STATIONARY SYSTEMS OF GAUSSIAN PROCESSES

BY ZAKHAR KABLUCHKO

Georg-August-Universität Göttingen

We describe all countable particle systems on $\mathbb{R}$ which have the following three properties: independence, Gaussianity and stationarity. More precisely, we consider particles on the real line starting at the points of a Poisson point process with intensity measure $\mathfrak{m}$ and moving independently of each other according to the law of some Gaussian process ξ. We classify all pairs $(\mathfrak{m}, \xi)$ generating a *stationary* particle system, obtaining three families of examples. In the first, trivial family, the measure $\mathfrak{m}$ is arbitrary, whereas the process ξ is stationary. In the second family, the measure $\mathfrak{m}$ is a multiple of the Lebesgue measure, and ξ is essentially a Gaussian stationary increment process with linear drift. In the third, most interesting family, the measure $\mathfrak{m}$ has a density of the form $\alpha e^{-\lambda x}$, where $\alpha > 0$, $\lambda \in \mathbb{R}$, whereas the process ξ is of the form $\xi(t) = W(t) - \lambda \sigma^2(t)/2 + c$, where W is a zero-mean Gaussian process with stationary increments, $\sigma^2(t) = \operatorname{Var} W(t)$, and $c \in \mathbb{R}$.

1. Introduction.

1.1. *Statement of the problem.* Stationary systems of particles evolving independently of each other according to the law of a Markov process have been extensively studied by many authors (see, e.g., the monographs [5], Chapter 1, [11], Chapter 1, [18], as well as the papers [3, 4, 7, 8, 13, 14], to cite only a few references). The aim of the present paper is to study systems of particles evolving independently of each other in a *Gaussian* rather than *Markovian* way. Our main result provides a classification of all those Gaussian particle systems which are *stationary*.

We are interested in at most countable systems of particles moving randomly on the real line in such a way that the following three requirements are satisfied:

(A1) The particles are independent of each other.
(A2) The law describing the motion of each particle is Gaussian and the same for all particles.
(A3) The particles are in an equilibrium.

The independence stated in requirement (A1) implies that the starting positions of particles should be scattered independently over $\mathbb{R}$, which, in more rigorous

terms, means that they should form a not necessarily homogeneous Poisson point process on $\mathbb{R}$. Requirement (A2) means that the stochastic processes describing the deviations of the particles from their starting positions should be Gaussian, having the same law for all particles, and, by requirement (A1), independent of each other.

In view of this, the meaning of the first two requirements may be described in rigorous terms as follows. Let $\{U_i, i \in \mathbb{N}\}$ be a Poisson point process on $\mathbb{R}$ with intensity measure $\mathfrak{m}$. We will always assume that $\mathfrak{m}$ satisfies the following integrability condition:

$$(1) \qquad \int_{\mathbb{R}} e^{-\varepsilon x^2} \mathfrak{m}(dx) < \infty \qquad \text{for every } \varepsilon > 0.$$

In most cases of interest, the measure $\mathfrak{m}$ will be infinite, and so let us agree to use $\mathbb{N}$ as an index set for the points U_i, even though the case where $\mathfrak{m}$ is finite (and, hence, a.s. only finitely many points U_i exist) is not formally excluded.

Let $\xi_i, i \in \mathbb{N}$, be independent copies of a Gaussian process $\{\xi(t), t \in \mathbb{R}^d\}$. We define $V_i(t)$, the position of ith particle at time $t \in \mathbb{R}^d$ (which we allow to be multidimensional), by

$$(2) \qquad V_i(t) = U_i + \xi_i(t).$$

DEFINITION 1.1. The random collection of functions $\mathfrak{P} = \{V_i, i \in \mathbb{N}\}$ will be called the *independent Gaussian particle system* (or simply *Gaussian system*) generated by the pair $(\mathfrak{m}, \xi)$. We use the notation $GS(\mathfrak{m}, \xi)$.

REMARK 1.1. It should be stressed that we *do not* assume the process ξ to have zero mean, which means that we allow for a deterministic component in the random motion of particles. In general, it also may happen that $\xi(0) \neq 0$, in which case the particles make nonzero jumps immediately after starting at U_i.

Let us turn to requirement (A3). Given $t_1, \ldots, t_n \in \mathbb{R}^d$, we define a point process $\mathfrak{P}_{t_1,\ldots,t_n}$ on $\mathbb{R}^n$ by recording the positions of particles at times $t_1, \ldots, t_n$. That is, we set

$$(3) \qquad \mathfrak{P}_{t_1,\ldots,t_n} = \{(V_i(t_1), \ldots, V_i(t_n)), i \in \mathbb{N}\}.$$

The family $\{\mathfrak{P}_{t_1,\ldots,t_n} : n \in \mathbb{N}, t_1, \ldots, t_n \in \mathbb{R}^d\}$ may be viewed as the family of "finite-dimensional distributions" of $\mathfrak{P}$.

DEFINITION 1.2. A Gaussian system $\mathfrak{P}$ is called *stationary* if for every $n \in \mathbb{N}$, every $t_1, \ldots, t_n \in \mathbb{R}^d$, and every $h \in \mathbb{R}^d$, we have the following equality of laws of point processes on $\mathbb{R}^n$:

$$(4) \qquad \mathfrak{P}_{t_1+h,\ldots,t_n+h} \stackrel{d}{=} \mathfrak{P}_{t_1,\ldots,t_n}.$$

The purpose of this paper is to provide a description of all stationary Gaussian systems. Let us stress that for Markovian particle systems, the corresponding question has a rather simple solution. Let the initial positions of the particles be chosen to form a Poisson point process with σ-finite intensity measure $\mathfrak{m}$ on some measurable space $(\Omega, \mathcal{A})$, and let the particles move independently of each other according to the law of some Markov process on Ω with transition kernel $P(x, dy)$. Then by a result of [3], the particle system is stationary if and only if the measure $\mathfrak{m}$ is P-invariant (see also [9], page 404, and [7], Theorem 2, for weaker results).

1.2. *Statement of the main result.* First we introduce some notation. For $\lambda \in \mathbb{R}$, we denote by $\mathfrak{e}_\lambda$ a measure on $\mathbb{R}$ with a density of the form $e^{-\lambda x}$ with respect to the Lebesgue measure. That is,

$$\text{(5)} \qquad \mathfrak{e}_\lambda(dx) = e^{-\lambda x}\, dx.$$

In particular, $\mathfrak{e}_0$ is the Lebesgue measure itself.

A function $f : \mathbb{R}^d \to \mathbb{R}$ is called *additive* if $f(t_1 + t_2) = f(t_1) + f(t_2)$ for every $t_1, t_2 \in \mathbb{R}^d$. Under minor additional assumptions, say, measurability, an additive function must be of the form $f(t) = \langle c, t \rangle$ for some $c \in \mathbb{R}^d$.

CONVENTION 1.1. All stationary processes and processes with stationary increments are always supposed to have *zero mean*.

The next theorem is our main result.

THEOREM 1.1. *Let $\mathcal{S}$ be the set of all pairs $(\mathfrak{m}, \xi)$, where $\mathfrak{m}$ is a measure satisfying* (1) *and $\{\xi(t), t \in \mathbb{R}^d\}$ is a Gaussian process, with the property that the particle system $GS(\mathfrak{m}, \xi)$ is stationary. Then*

$$\text{(6)} \qquad \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3,$$

where the sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are defined as follows:

1. *The set $\mathcal{S}_1$ consists of all pairs $(\mathfrak{m}, \xi)$, where $\mathfrak{m}$ is an arbitrary measure on $\mathbb{R}$ satisfying* (1), *and*

$$\{\xi(t), t \in \mathbb{R}^d\} \stackrel{d}{=} \{W(t) + c, t \in \mathbb{R}^d\}$$

for some stationary Gaussian process $\{W(t), t \in \mathbb{R}^d\}$ and some $c \in \mathbb{R}$.
2. *The set $\mathcal{S}_2$ consists of all pairs $(\mathfrak{m}, \xi)$, where*

$$\mathfrak{m} = \alpha \mathfrak{e}_0 \quad \text{and} \quad \{\xi(t), t \in \mathbb{R}^d\} \stackrel{d}{=} \{W(t) + f(t) + c, t \in \mathbb{R}^d\}$$

for some $\alpha > 0$, $c \in \mathbb{R}$, a Gaussian process $\{W(t), t \in \mathbb{R}^d\}$ with stationary increments, and an additive function $f : \mathbb{R}^d \to \mathbb{R}$.

3. *The set $\mathcal{S}_3$ consists of all pairs* $(\mathfrak{m}, \xi)$, *where*

$$\mathfrak{m} = \alpha \mathfrak{e}_\lambda \quad and \quad \{\xi(t), t \in \mathbb{R}^d\} \stackrel{d}{=} \{W(t) - \lambda \sigma^2(t)/2 + c, t \in \mathbb{R}^d\}$$

for some $\alpha > 0$, $\lambda \neq 0$, $c \in \mathbb{R}$, *and some Gaussian process* $\{W(t), t \in \mathbb{R}^d\}$ *with stationary increments and variance* $\sigma^2(t)$.

The stationarity of Gaussian systems of type $\mathcal{S}_1$ is a rather trivial fact and is due to the stationarity of the driving process ξ. Somewhat less trivial, but still rather appealing, is the fact that Gaussian systems of type $\mathcal{S}_2$ are stationary. An example of a Gaussian system of type $\mathcal{S}_2$ can be obtained by taking $\mathfrak{m}$ to be the Lebesgue measure on $\mathbb{R}$ and ξ to be a (fractional) Brownian motion with a linear drift.

Surprisingly, the class of stationary Gaussian systems is not exhausted by the two "trivial" families $\mathcal{S}_1$ and $\mathcal{S}_2$: there is one more, nontrivial, family $\mathcal{S}_3$. An example of a Gaussian system of type $\mathcal{S}_3$ can be obtained by taking

$$\mathfrak{m} = \mathfrak{e}_1 \quad and \quad \{\xi(t), t \in \mathbb{R}\} \stackrel{d}{=} \{W_\kappa(t) - |t|^\kappa, t \in \mathbb{R}\},$$

where $\{W_\kappa(t), t \in \mathbb{R}\}$ is a fractional Brownian motion with index $\kappa \in (0, 2]$, that is, a stationary increment Gaussian process with

$$\operatorname{Cov}(W_\kappa(t_1), W_\kappa(t_2)) = |t_1|^\kappa + |t_2|^\kappa - |t_1 - t_2|^\kappa, \qquad t_1, t_2 \in \mathbb{R}.$$

For $\kappa = 1$, this Gaussian system appeared in [2] in connection with maxima of independent Ornstein–Uhlenbeck processes. For general $\kappa \in (0, 2]$, the driving process $W_\kappa(t) - |t|^\kappa$ appeared in [16], also in connection with maxima of Gaussian processes. In a similar way, particle systems of type $\mathcal{S}_2$ appeared in [15] in connection with minima (in the absolute value sense) of independent Gaussian processes. The results of [2] were generalized in [10]. In particular, it was shown in Theorem 2 of [10] that Gaussian systems of type $\mathcal{S}_3$ with an additional requirement $\alpha = 1$, $\lambda = 1$, $c = 0$ were stationary. Gaussian systems of type $\mathcal{S}_3$ have some vague similarity with the "competing particle systems" studied in [19] (see also [1, 20]). Note that in contrast to our setting, the particles in [19] evolve by increments which are *independent* in time.

At a first sight, it may look that the family $\mathcal{S}_2$ can be included into the family $\mathcal{S}_3$ by allowing the parameter λ in the definition of $\mathcal{S}_3$ to be 0. However, this is not the case: the family $\mathcal{S}_2$ has an additional "degree of freedom" represented by the additive function f.

In view of particle systems interpretation of Theorem 1.1, of special interest are stationary Gaussian systems driven by a process ξ satisfying $\xi(0) = 0$. In the next corollary we provide a classification of such systems, excluding for convenience the noninteresting case in which ξ is a version of the zero process.

COROLLARY 1.1. *Let* $\mathfrak{m}$ *be a measure satisfying* (1), *and let* $\{\xi(t), t \in \mathbb{R}^d\}$ *be a Gaussian process with* $\xi(0) = 0$. *Assume that for some* t_0, $\xi(t_0)$ *is not a.s.* 0.

Then the particle system $GS(\mathfrak{m}, \xi)$ is stationary iff $\mathfrak{m} = \alpha \mathfrak{e}_\lambda$ for some $\alpha > 0$ and $\lambda \in \mathbb{R}$, and

$$\{\xi(t), t \in \mathbb{R}^d\} \stackrel{d}{=} \begin{cases} \{W(t) + f(t), t \in \mathbb{R}^d\}, & \text{if } \lambda = 0, \\ \{W(t) - \lambda\sigma^2(t)/2, t \in \mathbb{R}^d\}, & \text{if } \lambda \neq 0, \end{cases}$$

for some Gaussian process $\{W(t), t \in \mathbb{R}^d\}$ with stationary increments, variance $\sigma^2(t)$, $W(0) = 0$ and, eventually, an additive function $f : \mathbb{R}^d \to \mathbb{R}$.

1.3. *Organization of the paper.* Our main result, Theorem 1.1, will be proved in Section 2. Although Theorem 1.1 classifies all pairs $(\mathfrak{m}, \xi)$ generating a stationary Gaussian system, it does not tell how to decide whether two given pairs $(\mathfrak{m}', \xi')$, $(\mathfrak{m}'', \xi'')$ generate equal in law Gaussian systems or not. This gap will be filled in Section 3.

2. Proof of the main result.

2.1. *Idea of the proof.* In this section we prove Theorem 1.1. The "easy" part of Theorem 1.1 stating that Gaussian systems generated by the pairs $(\mathfrak{m}, \xi) \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ are stationary will be established in Proposition 2.1. The proof of the converse statement is much more difficult. The first step will be done in Proposition 2.2, where it is shown that a pair $(\mathfrak{m}, \xi)$ generating a stationary Gaussian system must belong to $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ provided that the measure $\mathfrak{m}$ is a linear combination of the Lebesgue measure $\mathfrak{e}_0$ and a measure of the form $\mathfrak{e}_\lambda$. Such linear combinations are well behaved under convolutions with Gaussian measures, which makes it possible to do explicit calculations with one- and two-dimensional distributions of $GS(\mathfrak{m}, \xi)$. The second step, carried out in Section 2.7, is to show that this additional assumption on the measure $\mathfrak{m}$ is satisfied for *most* (but not all!) pairs $(\mathfrak{m}, \xi)$ generating a stationary Gaussian system. Essentially, this is done by applying a result of Deny [6] and several related lemmas collected in Section 2.6 to the one-dimensional distributions of $GS(\mathfrak{m}, \xi)$. The pairs for which the additional assumption on $\mathfrak{m}$ is not satisfied are shown to belong to the family $\mathcal{S}_1$.

2.2. *Notation.* We start by introducing the notation. We always assume that $\mathfrak{m}$ is a measure on $\mathbb{R}$ satisfying the integrability condition (1), and that $\{\xi(t), t \in \mathbb{R}^d\}$ is a Gaussian process. The law of the process ξ is uniquely determined by its mean and covariance for which we use the notation

$$(7) \qquad \mu(t) = \mathbb{E}\xi(t), \qquad r(t_1, t_2) = \mathrm{Cov}(\xi(t_1), \xi(t_2)).$$

Further, we define the variance and the incremental variance of ξ by

$$(8) \qquad \sigma^2(t) = \mathrm{Var}\,\xi(t), \qquad \gamma(t_1, t_2) = \mathrm{Var}[\xi(t_1) - \xi(t_2)].$$

We will often use the identity

$$(9) \qquad r(t_1, t_2) = \tfrac{1}{2}\big(\sigma^2(t_1) + \sigma^2(t_2) - \gamma(t_1, t_2)\big).$$

Given $t_1, \ldots, t_n \in \mathbb{R}^d$, the law of the random vector $(\xi(t_1), \ldots, \xi(t_n))$ is denoted by $\mathfrak{n}_{t_1,\ldots,t_n}$.

Let $\mathcal{B}(\mathbb{R}^n)$ be the Borel σ-algebra of $\mathbb{R}^n$. For a set $B \subset \mathbb{R}^n$ and $x \in \mathbb{R}$, it will be convenient to define

$$B - x = B - (x, \ldots, x).$$

So, $B - x$ is obtained by shifting the set B "diagonally" in the direction of the vector $(1, \ldots, 1)$.

Define $\mathfrak{P}_{t_1,\ldots,t_n}$, the finite-dimensional distributions of $\mathfrak{P}$, as in (3). The transformation theory of Poisson point processes (see, e.g., Proposition 3.8 in [17]) tells that $\mathfrak{P}_{t_1,\ldots,t_n}$ is a Poisson point process on $\mathbb{R}^n$ with intensity measure $\mathfrak{m}_{t_1,\ldots,t_n}$ that is defined by

$$(10) \qquad \mathfrak{m}_{t_1,\ldots,t_n}(B) = \int_{\mathbb{R}} \mathbb{P}[(\xi(t_1), \ldots, \xi(t_n)) \in B - x]\mathfrak{m}(dx), \qquad B \in \mathcal{B}(\mathbb{R}^n).$$

In particular, we will often use that $\mathfrak{m}_t = \mathfrak{m} * \mathfrak{n}_t$ for every $t \in \mathbb{R}^d$, where $*$ denotes the convolution of measures. Note that condition (1) ensures that $\mathfrak{m}_{t_1,\ldots,t_n}(B)$ is finite for every bounded $B \in \mathcal{B}(\mathbb{R}^n)$.

We can restate Definition 1.2 as follows: A Gaussian system $\mathfrak{P}$ is stationary if for every $n \in \mathbb{N}$, every $t_1, \ldots, t_n, h \in \mathbb{R}^d$, and every $B \in \mathcal{B}(\mathbb{R}^n)$,

$$(11) \qquad \mathfrak{m}_{t_1,\ldots,t_n}(B) = \mathfrak{m}_{t_1+h,\ldots,t_n+h}(B).$$

We denote the one-dimensional Gaussian measure with expectation μ_0 and variance σ_0^2 by $\mathfrak{N}(\mu_0, \sigma_0^2)$. For future reference, let us recall the following formula for the Laplace transform of a Gaussian distribution:

$$(12) \qquad \text{if } N \sim \mathfrak{N}(\mu_0, \sigma_0^2), \qquad \text{then } \mathbb{E}e^{yN} = e^{\mu_0 y + \sigma_0^2 y^2/2}.$$

2.3. *Proof of the easy part of Theorem* 1.1. In the next proposition we prove that Gaussian systems of types $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are indeed stationary.

PROPOSITION 2.1. *Let* $\mathfrak{P} = GS(\mathfrak{m}, \xi)$, *where* $(\mathfrak{m}, \xi) \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. *Then* $\mathfrak{P}$ *is stationary.*

PROOF. Suppose that $(\mathfrak{m}, \xi) \in \mathcal{S}_1$. By definition of $\mathcal{S}_1$, we have the following equality of laws, valid for all $n \in \mathbb{N}, t_1, \ldots, t_n, h \in \mathbb{R}^d$:

$$(\xi(t_i))_{i=1}^n \stackrel{d}{=} (\xi(t_i + h))_{i=1}^n.$$

Let $B \subset \mathbb{R}^n$ be any Borel set. By (10), we have

$$\begin{aligned} \mathfrak{m}_{t_1,\ldots,t_n}(B) &= \int_{\mathbb{R}} \mathbb{P}[(\xi(t_1), \ldots, \xi(t_n)) \in B - z]\mathfrak{m}(dz) \\ &= \int_{\mathbb{R}} \mathbb{P}[(\xi(t_1 + h), \ldots, \xi(t_n + h)) \in B - z]\mathfrak{m}(dz) \\ &= \mathfrak{m}_{t_1+h,\ldots,t_n+h}(B). \end{aligned}$$

Hence, equation (11) holds and $\mathfrak{P}$ is stationary.

Suppose that $(\mathfrak{m}, \xi) \in \mathcal{S}_2$. By definition of $\mathcal{S}_2$, we have $\mathfrak{m} = \alpha \mathfrak{e}_0$ for some $\alpha > 0$, and

$$(13) \qquad \big(\xi(t_i) - \xi(t_1)\big)_{i=1}^n \overset{d}{=} \big(\xi(t_i + h) - \xi(t_1 + h)\big)_{i=1}^n$$

for all $n \in \mathbb{N}$, $t_1, \ldots, t_n, h \in \mathbb{R}^d$. Let $B \subset \mathbb{R}^n$ be any Borel set. Using (10) and (13), we obtain

$$\mathfrak{m}_{t_1,\ldots,t_n}(B)$$
$$= \alpha \int_{\mathbb{R}} \int_{\mathbb{R}^n} 1_{B-x}(y_1, \ldots, y_n) \mathfrak{n}_{t_1,\ldots,t_n}(dy_1, \ldots, dy_n)\, dx$$
$$= \alpha \int_{\mathbb{R}} \int_{\mathbb{R}^n} 1_{B-(x+y_1)}(0, y_2 - y_1, \ldots, y_n - y_1) \mathfrak{n}_{t_1,\ldots,t_n}(dy_1, \ldots, dy_n)\, dx$$
$$(14) \quad = \alpha \int_{\mathbb{R}} \int_{\mathbb{R}^n} 1_{B-z}(0, y_2 - y_1, \ldots, y_n - y_1) \mathfrak{n}_{t_1,\ldots,t_n}(dy_1, \ldots, dy_n)\, dz$$
$$= \alpha \int_{\mathbb{R}} \mathbb{P}\big[\big(\xi(t_i) - \xi(t_1)\big)_{i=1}^n \in B - z\big]\, dz$$
$$= \alpha \int_{\mathbb{R}} \mathbb{P}\big[\big(\xi(t_i + h) - \xi(t_1 + h)\big)_{i=1}^n \in B - z\big]\, dz$$
$$= \mathfrak{m}_{t_1+h,\ldots,t_n+h}(B).$$

Thus, equation (11) holds, and $\mathfrak{P}$ is stationary.

Suppose that $(\mathfrak{m}, \xi) \in \mathcal{S}_3$. In the particular case $\alpha = 1$, $\lambda = 1$ and $c = 0$, the stationarity of $\mathfrak{P}$ was proved in Theorem 2 of [10]. The general case follows by a straightforward application of affine transformations. $\quad \square$

2.4. *Two lemmas.* The next two lemmas are standard. We include their proofs only for completeness.

LEMMA 2.1. *The process $W(t) := \xi(t) - \mu(t)$ has stationary increments iff for all $t_1, t_2, h \in \mathbb{R}^d$,*

$$(15) \qquad \gamma(t_1, t_2) = \gamma(t_1 + h, t_2 + h).$$

PROOF. We prove only sufficiency since the necessity is evident. So, assume that (15) holds. Let $W_h(t) = W(t + h) - W(h)$. We have

$$\text{Cov}(W_h(t_1), W_h(t_2))$$
$$= r(t_1 + h, t_2 + h) + r(h, h) - r(h, t_1 + h) - r(h, t_2 + h)$$
$$= -\big(\gamma(t_1 + h, t_2 + h) - \gamma(h, t_1 + h) - \gamma(h, t_2 + h)\big)/2$$
$$= -\big(\gamma(t_1, t_2) - \gamma(0, t_1) - \gamma(0, t_2)\big)/2,$$

where the second equality follows from (9) and $\gamma(h, h) = 0$, and the last equality is a consequence of (15). Hence, the law of the process $\{W_h(t), t \in \mathbb{R}^d\}$ is independent of h, which proves the lemma. $\quad\square$

LEMMA 2.2. *Let $g : \mathbb{R}^d \to \mathbb{R}$ be a function satisfying*

$$(16) \qquad\qquad g(t_2 + h) - g(t_1 + h) = g(t_2) - g(t_1)$$

for all $t_1, t_2, h \in \mathbb{R}^d$. Then the following statements hold:

1. *The function $f(t) := g(t) - g(0)$ is additive.*
2. *Either $g \equiv$ const or the set of values of g is dense in $\mathbb{R}$.*

PROOF.   Inserting $t_2 := s_1$, $h := s_2$, $t_1 := 0$ into (16) yields $f(s_1 + s_2) = f(s_1) + f(s_2)$ and proves the first part of the lemma. To prove the second part, assume that g is not constant, which means that there is t with $f(t) \neq 0$. A standard inductive argument using the additivity of f gives $f(qt) = qf(t)$ for every rational number q. This implies that the set of values of the function f, and hence also the set of values of g, is dense in $\mathbb{R}$. $\quad\square$

2.5. *Proof of Theorem* 1.1: *Identifying the driving process ξ.*   In Section 2.3 we have shown that $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \subset \mathcal{S}$. Here we prove the more difficult converse inclusion under an additional assumption on the measure $\mathfrak{m}$. This is stated in the following proposition.

PROPOSITION 2.2.   *Let $\mathfrak{m}$ be a measure of the form $\mathfrak{m} = \alpha \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$ for some $\alpha \geq 0$, $\beta \geq 0$, $\lambda \neq 0$, and let $\{\xi(t), t \in \mathbb{R}^d\}$ be a Gaussian process. Assume that $\mathfrak{P} = GS(\mathfrak{m}, \xi)$ is stationary. Then $(\mathfrak{m}, \xi) \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, where $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are as in Theorem* 1.1.

We will need some technical lemmas on measures which are obtained by taking mixtures of diagonally shifted and exponentially weighted bivariate normal laws.

LEMMA 2.3.   *Let $\mathfrak{n}$ be the law of a bivariate Gaussian vector (X_1, X_2) with $\mathbb{E}X_i = \mu_i$, $\operatorname{Var} X_i = \sigma_i^2$ for $i = 1, 2$ and $\operatorname{Var}(X_1 - X_2) = \gamma$. Let $\mathfrak{l}$ be a measure on $\mathbb{R}^2$ defined for some $\kappa \in \mathbb{R}$ by*

$$(17) \qquad\qquad \mathfrak{l}(B) = \int_{\mathbb{R}} e^{-\kappa z} \mathfrak{n}(B - z) \, dz, \qquad B \in \mathcal{B}(\mathbb{R}^2).$$

Then there is a measure $\mathfrak{l}^{(\kappa)}$ concentrated on the line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$ such that the following representation holds:

$$(18) \qquad\qquad \mathfrak{l}(B) = \int_{\mathbb{R}} e^{-\kappa z} \mathfrak{l}^{(\kappa)}(B - z) \, dz, \qquad B \in \mathcal{B}(\mathbb{R}^2).$$

The Laplace transform of $\mathfrak{l}^{(\kappa)}$, *defined as* $\psi^{(\kappa)}(u) = \int_{\mathbb{R}^2} e^{ux_2} \mathfrak{l}^{(\kappa)}(dx_1, dx_2)$, *is given by*

$$(19) \quad \psi^{(\kappa)}(u) = \exp\{(\kappa - u)(\mu_1 + \tfrac{1}{2}\kappa\sigma_1^2) + u(\mu_2 + \tfrac{1}{2}\kappa\sigma_2^2) + \tfrac{1}{2}u(u - \kappa)\gamma\}.$$

REMARK 2.1. Equation (19) shows that the measure $\mathfrak{l}^{(\kappa)}$ is a multiple of a two-dimensional Gaussian measure.

REMARK 2.2. If the Gaussian measure $\mathfrak{n}$ has a density, then it is possible to compute the density of $\mathfrak{l}$ directly from its definition, equation (17). However, since $\mathfrak{n}$ (and also $\mathfrak{l}$) may fail to have a density, we use a somewhat more complicated representation of $\mathfrak{l}$ as an exponentially weighted shift of the essentially one-dimensional measure $\mathfrak{l}^{(\kappa)}$ given in (18).

PROOF OF LEMMA 2.3. Define

$$(20) \quad \mathfrak{l}^{(\kappa)}(B) = \int_{\mathbb{R}^2} e^{\kappa x_1} 1_B(0, x_2 - x_1) \mathfrak{n}(dx_1, dx_2), \qquad B \in \mathcal{B}(\mathbb{R}^2).$$

By construction, the measure $\mathfrak{l}^{(\kappa)}$ is concentrated on the line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$. Using transformations similar to those in [10] (see the proof of Proposition 6 therein), we obtain

$$
\begin{aligned}
\mathfrak{l}(B) &= \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{-\kappa z} 1_{B-z}(x_1, x_2) \mathfrak{n}(dx_1, dx_2) \, dz \\
(21) \quad &= \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{-\kappa(z+x_1)} e^{\kappa x_1} 1_{B-(z+x_1)}(0, x_2 - x_1) \mathfrak{n}(dx_1, dx_2) \, dz \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{-\kappa w} e^{\kappa x_1} 1_{B-w}(0, x_2 - x_1) \mathfrak{n}(dx_1, dx_2) \, dw.
\end{aligned}
$$

Applying (20) to the right-hand side of the above equation, we obtain (18).

Now we compute $\psi^{(\kappa)}(u)$, the Laplace transform of $\mathfrak{l}^{(\kappa)}$. The Laplace transform of $\mathfrak{n}$ is defined as

$$\psi(u_1, u_2) = \int_{\mathbb{R}^2} e^{u_1 x_1 + u_2 x_2} \mathfrak{n}(dx_1, dx_2).$$

By a two-dimensional analogue of (12), $\psi(u_1, u_2)$ is given by

$$(22) \quad \psi(u_1, u_2) = \exp\{\mu_1 u_1 + \mu_2 u_2 + \tfrac{1}{2}(\sigma_1^2 u_1^2 + 2r u_1 u_2 + \sigma_2^2 u_2^2)\},$$

where $r = \mathrm{Cov}(X_1, X_2) = (\sigma_1^2 + \sigma_2^2 - \gamma)/2$. It follows from (20) that

$$\psi^{(\kappa)}(u) = \int_{\mathbb{R}^2} e^{\kappa x_1} e^{u(x_2 - x_1)} \mathfrak{n}(dx_1, dx_2) = \psi(\kappa - u, u).$$

The above equation and (22) yield (19) after an elementary calculation. $\square$

LEMMA 2.4.   *Fix $\kappa \neq 0$. Let $\mathfrak{l}$ be a Radon measure on $\mathbb{R}^2$ admitting a decomposition*

$$(23) \quad \mathfrak{l}(B) = \int_{\mathbb{R}} e^{-\kappa z} \mathfrak{l}^{(\kappa)}(B - z)\,dz + \int_{\mathbb{R}} \mathfrak{l}^{(0)}(B - z)\,dz, \qquad B \in \mathcal{B}(\mathbb{R}^2),$$

where $\mathfrak{l}^{(\kappa)}$ and $\mathfrak{l}^{(0)}$ are measures concentrated on the line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$. Then the measures $\mathfrak{l}^{(\kappa)}$ and $\mathfrak{l}^{(0)}$ are determined uniquely.

PROOF.   Fix some bounded Borel set $A \subset \{0\} \times \mathbb{R}$. For $x > 0$, let B_x be a subset of $\mathbb{R}^2$ defined by $B_x = \bigcup_{y \in [0,x]} (A + y)$. Then (23) implies that

$$\mathfrak{l}(B_x) = \left(\int_0^x e^{-\kappa z}\,dz \right) \mathfrak{l}^{(\kappa)}(A) + x \mathfrak{l}^{(0)}(A).$$

The above is valid for every $x > 0$, and so, $\mathfrak{l}^{(\kappa)}(A)$ and $\mathfrak{l}^{(0)}(A)$ are determined uniquely. $\square$

PROOF OF PROPOSITION 2.2.   We start by proving three claims about the expectation $\mu(\cdot)$, the variance $\sigma^2(\cdot)$ and the incremental variance $\gamma(\cdot, \cdot)$ under various assumptions on α, β, λ.

CLAIM 2.1.   Assume that $\alpha > 0$. Then for all $t_1, t_2 \in \mathbb{R}^d$,

$$(24) \qquad \mu(t_2) - \mu(t_1) = -\frac{\lambda}{2}(\sigma^2(t_2) - \sigma^2(t_1)).$$

PROOF.   The measure $\mathfrak{m}_t = \mathfrak{m} * \mathfrak{n}_t$ has a density given by the convolution formula

$$\frac{\mathfrak{m}_t(dx)}{dx} = \int_{\mathbb{R}} (\alpha e^{-\lambda(x-y)} + \beta) \mathfrak{n}_t(dy) = \alpha e^{-\lambda x} \int_{\mathbb{R}} e^{\lambda y} \mathfrak{n}_t(dy) + \beta.$$

Applying (12) to the first term on right-hand side, we obtain

$$(25) \qquad \frac{\mathfrak{m}_t(dx)}{dx} = \alpha e^{-\lambda x} \exp\left\{ \mu(t)\lambda + \frac{1}{2}\sigma^2(t)\lambda^2 \right\} + \beta.$$

By stationarity of $\mathfrak{P}$, we must have $\mathfrak{m}_{t_1} = \mathfrak{m}_{t_2}$ for every $t_1, t_2 \in \mathbb{R}^d$. This leads to (24). $\square$

Let us turn to the "two-dimensional distributions" of $\mathfrak{P}$. Take $t_1, t_2 \in \mathbb{R}^d$ and recall that $\mathfrak{P}_{t_1,t_2} = \{(V_i(t_1), V_i(t_2)), i \in \mathbb{N}\}$ is a Poisson point process on $\mathbb{R}^2$. By (10), its intensity measure $\mathfrak{m}_{t_1,t_2}$ is given for $B \in \mathcal{B}(\mathbb{R}^2)$ by

$$
\begin{aligned}
\mathfrak{m}_{t_1,t_2}(B) &= \int_{\mathbb{R}} (\alpha e^{-\lambda x} + \beta) \mathfrak{n}_{t_1,t_2}(B - x)\,dx \\
&= \alpha \int_{\mathbb{R}} e^{-\lambda x} \mathfrak{n}_{t_1,t_2}(B - x)\,dx + \beta \int_{\mathbb{R}} \mathfrak{n}_{t_1,t_2}(B - x)\,dx.
\end{aligned}
$$
(26)

Applying Lemma 2.3 twice with $\kappa = \lambda$, $\mathfrak{n} = \mathfrak{n}_{t_1, t_2}$ and $\kappa = 0$, $\mathfrak{n} = \mathfrak{n}_{t_1, t_2}$, we obtain two measures on $\mathbb{R}^2$, called $\mathfrak{m}_{t_1, t_2}^{(\lambda)}$ and $\mathfrak{m}_{t_1, t_2}^{(0)}$, which are concentrated on the line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$ and have the property that for each Borel set $B \subset \mathbb{R}^2$,

$$(27) \qquad \mathfrak{m}_{t_1, t_2}(B) = \alpha \int_{\mathbb{R}} e^{-\lambda x} \mathfrak{m}_{t_1, t_2}^{(\lambda)}(B - x)\, dx + \beta \int_{\mathbb{R}} \mathfrak{m}_{t_1, t_2}^{(0)}(B - x)\, dx.$$

CLAIM 2.2. Assume that $\alpha > 0$. Then for all $t_1, t_2, h \in \mathbb{R}^d$,

$$(28) \qquad \gamma(t_1, t_2) = \gamma(t_1 + h, t_2 + h).$$

PROOF. By stationarity, $\mathfrak{m}_{t_1, t_2} = \mathfrak{m}_{t_1 + h, t_2 + h}$ for all $t_1, t_2, h \in \mathbb{R}^d$. Applying Lemma 2.4 to the decomposition (27), we obtain

$$\mathfrak{m}_{t_1, t_2}^{(\lambda)} = \mathfrak{m}_{t_1 + h, t_2 + h}^{(\lambda)}.$$

Recall that the measures $\mathfrak{m}_{t_1, t_2}^{(\lambda)}$ and $\mathfrak{m}_{t_1 + h, t_2 + h}^{(\lambda)}$ were constructed by means of Lemma 2.3 and thus have Laplace transforms given by the right-hand side of (19). So, we obtain that the expression (considered as a polynomial in u)

$$(\lambda - u)\left(\mu(t_1) + \frac{\lambda}{2}\sigma^2(t_1)\right) + u\left(\mu(t_2) + \frac{\lambda}{2}\sigma^2(t_2)\right) + \frac{1}{2}u(u - \lambda)\gamma(t_1, t_2)$$

does not change if we replace t_1, t_2 by $t_1 + h, t_2 + h$. Taking into account that by Claim 2.1,

$$\mu(t_i) + \frac{\lambda}{2}\sigma^2(t_i) = \mu(t_i + h) + \frac{\lambda}{2}\sigma^2(t_i + h), \qquad i = 1, 2,$$

we arrive at (28). $\quad\square$

CLAIM 2.3. Assume that $\beta > 0$. Then for all $t_1, t_2, h \in \mathbb{R}^d$,

$$(29) \qquad \mu(t_2) - \mu(t_1) = \mu(t_2 + h) - \mu(t_1 + h)$$

and

$$(30) \qquad \gamma(t_1, t_2) = \gamma(t_1 + h, t_2 + h).$$

PROOF. It follows from the decomposition (27) and Lemma 2.4 that

$$\mathfrak{m}_{t_1, t_2}^{(0)} = \mathfrak{m}_{t_1 + h, t_2 + h}^{(0)}.$$

Using the formula for the Laplace transform of $\mathfrak{m}_{t_1, t_2}^{(0)}$ and $\mathfrak{m}_{t_1 + h, t_2 + h}^{(0)}$ given in (19), we obtain that the expression (considered as a quadratic polynomial in u)

$$u\big(\mu(t_2) - \mu(t_1)\big) + \tfrac{1}{2}\gamma(t_1, t_2)u^2$$

remains unchanged if we replace t_1, t_2 by $t_1 + h, t_2 + h$. This yields (29) and (30). $\quad\square$

Now we are ready to complete the proof of Proposition 2.2. We distinguish three cases.

CASE 1.   Assume that $\alpha > 0$ and $\beta > 0$. We show that in this case, $(\mathfrak{m}, \xi) \in \mathcal{S}_1$. Combining Claims 2.1 and 2.3, we obtain

$$\sigma^2(t_2) - \sigma^2(t_1) = \sigma^2(t_2 + h) - \sigma^2(t_1 + h).$$

Since $\sigma^2(t) \geq 0$, it follows from part 2 of Lemma 2.2 that $\sigma^2(t)$ is a constant function. By Claim 2.1, $\mu(t)$ is constant as well. Finally, by (9) and Claim 2.2,

$$\begin{aligned}
r(t_1 + h, t_2 + h) &= \tfrac{1}{2}\big(\sigma^2(t_1 + h) + \sigma^2(t_2 + h) - \gamma(t_1 + h, t_2 + h)\big) \\
&= \tfrac{1}{2}\big(2\sigma^2(0) - \gamma(t_1, t_2)\big) \\
&= r(t_1, t_2).
\end{aligned}$$

This implies that the Gaussian process $W(t) := \xi(t) - \mu(t)$ is stationary. Hence, $(\mathfrak{m}, \xi) \in \mathcal{S}_1$.

CASE 2.   Assume that $\alpha = 0$ and $\beta > 0$. We show that in this case, $(\mathfrak{m}, \xi) \in \mathcal{S}_2$. First of all, note that in this case, $\mathfrak{m}$ is a multiple of $\mathfrak{e}_0$. By equation (30) of Claim 2.3 and Lemma 2.1, the process $W(t) := \xi(t) - \mu(t)$ has stationary increments. Further, the function $f(t) := \mu(t) - \mu(0)$ is additive by equation (29) of Claim 2.3 and part 1 of Lemma 2.2. So, we obtain a decomposition $\xi(t) = W(t) + f(t) + \mu(0)$ implying that $(\mathfrak{m}, \xi) \in \mathcal{S}_2$.

CASE 3.   Assume that $\alpha > 0$ and $\beta = 0$. We show that in this case, $(\mathfrak{m}, \xi) \in \mathcal{S}_3$. First, we have $\mathfrak{m} = \alpha \mathfrak{e}_\lambda$. Second, Claim 2.2 and Lemma 2.1 show that the process $W(t) := \xi(t) - \mu(t)$ has stationary increments. It follows from Claim 2.1 that

$$\mu(t) = -\lambda \sigma^2(t)/2 + \mu(0) + \lambda \sigma^2(0)/2 = -\lambda \sigma^2(t)/2 + c,$$

where $c = \mu(0) + \lambda \sigma^2(0)/2$. Hence, $(\mathfrak{m}, \xi) \in \mathcal{S}_3$.

The proof of Proposition 2.2 is complete.   □

2.6. *Lemmas on convolution equations.*   In this section we collect several auxiliary lemmas on solutions of convolution equations. These equations will arise in Section 2.7 when dealing with one-dimensional distributions of Gaussian systems. The proofs are based on explicit calculations with Laplace transforms and on the result of Deny [6].

LEMMA 2.5.   *Let* $\mathfrak{n}_0 = \mathfrak{N}(\mu_0, \sigma_0^2)$ *be a Gaussian measure on* $\mathbb{R}$. *Let* $\mathfrak{m}_1, \mathfrak{m}_2$ *be two measures satisfying* (1) *such that*

(31) $\mathfrak{m}_1 * \mathfrak{n}_0 = \mathfrak{m}_2 * \mathfrak{n}_0.$

Then $\mathfrak{m}_1 = \mathfrak{m}_2$.

PROOF. We assume that $\sigma_0^2 > 0$, since otherwise, the statement of the lemma is trivial. The density of the measure $\mathfrak{m}_i * \mathfrak{n}_0$, $i = 1, 2$, is given by the convolution formula

(32)
$$\frac{(\mathfrak{m}_i * \mathfrak{n}_0)(dx)}{dx} = \frac{1}{\sqrt{2\pi}\sigma_0} \int_{\mathbb{R}} e^{-(x-y-\mu_0)^2/(2\sigma_0^2)} \mathfrak{m}_i(dy)$$
$$= \frac{1}{\sqrt{2\pi}\sigma_0} e^{-x^2/(2\sigma_0^2)} e^{x\mu_0/\sigma_0^2}$$
$$\times \int_{\mathbb{R}} e^{xy/\sigma_0^2} e^{-(y+\mu_0)^2/(2\sigma_0^2)} \mathfrak{m}_i(dy).$$

Define new measures $\mathfrak{m}_1'$ and $\mathfrak{m}_2'$ by

(33)
$$\frac{\mathfrak{m}_i'(dy)}{\mathfrak{m}_i(dy)} = e^{-(y+\mu_0)^2/(2\sigma_0^2)}, \qquad i = 1, 2.$$

Let $\varphi_{\mathfrak{m}_i'}(x) = \int_{\mathbb{R}} e^{xy} \mathfrak{m}_i'(dy)$, $i = 1, 2$, be the Laplace transforms of $\mathfrak{m}_1'$ and $\mathfrak{m}_2'$. Note that by (1), $\varphi_{\mathfrak{m}_1'}(x)$ and $\varphi_{\mathfrak{m}_2'}(x)$ are finite for all $x \in \mathbb{R}$. We may rewrite (32) as follows:

(34)
$$\frac{(\mathfrak{m}_i * \mathfrak{n}_0)(dx)}{dx} = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-x^2/(2\sigma_0^2)} e^{x\mu_0/\sigma_0^2} \int_{\mathbb{R}} e^{xy/\sigma_0^2} \mathfrak{m}_i'(dy)$$
$$= \frac{1}{\sqrt{2\pi}\sigma_0} e^{-x^2/(2\sigma_0^2)} e^{x\mu_0/\sigma_0^2} \varphi_{\mathfrak{m}_i'}\left(\frac{x}{\sigma_0^2}\right).$$

By (31), the densities of the measures $\mathfrak{m}_1 * \mathfrak{n}_0$ and $\mathfrak{m}_2 * \mathfrak{n}_0$ must be equal. Taking into account (34), this yields

$$\varphi_{\mathfrak{m}_1'}(x) = \varphi_{\mathfrak{m}_2'}(x) \qquad \forall x \in \mathbb{R}.$$

By the uniqueness of the Laplace transform, $\mathfrak{m}_1' = \mathfrak{m}_2'$. Recalling (33) yields that $\mathfrak{m}_1 = \mathfrak{m}_2$. This proves the lemma. $\square$

LEMMA 2.6. *Let* $\mathfrak{n}_1 = \mathfrak{N}(\mu_1, \sigma_1^2)$ *and* $\mathfrak{n}_2 = \mathfrak{N}(\mu_2, \sigma_2^2)$ *be two Gaussian measures on* $\mathbb{R}$ *such that* $\sigma_1^2 \leq \sigma_2^2$. *Let* $\mathfrak{m}_1$ *and* $\mathfrak{m}_2$ *be two measures satisfying* (1) *such that*

(35)
$$\mathfrak{m}_1 * \mathfrak{n}_1 = \mathfrak{m}_2 * \mathfrak{n}_2.$$

Then $\mathfrak{m}_1 = \mathfrak{m}_2 * \mathfrak{N}(\mu_2 - \mu_1, \sigma_2^2 - \sigma_1^2)$.

PROOF. We may rewrite (35) as

$$\mathfrak{m}_1 * \mathfrak{n}_1 = \left(\mathfrak{m}_2 * \mathfrak{N}(\mu_2 - \mu_1, \sigma_2^2 - \sigma_1^2)\right) * \mathfrak{n}_1.$$

The proof is completed by applying Lemma 2.5. $\square$

LEMMA 2.7.   *Let $\mathfrak{m}$ be a measure satisfying* (1), *and let $\mathfrak{n}_0 = \mathfrak{N}(\mu_0, \sigma_0^2)$ be a Gaussian measure such that for some $\alpha \geq 0$, $\beta \geq 0$, $\lambda \neq 0$,*

$$\tag{36} \mathfrak{m} * \mathfrak{n}_0 = \alpha \mathfrak{e}_\lambda + \beta \mathfrak{e}_0.$$

Then $\mathfrak{m} = \alpha e^{-\lambda^2 \sigma_0^2/2} e^{-\lambda \mu_0} \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$.

PROOF.   Define a measure $\mathfrak{m}_1 = \alpha e^{-\lambda^2 \sigma_0^2/2} e^{-\lambda \mu_0} \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$. Then the density of the measure $\mathfrak{m}_1 * \mathfrak{n}_0$ can be computed by means of the convolution formula:

$$\frac{(\mathfrak{m}_1 * \mathfrak{n}_0)(dx)}{dx} = \int_{\mathbb{R}} (\alpha e^{-\lambda^2 \sigma_0^2/2} e^{-\lambda \mu_0} e^{-\lambda(x-y)} + \beta) \mathfrak{n}_0(dy) = \alpha e^{-\lambda x} + \beta,$$

where the second equality follows from (12). Hence,

$$\mathfrak{m} * \mathfrak{n}_0 = \mathfrak{m}_1 * \mathfrak{n}_0.$$

By Lemma 2.5, we have $\mathfrak{m} = \mathfrak{m}_1$. The proof is complete.   □

LEMMA 2.8.   *Let $\mathfrak{n}_1 = \mathfrak{N}(\mu_1, \sigma_1^2)$ and $\mathfrak{n}_2 = \mathfrak{N}(\mu_2, \sigma_2^2)$ be two Gaussian measures on $\mathbb{R}$ such that $\sigma_1^2 \neq \sigma_2^2$. Let $\mathfrak{m}$ be a measure satisfying* (1) *such that*

$$\tag{37} \mathfrak{m} * \mathfrak{n}_1 = \mathfrak{m} * \mathfrak{n}_2.$$

Then $\mathfrak{m} = \alpha \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$ for some $\alpha \geq 0$, $\beta \geq 0$ and $\lambda \neq 0$.

PROOF.   By symmetry, we may assume that $\sigma_1^2 < \sigma_2^2$. Then Lemma 2.5 implies that

$$\tag{38} \mathfrak{m} = \mathfrak{m} * \mathfrak{n}_0,$$

where $\mathfrak{n}_0 = \mathfrak{N}(\mu_2 - \mu_1, \sigma_2^2 - \sigma_1^2)$. By Theorem 3′ of [6], every solution $\mathfrak{m}$ of (38) can be represented as a mixture of exponentials; that is, we may write

$$\frac{\mathfrak{m}(dy)}{dy} = \int_E e^{-\lambda y} \rho(d\lambda),$$

where ρ is a finite Borel measure on the set $E = \{\lambda \in \mathbb{R} : \int_{\mathbb{R}} e^{\lambda x} \mathfrak{n}_0(dx) = 1\}$. Now, in our case the measure $\mathfrak{n}_0$ is Gaussian, and so (12) shows that E consists of at most two points. One of them is always 0, and the second is denoted by λ (if $E = \{0\}$, let $\lambda \neq 0$ be arbitrary). Taking $\alpha = \rho(\{\lambda\})$ and $\beta = \rho(\{0\})$, we obtain $\mathfrak{m} = \alpha \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$. This completes the proof.   □

2.7. *Proof of Theorem* 1.1: *Identifying the measure* $\mathfrak{m}$. In this section we complete the proof of the inclusion $\mathcal{S} \subset \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. Let $(\mathfrak{m}, \xi)$ be a pair generating a stationary Gaussian system $\mathfrak{P} = GS(\mathfrak{m}, \xi)$. Our goal is to show that

$$(39) \qquad (\mathfrak{m}, \xi) \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3.$$

The idea of the proof is to show, whenever possible, that the measure $\mathfrak{m}$ is of the form $\alpha \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$ and then to apply Proposition 2.2. In all other cases, we will prove that $(\mathfrak{m}, \xi) \in \mathcal{S}_1$.

Assume for a moment that $\xi(0) = 0$ and $\mathrm{Var}\, \xi(t_0) > 0$ for some $t_0 \in \mathbb{R}^d$. Under this restriction, the proof takes the following particularly simple form. By stationarity, we have $\mathfrak{m}_0 = \mathfrak{m}_{t_0}$. Using $\xi(0) = 0$, this can be written as $\mathfrak{m} = \mathfrak{m} * \mathfrak{n}_{t_0}$. Applying to this convolution equation the result of Deny [6] as in the proof of Lemma 2.8, we conclude that $\mathfrak{m}$ must be of the form $\alpha \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$. Hence, Proposition 2.2 is applicable and (39) is proved.

Let us now consider Theorem 1.1 in its full generality. We will distinguish between different cases.

CASE 1. Assume that the function σ^2 is not constant. So, there are $t_1, t_2 \in \mathbb{R}^d$ such that

$$(40) \qquad \sigma^2(t_1) \neq \sigma^2(t_2).$$

By stationarity of $\mathfrak{P}$, we must have $\mathfrak{m}_{t_1} = \mathfrak{m}_{t_2}$ and hence,

$$\mathfrak{m} * \mathfrak{N}(\mu(t_1), \sigma^2(t_1)) = \mathfrak{m} * \mathfrak{N}(\mu(t_2), \sigma^2(t_2)).$$

Then Lemma 2.8, which is applicable in view of (40), implies that $\mathfrak{m} = \alpha \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$ for some $\alpha \geq 0$, $\beta \geq 0$, $\lambda \neq 0$. An application of Proposition 2.2 shows that (39) holds.

CASE 2. Assume that $\sigma^2(t) = \sigma^2 \geq 0$ is constant. Take some $t_1, t_2 \in \mathbb{R}^d$ and fix some $\vartheta \in [0, 1]$. Consider $\tilde{\mathfrak{P}}_{t_1, t_2}$, a point process on $\mathbb{R}$ defined by

$$(41) \qquad \tilde{\mathfrak{P}}_{t_1, t_2} = \{U_i + \vartheta \xi_i(t_1) + (1 - \vartheta)\xi_i(t_2), i \in \mathbb{N}\},$$

where the U_i's and the ξ_i's are as in Section 1.1. Recalling from (2) that $V_i(t) = U_i + \xi_i(t)$, we may rewrite (41) as

$$(42) \qquad \tilde{\mathfrak{P}}_{t_1, t_2} = \{\vartheta V_i(t_1) + (1 - \vartheta)V_i(t_2), i \in \mathbb{N}\}.$$

By Proposition 3.8 of [17], $\tilde{\mathfrak{P}}_{t_1, t_2}$ is a Poisson point process whose intensity measure $\tilde{\mathfrak{m}}_{t_1, t_2}$ is given by the formula

$$\tilde{\mathfrak{m}}_{t_1, t_2} = \mathfrak{m} * \mathfrak{N}(\tilde{\mu}(t_1, t_2), \tilde{\sigma}^2(t_1, t_2)),$$

where

$$(43) \qquad \tilde{\mu}(t_1, t_2) = \vartheta \mu(t_1) + (1 - \vartheta)\mu(t_2)$$

and

$$(44) \qquad \tilde{\sigma}^2(t_1, t_2) = (\vartheta^2 + (1-\vartheta)^2)\sigma^2 + 2\vartheta(1-\vartheta)r(t_1, t_2).$$

The stationarity of the particle system $\mathfrak{P}$ together with representation (42) implies that for every $t_1, t_2, h \in \mathbb{R}^d$, the point processes $\tilde{\mathfrak{P}}_{t_1, t_2}$ and $\tilde{\mathfrak{P}}_{t_1+h, t_2+h}$ must have the same law. Hence, $\tilde{\mathfrak{m}}_{t_1, t_2} = \tilde{\mathfrak{m}}_{t_1+h, t_2+h}$ and consequently,

$$(45) \qquad \mathfrak{m} * \mathfrak{N}(\tilde{\mu}(t_1, t_2), \tilde{\sigma}^2(t_1, t_2)) = \mathfrak{m} * \mathfrak{N}(\tilde{\mu}(t_1+h, t_2+h), \tilde{\sigma}^2(t_1+h, t_2+h)).$$

The proof will be completed after we have considered two subcases.

SUBCASE 2A.   Assume that for some $t_1, t_2, h \in \mathbb{R}^d$,

$$(46) \qquad\qquad r(t_1, t_2) \neq r(t_1 + h, t_2 + h).$$

Take $\vartheta = 1/2$ in the definition of the point process $\tilde{\mathfrak{P}}_{t_1, t_2}$. Then (44) and (46) imply that

$$\tilde{\sigma}^2(t_1, t_2) \neq \tilde{\sigma}^2(t_1 + h, t_2 + h).$$

By Lemma 2.8, applied to (45), the measure $\mathfrak{m}$ is of the form $\alpha \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$ for some $\alpha \geq 0$, $\beta \geq 0$, $\lambda \neq 0$. An application of Proposition 2.2 shows that (39) holds.

SUBCASE 2B.   Assume that for all $t_1, t_2, h \in \mathbb{R}^d$,

$$(47) \qquad\qquad r(t_1, t_2) = r(t_1 + h, t_2 + h).$$

This implies that the process $W(t) := \xi(t) - \mu(t)$ is stationary.

If the function μ is constant, then $(\mathfrak{m}, \xi) \in \mathcal{S}_1$. Therefore, let us assume that μ is not constant. We will show that this implies that $\mathfrak{m}$ is a multiple of the Lebesgue measure. Let

$$G = \{g \in \mathbb{R} : \mathfrak{m} * \delta_g = \mathfrak{m}\}$$

be the set of "periods" of $\mathfrak{m}$, where δ_g is the Dirac measure concentrated at g. Clearly, G is an additive subgroup of $\mathbb{R}$.

By stationarity of $\mathfrak{P}$, we have $\mathfrak{m}_{t_1} = \mathfrak{m}_{t_2}$ for every $t_1, t_2 \in \mathbb{R}^d$. Equivalently,

$$\mathfrak{m} * \mathfrak{N}(\mu(t_1), \sigma^2) = \mathfrak{m} * \mathfrak{N}(\mu(t_2), \sigma^2).$$

By Lemma 2.6, this implies that

$$(48) \qquad\qquad \mu(t_1) - \mu(t_2) \in G \qquad \forall t_1, t_2 \in \mathbb{R}^d.$$

Since μ is assumed to be nonconstant, equation (48) implies that $G \neq \{0\}$, which means that $\mathfrak{m}$ has a nontrivial period. Of course, this is not sufficient to conclude that $\mathfrak{m}$ is a multiple of the Lebesgue measure, and so, let us use the stationarity of the two-dimensional distributions of $\mathfrak{P}$. Recalling (44) and taking into account (47), we obtain that for every $t_1, t_2, h \in \mathbb{R}^d$,

$$\tilde{\sigma}^2(t_1, t_2) = \tilde{\sigma}^2(t_1 + h, t_2 + h).$$

Applying Lemma 2.6 to (45), we obtain

$$\tilde{\mu}(t_1, t_2) - \tilde{\mu}(t_1 + h, t_2 + h) \in G \qquad \forall t_1, t_2, h \in \mathbb{R}^d.$$

Recalling a formula for $\tilde{\mu}$ given in (43), we arrive at

$$\vartheta \cdot \big(\mu(t_1) - \mu(t_2) - \mu(t_1 + h) + \mu(t_2 + h)\big) + \big(\mu(t_2) - \mu(t_2 + h)\big) \in G.$$

Note that this is valid for every $\vartheta \in [0, 1]$. Assume that in the above expression, ϑ appears with a nonzero coefficient for some t_1, t_2, h. Then G contains a nontrivial interval, and so, we must have $G = \mathbb{R}$. In other words, the measure $\mathfrak{m}$ is translation invariant. Since by (1), $\mathfrak{m}$ is finite on bounded intervals, this implies that $\mathfrak{m}$ is a multiple of the Lebesgue measure.

So, let us assume that for every $t_1, t_2, h \in \mathbb{R}^d$,

(49) $$\mu(t_1) - \mu(t_2) = \mu(t_1 + h) - \mu(t_2 + h).$$

Recall also that we assume that μ is nonconstant. Hence, by part 2 of Lemma 2.2, the set of values of the function μ is dense in $\mathbb{R}$. By (48), the group G must be dense in $\mathbb{R}$.

We claim that in fact, $G = \mathbb{R}$. To prove this, we need to show that G is closed. First of all, the measure $\mathfrak{m}$ is atomless, since if it would have an atom, then the invariance under G would imply that $\mathfrak{m}$ has a dense set of atoms of equal mass, which would contradict (1). Now, let $g_1, g_2, \ldots$ be a sequence in G converging to some $g \in \mathbb{R}$. We claim that $g \in G$. Indeed, for every interval $[a, b] \subset \mathbb{R}$, we have

$$\mathfrak{m}([a - g, b - g]) = \lim_{i \to \infty} \mathfrak{m}([a - g_i, b - g_i]) = \lim_{i \to \infty} \mathfrak{m}([a, b]) = \mathfrak{m}([a, b]),$$

where the first equality holds since $\mathfrak{m}$ is atomless, and the second equality follows from $g_i \in G$. This proves that $g \in G$. Therefore, the group G, being dense and closed, must be equal to $\mathbb{R}$.

The fact that $G = \mathbb{R}$ means that the measure $\mathfrak{m}$ is translation invariant and thus, must be a multiple of the Lebesgue measure. Therefore, we can apply Proposition 2.2 which shows that (39) holds.

The proof of Theorem 1.1 is complete.

3. Pairs generating equal in law Gaussian systems. In this section we give an answer to the following question: Given two pairs $(\mathfrak{m}', \xi')$ and $(\mathfrak{m}'', \xi'')$ in $\mathcal{S}$, determine whether $GS(\mathfrak{m}', \xi')$ has the same law as $GS(\mathfrak{m}'', \xi'')$. The next proposition is a first step in this direction.

PROPOSITION 3.1. *The decomposition $\mathcal{S} = \mathcal{S}_1^* \cup \mathcal{S}_2 \cup \mathcal{S}_3$, where $\mathcal{S}_1^* = \mathcal{S}_1 \backslash (\mathcal{S}_2 \cup \mathcal{S}_3)$, is disjoint. Pairs belonging to different sets in this decomposition generate different in law Gaussian systems.*

PROOF.   We will show that Gaussian systems generated by pairs belonging to different sets in the decomposition $\mathcal{S} = \mathcal{S}_1^* \cup \mathcal{S}_2 \cup \mathcal{S}_3$ differ by their one-dimensional distributions. If $(\mathfrak{m}, \xi) \in \mathcal{S}_2$, then $\mathfrak{m} = \alpha \mathfrak{e}_0$ for some $\alpha > 0$, and consequently, $\mathfrak{m}_t = \mathfrak{m} * \mathfrak{n}_t = \alpha \mathfrak{e}_0$ for every $t \in \mathbb{R}^d$. If $(\mathfrak{m}, \xi) \in \mathcal{S}_3$, then $\mathfrak{m} = \alpha \mathfrak{e}_\lambda$ for some $\alpha > 0$ and $\lambda \neq 0$. Hence, in this case, $\mathfrak{m}_t = \mathfrak{m} * \mathfrak{n}_t = \tilde{\alpha} \mathfrak{e}_\lambda$ for some $\tilde{\alpha} > 0$. Finally, if $(\mathfrak{m}, \xi) \in \mathcal{S}_1^*$, then $\mathfrak{m}_t$ is not a multiple of $\mathfrak{e}_\lambda, \lambda \in \mathbb{R}$. Otherwise, Lemma 2.7 would imply that the same is true for $\mathfrak{m}$, which contradicts the assumption $(\mathfrak{m}, \xi) \in \mathcal{S}_1^*$. □

In the sequel, we concentrate on pairs belonging to the same set in the decomposition $\mathcal{S} = \mathcal{S}_1^* \cup \mathcal{S}_2 \cup \mathcal{S}_3$. Let us call a pair $(\mathfrak{m}, \xi)$ belonging to $\mathcal{S}_2$ or $\mathcal{S}_3$ *canonical* if $\xi(0) = 0$. A classification of such pairs was given in Corollary 1.1.

PROPOSITION 3.2.   *For every* $(\mathfrak{m}, \xi) \in \mathcal{S}_2$ *there is a unique canonical pair* $(\tilde{\mathfrak{m}}, \tilde{\xi}) \in \mathcal{S}_2$ *generating the same Gaussian system as* $(\mathfrak{m}, \xi)$.

PROOF.   To show the existence, set $\tilde{\mathfrak{m}} = \mathfrak{m}$ and $\tilde{\xi}(t) = \xi(t) - \xi(0)$. Applying (14) two times, we obtain that for every $B \in \mathcal{B}(\mathbb{R}^n)$,

$$\mathfrak{m}_{t_1, \ldots, t_n}(B) = \alpha \int_{\mathbb{R}} \mathbb{P}\big[(\xi(t_i) - \xi(t_1))_{i=1}^n \in B - z \big] \, dz$$

$$= \alpha \int_{\mathbb{R}} \mathbb{P}\big[(\tilde{\xi}(t_i) - \tilde{\xi}(t_1))_{i=1}^n \in B - z \big] \, dz$$

$$= \tilde{\mathfrak{m}}_{t_1, \ldots, t_n}(B),$$

where $\tilde{\mathfrak{m}}_{t_1, \ldots, t_n}$ are the finite-dimensional intensities of $GS(\tilde{\mathfrak{m}}, \tilde{\xi})$ [cf. (10)]. Hence, $(\mathfrak{m}, \xi)$ and $(\tilde{\mathfrak{m}}, \tilde{\xi})$ generate equal in law Gaussian systems.

We prove the uniqueness part. Let $(\mathfrak{m}, \xi)$ be a canonical pair. Then $\mathfrak{m} = \alpha \mathfrak{e}_0$ and $\xi(t) = W(t) + f(t)$ (see Theorem 1.1). We will show that the triple (α, W, f) is uniquely determined by the finite-dimensional distributions of $\mathfrak{P} = GS(\mathfrak{m}, \xi)$.

First, we have $\mathfrak{m}_t = \mathfrak{m} * \mathfrak{n}_t = \alpha \mathfrak{e}_0$ for every $t \in \mathbb{R}^d$, and so, α is uniquely determined. Let us turn to the two-dimensional distributions of $\mathfrak{P}$. By (10), we have

$$\mathfrak{m}_{0,t}(B) = \alpha \int_{\mathbb{R}} \mathfrak{n}_{0,t}(B - z) \, dz.$$

By Lemma 2.3, there is a representation

$$\mathfrak{m}_{0,t}(B) = \alpha \int_{\mathbb{R}} \mathfrak{m}_{0,t}^{(0)}(B - z) \, dz$$

for some measure $\mathfrak{m}_{0,t}^{(0)}$ concentrated on the line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$ and having the Laplace transform $\exp\{f(t)u + 1/2\gamma(0, t)u^2\}$. By Lemma 2.4, this shows that the two-dimensional distributions of $\mathfrak{P}$ determine $f(t)$ and $\gamma(0, t)$ uniquely. To

see that $\gamma(0, t)$ determines the law of W uniquely, recall that $W(0) = 0$ and hence, we may write the covariance function of W in the form

$$r(t_1, t_2) = \tfrac{1}{2}(\gamma(0, t_1) + \gamma(0, t_2) - \gamma(0, t_1 - t_2)).$$

This completes the proof of the uniqueness part. $\square$

PROPOSITION 3.3. *For every $(\mathfrak{m}, \xi) \in \mathcal{S}_3$ there is a unique canonical pair $(\tilde{\mathfrak{m}}, \tilde{\xi}) \in \mathcal{S}_3$ generating the same Gaussian system as $(\mathfrak{m}, \xi)$.*

PROOF. All necessary ingredients are contained in [10]. Take $\tilde{\xi}(t) = \xi(t) - \xi(0)$ and $\tilde{\mathfrak{m}} = \mathfrak{m} * \delta_c$, where c is as in Theorem 1.1. The fact that $(\tilde{\mathfrak{m}}, \tilde{\xi})$ and $(\mathfrak{m}, \xi)$ generate equal Gaussian systems was essentially shown in Proposition 11 of [10]. The uniqueness part follows under the additional assumption $\lambda = 1$ from Remark 24 of [10]. The general case is analogous. $\square$

The next proposition gives a necessary and sufficient condition on two pairs belonging to $\mathcal{S}_1^*$ to generate equal in law Gaussian systems.

PROPOSITION 3.4. *Let $(\mathfrak{m}', \xi')$ and $(\mathfrak{m}'', \xi'')$ be two pairs, both belonging to $\mathcal{S}_1^*$ and generating Gaussian systems $\mathfrak{P}'$ and $\mathfrak{P}''$. Then*

$$\mathfrak{P}' \overset{d}{=} \mathfrak{P}'' \tag{50}$$

iff the following holds: There is a Gaussian variable N_0 whose distribution on $\mathbb{R}$ is denoted by $\mathfrak{n}_0$ and which is independent of ξ', ξ'', such that

$$\mathfrak{m}' = \mathfrak{m}'' * \mathfrak{n}_0 \quad and \quad \{\xi''(t), t \in \mathbb{R}^d\} \overset{d}{=} \{\xi'(t) + N_0, t \in \mathbb{R}^d\}, \tag{51}$$

or

$$\mathfrak{m}'' = \mathfrak{m}' * \mathfrak{n}_0 \quad and \quad \{\xi'(t), t \in \mathbb{R}^d\} \overset{d}{=} \{\xi''(t) + N_0, t \in \mathbb{R}^d\}. \tag{52}$$

PROOF. Introduce the notation μ', r', μ'', r'', etc. as in Section 2.2. By definition of the family $\mathcal{S}_1^*$, the functions μ', σ'^2, μ'', σ''^2 are constant. Therefore, we write, say, μ' instead of $\mu'(t)$. We may rewrite (9) as follows:

$$\gamma'(t_1, t_2) = 2(\sigma'^2 - r'(t_1, t_2)) \quad and \quad \gamma''(t_1, t_2) = 2(\sigma''^2 - r''(t_1, t_2)). \tag{53}$$

We start by proving the "if" part of the proposition. Assume for concreteness that (51) holds. Then, by (10),

$$\begin{aligned}
\mathfrak{m}''_{t_1, \ldots, t_n}(B) &= \int_{\mathbb{R}} \mathbb{P}[(\xi''(t_1), \ldots, \xi''(t_n)) \in B - z]\mathfrak{m}''(dz) \\
&= \int_{\mathbb{R}} \mathbb{P}[(\xi'(t_1) + N_0, \ldots, \xi'(t_n) + N_0) \in B - z]\mathfrak{m}''(dz) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{P}[(\xi'(t_1), \ldots, \xi'(t_n)) \in B - (z + y)]\mathfrak{m}''(dz)\mathfrak{n}_0(dy).
\end{aligned}$$

For every nonnegative function $f : \mathbb{R}^d \to \mathbb{R}$ the following formula holds:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(z+y) \mathfrak{m}''(dz) \mathfrak{n}_0(dy) = \int_{\mathbb{R}} f(x)(\mathfrak{m}'' * \mathfrak{n}_0)(dx).$$

Hence,

$$\mathfrak{m}''_{t_1,\ldots,t_n}(B) = \int_{\mathbb{R}} \mathbb{P}[(\xi'(t_1), \ldots, \xi'(t_n)) \in B - x](\mathfrak{m}'' * \mathfrak{n}_0)(dx)$$

$$= \int_{\mathbb{R}} \mathbb{P}[(\xi'(t_1), \ldots, \xi'(t_n)) \in B - x] \mathfrak{m}'(dx)$$

$$= \mathfrak{m}'_{t_1,\ldots,t_n}(B).$$

This proves (50).

Now we prove the "only if" part of the proposition. Assume that (50) holds. Without loss of generality we assume that $\sigma'^2 \le \sigma''^2$. Define

$$\mathfrak{n}_0 = \mathfrak{N}(\mu'' - \mu', \sigma''^2 - \sigma'^2),$$

and let $N_0 \sim \mathfrak{n}_0$ be a Gaussian variable independent of ξ' and ξ''. We will show that (51) holds.

We start by proving the first equality in (51). It follows from (50) that $\mathfrak{m}'_t = \mathfrak{m}''_t$ for all $t \in \mathbb{R}^d$. Equivalently,

$$\mathfrak{m}' * \mathfrak{N}(\mu', \sigma'^2) = \mathfrak{m}'' * \mathfrak{N}(\mu'', \sigma''^2).$$

Then, by Lemma 2.6, $\mathfrak{m}' = \mathfrak{m}'' * \mathfrak{n}_0$. This proves the first equality in (51).

We claim that the second equality in (51) follows from the following statement: for all $t_1, t_2 \in \mathbb{R}^d$,

(54) $\gamma'(t_1, t_2) = \gamma''(t_1, t_2).$

To see this, set for a moment $\tilde{\xi}'(t) = \xi'(t) + N_0$. Then

$$\mathbb{E}\tilde{\xi}'(t) = \mu' + (\mu'' - \mu') = \mu'' = \mathbb{E}\xi''(t).$$

Elementary transformations using (53) and (54) yield

$$\mathrm{Cov}(\tilde{\xi}'(t_1), \tilde{\xi}'(t_2)) = r'(t_1, t_2) + (\sigma''^2 - \sigma'^2) = r''(t_1, t_2) = \mathrm{Cov}(\xi''(t_1), \xi''(t_2)).$$

From now on, we are proving (54). We need to consider two cases.

CASE 1. Assume that $\mathfrak{m}' = \alpha' \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$ for some $\alpha' > 0$, $\beta > 0$, $\lambda \ne 0$. It follows from $\mathfrak{m}'_t = \mathfrak{m}''_t$ that

$$\mathfrak{m}' * \mathfrak{N}(\mu', \sigma'^2) = \mathfrak{m}'' * \mathfrak{N}(\mu'', \sigma''^2).$$

The left-hand side of the above equation is of the form $\alpha \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$ for some $\alpha > 0$. Hence, using Lemma 2.7, we conclude that $\mathfrak{m}'' = \alpha'' \mathfrak{e}_\lambda + \beta \mathfrak{e}_0$ for some $\alpha'' > 0$.

Let us consider the two-dimensional distributions of $\mathfrak{P}'$. By (10),

$$\mathfrak{m}'_{t_1,t_2}(B) = \alpha' \int_{\mathbb{R}} e^{-\lambda z} \mathfrak{n}'_{t_1,t_2}(B - z)\, dz + \beta \int_{\mathbb{R}} \mathfrak{n}'_{t_1,t_2}(B - z)\, dz, \qquad B \in \mathcal{B}(\mathbb{R}^2).$$

Applying Lemma 2.3 twice, we get two measures $\mathfrak{m}'^{(\lambda)}_{t_1,t_2}$ and $\mathfrak{m}'^{(0)}_{t_1,t_2}$ concentrated on $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$ such that the following decomposition is valid:

$$\mathfrak{m}'_{t_1,t_2}(B) = \alpha' \int_{\mathbb{R}} e^{-\lambda z} \mathfrak{m}'^{(\lambda)}_{t_1,t_2}(B - z)\, dz + \beta \int_{\mathbb{R}} \mathfrak{m}'^{(0)}_{t_1,t_2}(B - z)\, dz, \qquad B \in \mathcal{B}(\mathbb{R}^2).$$

Furthermore, $\psi'_{t_1,t_2}(u)$, the Laplace transform of $\mathfrak{m}'^{(0)}_{t_1,t_2}$, is given by

$$(55) \qquad \qquad \psi'_{t_1,t_2}(u) = e^{\gamma'(t_1,t_2)u^2/2}.$$

Similar calculations can be done for $\mathfrak{m}''_{t_1,t_2}$. By (50), we must have $\mathfrak{m}'_{t_1,t_2} = \mathfrak{m}''_{t_1,t_2}$. By Lemma 2.4, this implies

$$\mathfrak{m}'^{(0)}_{t_1,t_2} = \mathfrak{m}''^{(0)}_{t_1,t_2}.$$

Comparing the Laplace transforms, we obtain (54).

CASE 2. Assume that the condition of Case 1 is not satisfied. We define a point process $\tilde{\mathfrak{P}}'_{t_1,t_2}$ as in (41) and (42) with $\vartheta = 1/2$: we set

$$\tilde{\mathfrak{P}}'_{t_1,t_2} = \{U'_i + \xi'_i(t_1)/2 + \xi'_i(t_2)/2, i \in \mathbb{N}\},$$

where $\{U'_i, i \in \mathbb{N}\}$ and ξ'_i, $i \in \mathbb{N}$, are the starting points and the driving processes of the Gaussian system $\mathfrak{P}'$. Then $\tilde{\mathfrak{P}}'_{t_1,t_2}$ is a Poisson point process on $\mathbb{R}$ whose intensity measure $\tilde{\mathfrak{m}}'_{t_1,t_2}$ is given by the formula

$$(56) \qquad \qquad \tilde{\mathfrak{m}}'_{t_1,t_2} = \mathfrak{m}' * \mathfrak{N}(\mu', \tfrac{1}{2}\sigma'^2 + \tfrac{1}{2}r'(t_1, t_2)).$$

A simple calculation using (53) shows that

$$\mathfrak{m}'_{t_1} = \tilde{\mathfrak{m}}'_{t_1,t_2} * \mathfrak{N}(0, \tfrac{1}{4}\gamma'(t_1, t_2)).$$

Similar calculations can be done for the pair $(\mathfrak{m}'', \xi'')$. By (50), we must have $\tilde{\mathfrak{m}}'_{t_1,t_2} = \tilde{\mathfrak{m}}''_{t_1,t_2}$. Denoting these equal measures for a moment by $\tilde{\mathfrak{m}}_{t_1,t_2}$, we obtain

$$\tilde{\mathfrak{m}}_{t_1,t_2} * \mathfrak{N}(0, \tfrac{1}{4}\gamma'(t_1, t_2)) = \tilde{\mathfrak{m}}_{t_1,t_2} * \mathfrak{N}(0, \tfrac{1}{4}\gamma''(t_1, t_2)).$$

Now assume that (54) does not hold for some $t_1, t_2 \in \mathbb{R}^d$. Then Lemma 2.8 implies that $\tilde{\mathfrak{m}}_{t_1,t_2}$ is of the form $\tilde{\alpha}\mathfrak{e}_\lambda + \tilde{\beta}\mathfrak{e}_0$ for some $\tilde{\alpha} \geq 0$, $\tilde{\beta} \geq 0$ and $\lambda \neq 0$. Further, Lemma 2.7 applied to (56) yields that $\mathfrak{m}'$ is of the form $\alpha'\mathfrak{e}_\lambda + \beta'\mathfrak{e}_0$ for some $\alpha' \geq 0$, $\beta' \geq 0$ and $\lambda \neq 0$. In fact, the assumption $(\mathfrak{m}', \xi') \in \mathcal{S}^*_1$ implies that even $\alpha' > 0$, $\beta' > 0$. Hence, we are in the situation of Case 1, which is a contradiction.

The proof of Proposition 3.4 is complete. $\square$

4. Open questions. We have considered only particles moving on the one-dimensional real line (although we allowed for a multidimensional time). An interesting question is whether it is possible to obtain an analogue of Theorem 1.1 for particles moving in a multidimensional Euclidean space.

Another problem is to classify all stationary systems of particles driven by independent Gaussian processes and starting at the points of an arbitrary point process (rather than a Poisson point process). It seems that to gain information from the stationarity of the one-dimensional distributions of such particle systems, the results of [12] should be used instead of that of [6].

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INSTITUT FÜR MATHEMATISCHE STOCHASTIK
GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN
GOLDSCHMIDTSTR. 7
D-37077 GÖTTINGEN
GERMANY
E-MAIL: kabluch@math.uni-goettingen.de