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LIMIT DISTRIBUTIONS FOR LARGE PÓLYA URNS[1]

BY BRIGITTE CHAUVIN, NICOLAS POUYANNE AND REDA SAHNOUN

INRIA Rocquencourt and Université de Versailles, Université de Versailles and Université de Versailles

We consider a two-color Pólya urn in the case when a fixed number S of balls is added at each step. Assume it is a *large* urn that is, the second eigenvalue m of the replacement matrix satisfies $1/2 < m/S \leq 1$. After n drawings, the composition vector has asymptotically a first deterministic term of order n and a second random term of order $n^{m/S}$. The object of interest is the limit distribution of this random term.

The method consists in embedding the discrete-time urn in continuous time, getting a two-type branching process. The dislocation equations associated with this process lead to a system of two differential equations satisfied by the Fourier transforms of the limit distributions. The resolution is carried out and it turns out that the Fourier transforms are explicitly related to Abelian integrals over the Fermat curve of degree m. The limit laws appear to constitute a new family of probability densities supported by the whole real line.

1. Introduction. Consider a two-color Pólya–Eggenberger urn random process, with replacement matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: the urn starts with a finite number of red and black balls as initial composition (possibly monochromatic). At each discrete time n, one draws a ball uniformly at random, notices its color, puts it back into the urn and adds balls according to the following rule: if the ball drawn is red, one adds a red balls and b black balls; if the ball drawn is black, one adds c red balls and d black balls. The integers a, b, c, d are assumed to be nonnegative[2] and the urn is assumed to be *balanced*, which means that the total number of balls added at each step is a constant $S = a + b = c + d$. The composition vector of the urn at time n is denoted by

$$U^{DT}(n) = \begin{pmatrix} \text{\# red balls at time } n \\ \text{\# black balls at time } n \end{pmatrix}.$$

It is a random vector and the article deals with its asymptotics when n tends to infinity. Throughout the paper, the qualifier DT is used to refer to *discrete-time* objects while CT will refer to *continuous-time* ones.

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[2]One admits classically negative values for a and d, together with arithmetical conditions on c and b. Nevertheless, the paper deals with so-called *large* urns, for which this never happens.

Since Pólya's original paper [13], this question has been extensively studied so that citing all contributions is hopeless. The following references give however a good idea of the variety of methods: combinatorics with many papers by Mahmoud (see his recent book [12]), probabilistic methods by means of embedding the process in continuous time (see Janson [9]), analytic combinatorics by Flajolet et al. [7] and a more algebraic approach in [14]. The union of these papers is sufficiently well documented, guiding the reader to a quasi exhaustive bibliography.

The asymptotic behavior of $U^{DT}(n)$ is closely related to the spectral decomposition of the replacement matrix. In case of two colors, R is equivalent to $\left(\begin{smallmatrix} S & 0 \\ 0 & m \end{smallmatrix}\right)$, where the largest eigenvalue is the balance S and the smallest eigenvalue is the integer $m = a - c = d - b$. We denote by σ the ratio between the two eigenvalues:

$$\sigma = \frac{m}{S} \leq 1.$$

It is well known that the asymptotics of the process has two different behaviors depending on the position of σ with respect to the value $1/2$. Briefly:

(i) when $\sigma < \frac{1}{2}$, the urn is called *small* and, except when R is triangular, the composition vector is asymptotically Gaussian[3]:

$$\frac{U^{DT}(n) - nv_1}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \Sigma^2),$$

where v_1 is a suitable eigenvector of ${}^tR$ relative to S and Σ^2 has a simple closed form;

(ii) when $\frac{1}{2} < \sigma < 1$, the urn is called *large* and the composition vector has a quite different strong asymptotic form:

$$U^{DT}(n) = nv_1 + n^{\sigma} W^{DT} v_2 + o(n^{\sigma}), \tag{1}$$

where v_1, v_2 are suitable eigenvectors of ${}^tR$ relative to the eigenvalues S and m, W^{DT} is a real-valued random variable arising as the limit of a martingale, the little o being almost sure and in any L^p, $p \geq 1$. The moments of W^{DT} can be recursively calculated but they have no known global closed form [14].

The particular case $\sigma = 1$ is the original Pólya urn; it corresponds to taking $R = S\,\mathrm{Id}$ as replacement matrix. It has been well known, since Gouet [8], that the composition vector admits an almost sure asymptotics of order one: $U^{DT}(n) = nD + o(n)$ where the random vector D has a Dirichlet density (explicitly given in [8]).

In the present article, the object of interest is the distribution of W^{DT} for large urns.

The first step consists in classically embedding the discrete-time process $(U^{DT}(n))_{n\geq 0}$ into a continuous-time Markov process $(U^{CT}(t))_{t\geq 0}$, by equipping

[3]The case $\sigma = 1/2$ is similar to this one, the normalization being $\sqrt{n \log n}$ instead of $\sqrt{n}$.

each ball with an exponential clock. At any nth jump time τ_n, the continuous-time process $U^{CT}(\tau_n)$ has the same distribution as $U^{DT}(n)$. This connection between both processes is the key point, allowing us to work on the continuous-time process, where independence properties have been gained.

In Theorem 3.2, we show that, in the case of large urns, the continuous-time process satisfies, when t tends to infinity, the following asymptotics,

$$U^{CT}(t) = e^{St}\xi v_1(1 + o(1)) + e^{mt} W^{CT} v_2(1 + o(1)), \tag{2}$$

where v_1 and v_2 are the same vectors as above, ξ and W^{CT} are real-valued random variables that arise as limits of martingales, with $o(\cdot)$ meaning "almost sure and in any L^p, $p \geq 1$." Moreover, we prove that ξ is Gamma-distributed. These results are based on the spectral decomposition of the infinitesimal generator of the continuous-time process on spaces of two-variable polynomials.

Thanks to the embedding connection, the two random variables W^{DT} and W^{CT} are connected (Theorem 3.10):

$$W^{CT} = \xi^\sigma W^{DT} \qquad \text{a.s.},$$

ξ and W^{DT} being independent. Since ξ^σ is invertible,[4] the attention is focused on the determination of the distribution of W^{CT}.

Because of the nonnegativity of R entries, $(U^{CT}(t))_{t\geq 0}$ is a two-type branching process, visualized as a random tree: the branching property gives rise to dislocation equations on $U^{CT}(t)$. If one denotes by $\mathcal{F}$ (resp., $\mathcal{G}$) the characteristic function of W^{CT} starting from one red ball and no black ball (resp., no red ball, one black ball), the independence of the subtrees in the branching process implies that the characteristic function of *any* W^{CT} starting from α red balls and β black balls is the product $\mathcal{F}^\alpha\mathcal{G}^\beta$. Furthermore, the dislocation equations on $U^{CT}(t)$ lead to the following differential system

$$\begin{cases} \mathcal{F}(x) + mx\mathcal{F}'(x) = \mathcal{F}(x)^{a+1}\mathcal{G}(x)^b, \\ \mathcal{G}(x) + mx\mathcal{G}'(x) = \mathcal{F}(x)^c\mathcal{G}(x)^{d+1}, \end{cases} \tag{3}$$

together with suitable boundary conditions. Notice that the corresponding exponential moment generating series (Laplace series) are also solutions of (3), but their radius of convergence is equal to 0. This is detailed in Section 8.2.

The solution of system (3) is obtained in Section 6, where it is shown that $\mathcal{F}$ and $\mathcal{G}$ can be made explicit in terms of inverse functions of Abelian integrals over the Fermat curve of degree m. Indeed, for any complex z in a suitable open subset of $\mathbb{C}$, let

$$I_{m,S,b}(z) = \int_{[z,z\infty)} (1 + u^m)^{b/m} \frac{du}{u^{S+1}},$$

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[4]A probability distribution A is called *invertible* when, for any probability distributions A and B, the equation $AX = B$ admits a unique solution X independent of A, see, for instance, Chaumont and Yor [4]. The invertibility of any power of a Gamma distribution can be shown by elementary considerations on Fourier transforms.

where $[z, z\infty)$ denotes the ray $\{tz, t \geq 1\}$. The function $I_{m,S,b}$ defines a conformal mapping on the open sector $\mathcal{V}_m = \{z \neq 0, 0 < \arg(z) < \pi/m\}$. If $J_{m,S,b}$ denotes the holomorphic function, defined on the lower half-plane as left inverse function of $I_{m,S,b}$ and extended to the slit plane by conjugacy, the closed form of $\mathcal{F}$ and $\mathcal{G}$ are given in the following result.

THEOREM. *For any* $x > 0$,

$$\begin{cases} \mathcal{F}(x) = Kx^{-1/m} J_{m,S,b}\left(C_0 + \dfrac{K^S}{S} x^{-S/m}\right), \\ \mathcal{G}(x) = Kx^{-1/m} J_{m,S,c}\left(C_0 + \dfrac{K^S}{S} x^{-S/m}\right), \end{cases}$$

where $K \in \mathbb{C}$ *and* $C_0 < 0$ *are explicit constants.*

For precise statements and proofs, see Section 6.3 and Theorem 6.7.

The solution of system (3) is effected by a ramified change of variable and functions, leading to the following monomial system:

$$\begin{cases} f' = f^{a+1} g^b, \\ g' = f^c g^{d+1}. \end{cases} \tag{4}$$

This remarkable fact is evocative of the case of *small* urns and discrete time, as considered in a beautiful study of Flajolet et al. [6]. The method of [6] leads directly to the same system (4) on generating functions. The assumption $\sigma < 1/2$, when expressed in term of the four parameters a, b, c and d, does not fundamentally affect the system but requires completely different analytic considerations.

The limit laws of the W^{CT} appear to constitute a new family of probability distributions, indexed by three parameters S, m, b subject to assumptions (11) and by initial conditions α, β. We prove in Section 7 that they admit densities that can be expressed by means of the inverse Fourier transforms of their characteristic function derivatives. Furthermore, the laws are infinitely divisible and their support is the whole real line, the radius of convergence of their exponential moment generating series being equal to 0.

Many questions remain open. For instance, are these distributions characterized by their moments? What is the precise asymptotics of their densities at infinity (tails)? It is shown in [15] that, for triangular and nondiagonal replacement matrices, the discrete-time limit law W^{DT} is never infinitely divisible; does this situation persist in the present nontriangular case?

2. The model.

2.1. *Definition of the process.* Let a, b, c and d be nonnegative integers such that $a + b = c + d =: S$ and R be the matrix

$$R := \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & S-a \\ S-d & d \end{pmatrix}. \tag{5}$$

The discrete-time Pólya–Eggenberger urn process associated with the replacement matrix R, which has been informally described in the Introduction, is the Markov chain $(U^{DT}(n), n \in \mathbb{N})$, having $\mathbb{N}^2 \setminus \{(0,0)\}$ as state space and

$$\frac{x}{x+y}\delta_{(x+a,y+b)} + \frac{y}{x+y}\delta_{(x+c,y+d)} \tag{6}$$

as transition probability at any nonzero point $(x, y) \in \mathbb{N}^2$. In this formula, δ_v denotes Dirac point mass at $v \in \mathbb{N}^2$. This means that $(U^{DT}(n), n \in \mathbb{N})$ is a random walk in $\mathbb{N}^2 \setminus \{(0,0)\}$ (or in the two-dimensional one-column nonzero matrices with nonnegative integer entries, we will use both notations) recursively defined by the conditional probabilities

$$\begin{cases} \mathrm{P}\Big(U^{DT}(n+1) = U^{DT}(n) + \begin{pmatrix} a \\ b \end{pmatrix} | U^{DT}(n) = \begin{pmatrix} x \\ y \end{pmatrix}\Big) = \dfrac{x}{x+y}, \\ \mathrm{P}\Big(U^{DT}(n+1) = U^{DT}(n) + \begin{pmatrix} c \\ d \end{pmatrix} | U^{DT}(n) = \begin{pmatrix} x \\ y \end{pmatrix}\Big) = \dfrac{y}{x+y}. \end{cases}$$

In the sequel,

$$\big(U^{DT}_{(\alpha,\beta)}(n), n \geq 0\big)$$

will denote the process starting from the nonzero vector (α, β) and

$$u := \alpha + \beta$$

will denote the total number of balls at time 0. Notice that the balance property $S = a + b = c + d$ implies that the total number of balls at time n, when $U^{DT}(n) = (x, y)$, is the (nonrandom) number $x + y = u + nS$.

Denote by $w_1 = \binom{a}{b}$ and $w_2 = \binom{c}{d}$ the increment vectors of the walk. The transition operator Φ is defined, for any function f on $\mathbb{N}^2$ and for any $v = \binom{x}{y}$, by

$$\Phi(f)(v) = x[f(v + w_1) - f(v)] + y[f(v + w_2) - f(v)].$$

Conditioning on $(\mathcal{F}_n, n \geq 0)$, which is the filtration associated with the process $(U^{DT}(n), n \geq 0)$, one gets

$$\mathbb{E}^{\mathcal{F}_n}\big(f\big(U^{DT}(n+1)\big)\big) = \Big(I + \frac{\Phi}{u+nS}\Big)(f)(U^{DT}(n)).$$

In particular,

$$\mathbb{E}^{\mathcal{F}_n}\big(U^{DT}(n+1)\big) = \Big(I + \frac{A}{u+nS}\Big)U^{DT}(n), \tag{7}$$

where I denotes the two-dimensional identity matrix and

$$A := {}^tR = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

2.2. *Asymptotics of the discrete-time process $U^{DT}(n)$.* As mentioned in Section 1 and briefly recalled hereunder, a discrete-time Pólya–Eggenberger urn process has two different kinds of asymptotics depending on the ratio of the eigenvalues of its replacement matrix R. With our notation, these eigenvalues are S and

$$m := a - c = d - b.$$

Let us denote by u_1 and u_2 the two following linear eigenforms[5] of A, respectively associated with the eigenvalues S and m, which means that $u_1 \circ A = Su_1$ and $u_2 \circ A = mu_2$:

$$u_1(x, y) = \frac{1}{S}(x + y), \qquad u_2(x, y) = \frac{1}{S}(bx - cy), \tag{8}$$

and denote by (v_1, v_2) the dual basis of (u_1, u_2):

$$v_1 = \frac{S}{(b+c)} \begin{pmatrix} c \\ b \end{pmatrix}, \qquad v_2 = \frac{S}{(b+c)} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{9}$$

The vectors v_k are eigenvectors of A and the projections on the eigenlines are $u_1 v_1$ and $u_2 v_2$.

For any positive real x and any nonnegative integer n, if one denotes by $\gamma_{x,n}$ the polynomial

$$\gamma_{x,n}(t) := \prod_{k=0}^{n-1} \left(1 + \frac{t}{x + k}\right),$$

the matrix $\gamma_{\frac{m}{S},n}(\frac{A}{S})$ in nonsingular and it is immediate from (7) that

$$\gamma_{\frac{m}{S},n}\left(\frac{A}{S}\right)^{-1} U^{DT}(n)$$

is a (vector-valued) martingale.

As indicated in the Introduction, the ratio of R eigenvalues is denoted by

$$\sigma := \frac{m}{S} \le 1.$$

The case of *small* urn processes (i.e., when $\sigma \le 1/2$) has been well studied; in this case, when R is not triangular, the random vector admits a Gaussian central limit theorem (see Janson [9]). Triangular replacement matrices impose a particular treatment and lead most often to a nonnormal second-order limit (see Janson [10] or [15]).

Our subject of interest is the case of *large* urns, that is, when $\sigma > 1/2$. In this case, $\frac{1}{S} U^{DT}(n)$ is a large Pólya process with replacement matrix $\frac{1}{S} R$ in the sense of [14]. As a matter of consequence, the projections of the above vector-valued martingale on the eigenlines of A, which are of course also martingales, converge

[5] An eigenform of an endomorphism f is an eigenvector of ${}^t f$; some authors call these linear forms left eigenvectors of f, referring to matrix operations.

in any L^p, $p \geq 1$ (and a.s.). In particular (second projection),

$$M^{DT}(n) := \frac{1}{\gamma_{\frac{u}{S},n}(\sigma)} u_2(U^{DT}(n))$$

is a convergent martingale; since $\gamma_{u,n}(\sigma) = n^\sigma \frac{\Gamma(u)}{\Gamma(u+\sigma)}(1 + o(1))$, denoting by

$$W^{DT} := \lim_{n\to+\infty} \frac{1}{n^\sigma} u_2(U^{DT}(n)), \tag{10}$$

a slight adaptation of [14] leads to the following theorem. Note that this theorem was essentially proven by Athreya and Karlin [1] and Janson [9] for random replacement matrices. The convergence in L^p-spaces when R is nonrandom is shown by the indicated adaptation of [14].

THEOREM 2.1. *Suppose that $\sigma \in]1/2, 1[$. Then, as n tends to infinity,*

$$U^{DT}(n) = nv_1 + n^\sigma W^{DT} v_2 + o(n^\sigma),$$

where v_1 and v_2, defined in (9), are eigenvectors associated with the eigenvalues S and m; W^{DT} is defined by (10); $o(\cdot)$ means a.s. and in any L^p, $p \geq 1$.

2.3. *Parametrization and hypotheses.* The subject of the paper is W^{DT} distribution in Theorem 2.1 so that the Pólya urn process will be supposed large. Furthermore, the replacement matrix R is supposed to be not triangular because this case has to be treated separately with regard to its limit law, as attested by Janson [10], Flajolet et al. [7], [15] and the present paper.

Under these conditions, the assumptions on the replacement matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are: $a + b = c + d = S$ (balance condition), $S/2 < m = a - c = d - b < S$ (large urn) and $b, c \geq 1$ (not triangular). Because of the balance condition, the parametrization of Pólya urns is governed by three free parameters. The computation of Fourier transforms will show in Section 6.3 that a natural choice of free parameters is (m, S, b). The assumption "large and nontriangular" is equivalent, in terms of these data, to the following:

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} S - b & b \\ S - m - b & m + b \end{pmatrix}$$

with

$$\begin{cases} m + 2 \leq S \leq 2m - 1, \\ 1 \leq b \leq S - m - 1. \end{cases} \tag{11}$$

Note that these inequalities imply $S \geq 5$ and $m \geq 3$ and that, for a given m, the point (m, b) belongs to a triangle as represented in Figure 1.

For small values of S, large urn processes have the following possible replacement matrices: for $S \in \{1, 2\}$, only $R = S\,\mathrm{Id}_2$ defines a large urn; for $S \in \{3, 4\}$, all

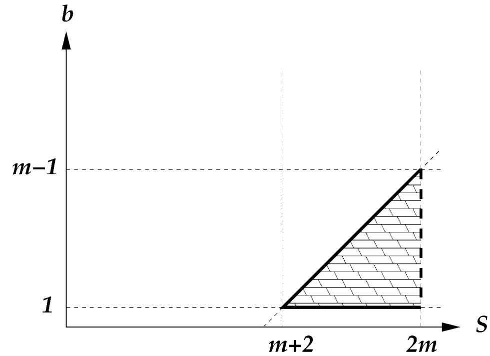

FIG. 1. *Parameters* (b, S) *for a given* m.

large urns have triangular matrices. For $S \in \{5, 6\}$, only $R = \begin{pmatrix} S-1 & 1 \\ 1 & S-1 \end{pmatrix}$ defines a nontriangular large urn. For $S = 7$, apart from triangular or symmetric matrices, there are only two replacement matrices that define large urns: $\begin{pmatrix} 6 & 1 \\ 2 & 5 \end{pmatrix}$ and the other one derived from it by permutation of coordinates.

3. Embedding in continuous time and martingale connection.

3.1. *Embedding*. The idea of embedding discrete urn models in continuous-time branching processes goes back at least to Athreya and Karlin [1]. A description is given in Athreya and Ney [2], Section 9. The method has been recently revisited and developed by Janson [9].

We define the continuous-time Markov branching process $(U^{CT}(t), t \geq 0)$ as being the embedded process of $(U^{DT}(n), n \geq 0)$. Following, for instance, Bertoin [3], Section 1.1, this means that it is defined as the continuous-time Markov chain having as jump rate, at any nonzero point $(x, y) \in \mathbb{N}^2$, the finite measure given by the transition probability of the discrete-time process [formula (6)]. One gets this way a branching process having the following dynamical description in terms of red and black balls. In the urn, at any moment, each ball is equipped with an $\mathcal{E}xp(1)$-distributed[6] random clock, all the clocks being independent. When the clock of a red ball rings, a red balls and b black balls are added in the urn; when the ringing clock belongs to a black ball, one adds c red balls and d black balls, so that the replacement rules are the same as in the discrete-time urn process.

The successive jumping times of $(U^{CT}(t), t \geq 0)$, will be denoted by

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots.$$

The nth jumping time is the time of the nth dislocation of the branching process. The process is thus constant on any interval $[\tau_n, \tau_{n+1}[$.

[6]For any positive real a, $\mathcal{E}xp(a)$ denotes the exponential distribution with parameter a.

In the sequel,

$$\left(U^{CT}_{(\alpha,\beta)}(t), t \geq 0\right)$$

will denote the process starting from the nonzero vector (α, β). Thus, for any initial condition (α, β), for any $t \geq 0$,

$$U^{CT}_{(\alpha,\beta)}(t) = U^{DT}_{(\alpha,\beta)}(a(t)),$$

where

$$a(t) := \inf\{n \geq 0, \tau_n \geq t\}.$$

LEMMA 3.1. *(i) for $n \geq 0$, the distribution of $\tau_{n+1} - \tau_n$ is $\mathcal{E}xp(u + Sn)$ where u denotes the total number of balls at time* 0;
(ii) *the processes $(\tau_n)_{n\geq 0}$ and $(U^{CT}(\tau_n))_{n\geq 0}$ are independent*;
(iii) *the processes $(U^{CT}(\tau_n))_{n\geq 0}$ and $(U^{DT}(n))_{n\geq 0}$ have the same distribution*.

PROOF. The total number of balls at time $t \in [\tau_n, \tau_{n+1}[$ is $u + Sn$. Therefore, (i) is a consequence of the fact that the minimum of k independent random variables $\mathcal{E}xp(1)$-distributed is $\mathcal{E}xp(k)$-distributed. (ii) is the classical independence between the jump chain and the jump times in such Markov processes. The initial states and evolution rules of both Markov chains in discrete time and in continuous time are the same ones, so that (iii) holds. □

Convention: From now on, thanks to (iii) of Lemma 3.1, we will classically consider that the discrete-time process and the continuous-time process are built on the *same* probability space on which

$$(U^{CT}(\tau_n))_{n\geq 0} = (U^{DT}(n))_{n\geq 0} \qquad \text{a.s.} \tag{12}$$

3.2. *Asymptotics of the continuous-time process $U^{CT}(t)$.* Let v_1 and v_2 the linearly independent eigenvectors of A defined by (9). In the case of large urns, the asymptotics of the continuous-time process $(U^{CT}(t))_{t\geq 0}$ is given in the following theorem.

THEOREM 3.2 (Asymptotics of continuous-time process). *When t tends to infinity,*

$$U^{CT}(t) = e^{St}\xi v_1\bigl(1 + o(1)\bigr) + e^{mt}W^{CT}v_2\bigl(1 + o(1)\bigr), \tag{13}$$

where ξ and W^{CT} are real-valued random variables, the little o being almost sure and in any L^p-space, $p \geq 1$. Furthermore, ξ is Gamma(u/S)*-distributed, where $u = \alpha + \beta$ is the total number of balls at time* 0.

REMARK 3.3. Another formulation of this Theorem is: in the basis (v_1, v_2), the coordinate of $U^{CT}(t)$ along v_1 has $e^{St}\xi$ as its dominant term while the coordinate of $U^{CT}(t)$ along v_2 has $e^{mt}W^{CT}$ as its dominant term.

PROOF OF THEOREM 3.2. The embedding in continuous time has been studied in Athreya and Karlin [1] and in Janson [9]. It has become classical that the process

$$(e^{-tA}U^{CT}(t))_{t\geq 0}$$

is a vector-valued martingale and that, in the case of large urns ($\sigma > 1/2$), this martingale is bounded in L^2, thus converges. Its projections on the eigenlines $\mathbb{R}v_1$ and $\mathbb{R}v_2$, that is, respectively,

$$e^{-St}u_1(U^{CT}(t)) \quad \text{and} \quad e^{-mt}u_2(U^{CT}(t))$$

are also L^2-convergent real valued martingales, thus converge almost surely. Their respective limits are named ξ and W^{CT}. What still has to be proved is that these martingales converge in fact in any L^p, $p \geq 1$. The identification of ξ distribution will be a consequence of this proof.

The infinitesimal generator of the Markov process $(U^{CT}(t))_t$ is the finite-difference operator

$$\Phi(f)(x,y) = x\{f(x+a,y+b) - f(x,y)\} + y\{f(x+c,y+d) - f(x,y)\},$$

defined for any (measurable) function f and any $(x,y) \in \mathbb{R}^2$. For a very synthetic reference on semi-groups of Markov continuous-time processes, one can refer to Bertoin [3], Chapter 1. This operator Φ acts on two-variable polynomials. This action has been studied in detail in [14] in a more general framework. More precisely, for any integer $d \geq 1$, the operator Φ acts on the finite-dimensional space of polynomials of degree less than d, so that, for any two-variable polynomial P and for any $t \geq 0$,

$$\mathbb{E}(P(U^{CT}(t))) = \exp(t\Phi)(P)((U^{CT}(0))), \tag{14}$$

where, in this formula, Φ denotes the restriction of Φ itself on any finite-dimensional polynomials space containing P. The properties of Φ listed in the following lemma are proven in [14] and will be used here.

LEMMA 3.4. *There exists a unique family of polynomials $Q_{p,q} \in \mathbb{R}[x,y]$, p,q nonnegative integers, called* reduced polynomials, *such that*:

(1) $Q_{0,0} = 1$, $Q_{1,0} = u_1$ *and* $Q_{0,1} = u_2$ [*see* (8) *for a definition of eigenforms* u_1 *and* u_2];

(2) $\Phi(Q_{p,q}) = (pS + qm)Q_{p,q}$ *for all nonnegative integers* p,q;

(3) $u_1^p u_2^q - Q_{p,q} \in \mathrm{Span}\{Q_{p',q'}, p'S + q'm < pS + qm\}$ *for all nonnegative integers* p,q.

Note that the reduced polynomial $Q_{p,q}$ is in fact the projection of $u_1^p u_2^q$ on a suitable characteristic subspace of Φ restriction to some finite-dimensional polynomial space, and that this spectral decomposition of Φ on polynomials has a particularly simple form (it is diagonalizable) because the urn is large and two-dimensional. See [14] for more details.

Formula (14) and property (ii) of Lemma 3.4 lead to

$$\forall (p,q) \in \mathbb{Z}_{\geq 0}^2 \qquad \mathbb{E}(Q_{p,q}(U^{CT}(t))) = e^{t(pS+qm)} Q_{p,q}(U^{CT}(0)).$$

This implies straightforwardly, with (iii) of Lemma 3.4, that, for any (p,q),

$$\mathbb{E}(u_1^p u_2^q(U^{CT}(t))) = e^{t(pS+qm)} Q_{p,q}(U^{CT}(0)) + o\big(e^{t(pS+qm)}\big). \tag{15}$$

In particular, the martingales $e^{-St}u_1(U^{CT}(t))$ and $e^{-mt}u_2(U^{CT}(t))$ are L^p-bounded for any $p \geq 1$ and their respective limits, namely ξ and W^{CT} satisfy, for any nonnegative integer p,

$$\mathbb{E}\xi^p = Q_{p,0}(U^{CT}(0)) \quad \text{and} \quad \mathbb{E}(W^{CT})^p = Q_{0,p}(U^{CT}(0)). \tag{16}$$

The convergence part of the theorem follows now from the spectral decomposition of A: for any $t \geq 0$,

$$U^{CT}(t) = u_1(U^{CT}(t)) \cdot v_1 + u_2(U^{CT}(t)) \cdot v_2.$$

Besides, it is proven in [14], or one can check it after an easy computation, that the reduced polynomials corresponding to the powers of u_1 have the following closed form expression

$$Q_{p,0} = u_1(u_1+1)(u_1+2)\cdots(u_1+p-1).$$

Thanks to formula (16), this shows that the pth moment of ξ is, for any integer $p \geq 0$,

$$\mathbb{E}\xi^p = \frac{u}{S}\left(\frac{u}{S}+1\right)\left(\frac{u}{S}+2\right)\cdots\left(\frac{u}{S}+p-1\right) = \frac{\Gamma(\frac{u}{S}+p)}{\Gamma(\frac{u}{S})},$$

where u is the total number of balls at time 0 [remember that $u_1(U^{CT}(0)) = u/S$, see (8)]. One identifies this way the required Gamma(u/S) distribution, characterized by its moments. $\square$

REMARK 3.5. Notice that the distribution of ξ has been given by Janson [9] calculating first the distribution of $u_1(U^{CT}(t))$for every t:

$$u_1(U^{CT}(t)) = \frac{u}{S} + Z(t),$$

where $Z(t)$ is a negative binomial distribution.

REMARK 3.6. Reduced polynomials $Q_{0,p}$ do not have a known closed form, so that reproducing the above method in order to compute the moments of W^{CT} fails.

REMARK 3.7. It follows from the proof that the real-valued random variables ξ and W^{CT} are respective limits of the martingales

$$\xi = \lim_{t\to+\infty} e^{-St} u_1(U^{CT}(t)),$$
$$W^{CT} = \lim_{t\to+\infty} e^{-mt} u_2(U^{CT}(t)).$$

They are not independent and their joint moments are computed from formula (15): for any nonnegative integers p, q,

$$E[(\xi)^p (W^{CT})^q] = Q_{p,q}(U^{CT}(0)).$$

For example, their respective means are $E\xi = u_1(U^{CT}(0)) = \frac{1}{S}(\alpha+\beta)$ and $EW^{CT} = u_2(U^{CT}(0)) = \frac{1}{S}(b\alpha - c\beta)$, whereas

$$E(\xi W^{CT}) = \frac{(\alpha+\beta+m)(b\alpha - c\beta)}{S^2}$$

as can be shown by computation of the reduced polynomial $Q_{1,1} = (u_1+\sigma)u_2$ (one can directly check that this polynomial is an eigenvector of Φ, associated with the eigenvalue $S+m$).

REMARK 3.8. When the urn is small ($\sigma < 1/2$), the same method shows that the result on the first projection is still valid: the martingale $(e^{-St}u_1(U^{CT}(t)))_t$ converges in any L^p $(p \geq 1)$ to a Gamma(u/S) distributed random variable. On the contrary, the martingale $(e^{-mt}u_2(U^{CT}(t)))_t$ diverges and it is shown in Janson [9] that the second projection satisfies a central limit theorem: when $\sigma = \frac{m}{S} < 1/2$,

$$e^{-\frac{S}{2}t} u_2(U^{CT}(t)) \xrightarrow[t\to+\infty]{\mathcal{D}} \mathcal{N},$$

where $\mathcal{N}$ is a normal distribution. In the case $\sigma = 1/2$, the normalization must be modified and one gets the convergence in law of $\sqrt{t}e^{-St/2}u_2(U^{CT}(t))$ to a normal distribution.

REMARK 3.9. The distributions of the W^{CT} are infinitely divisible, because they are limits of infinitely divisible ones, obtained by scaling and projection of infinitely divisible ones. Indeed, in finite time, the distributions of the $U^{CT}_{(\alpha,\beta)}(t)$ are infinitely divisible. It has been said by Janson [9], proof of Theorem 3.9. With our notations, especially the one of (19), it relies on the fact that

$$U^{CT}_{(\alpha,\beta)}(t) \stackrel{\mathcal{L}}{=} [n] U^{CT}_{(\alpha/n,\beta/n)}(t),$$

where a continuous-time branching process (with the same branching dynamics as before), starting from real (nonnecessary integer) conditions, is suitably defined.

3.3. *DT and CT connections.* Apply the first projection to the embedding principle (12):

$$u_1(U^{CT}(\tau_n)) = u_1(U^{DT}(n)) \qquad \text{a.s.}$$

By definition (8) of u_1, this number is $\frac{1}{S}$ times the number of balls in the urn at time n, which equals $\frac{1}{S}(u + Sn) = n(1 + o(1))$. Since stopping times τ_n tend to $+\infty$, renormalizing by $e^{-S\tau_n}$ and applying the convergence result of Section 3.2 leads to

$$\xi = \lim_{n\to+\infty} ne^{-S\tau_n}. \tag{17}$$

Apply now the second projection to the embedding principle (12):

$$u_2(U^{CT}(\tau_n)) = u_2(U^{DT}(n)) \qquad \text{a.s.}$$

Renormalizing by $e^{-m\tau_n}$ implies that

$$e^{-m\tau_n}u_2(U^{CT}(\tau_n)) = W^{CT}(\tau_n) = e^{-m\tau_n}\gamma_{\frac{u}{S},n}(\sigma)M^{DT}(n) \qquad \text{a.s.}$$

which is a "martingale connection" in finite time.

Thanks to (17) and Theorem 3.2, passing to the limit $n \to \infty$ leads to the following theorem, already mentioned in Janson [9] in a more general framework. Note that the independence between ξ and W^{DT} comes from Lemma 3.1(ii).

THEOREM 3.10 (Martingale connection).

$$W^{CT} = \xi^{\sigma} W^{DT} \qquad a.s. \tag{18}$$

ξ and W^{DT} being independent.

The distribution of ξ^{σ} is invertible (see footnote in the Introduction), so that any information on W^{CT} can be pulled back to W^{DT} thanks to connection (18).

4. Dislocation equations for continuous urns.

4.1. *Vectorial finite time dislocation equations.* By embedding in continuous time, the previous section provided a branching process $(U^{CT}_{(\alpha,\beta)}(t), t \geq 0)$. The independence properties of this process imply that it is equal to the sum of α copies of $U^{CT}_{(1,0)}(t)$ (the process starting from one red ball) and β copies of $U^{CT}_{(0,1)}(t)$ (the process starting from one black ball). We are led to study these two $\mathbb{R}^2$-valued processes.

Let us now apply the strong Markov branching property to these processes: let us denote by τ the first splitting time for any of these processes (they have the same $\mathcal{E}xp(1)$ distribution). We get the following vectorial finite time dislocation equations:

$$\forall t > \tau \begin{cases} U^{CT}_{(1,0)}(t) \overset{\mathcal{L}}{=} [a+1]U^{CT}_{(1,0)}(t-\tau) + [b]U^{CT}_{(0,1)}(t-\tau), \\ U^{CT}_{(0,1)}(t) \overset{\mathcal{L}}{=} [c]U^{CT}_{(1,0)}(t-\tau) + [d+1]U^{CT}_{(0,1)}(t-\tau), \end{cases} \tag{19}$$

where the notation $[n]X + [m]Y$ stands for the sum of n copies of the random variable X and m copies of the random variable Y (n and m are nonnegative integers).

REMARK 4.1. The above equations could be written with a.s. equalities. Taking a probability space of trees is more convenient. The price to pay is just to write the different processes for each subtree with different indexes and to distinguish the two splitting times for the two starting situations.

4.2. *Limit dislocation equations.* Remember that $(e^{-mt}u_2(U^{CT}_{(1,0)}(t)))_t$ and $(e^{-mt}u_2(U^{CT}_{(0,1)}(t)))_t$ are martingales having, respectively, $u_2(U^{CT}_{(1,0)}(0)) = b/S$ and $u_2(U^{CT}_{(0,1)}(0)) = -c/S$ as expectations. They converge in L^p for every nonnegative integer $p \geq 1$. We are interested in the probability distributions of

$$X := \lim_{t\to+\infty} e^{-mt}u_2\big(U^{CT}_{(1,0)}(t)\big) \quad \text{and} \quad Y := \lim_{t\to+\infty} e^{-mt}u_2\big(U^{CT}_{(0,1)}(t)\big). \tag{20}$$

Projecting along the second eigenline, scaling and passing to the limit in system (19) lead straightforwardly to the following proposition.

PROPOSITION 4.2. *The limit random variables X and Y are solution of the following* (*scalar*) *limit dislocation equations*:

$$\begin{cases} X \overset{\mathcal{L}}{=} e^{-m\tau}([a+1]X + [b]Y), \\ Y \overset{\mathcal{L}}{=} e^{-m\tau}([c]X + [d+1]Y), \end{cases} \tag{21}$$

with

$$\mathbb{E}(X) = \frac{b}{S}, \qquad \mathbb{E}(Y) = -\frac{c}{S}, \tag{22}$$

where all the mentioned variables are independent.

REMARK 4.3. Janson [9] in his Theorem 3.9 gets the same limit dislocation equations. He obtains the unicity of the solution in L^2 by a fixed point method. Hereunder in Section 6.3, calculating explicitly the solution of the fixed point system (21) together with conditions (22), we give in passing another proof of the unicity in L^2.

5. Characteristic functions: fundamental differential system. Let $\mathcal{F}$ and $\mathcal{G}$ be respectively, the characteristic functions of X and Y:

$$\forall x \in \mathbb{R} \qquad \mathcal{F}(x) = \mathbb{E}(e^{ixX}) = \int_{-\infty}^{+\infty} e^{ixt}\, d\mu_X(t)$$

with a similar formula for $\mathcal{G}$. Since X and Y admit moments of all orders, $\mathcal{F}$ and $\mathcal{G}$ are infinitely differentiable on $\mathbb{R}$.

PROPOSITION 5.1. *The characteristic functions $\mathcal{F}$ and $\mathcal{G}$ are solutions of the differential system*

$$\begin{cases} \mathcal{F}(x) + mx\mathcal{F}'(x) = \mathcal{F}(x)^{a+1}\mathcal{G}(x)^b, \\ \mathcal{G}(x) + mx\mathcal{G}'(x) = \mathcal{F}(x)^c\mathcal{G}(x)^{d+1}, \end{cases} \tag{23}$$

and satisfy the boundary conditions at the origin

$$\begin{cases} \mathcal{F}(x) = 1 + i\dfrac{b}{S}x + O(x^2), \\ \mathcal{G}(x) = 1 - i\dfrac{c}{S}x + O(x^2). \end{cases} \tag{24}$$

PROOF. Conditioning on τ the distribution of which is exponential with mean 1, the first dislocation equation (21) implies successively that, for any $x \in \mathbb{R}$,

$$\begin{aligned} \mathcal{F}(x) &= \mathbb{E}\big(\mathbb{E}\big(\exp\big(ixe^{-m\tau}([a+1]X + [b]Y)|\tau\big)\big)\big) \\ &= \int_0^{+\infty} \mathcal{F}^{a+1}\big(xe^{-mt}\big)\mathcal{G}^b\big(xe^{-mt}\big)e^{-t}\, dt. \end{aligned}$$

After a change of variable under the integral, this functional equation can be written

$$\forall x \neq 0 \qquad \mathcal{F}(x) = \frac{x}{m|x|^{1+1/m}} \int_0^x \mathcal{F}^{a+1}(t)\mathcal{G}^b(t)\frac{dt}{|t|^{1-1/m}}.$$

Differentiation of this equality and the similar one obtained from the second dislocation equation in (21) lead to the result. The boundary conditions come elementarily from the computation of the means of X and Y and from the existence of their second moment (Taylor expansion of $\mathcal{F}$ and $\mathcal{G}$ at 0). □

REMARK 5.2. The differential system (23) is singular at 0 so that the unicity of its solution that satisfies the boundary condition (24) is not an elementary consequence of general theorems for ordinary differential equations.

6. Resolution of the fundamental differential system.

6.1. *Change of functions*: *Heuristics*. Formally, without carefully checking which mth roots should be considered, if the variables $x \in \mathbb{R}$ and $w \in \mathbb{C}$ are related by $x^S w^m = 1$, the change of functions

$$\begin{cases} f(w) = x^{1/m}\mathcal{F}(x), \\ g(w) = x^{1/m}\mathcal{G}(x) \end{cases}$$

reduces the problems (23) and (24) to the regular differential system

$$\begin{cases} f' = \dfrac{-1}{S} f^{a+1} g^b, \\ g' = \dfrac{-1}{S} f^c g^{d+1}, \end{cases} \tag{25}$$

with boundary conditions at infinity

$$\begin{cases} f(w) = w^{-1/S} + i\dfrac{b}{S} w^{-(1+m)/S} + O\big(|w|^{-(1+2m)/S}\big), \\ g(w) = w^{-1/S} - i\dfrac{c}{S} w^{-(1+m)/S} + O\big(|w|^{-(1+2m)/S}\big). \end{cases} \tag{26}$$

The basic fact for the resolution of (25) is that it admits $1/g^m - 1/f^m$ as first integral: if K is any complex number such that the constant function $1/g^m - 1/f^m$ equals $1/K^m$, then g^m can be straightforwardly expressed as a function of f and (25) implies that f is solution of the ordinary differential equation

$$f' \times \frac{(1 + (f/K)^m)^{b/m}}{(f/K)^{S+1}} = -\frac{K^{S+1}}{S} \tag{27}$$

with boundary conditions coming from (26).

This leads to consider a primitive of the function $z \mapsto (1+z^m)^{b/m}/z^{S+1}$ in the complex field.

6.2. *Abelian integral I and its inverse J.* For all integers m, S and b that satisfy $S \geq 5$, $S/2 < m < S$, $1 \leq b < S/2$, we denote by $I = I_{m,S,b}$ the function

$$I(z) = \int_{[z,z\infty)} (1+u^m)^{b/m} \frac{du}{u^{S+1}} = \frac{1}{z^S} \int_1^{+\infty} [1 + (tz)^m]^{b/m} \frac{dt}{t^{S+1}},$$

where $[z, z\infty)$ denotes the ray $\{tz, t \geq 1\}$ and where the power $1/m$ is used for the principal determination of the mth root. The function I is an Abelian integral on the curve $x^m - y^m = 1$ (which is isomorphic to the famous Fermat curve $x^m + y^m = 1$ by a straightforward linear change of variables), defined on the open set

$$\mathcal{O}_m = \mathbb{C} \setminus \bigcup_{p \in \{0,\dots,m-1\}} \mathbb{R}_{\geq 0} e^{(i\pi/m)(1+2p)}.$$

Note that the integral is convergent because $S - b + 1 \geq 3$. Let $\mathcal{S}_m$ be the open sector of the complex plane defined by

$$\mathcal{S}_m = \left\{ z \in \mathbb{C} \setminus \{0\},\ -\frac{\pi}{m} < \arg(z) < \frac{\pi}{m} \right\}.$$

The open set $\mathcal{O}_m$ is the union of the images of $\mathcal{S}_m$ under all rotations of angles $2k\pi/m$ around the origin, $k \in \mathbb{Z}$.

In the following, the notation $\binom{b/m}{n}$ denotes the ordinary binomial coefficient, generalized for rational (or even complex) values of b/m by Euler's Gamma function. As everywhere else in the paper, the positive integer a is $a = S - b$.

PROPOSITION 6.1 (Properties of I).

(i) *I is holomorphic on $\mathcal{O}_m$ and for any $z \in \mathcal{O}_m$,*

$$I'(z) = -\frac{(1+z^m)^{b/m}}{z^{S+1}}. \tag{28}$$

(ii) *For any mth root of unity ω and for any $z \in \mathcal{O}_m$,*

$$I(\omega z) = \omega^{-S} I(z). \tag{29}$$

(iii) *The function I admits a power series expansion in the neighborhood of infinity in any connected component of $\mathcal{O}_m$. On $\mathcal{S}_m$, this expansion is given by the formula*

$$I(z) = \sum_{n \geq 0} \frac{1}{a+mn} \binom{b/m}{n} z^{-a-mn} = \frac{1}{az^a} + \frac{b}{m(a+m)} \frac{1}{z^{a+m}} + \cdots, \tag{30}$$

valid for any $z \in \mathcal{S}_m$, $|z| \geq 1$.

(iv) *The function I admits a Laurent series expansion in the neighborhood of the origin in any connected component of $\mathcal{O}_m$. On $\mathcal{S}_m$, this expansion is given by the formula*

$$I(z) = \frac{1}{Sz^S} + \frac{b}{m(S-m)} \frac{1}{z^{S-m}} + C_0 - \sum_{n \geq 2} \binom{b/m}{n} \frac{z^{mn-S}}{mn-S}, \tag{31}$$

where C_0 is the constant

$$C_0 = \sum_{n \geq 0} \binom{b/m}{n} \left(\frac{1}{a+mn} + \frac{1}{mn-S} \right). \tag{32}$$

Formula (31) is valid for any $z \in \mathcal{S}_m$, $|z| \leq 1$.

(v) $C_0 < 0$.

PROOF. (i) and (ii) are direct consequences of the definition of I. Expansion (30) and its validity for $z \in \mathcal{S}_m$, $|z| > 1$ comes directly from the power series expansion of $\zeta \mapsto (1+\zeta)^{b/m}$ in the definition of I. Its validity for $|z| = 1$ is given by the convergence of the series at such a z and application of Abel's theorem,[7] proving (iii). To prove expansion (31), notice first that I is holomorphic on the simply connected domain $\mathcal{S}_m$ and $I'(z)$ tends to 0 as z tends to infinity, so that

[7]We refer to the following theorem of Abel: if a series $\sum_n a_n$ is convergent, then the power series $\sum_n a_n z^n$ converges uniformly on the segment [0, 1].

integration on the ray $[z, z\infty)$ is equivalent to integration on $[z,1]$ followed by integration on $[1,+\infty)$. Thus,

$$I(z) = I(1) + \int_{[z,1]} (1+u^m)^{b/m} \frac{du}{u^{S+1}}.$$

Power series expansion of $u \mapsto (1+u)^{b/m}$ under this last integral leads then to (31). The proof of (iv) is again made complete by application of Abel's theorem. Note that, since S is not a multiple of m because of our assumptions on the parameters, the denominators in Formula (31) do not vanish. Finally, if α_n denotes the general term of the series (32), a straightforward computation shows that

$$\alpha_0 + \alpha_1 = \frac{(S-a)(m^2+aS)(S-a-m)}{amS(a+m)(S-m)} < 0,$$

the last inequality coming from $S-a-m < S - S/2 - S/2 = 0$ and from the other hypotheses on the parameters. Furthermore, $\alpha_{2n} + \alpha_{2n+1} < 0$ for any $n \geq 1$, which concludes the proof. [Hint: compute $\alpha_{2n}+\alpha_{2n+1}$, factorize $\binom{b/m}{2n}$ by $\binom{b/m}{2n+1}$, use the fact $(2n+1)/(2n-b/m) > 1$, notice that $\binom{b/m}{2n+1} > 0$ because $0 < b/m = (S-a)/m < S/2m < 1$.] $\square$

Let $\mathbb{H}$ denote Poincaré half-plane:

$$\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\} \quad \text{and} \quad \overline{\mathbb{H}} = \{\overline{z}, z \in \mathbb{H}\}.$$

PROPOSITION 6.2. *The analytic function $I : \mathcal{S}_m \bigcap \mathbb{H} \to \mathbb{C}$ is a conformal mapping onto the open subset*

$$\mathcal{U} = \left\{z, -\frac{a\pi}{m} < \arg(z) < 0\right\} \cup \left(I_1 + \left\{z,\ -\frac{S\pi}{m} < \arg(z) < -\frac{a\pi}{m}\right\}\right)$$

(*see Figure* 2), *where*

$$I_1 := \frac{1}{m} B\left(\frac{a}{m}, \frac{d}{m}\right) e^{-(ia\pi)/m}, \tag{33}$$

and where B denotes Euler's Beta function $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

PROOF. Let $\zeta_m = \exp(i\pi/m)$. We show hereunder that the restriction of I to the sector $\mathcal{S}_m \cap Cl(\mathbb{H})$ (where $Cl(\mathbb{H})$ denotes the topological closure of $\mathbb{H}$) admits a continuous continuation to the ray $\{t\zeta_m, t > 0\}$ and that this continuation maps homeomorphically the boundary of the sector $\mathcal{S}_m \cap \mathbb{H}$ onto the boundary of $\mathcal{U}$. The result is then a consequence of elementary geometrical conformal theory (see, for example, Saks and Zygmund [17]).

Let $h \in \mathbb{H}$, $r > 0$, $t > 1$ and $z = r(1-h)\zeta_m$. When h tends to 0, then $1+(tz)^m = 1 - r^m t^m + m r^m t^m h + O(h^2)$ so that the value of mth root principal determination of $1+(tz)^m$ according to the sign of $1-(rt)^m$ leads to the respective limits in terms of Beta incomplete functions:

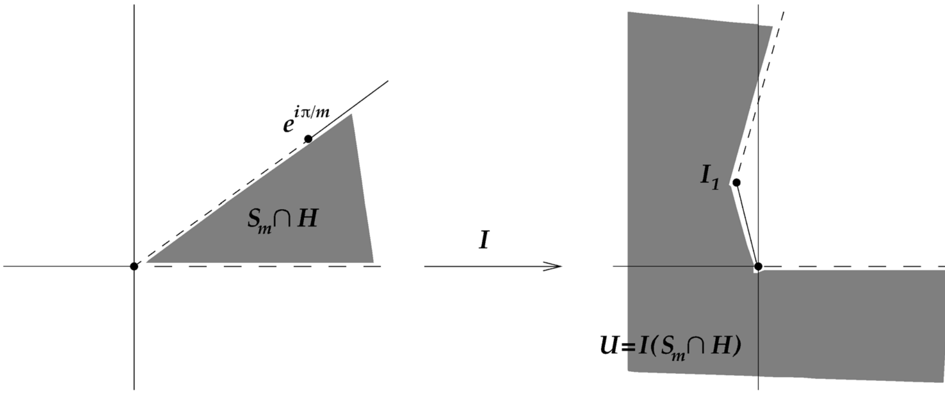

FIG. 2. *Domain $\mathcal{S}_m \bigcap \mathbb{H}$ and its image by I.*

- if $r \geq 1$, then

$$\lim_{z \to r\zeta_m, z \in \mathcal{S}_m} I(z) = \frac{1}{m} \zeta_m^{-a} \int_0^{1/r^m} (1-u)^{b/m} u^{c/m} \, du; \tag{34}$$

- if $r \leq 1$, then

$$\lim_{z \to r\zeta_m, z \in \mathcal{S}_m} I(z) = I_1 + \frac{1}{m} \zeta_m^{-S} \int_1^{1/r^m} (u-1)^{b/m} u^{c/m} \, du. \tag{35}$$

The complex number I_1 is simply

$$I_1 = \lim_{z \to \zeta_m, z \in \mathcal{S}_m} I(z);$$

formula (33) is a consequence of the integral representation of Euler Beta function $B(\alpha, \beta) = \int_0^1 (1-u)^{\alpha-1} u^{\beta-1} \, du$. The monotonicity of real integrals (34) and (35) with respect to r show that the continuous continuation of I defined by these formulae maps decreasingly the ray $]0, +\infty[$ onto itself and respectively, the ray $]0, \zeta_m]$ onto the ray $\{I_1 + t\zeta_m^{-S}, t \geq 0\}$ and the ray $[\zeta_m, \zeta_m \infty)$ onto $[I_1, 0[$. $\square$

REMARK 6.3. By computation in the realm of hypergeometric functions, one shows that the numbers C_0 defined by (32) and I_1 defined by (33) are related by

$$C_0 = -\frac{\sin \pi (1 + b/m)}{\sin \pi (1 + S/m)} |I_1| = -\frac{1}{m} \frac{\sin \pi (1 + b/m)}{\sin \pi (1 + S/m)} B\left(\frac{S-b}{m}, \frac{m+b}{m}\right).$$

DEFINITION 6.4. Let $J = J_{m,S,b} : \mathbb{C} \backslash] - \infty, 0] \to \mathcal{S}_m$ the only continuous function defined by:

- $\forall z \in \overline{\mathbb{H}}$, $J(z) = I^{-1}(z)$ in the sense of Proposition 6.2 ($\overline{\mathbb{H}}$ is an open subset of $\mathcal{U}$ so that this functional inverse exists);

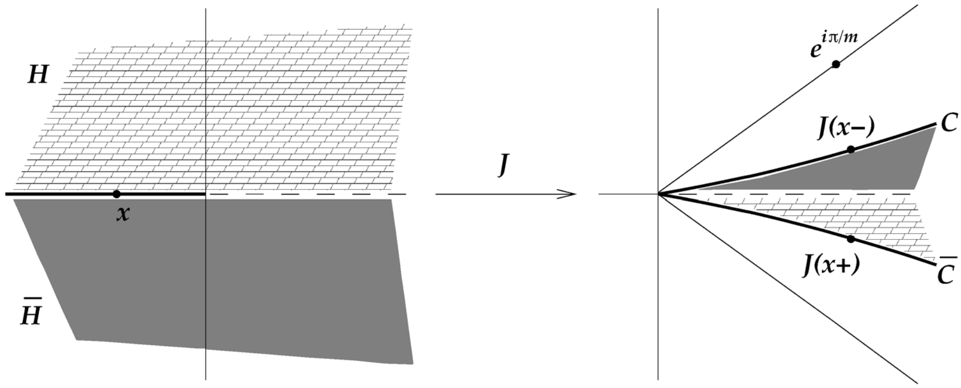

FIG. 3. *Action of J on the slit plane $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.*

- $\forall z \in \mathbb{H}$, $J(z) = \overline{J(\overline{z})}$ (complex conjugacy).

The properties of I shown in Propositions 6.1 and 6.2 imply that J is a conformal mapping between $\mathbb{C}\backslash]-\infty, 0]$ and an open subset of $\mathcal{S}_m$ (use Schwarz reflection principle), that maps $\mathbb{H}$ into $\mathcal{S}_m \cap \overline{\mathbb{H}}$ and $\overline{\mathbb{H}}$ into $\mathcal{S}_m \cap \mathbb{H}$. If $\mathcal{C}$ denotes the inverse of the negative real axis by the restriction of I to $\mathcal{S}_m \cap \mathbb{H}$, then the boundary of the image of J is $\mathcal{C} \cup \overline{\mathcal{C}} \cup \{0\}$ (see Figure 3). Furthermore, the restriction of J to the positive real half-line is the inverse of I's and J is the unique analytic expansion of $(I_{|]0,+\infty[})^{-1}$ to the slit plane. Naturally, the formula $J(\overline{z}) = \overline{J(z)}$ is valid when z is any nonnegative complex number.

PROPOSITION 6.5. *For any negative real number x, both limits*

$$\lim_{z \to x, z \in \mathbb{H}} J(z) \quad \textit{and} \quad \lim_{z \to x, z \in \overline{\mathbb{H}}} J(z)$$

exist, are nonreal and conjugate (thus, different).

PROOF. Direct consequence of the preceding properties of J and Proposition 6.2 (see Figure 3). $\square$

We adopt the following notation:

$$\forall x < 0 \qquad \begin{cases} J(x-) = \lim\limits_{z \to x, z \in \overline{\mathbb{H}}} J(z) \in \mathcal{S}_m \cap \mathbb{H}, \\ J(x+) = \lim\limits_{z \to x, z \in \mathbb{H}} J(z) \in \mathcal{S}_m \cap \overline{\mathbb{H}}. \end{cases} \tag{36}$$

PROPOSITION 6.6. *The function J admits, as z tends to infinity in the slit plane $\mathbb{C} \setminus \mathbb{R}_-$, an asymptotic Puiseux series expansion at any order in the scale*

$$\left(\frac{1}{z}\right)^{1/S + p\sigma + q}, \qquad (p, q) \in \mathbb{N}^2,$$

where all fractional powers denote principal determination. The beginning of this asymptotic expansion is

$$J(z) = \left(\frac{1}{Sz}\right)^{1/S} + \frac{b}{m(S-m)}\left(\frac{1}{Sz}\right)^{(m+1)/S} + C_0\left(\frac{1}{Sz}\right)^{(S+1)/S} + o\left(\frac{1}{z}\right)^{(S+1)/S}. \tag{37}$$

PROOF. Expansion (31) leads to (37) using the reversion formula $J \circ I = \mathrm{Id}$. □

6.3. *Computation of characteristic functions.* This section gives an explicit closed form of characteristic functions $\mathcal{F}$ and $\mathcal{G}$ for the elementary continuous-time urn processes X and Y [defined in (20)] associated with the replacement matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, in terms of the just defined functions J. Remember: the urn is supposed to be large and nontriangular so that $b > 0$ and $c > 0$. Let κ be the positive number defined by

$$\kappa = \sqrt[m]{\frac{S}{m(S-m)}}. \tag{38}$$

THEOREM 6.7. *The characteristic functions $\mathcal{F}$ and $\mathcal{G}$ are the unique solutions of the differential system* (23) *that satisfy boundary conditions* (24). *They are given by the formulae*

$$\forall x > 0 \qquad \begin{cases} \mathcal{F}(x) = \kappa e^{-i\pi/(2m)} x^{-1/m} J_{m,S,b}\left(C_0 + \dfrac{\kappa^S e^{-i\pi S/(2m)}}{S} x^{-S/m}\right), \\ \mathcal{G}(x) = \kappa e^{i\pi/(2m)} x^{-1/m} J_{m,S,c}\left(C_0 + \dfrac{\kappa^S e^{i\pi S/(2m)}}{S} x^{-S/m}\right) \end{cases} \tag{39}$$

and

$$\forall x \in \mathbb{R} \qquad \mathcal{F}(-x) = \overline{\mathcal{F}(x)}, \qquad \mathcal{G}(-x) = \overline{\mathcal{G}(x)}. \tag{40}$$

PROOF. (1) We first solve (23) on $\mathbb{R}_{>0}$. Let F and G be solutions of (23) that satisfy (24). Lets do the change of variable $x \in \mathbb{R}_{>0} \to w = x^{-S/m} \in \mathbb{R}_{>0}$ and the change of functions

$$f(w) = w^{-1/S} F(w^{-m/S}) \quad \text{and} \quad g(w) = w^{-1/S} G(w^{-m/S})$$

that is straightforwardly reversed by the formula $F(x) = x^{-1/m} f(x^{-S/m})$ and a similar one for G and g. Then f and g are solutions of (25) on $\mathbb{R}_{>0}$ and satisfy boundary conditions (26) at $+\infty$. In particular, since (25) is a nonsingular differential system, Cauchy–Lipschitz theorem guarantees that if (f, g) is any solution, then f (resp., g) is identically zero or does not vanish. This implies that f and g

do not vanish on $\mathbb{R}_{>0}$. Because of the balance conditions $a+b=c+d$, differentiation of $1/g^m - 1/f^m$ leads to the fact that this function is constant on $\mathbb{R}_{>0}$ (first integral). Furthermore, boundary conditions at $+\infty$ (26) imply that this constant value is $i\frac{m}{S}(b+c)$. If K denotes the complex number

$$K = \kappa \exp\left(-\frac{i\pi}{2m}\right)$$

[$\kappa > 0$ has been defined by formula (38)], this shows that

$$\forall w > 0 \qquad \frac{1}{g^m(w)} - \frac{1}{f^m(w)} = \frac{1}{K^m}. \tag{41}$$

Since f/g is continuous on $\mathbb{R}_{>0}$, does not vanish and tends to 1 at $+\infty$ (26), relation $(f/g)^m = 1 + (f/K)^m$ implies that, on $\mathbb{R}_{>0}$,

$$g = \frac{f}{(1+(f/K)^m)^{1/m}} \tag{42}$$

(principal determination of the mth root). Reporting in the first equation of (25) shows that f is necessarily a solution of equation (27) on $\mathbb{R}_{>0}$. Boundary conditions (26) imply that, when w tends to $+\infty$, $\frac{1}{K} f(w) \sim \frac{1}{\kappa} e^{i\pi/2m} w^{-1/S} \in \mathcal{S}_m$, so that equation (27) can be written

$$\frac{d}{dw} I_{m,S,b}\left(\frac{f(w)}{K}\right) = \frac{K^S}{S}$$

in a neighborhood of $+\infty$. Integration of this equation shows that

$$I_{m,S,b} \circ \left(\frac{f}{K}\right)(w) = \frac{K^S}{S} w + C_1$$

in a neighborhood of $w = +\infty$, for a suitable complex constant C_1. The determination of C_1 is made by means of local expansions: since f tends to 0 at $+\infty$, using (31) and previous equality leads to

$$C_1 + \frac{K^S}{S} w = \frac{K^S}{Sf(w)^S} + \frac{b}{m(S-m)} \frac{K^{S-m}}{f(w)^{S-m}} + C_0 + o(1),$$

when w tends to $+\infty$, so that boundary conditions (26) lead to the equality $C_1 = C_0$. Note that this computation makes use of the big-O in (26), of the assumption $1 - 2m/S < 0$ (large urn) and of the relation $S - m = b + c$. Thus, necessarily,

$$f(w) = K\, J_{m,S,b}\left(C_0 + \frac{K^S}{S} w\right) \tag{43}$$

for any w in a neighborhood of $+\infty$. The function $w \to K\, J_{m,S,b}(C_0 + K^S w/S)$ is well defined on $\mathbb{R}_{>0}$ because $C_0 < 0$ [Proposition 6.1(5)] and $-\pi < \arg(K^S) <$

$-\pi/2$, so that it is the only maximal solution on $\mathbb{R}_{>0}$ of equation (27) that satisfies the first equation of (26). This shows finally that

$$\forall x > 0 \qquad F(x) = Kx^{-1/m} J_{m,S,b}\left(C_0 + \frac{K^S}{S} x^{-S/m}\right).$$

Since $-K^m = \overline{K}^m$, the same arguments show that, for any $w > 0$,

$$g(w) = \overline{K} J_{m,S,c}\left(C_0 + \frac{\overline{K}^S}{S} w\right),$$

which shows completely formula (39).

(2) The resolution on $\mathbb{R}_{<0}$ is made the same way. To this effect, lets do the new change of variable $x \in \mathbb{R}_{<0} \to w = |x|^{-S/m} e^{i\pi S/m} \in \mathbb{R}_{>0} e^{i\pi S/m}$. Lets do as well the change of functions

$$f(w) = e^{-i\pi/m} |w|^{-1/S} F(-|w|^{-m/S})$$

and

$$g(w) = e^{-i\pi/m} |w|^{-1/S} G(-|w|^{-m/S}).$$

These changes of variable and functions are reversed by the formulae $x = -|w|^{-m/S}$ and $F(x) = e^{i\pi/m}|x|^{-1/m} f(e^{i\pi S/m}|x|^{-S/m})$ with a similar formula for G and g. Functions f and g are still solutions of (25) but boundary conditions become, as $|w|$ tends to infinity,

$$\begin{cases} f(w) = e^{-i(\pi/m)}|w|^{-1/S}\left(1 - i\dfrac{b}{S}|w|^{-m/S} + O(|w|^{-2m/S})\right), \\ g(w) = e^{-i(\pi/m)}|w|^{-1/S}\left(1 + i\dfrac{c}{S}|w|^{-m/S} + O(|w|^{-2m/S})\right). \end{cases} \tag{44}$$

This implies that First integral (41) is still valid (same K) and, since f and g are still equivalent at infinity, relation (42) is satisfied. Boundary conditions (44) imply that, when w tends to $+\infty$, $\frac{1}{K} f(w) \sim \frac{1}{\kappa}|w|^{-1/S} e^{-i\pi/2m} \in \mathcal{S}_m$. Consequently, the same arguments as before show that formula (43) remains valid (note that $C_0 + wK^S/S \in \mathbb{H}$ so that this formula is well defined for any w). This shows that

$$\forall x < 0 \qquad F(x) = Ke^{i\pi/m}|x|^{-1/m} J_{m,S,b}\left(C_0 + \frac{K^S}{S} e^{i\pi(S/m)}|x|^{-S/m}\right).$$

Since $Ke^{i\pi/m} = \overline{K}$, one gets finally $F(-x) = \overline{F(x)}$ for any real number x. The proof of the whole theorem is made complete by the same arguments for G. $\square$

REMARK 6.8. Formula (40) on characteristic functions comes directly from the fact that X and Y are real-valued random variables.

We want to know more about the analyticity properties of $\mathcal{F}$ and $\mathcal{G}$ around 0. Let $\varphi = \varphi_{m,S,b}$ be the function defined by the formula

$$\varphi(z) = \kappa z^{-1/m} J_{m,S,b}\left(C_0 + \frac{\kappa^S}{S}(z^{-1/m})^S\right), \tag{45}$$

where the power $1/m$ denotes the principal determination of the mth root. Note that κ and C_0, respectively, defined by formulas (38) and (32) are functions of m, S and b too. If ρ denotes the positive number

$$\rho = \left(\frac{S|C_0|}{\kappa^S}\right)^{-m/S} = \frac{S^{1-S/m}|C_0|^{-m/S}}{m(S-m)},$$

it follows from the properties of J that φ is defined and holomorphic on the open set

$$\mathcal{V} = \mathbb{C} \setminus \{(-\infty, 0] \cup [\rho, +\infty)\}.$$

Furthermore, the characteristic functions $\mathcal{F}$ and $\mathcal{G}$ are restrictions of φ functions on the imaginary axis: for any $x \in \mathbb{R}$,

$$\mathcal{F}(x) = \varphi_{m,S,b}(ix) \quad \text{and} \quad \mathcal{G}(x) = \varphi_{m,S,c}(-ix).$$

Note that κ is a function of (m, S) so that the same κ appears in both functions $\varphi_{m,S,b}$ and $\varphi_{m,S,c}$ (respective constants C_0 and ρ are however different).

PROPOSITION 6.9. *The function φ, holomorphic on $\mathcal{V}$, cannot be analytically extended on a larger subset of $\mathbb{C}$. However, setting $\varphi(0) = 1$ defines a continuously differentiable extension of φ on $\mathcal{V} \cup \{0\}$.*

PROOF. The half-line $[\rho, +\infty)$ is the locus of complex z such that $C_0 + \frac{\kappa^S}{S}(z^{-1/m})^S$ is a real nonpositive number (remember that $m < S < 2m$). Since the principal determination of the mth root is well defined and nonzero in a neighbourhood of this half-line, Proposition 6.5 implies that φ cannot be continuously extended at any point of $[\rho, +\infty)$.

If x is a negative number, definition of the principal determination of the mth root leads to the existence of both limits

$$\begin{cases} \lim\limits_{z \to x, z \in \mathbb{H}} \varphi(z) = \kappa e^{-i(\pi/m)}|x|^{-1/m} J\left(C_0 + \dfrac{\kappa^S}{S} e^{-i(\pi S/m)}|x|^{-S/m}\right) := \varphi(x+), \\ \lim\limits_{z \to x, z \in \overline{\mathbb{H}}} \varphi(z) := \varphi(x-) = \overline{\varphi(x+)}. \end{cases}$$

Since the image of J is included in $\mathcal{S}_m$, the limit $\varphi(x+)$ belongs to the open sector $e^{-i(\pi/m)}\mathcal{S}_m$ which contains no real number, so that $\varphi(x+) \neq \varphi(x-)$. This shows that φ cannot be continuously extended at any point of $\mathbb{R}_{<0}$.

When z tends to 0 in the slit plane $\mathbb{C} \setminus \mathbb{R}_{<0}$, Proposition 6.6 shows that $\varphi(z)$ tends to 1. One step more, computing the derivative of φ in terms of J using the algebraic expression of I' (28) implies, with expansion (37), that

$$\lim_{z\to 0, z\in\mathbb{C}\setminus\mathbb{R}_{\leq 0}} \varphi'(z) = \frac{b}{S}. \qquad \square$$

COROLLARY 6.10. *The exponential moment generating series*

$$\sum \frac{\mathbb{E}(X^p)}{p!} T^p \quad \textit{and} \quad \sum \frac{\mathbb{E}(Y^p)}{p!} T^p$$

have a radius of convergence equal to 0.

PROOF. These series are the Taylor series of $\varphi_{m,S,b}$ and $\varphi_{m,S,c}$ at 0. If these radii were positive, these functions could be analytically extended to a neighborhood of the origin. $\square$

REMARK 6.11. The singularity of φ at the origin is thus not due to ramification but to a divergent Taylor series phenomenon. Indeed, the apparent ramification coming from the mth root at the origin in formula (45) is compensated by both Puiseux expansion (37) and the Sth power of the mth root appearing in the argument of J in formula (45).

7. Density of W^{CT}. Notice, with the notation (36) that

$$\mathcal{F}(x) \underset{x\to+\infty}{\sim} \kappa J(C_0-)x^{-1/m}, \tag{46}$$

where the nonreal complex number $J(C_0-)$ is different from 0 (see Figure 3).

A first consequence is that $\mathcal{F}(x)$ tends to 0 when x tends to $+\infty$. Hence, the probability distribution function of W^{CT} is continuous so that the law of W^{CT} has no point mass.

A second consequence is that $\mathcal{F}$ is not in L^1 so that W^{CT} distribution cannot be obtained by classical Fourier inversion. Nevertheless, we obtain in Section 7.3 an expression of this density using the derivative of the characteristic function $\mathcal{F}$. Before, we need firstly to ensure that the support of W^{CT} is the whole real line $\mathbb{R}$ which is proven in Section 7.1 and secondly to ensure that W^{CT} admits a density which is proven in Section 7.2 using the martingale connection (18). As usually, this kind of connection induces a smoothing phenomenon between W^{DT} and W^{CT}, allowing us to prove that W^{CT} has a density, whatever W^{DT} distribution is.

7.1. *Support of W^{CT}.*

PROPOSITION 7.1. *The support of W^{CT} is $\mathbb{R}$.*

PROOF. As in (20), let X denote the random variable W^{CT} starting from one red ball. Because of the branching property (see beginning of Section 4.1), it suffices to prove that the support of X is the whole real line $\mathbb{R}$. General results on infinite divisibility (see, for instance, Steutel and van Harn [16], page 186) ensure that the support of an infinitely divisible random variable having a continuous probability distribution function is either a half-line or $\mathbb{R}$. Suppose that the support of X is $[\alpha, +\infty[$ for a given real number α. Then denoting X distribution by μ_X,

$$\mathbb{E}(e^{-sX}) = \int_{\alpha}^{+\infty} e^{-st}\, d\mu_X(t) = e^{-s\alpha} \int_{\alpha}^{+\infty} e^{-s(t-\alpha)}\, d\mu_X(t)$$

exists for every real number $s \geq 0$. Hence, the function $L : s \to \mathbb{E}(e^{-sX})$ is analytic on the half-plane $\{\Re z > 0\}$, continuous on the boundary of this half-plane and $\lim_{t\to\pm\infty} \mathbb{E}(e^{itX}) = 0$. By unicity of the analytic continuation, necessarily:

$$L(s) = \varphi(-s) \qquad \forall s, \Re(s) \geq 0,$$

where φ has been introduced in (45). But it has been proven in Proposition 6.9 that φ cannot be analytically extended on the half-plane $\{\Re z < 0\}$. There is a contradiction: the support of X cannot be a half-line $[\alpha, +\infty[$.

In the same way, if we suppose that the support of W^{CT} is $]-\infty, \beta]$ for a given real number β, we are led to a contradiction, because φ cannot be analytically extended on the whole half-plane $\{\Re z > 0\}$ (Proposition 6.9). $\square$

7.2. *Connection between the distribution of W^{DT} and the density of W^{CT}.*

PROPOSITION 7.2. *Let μ be the distribution of W^{DT} (it is a probability measure on $\mathbb{R}$).*

(1) *W^{CT} admits a density p on $\mathbb{R}$ given by*

$$\begin{cases} \forall w > 0 \qquad p(w) = \dfrac{1}{\sigma}\dfrac{1}{\Gamma(1/S)} w^{-1+\frac{1}{m}} \displaystyle\int_{]0,+\infty[} v^{-1/m} e^{-(w/v)^{1/\sigma}}\, d\mu(v), \\ \forall w < 0 \qquad p(w) = \dfrac{1}{\sigma}\dfrac{1}{\Gamma(1/S)} |w|^{-1+\frac{1}{m}} \displaystyle\int_{]-\infty,0[} |v|^{-1/m} e^{-(w/v)^{1/\sigma}}\, d\mu(v). \end{cases}$$

(2) *The density p is infinitely differentiable and increasing on $\mathbb{R}_{<0}$, infinitely differentiable and decreasing on $\mathbb{R}_{>0}$; it is not continuous at* 0: $\lim_{w\to 0, w\neq 0} p(w) = +\infty$. *In particular, the distribution is unimodal, the mode is* 0.

PROOF. (1) To exhibit a density, let us take any real-valued bounded continuous function h defined on $\mathbb{R}$ and, thanks to the martingale connection (18), compute

$$\mathbb{E}(h(W^{CT})) = \int_{\mathbb{R}} \int_0^{+\infty} h(uv) g(u)\, du\, d\mu(v),$$

where g is the density of ξ^σ. After the change of variable $w = uv$, we get

$$\mathbb{E}(h(W^{CT})) = \int_{]-\infty,0[} \frac{d\mu(v)}{|v|} \int_{-\infty}^{0} h(w) g\left(\frac{w}{v}\right) dw + \mu(\{0\}) h(0) + \int_{]0,+\infty[} \frac{d\mu(v)}{v} \int_{0}^{+\infty} h(w) g\left(\frac{w}{v}\right) dw.$$

Recall that W^{CT} has no point mass (see Section 7, introductory paragraph), so we get that W^{CT} admits a density given by

$$p(w) = \mathbf{1}_{\mathbb{R}_{<0}}(w) \int_{]-\infty,0[} g\left(\frac{w}{v}\right) \frac{d\mu(v)}{|v|} + \mathbf{1}_{\mathbb{R}_{>0}}(w) \int_{]0,+\infty[} g\left(\frac{w}{v}\right) \frac{d\mu(v)}{v}. \tag{47}$$

The only point to verify is that the integrals in formula (47) are well defined. The density g is explicit. To simplify the notation, we consider the case when we start from one ball ($u = 1$). In this case,

$$g(x) = \frac{1}{\sigma} \frac{1}{\Gamma(1/S)} x^{-1+\frac{1}{m}} e^{-x^{1/\sigma}} \mathbf{1}_{x>0}, \tag{48}$$

so that, for any nonzero w,

$$\frac{1}{|v|} g\left(\frac{w}{v}\right) = C|w|^{-1+\frac{1}{m}} |v|^{-1/m} e^{-|w|^{1/\sigma}|v|^{-1/\sigma}}$$

is bounded as a function of v.

(2) Let us prove that $\lim_{w\to 0^+} p(w) = +\infty$, looking at

$$\lim_{w\to 0^+} w^{-1+\frac{1}{m}} \int_{]0,+\infty[} v^{-1/m} e^{-(w/v)^{1/\sigma}} \, d\mu(v).$$

The last integral, for any $w < 1$, is greater than

$$\int_{]0,+\infty[} v^{-1/m} e^{-(1/v)^{1/\sigma}} \, d\mu(v),$$

so that it is sufficient to prove that this integral is a positive constant. If not, this integral would be equal to zero, and this happens only if the support of μ is included in $]-\infty, 0]$. By the martingale connection (18), this would imply that the support of W^{CT} is included in $]-\infty, 0]$, which is not the case because of Proposition 7.1.

The result on the limit of p at 0^- is proved the same way. Differentiability is immediate by dominated convergence and monotonicity comes from derivation of formula (47). □

REMARK 7.3. The distribution of W^{CT} is not symmetric around 0 (the expectation equals $\frac{b}{S} \neq 0$ when one starts with only one red ball).

7.3. *Fourier inversion.* The characteristic function $\mathcal{F}$ is not integrable. Nevertheless, formulas (23) and (46), imply straightforwardly that, for any real $x \neq 0$,

$$\mathcal{F}'(x) = \frac{1}{mx}\mathcal{F}(x)[\mathcal{F}^a(x)\mathcal{G}^b(x) - 1]$$

and that $\mathcal{F}'$ is in L^1. Theorem 7.4 gives an explicit expression of the density of W^{CT} by means of inverse Fourier transform of $\mathcal{F}'$, completing Proposition 7.2.

THEOREM 7.4. *The density p on $\mathbb{R}$ of the random variable W^{CT} is given, for any $x \neq 0$, by*

$$p(x) = \frac{1}{2i\pi x}\int_{\mathbb{R}} e^{-itx}\mathcal{F}'(t)\,dt. \tag{49}$$

PROOF. Let F be the probability distribution function of W^{CT}. We are going to show that $\forall x \neq 0$,

$$\lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = \frac{1}{2i\pi x}\int_{\mathbb{R}} e^{-itx}\mathcal{F}'(t)\,dt, \tag{50}$$

which is sufficient to prove that W^{CT} admits a continuous density given by (49).

For any $h \neq 0$, let d_h be the function defined on $\mathbb{R}\setminus\{0\}$ by

$$d_h(t) := \frac{1 - e^{-ith}}{ith}$$

and continuated by continuity at 0. It follows from the general Fourier inversion theorem (see, for instance, Lukacs [11], Theorem 3.2.1, page 38) that $\forall x \in \mathbb{R}$, $\forall h \neq 0$, since x and $x+h$ are continuity points of F (remember that F is continuous because its characteristic function tends to 0 at infinity),

$$\frac{F(x+h) - F(x)}{h} = \lim_{T\to+\infty} I_{T,h}(x),$$

where

$$I_{T,h}(x) := \frac{1}{2\pi}\int_{-T}^{T} e^{-itx}d_h(t)\mathcal{F}(t)\,dt.$$

Integrating by parts implies that, for any $x \neq 0$,

$$I_{T,h}(x) = I_{T,h}^{(1)}(x) + I_{T,h}^{(2)}(x) + I_{T,h}^{(3)}(x)$$

where

$$\begin{cases} I_{T,h}^{(1)}(x) = \dfrac{1}{2\pi}\Big[-\dfrac{e^{-iTx}}{ix}d_h(T)\mathcal{F}(T) + \dfrac{e^{iTx}}{ix}d_h(-T)\mathcal{F}(-T)\Big], \\ I_{T,h}^{(2)}(x) = \dfrac{1}{2i\pi x}\int_{-T}^{T} e^{-itx}d_h(t)\mathcal{F}'(t)\,dt, \\ I_{T,h}^{(3)}(x) = \dfrac{1}{2i\pi x}\int_{-T}^{T} e^{-itx}d_h'(t)\mathcal{F}(t)\,dt. \end{cases}$$

It is elementary to see that $d_h(t)$ has the following properties: $\forall h \neq 0, \forall t \neq 0$,

$$|d_h(t)| = \left|\frac{\sin th/2}{th/2}\right| \leq \min\left\{1, \frac{2}{|th|}\right\}, \tag{51}$$

$$|d_h'(t)| \leq \min\left\{\frac{|h|}{2}, \frac{2}{|t|}\right\}. \tag{52}$$

Since $\mathcal{F}$ is bounded (it is a characteristic function) and since d_h tends to 0 at infinity,

$$\lim_{T\to+\infty} I_{T,h}^{(1)}(x) = 0.$$

Since $\mathcal{F}' \in \mathrm{L}^1$, (51) and Lebesgue dominated convergence theorem lead to

$$\lim_{T\to+\infty} I_{T,h}^{(2)}(x) = \frac{1}{2i\pi x}\int_{\mathbb{R}} e^{-itx} d_h(t)\mathcal{F}'(t)\,dt.$$

At least, (52) implies that $d_h'\mathcal{F} \in \mathrm{L}^1$ so that, by dominated convergence,

$$\lim_{T\to+\infty} I_{T,h}^{(3)}(x) = \frac{1}{2i\pi x}\int_{\mathbb{R}} e^{-itx} d_h'(t)\mathcal{F}(t)\,dt.$$

So, for any $x \neq 0$ and $h \neq 0$,

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{2i\pi x}\int_{\mathbb{R}} e^{-itx} d_h(t)\mathcal{F}'(t)\,dt + \frac{1}{2i\pi x}\int_{\mathbb{R}} e^{-itx} d_h'(t)\mathcal{F}(t)\,dt.$$

To get (50), it is now sufficient to take the limit when $h \to 0$, using dominated convergence and (52). □

REMARK. We have not found the following result in the literature but the arguments of this proof lead to the following proposition.

PROPOSITION 7.5. *Let $\mathcal{F}$ be the characteristic function of a probability distribution function F. Suppose that $\mathcal{F}$ is derivable, $\mathcal{F}' \in \mathrm{L}^1$ ($\mathcal{F}$ is not necessarily in L^1) and $\frac{\mathcal{F}(t)}{t} \in \mathrm{L}^1$. Then F admits a density p given for all $x \neq 0$ by*

$$p(x) = \frac{1}{2i\pi x}\int_{\mathbb{R}} e^{-itx}\mathcal{F}'(t)\,dt.$$

8. Concluding remarks.

8.1. *More colors.* The same questions arise naturally for limit laws of large urn processes with any finite number of colors. Embedding in continuous time, martingale connection, dislocation equations on elementary limit distributions and differential system (23) on Fourier transforms or on formal Laplace power series can be generalized. However, the resolution of (23) relies on the question of its integrability, even if an explicit closed form of its solutions may not be necessary to derive properties of the corresponding distributions.

The space requirements of an m-ary search tree is a special case of Pólya–Eggenberger urn process with $m-1$ colors (see [5], for example). Because of the negativeness of the diagonal entries $-1, -2, \dots, -(m-1)$ of its replacement matrix, the corresponding continuous-time Markov process is not a branching process. However, the discrete-time node process of an m-ary search tree can be embedded into a branching process. When $m \geq 27$, the corresponding limit laws can be studied with the same method as in the present paper. This is the subject of a forthcoming companion paper.

8.2. *Laplace series*. Remember from Section 4.2 that X (resp., Y) is the martingale limit W^{CT} of the continuous-time urn process starting from $(1,0)$ [resp., from $(0,1)$]. For $n \geq 0$, let

$$a_n = \mathbb{E}(X^n) \quad \text{and} \quad b_n = \mathbb{E}(Y^n),$$

and let F and G be the Laplace series of X and Y, that is, the formal exponential series of the moments:

$$F(T) = \sum_{n\geq 0} \frac{a_n}{n!} T^n \quad \text{and} \quad G(T) = \sum_{n\geq 0} \frac{b_n}{n!} T^n \in \mathbb{R}[[T]].$$

From equations (21), we write recursion formulae relating $(a_k)_{0\leq k\leq n}$ and $(b_k)_{0\leq k\leq n}$. Thanks to the multinomial formula, they arrange themselves into the differential system with boundary conditions:

$$\begin{cases} F(T) + mTF'(T) = F(T)^{a+1}G(T)^b, \\ G(T) + mTG'(T) = F(T)^c G(T)^{d+1}, \\ F(0) = G(0) = 1, \\ F'(0) = \dfrac{b}{S} \quad \text{and} \quad G'(0) = -\dfrac{c}{S}. \end{cases} \tag{53}$$

The fact that the urn is large implies that equations (53) characterize the moments of X and Y. Indeed, proceed by recursion: for any $n \geq 2$, $v_n = (a_n, b_n)$ is the solution of a linear system of the form $(R - nmI)(v_n) =$ [polynomial function of $v_1, \dots, v_{n-1}$], R being the replacement matrix of the process (5). Since the urn is large, $nm > nS/2 \geq S$ so that nm is not an eigenvalue of R.

A remarkable fact, which explains why we have worked with characteristic functions and not with Laplace transforms, is that, for nontriangular urns, that is, when $bc \neq 0$, series F and G have a radius of convergence equal to 0 (Corollary 6.10).

8.3. *Question*. The main theorem provides a family of distributions, those of the W^{CT}, indexed by the three parameters S, m, b of the urn and by the initial condition (α, β). A challenging question is: can the physical relations between these distributions be translated into relations between the Abelian integrals? In otherwords, can the addition formulas between Abelian integrals be interpreted by a combinatorial/probabilistic approach using these distributions?

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B. CHAUVIN
INRIA ROCQUENCOURT, PROJECT ALGORITHMS
DOMAINE DE VOLUCEAU B.P.105
78153 LE CHESNAY CEDEX
FRANCE
AND
LABORATOIRE DE MATHÉMATIQUES
DE VERSAILLES
CNRS, UMR 8100
UNIVERSITÉ DE VERSAILLES, ST. QUENTIN
45, AVENUE DES ETATS-UNIS
78035 VERSAILLES CEDEX
FRANCE
E-MAIL: chauvin@math.uvsq.fr
URL: http://www.math.uvsq.fr/~chauvin/

N. POUYANNE
LABORATOIRE DE MATHÉMATIQUES
DE VERSAILLES
CNRS, UMR 8100
UNIVERSITÉ DE VERSAILLES, ST. QUENTIN
45, AVENUE DES ETATS-UNIS
78035 VERSAILLES CEDEX
FRANCE
E-MAIL: pouyanne@math.uvsq.fr
URL: http://www.math.uvsq.fr/~pouyanne/

R. SAHNOUN
LABORATOIRE DE MATHÉMATIQUES DE VERSAILLES
CNRS, UMR 8100
UNIVERSITÉ DE VERSAILLES, ST. QUENTIN
45, AVENUE DES ETATS-UNIS
78035 VERSAILLES CEDEX
FRANCE
E-MAIL: sahnoun@math.uvsq.fr