financed bond portfolio will have properties to be handled with care and which can even limit the mathematical and practical usefulness of generalized portfolios. In fact, (see Theorem 3.2):

(a) There exist bounded smooth random variables, hedgeable in the sense of [6] by a unique generalized self-financed bond portfolio (x, μ), whose risky part μ^1 is unique and is a positive C^∞ density. The price of μ^1 is $+\infty$ at each time. Equivalently, it requires to hold a loan of infinite amount, at each time.

(b) For all "admissible" utility functions, there exists a unique well-defined optimal wealth $\hat{X}$, solution of the optimal expected utility problem. $\hat{X}$ is hedgeable in the sense of [6] by a a unique generalized portfolio (x, μ). Also here this generalized portfolio requires to hold a loan of infinite amount, at each time.

(c) In each one of the cases (a) and (b), approximate portfolios converging to (x, μ) can be chosen such that the sequence of the prices of their risky part converges to any given extended real number in $[-\infty, \infty]$.

Theorem 3.2 gives counter examples to some statements in [6] (see Remark 3.3). Results analogous to those of this paper should apply to other infinite dimensional markets, as in [9].

The present article is a motivation for future research on the hedging problem in bond markets treated as a super-replication problem under constraints instead of replication by "standard" or generalized portfolios.

2. Mathematical set-up and the market model. We shall use a simple case of the Hilbert space Zero-Coupon Bond models of [7] and [8]. The Zero-Coupon Bond price curves belong to a Hilbert space H, of continuous functions on $[0, \infty[$. In this paper, we choose $H = H^1([0, \infty[)$, the Sobolev space of order 1 of real-valued functions on $[0, \infty[$. Let $\mathcal{L}$ be the contraction semi-group of left translations in $L^2([0, \infty[)$, let ∂ be its infinitesimal generator and for a positive integer $n \geq 0$

let $H^n([0,\infty[)$ be the subspace of functions f such that $[0,\infty[\ni a \mapsto \mathcal{L}_a f \in L^2([0,\infty[)$ is n-times continuously differentiable. $H^n([0,\infty[)$ is a Hilbert space for the norm defined by

$$\|f\|_{H^n} = \left(\int_0^\infty \sum_{i=0}^{n} |\partial^i f(x)|^2\,dx\right)^{1/2} \tag{2.1}$$

and $\mathcal{L}$ (restricted to $H^n([0,\infty[)$) is a contraction semi-group in $H^n([0,\infty[)$. Pointwise multiplication $H^n([0,\infty[) \times H^n([0,\infty[) \ni (f,g) \mapsto fg \in H^n([0,\infty[)$ is continuous for $n \geq 1$.

A real-valued bi-linear form $\langle\cdot,\cdot\rangle$, where

$$\langle f, g\rangle = \int_0^\infty f(x)g(x)\,dx, \tag{2.2}$$

is first defined for (real) tempered distributions f with support contained in $[0,\infty[$ and for (real) tempered test functions g on $\mathbb{R}$. $H^{-n}([0,\infty[)$ is the subset of all such f, for which the mapping $g \mapsto \langle f, g\rangle$ has a continuous extension to $H^n([0,\infty[)$. The dual $(H^n([0,\infty[))'$ of $H^n([0,\infty[)$ is identified with $H^{-n}([0,\infty[)$ and we write $H' = (H^1([0,\infty[))'$.

We consider a time interval $\mathbb{T} = [0,\bar{T}]$, where $\bar{T} > 0$ is a finite time-horizon. The random source is an infinite dimensional ℓ^2-cylindrical Brownian motion $W = (W^1,\ldots,W^n,\ldots)$ on a complete filtered probability space $(\Omega, P, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}})$, where $\mathcal{F} = \mathcal{F}_{\bar{T}}$ and the filtration is generated by the independent Brownian motions W^n, $n \geq 1$.

The price at time $t \in \mathbb{T}$ of a Zero-Coupon Bond with time to maturity $x \geq 0$ is denoted $\tilde{p}_t(x)$ and the corresponding discounted price $p_t(x)$. By convention, $\tilde{p}_t(0) = 1$.

In this paper, we shall use a time independent volatility operator $\sigma \in L_2(\ell^2, H^2)$, the space of Hilbert–Schmidt operators from ℓ^2 to $H^2([0,\infty[)$. For $z \in \ell^2$, $\sigma z = \sum_{i\geq 1}\sigma^i z^i$, where the functions σ^i satisfy $\sigma^i(0) = 0$. Moreover, we impose that $\sigma^i \in C^\infty([0,\infty[)$, that σ^i has compact support, that the set $\{\sigma^1,\ldots,\sigma^i,\ldots\}$ is linearly independent and total in the subspace of functions $f \in H^1([0,\infty[)$ satisfying $f(0) = 0$. In particular, it follows that σ is injective.

A drift function m is given such that $m = \sigma\gamma$, for a time independent market price of risk $\gamma \in \ell^2$. In particular, it follows that $m \in H^2([0,\infty[)$, since $\sigma \in L_2(\ell^2, H^2)$, and that $m(0) = 0$.

The discounted price p is a continuous H-valued process satisfying

$$p_t = \mathcal{L}_t p_0 + \int_0^t \mathcal{L}_{t-s}(p_s m)\,ds + \int_0^t \mathcal{L}_{t-s}(p_s\sigma)\,dW_s, \tag{2.3}$$

where $p_0 \in H^2([0,\infty[)$ is a strictly positive function with $p_0(0) = 1$. Here, the notations of pointwise multiplication are used, so explicitly for the integrand in the second integral: $(\mathcal{L}_a(p_s\sigma)z)(x) = \sum_{i\geq 1} p_s(x+a)\sigma^i(x+a)z^i$ for all $z \in \ell^2$ and $a, x \geq 0$.

Equation (2.3) has a unique H-valued mild solution p (see [7] and [8] for properties of the solution of (2.3)). This solution is a strong solution and it satisfies the following equation in H, which shows that p is a H-valued semi-martingale:

$$dp_t = (\partial p_t + p_t m)\,dt + p_t\sigma\,dW_t. \tag{2.4}$$

For later reference, we note that it follows from Theorem 2.2 of [7], that the mapping $[0,\infty[\ni x\mapsto p(x)$ is a continuous mapping into the space of real semimartingales $S(P)$ endowed with the semimartingale topology, cf. [10].

A portfolio, also called "standard portfolio" in this paper, is an H'-valued progressively measurable process θ defined on $\mathbb{T}$. If θ is a portfolio, then its discounted value at time t is

$$V_t(\theta) = \langle\theta_t, p_t\rangle. \tag{2.5}$$

θ is an *admissible portfolio* if[2]

$$\|\theta\|_{\mathsf{P}}^2 = E\biggl(\int_0^{\bar T}(\|\theta_t\|_{H'}^2 + \|\sigma'\theta_t p_t\|_{\ell^2}^2)\,dt + \biggl(\int_0^{\bar T}|\langle\theta_t, p_t m\rangle|\,dt\biggr)^2\biggr) < \infty, \tag{2.6}$$

where σ' is the adjoint of σ defined by $\langle f, \sigma x\rangle = (\sigma' f, x)_{\ell^2}$, for all $f\in H'$ and $x\in\ell^2$. Explicitly, we have:

$$\sigma' f = (\langle f,\sigma^1\rangle,\dots,\langle f,\sigma^i\rangle,\dots). \tag{2.7}$$

The set of all admissible portfolios defines a Banach space P for the norm $\|\cdot\|_{\mathsf{P}}$. A portfolio $\theta\in\mathsf{P}$ is by definition *self-financed* if

$$dV_t(\theta) = \langle\theta_t, p_t m\rangle\,dt + \sum_{i\in\mathbb{N}^*}\langle\theta_t, p_t\sigma^i\rangle\,dW_t^i. \tag{2.8}$$

There is a unique e.m.m. (equivalent martingale measure) Q. It is given by $dQ/dP = \xi_{\bar T}$, where

$$\xi_t = \exp\bigl((\gamma, W_t)_{\ell^2} - \tfrac{1}{2}\|\gamma\|_{\ell^2}^2 t\bigr).$$

By Girsanov's theorem the $\bar W^i$, $i\ge 1$, where $\bar W_t^i = W_t^i + \gamma^i t$, are independent Q-B.m. Obviously,

$$p_t = \mathcal{L}_t p_0 + \int_0^t \mathcal{L}_{t-s}(p_s\sigma)\,d\bar W_s. \tag{2.9}$$

We shall only consider derivative products with *discounted pay-off* belonging to the (Fréchet) space D_0, which by definition is the intersection of all the spaces

[2] In this paper, all considered admissible portfolios will also satisfy $V_t(\theta)\ge C$ a.e. (t,ω) for some $C\in\mathbb{R}$ depending on θ.

$L^p(\Omega, Q, \mathcal{F})$, $1 \le p < \infty$. Such a derivative X has a unique decomposition as a stochastic integral w.r.t. $\bar{W}$ (cf. [4] and Lemma 3.2 of [11])

$$X = E_Q[X] + \int_0^{\bar{T}} (x_t, d\bar{W}_t), \tag{2.10}$$

where x is a progressively measurable ℓ^2-valued process satisfying

$$x \in L^p(\Omega, Q, L^2(\mathbb{T}, \ell^2)), \qquad 1 \le p < \infty. \tag{2.11}$$

It is important to have information about the decay properties of x_t^n for large n, to study hedging properties of X. We therefore also introduce the spaces of derivative products D_s, $s > 0$. D_s is the subspace of all $X \in \mathsf{D}_0$ such that the integrand x in (2.10) satisfies

$$\left(\int_0^{\bar{T}} \|x_t\|^2_{\ell^{s,2}}\, dt\right)^{1/2} \in \mathsf{D}_0, \tag{2.12}$$

where $\ell^{s,2} \equiv \ell^{s,2}(q)$ is the Hilbert space of real sequences endowed with the norm

$$\|y\|_{\ell^{s,2}} = \left(\sum_{i \in \mathbb{N}^*} q_i^{2s} |y^i|^2\right)^{1/2}, \tag{2.13}$$

where $q_i \ge 1$ is a given increasing unbounded sequence of real numbers. (See Remark 4.7 of [11], where $q_i = (1 + i^2)^{1/2}$ was used.)

Later we shall also impose X to be smooth in the sense of Malliavin.

A *hedging portfolio* θ of X is a self-financed portfolio $\theta \in \mathsf{P}$ such that $\langle \theta_{\bar{T}}, p_{\bar{T}} \rangle = X$, which then is called replicable.

Bounded and smooth X are not always replicable, see Remark 4.6 and Theorem 4.1 of [11]. By the definition of self-financed portfolio, it follows that a portfolio $\theta \in \mathsf{P}$ is a hedging portfolio of X satisfying (2.11) iff $\forall t \in \mathbb{T}$ and $i \ge 1$

$$\langle \theta_t, p_t \rangle = E_Q[X | \mathcal{F}_t], \qquad \langle \theta_t, p_t \sigma^i \rangle = x_t^i, \tag{2.14}$$

where x is given by formula (2.10). When it exists, the solution $\theta \in \mathsf{P}$ is unique. In fact, if θ and ϕ are two solutions, then the second formula in (2.14) gives that $\langle \theta_t - \phi_t, p_t \sigma^i \rangle = 0$, for all $i \ge 1$. Since the set of the σ^i is total in the subspace of functions vanishing at 0 in $H^1([0, \infty[)$, it follows that $\theta_t - \phi_t = b_t \delta_0$ for some real process b, where $\langle \delta_a, f \rangle = f(a)$, for $a \ge 0$. The first formula of (2.14) then gives that $0 = \langle \theta_t - \phi_t, p_t \rangle = b_t p_t(0)$. So $b_t = 0$, since $p_t(0) > 0$.

A self-financed discounted risk-free investment, with discounted value $V_t(\theta) = 1$, is realized by the portfolio

$$\theta_t = \exp\left(\int_0^t R_s(0)\, ds\right) \delta_0, \tag{2.15}$$

where the instantaneous forward rate $R_t(x)$ at t for time to maturity x is defined by

$$R_t(x) = -(\partial \ln p_t)(x).$$

In certain cases a portfolio θ can be separated into a risk-free part and a risky part. This is the case when 0 is not in the singular support of θ or is an isolated point in the singular support a.e. (t, ω). Then θ has a unique decomposition into a risk-free part ψ^0 and a risky part ψ^1, such that

$$\theta = \psi^0 + \psi^1, \qquad \psi_t^0 = a_t \delta_0, \qquad \operatorname{sing\,supp} \psi_t^1 \subset]0, \infty[, \tag{2.16}$$

where a is progressively measurable real-valued process. So here $\langle \psi_t^0, p_t \rangle = a_t p_t(0)$ and $\langle \psi_t^1, p_t \rangle$ are respectively the discounted risk-free and risky investments at t corresponding to θ.

The notion of *generalized bond portfolio* was introduced in ref. [6], in an attempt to circumvent the problem of the existence of bond markets with a unique e.m.m., but which are not complete in the sense that every sufficiently integrable r.v. is replicable (by a self-financed admissible bond portfolio).

Let the product-space $\mathbb{R}^{\mathbb{R}_+}$ be given its natural product-topology and let $\mathcal{U}$ be the set of all (bounded and unbounded) linear forms on $\mathbb{R}^{\mathbb{R}_+}$. Each element $l \in \mathcal{U}$ is defined by its domain $\mathcal{D}(l)$ and its values $l(f)$ for $f \in \mathcal{D}(l)$. Adapted to our mathematical set-up, a *generalized self-financed bond portfolio* (see Definition 3.1 of [6]) is a pair (x, μ), where x is a real number (the value of the generalized portfolio at $t = 0$) and where μ is a generalized integrand in the sense that μ is a $\mathcal{U}$-valued weakly predictable process and there exist simple integrands $\mu^{(n)}$, that is, $\mu^{(n)} = \sum_i h^{n,i} \delta_{x_{n,i}}$ where the sum is finite and $h^{n,i}$ are bounded predictable real processes, such that

(C_1) $\mu^{(n)}$ converges to μ a.s. in $\mathcal{U}$ (pointwise),

(C_2) the sequence Y^n, where $Y_t^n = \int_0^s \sum_i \langle \mu_s^{(n)}, p_t \sigma^i \rangle \, d\bar{W}_s^i$ converges to a limit process $Y \in S(P)$. Y_t is also denoted

$$Y_t = \int_0^t \sum_i \langle \mu_s, p_s \sigma^i \rangle \, d\bar{W}_s^i. \tag{2.17}$$

The limit Y of Y^n is independent of the sequence $(\mu^{(n)})_{n \geq 1}$. We recall that, more generally (see Theorem 2.4 of [6]), if $\mu^{(n)}$ is a sequence of generalized integrands satisfying (C_2) then there exists a generalized integrand μ such that equality (2.17) is satisfied.

The discounted value process of the generalized portfolio (x, μ) is by definition $x + Y$. For every $x \in \mathbb{R}$ and portfolio $\mu \in \mathsf{P}$, (x, μ) is a generalized self-financed bond portfolio. A generalized self-financed bond portfolio (x, μ) is called *generalized hedging portfolio* of X when

$$X = x + \int_0^{\bar{T}} \sum_i \langle \mu_t, p_t \sigma^i \rangle \, d\bar{W}_t^i. \tag{2.18}$$

3. Main results. A natural question is what are the sequences of risk-free and risky investments permitting to realize a sequence of approximations $(x,\mu^{(n)})$, satisfying (C_1) and (C_2), of a generalized self-financed bond portfolio (x,μ). What are the limits of these sequences, if they exist, and are they independent of the choice of the approximating sequence? More precisely and generally (cf. Theorem 2.4 of [6]), let $(\mu^{(n)})_{n\geq 1}$ be a sequence of integrands in P (i.e., portfolios) satisfying (C_1) and (C_2), with the corresponding sequence $(Y^n)_{n\geq 1}$. Self-financed portfolios $\theta^{(n)} \in \mathsf{P}$ are then defined by (cf. Proposition 2.5 of [7])

$$\theta^{(n)} = b^n\delta_0 + \mu^{(n)}, \qquad b_t^n = \big(x + Y_t^n - \langle \mu_t^{(n)}, p_t\rangle\big)/p_t(0). \tag{3.1}$$

If the decomposition (2.16) applies to the portfolios $\mu^{(n)}$, with risky part $\mu^{(n)1}$, then it follows that the self-financed portfolio $\theta^{(n)}$ has a unique decomposition

$$\begin{aligned} &\theta^{(n)} = \theta^{(n)0} + \theta^{(n)1}, \\ &\theta_t^{(n)0} = a_t^n\delta_0, \theta^{(n)1} = \mu^{(n)1}, \qquad \operatorname{sing\,supp} \theta_t^{(n)1} \subset]0,\infty[. \end{aligned} \tag{3.2}$$

The real-valued process a^n, which is the investment in the risk-free asset, is then given by

$$a_t^n = \big(x + Y_t^n - \langle \mu_t^{(n)1}, p_t\rangle\big)/p_t(0). \tag{3.3}$$

We will come back later to the above questions concerning the possible limits of the sequence a_t^n of risk-free investments, by studying the sequence $r_t^n = \langle \mu_t^{(n)1}, p_t\rangle$ of discounted risky investments.

Another natural question is what risk-free and risky investments are required to realize the generalized self-financed portfolio (x,μ). Suppose that the risky part, lets call it μ^1, is well-defined. Then if μ_t^1 is (a.s.) a positive density, its discounted value $\langle \mu_t^1, p_t\rangle \in [0,\infty]$ (a.s.) is well-defined. This can easily be generalized to the case where the limit of $\int_0^x \mu_t^1(y)p_t(y)$ as $x\to\infty$ makes sense. In these cases, the risk-free investment is obtained as in (3.3)

$$a_t = (x + Y_t - \langle \mu_t^1, p_t\rangle)/p_t(0).$$

We shall construct a bond market and generalized self-financed bond portfolios (x,μ), whose realization require an infinite short position in the risk-free asset (i.e., loan) at each instant $t\in\mathbb{T}$. More precisely, to have a clear separation between the investment into the risk-free asset and the risky assets, we construct a market and generalized portfolios (x,μ) satisfying:

(P_1) ν is an element in $\mathcal{U}$ with domain (ls denotes linear span)

$$\mathcal{D}(\nu) = \operatorname{ls}\big(C_0^\infty(]0,\infty[) \cup \{p_0\}\big). \tag{3.4}$$

The restriction of ν to $C_0^\infty(]0,\infty[)$ is a function $\nu^1 \in C^\infty([0,\infty[)$, $\operatorname{supp}\nu^1 \subset [3/4,\infty[$, $\langle \nu, p_0\rangle = 0$ and

$$\lim_{x\to\infty}\int_0^x \nu^1(y)p_0(y)\,dy = \infty. \tag{3.5}$$

$\mu_t \in \mathcal{U}$ a.s. has domain

$$\mathcal{D}(\mu_t) = \mathrm{ls}\big(C_0^\infty(]0,\infty[) \cup \{p_t\}\big) \tag{3.6}$$

and

$$\langle \mu_t, f\rangle = \alpha_t \left\langle \nu, f\frac{p_0}{p_t}\right\rangle, \qquad f \in \mathcal{D}(\mu_t), \tag{3.7}$$

where α is a strictly positive continuous adapted uniformly bounded (in t and ω) process. The discounted total risky investment is

$$\lim_{x\to\infty} \int_0^x \mu_t^1(y) p_t(y)\,dy = \infty \qquad \text{a.e. } (t,\omega). \tag{3.8}$$

(P_2) For all $C \in [-\infty,\infty]$, μ is the limit in the sense of (C_1) and (C_2) of a sequence $(\mu^{(n)})_{n\geq 1}$ of continuous linear functionals on H such that, a.s.

$$\begin{aligned} &\mu^{(n)} = \mu^{(n)0} + \mu^{(n)1}, \qquad \forall t \in \mathbb{T} \quad \operatorname{supp} \mu_t^{(n)0} \subset \{0\},\\ &\mu_t^{(n)1} \in C_0^\infty(]1/2,\infty[) \end{aligned} \tag{3.9}$$

and

$$\lim_{n\to\infty} a_0^n = C, \tag{3.10}$$

where a^n is defined by (3.3). Moreover, if $C = -\infty$ (resp. C is finite and $C = +\infty$) then

$$\forall t \in \mathbb{T}, \qquad \lim_{n\to\infty} a_t^n = -\infty \qquad (\text{resp. finite and } +\infty). \tag{3.11}$$

REMARK 3.1. The definition of ν makes sense since p_0 is not in $K = \{f \in H \mid f(0) = 0\}$, the closure of $C_0^\infty(]0,\infty[)$ in H. The formula (3.7) makes sense since, according to Theorem 21 of [8], $\|\ell_t/p_t\|_{L^\infty} < \infty$, where $\ell_t = \mathcal{L}_t p_0$. So $f\ell_t/p_t \in \mathcal{D}_0$ a.s.

An admissible utility function U is (in this article), a strictly concave and increasing C^2 function on $]0,\infty[$ satisfying conditions, stated in (3.12), strengthening the Inada conditions. Let I be the inverse of U' and assume that there exists $C, p > 0$ such that

$$\begin{aligned} &U'(]0,\infty[) =]0,\infty[\quad \text{and}\\ &|\mathrm{I}(x)| + |x\mathrm{I}'(x)| \leq C(x^p + x^{-p}), \qquad x > 0. \end{aligned} \tag{3.12}$$

We shall consider the optimal portfolio problem. For an admissible utility function U and an initial investment of $E_Q[\mathrm{I}(y\xi_{\bar{T}})]$, for given $y > 0$, the optimal final wealth is given by

$$\hat{X} = \mathrm{I}(y\xi_{\bar{T}}) \tag{3.13}$$

and $\hat{X} \in L^q$, for all $1 \leq q < \infty$, cf. Theorem 3.3 of [7].

We can now state the main results (in which risky means that 0 is not in the singular support).

THEOREM 3.2. *One can choose an initial condition p_0, a time-independent volatility operator σ and a time-independent drift function m such that*:

A. *The $\sigma^i \in C_0^\infty(]0, \infty[)$, $\sigma \in L_2(\ell^2, H^2([0, \infty[))$ is injective and $p_0(x) = e^{-ax}$, for some $a > 0$. The drift $m \in H^2([0, \infty[)$ and the market price of risk $\gamma \in \ell^2$.*

B. *For all admissible utility functions U and $y > 0$, $\hat{X}$ given by (3.13) has a generalized hedging portfolio $(E_Q[\hat{X}], \mu)$ satisfying (2.18) and with the properties* (P_1) *and* (P_2). *The risky part of $(E_Q[\hat{X}], \mu)$ is unique.*

C. *There exists a bounded smooth r.v. X having a generalized hedging portfolio $(E_Q[X], \mu)$ satisfying (2.18) and satisfying* (P_1) *and* (P_2) *with ν^1 positive. The risky part of $(E_Q[X], \mu)$ is unique and positive.*

REMARK 3.3. If (x, μ) is a generalized hedging portfolio given by Theorem 3.2 then:

1. Since (P_1) is satisfied it follows that the value of the risky part of (x, μ), is infinite and that the realization of (x, μ) requires an infinite short position in the risk-free asset (i.e., loan) at each instant $t \in \mathbb{T}$.
2. According to (P_2), the sequence of prices, at $t = 0$, r_0^n of the risky part (or a_0^n of the risk-free part) of approximating sequences $(x, \mu^{(n)})$ give no information concerning the value of the risky part (or of the risk-free part) of (x, μ). As matter of fact for the given (x, μ), one can choose an approximating sequence $(x, \mu^{(n)})$ such that the limit of a_0^n is equal to any extended real number in $[-\infty, \infty]$.
3. $p_t \in \mathcal{D}(\mu_t)$ [in fact $\langle \mu_t, p_t \rangle = 0$ a.e., according to (P_1)], which is a condition in a discussion in [6] (second paragraph after Definition 3.1). The preceding points 1 and 2 of this remark are counter examples the conclusions of that discussion.

4. Proofs. Following [11], we introduce for $t \in \mathbb{T}$, the operator $B_t = \ell_t \sigma \in L_2(\ell^2, H)$, where $\ell_t = \mathcal{L}_t p_0$. Here, B_t is deterministic. Let B_t^* be the adjoint of B_t with respect to the scalar product $(\cdot, \cdot)_H$ in H. We also introduce

$$A_t = B_t^* B_t, \tag{4.1}$$

which is a strictly positive self-adjoint trace-class operator in ℓ^2. We shall impose the following condition [to be verified after (4.27)] on the operators A_t: There exists $s > 0$ and $k > 0$ such that for all $t \in \mathbb{T}$ and $x \in \ell^2$,

$$\|x\|_{\ell^2} \leq k \|(A_t)^{1/2} x\|_{\ell^{s,2}}. \tag{4.2}$$

When this condition is satisfied, the contingent claims in D_s are replicable by self-financed portfolios in P (Theorem 4.3 of [11]).

Let $\mathcal{S}$ be the canonical isomorphism of H onto H' defined by

$$\forall f, g \in H, \qquad (f, g)_H = \langle \mathcal{S} f, g\rangle \tag{4.3}$$

and let S_t be the isometric embedding of ℓ^2 into H equal to the closure of $B_t(A_t)^{-1/2}$. We note that if $f \in H$ is C^2, then

$$\mathcal{S} f = f - \partial^2 f - (\partial f)(0)\delta_0. \tag{4.4}$$

If $X \in \mathsf{D}^s$, with $s > 0$ as in (4.2), then the equations (2.14) have a unique solution $\theta \in \mathsf{P}$ and $\theta = \theta^0 + \theta^1$, $\theta^0, \theta^1 \in \mathsf{P}$, where

$$\theta_t^1 = (l_t/p_t)\mathcal{S}\eta_t, \qquad \eta_t(\omega) = S_t(A_t)^{-1/2} x_t(\omega) \tag{4.5}$$

and

$$\theta_t^0 = b_t \delta_0, \qquad b_t = (E_Q[X|\mathcal{F}_t] - \langle \theta_t^1, p_t\rangle)/p_t(0). \tag{4.6}$$

For such X,

$$\langle \theta_t^1, p_t\rangle = (S_t(A_t)^{-1/2} x_t, l_t)_H. \tag{4.7}$$

We shall now construct the volatility operator σ and drift function m of Theorem 3.2. For a given $a > 0$, we define p_0 by

$$p_0(x) = \exp(-ax). \tag{4.8}$$

$C_0^\infty(]0, \infty[)$ is dense in the (closed) subspace K of functions $f \in H$, satisfying $f(0) = 0$. Let $h_1 \in C_0^\infty(]0, \infty[)$ be such that $h_1 \geq 0$, $\operatorname{supp} h_1 \subset [3/4, 5/4]$ and $\|h_1\|_H = 1$. $h_{2n-1} \in C_0^\infty(]0, \infty[)$, $n > 1$ is defined by $h_{2n-1}(x) = h_1(x - 2n + 2)$ if $x \geq 2n - 2$ and $h_n(x) = 0$ if $0 \leq x < 2n - 2$. The set of functions $\{h_{2n-1}\}_{n\geq 1}$ is orthonormal in K and

$$\operatorname{supp} h_n \subset \left[n - \tfrac{1}{4}, n + \tfrac{1}{4}\right], \qquad n \text{ odd}.$$

We complete it to an orthonormal basis $\{h_i\}_{i=1}^\infty \subset C_0^\infty(]0, \infty[)$ of K. Then $h_i/p_0 \in K$.

Let the volatility functions satisfy

$$\sigma^i = k_i h_i/p_0, \qquad k_i \neq 0, \qquad \text{s.t.} \quad \sum_{i\geq 1} i^2 k_i^2 (1 + \|h_i/p_0\|_{H^2}^2) < \infty. \tag{4.9}$$

The conditions $\sigma \in L_2(\ell^2, H^2)$ and $\sigma^i(0) = 0$ are then satisfied and the set $\{\sigma_i\}_{i=1}^\infty$ is by construction linearly independent and total in K.

The definition of B_t gives

$$B_t y = e^{-at} \sum_{i\geq 1} k_i h_i y_i \quad \text{and} \quad (B_t^* f)^i = e^{-at} k_i (h_i, f)_H. \tag{4.10}$$

It follows that

$$(A_t y)^i = e^{-2at} k_i^2 y_i \quad \text{and} \quad (A_t^{1/2} y)^i = e^{-at} |k_i| y_i. \tag{4.11}$$

It then follows that $(A_t)^{-1/2}$ and $(A_0)^{-1/2}$ have the same domain and that after closure

$$S_t y = \sum_i \operatorname{sgn}(k_i) h_i y_i, \qquad y \in \ell^2. \tag{4.12}$$

So for $y \in \mathcal{D}((A_0)^{-1/2})$

$$\begin{aligned} S_t(A_t)^{-1/2} y &= e^{at} \sum_{i\geq 1} \frac{1}{k_i} h_i y_i \quad \text{and} \\ \|S_t(A_t)^{-1/2} y\|_H^2 &= e^{2at} \sum_{i\geq 1} \left(\frac{y_i}{k_i}\right)^2. \end{aligned} \tag{4.13}$$

This gives for $y \in \mathcal{D}((A_0)^{-1/2})$:

$$(S_t(A_t)^{-1/2} y, \ell_t)_H = (S_0(A_0)^{-1/2} y, p_0)_H = \sum_{i\geq 1} \frac{1}{k_i} (h_i, p_0)_H y_i. \tag{4.14}$$

We define

$$m = \sigma\gamma, \qquad \gamma^i = \frac{1}{i} \quad \text{if } i \text{ odd} \quad \text{and} \quad \gamma^i = 0 \quad \text{if } i \text{ even}. \tag{4.15}$$

In the following lemma, H_c stands for the complex linear Hilbert space $H^1([0, \infty[, \mathbb{C})$. The function $[0, \infty[\ni x \mapsto e^{-ax}$ is in H_c for $\Re a > 0$.

LEMMA 4.1. *For every i the function $a \mapsto (h_i, e^{-a\cdot})_{H_c}$, $\Re a > 0$, extends to an entire analytic function on $\mathbb{C}$. There is only a countable number of $a \in \mathbb{C}$ such that*

$$(h_i, e^{-a\cdot})_{H_c} = 0 \qquad \textit{for some } i \geq 1. \tag{4.16}$$

PROOF. For $\Re a > 0$, $F(a) \in H_c$, where $(F(a))(x) = e^{-ax}$. With an obvious extension of $\langle \cdot, \cdot \rangle$ and recalling that h is real-valued, we have

$$\lambda_i(a) \equiv (h_i, F(a))_{H_c} = \langle \mathcal{S} h_i, F(a) \rangle.$$

According to (4.4), the distribution $\mathcal{S} h_i$ has compact support. The Fourier–Laplace transformation λ_i of $\mathcal{S} h_i$ therefore defines an entire analytic function in $\mathbb{C}$. Since $\mathcal{S} h_i \neq 0$, the set of zeros A_i of the function λ_i in $\mathbb{C}$ is countable. The set

$$A = \bigcup_{i\geq 1} A_i$$

is then countable, since it is a countable union of countable sets. A is the set of a that satisfies (4.16). $\square$

PROOF OF THEOREM 3.2. Obviously, p_0 and the σ^i are as stated in the theorem. By construction and $\gamma \in \ell^2$ according to (4.15), so statement A is true.

To prove the statement B, we follow Remark 4.6 of [11]. Since

$$\xi_t = \exp\big((\gamma, \bar{W}_t)_{\ell^2} + \tfrac{1}{2}\|\gamma\|_{\ell^2}^2 t\big),$$

$\hat{X}$ has the representation

$$\hat{X} = E_Q[\hat{X}] + \int_0^{\bar{T}} E_Q[y\xi_{\bar{T}} \mathrm{I}'(y\xi_{\bar{T}})|\mathcal{F}_t] \sum_{i\geq 1}(-\gamma^i)\, d\bar{W}_t^i.$$

Let $c = 1/\|\gamma\|_{\ell^2}$ [see (4.15)] and $e = c\gamma$, $Z = \sum_{i\geq 1} e^i \bar{W}_{\bar{T}}^i$. This proof is based on the fact that $e \notin \mathcal{D}((A_0)^{-1/2})$, according to (4.9) and (4.11). The real-valued function g is defined by

$$g(z) = -\frac{1}{c} h(z) I'(h(z)), \qquad h(z) = y \exp\big(z/c + \bar{T}/(2c^2)\big), \qquad z > 0,$$

g is strictly positive on $]0, \infty[$. Then

$$\hat{X} = E_Q[\hat{X}] + \int_0^{\bar{T}} \sum_{i\geq 1} x_t^i \, d\bar{W}_t^i, \qquad x_t = \alpha_t e, \qquad \alpha_t = E_Q[g(Z)|\mathcal{F}_t], \tag{4.17}$$

where the r.v. α_t is strictly positive.

The unbounded linear functional $\nu \in \mathcal{U}$ is defined by its domain given by formula (3.4) and by

$$\langle \nu, f\rangle = \left\langle \sum_{i\geq 1} \frac{e^i}{k_i} \mathcal{S}h_i, f \right\rangle \qquad \text{if } f \in C_0^\infty(]0, \infty[) \text{ and } \langle \nu, p_0\rangle = 0. \tag{4.18}$$

This definition makes sense, since for given f the sum has only a finite number of terms and since $p_0 \notin K$, the closure of $C_0^\infty(]0, \infty[)$ in H. We define ν^1 by the sum

$$\nu^1(x) = \sum_{i\geq 1} \frac{e^i}{k_i} \mathcal{S}h_i(x), \qquad x \geq 0. \tag{4.19}$$

Here, at most one term is nonvanishing and it must be a term with an odd index i. Due to the properties of h_i for odd i and (4.4), we have $\nu^1 \in C^\infty([0, \infty[)$ and $\operatorname{supp} \nu^1 \subset [3/4, \infty[$. Obviously, ν^1 is the restriction of ν to $C_0^\infty(]0, \infty[)$.

In order to construct a generalized self-financed bond portfolio $(E_Q[\hat{X}], \mu)$, with value process Y, where $Y_t = E_Q[\hat{X}|\mathcal{F}_t]$, we define μ a.e. $dt \times dP$ by formulas (3.6) and (3.7) and with α given by (4.17). This makes sense since $f \mapsto f\frac{p_0}{p_t}$ maps $\mathcal{D}(\mu_t)$ into $\mathcal{D}(\nu)$. Property (P_1) is then satisfied.

The sequence $\{e^{(n)}\}_{n\geq 1}$ in ℓ^2 is defined by $(e^{(n)})^i = e^i$ for $1 \leq i \leq n$ and $(e^{(n)})^i = 0$ for $i > n$. Let

$$X^n = E_Q[\hat{X}] + \int_0^{\bar{T}} \alpha_t \sum_{i\geq 1} (e^{(n)})^i \, d\bar{W}_t^i, \qquad Y_t^n = E_Q[X^n|\mathcal{F}_t].$$

As $e^{(n)}$ belongs to the domain of $(A_t)^{-1/2}$ we can proceed as in Remark 4.8 and Theorem 4.3 of [11] to construct the unique hedging portfolio $\theta^{(n)} = \theta^{(n)0} + \theta^{(n)1}$, where $\theta^{(n)0}, \theta^{(n)1} \in \mathsf{P}$ are given by (4.6) and (4.5). Applying (4.13) and (4.14), we obtain

$$\theta_t^{(n)0} = a_t^n \delta_0, \qquad a_t^n = \Big(E_Q[X^n | \mathcal{F}_t] - \alpha_t \sum_{1 \le i \le n} \frac{e^i}{k_i} (h_i, p_0)_H \Big) \Big/ p_t(0) \tag{4.20}$$

and

$$\theta_t^{(n)1} = \frac{p_0}{p_t} \alpha_t \sum_{1 \le i \le n} \frac{e^i}{k_i} \mathcal{S} h_i. \tag{4.21}$$

The sequence $\nu^{(n)} \in H'$ is defined by $\langle \nu^{(n)}, f \rangle$, $f \in H$, where

$$\langle \nu^{(n)}, f \rangle = \Big\langle \sum_{1 \le i \le n} \frac{e^i}{k_i} \mathcal{S} h_i, f \Big\rangle, \qquad \text{if } f(0) = 0 \text{ and } \langle \nu^{(n)}, p_0 \rangle = 0. \tag{4.22}$$

We note that $\nu^{(n)}$ converges to ν in $\mathcal{U}$:

$$\forall f \in \mathcal{D}(\nu), \qquad \lim_{n \to \infty} \langle \nu^{(n)}, f \rangle = \langle \nu, f \rangle. \tag{4.23}$$

Let $\nu^{(n)1}$ be the restriction of $\nu^{(n)}$ to $C_0^\infty(]0, \infty[)$. Due to the properties of h_i for odd i and (4.4), $\nu^{(n)1} \in C^\infty([0, \infty[)$ has compact support,

$$\operatorname{supp} \nu^{(n)}, \operatorname{supp} \nu^{(n+1)} \subset \big[3/4, n + \tfrac{1}{4}\big[, \qquad n \text{ odd}.$$

We have the decomposition $\nu^{(n)} = \nu^{(n)0} + \nu^{(n)1}$, where $\nu^{(n)0}, \nu^{(n)1} \in \mathsf{P}$ are given by

$$\nu^{(n)1} = \sum_{1 \le i \le n} \frac{e^i}{k_i} \mathcal{S} h_i, \qquad \nu_t^{(n)0} = b^n \delta_0, \qquad b^n = - \sum_{1 \le i \le n} \frac{e^i}{k_i} (h_i, p_0)_H. \tag{4.24}$$

We define $\mu_t^{(n)} \in H'$ a.e. (t, ω) by

$$\mu_t^{(n)} = \alpha_t \frac{p_0}{p_t} \nu^{(n)}. \tag{4.25}$$

We have the decomposition $\mu^{(n)} = \mu^{(n)0} + \mu^{(n)1}$, where

$$\mu_t^{(n)0} = \alpha_t \frac{1}{p_t(0)} \nu^{(n)0}, \qquad \mu_t^{(n)1} = \alpha_t \frac{p_0}{p_t} \nu^{(n)1} = \theta^{(n)1}. \tag{4.26}$$

It follows from formulas (3.6), (3.7) and (4.25) that $\mu_t^{(n)}$ converges a.e. (t, ω) to μ_t in $\mathcal{U}$, so (C_1) is satisfied. Since

$$\langle \mu_t^{(n)}, p_t \sigma^i \rangle = \alpha_t \big(e^{(n)}\big)^i,$$

it follows that Y^n converges to Y in the topology of square integrable martingales, which is stronger than the semi-martingale topology. So also (C_2) is satisfied. Therefore, $(E_Q[X], \mu)$ is a generalized hedging-portfolio of X.

We now fix a and the k_i. $a > 0$ is chosen such that $\lambda_i(a) \equiv (h_i, e^{-a\cdot})_H \neq 0$ for all $i \geq 1$, which is possible according to Lemma 4.1. Let

$$\text{(4.27)} \qquad \begin{aligned} &\operatorname{sgn}(k_i) = \operatorname{sgn}(\lambda_i(a)), \\ &0 < |k_i| \leq \min\big(|\lambda_i(a)|, (1 + \|h_i/p_0\|^2_{H^2})^{-1/2}\big)/i^2. \end{aligned}$$

The condition in (4.9) is then satisfied.

The sequence $E_Q[X^n|\mathcal{F}_t]$ converges to $E_Q[X|\mathcal{F}_t]$ in $L^2(\Omega, Q)$ as $n \to \infty$. We have

$$\sum_{\substack{1 \leq i \leq n \\ i \text{ odd}}} \frac{1}{ik_i}(h_i, p_0)_H \geq \sum_{\substack{1 \leq i \leq n \\ i \text{ odd}}} i.$$

The last sum goes to $+\infty$ when $n \to \infty$. Since $c > 0$ and $\alpha_t > 0$ a.s. it follows from (4.7) that (3.10) is satisfied in the case of $C = -\infty$.

We shall impose supplementary conditions on the k_i to ensure that (3.10) is satisfied also for C finite and $C = +\infty$. Let $J : \mathbb{N}^* \to 2\mathbb{N}^* + 1$ be defined by

$$J(n) = n + 2 \qquad \text{if } n \text{ is odd} \quad \text{and} \quad J(n) = n + 1 \qquad \text{if } n \text{ is even}.$$

For n odd let $d^{(n)} \in \mathbb{R}$, for n even let $d^{(n)} = d^{(n-1)}$ and define for $n \in \mathbb{N}^*$

$$\tilde{\nu}^{(n)1} = \nu^{(n)1} + d^{(n)}\mathcal{S}h_{J(n)}.$$

We define $\tilde{\nu}^{(n)} \in H'$ by

$$\langle \tilde{\nu}^{(n)}, f \rangle = \langle \tilde{\nu}^{(n)1}, f \rangle \qquad \text{for } f \in K \text{ and } \langle \tilde{\nu}^{(n)}, p_0 \rangle = 0.$$

$\tilde{\nu}^{(n)}$ converges to ν in $\mathcal{U}$:

$$\text{(4.28)} \qquad \forall f \in \mathcal{D}(\nu), \qquad \lim_{n \to \infty} \langle \tilde{\nu}^{(n)}, f \rangle = \langle \nu, f \rangle.$$

Since $\langle d^{(n)}\mathcal{S}h_{J(n)}, p_0\sigma^j \rangle = d^{(n)}k_j\delta_{jJ(n)}$, it follows that

$$\text{(4.29)} \qquad \sum_{j=1}^{\infty} \big(\langle d^{(n)}\mathcal{S}h_{J(n)}, p_0\sigma^j \rangle\big)^2 = \big(d^{(n)}\big)^2 \big(k_{J(n)}\big)^2.$$

We impose the following condition, which we for the moment suppose is possible:

$$\text{(4.30)} \qquad \lim_{n \to \infty} d^{(n)}k_{J(n)} = 0.$$

$\tilde{\mu}^{(n)}_t$ is defined as in (4.25), but with $\tilde{\nu}$ instead of ν. Formulas (4.28) and (4.30) imply that $(x, \tilde{\mu}^{(n)})$ is an approximating sequence for the generalized portfolio (x, μ).

We note that

$$\langle d^{(n)}\mathcal{S}h_{J(n)}, p_0\rangle = d^{(n)}e^{-aJ(n)}(h_1, p_0)_H,$$

which gives

$$\langle\tilde{\nu}^{(n)1}, p_0\rangle = \sum_{1\le i\le n}\frac{e^i}{k_i}(h_i, p_0)_H + d^{(n)}e^{-aJ(n)}(h_1, p_0)_H.$$

Similarly as in (4.20), we introduce [recalling that $p_0(0) = 1$] $\tilde{a}_0^n = E_Q[\hat{X}] - \alpha_0\langle\tilde{\nu}^{(n)1}, p_0\rangle$, which gives

$$\tilde{a}_0^n = E_Q[\hat{X}] - \alpha_0\biggl(\sum_{1\le i\le n}\frac{e^i}{k_i}(h_i, p_0)_H + d^{(n)}e^{-aJ(n)}(h_1, p_0)_H\biggr).$$

For given $\tilde{a}_0^n$, this is an equation for $d^{(n)}$. We now define for $n \ge 1$:

$$\tilde{a}_0^n = C \qquad \text{if } C \text{ is finite} \quad \text{and}$$

$$\tilde{a}_0^n = E_Q[\hat{X}] + \alpha_0\sum_{1\le i\le n}\frac{e^i}{k_i}(h_i, p_0)_H \qquad \text{if } C = \infty.$$

In both cases, $\lim_{n\to\infty}\tilde{a}_0^n = C$. For C finite, $d^{(n)}$ is then given by

$$d^{(n)} = \biggl(E_Q[\hat{X}] - C - \alpha_0\sum_{1\le i\le n}\frac{e^i}{k_i}(h_i, p_0)_H\biggr)\frac{e^{aJ(n)}}{\alpha_0(h_1, p_0)_H}$$

and for $C = +\infty$ by

$$d^{(n)} = -2\frac{e^{aJ(n)}}{(h_1, p_0)_H}\sum_{1\le i\le n}\frac{e^i}{k_i}(h_i, p_0)_H.$$

The property $d^{(n)} = d^{(n+1)}$ is then satisfied for n odd. For odd n, we choose $|k_{n+2}| > 0$ sufficiently small so that $|d^{(n)}k_{n+2}| \le 1/n$. Condition (4.30) is then satisfied. This proves B.

To prove C, let ν^1 be a positive function satisfying $\nu^1 \in C^\infty([0,\infty[)$, $\operatorname{supp}\nu^1 \subset [3/4,\infty[$, (3.5) and

$$\sum_i(\langle\nu^1, p_0\sigma^i\rangle)^2 < \infty, \tag{4.31}$$

which is possible since the σ^i have compact support and by possibly choosing the $|k_{n+2}| > 0$ even smaller. For this given ν^1, $\nu \in \mathcal{U}$ is defined as in (P_1).

Let $F \in C^\infty(\mathbb{R})$ be a positive function satisfying $\operatorname{supp} F \subset [0,2]$ and $F(1) = 1$. Y is the unique $(\mathcal{F}_t)$-adapted process satisfying

$$Y_t = 1 + \int_0^t F(Y_s)\,dM_s, \qquad t \in \mathbb{T}, \tag{4.32}$$

where M is the square integrable Q-martingale defined by

$$M_t = \sum_i \langle \nu^1, p_0\sigma^i \rangle \bar{W}_t^i. \tag{4.33}$$

$X = Y_{\bar{T}}$ is a positive bounded smooth $\mathcal{F}$ measurable random variable.

μ is defined by formulas (3.6) and (3.7), with $\alpha_t = F(Y_t)$, and it is a generalized integrand. This easily follows by introducing the sequence $\nu^{(n)} \in H'$ defined by $\langle \nu^{(n)}, f \rangle$, $f \in H$, where

$$\langle \nu^{(n)}, f \rangle = \langle \nu^{(n)1}, f \rangle, \qquad \text{if } f \in K \text{ and } \langle \nu^{(n)}, p_0 \rangle = 0 \tag{4.34}$$

and where $\nu^{(n)1} = \nu^1 g_n$ for a sequence of positive C^∞ cut-off functions g_n. We here choose $g_n(x) = 1$ for $0 \le x \le n$ and $g_n(x) = 0$ for $x \ge n + 1$. The sequence $\nu^{(n)} \in H'$ then satisfies (C_1) and (C_2), which follows similarly as in the proof of B.

The decomposition (2.18) of X is valid with $x = 1$, so $(1, \mu)$ is a generalized hedging portfolio of X.

The discounted risk-free investment at t, given by the generalized portfolio $(1, \mu^{(n)})$ is

$$a_t^n p_t(0) = Y_t - \langle \mu_t^{(n)}, p_t \rangle = Y_t - \alpha_t \langle \nu^{(n)1}, p_0 \rangle. \tag{4.35}$$

We now choose ν^1 and possibly further restrict the k_i, which is possible, such that

$$\lim_{n\to\infty} \langle \nu^{(n)1}, p_0 \rangle = \infty \tag{4.36}$$

and such that the condition in (4.31) is satisfied. This proves the part $C = -\infty$ of (P_2). The statements for C finite and $C = +\infty$ are proved so similarly to those in B, that we omit the proof. □

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EISTI
AVENUE DU PARC
95011 CERGY
FRANCE
E-MAIL: taflin@eisti.fr