

## DIFFUSION LIMIT FOR MANY PARTICLES IN A PERIODIC STOCHASTIC ACCELERATION FIELD

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The one-dimensional motion of any number  $\mathcal{N}$  of particles in the field of many independent waves (with strong spatial correlation) is formulated as a second-order system of stochastic differential equations, driven by two Wiener processes. In the limit of vanishing particle mass  $m \rightarrow 0$ , or, equivalently, of large noise intensity, we show that the momenta of all  $\mathcal{N}$  particles converge weakly to  $\mathcal{N}$  independent Brownian motions, and this convergence holds even if the noise is periodic. This justifies the usual application of the diffusion equation to a family of particles in a unique stochastic force field. The proof rests on the ergodic properties of the relative velocity of two particles in the scaling limit.

**1. Introduction.** The motion of a particle in the field of many waves [8, 9, 30] is a fundamental process in classical physics, the understanding of which is a prerequisite to the analysis of many plasma and fluid phenomena. In one space dimension, it can be described by the Hamiltonian model

$$(1.1) \quad H = \frac{p^2}{2m} + \sum_{m=1}^{\mathcal{M}} A_m \cos(k_m q - \omega_m t - \varphi_m),$$

where the particle with mass  $m$  has position  $q$  and momentum  $p$ , while the force field derives from a potential with time Fourier components  $A_m e^{i\varphi_m}$ . The wave field comprises  $\mathcal{M}$  waves, with a smooth dispersion relation associating a wavenumber  $k_m$ , a pulsation  $\omega_m$  and a phase velocity  $v_m = \omega_m / k_m$  to each wave—usually determined by fixed properties of the environment, such as the geometry of the domain where waves propagate (then wavenumbers  $k_m$  and pulsations  $\omega_m$  are discrete). The complex amplitudes  $A_m e^{i\varphi_m}$  are more easily tuned by the experimenter or affected by simple changes in the environment.

The dynamical systems approach to this problem discusses the particle motion after prescribing a single choice for each wave complex amplitude. As it would be quite exceptional to control all waves (though this is, e.g., the assumption underlying the standard map, see [3] for a discussion), physicists often turn to a

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probabilistic description of the dynamics, considering an “ensemble” of realizations  $(A_m, \varphi_m)$ . Various arguments are then invoked to reduce the particle evolution equations to a stochastic differential equation, often driven by a “white noise.” This results in somewhat tractable models (see, e.g., [22] about the validity of such derivations).

In this paper we focus on two issues. First, a random field characterized by  $(A_m, \varphi_m)$  for a given dispersion relation with discrete frequency spectrum may be periodic in time: may the force on the particle be considered as independent over several time periods in a genuine limit? Second, may one consider several particles subject to the same wavefield as independent in a genuine limit? The latter issue underlies the frequent application of the Fokker–Planck equation to the evolution of a family of particles in a single turbulent wavefield—though a priori one can only grant that the diffusion equation describes the evolution of the distribution of a single particle for an ensemble of wavefield samples.

From a more general perspective, this work also relates to the issue of “propagation of chaos” in statistical physics [18, 19], an aspect of Hilbert’s 6th problem: how does chaotic dynamics enable a system, in which initial data are independent (“random”) but the evolution may generate correlations, to behave as if the evolution regenerated independence (“randomness”) or destroyed correlations? Here, how do two Wiener processes, fully describing a prescribed “turbulent” environment, generate  $\mathcal{N}$  independent Brownian motions for particles?

A further motivation for the present work is that physics literature most often focuses on the evolution of particle distribution functions, for example, by showing that they obey a Fokker–Planck equation, and on instantaneous observables such as  $p(t_1)$  for given  $t_1$  (pointwise in time). However, the notion of a diffusion process implies rather a measure on the set of trajectories, namely, functions  $p(\cdot)$  (globally in time). Here we shall show how our model implies that an arbitrary number of trajectories in a single realization of the dynamics do, jointly, admit the Wiener measure description.

In Sections 2 and 3 we motivate the mathematical model more precisely and state our main results, which are proved in the subsequent sections. The crucial Theorem 3.2 is an ergodic theorem for a rescaled process, implying that a process  $V_t$ , describing the relative velocity of one particle with respect to another one (or to its own earlier motion), converges weakly to a Brownian motion. The difficulty in proving the ergodic theorem is that the invariant measure of the diffusion process is infinite (it is the Lebesgue measure  $dx dy$ ), and we must estimate a continuous additive functional generated by a function [namely  $f(x, y) = \sin^2 x$ ] which is not integrable with respect to this measure (it is only locally integrable). This weak convergence then implies the final many-particle result (Theorem 3.1) by a straightforward application of the Lévy characterization of Brownian motion. Section 8 outlines implications and possible extensions to this work.

**2. Physical background.** Because the waves have different frequencies and velocities, it is generally unrealistic to assume their phases to be correlated. Their intensities are more easily observed, but, both in nature and in the laboratory, the accumulation of statistical data on waves often involves only their average power spectra, not the detailed intensity data for each measurement run. We assume here that these complex amplitudes are random data, and investigate the statistics of the particle motion in the resulting time-dependent random field. This dynamics is a “stochastic acceleration problem” for a “passive particle” in weak plasma turbulence [6, 7, 23, 28, 31], and its understanding is a prerequisite to a proper analysis of the case where the particle motion feeds back on the wave evolution [10, 14].

The Hamiltonian (1.1) generates equations of motion

$$(2.1) \quad \dot{q} = p/m,$$

$$(2.2) \quad \dot{p} = \sum_m k_m A_m \sin(k_m q - \omega_m t - \varphi_m).$$

An important observation [5, 15] on the motion of a particle in the field (2.2) is locality in velocity: the evolution of the particle when it has velocity  $\dot{q} = v$  depends only weakly on the waves with a Doppler-shifted frequency  $\omega_m - k_m v$  much larger than their trapping oscillation frequency  $k_m \sqrt{A_m/m}$ . In particular, for a two-wave system the resonance overlap parameter

$$s_{1,2} = \frac{2\sqrt{A_1/m} + 2\sqrt{A_2/m}}{|\omega_2/k_2 - \omega_1/k_1|}$$

becomes unity when there exists a velocity  $u = \omega_2/k_2 - 2\sqrt{A_2/m} = \omega_1/k_1 + 2\sqrt{A_1/m}$  (with  $k_1 > 0, k_2 > 0, \omega_2 > \omega_1$ ). For many waves with overlap parameters  $s \gg 1$ , the relevant phase velocity range for waves influencing the particle is a “resonance box,” with a width scaling as  $(A/m)^{2/3}$  [1, 2].

A good approximation to typical wave dispersion relations in the strong overlap limit, after a Galileo change of reference frame (see, e.g., Section 6.7 in [14]), is

$$(2.3) \quad k_m = k_0, \quad \omega_m = 2\pi(m - \mathcal{M}/2)/T$$

for some  $k_0, T$ . Then, in the limit  $\mathcal{M} \rightarrow \infty$ , the equations of motion yield for  $A_m e^{i\varphi_m} = A_0$  (with real  $A_0$ ) the well-known standard map [3]. The case where phases  $\varphi_m$  are independent random variables uniformly distributed on the circle  $[0, 2\pi]$ , while  $A_m = A_0$  is given, was investigated notably by Cary et al. [1, 4, 14] and occurs in the context of the random phase approximation.

To the extent that the phases and amplitudes of the waves are independent random variables, the physicist usually views the force (2.2) as a mollification of a white noise, with amplitude  $\sigma = \sqrt{\mathbb{E}(k_m^2 A_m^2)} = |k_m| \sqrt{\mathbb{E}A_m^2}$ , where the relevant mode  $m$  is the one nearest to the current particle velocity (the mathematical expectation  $\mathbb{E}$  is called ensemble average with respect to wave amplitudes and phases).<sup>2</sup>

<sup>2</sup>Phases do not appear in  $\sigma$  (nor in  $s$ ) because the relative phase of two waves  $\varphi_m + \omega_m t - \varphi_n - \omega_n t$  varies uniformly over time (hence  $\varphi_m - \varphi_n$  can be absorbed in the choice of the time origin).

This is the core of quasilinear theory [11, 25, 27, 32]. With some care, one interprets (2.1) and (2.2) as a stochastic differential equation; this applies in the case  $m \in \mathbb{Z}$  for dispersion relation (2.3) with Gaussian independent complex amplitudes such that  $\mathbb{E}A_m^2 = k_0^{-2}\sigma^2$ . The particle velocity then has a Brownian evolution, so that for  $t, t' \in [0, T]$

$$(2.4) \quad \mathbb{E}(p_t - p_0)(p_{t'} - p_0) = D_{\text{QL}} \min(t, t')$$

with the quasilinear diffusion coefficient  $D_{\text{QL}} = \sigma^2 T / 2$ . However, the particle evolution for  $t > T$  may show a strong correlation to its motion for  $0 \leq t \leq T$  because the waves are periodic in time [12, 13], and the dispersion relation (2.3) may generate a strong spatial correlation between the motions of two particles because all waves acting on a particle at any time have the same wavelength.<sup>3</sup>

Set  $k_0 = 1$  and  $T = 2\pi$  by the choice of space and time units. The particle phase space is  $\mathbb{T} \times \mathbb{R}$ , where  $\mathbb{T}$  is the circle modulo  $2\pi$ , and the particle equation of motion reads, with initial data  $(q(0), \dot{q}(0)) = (q_0, \dot{q}_0)$ ,

$$(2.5) \quad \begin{aligned} \ddot{q} &= m^{-1} \sum_{m=-\mu}^{\mu'-1} A_m \sin(q - mt + \varphi_m) \\ &= m^{-1} \sum_m A_m \cos(mt - \varphi_m) \sin q - m^{-1} \sum_m A_m \sin(mt - \varphi_m) \cos q \end{aligned}$$

in a galilean frame moving at a velocity inside the spectrum of wave phase velocities ( $\mu + \mu' = \mathcal{M}$ ). We assume  $\mu \gg 1, \mu' \gg 1$ . For  $A_m = A_0$  given and independent random phases (uniform on the circle), in the limit  $A_0/m \rightarrow \infty$ , with  $(m/A_0)^{2/3} \mathcal{M} \gtrsim 10$ , Bénisti and Escande [1, 2] have shown that the particle momentum  $p = m\dot{q}$  follows essentially a Brownian motion, with diffusion constant given by the quasilinear estimate

$$D_{\text{QL}} = \pi A_0^2$$

as long as the motion does not approach the boundaries of the wave velocity spectrum. Then the particle momenta  $p$  for an ensemble of independent realizations of the system will be described by a distribution function verifying the Fokker–Planck equation

$$(2.6) \quad \partial_t f = \frac{D_{\text{QL}}}{2} \partial_p^2 f$$

even for  $t > T$ . Numerical simulations [13] show similar behavior for i.i.d. complex Gaussian random variables  $A_m e^{i\varphi_m}$ . The present work establishes a rigorous version of this result in the frame of stochastic processes.

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<sup>3</sup>There is a large body of literature on the case of incoherent waves with no dispersion relation. Then the sum  $\sum_m$  becomes a double sum  $\sum_{m,n}$  and one varies wavenumbers  $k_n$  independently from pulsations  $\omega_m$ . This space–time stochastic environment is more noisy than our model and may also be considered to motivate a quasilinear approximation.

**3. Main results.** We first let  $\min(\mu, \mu') \rightarrow \infty$  in the model, taking  $A_m \times \cos \varphi_m, A_m \sin \varphi_m$  as i.i.d. Gaussian random variables with zero expectation and  $\mathbb{E}(A_m \cos \varphi_m)^2 = \mathbb{E}(A_m \sin \varphi_m)^2 = 1/2$  for all  $m \in \mathbb{Z}$ . In particular this implies that phases  $\varphi_m$  are i.i.d. uniformly on  $\mathbb{T}$ . To follow earlier practice, we now set  $m = 1/A$ . Then formally (2.5) becomes the Stratonovich stochastic differential equation for  $0 \leq t \leq 2\pi$

$$(3.1) \quad dQ_t = AP_t dt,$$

$$(3.2) \quad dP_t = \sin(Q_t) \circ dC_t + \cos(Q_t) \circ dS_t$$

with initial data  $Q_0 = q_0, P_0 = p_0 = m\dot{q}_0 = \dot{q}_0/A$ ; here, from [20],  $\pi^{-1/2}(C, S)$  is a standard two-dimensional Brownian motion. In other words,  $C$  and  $S$  are martingales, with  $C_0 = S_0 = 0$  and

$$(3.3) \quad \langle C \rangle_t = \langle S \rangle_t = \pi t, \quad \langle C, S \rangle_t = 0.$$

Note that the Stratonovich and Itô integrals define the same solutions for this system, and that the vector fields  $(\sin q)\partial_p$  and  $(\cos q)\partial_p$  commute.

For  $0 \leq t \leq 2\pi$ , it is clear that  $P$  is a Brownian motion for any value of  $A \geq 0$  (see the first lines of Section 7). However, model (2.5) defines a dynamical system for all times  $0 \leq t < \infty$ , and one may wonder how the initial stochastic behavior over  $[0, 2\pi]$  extends for longer times. Formally, one solves then (3.1) and (3.2) with the periodized field, that is, with the continuous processes defined by

$$(3.4) \quad dC_{t+2k\pi} = dC_t, \quad dS_{t+2k\pi} = dS_t$$

for  $k \in \mathbb{Z}$ . In other words,  $C_t - \frac{t}{2\pi}C_{2\pi}$  and  $S_t - \frac{t}{2\pi}S_{2\pi}$  are independent Brownian bridges repeated periodically for  $t \in \mathbb{R}$ , while  $C_{2\pi}$  and  $S_{2\pi}$  are independent Gaussian random variables with expectation 0 and variance  $2\pi^2$ .

For this extended process, the wave field acting on the particle for  $t \notin [0, 2\pi]$  is not stochastically independent from the wave field acting during  $[0, 2\pi]$ . Therefore one does not expect the particle momentum to proceed as a Brownian motion for all times, and indeed for  $A$  small enough the velocity  $\dot{q}$  may remain bounded in a narrow interval for all times. This is easily seen numerically and can be attributed to the existence of Kolmogorov–Arnold–Moser invariant tori in the three-dimensional extended phase space with coordinates  $(t, q, \dot{q})$ .

On the other hand, for large  $A$ , the dynamics viewpoint [1, 2, 14] suggests that the nonlinearity in the equations of motion (due to trigonometric functions of  $Q$ ) may enable a decorrelation of the force over the period  $\mathcal{T} = 2\pi$ , so that the long-time evolution of the velocity would also be close to Brownian. This is what we shall show.

An intimately related issue is the relative motion of several particles, released in the same realization of the wave field. Even though each particle velocity diffuses for  $t \in [0, 2\pi]$ , their motions are not independent. We shall also show that for large  $A$  the motions of any finite family of particles released at initial data  $(Q_0^{(v)}, P_0^{(v)})$ ,

$1 \leq \nu \leq \mathcal{N}$ , approaches a family of  $\mathcal{N}$  independent processes. This can also be expected from the consideration of the top Lyapunov exponent of the dynamics (3.1) and (3.2) in the limit  $A \rightarrow \infty$ .

**THEOREM 3.1.** *For any  $\mathcal{N} > 0$ , the momentum processes  $P^{(\nu)}$  defined by*

$$(3.5) \quad dQ_t^{(\nu)} = AP_t^{(\nu)} dt, \quad Q_0^{(\nu)} = q_0^{(\nu)},$$

$$(3.6) \quad dP_t^{(\nu)} = (\sin Q_t^{(\nu)}) dC_t + (\cos Q_t^{(\nu)}) dS_t, \quad P_0^{(\nu)} = p_0^{(\nu)},$$

with  $\mathcal{N}$  different initial data  $(q_0^{(\nu)}, p_0^{(\nu)}) \in \mathbb{T} \times \mathbb{R}$ ,  $1 \leq \nu \leq \mathcal{N}$ , converge as  $A \rightarrow \infty$  to  $\mathcal{N}$  independent Wiener processes with variance  $\pi t$ , and convergence is in law in  $C(\mathbb{R}_+, \mathbb{R}^{\mathcal{N}})$ .

The key argument in the proof is the following weak convergence theorem, where we write now  $n = \pi^{1/2}A$ . Consider the two-dimensional diffusion process indexed by  $n \geq 1$ , solution of the SDE on  $\mathbb{R}^2$ ,

$$(3.7) \quad \begin{cases} \frac{dU_t^n}{dt} = nV_t^n, & U_0 = u, \\ dV_t^n = \sin(U_t^n) dW_t, & V_0 = v, \end{cases}$$

where  $(u, v) \notin \{(k\pi, 0), k \in \mathbb{Z}\}$  and  $W$  is a standard Brownian motion. We prove

**THEOREM 3.2.** *As  $n \rightarrow \infty$ ,*

$$V^n \Rightarrow v + \frac{1}{\sqrt{2}}B,$$

where  $\{B_t, t \geq 0\}$  is a standard one-dimensional Brownian motion, and the convergence is in law in  $C(\mathbb{R}_+, \mathbb{R})$ .

**4. A change of time scale.** Note that for any  $n \geq 1$ , the law of  $\{(U_t^n, V_t^n), t \geq 0\}$ , the solution of (3.7), is characterized by the statement

$$\begin{cases} \frac{dU_t^n}{dt} = nV_t^n, & U_0 = u, \\ V^n \text{ is a martingale,} & \frac{d\langle V^n \rangle_t}{dt} = \sin^2(U_t^n), & V_0^n = v. \end{cases}$$

Now define (like Bénisti and Escande [1, 2])

$$(4.1) \quad X_t = U_{n^{-2/3}t}^n, \quad Y_t = n^{1/3}V_{n^{-2/3}t}^n.$$

We first note that  $X_0 = u$ ,  $Y_0 = n^{1/3}v$ ,  $Y$  is a martingale, and

$$\begin{cases} \frac{dX_t}{dt} = n^{-2/3} \frac{dU^n}{dt}(n^{-2/3}t) = n^{1/3}V_{n^{-2/3}t}^n = Y_t, \\ \langle Y \rangle_t = n^{2/3} \langle V^n \rangle_{n^{-2/3}t}, & \frac{d\langle Y \rangle_t}{dt} = \sin^2(X_t). \end{cases}$$

Using a well-known martingale representation theorem, we can pretend that there exists a standard Brownian motion  $\{B_t, t \geq 0\}$  such that

$$(4.2) \quad \begin{cases} \frac{dX_t}{dt} = Y_t, & X_0 = u, \\ dY_t = \sin(X_t) dB_t, & Y_0 = n^{1/3}v. \end{cases}$$

Note that the process  $\{(X_t, Y_t), t \geq 0\}$  still depends upon  $n$ , but only through the value of  $Y_0$ .

On the other hand,  $V_t^n = n^{-1/3}Y_{n^{2/3}t}$ . Hence

$$V_t^n = v + n^{-1/3} \int_0^{n^{2/3}t} \sin(X_s) dB_s,$$

in other words  $V^n$  is a martingale such that  $V_0^n = v$  and

$$\langle V^n \rangle_t = n^{-2/3} \int_0^{n^{2/3}t} \sin^2(X_s^n) ds.$$

Here we recall the fact that the process  $X$  depends upon  $n$  (through the initial condition of  $Y$ ), unless  $v = 0$ . Consequently

$$(4.3) \quad \lim_{n \rightarrow \infty} \langle V^n \rangle_t = t \times \lim_{n \rightarrow \infty} \frac{1}{n^{2/3}t} \int_0^{n^{2/3}t} \sin^2(X_s^n) ds,$$

and in order to prove Theorem 3.2 it suffices to show that the above limit is  $t/2$ .

**5. Qualitative properties of the solution of (4.2).** We now consider the two-dimensional diffusion process

$$(5.1) \quad \begin{cases} \frac{dX_t}{dt} = Y_t, & X_0 = x, \\ dY_t = \sin(X_t) dB_t, & Y_0 = y, \end{cases}$$

with values in the state-space  $E = [0, 2\pi) \times \mathbb{R} \setminus \{(0, 0), (\pi, 0)\}$ , where  $2\pi$  is identified with 0. We first prove that the process  $\{(X_t, Y_t), t \geq 0\}$  is a conservative  $E$ -valued diffusion. Indeed,

PROPOSITION 5.1. *Whenever the initial condition  $(x, y)$  belongs to  $E$ ,*

$$\inf\{t > 0, (X_t, Y_t) \in \{(0, 0), (\pi, 0)\}\} = +\infty \quad a.s.$$

PROOF. We define the stopping time

$$\tau = \inf\{t, (X_t, Y_t) = (0, 0)\}.$$

Let  $R_t = X_t^2 + Y_t^2$ ,  $Z_t = \log R_t$ ,  $t \geq 0$ . A priori,  $Z_t$  takes its values in  $[-\infty, +\infty)$ . Itô calculus on the interval  $[0, \tau)$  yields

$$\begin{aligned} dX_t^2 &= 2X_t Y_t dt, \\ dY_t^2 &= 2Y_t \sin(X_t) dB_t + \sin^2(X_t) dt, \\ dZ_t &= \frac{dR_t}{R_t} - \frac{d\langle R \rangle_t}{2R_t^2} \\ &= \frac{2Y_t X_t + \sin^2(X_t)}{R_t} dt - 2 \frac{Y_t^2 \sin^2(X_t)}{R_t^2} dt + 2 \frac{Y_t \sin(X_t)}{R_t} dB_t. \end{aligned}$$

Now clearly  $|\sin(x)| \leq |x|$ ,  $\sin^2(x) \leq x^2$ , and it follows from the above and standard inequalities that on the time interval  $[0, \tau)$ ,

$$Z_t \geq Z_0 - 2t + \int_0^t \varphi_s dB_s,$$

where  $|\varphi_s| \leq 1$ . Hence the process  $\{Z_t, t \geq 0\}$  is bounded from below on any finite time interval, which implies that  $\tau = +\infty$  a.s., since  $\tau = \inf\{t, Z_t = -\infty\}$ . A similar argument shows that  $\tau' = +\infty$  a.s., where

$$\tau' = \inf\{t, (X_t, Y_t) \in \{(0, 0), (\pi, 0)\}\}. \quad \square$$

We next prove (here and below  $\mathcal{B}_E$  stands for the  $\sigma$ -algebra of Borel subsets of  $E$ )

PROPOSITION 5.2. *The transition probabilities*

$$\{p((x, y); t, A) := \mathbb{P}_{x,y}((X_t, Y_t) \in A), (x, y) \in E, t > 0, A \in \mathcal{B}_E\}$$

have smooth densities  $p((x, y); t, (x', y'))$  with respect to Lebesgue's measure  $dx' dy'$  on  $E$ .

PROOF. Consider the Lie algebra of vector fields on  $E$  generated by  $X_1 = \sin(x) \frac{\partial}{\partial y}$ ,  $X_2 = [X_0, X_1]$  and  $X_3 = [X_0, [X_0, X_1]]$ , where  $X_0 = y \frac{\partial}{\partial x}$ . This Lie algebra has rank 2 at each point of  $E$ . The result is now a standard consequence of the well-known Malliavin calculus (see, e.g., Nualart [24]).  $\square$

PROPOSITION 5.3. *The  $E$ -valued diffusion process  $\{(X_t, Y_t), t \geq 0\}$  is topologically irreducible, in the sense that for all  $(x, y) \in E$ ,  $t > 0$ ,  $A \in \mathcal{B}_E$  with non-empty interior*

$$\mathbb{P}_{x,y}((X_t, Y_t) \in A) > 0.$$

PROOF. From Stroock–Varadhan’s support theorem (see, e.g., Ikeda–Watanabe [17]) the support of the law of  $(X_t, Y_t)$  starting from  $(X_0, Y_0) = (x, y)$  is the closure of the set of points which the following controlled ordinary differential equation can reach at time  $t$  by varying the control  $\{u(s), 0 \leq s \leq t\}$  in the class of piecewise continuous functions

$$(5.2) \quad \begin{cases} \frac{dx}{ds}(s) = y(s), & x(0) = x, \\ \frac{dy}{ds}(s) = \sin(x(s))u(s), & y(0) = y. \end{cases}$$

It is not hard to show that the set of accessible points at time  $t > 0$  by the solution of (5.2) is dense in  $E$ . The result now follows from the fact that the transition probability is absolutely continuous with respect to Lebesgue’s measure (see Proposition 5.2).  $\square$

We next prove:

LEMMA 5.4.

$$\mathbb{P}(|Y_t| \rightarrow \infty, \text{ as } t \rightarrow \infty) = 0.$$

PROOF. The lemma follows readily from the fact that

$$Y_t = W\left(\int_0^t \sin^2(X_s) ds\right),$$

where  $\{W(t), t \geq 0\}$  is a scalar Brownian motion. Then either  $\int_0^t \sin^2(X_s) ds$  is bounded and  $Y_t$  is finite, or the integral diverges and  $Y_t$  is finite anyway because  $W$  is recurrent.  $\square$

Hence the topologically irreducible  $E$ -valued Feller process  $\{(X_t, Y_t), t \geq 0\}$  is recurrent. Its unique (up to a multiplicative constant) invariant measure is the Lebesgue measure on  $E$ , so that in particular the process is null-recurrent. It then follows from (ii) in Theorem 20.21 from Kallenberg [21].

LEMMA 5.5. For all  $M > 0$ , as  $t \rightarrow \infty$ ,

$$\frac{1}{t} \int_0^t \mathbf{1}_{\{|Y_s| \leq M\}} ds \rightarrow 0 \quad a.s.$$

**6. A path decomposition of the process  $\{(X_t, Y_t), t \geq 0\}$ .** We first define two sequences of stopping times. Let  $T_0 = 0$  and

$$\begin{aligned} \text{for } \ell \text{ odd} & \quad T_\ell = \inf\{t > T_{\ell-1}, |Y_t| \geq M + 1\}, \\ \text{for } \ell \text{ even} & \quad T_\ell = \inf\{t > T_{\ell-1}, |Y_t| \leq M\}. \end{aligned}$$

Let now  $\tau_0 = T_1$ . We next define recursively  $\{\tau_k, k \geq 1\}$  as follows. Given  $\tau_{k-1}$ , we first define

$$L_k = \sup\{\ell \geq 0, \tau_{k-1} \geq T_{2\ell+1}\}.$$

Now let

$$\eta_k = \begin{cases} \tau_{k-1}, & \text{if } \tau_{k-1} < T_{2L_k+2}, \\ T_{2L_k+3}, & \text{if } \tau_{k-1} \geq T_{2L_k+2}. \end{cases}$$

We now define

$$\tau_k = \inf\{t > \eta_k, |X_t - X_{\eta_k}| = 2\pi\} \wedge \inf\{t > \eta_k, |Y_t - Y_{\eta_k}| > 1\}.$$

It follows from the above definitions that

$$\int_0^t \mathbf{1}_{\{|Y_s| \geq M+1\}} \sin^2(X_s) ds \leq \sum_{k=1}^{\infty} \int_{\eta_k \wedge t}^{\tau_k \wedge t} \sin^2(X_s) ds \leq \int_0^t \sin^2(X_s) ds,$$

a statement which will be refined in the proof of Proposition 6.3. Define

$$\begin{aligned} K^0 &= \{k \geq 1, |Y_{\tau_k} - Y_{\eta_k}| < 1\}, \\ K^1 &= \{k \geq 1, |Y_{\tau_k} - Y_{\eta_k}| = 1\}, \\ K_t &= \{k \geq 1, \eta_k < t\}, \\ K_t^0 &= K^0 \cap K_t, \quad K_t^1 = K^1 \cap K_t. \end{aligned}$$

We first prove:

LEMMA 6.1.

$$\frac{1}{t} \sum_{k \in K_t^1} (\tau_k - \eta_k) \rightarrow 0$$

in  $L^1(\Omega)$  as  $M \rightarrow \infty$ , uniformly in  $t > 0$ .

PROOF. We shall use repeatedly the fact that since  $|Y_{\eta_k}| \geq M > 2$ ,  $|Y_{\eta_k}| - 1 \geq |Y_{\eta_k}|/2$ . We have that (see the Appendix below), since  $\tau_k - \eta_k \leq 4\pi/|Y_{\eta_k}|$ ,

$$\begin{aligned} \mathbb{P}(k \in K^1 | \mathcal{F}_{\eta_k}) &\leq \mathbb{P}\left(\sup_{\eta_k \leq t \leq \tau_k} |Y_t - Y_{\eta_k}| \geq 1 | \mathcal{F}_{\eta_k}\right) \\ &\leq 2 \exp(-|Y_{\eta_k}|/(8\pi)). \end{aligned}$$

Consequently, using again the inequality  $\tau_k - \eta_k \leq 4\pi/|Y_{\eta_k}|$ , we deduce that

$$\begin{aligned} \mathbb{E}[(\tau_k - \eta_k) \mathbf{1}_{\{k \in K^1\}} | \mathcal{F}_{\eta_k}] &\leq \frac{8\pi}{|Y_{\eta_k}|} \exp(-|Y_{\eta_k}|/(8\pi)) \\ &\leq \frac{8\pi}{|Y_{\eta_k}|} \exp(-M/(8\pi)). \end{aligned}$$

On the other hand, whenever  $k \in K^0$ ,

$$\tau_k - \eta_k \geq 2\pi/(|Y_{\eta_k}| + 1) \geq \pi/|Y_{\eta_k}|.$$

Now, provided  $t \geq 4\pi/M$ ,

$$\begin{aligned} 2t &\geq t + \frac{4\pi}{M} \geq \mathbb{E}\left[\sum_{k \in K_t^0} (\tau_k - \eta_k)\right] \\ &\geq \pi \mathbb{E}\left[\sum_{k \in K_t} \mathbf{1}_{\{k \in K^0\}} \frac{1}{|Y_{\eta_k}|}\right] \\ &\geq \frac{\pi}{2} \mathbb{E}\left[\sum_{k \in K_t} \frac{1}{|Y_{\eta_k}|}\right], \end{aligned}$$

since

$$\begin{aligned} \mathbb{P}(k \in K^0 | \mathcal{F}_{\eta_k}) &= 1 - \mathbb{P}(k \in K^1 | \mathcal{F}_{\eta_k}) \\ &\geq 1 - 2 \exp(-M/(8\pi)) \\ &\geq 1/2, \end{aligned}$$

provided  $M$  is large enough. Finally

$$\begin{aligned} \frac{1}{t} \mathbb{E}\left[\sum_{k \in K_t^1} (\tau_k - \eta_k)\right] &\leq 32 \exp(-M/(8\pi)) \frac{\mathbb{E}[\sum_{k \in K_t} |Y_{\eta_k}|^{-1}]}{\mathbb{E}[\sum_{k \in K_t} |Y_{\eta_k}|^{-1}]} \\ &= 32 \exp(-M/(8\pi)) \\ &\rightarrow 0, \end{aligned}$$

as  $M \rightarrow \infty$ , uniformly in  $t$ .  $\square$

Now, for any  $k \in K^0$ ,

$$\begin{aligned} (6.1) \quad \int_{\eta_k}^{\tau_k} \sin^2(X_s) ds &= \frac{\tau_k - \eta_k}{2\pi} \int_0^{2\pi} \sin^2(x) dx \\ &\quad + \int_{\eta_k}^{\tau_k} \sin^2(X_s) \left[1 - \frac{Y_s(\tau_k - \eta_k)}{2\pi}\right] ds, \end{aligned}$$

and we have

$$\begin{aligned} \left| \int_{\eta_k}^{\tau_k} \sin^2(X_s) \left[1 - \frac{Y_s(\tau_k - \eta_k)}{2\pi}\right] ds \right| &= \left| \int_{\eta_k}^{\tau_k} \int_{\eta_k}^{\tau_k} \sin^2(X_s) \frac{Y_r - Y_s}{2\pi} dr ds \right| \\ &\leq \frac{1}{2\pi} \int_{\eta_k}^{\tau_k} \int_{\eta_k}^{\tau_k} |Y_r - Y_s| dr ds. \end{aligned}$$

Finally we have:

LEMMA 6.2. *Uniformly in  $t > 0$ ,*

$$\frac{\sum_{k \in K_t^0} \int_{\eta_k}^{\tau_k} \int_{\eta_k}^{\tau_k} |Y_r - Y_s| dr ds}{\sum_{k \in K_t^0} (\tau_k - \eta_k)} \rightarrow 0$$

a.s., as  $M \rightarrow \infty$ .

PROOF. Since  $|Y_t - Y_{\eta_k}| \leq 1$  for  $\eta_k \leq t \leq \tau_k$ ,

$$\begin{aligned} \frac{\sum_{k \in K_t^0} \int_{\eta_k}^{\tau_k} \int_{\eta_k}^{\tau_k} |Y_r - Y_s| dr ds}{\sum_{k \in K_t^0} (\tau_k - \eta_k)} &\leq 2 \sup_{k \in K_t^0} (\tau_k - \eta_k) \\ &\leq 8\pi/M \\ &\rightarrow 0, \end{aligned}$$

as  $M \rightarrow \infty$ , uniformly in  $t$ .  $\square$

We are now in a position to prove the following ergodic-type theorem, from which Theorem 3.2 will follow:

PROPOSITION 6.3. As  $t \rightarrow \infty$ ,

$$\frac{1}{t} \int_0^t \sin^2(X_s) ds \rightarrow \frac{1}{2}$$

in probability.

PROOF. We first note that

$$[0, t] = B_t^0 \cup B_t^1 \cup C_t,$$

where

$$B_t^0 = [0, t] \cap \left( \bigcup_{k \in K_t^0} [\eta_k, \tau_k] \right),$$

$$B_t^1 = [0, t] \cap \left( \bigcup_{k \in K_t^1} [\eta_k, \tau_k] \right),$$

$$C_t = [0, t] \setminus (B_t^0 \cup B_t^1).$$

We have

$$\begin{aligned} \frac{1}{t} \int_0^t \sin^2(X_s) ds &= \frac{1}{t} \int_0^t \mathbf{1}_{B_t^0}(s) \sin^2(X_s) ds + \frac{1}{t} \int_0^t \mathbf{1}_{B_t^1}(s) \sin^2(X_s) ds \\ &\quad + \frac{1}{t} \int_0^t \mathbf{1}_{C_t}(s) \sin^2(X_s) ds. \end{aligned}$$

Now  $C_t \subset \{s \in [0, t], |Y_s| \leq M + 1\}$ , so for each fixed  $M > 0$ , it follows from Lemma 5.5 that the last term can be made arbitrarily small, by choosing  $t$  large enough. The second term goes to zero as  $M \rightarrow \infty$ , uniformly in  $t$ , from Lemma 6.1. Finally the first term equals the searched limit, plus an error term which goes to 0 as  $M \rightarrow \infty$ , uniformly in  $t$ , as follows from Lemma 6.2 and the

following fact, which follows from the combination of Lemma 6.1 and Lemma 5.5

$$\frac{1}{t} \sum_{k \in K_t^0} (\tau_k - \eta_k) \rightarrow 1$$

in probability, as  $n \rightarrow \infty$ .  $\square$

We can finally proceed with:

PROOF OF THEOREM 3.2. All we have to show is that [see (4.3)]

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2/3}t} \int_0^{n^{2/3}t} \sin^2(X_s^n) ds = \frac{1}{2\pi} \int_0^{2\pi} \sin^2(x') dx' = \frac{1}{2}$$

in probability. In the case  $v = 0$ , the process  $\{(X_t^n, Y_t^n)\}$  does not depend upon  $n$ , and the result follows precisely from Proposition 6.3. Now suppose that  $v \neq 0$ . In that case, the result can be reformulated equivalently as follows. For some  $x \in \mathbb{R}$ ,  $y \neq 0$ , each  $t > 0$ , define the process  $\{(X_s^t, Y_s^t), 0 \leq s \leq t\}$  as the solution of the SDE

$$\begin{cases} \frac{dX_s^t}{ds} = Y_s^t, & X_0^t = x, \\ dY_s^t = \sin(X_s^t) dW_s, & Y_0^t = \sqrt{t}y. \end{cases}$$

We need to show that

$$\frac{1}{t} \int_0^t \sin^2(X_s^t) ds \rightarrow \frac{1}{2\pi} \int_0^{2\pi} \sin^2(x') dx'$$

in probability, as  $t \rightarrow \infty$ . Note that in time  $t$ , the process  $Y^t$  starting from  $\sqrt{t}y$  can come back near the origin.

It is easily seen, by introducing the Markov time  $\tau_M^t = \inf\{s > 0, |Y_s^t| \leq M\}$  and exploiting the strong Markov property, that

$$\frac{1}{t} \int_0^t \mathbf{1}_{\{|Y_s^t| \leq M\}} ds \rightarrow 0 \quad \text{a.s.}$$

follows readily from Lemma 5.5. The rest of the argument leading to Proposition 6.3 is based upon limits as  $M \rightarrow \infty$ , uniformly with respect to  $t$ . It thus remains to check that the fact that  $Y_0^t$  now depends upon  $t$  does not spoil this uniformity, which is rather obvious.  $\square$

REMARK 6.4. This proof holds uniformly with respect to initial data  $(u, v)$  satisfying  $|v| \geq a$  for any  $a > 0$ .

**7. Proof of Theorem 3.1.** Our Theorem 3.1 now appears as a simple corollary of Theorem 3.2.

PROOF OF THEOREM 3.1. We first prove the theorem for  $t \in [0, 2\pi]$ . Then the vector  $\mathbf{P} = (P^{(1)}, \dots, P^{(N)})$  is a martingale in  $\mathbb{R}^N$ , and to prove our claim it suffices to show that its quadratic variation matrix converges to  $\pi t$  times the identity matrix. The diagonal elements of the matrix are

$$\langle P^{(v)} \rangle_t = \int_0^t (\sin^2 Q_s^{(v)} + \cos^2 Q_s^{(v)}) \pi ds = \pi t$$

and we only need to compute the cross-variation

$$\begin{aligned} \langle P^{(v)}, P^{(v')} \rangle_t &= \int_0^t (\sin Q_s^{(v)} \sin Q_s^{(v')} + \cos Q_s^{(v)} \cos Q_s^{(v')}) \pi ds \\ (7.1) \qquad &= \int_0^t \cos(Q_s^{(v)} - Q_s^{(v')}) \pi ds. \end{aligned}$$

Now, define (with  $n = \pi^{1/2} A$ )

$$\begin{aligned} U_t^n &= \frac{1}{2}(Q_t^{(v)} - Q_t^{(v')}), & V_t^n &= n^{-1} \frac{dU_t^n}{dt}, \\ U_t^m &= \frac{1}{2}(Q_t^{(v)} + Q_t^{(v')}), & V_t^m &= n^{-1} \frac{dU_t^m}{dt}. \end{aligned}$$

These processes solve the stochastic differential equation

$$(7.2) \qquad dU_t^n = nV_t^n dt, \qquad U_0^n = \frac{q_0^{(v)} - q_0^{(v')}}{2},$$

$$(7.3) \qquad V_0^n = \frac{p_0^{(v)} - p_0^{(v')}}{2\pi^{1/2}},$$

$$\begin{aligned} (7.4) \qquad dV_t^n &= \frac{1}{2\sqrt{\pi}} (\sin(U_t^m + U_t^n) - \sin(U_t^m - U_t^n)) dC_t \\ &+ \frac{1}{2\sqrt{\pi}} (\cos(U_t^m + U_t^n) - \cos(U_t^m - U_t^n)) dS_t \\ &= \sin U_t^n dW_t^n, \end{aligned}$$

where the process  $W^n$  is the martingale defined by  $W_0^n = 0$  and

$$dW_t^n = \pi^{-1/2} (\cos U_t^m dC_t - \sin U_t^m dS_t).$$

The quadratic variation of  $W^n$  is

$$\langle W^n \rangle_t = \pi^{-1} \int_0^t (\cos^2 U_s^m + \sin^2 U_s^m) \pi ds = t$$

in view of the quadratic and cross variations (3.3) of  $(C, S)$ , and this result does not depend on the process  $U^m$  (which follows the center of mass of the two particles  $v$  and  $v'$ ). Thus  $W^n$  is a standard Wiener process, and  $(U^n, V^n)$  defined by (7.2),

(7.3) and (7.4) satisfies the hypotheses of Theorem 3.2. Hence, with  $X$  defined by (4.1),

$$\begin{aligned} \langle P^{(\nu)}, P^{(\nu')} \rangle_t &= \int_0^t \cos(2U_s^n) \pi \, ds = \int_0^t (1 - 2 \sin^2 U_s^n) \pi \, ds \\ &= \pi t \left( 1 - \frac{2}{n^{2/3} t} \int_0^{n^{2/3} t} \sin^2 X_{s'}^n \, ds' \right), \end{aligned}$$

which converges in probability to 0 for  $n \rightarrow \infty$  as shown in the proof of the theorem.

Now we consider the process over the interval  $[0, 4\pi]$ , taking into account that  $(C, S)$  over the whole interval is neither a martingale nor Markov. From the given initial data, amplitude  $A$  and realization of  $(C, S)$ , we define a subsidiary set of particles  $\mathcal{N} + 1 \leq \nu \leq 2\mathcal{N}$ , with initial data

$$Q_0^{(\nu)} = Q_{2\pi}^{(\nu-\mathcal{N})}, \quad P_0^{(\nu)} = P_{2\pi}^{(\nu-\mathcal{N})},$$

of which a.s. none coincides (modulo  $2\pi$  for  $q$ ) with any of the initial data  $(Q_0^{(\nu)}, P_0^{(\nu)})$ . Recalling that at  $t = 2\pi$  the law of any  $P_{2\pi}^{(\nu)}$  (for  $1 \leq \nu \leq \mathcal{N}$ ) is Gaussian with variance  $2\pi^2$  [so that its probability density is bounded by  $1/\sqrt{2\pi 2\pi^2} < 1/(2\pi)$ ], this ensures that, given  $\varepsilon > 0$ ,

$$\begin{aligned} &\mathbb{P} \left( \min_{1 \leq \nu, \nu' \leq \mathcal{N}} |P_0^{(\nu)} - P_{2\pi}^{(\nu')}| < \varepsilon \text{ or } \min_{1 \leq \nu < \nu' \leq \mathcal{N}} |P_{2\pi}^{(\nu)} - P_{2\pi}^{(\nu')}| < \varepsilon \right) \\ &\leq \frac{3\varepsilon \mathcal{N}^2}{\sqrt{2\pi 2\pi^2}} < \varepsilon \mathcal{N}^2 / 2. \end{aligned}$$

For processes such that  $\min_{\nu, \nu'} |P_0^{(\nu)} - P_{2\pi}^{(\nu')}| \geq \varepsilon$  and  $\min_{\nu < \nu'} |P_{2\pi}^{(\nu)} - P_{2\pi}^{(\nu')}| \geq \varepsilon$ , the above proof holds (uniformly with respect to initial data) for the full set of  $2\mathcal{N}$  particles defined here. It follows that a.s. the  $P_t^{(\nu)} - P_0^{(\nu)}$ ,  $t \in [0, 2\pi]$ ,  $1 \leq \nu \leq 2\mathcal{N}$ , are mutually independent Wiener processes, each with variance  $\pi t$ . This implies that  $P_t^{(\nu)} - P_0^{(\nu)}$ ,  $t \in [0, 4\pi]$ ,  $1 \leq \nu \leq \mathcal{N}$ , are also mutually independent Wiener processes.

For any interval  $[0, 2k\pi]$  with  $k \in \mathbb{N}_0$  the same argument reduces the  $\mathcal{N}$ -particle problem to  $k\mathcal{N}$  particles over  $[0, 2\pi]$ , and the proof is complete.  $\square$

REMARK 7.1. Our theorem allows that  $P_0^{(\nu)} = P_0^{(\nu')}$  for some  $\nu \neq \nu'$  with  $1 \leq \nu, \nu' \leq \mathcal{N}$ .

**8. Perspectives.** The implications of Theorem 3.1 are twofold. First, for  $\mathcal{N} = 1$ , they support the observation [1, 2, 4, 13] that, in systems with finite  $\mathcal{M} \gg 1$  and  $A_0/m \gg 1$  (with appropriate scaling), the long-time behavior of a single particle in a periodic wave field exhibits statistical properties approaching those of the Brownian motion. To complete the connection with physics literature, one must

now discuss how the finite sums in (2.2) approach the right-hand side of (3.2), and how this implies that the solutions of the first equation approach the solutions of the latter equation [29]. This will be discussed separately.

Second, and this is conceptually more fundamental, for a single sample of the Wiener wavefield (with  $\mathcal{M} = \infty$  formally), with strong spatial correlations (thanks to the single wavevector  $k_0$  in the model), the limit  $A_0/m \rightarrow \infty$  leads to independence of the evolutions of all  $\mathcal{N}$  particles, which can then be collectively described by the diffusion equation. While such an independence is often admitted without proof in physics practice, our work provides an explicit justification to it. We even prove a little more than usual one- or two-time statements, as in our limit  $p^{(\mathcal{N})}$  is independent in law from even the full evolution data  $\{p^{(\nu)}(t), 0 \leq t \leq T, 1 \leq \nu \leq \mathcal{N} - 1\}$ .

This second implication is an important issue, as the acceleration of a passive particle is a Hamiltonian process while the diffusion equation is irreversible. While the key to this irreversibility is clearly the fact that the diffusion process only relates to the momentum component of particle evolution, we shall further investigate the interplay of limits  $\mathcal{N} \rightarrow \infty$  and  $A \rightarrow \infty$ , or  $m \rightarrow 0$ , with stochasticity versus Hamiltonian “conservativeness” in  $(Q, P)$  variables in future work.

Finally, we followed general practice in discussing the stochastic acceleration problem in only one space dimension [5, 16]. This is rather classical, and it applies, for example, to particle motion along magnetic field lines in strongly magnetized plasmas; higher-dimensional motions may call for different elementary models.

APPENDIX

For the convenience of the reader, we prove the following well-known result (see, e.g., Exercise IV.3.16 in [26]).

PROPOSITION A.1. *Let  $\eta$  and  $\tau$  be two stopping times such that  $0 \leq \eta \leq \tau \leq \eta + T$  and  $M_t = \int_0^t \varphi_s dB_s$ , where  $\{B_t, t \geq 0\}$  is a standard Brownian motion, and  $\{\varphi_t, t \geq 0\}$  is progressively measurable and satisfies  $\int_0^t \varphi_s^2 ds \leq k^2 t$  for all  $t \geq 0$ . Then for all  $b > 0$ ,*

$$\mathbb{P}\left(\sup_{\eta \leq t \leq \tau} |M_t - M_\eta| \geq b\right) \leq 2 \exp\left(-\frac{b^2}{2k^2 T}\right).$$

PROOF. From the optional stopping theorem, it suffices to treat the case  $\eta = 0, \tau = T$ . We have

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |M_t| \geq b\right) \leq \mathbb{P}\left(\sup_{0 \leq t \leq T} M_t \geq b\right) + \mathbb{P}\left(\inf_{0 \leq t \leq T} M_t \leq -b\right).$$

We estimate the first term on the right. The second one is bounded by the same quantity. Define for all  $\lambda > 0$

$$\mathcal{M}_t^\lambda = \exp\left(\lambda M_t - \frac{\lambda^2}{2} \int_0^t \varphi_s^2 ds\right).$$

Then

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq T} M_t \geq b\right) &\leq \mathbb{P}\left(\sup_{0 \leq t \leq T} \mathcal{M}_t^\lambda \geq \exp(\lambda b - \lambda^2 k^2 T/2)\right) \\ &\leq \exp(\lambda^2 k^2 T/2 - \lambda b) \end{aligned}$$

from Doob's inequality, since  $\{\mathcal{M}_t^\lambda, t \geq 0\}$  is a martingale with mean one. Optimizing the value of  $\lambda$ , we deduce that

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} M_t \geq b\right) \leq \exp\left(-\frac{b^2}{2k^2 T}\right)$$

from which the result follows.  $\square$

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## REFERENCES

- [1] BÉNISTI, D. and ESCANDE, D. F. (1997). Origin of diffusion in Hamiltonian dynamics. *Phys. Plasmas* **4** 1576–1581. [MR1465781](#)
- [2] BÉNISTI, D. and ESCANDE, D. F. (1998). Finite range of large perturbations of Hamiltonian dynamics. *J. Stat. Phys.* **92** 909–972. [MR1657848](#)
- [3] BÉNISTI, D. and ESCANDE, D. F. (1998). Nonstandard diffusion properties of the standard map. *Phys. Rev. Lett.* **80** 4871–4874. [MR1625821](#)
- [4] CARY, J. R., ESCANDE, D. F. and VERGA, A. D. (1990). Non quasilinear diffusion far from the chaotic threshold. *Phys. Rev. Lett.* **65** 3132–3135.
- [5] CHIRIKOV, B. V. (1979). A universal instability of many-dimensional oscillator systems. *Phys. Rep.* **52** 263–379. [MR536429](#)
- [6] DIMITS, A. M. and KROMMES, J. M. (1986). Stochastic particle acceleration and statistical closures. *J. Stat. Phys.* **44** 879–906.
- [7] DOVEIL, F. and GRÉSILLON, D. (1982). Statistics of charged particles in external random longitudinal electric fields. *Phys. Fluids* **25** 1396–1402.
- [8] DOVEIL, F. and GUYOMARCH, D. (2003). Diffusion of a test electron beam in a discrete spectrum of waves propagating along a traveling wave tube. *Commun. Nonlinear Sci. Numer. Simul.* **8** 529–536.
- [9] DOVEIL, F. and MACOR, A. (2006). Wave-particle interaction and Hamiltonian dynamics investigated in a traveling wave tube. *Phys. Plasmas* **13** 055704. [MR2243627](#)
- [10] DOXAS, I. and CARY, J. R. (1997). Numerical observation of turbulence enhanced growth rates. *Phys. Plasmas* **4** 2508–2518.
- [11] DRUMMOND, W. E. and PINES, D. (1962). Nonlinear stability of plasma oscillations. *Nuclear Fusion Suppl.* **3** 1049–1057.
- [12] ELSKENS, Y. (2007). Quasilinear limit for particle motion in a prescribed spectrum of random waves. *Physics AUC* **17** 109–121.
- [13] ELSKENS, Y. (2010). Nonquasilinear evolution of particle velocity in incoherent waves with random amplitudes. *Commun. Nonlinear Sci. Numer. Simul.* **15** 10–15.

- [14] ELSKENS, Y. and ESCANDE, D. (2003). Microscopic dynamics of plasmas and chaos. IoP publishing, Bristol, UK.
- [15] ESCANDE, D. F. (1985). Stochasticity in classical Hamiltonian systems: Universal aspects. *Phys. Rep.* **121** 165–261. [MR791722](#)
- [16] HAZELTINE, R. D. and WAELEBROECK, F. L. (2004). *The Framework of Plasma Physics*. Westview Press, Boulder, CO.
- [17] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam. [MR1011252](#)
- [18] KAC, M. (1956). Foundations of kinetic theory. In *Proc. Third Berkeley Sympos. Math. Statist. Probab. Vol. III* 171–197. Univ. California, Berkeley. [MR0084985](#)
- [19] KAC, M. (1959). *Probability and Related Topics in Physical Sciences*. Amer. Math. Soc., Providence, RI. [MR0102849](#)
- [20] KAHANE, J.-P. (1985). *Some Random Series of Functions*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **5**. Cambridge Univ. Press, Cambridge. [MR833073](#)
- [21] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. Springer, New York. [MR1876169](#)
- [22] KROMMES, J. A. (2002). Fundamental statistical descriptions of plasma turbulence in magnetic fields. *Phys. Rep.* **360** 1–352.
- [23] MAASJOST, W. and ELSÄSSER, K. (1982). Stochastic particle acceleration. *J. Stat. Phys.* **28** 793–814.
- [24] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. Springer, Berlin. [MR2200233](#)
- [25] PESME, D. (1994). The Fokker–Planck description in the context of quasilinear theory. *Physica Scr.* **T50** 7–14.
- [26] REVUZ, D. and YOR, M. (2005). *Continuous Martingales and Brownian Motion*, 3rd ed. Springer, Berlin.
- [27] ROMANOV, Y. A. and FILIPPOV, G. F. (1961). The interaction of fast electron beams with longitudinal plasma waves. *J. Eksp. Theoret. Phys. USSR* **40** 123–132; *Sov. Phys. JETP* **13** 87–92.
- [28] STURROCK, P. A. (1966). Stochastic acceleration. *Phys. Rev.* **141** 186–191.
- [29] SUSSMANN, H. J. (1978). On the gap between deterministic and stochastic ordinary differential equations. *Ann. Probab.* **6** 19–41. [MR0461664](#)
- [30] TSUNODA, S. I., DOVEIL, F. and MALMBERG, J. H. (1991). Experimental test of quasilinear theory. *Phys. Fluids* **B3** 2747–2757.
- [31] VANDEN EIJNDEN, E. (1997). Some remarks on the quasilinear treatment of the stochastic acceleration problem. *Phys. Plasmas* **4** 1486–1488.
- [32] VEDENOV, A. A., VELIKHOV, E. P. and SAGDEEV, R. Z. (1962). Quasilinear theory of plasma oscillations. *Nuclear Fusion Suppl.* **2** 465–475.

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