

ON THE WIENER DISORDER PROBLEM

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In the Wiener disorder problem, the drift of a Wiener process changes suddenly at some unknown and unobservable disorder time. The objective is to detect this change as quickly as possible after it happens. Earlier work on the Bayesian formulation of this problem brings optimal (or asymptotically optimal) detection rules assuming that the prior distribution of the change time is given at time zero, and additional information is received by observing the Wiener process only. Here, we consider a different information structure where possible causes of this disorder are observed. More precisely, we assume that we also observe an arrival/counting process representing external shocks. The disorder happens because of these shocks, and the change time coincides with one of the arrival times. Such a formulation arises, for example, from detecting a change in financial data caused by major financial events, or detecting damages in structures caused by earthquakes. In this paper, we formulate the problem in a Bayesian framework assuming that those observable shocks form a Poisson process. We present an optimal detection rule that minimizes a linear Bayes risk, which includes the expected detection delay and the probability of early false alarms. We also give the solution of the “variational formulation” where the objective is to minimize the detection delay over all stopping rules for which the false alarm probability does not exceed a given constant.

1. Introduction. Suppose that at time $t = 0$ we start observing a Wiener process X and a simple Poisson process N with arrival times $(T_n)_{n \geq 0}$. The Poisson process is assumed to apply external shocks on X , and these shocks will eventually cause a change in the drift of X . The time Θ , at which the drift changes is unknown and unobservable. We only know that it coincides with one of the arrival times according to the prior distribution

$$(1.1) \quad \mathbb{P}\{\Theta = 0\} = \pi, \quad \mathbb{P}\{\Theta = T_n\} = (1 - \pi)(1 - p)^{n-1}p \quad \text{for all } n \geq 1$$

for some known $\pi \in [0, 1)$ and $p \in (0, 1]$. We also assume that pre- and post-disorder drifts μ_0 and μ_1 are given, and the arrival rate λ of the Poisson process is known.

Our aim is to detect the time Θ as quickly as possible after it happens, and by using our observations from the processes X and N only. More precisely, if we

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let $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t \geq 0}$ be the observation filtration, our objective is to find an \mathbb{F} -stopping time τ that minimizes the Bayes risk

$$(1.2) \quad R(\tau, \pi) := \mathbb{P}\{\tau < \Theta\} + c \cdot \mathbb{E}(\tau - \Theta)^+$$

for some delay cost $c > 0$. If such a stopping time exists, then it resolves optimally the trade-off between early false alarms and detection delay.

We also consider an alternative but related formulation, in which the objective is to minimize the detection delay $\mathbb{E}(\tau - \Theta)^+$ over all \mathbb{F} -stopping times, for which the false alarm frequency $\mathbb{P}\{\tau < \Theta\}$ is bounded above by a given constant $\alpha \in (0, 1)$. Needless to say, this formulation is more desirable if frequent false alarms cannot be tolerated.

Change detection problems have been studied in the literature with numerous applications in different contexts. These applications include quality control and fault detection in industrial processes, detection of onset of an epidemic in biomedical signal processing, target identification in national defense, intrusion detection in computer networks and security systems, threat detection in national security, pattern recognition in seismology, detection of change in the riskiness of financial assets, and many others. The reader may refer to [1, 3, 11–13, 23–27], and the references therein for an extensive discussion on these and other applications.

Earlier foundational studies on change detection problems include [14] and [16] on non-Bayesian settings; and [9] and [22] on Bayesian formulations respectively. In particular, [22] gives the solution of the Bayesian formulation of the Wiener disorder problem for the Bayes risk in (1.2) assuming that the change time has an exponential prior distribution (see also [20, 21]). Later, following [19], this problem is reconsidered by [6] for a different Bayes risk including an exponential penalty term (which is more suitable for financial applications). Recently, [8] obtained the solution of the finite-horizon version of the original formulation of [22] (see also [17], Chapter 6.22). The extension to the case where observations consist of multiple Wiener processes is given by [7].

The common assumption in this line of work is that the change-time has (zero-modified) exponential distribution. Under this assumption, the sufficient statistic (i.e., *conditional probability process*) is one dimensional, and it is possible to obtain explicit results. In addition to this analytical advantage, the exponential distribution can be regarded as a reasonable choice for highly reliable systems considering the asymptotic approximation of the exponential distribution with geometric distribution. That is, if we perform independent experiments at times $\delta, 2\delta, 3\delta, \dots$, for $\delta > 0$, where the failure (disorder) probability is $\lambda\delta$, then as $\delta \rightarrow 0^+$ we have $\mathbb{P}(\text{time to first failure} > t) \rightarrow e^{-\lambda t}$.

In other settings where the prior distribution is not exponential, the literature offers asymptotically optimal Bayes rules. When the prior distribution is not exponential, sufficient statistics are not one-dimensional anymore, and explicit results

are difficult to obtain, in which case asymptotically optimal rules prove useful for online implementation. The reader may refer to, for example, [4] and [5] for such asymptotical results including explicit expansions of the optimal Bayes risks [which are modified versions of (1.2)]; see also [18] for related results. We refer the reader to the recent work [2] for a comprehensive asymptotical analysis of more general continuous-time models (including the Wiener disorder problem). The same work [2] can also be consulted for a brief survey and overview of the earlier work on asymptotical detection theory.

In the aforementioned models, the observed Wiener process is the only source of information for detecting the change time. However, it is sometimes possible to observe the external factors that are responsible for the disorder. This is usually the case if we would like to detect, for example, a sudden change in financial data caused by major financial events/news, or damages in structures caused by earthquakes using continuously acquired vibration measurements (see [3], Chapters 1.2.5 and 11.1.4, for a discussion on vibration monitoring in mechanical systems). Here, we consider such a setting where the underlying system is exposed to observable shocks/impulses, and the disorder happens at one of these shocks.

Such a formulation is considered for the first time by [15] for a Brownian motion in a non-Bayesian framework, and under the assumption that these shocks form a Poisson process. Sections 4 and 5 in [15] derive an optimal solution for an (extended) Lorden criteria in terms of the (extended) CUSUM process. However, to our knowledge, no Bayesian formulation of this problem has been given yet. This formulation and its solution are the contributions of the current paper. It should be noted that under the distribution in (1.1), the unconditional distribution of Θ is (zero-modified) exponential with parameter λp . Hence, our model can also be considered as a modification of the original formulation in [22]. The major difference is that we not only observe the underlying Wiener process but also the external causes of the disorder. In this “more informed” setting, the detection decision may improve greatly and this is indeed confirmed by our numerical example in Figure 1.

As an additional remark, we would like to note that although the change can happen only at discrete points in time, a detection decision can be made at any time. Hence, the problem is rather a continuous-time problem as expected. It is essentially composed of a sequence of hypothesis-testing problems: between two arrivals of the Poisson process, the observer tests the hypotheses

$$H_0 : \text{drift} = \mu_0 \quad \text{vs.} \quad H_1 : \text{drift} = \mu_1$$

using the observations received from the Brownian motion. Indeed, on every inter-arrival period (T_n, T_{n+1}) , the conditional probability process $\Pi_t := \mathbb{P}\{\Theta \leq t | \mathcal{F}_t\}$, for $t \geq 0$, follows the same dynamics as those of the sufficient statistic $\bar{\Pi}_t := \mathbb{P}\{H_1 \text{ is true} | \mathcal{F}_t\}$, for $t \geq 0$, of the sequential hypothesis-testing problem in [22], Section 4.2; see Remark 2.1. If a decision has not been made by the next arrival

time T_{n+1} , then the conditional probabilities are updated and the hypothesis-testing problem restarts again with new (updated) prior likelihoods.

In this paper, we show that the problem of minimizing the Bayes risk in (1.2) is equivalent to an optimal stopping problem in terms of the conditional probability process $\Pi \equiv \{\Pi_t\}_{t \geq 0}$, and it is optimal to stop the first time the process Π exceeds a threshold π_∞ . The conditional probability process Π is a jump-diffusion jointly driven by the observed Wiener process and the Poisson process [see (2.9) for its dynamics]. To compute the optimal threshold π_∞ and the optimal Bayes risk, we transform the corresponding optimal stopping problem into a sequence of stopping problems for the diffusive part of the process Π . Each of these sub-problems are solved by studying a free-boundary problem under a smooth fit principle, and these solutions are then combined using a jump operator; see Sections 3 and 4 below for details. This approach is introduced for the first time by [7] in order to solve an optimal stopping problem involving a discounted running cost only. In our setting, the problem includes a running cost and a terminal cost, and involves no discounting. This requires nontrivial modifications of their arguments as illustrated in Sections 3 and 4.

In Section 2 below, we formulate the problem as an optimal stopping problem for the conditional probability process Π , and we study the dynamics of this process. In Section 3, we introduce a jump operator whose role is to incorporate the information generated by the Poisson process at every arrival time. Using this operator, we construct the optimal Bayes risk sequentially in Section 4, and we identify an optimal Bayes rule. Finally, in Section 5, we solve the variational formulation using the properties of the optimal solution given in Section 4. Appendices at the end include some of the lengthy derivations.

2. Problem description. Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space hosting a Wiener process W and a simple Poisson process N with arrival times $(T_n)_{n \geq 0}$ and rate $\lambda > 0$. On this space, we have also an independent random variable ζ with the zero-modified geometric distribution

$$(2.1) \quad \mathbb{P}\{\zeta = 0\} = \pi, \quad \mathbb{P}\{\zeta = n\} = (1 - \pi)(1 - p)^{n-1}p \quad \text{for all } n \in \mathbb{N}$$

for some $\pi \in [0, 1)$ and $p \in (0, 1]$. In terms of these elements, we introduce a new \mathbb{R}_+ -valued variable

$$(2.2) \quad \Theta := \sum_{i=0}^{\infty} T_i 1_{\{\zeta=i\}}$$

representing the disorder time. Then, our observation process $X = \{X_t\}_{t \geq 0}$ can be defined as

$$(2.3) \quad X_t := W_t + \mu \cdot (t - \Theta)^+ \quad \text{for all } t \geq 0.$$

In other words, as described in Section 1, the process X is a Brownian motion gaining a drift μ at time Θ , and the change time Θ has zero-modified geometric distribution on the Poissonian clock. With the notation in Section 1, we assume that $\mu_0 = 0$ and $\mu_1 = \mu \neq 0$ without loss of generality.

Let $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t \geq 0}$ be the filtration of the observed pair (X, N) ; that is, $\mathcal{F}_t := \sigma\{X_s, N_s : s \leq t\}$, for $t \geq 0$. For an \mathbb{F} -stopping time τ , let $R(\tau, \pi)$ denote the Bayes risk

$$R(\tau, \pi) := \mathbb{P}^\pi\{\tau < \Theta\} + c \cdot \mathbb{E}^\pi(\tau - \Theta)^+,$$

in which \mathbb{P}^π is the probability measure \mathbb{P} where ζ has the distribution in (2.1). The Bayes risk above includes the false alarm probability and the expected detection delay cost for some $c > 0$. Our objective in this problem is to compute

$$(2.4) \quad V(\pi) := \inf_{\tau \in \mathbb{F}} R(\tau, \pi),$$

and if exists, find a stopping time attaining this infimum.

Using the standard arguments in [22], Chapter 4, we can transform the problem in (2.4) into an optimal stopping problem for the *conditional probability process* defined as

$$(2.5) \quad \Pi_t := \mathbb{P}\{\Theta \leq t | \mathcal{F}_t\}, \quad t \geq 0.$$

More precisely, the minimal Bayes risk in (2.4) is the value function of the optimal stopping problem

$$(2.6) \quad V(\pi) = \inf_{\tau \in \mathbb{F}} \mathbb{E}^\pi \left[\int_0^\tau g(\Pi_s) ds + h(\Pi_\tau) \right],$$

where $g(\pi) := c \cdot \pi$ and $h(\pi) := 1 - \pi$.

In Appendix A, we show that the process Π has the characterization

$$(2.7) \quad \begin{aligned} \Pi_t &= \frac{\Phi_t}{1 + \Phi_t}, \quad \text{where} \\ \Phi_t &:= \frac{L_t}{(1-p)^{N_t}} \left(\frac{\pi}{1-\pi} + \sum_{i=1}^{N_t} \frac{(1-p)^{i-1} p}{L_{T_i}} \right) \end{aligned}$$

in terms of

$$(2.8) \quad L_t := \exp \left\{ \mu X_t - \frac{\mu^2}{2} t \right\} \quad \text{for } t \geq 0.$$

Using (2.7) and (2.8), we obtain

$$(2.9) \quad d\Pi_t = \mu \Pi_{t-} (1 - \Pi_{t-}) d\widehat{W}_t + p(1 - \Pi_{t-}) dN_t,$$

where $\widehat{W}_t := X_t - \mu \int_0^t \Pi_s ds$, for $t \geq 0$, is a (\mathbb{P}, \mathbb{F}) -Wiener process. In Appendix B, we also show that for $t \leq s_1$ and $t \leq s_2$ and for $r, q \in \mathbb{R}$, we have

$$(2.10) \quad \begin{aligned} &\mathbb{E}[\exp\{ir(\widehat{W}_{s_1} - \widehat{W}_t) + iq(N_{s_2} - N_t)\} | \mathcal{F}_t] \\ &= \exp\left\{-\frac{1}{2}r^2(s_1 - t) + \lambda(e^{iq} - 1)(s_2 - t)\right\}, \end{aligned}$$

which implies that \widehat{W} and N are independent. This further implies that the process Π whose dynamics are given in (2.9) is a strong Markov process.

REMARK 2.1. Between two arrival times, the process Π satisfies $d\Pi_t = \mu\Pi_{t-}(1 - \Pi_{t-})d\widehat{W}_t$, which coincides with the dynamics of the conditional probability process in the sequential hypothesis-testing problem considered in [22], Section 4.2. In that problem, an observer is given two hypotheses

$$(2.11) \quad H_0 : \text{drift} = 0 \quad \text{and} \quad H_1 : \text{drift} = \mu$$

about an observed Wiener process. The hypotheses have prior likelihoods $1 - \pi$ and π respectively, and the aim is to identify the correct one as soon as possible.

In our problem, the change can happen only at one of the arrival times of the Poisson process. Hence, between two arrival times $[T_n, T_{n+1})$ the role of the process Π is to indicate the posterior likelihood of the hypothesis H_1 , whose initial prior is Π_{T_n} as of time T_n . In this setting, if a decision is made by the next arrival time, the hypothesis-testing problem terminates. Otherwise, it restarts with new priors $1 - \Pi_{T_{n+1}}$ and $\Pi_{T_{n+1}}$, respectively.

REMARK 2.2. Using its definition in (2.5), it can easily be verified that the process Π is a bounded submartingale with a last element $\Pi_\infty \leq 1$. Moreover, thanks to bounded convergence theorem we have

$$\begin{aligned} 1 &\geq \mathbb{E}^\pi \Pi_\infty = \lim_{t \rightarrow \infty} \mathbb{E}^\pi \Pi_t = \lim_{t \rightarrow \infty} \mathbb{E}^\pi [1_{\{\Theta \leq t\}}] \\ &= \lim_{t \rightarrow \infty} \mathbb{E}^\pi [\mathbb{E}[1_{\{\Theta \leq t\}} | N_u : u \leq t]] \\ &= \lim_{t \rightarrow \infty} \mathbb{E}^\pi [1 - (1 - \pi)(1 - p)^{N_t}] = 1, \end{aligned}$$

which implies that $\Pi_\infty = 1$, \mathbb{P}^π -a.s., for all $\pi \in [0, 1]$.

The limiting behavior of Π implies that the exit time $\tilde{\tau}_r$ of Π from an interval $[0, r)$, for $r \in [0, 1)$, is finite \mathbb{P}^π -almost surely, for $\pi \in [0, 1]$. Indeed, the dynamics in (2.9) give

$$\begin{aligned} \mathbb{E}^\pi \Pi_{t \wedge \tilde{\tau}_r} &= \pi + \mathbb{E}^\pi \left[\int_0^{t \wedge \tilde{\tau}_r} \mu \Pi_{u-}(1 - \Pi_{u-}) d\widehat{W}_u \right. \\ &\quad \left. + \int_0^{t \wedge \tilde{\tau}_r} p(1 - \Pi_{u-})(dN_u - \lambda du) \right. \\ &\quad \left. + \int_0^{t \wedge \tilde{\tau}_r} \lambda p(1 - \Pi_{u-}) du \right]. \end{aligned}$$

Since Π is bounded, the first two integral has zero expectations. Moreover, for $u \leq \tilde{\tau}_r$, we have $1 - \Pi_u \geq 1 - r$, and this yields

$$(2.12) \quad 1 \geq \mathbb{E}^\pi \Pi_{t \wedge \tilde{\tau}_r} \geq \lambda p(1 - r) \mathbb{E}^\pi t \wedge \tilde{\tau}_r,$$

showing that $\mathbb{E}^\pi \tilde{\tau}_r$ is uniformly bounded, for all $\pi \in [0, 1]$, thanks to monotone convergence theorem.

3. Dynamic programming operator. The first arrival time T_1 is a regeneration time of the conditional probability process Π . Therefore, if the process Π has not been stopped yet, the minimal Bayes risk that one can attain starting from T_1 should be computed by evaluating the function $V(\cdot)$ at Π_{T_1} . This tells us that the value function should satisfy the dynamic programming equation

$$(3.1) \quad V(\pi) = \inf_{\tau \in \mathbb{F}} \mathbb{E}^\pi \left[\int_0^{\tau \wedge T_1} g(\Pi_t) dt + 1_{\{\tau < T_1\}} h(\Pi_\tau) + 1_{\{\tau \geq T_1\}} V(\Pi_{T_1}) \right].$$

Until the first arrival time, Π coincides with a diffusion starting from $Y_0 = \pi$ and satisfying

$$(3.2) \quad dY_t = \mu Y_t(1 - Y_t) d\widehat{W}_t \quad \text{for } t \geq 0.$$

Hence, a given stopping time $\tau \in \mathbb{F}$ should coincide on the event $\{\tau < T_1\}$ with another stopping time of the process Y . This observation suggests that the function $V(\cdot)$ should be a fixed point of the operator

$$(3.3) \quad J[w](\pi) := \inf_{\tau \in \mathbb{F}^Y} \mathbb{E}^\pi \left[\int_0^{\tau \wedge T_1} g(Y_t) dt + 1_{\{\tau < T_1\}} h(Y_\tau) + 1_{\{\tau \geq T_1\}} w(Y_{T_1} + p(1 - Y_{T_1})) \right],$$

which is obtained by replacing \mathbb{F} in (3.1) with the filtration \mathbb{F}^Y of the process Y , and $V(\cdot)$ with a bounded function $w(\cdot)$ on $[0, 1]$. Using the independence of \widehat{W} and N , and the distribution of T_1 we can rewrite this operator

$$(3.4) \quad J[w](\pi) = \inf_{\tau \in \mathbb{F}^Y} \mathbb{E}^\pi \left[\int_0^\tau e^{-\lambda t} [g(Y_t) + \lambda w(\mathbb{S}(Y_t))] dt + e^{-\lambda \tau} h(Y_\tau) \right],$$

where $\mathbb{S}(\pi) := \pi + p(1 - \pi)$.

In this section, we study the properties of the operator J for a suitable class of function $w(\cdot)$'s. Under certain assumptions on $w(\cdot)$, we show that the infimum in (3.4) is attained by the exit time of the process Y from an interval of the form $[0, r)$, and that the function $J[w](\cdot)$ solves the variational inequalities of the optimal stopping problem in (3.4). Using the results of this section, we show in Section 4 that the function $V(\cdot)$ indeed satisfies $V(\cdot) = J[V](\cdot)$ as expected.

REMARK 3.1. For a bounded function $w: [0, 1] \mapsto \mathbb{R}_+$, we have $0 \leq J[w](\cdot) \leq h(\cdot)$. Moreover for two bounded functions $w_1(\cdot) \leq w_2(\cdot)$, we have $J[w_1](\cdot) \leq J[w_2](\cdot)$.

PROOF. The upper bound $J[w](\cdot) \leq h(\cdot)$ follows by taking $\tau = 0$ in (3.4). Nonnegativity of $J[w](\cdot)$ and the monotonicity of $w \mapsto J[w]$ are obvious. \square

3.1. *Solving the optimal stopping problem in (3.4).* Below we solve the minimization problem in (3.4) under the following assumption.

ASSUMPTION A1. *The function $w(\cdot)$ is an arbitrary (but fixed) nonnegative and continuous function on $[0, 1]$ bounded above by $h(\cdot)$.*

Let us define the functions

$$(3.5) \quad \begin{aligned} \psi(\pi) &:= \pi^{m_1} (1 - \pi)^{1-m_1} \quad \text{and} \\ \eta(\pi) &:= \pi^{m_2} (1 - \pi)^{1-m_2} \quad \text{for } \pi \in [0, 1], \end{aligned}$$

where $m_1 > 1$ and $m_2 < 0$ are the roots of the quadratic equation

$$(3.6) \quad m(m-1) = \frac{2\lambda}{\mu^2}.$$

The functions $\psi(\cdot)$ and $\eta(\cdot)$ are respectively, the increasing and decreasing solutions (up to multiplication by a constant) of the equation $\mathcal{A}_0 f(\pi) = \lambda f(\pi)$, where \mathcal{A}_0 is the infinitesimal generator of the diffusion process Y in (3.2); that is, $\mathcal{A}_0 f(\pi) := \frac{1}{2} \mu^2 \pi^2 (1 - \pi)^2 f''(\pi)$. It is easy to verify that the functions $\psi(\cdot)$ and $\eta(\cdot)$ satisfy the boundary conditions

$$(3.7) \quad \begin{aligned} \psi(0+) &= \psi'(0+) = 0 = \eta'(1-) = \eta(1-), \\ \psi(1-) &= \psi'(1-) = \infty = \eta'(0+) = \eta(0+), \end{aligned}$$

and that their Wronskian is $m_1 - m_2$.

In terms of the drift $r(\cdot) \equiv 0$ and the volatility $\sigma(\pi) = \mu\pi(1 - \pi)$ of the process Y [see the dynamics in (3.2)], let $S(\cdot)$ denote the *scale function*

$$S(\pi) := \int_d^\pi S(dy), \quad \text{where } S(dy) := \exp\left\{\int_c^y \frac{r(z)}{\sigma^2(z)} dz\right\} dy = dy$$

for arbitrary $c, d \in (0, 1)$, and let $M(\cdot)$ be the *speed measure*

$$M(dy) := \frac{dy}{\sigma^2(y)S'(y)} = \frac{dy}{\mu^2 \pi^2 (1 - \pi)^2}$$

for $y \in (0, 1)$. Feller boundary test at the right boundary $\{1\}$ gives

$$(3.8) \quad \int_c^1 \int_y^1 S(dz) M(dy) = \infty, \quad \int_c^1 \int_y^1 M(dz) S(dy) = \infty,$$

and according to [10], Table 6.2, we conclude that the right boundary is *natural*. This implies that the process Y cannot reach the right boundary in finite time. On the other hand, the process $\{1 - Y_t\}_{t \geq 0}$ has the same dynamics in (3.2) and by symmetry the left boundary $\{0\}$ is also natural for Y . Indeed, by a change of variable in (3.8) as $u = 1 - y$ and $q = 1 - z$ we get the Feller boundary test at $\{0\}$

$$\begin{aligned} \infty &= \int_c^1 \int_y^1 S(dz)M(dy) = \int_0^{1-c} \int_0^u dq \frac{du}{\mu^2 u^2 (1-u)^2} \\ &= \int_0^{1-c} \int_0^u S(dq)M(du), \\ \infty &= \int_c^1 \int_y^1 M(dz)S(dy) = \int_0^{1-c} \int_0^u \frac{dq}{\mu^2 q^2 (1-q)^2} du \\ &= \int_0^{1-c} \int_0^u M(dq)S(du), \end{aligned}$$

which gives the same conclusion for the left boundary.

REMARK 3.2. The process Y is a bounded martingale [see (3.2)], and we have

$$\begin{aligned} \mathbb{E}^\pi \left[\int_0^\infty e^{-\lambda t} [g(Y_t) + \lambda w(\mathbb{S}(Y_t))] dt \right] &\leq \mathbb{E}^\pi \left[\int_0^\infty e^{-\lambda t} [g(Y_t) + \lambda \|w\|] dt \right] \\ &= \int_0^\infty e^{-\lambda t} [\mathbb{E}^\pi g(Y_t) + \lambda \|w\|] dt \\ &= \int_0^\infty e^{-\lambda t} [g(\pi) + \lambda \|w\|] dt < \infty. \end{aligned}$$

LEMMA 3.1. For $0 < l \leq r < 1$, and let $\tau_{l,r}$ be the exit time of the process Y from the interval (l, r) . The expectation

$$(3.9) \quad H_{l,r}[w](\pi) := \mathbb{E}^\pi \left[\int_0^{\tau_{l,r}} e^{-\lambda t} [g(Y_t) + \lambda w(\mathbb{S}(Y_t))] dt + e^{-\lambda \tau_{l,r}} h(Y_{\tau_{l,r}}) \right],$$

has the explicit form

$$\begin{aligned} H_{l,r}[w](\pi) &= \psi(\pi) \left[C_1 + \int_\pi^r u_1[w](y) dy \right] \\ (3.10) \quad &+ \eta(\pi) \left[C_2 - \int_\pi^r u_2[w](y) dy \right] \\ &+ h(l) \frac{\psi(\pi)\eta(r) - \psi(r)\eta(\pi)}{\psi(l)\eta(r) - \psi(r)\eta(l)} + h(r) \frac{\psi(l)\eta(\pi) - \psi(\pi)\eta(l)}{\psi(l)\eta(r) - \psi(r)\eta(l)} \end{aligned}$$

for $\pi \in (l, r)$, in terms of

$$\begin{aligned}
 u_1[w](y) &:= 2 \frac{g(y) + \lambda w(\mathbb{S}(y))}{(m_1 - m_2)\sigma^2(y)} \eta(y), \\
 u_2[w](y) &:= 2 \frac{g(y) + \lambda w(\mathbb{S}(y))}{(m_1 - m_2)\sigma^2(y)} \psi(y), \\
 (3.11) \quad C_2 &:= \frac{\eta(l) \int_l^r u_2[w](y) dy - \int_l^r u_2[w](y) dy \psi(l)}{\psi(r)\eta(l) - \psi(l)\eta(r)} \psi(r), \\
 C_1 &:= -\frac{\eta(r)}{\psi(r)} C_2.
 \end{aligned}$$

Clearly, $H_{l,r}[w](\cdot)$ is nonnegative, and we have $H_{l,r}[w](\pi) = h(\pi)$, for $\pi \notin (l, r)$.

PROOF. Nonnegativity of $H_{l,r}[w](\cdot)$ and the identity $H_{l,r}[w](\cdot) = h(\cdot)$, on $[0, 1] \setminus (l, r)$, are obvious. For $\pi \in (l, r)$, let $f(\cdot)$ denote the function on the right-hand side in (3.10). Direct computation shows that $f(\cdot)$ satisfies

$$(-\lambda + \mathcal{A}_0)f(\pi) + g(\pi) + \lambda w(\mathbb{S}(y)) = 0 \quad \text{on } \pi \in (l, r),$$

with boundary conditions $f(l+) = h(l)$ and $f(r-) = h(r)$. Moreover, its derivative (with respect to π) is

$$\begin{aligned}
 &\psi'(\pi) \left[C_1 + \int_\pi^r u_1[w](y) dy \right] + \eta'(\pi) \left[C_2 - \int_\pi^r u_2[w](y) dy \right] \\
 &+ h(l) \frac{\psi'(\pi)\eta(r) - \psi(r)\eta'(\pi)}{\psi(l)\eta(r) - \psi(r)\eta(l)} + h(r) \frac{\psi(l)\eta'(\pi) - \psi'(\pi)\eta(l)}{\psi(l)\eta(r) - \psi(r)\eta(l)},
 \end{aligned}$$

which is bounded on $[l, r]$. Also, observe that the exit time $\tau_{l,r}$ of the regular diffusion Y is finite and $h(Y_{\tau_{l,r}}) = f(Y_{\tau_{l,r}})$, \mathbb{P}^π -almost surely for all $\pi \in (0, 1)$. Then, by applying Itô's rule, we obtain

$$\begin{aligned}
 \mathbb{E}^\pi e^{-\lambda \tau_{l,r}} h(Y_{\tau_{l,r}}) &= \mathbb{E}^\pi e^{-\lambda \tau_{l,r}} f(Y_{\tau_{l,r}}) \\
 &= f(\pi) + \mathbb{E}^\pi \int_0^{\tau_{l,r}} e^{-\lambda t} (-\lambda + \mathcal{A}_0) f(Y_u) du \\
 &= f(\pi) - \mathbb{E}^\pi \int_0^{\tau_{l,r}} e^{-\lambda t} [g(Y_u) + \lambda w(\mathbb{S}(Y_u))] du,
 \end{aligned}$$

and this shows $f(\cdot) = H_{l,r}[w](\cdot)$ on (l, r) . \square

LEMMA 3.2. For $0 < r < 1$, and $\tau_r := \inf\{t \geq 0 : Y_t \geq r\}$, let us define

$$\begin{aligned}
 H_r[w](\pi) &:= \mathbb{E}^\pi \left[\int_0^{\tau_r} e^{-\lambda t} [g(Y_t) + \lambda w(\mathbb{S}(Y_t))] dt \right. \\
 (3.12) \quad &\quad \left. + e^{-\lambda \tau_r} h(Y_{\tau_r}) \right], \quad \pi \in [0, 1],
 \end{aligned}$$

which clearly equals $h(\cdot)$, for $\pi \geq r$. For $\pi \in (0, r)$, the function $H_r[w](\cdot)$ can be computed by taking the limit of (3.10) as $l \searrow 0$. That is,

$$\begin{aligned} H_r[w](\pi) &= \lim_{l \searrow 0} H_{l,r}[w](\pi) \\ (3.13) \quad &= \psi(\pi) \left(-\frac{\eta(r)}{\psi(r)} \int_0^r u_2[w](y) dy + \int_\pi^r u_1[w](y) dy + \frac{h(r)}{\psi(r)} \right) \\ &\quad + \eta(\pi) \int_0^\pi u_2[w](y) dy. \end{aligned}$$

The expression in (3.13) is twice-continuously differentiable [on $(0, r)$] and solves

$$(3.14) \quad (-\lambda + \mathcal{A}_0)H_r[w](\pi) + g(\pi) + \lambda w(\mathbb{S}(y)) = 0.$$

Moreover, the function $H_r(\cdot)$ is continuous on $[0, 1]$ with

$$(3.15) \quad \lim_{\pi \searrow 0} H_r[w](\pi) = w(p) = H_r[w](0).$$

PROOF. The point $\{0\}$ is a natural boundary for Y ; therefore, we have $\tau_r = \lim_{l \searrow 0} \tau_{l,r}$, \mathbb{P}^π -almost surely, for $\pi \in (0, r)$. Then, the dominated convergence theorem (see Remark 3.2) implies that $H_r[w](\pi) = \lim_{l \searrow 0} H_{l,r}[w](\pi)$.

To compute the limit of $H_{l,r}[w](\pi)$ as $l \searrow 0$, we first observe

$$\begin{aligned} (3.16) \quad &\lim_{l \searrow 0} h(l) \frac{\psi(\pi)\eta(r) - \psi(r)\eta(\pi)}{\psi(l)\eta(r) - \psi(r)\eta(l)} + h(r) \frac{\psi(l)\eta(\pi) - \psi(\pi)\eta(l)}{\psi(l)\eta(r) - \psi(r)\eta(l)} \\ &= \psi(\pi) \frac{h(r)}{\psi(r)}. \end{aligned}$$

Moreover, since $0 \leq g(\cdot) + \lambda w(\mathbb{S}(\cdot)) \leq c + \lambda \|w\|$, we have

$$\begin{aligned} \int_0^\pi u_2[w](y) dy &\leq \frac{c + \lambda \|w\|}{m_1 - m_2} \int_0^\pi \frac{2\psi(y)}{\sigma^2(y)} dy \\ &= \frac{c + \lambda \|w\|}{m_1 - m_2} \int_0^\pi \frac{\psi''(y)}{\lambda} dy \\ &= \frac{(c/\lambda) + \|w\|}{m_1 - m_2} \psi'(\pi) < \infty \quad \text{for } \pi < 1, \end{aligned}$$

and using (3.11) we get

$$(3.17) \quad \lim_{l \searrow 0} C_2 = \int_0^r u_2[w](y) dy \quad \text{and} \quad \lim_{l \searrow 0} C_1 = \frac{\eta(r)}{\psi(r)} \int_0^r u_2[w](y) dy.$$

Finally letting $l \searrow 0$ in (3.10) and using the limits found in (3.16) and (3.17), we obtain the expression in (3.13). It is evident that this expression is twice-continuously differentiable. Moreover, by direct computation [using (3.11)] it can be verified easily that it solves the equation in (3.14).

Clearly, $H_r[w](\cdot)$ is continuous on $(0, r)$ and $(r, 1)$. The continuity at $\{r\}$ can be checked by letting $\pi \nearrow r$ in the expression given in (3.13), which goes to $h(r)$. To establish (3.15), we first note that $Y_t = 0$, for all $t > 0$, if $Y_0 = \pi = 0$. This implies

$$H_r[w](0) = \mathbb{E}^\pi \left[\int_0^\infty e^{-\lambda t} [g(0) + \lambda w(S(0))] dt \right] = w(p).$$

On the other hand, applying L'Hôpital rule and using the explicit form of $\psi(\cdot)$ and $\eta(\cdot)$, we obtain

$$\begin{aligned} \lim_{\pi \searrow 0} \psi(\pi) \int_\pi^r u_1[w](y) dy &= \frac{2\lambda w(p)}{\mu^2 m_1(m_1 - m_2)}, \\ \lim_{\pi \searrow 0} \eta(\pi) \int_0^\pi u_2[w](y) dy &= -\frac{2\lambda w(p)}{\mu^2 m_2(m_1 - m_2)}. \end{aligned}$$

Since $m_1 \cdot m_2 = -2\mu^2/\lambda$ [see (3.6)], taking the limit in (3.13) gives $w(p)$, and this concludes the proof. \square

Lemma 3.2 shows that at the point $\pi = r$, we have $(H_r[w])'(r+) = -1$ and

$$\begin{aligned} (H_r[w])'(r-) &= \psi'(r) \left(-\frac{\eta(r)}{\psi(r)} \int_0^r u_2[w](y) dy + \frac{h(r)}{\psi(r)} \right) \\ &\quad + \eta'(r) \int_0^r u_2[w](y) dy. \end{aligned}$$

Since the Wronskian $\psi'(r)\eta(r) - \psi(r)\eta'(r)$ equals $m_1 - m_2$, we can rewrite the left derivative as

$$(H_r[w])'(r-) = \frac{1}{\psi(r)} \left(-\int_0^r 2 \frac{\psi(y)}{\sigma^2(y)} [g(y) + \lambda w(\mathbb{S}(y))] dy + \psi'(r)h(r) \right).$$

Hence, the derivative is continuous at $\pi = r$ if and only if

$$\begin{aligned} &-\int_0^r 2 \frac{\psi(y)}{\sigma^2(y)} [g(y) + \lambda w(\mathbb{S}(y))] dy + \psi'(r)h(r) + \psi(r) = 0 \\ (3.18) \quad &\iff B[w](r) := \int_0^r \frac{2\psi(y)}{\sigma^2(y)} [-g(y) - \lambda w(\mathbb{S}(y)) + \lambda h(y)] dy = 0, \end{aligned}$$

where the second equation follows after noting that

$$\psi'(r)h(r) + \psi(r) = \int_0^r h(y) [2\lambda\psi(y)/\sigma^2(y)] dy,$$

which can be verified using $\lambda\psi(\cdot) = \mathcal{A}_0\psi(\cdot)$.

LEMMA 3.3. *If $w(\cdot)$ is concave, then the function $-g(\pi) - \lambda w(\mathbb{S}(\pi)) + \lambda h(\pi)$ has a unique root $d[w] \in (0, 1)$. The function $\pi \mapsto B[w](\pi)$ equals zero for $\pi = 0$, strictly increases on $(0, d[w])$ and strictly decreases on $(d[w], 1)$ with $\lim_{\pi \nearrow 1} B[w](\pi) = -\infty$. Hence, there exists a unique point $r[w] \in (d[w], 1)$ at which $B[w](r[w]) = 0$.*

PROOF. The function $\pi \mapsto -g(\pi) - \lambda w(\mathbb{S}(\pi)) + \lambda h(\pi) = c\pi - \lambda w(\mathbb{S}(\pi)) + \lambda(1 - \pi)$ is convex and continuous on $[0, 1]$. At the point $\pi = 0$, it equals $\lambda(-w(p) + 1) \geq \lambda(-h(p) + 1) > 0$; and at $\pi = 1$, its value is $-c - \lambda w(p) < 0$. Hence, there exists a single point $d[w] \in (0, 1)$ at which it is zero. To the left of this point it is positive, and to the right it is negative. Therefore, $\pi \mapsto B[w](\pi)$ is zero at $\pi = 0$, strictly increases on $(0, d[w])$ and strictly decreases on $(d[w], 1)$. Also, observe that, for $\pi < 1$,

$$|B[w](\pi)| \leq (c + 2\lambda) \int_0^\pi \frac{2\psi(y)}{\sigma^2(y)} dy = \frac{c + 2\lambda}{\lambda} \int_0^\pi \psi''(y) dy = \frac{c + 2\lambda}{\lambda} \psi'(\pi) < \infty$$

and

$$\begin{aligned} & \int_{d[w]+\delta}^1 \frac{2\psi(y)}{\sigma^2(y)} [-g(y) - \lambda w(\mathbb{S}(y)) + \lambda h(y)] dy \\ & \leq \left(\min_{\pi \in [d[w]+\delta, 1]} \{-g(\pi) - \lambda w(\mathbb{S}(\pi)) + \lambda h(\pi)\} \right) \cdot \int_{d[w]+\delta}^1 \frac{2\psi(y)}{\sigma^2(y)} dy \\ & = \left(\min_{\pi \in [d[w]+\delta, 1]} \{-g(\pi) - \lambda w(\mathbb{S}(\pi)) + \lambda h(\pi)\} \right) \cdot [\psi'(\pi)]_{d[w]+\delta}^1 = -\infty \end{aligned}$$

for all $\delta \in (0, 1 - d[w])$, where the last equality follows using (3.5). Hence, we conclude that $B[w](\pi)$ goes to $-\infty$ as $\pi \rightarrow 1$, and this implies that it has a unique root $r[w] \in (d[w], 1)$. \square

REMARK 3.3. For two concave functions $w_1(\cdot) \leq w_2(\cdot)$ satisfying Assumption A1, we have $B[w_1](\cdot) \geq B[w_2](\cdot)$; therefore $r[w_1] \geq r[w_2]$. If we select the zero function (which equals zero on $[0, 1]$), direct computation yields

$$B[0](\pi) = \frac{\psi(\pi)}{\pi(1-\pi)} \left[-\pi \left((m_1 - 1) \frac{c}{\lambda} + m_1 \right) + m_1 \right],$$

and for $h(\pi) = 1 - \pi$, we get

$$B[h](\pi) = \frac{\psi(\pi)}{\pi(1-\pi)} \left[-\pi \left((m_1 - 1) \frac{c}{\lambda} + m_1 p \right) + m_1 p \right].$$

Hence, we have the bounds

$$(3.19) \quad \frac{m_1 p}{(m_1 - 1)(c/\lambda) + m_1 p} \leq r[w] \leq \frac{m_1}{(m_1 - 1)(c/\lambda) + m_1}.$$

Observe that, if the function $w(\cdot)$ is concave, $H_{r[w]}[w](\cdot)$ is continuously differentiable on $(0, 1)$. On $(r[w], 1)$, $H_{r[w]}[w](\cdot)$ coincides with $h(\cdot)$, and

$$(-\lambda + \mathcal{A}_0)H_{r[w]}[w](\pi) + g(\pi) + \lambda w(\mathbb{S}(\pi)) = -\lambda h(\pi) + g(\pi) + \lambda w(\mathbb{S}(\pi)) > 0,$$

since $d[w] < r[w]$.

On $(0, r[w])$, the function $H_{r[w]}[w](\cdot)$ solves $(-\lambda + \mathcal{A}_0)H_{r[w]}[w](\pi) + g(\pi) + \lambda w(\mathbb{S}(\pi)) = 0$. In Appendix B, we also show that

$$(3.20) \quad \lambda H_{r[w]}[w](\pi) - g(\pi) - \lambda w(\mathbb{S}(\pi)) < 0 \quad \text{for } 0 < \pi < r[w].$$

Since $\mathcal{A}_0 H_{r[w]}[w](\pi) = (\sigma^2(\pi)/2) \cdot (H_{r[w]}[w](\pi))''$, the inequality in (3.20) implies that $H_{r[w]}[w](\cdot)$ is strictly concave and $H_{r[w]}[w](\cdot) < h(\cdot)$ on $(0, r[w])$.

Finally, the (strict) concavity on $(0, r[w])$ and the “smooth-fit” at $\{r[w]\}$ imply that $H_{r[w]}[w](\cdot)$ is also concave on $(0, 1)$. The following remark is a summary of the analytical properties of $H_{r[w]}[w](\cdot)$ described above.

REMARK 3.4. Suppose that the function $w(\cdot)$ is concave. Then, $H_{r[w]}[w](\cdot)$ is nonnegative, continuous and concave on $[0, 1]$. It is continuously differentiable on $(0, 1)$, twice-continuously differentiable on $(0, 1) \setminus \{r[w]\}$, and it satisfies

$$(3.21) \quad \begin{cases} \left\{ \begin{array}{l} H_{r[w]}[w](\pi) = h(\pi) \\ (-\lambda + \mathcal{A}_0)H_{r[w]}[w](\pi) + g(\pi) + \lambda w(\mathbb{S}(\pi)) > 0 \end{array} \right\}, & \pi \in (r[w], 1), \\ \left\{ \begin{array}{l} H_{r[w]}[w](\pi) < h(\pi) \\ (-\lambda + \mathcal{A}_0)H_{r[w]}[w](\pi) + g(\pi) + \lambda w(\mathbb{S}(\pi)) = 0 \end{array} \right\}, & \pi \in (0, r[w]). \end{cases}$$

LEMMA 3.4. If $w(\cdot)$ is concave, we have $J[w](\cdot) = H_{r[w]}[w](\cdot)$, and $\tau_{r[w]} := \inf\{t \geq 0 : Y_t \geq r[w]\}$ is an optimal stopping time for (3.4).

PROOF. For $\pi \in (0, 1)$, let τ be an \mathbb{F}^Y -stopping time, and $\tau_{l,r}$ be the exit time of Y from (l, r) for $0 < l \leq r < 1$. Then, by Itô's rule

$$\begin{aligned} & e^{-\lambda \cdot \tau \wedge \tau_{l,r}} H_{r[w]}[w](Y_{\tau \wedge \tau_{l,r}}) \\ &= H_{r[w]}[w](\pi) + \int_0^{\tau \wedge \tau_{l,r}} e^{-\lambda t} (-\lambda + \mathcal{A}_0) H_{r[w]}[w](Y_t) dt \\ & \quad + \int_0^{\tau \wedge \tau_{l,r}} e^{-\lambda t} \sigma(Y_t) (H_{r[w]}[w])'(Y_t) d\widehat{W}_t. \end{aligned}$$

The function $H_{r[w]}[w](\cdot)$ is continuously differentiable on $(0, 1)$. Its derivative is therefore bounded on $[l, r]$ and $\|\sigma(\cdot)\| \leq |\mu|$. Then, taking expectations above gives

$$\begin{aligned} & \mathbb{E}^\pi e^{-\lambda \cdot \tau \wedge \tau_{l,r}} H_{r[w]}[w](Y_{\tau \wedge \tau_{l,r}}) \\ &= H_{r[w]}[w](\pi) + \mathbb{E}^\pi \int_0^{\tau \wedge \tau_{l,r}} e^{-\lambda t} (-\lambda + \mathcal{A}_0) H_{r[w]}[w](Y_t) dt \\ &\geq H_{r[w]}[w](\pi) - \mathbb{E}^\pi \int_0^{\tau \wedge \tau_{l,r}} e^{-\lambda t} (g(Y_t) + \lambda w(\mathbb{S}(Y_t))) dt, \end{aligned}$$

where the inequality is due to (3.21). Since both boundaries are natural, we first let $r \nearrow 1$ and then $l \searrow 0$ to obtain

$$(3.22) \quad \begin{aligned} \mathbb{E}^\pi e^{-\lambda \cdot \tau} h(Y_\tau) &\geq \mathbb{E}^\pi e^{-\lambda \cdot \tau} H_{r[w]}[w](Y_\tau) \\ &\geq H_{r[w]}[w](\pi) - \mathbb{E}^\pi \int_0^\tau e^{-\lambda t} (g(Y_t) + \lambda w(\mathbb{S}(Y_t))) dt \end{aligned}$$

thanks to dominated convergence theorem (see Remark 3.2), and this shows $H_{r[w]}[w](\cdot) \leq J[w](\cdot)$ on $(0, 1)$.

When we repeat the steps above with $\tau = \tau_{r[w]} = \inf\{t \geq 0 : Y_t \geq r[w]\}$, the inequalities become equalities again by (3.21). Hence, $J[w](\cdot) = H_{r[w]}[w](\cdot)$ on $(0, 1)$.

If $Y_0 = \pi = 0$; then $Y_t = 0$, for $t \geq 0$, and

$$\begin{aligned} J[w](0) &= \inf_\tau \mathbb{E}^\pi [\lambda w(\mathbb{S}(0))(1 - e^{-\lambda \tau}) + e^{-\lambda \tau} h(0)] \\ &= w(\mathbb{S}(0)) = w(p) = H_{r[w]}(0), \end{aligned}$$

thanks to Lemma 3.2 [note that $w(p) \leq h(p) < h(0)$]. Moreover, this value is attained by selecting $\tau = \infty = \tau_{r[w]}$. Similarly, if $Y_0 = 1$, we have $J[w](1) = h(1) = 0 = H_{r[w]}(1)$, which is attained by $\tau = 0 = \tau_{r[w]}$. \square

4. The value function and an optimal detection rule. Using the dynamic programming operator J , let us define the sequence of functions

$$v_0 \equiv h(\cdot) \quad \text{and} \quad v_{n+1}(\cdot) := J[v_n](\cdot) = H_{r[v_n]}[v_n](\cdot) \quad \text{for } n \in \mathbb{N}.$$

REMARK 4.1. The sequence $(v_n)_{n \in \mathbb{N}}$ is nonincreasing, and each element of the sequence is a nonnegative, continuous and concave function on $[0, 1]$.

PROOF. We have $v_1(\cdot) = J[h](\cdot) \leq h(\cdot) = v_0(\cdot)$, where the inequality follows from the definition of the operator J in (3.4). Next, assume that $v_n(\cdot) \leq v_{n-1}(\cdot)$, for some $n \in \mathbb{N}$. Then Remark 3.1 implies $v_{n+1}(\cdot) = J[v_n](\cdot) \leq J[v_{n-1}](\cdot) = v_n(\cdot)$, and this shows that the sequence $(v_n)_{n \in \mathbb{N}}$ is nonincreasing by induction. Finally, since $v_0(\cdot) = h(\cdot)$ is nonnegative, continuous and concave, these properties also hold for each $v_n(\cdot)$, $n \in \mathbb{N}$, by induction thanks to Remark 3.4. \square

For $n \in \mathbb{N}$, let $\pi_n := r[v_{n-1}]$ be the solution of the equation $B[v_{n-1}](r) = 0$ [see (3.18)]. Since $v_{n-1}(\cdot)$ is concave and satisfies Assumption A1, this equation has a unique root on $(0, 1)$ thanks to Lemma 3.3. Moreover, Remark 3.4 and Lemma 3.4 imply that $v_n(\cdot)$ is continuously differentiable on $(0, 1)$, twice-continuously differentiable on $(0, 1) \setminus \{\pi_n\}$ and solves the variational inequalities

$$\begin{cases} v_n(\pi) = h(\pi) \\ (-\lambda + \mathcal{A}_0)v_n(\pi) + g(\pi) + \lambda v_{n-1}(\mathbb{S}(\pi)) > 0 \end{cases}, \quad \pi \in (\pi_n, 1),$$

$$\begin{cases} v_n(\pi) < h(\pi) \\ (-\lambda + \mathcal{A}_0)v_n(\pi) + g(\pi) + \lambda v_{n-1}(\mathbb{S}(\pi)) = 0 \end{cases}, \quad \pi \in (0, \pi_n).$$

Observe that $\pi_n = \inf\{\pi \in [0, 1] : v_n(\pi) = h(\pi)\}$; hence, $\{\pi_n\}_{n \in \mathbb{N}}$ is nondecreasing.

Let $v_\infty(\cdot) := \inf_{n \in \mathbb{N}} v_n(\cdot)$ be the pointwise limit of $(v_n)_{n \in \mathbb{N}}$. Then dominated convergence theorem gives

$$\begin{aligned}
 v_\infty(\pi) &= \inf_{n \in \mathbb{N}} J[v_{n-1}](\pi) \\
 &= \inf_{n \in \mathbb{N}} \inf_{\tau \in \mathbb{F}^Y} \mathbb{E}^\pi \left[\int_0^\tau e^{-\lambda t} [g(Y_t) + \lambda v_{n-1}(\mathbb{S}(Y_t))] dt + e^{-\lambda \tau} h(\Pi_\tau^\pi) \right] \\
 (4.1) \quad &= \inf_{\tau \in \mathbb{F}^Y} \inf_{n \in \mathbb{N}} \mathbb{E}^\pi \left[\int_0^\tau e^{-\lambda t} [g(Y_t) + \lambda v_{n-1}(\mathbb{S}(Y_t))] dt + e^{-\lambda \tau} h(\Pi_\tau^\pi) \right] \\
 &= \inf_{\tau \in \mathbb{F}^Y} \mathbb{E}^\pi \left[\int_0^\tau e^{-\lambda t} [g(Y_t) + \lambda v_\infty(\mathbb{S}(Y_t))] dt + e^{-\lambda \tau} h(\Pi_\tau^\pi) \right] \\
 &= J[v_\infty](\pi),
 \end{aligned}$$

which shows that the function $v_\infty(\cdot)$ is a fixed point of the operator J .

LEMMA 4.1. *The sequence $(v_n)_{n \geq 1}$ converges to $v_\infty(\cdot)$ uniformly on $[0, 1]$. More precisely, we have*

$$(4.2) \quad v_\infty(\pi) \leq v_n(\pi) \leq v_\infty(\pi) + (1-p)^n(1-\pi)$$

for all $n \in \mathbb{N}$.

PROOF. The first inequality in (4.2) is immediate since the sequence $(v_n)_{n \in \mathbb{N}}$ is nonincreasing. The second inequality is also obvious for $n = 0$ as $v_\infty(\cdot) = \inf_{n \in \mathbb{N}} v_n(\cdot) \geq 0$. Assume the second inequality holds for some $n \in \mathbb{N}$. This implies that $v_n(\mathbb{S}(\pi)) = v_n(\pi + p(1-\pi)) \leq v_\infty(\pi + p(1-\pi)) + (1-p)^n(1-\pi - p(1-\pi)) = v_\infty(\mathbb{S}(\pi)) + (1-p)^{n+1}(1-\pi)$. Then we have

$$\begin{aligned}
 v_{n+1}(\pi) &= J[v_n](\pi) \\
 &= \inf_{\tau \in \mathbb{F}^Y} \mathbb{E}^\pi \left[\int_0^\tau e^{-\lambda t} [g(Y_t) + \lambda v_n(\mathbb{S}(Y_t))] dt + e^{-\lambda \tau} h(\Pi_\tau^\pi) \right] \\
 &\leq \inf_{\tau \in \mathbb{F}^Y} \mathbb{E}^\pi \left[\int_0^\tau e^{-\lambda t} [g(Y_t) + \lambda v_\infty(\mathbb{S}(Y_t)) + \lambda(1-p)^{n+1}(1-Y_t)] dt \right. \\
 &\quad \left. + e^{-\lambda \tau} h(\Pi_\tau^\pi) \right] \\
 &\leq \inf_{\tau \in \mathbb{F}^Y} \mathbb{E}^\pi \left[\int_0^\tau e^{-\lambda t} [g(Y_t) + \lambda v_\infty(\mathbb{S}(Y_t))] dt + e^{-\lambda \tau} h(\Pi_\tau^\pi) \right] \\
 &\quad + \mathbb{E}^\pi \left[\int_0^\infty e^{-\lambda t} \lambda(1-p)^{n+1}(1-Y_t) dt \right].
 \end{aligned}$$

Since $v_\infty(\cdot)$ satisfies $v_\infty(\cdot) = J[v_\infty](\cdot)$, the last inequality gives

$$\begin{aligned} v_{n+1}(\pi) &\leq v_\infty(\pi) + \int_0^\infty e^{-\lambda t} \lambda (1-p)^{n+1} \mathbb{E}^\pi [1 - Y_t] dt \\ &= v_\infty(\pi) + (1-p)^{n+1} (1-\pi), \end{aligned}$$

where we used the martingale property of Y to justify the last equality. This shows the second inequality in (4.2) for $n+1$, and the proof is complete by induction. \square

COROLLARY 4.1. *The uniform convergence in Lemma 4.1 implies that $v_\infty(\cdot)$ is continuous on $[0, 1]$. Moreover, as the infimum of nonnegative concave functions $v_n(\cdot)$'s, it is also nonnegative and concave.*

Corollary 4.1 and the identity $v_\infty(\cdot) = J[v_\infty](\cdot)$ [see (4.1)] allow us to conclude that $v_\infty(\cdot)$ is continuously differentiable on $(0, 1)$ and twice-continuously differentiable on $(0, 1) \setminus \{\pi_\infty\}$, where $\pi_\infty := r[v_\infty]$ is the unique root of the equation $B[v_\infty](r) = 0$ defined in (3.18). Furthermore, $v_\infty(\cdot)$ satisfies

$$(4.3) \quad \begin{cases} \left\{ \begin{array}{l} v_\infty(\pi) = h(\pi) \\ (-\lambda + \mathcal{A}_0)v_\infty(\pi) + g(\pi) + \lambda v_\infty(\mathbb{S}(\pi)) > 0 \end{array} \right\}, & \pi \in (\pi_\infty, 1), \\ \left\{ \begin{array}{l} v_\infty(\pi) < h(\pi) \\ (-\lambda + \mathcal{A}_0)v_\infty(\pi) + g(\pi) + \lambda v_\infty(\mathbb{S}(\pi)) = 0 \end{array} \right\}, & \pi \in (0, \pi_\infty), \end{cases}$$

which also implies that $\pi_\infty = \inf\{\pi \in [0, 1] : v_\infty(\pi) = h(\pi)\}$. Since $v_n(\cdot) \searrow v(\cdot)$, we have $\pi_n \nearrow \pi_\infty$.

PROPOSITION 4.1. *The function $v_\infty(\cdot)$ is the value function $V(\cdot)$ of the optimal stopping problem in (2.6), and the first entrance time $\tilde{\tau}_{\pi_\infty}$ of the process Π to the interval $[\pi_\infty, 1]$ is an optimal solution for the change-detection problem in (2.4).*

PROOF. The claim is obvious if $\Pi_0 = \pi = 1$; both $v_\infty(1)$ and $V(1)$ are nonnegative and bounded by $h(1) = 0$, which is also the expected reward in (2.6) by stopping immediately.

For $\pi \in (0, 1)$ and $0 < l \leq r < 1$, let $\tilde{\tau}_{[0,l]}$ and $\tilde{\tau}_{[r,1]}$ be respectively, the entrance times of the process Π to the intervals $[0, l]$ and $[r, 1]$. Also, define $\tilde{\tau}_{l,r} := \tilde{\tau}_{[0,l]} \wedge \tilde{\tau}_{[r,1]}$. Then for an \mathbb{F} -stopping time τ , Itô's rule gives

$$\begin{aligned} v_\infty(\Pi_{\tau \wedge \tilde{\tau}_{l,r}}) &= v_\infty(\pi) + \int_0^{\tau \wedge \tilde{\tau}_{l,r}} [(-\lambda + \mathcal{A}_0)v_\infty(\Pi_{u-}) + \lambda v_\infty(\mathbb{S}(\Pi_{u-}))] du \\ &\quad + \int_0^{\tau \wedge \tilde{\tau}_{l,r}} \mu \Pi_{u-} (1 - \Pi_{u-}) v'_\infty(\Pi_{u-}) d\widehat{W}_u \\ &\quad + \int_0^{\tau \wedge \tilde{\tau}_{l,r}} [v_\infty(\mathbb{S}(\Pi_{u-})) - v_\infty(\Pi_{u-})] (dN_u - \lambda du). \end{aligned}$$

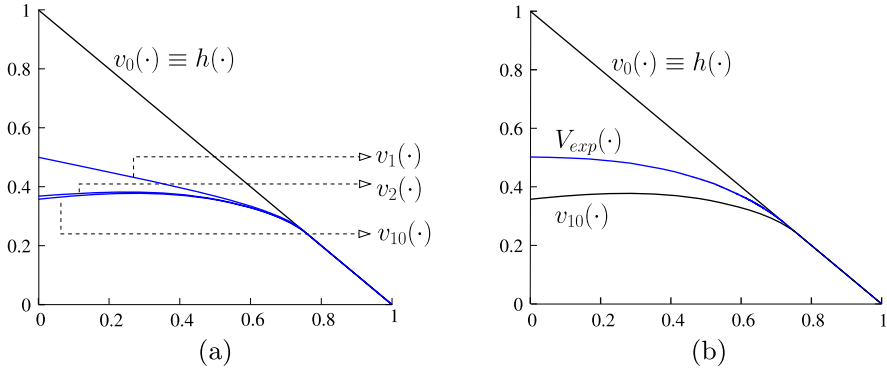


FIG. 1. We present a numerical example where $\mu = 1$, $\lambda = 2$, $p = 0.5$ and $c = 0.5$. Panel (a) illustrates the sequential approximation of the optimal Bayes risk. The functions $v_n(\cdot)$, for $n \leq 10$, are computed by first finding the threshold π_n and then evaluating the exit time expectation $H_{\pi_n}[v_n](\cdot)$ in (3.13) for $\pi \leq \pi_n$. The convergence is uniformly fast as given by Lemma 4.1. For $n = 10$, we have $\|V - v_{10}\| \leq 9.76 \cdot 10^{-4}$. Panel (b) compares two information levels on this detection problem. Recall that unconditional distribution of the change time is exponential with parameter $\lambda p = 1$. If we only observe the Wiener process (given this prior distribution) without observing the Poisson process N , then we are in the framework considered by [17], Section 6.22. The function $V_{exp}(\cdot)$ is the value function corresponding to this “less information” setting. It is computed by evaluating the expressions in [17], pages 311–312, with the values of μ and c given above. The figure illustrates that another observer who is also presented the process N performs significantly better in detecting the change.

Since the function v_∞ is bounded, the stochastic integral with respect to the martingale $\{N_t - \lambda t\}_{t \geq 0}$ is a square-integrable martingale stopped at $\tau \wedge \tilde{\tau}_{l,r}$ [whose expectation is finite due to (2.12)]. Similarly, so is the integral with respect to \widehat{W} as v'_∞ is continuous and bounded on $[l, r]$. Then taking expectations, we obtain

$$\begin{aligned}
 & \mathbb{E}^\pi v_\infty(\Pi_{\tau \wedge \tilde{\tau}_{l,r}}) \\
 &= v_\infty(\pi) + \mathbb{E}^\pi \int_0^{\tau \wedge \tilde{\tau}_{l,r}} [(-\lambda + \mathcal{A}_0)v_\infty(\Pi_{u-}) + \lambda v_\infty(\mathbb{S}(\Pi_{u-}))] du \\
 &\geq v_\infty(\pi) - \mathbb{E}^\pi \int_0^{\tau \wedge \tilde{\tau}_{l,r}} g(\Pi_{u-}) du \\
 &= v_\infty(\pi) - \mathbb{E}^\pi \int_0^{\tau \wedge \tilde{\tau}_{l,r}} g(\Pi_u) du
 \end{aligned}
 \tag{4.4}$$

thanks to the inequalities in (4.3).

The left boundary $\{0\}$ is natural for the diffusion in (3.2). Between two arrivals of N , the process Π follows these dynamics, and at an arrival time T_n it jumps to the right by an amount of $p(1 - \Pi_{T_n})$. Hence, as we let $l \searrow 0$, $\tilde{\tau}_{[0,l]}$ goes to ∞

thanks to strong Markov property, and $\tilde{\tau}_{l,r} \nearrow \tilde{\tau}_{[r,1]}$. Moreover, $\lim_{t \rightarrow \infty} \Pi_t = 1$, and $\Pi_t < 1$ [since $\Phi_t < \infty$ in (2.7)] for finite t , if $\pi < 1$. Hence, as $r \nearrow 1$, we have $\tilde{\tau}_{[r,1]} \nearrow \infty$. Therefore, when we let $l \searrow 0$ and $r \nearrow 1$ in (4.4), bounded convergence and monotone convergence theorems give

$$(4.5) \quad \mathbb{E}^\pi v_\infty(\Pi_\tau) \geq v_\infty(\pi) - \mathbb{E}^\pi \int_0^\tau g(\Pi_u) du.$$

Also note that we have $\mathbb{E}^\pi h(\Pi_\tau) \geq \mathbb{E}^\pi v_\infty(\Pi_\tau)$. Then we obtain

$$\mathbb{E}^\pi \left[\int_0^\tau g(\Pi_u) du + h(\Pi_\tau) \right] \geq v_\infty(\pi),$$

which implies that $v_\infty(\pi) \leq V(\pi)$ on $(0, 1)$.

When we replace τ in (4.4) with the entrance time $\tilde{\tau}_{\pi_\infty}$, the inequality in (4.5) becomes an equality. Then the equality $\mathbb{E}^\pi h(\Pi_{\tilde{\tau}_{\pi_\infty}}) = \mathbb{E}^\pi v_\infty(\Pi_{\tilde{\tau}_{\pi_\infty}})$ yields

$$(4.6) \quad v_\infty(\pi) = \mathbb{E}^\pi \left[\int_0^{\tilde{\tau}_{[\pi_\infty, 1]}} g(\Pi_u) du + h(\Pi_{\tilde{\tau}_{[\pi_\infty, 1]}}) \right],$$

and this implies $V(\pi) = v_\infty(\pi)$, for $\pi \in (0, 1)$.

To show the same equality for $\Pi_0 = \pi = 0$, we first note that $\Pi_t = 0$ for $t < T_1$, and $\Pi_{T_1} = p$ if the process Π starts from the point $\{0\}$. Also note that the identity $v_\infty(\cdot) = J[v_\infty](\cdot)$ implies $v_\infty(0) = v_\infty(p)$ [see (3.3)]. Then for an \mathbb{F} -stopping time τ , by modifying the arguments above, we get

$$\begin{aligned} & \mathbb{E}^0 v_\infty(\Pi_{\tau \wedge (\tilde{\tau}_{l,r} \circ \theta_{T_1})}) \\ &= v_\infty(0) + \mathbb{E}^0 1_{\{\tau \geq T_1\}} \int_{T_1}^{\tau \wedge (\tilde{\tau}_{l,r} \circ \theta_{T_1})} [(-\lambda + \mathcal{A}_0)v_\infty(\Pi_u) \\ & \quad + \lambda v_\infty(\mathbb{S}(\Pi_u-))] du \\ & \geq v_\infty(0) - \mathbb{E}^0 1_{\{\tau \geq T_1\}} \int_{T_1}^{\tau \wedge (\tilde{\tau}_{l,r} \circ \theta_{T_1})} g(\Pi_u) du \\ &= v_\infty(0) - \mathbb{E}^0 \int_0^{\tau \wedge (\tilde{\tau}_{l,r} \circ \theta_{T_1})} g(\Pi_u) du, \end{aligned} \quad (4.7)$$

where θ is the time-shift operator. Letting $l \searrow 0$ and $r \nearrow 1$ in (4.7), and using the inequality $\mathbb{E}^0 h(\Pi_\tau) \geq \mathbb{E}^0 v_\infty(\Pi_\tau)$, we obtain $v_\infty(0) \leq V(0)$.

Replacing τ above with $\tilde{\tau}_{[\pi_\infty, 1]}$, we get equalities in (4.7). Then, letting $l \searrow 0$, $r \nearrow 1$, and using the equality $\mathbb{E}^0 h(\Pi_{\tilde{\tau}_{[\pi_\infty, 1]}}) = \mathbb{E}^0 v_\infty(\Pi_{\tilde{\tau}_{[\pi_\infty, 1]}})$ we obtain (4.6) for $\pi = 0$. Hence, we have $v_\infty(0) = V(0)$, and this concludes the proof. \square

REMARK 4.2. For $\varepsilon > 0$, let us fix $n \in \mathbb{N}$ such that $n \geq \ln(\varepsilon)/\ln(1-p)$ and $v_n(\cdot) \leq v_\infty(\cdot) + \varepsilon$, on $[0, 1]$. The exit time $\tilde{\tau}_{\pi_n}$ of Π from the interval $[0, \pi_n) = \{\pi \in [0, 1] : v_n(\pi) < h(\pi)\}$ is ε -optimal for the problem in (2.6). That is,

$$(4.8) \quad \mathbb{E}^\pi \left[\int_0^{\tilde{\tau}_{\pi_n}} g(\Pi_t) dt + h(\Pi_{\tilde{\tau}_{\pi_n}}) \right] \leq V(\pi) + \varepsilon \quad \text{for all } \pi \in [0, 1].$$

PROOF. For $\pi > 0$, a localization argument and Itô's rule (as in the proof of Proposition 4.1) give

$$\begin{aligned}\mathbb{E}^\pi v_\infty(\Pi_{\tilde{\tau}_{\pi_n}}) &= v_\infty(\pi) + \mathbb{E}^\pi \int_0^{\tilde{\tau}_{\pi_n}} [(-\lambda + \mathcal{A}_0)v_\infty(\Pi_{u-}) + \lambda v_\infty(\mathbb{S}(\Pi_{u-}))] du \\ &= v_\infty(\pi) - \mathbb{E}^\pi \int_0^{\tilde{\tau}_{\pi_n,1}} g(\Pi_u) du,\end{aligned}$$

where the last equality follows from (4.3) (recall that $\pi_n \leq \pi_\infty$). Note that $\tilde{\tau}_{\pi_n} < \infty$ and $v_n(\Pi_{\tilde{\tau}_{\pi_n}}) = h(\Pi_{\tilde{\tau}_{\pi_n}})$, \mathbb{P}^π -almost surely. Then the inequality $v_n(\cdot) \leq v_\infty(\cdot) + \varepsilon$ yields

$$\begin{aligned}\mathbb{E}^\pi h(\Pi_{\tilde{\tau}_{\pi_n}}) - \varepsilon &= \mathbb{E}^\pi v_n(\Pi_{\tilde{\tau}_{\pi_n}}) - \varepsilon \\ &\leq \mathbb{E}^\pi v_\infty(\Pi_{\tilde{\tau}_{\pi_n}}) \\ &= v_\infty(\pi) - \mathbb{E}^\pi \int_0^{\tilde{\tau}_{\pi_n}} g(\Pi_u) du,\end{aligned}$$

and (4.8) follows.

For $\pi = 0$, we have

$$\begin{aligned}\mathbb{E}^0 \int_0^{\tilde{\tau}_{\pi_n}} g(\Pi_u) du + h(\Pi_{\tilde{\tau}_{\pi_n}}) &= \mathbb{E}^0 \int_{T_1}^{\tilde{\tau}_{\pi_n} \circ \theta_{T_1}} g(\Pi_u) du + h(\Pi_{\tilde{\tau}_{\pi_n} \circ \theta_{T_1}}) \\ &\leq V(p) + \varepsilon = V(0) + \varepsilon,\end{aligned}$$

where the inequality is due the strong Markov property (and also the result already proved above for $\pi = p > 0$), and the last equality follows from the identity $V(0) = J[V](0) = V(p)$. \square

5. Variational formulation. In this section, we solve the variational formulation of the problem where the objective is to minimize the expected detection delay $\mathbb{E}^\pi(\tau - \Theta)^+$ over all \mathbb{F} -stopping times for which the false alarm probability $\mathbb{P}^\pi(\tau < \Theta)$ is less than or equal to some predetermined value $\alpha \in (0, 1)$. The optimality of $\tau = 0$ is immediate when $\pi = 1$; hence, this case is excluded below.

When $\pi \in (0, 1)$, $\tau = 0$ is also an optimal solution if $\alpha \geq 1 - \pi$. On the other hand, if $\pi = 0$ and $\alpha \geq 1 - p$, the first arrival time T_1 of N yields a false alarm probability of $1 - p$ and its expected delay is still zero [see (2.1)–(2.2)].

If none of these trivial cases hold, we can find an optimal stopping time (for the variational formulation) using the solution of the problem in (2.4) as explained in [22]. More precisely, let $\pi_\infty(c)$ be the optimal threshold found in Section 4 as a function of c , and let $\tilde{\tau}_{\pi_\infty(c)}$ be the corresponding exit time of the process Π . For a given value of α , assume there exists a value of $c > 0$ such that the false alarm probability $\mathbb{P}^\pi(\tilde{\tau}_{\pi_\infty(c)} < \Theta) = \mathbb{E}^\pi h(\Pi_{\tilde{\tau}_{\pi_\infty(c)}})$ equals α . Then $\tilde{\tau}_{\pi_\infty(c)}$ solves the variational formulation. Indeed, the optimality of $\tilde{\tau}_{\pi_\infty(c)}$ for the original prob-

lem in (2.4) implies that, for any \mathbb{F} -stopping time τ , we have

$$c\mathbb{E}^\pi(\tilde{\tau}_{\pi_\infty(c)} - \Theta)^+ + \mathbb{P}^\pi(\tilde{\tau}_{\pi_\infty(c)} < \Theta) \leq c\mathbb{E}^\pi(\tau - \Theta)^+ + \mathbb{P}^\pi(\tau < \Theta).$$

Since $\mathbb{P}^\pi(\tilde{\tau}_{[\pi_\infty(c), 1]} < \Theta) = \alpha$, its expected detection delay has to be minimal compared to other stopping time τ 's for which $\mathbb{P}^\pi(\tau < \Theta) \leq \alpha$.

In this section, we show that $c \mapsto \mathbb{E}^\pi h(\Pi_{\tilde{\tau}_{\pi_\infty(c)}})$ is a continuous function of $c \in (0, \infty)$ with limits 0 (0) and $1 - \pi$ ($1 - p$) as $c \searrow 0$ and $c \nearrow \infty$, respectively if $\pi > 0$ ($\pi = 0$). Hence, for a given pair (π, α) the arguments in [22] work, and $\tilde{\tau}_{\pi_\infty(c)}$ is optimal for the value of c , for which $\mathbb{E}^\pi h(\Pi_{\tilde{\tau}_{\pi_\infty(c)}}) = \alpha$.

5.1. False alarm probabilities. For a given threshold $r \in (0, 1)$, let $\tilde{\tau}_r := \inf\{t \geq 0 : \Pi_t \geq r\}$ be the exit time of Π from the interval $[0, r)$, and let

$$(5.1) \quad F_r(\pi) := \mathbb{P}^\pi(\tilde{\tau}_r < \Theta) = \mathbb{E}^\pi h(\Pi_{\tilde{\tau}_r})$$

be the corresponding false alarm probability. On the event $\{\tilde{\tau}_r < T_1\}$, the exit time of Π coincides with the exit time τ_r of the process Y [in (3.2)], and we have $h(\Pi_{\tilde{\tau}_r}) = h(Y_{\tau_r})$. On the other hand, conditioned on $\{\tilde{\tau}_r \geq T_1\}$, strong Markov property implies that the false alarm probability should be computed by evaluating the function $F_r(\cdot)$ at the point Π_{T_1} . Therefore, we expect the function $F_r(\cdot)$ to solve

$$\begin{aligned} F_r(\pi) &= \mathbb{E}^\pi[1_{\{\tau_r < T_1\}} h(Y_{\tau_r}) + 1_{\{\tau_r \geq T_1\}} \cdot F_r(Y_{T_1-} + p(1 - Y_{T_1-}))] \\ &= \mathbb{E}^\pi \left[e^{-\lambda \tau_r} h(Y_{\tau_r}) + \int_0^{\tau_r} \lambda e^{-\lambda t} \cdot F_r(\mathbb{S}(Y_t)) dt \right] =: H_r^{(0)}[F_r](\pi), \end{aligned}$$

where $H_r^{(0)}\cdot$ denotes the operator $H_r\cdot$ in (3.12)–(3.13) with $c = 0$ [see also (3.11)]. Hence, if we apply the operator $H_r^{(0)}\cdot$ successively starting with a suitably selected initial function, the sequence that we obtain should converge to the function $F_r(\cdot)$. Indeed, in Appendix A.2, we show that the sequence constructed as

$$(5.2) \quad u_{0,r}(\cdot) = h(\cdot) \quad \text{and} \quad u_{n+1,r}(\cdot) = H_r^{(0)}[u_{n,r}](\cdot) \quad \text{for } n \in \mathbb{N},$$

is nonincreasing and converges uniformly to $F_r(\cdot)$ with error bounds

$$(5.3) \quad 0 \leq F_r(\pi) \leq u_{n,r}(\pi) \leq F_r(\pi) + (1 - p)^n(1 - \pi) \quad \text{for all } n \in \mathbb{N}.$$

It can easily be verified that the results in Lemmas 3.1 and 3.2 still hold for $c = 0$. Hence, on the region $\{(\pi, r) : \pi < r\}$, $u_{n,r}(\pi)$ has the form

$$\begin{aligned} (5.4) \quad \psi(\pi) &\left(-\frac{\eta(r)}{\psi(r)} \int_0^r \frac{u_{n-1,r}(\mathbb{S}(y))}{m_1 - m_2} \psi''(y) dy \right. \\ &\quad \left. + \int_\pi^r \frac{u_{n-1,r}(\mathbb{S}(y))}{m_1 - m_2} \eta''(y) dy + \frac{h(r)}{\psi(r)} \right) \\ &\quad + \eta(\pi) \int_0^\pi \frac{u_{n-1,r}(\mathbb{S}(y))}{(m_1 - m_2)} \psi''(y) dy, \end{aligned}$$

thanks to identities $\mathcal{A}_0\psi(\cdot) = \lambda\psi(\cdot)$ and $\mathcal{A}_0\eta(\cdot) = \lambda\eta(\cdot)$. On the region $\{(\pi, r) : \pi \geq r\}$, obviously, we have $u_{n,r}(\pi) = 1 - \pi$.

LEMMA 5.1. *For each $\pi \in [0, 1]$, the functions $r \mapsto u_{n,r}(\pi)$, for $n \in \mathbb{N}$, and $r \mapsto F_r(\pi)$ are continuous on $r \in (0, 1)$.*

PROOF. The result is obvious for $\pi = 1$, since $u_{n,r}(1) = F_r(1) = 1$, for all $r \in (0, 1)$. To prove the result for $\pi < 1$, we will show that $(\pi, r) \mapsto u_{n,r}(\pi)$ is jointly continuous on $(0, 1) \times (0, 1)$. However, observe that $u_{n,r}(\pi) = 1 - \pi$, for $\pi \leq r$; and $u_{n,r}(0) = u_{n-1,r}(p)$, for $r > \pi = 0$ thanks to (5.4) [see also (3.15) in Lemma 3.2]. Then direct computation gives

$$\lim_{r \rightarrow 0^+} u_{n,r}(0) = \lim_{r \rightarrow 0^+} u_{n-1,r}(p) = 1 - p < 1 = u_{n,0}(0),$$

which shows that $(\pi, r) \mapsto u_{n,r}(\pi)$ is not continuous at $(0, 0)$.

Clearly, $(\pi, r) \mapsto u_{0,r}(\pi) = 1 - \pi$ is continuous on $(0, 1) \times (0, 1)$. Suppose that the result holds for some $n \in \mathbb{N}$. On the region $\{(\pi, r) : \pi \geq r\}$, $u_{n+1,r}(\pi)$ again equals $h(\pi)$, and continuity is immediate.

Also, using the joint continuity on $(0, 1) \times (0, 1)$ of the bounded function $u_{n,r}(\pi)$ [and the boundary conditions $\psi'(0+) = 0$ and $\eta'(1-) = 0$], it can be verified that the expression in (5.4) is jointly continuous on $\{(\pi, r) \in (0, 1) \times (0, 1) : \pi \leq r\}$. When we let $r \rightarrow \pi$ in (5.4), direct computation gives

$$\begin{aligned} \psi(\pi) & \left(-\frac{\eta(\pi)}{\psi(\pi)} \int_0^\pi \frac{u_{n,r}(\mathbb{S}(y))}{(m_1 - m_2)} \psi''(y) dy + \frac{h(\pi)}{\psi(\pi)} \right) \\ & + \eta(\pi) \int_0^\pi \frac{u_{n,r}(\mathbb{S}(y))}{(m_1 - m_2)} \psi''(y) dy = h(\pi). \end{aligned}$$

This implies that $u_{n+1,r}(\pi)$ is jointly continuous on $(0, 1) \times (0, 1)$, and the result is true all $n \in \mathbb{N}$ by induction.

For $\pi = 0$ and $n \in \mathbb{N}$, we have $u_{n+1,r}(0) = u_{n,r}(p)$, and the continuity of $r \mapsto u_{n+1,r}(0)$ follows from the first part of the proof. Finally, the uniform convergence in (5.3) imply that $r \mapsto F_r(\pi)$ is also continuous, for each $\pi \in [0, 1]$, and this concludes the proof. \square

By the definition of $F_r(\pi)$ given in (5.1), we have

$$(5.5) \quad \begin{aligned} \lim_{r \rightarrow 0^+} F_r(\pi) &= 1 - \pi \quad \text{for } \pi > 0 \quad \text{and} \\ \lim_{r \rightarrow 0^+} F_r(0) &= 1 - p, \end{aligned}$$

where the second limit follows from the behavior of the process Π at $\{0\}$. That is, if $\Pi_0 = 0$, the process remains at this point until the first arrival time T_1 , and then

it jumps to the point $\{p\}$ [see (2.9)]. Also note that, for all $\pi \in [0, 1]$ and $r < 1$, the exit time $\tilde{\tau}_r$ is finite \mathbb{P}^π -almost surely, and $\Pi_{\tilde{\tau}_r} \in (r, r + p(1 - r))$. Hence,

$$(5.6) \quad \lim_{r \rightarrow 1^-} F_r(\pi) = \lim_{r \rightarrow 1^-} \mathbb{E}^\pi[1 - \Pi_{\tilde{\tau}_r}] = 0 \quad \text{for } \pi \geq 0.$$

REMARK 5.1. The optimal threshold of the Bayesian formulation is a nonincreasing and continuous function of the cost parameter c . If we let $\pi_\infty(c)$ denote the optimal threshold as a function of c , we have

$$(5.7) \quad \lim_{c \rightarrow 0^+} \pi_\infty(c) = 1 \quad \text{and} \quad \lim_{c \rightarrow \infty} \pi_\infty(c) = 0.$$

The limits in (5.7) can be obtained using the bounds in (3.19). Monotonicity of $\pi_\infty(c)$ in c is also obvious and follows from (1.2) and Remark 3.3. For the proof of the continuity of $c \mapsto \pi_\infty(c)$, Appendix B can be consulted.

Lemma 5.1 and Remark 5.1 imply that $F_{\pi_\infty(c)}(\pi)$ is continuous with respect to c on $(0, \infty)$. Moreover, thanks to (5.5)–(5.6) we have

$$\begin{aligned} \lim_{c \rightarrow 0^+} F_{\pi_\infty(c)}(\pi) &= \lim_{r \rightarrow 1^-} F_r(\pi) = 0, & \text{with} \\ \lim_{c \rightarrow \infty} F_{\pi_\infty(c)}(\pi) &= \lim_{r \rightarrow 0^+} F_r(\pi) = 1 - \pi \end{aligned}$$

for $\pi > 0$, and

$$\begin{aligned} \lim_{c \rightarrow 0^+} F_{\pi_\infty(c)}(0) &= \lim_{r \rightarrow 1^-} F_r(0) = 0, & \text{with} \\ \lim_{c \rightarrow \infty} F_{\pi_\infty(c)}(0) &= \lim_{r \rightarrow 0^+} F_r(0) = 1 - p. \end{aligned}$$

Hence (excluding the trivial cases) it is possible to pick a value of c such that the exit time $\tilde{\tau}_{\pi_\infty(c)}$ has a false alarm probability α and solves the variational formulation.

APPENDIX A: ON THE CONDITIONAL PROBABILITY PROCESS

A.1. An auxiliary probability measure and the proof of (2.7). Let $(\Omega, \mathcal{H}, \mathbb{P}_0)$ be a probability space hosting the following independent stochastic elements:

- a Wiener process X (with $\mu = 0$),
- a simple Poisson process N with arrival rate λ and arrival times $(T_n)_{n \geq 0}$,
- an integer valued random variable with distribution $\mathbb{P}_0\{\zeta = 0\} = \pi$ and $\mathbb{P}_0\{\zeta = n\} = (1 - \pi)(1 - p)^{n-1}p$ for $n \in \mathbb{N}$,
- a random variable Θ defined as in (2.2).

Let $\mathbb{G} \equiv \{\mathcal{G}_t\}_{t \geq 0}$ be an extended filtration such that $\mathcal{G}_t := \sigma\{X_s, N_s, \zeta : s \leq t\}$. In terms of the process $L_t = \exp\{\mu X_t - \mu^2 t/2\}$, we introduce a new probability measure \mathbb{P} whose Radon–Nykodym derivative is

$$Z_t := \frac{d\mathbb{P}}{d\mathbb{P}_0} \Big|_{\mathcal{G}_t} = 1_{\{\Theta > t\}} + 1_{\{\Theta \leq t\}} \frac{L_t}{L_\Theta}.$$

Under the new measure, the process X is a Brownian motion that gains a drift μ at Θ . The random variables ζ and Θ have the same distribution under \mathbb{P} since $\zeta \in \mathcal{G}_0$ and $Z_0 = 1$. In other words, we have the same setup described in Sections 1 and 2.

Let us now define the likelihood ratio process

$$\Phi_t := \frac{\mathbb{P}\{\Theta \leq t | \mathcal{F}_t\}}{\mathbb{P}\{\Theta > t | \mathcal{F}_t\}} = \frac{\mathbb{E}_0[Z_t 1_{\{\Theta \leq t\}} | \mathcal{F}_t]}{\mathbb{E}_0[Z_t 1_{\{\Theta > t\}} | \mathcal{F}_t]},$$

where the equality follows from Bayes' rule. Using the independence of X , N and ζ under \mathbb{P}_0 , we obtain

$$\mathbb{E}_0[Z_t 1_{\{\Theta \leq t\}} | \mathcal{F}_t] = \pi L_t + (1 - \pi) \sum_{i=1}^{N_t} (1 - p)^{i-1} p \frac{L_t}{L_{T_i}}$$

and

$$\mathbb{E}_0[Z_t 1_{\{\Theta > t\}} | \mathcal{F}_t] = \mathbb{P}_0[1_{\{\Theta > t\}} | \mathcal{F}_t] = (1 - \pi)(1 - p)^{N_t}.$$

Therefore, we have

$$\Phi_t = \frac{L_t}{(1 - p)^{N_t}} \left(\frac{\pi}{1 - \pi} + \sum_{i=1}^{N_t} \frac{(1 - p)^{i-1} p}{L_{T_i}} \right),$$

and this proves (2.7).

A.2. Constructing the exit time (false alarm) probabilities. Let $H_r^{(0)}$ denotes H_r defined in (3.12) with $c = 0$. It should be noted that the proofs of Lemmas 3.1 and 3.2 use only the continuity of the given function $w(\cdot)$ and the bounds $0 \leq w(\cdot) \leq h(\cdot)$. Hence, they also cover the case $c = 0$.

REMARK A.1. The operator $H_r^{(0)}$ is monotone in $w(\cdot)$; that is for $w_1(\cdot) \leq w_2(\cdot)$, we have $H_r^{(0)}[w_1](\cdot) \leq H_r^{(0)}[w_2](\cdot)$. Moreover, if $w(\cdot)$ is a continuous function bounded as $0 \leq w(\cdot) \leq h(\cdot)$, then so is $H_r^{(0)}[w](\cdot)$.

PROOF. The claim on monotonicity is obvious. Given $w(\cdot)$ continuous and bounded as $0 \leq w(\cdot) \leq h(\cdot)$, $H_r^{(0)}[w](\cdot)$ is again continuous by Lemma 3.2.

Since the process Y in (3.2) is a bounded martingale, we have

$$\begin{aligned} \ell(\pi) &:= \mathbb{E}^\pi \int_0^\infty e^{-\lambda t} \lambda h(\mathbb{S}(Y_t)) dt \\ &= \mathbb{E}^\pi \int_0^\infty e^{-\lambda t} \lambda (1 - p)(1 - Y_t) dt \\ &= (1 - p)(1 - \pi). \end{aligned}$$

Then, for a function $w(\cdot)$ bounded as $0 \leq w(\cdot) \leq h(\cdot)$, strong Markov property gives

$$\begin{aligned} 0 &\leq H_r^{(0)}[w](\pi) \leq \mathbb{E}^\pi \left[e^{-\lambda \tau_r} h(Y_{\tau_r}) + \int_0^{\tau_r} e^{-\lambda t} \lambda h(\mathbb{S}(Y_t)) dt \right] \\ &= \ell(\pi) + \mathbb{E}^\pi e^{-\lambda \tau_r} [h(Y_{\tau_r}) - \ell(Y_{\tau_r})] \\ &= \ell(\pi) + \mathbb{E}^\pi e^{-\lambda \tau_r} p(1 - Y_{\tau_r}) \leq \ell(\pi) + \mathbb{E}^\pi p(1 - Y_{\tau_r}) = h(\cdot). \end{aligned}$$

Hence, $0 \leq H_r^{(0)}[w](\cdot) \leq h(\cdot)$ again. \square

Using Remark A.1 above, it can be shown by induction (as in the proof of Remark 4.1) that the sequence

$$(A.1) \quad u_{0,r}(\cdot) = h(\cdot) \quad \text{and} \quad u_{n+1,r}(\cdot) = H_r^{(0)}[u_{n,r}](\cdot) \quad \text{for } n \in \mathbb{N},$$

is nonincreasing, and each function is nonnegative, continuous and bounded above by $h(\cdot)$. The pointwise limit $u_{\infty,r}(\cdot) := \inf_{n \in \mathbb{N}} u_{n,r}(\cdot)$ exists and it is bounded as $0 \leq u_{\infty,r}(\cdot) \leq h(\cdot)$.

REMARK A.2. The limit function $u_{\infty,r}(\cdot)$ solves $u_{\infty,r}(\cdot) = H_r^{(0)}[u_{\infty,r}](\cdot)$, on $[0, 1]$.

PROOF. The proof follows from a straightforward modification of (4.1) by replacing v_∞, v_n, τ with $u_{\infty,r}, u_{n,r}, \tau_r$ respectively. \square

REMARK A.3. The sequence defined in (A.1) converges uniformly on $[0, 1]$, and we have the explicit error bounds

$$(A.2) \quad 0 \leq u_{n,r}(\pi) - u_{\infty,r}(\pi) \leq (1-p)^n(1-\pi) \quad \text{for } n \in \mathbb{N}.$$

PROOF. We will establish the inequalities above by modifying the proof of Lemma 4.1.

The first inequality in (5.3) is obvious. The second inequality follows immediately for $n = 0$ since $0 \leq u_{\infty,r}(\cdot) \leq h(\cdot)$. Assume it holds for some $n \in \mathbb{N}$. Then using the induction hypothesis and the identity $u_{\infty,r}(\cdot) = H_r^{(0)}[u_{\infty,r}](\cdot)$, we have

$$\begin{aligned} u_{n+1,r}(\pi) &= H_r^{(0)}[u_{n,r}](\pi) \\ &\leq \mathbb{E}^\pi \left[e^{-\lambda \tau_r} h(Y_{\tau_r}) \right. \\ &\quad \left. + \int_0^{\tau_r} e^{-\lambda t} \lambda [u_{\infty,r}(\mathbb{S}(Y_t)) + (1-p)^{n+1}(1-Y_t)] dt \right] \\ &\leq u_{\infty,r}(\cdot) + \mathbb{E}^\pi \left[\int_0^\infty e^{-\lambda t} \lambda [(1-p)^{n+1}(1-Y_t)] dt \right] \\ &= u_{\infty,r}(\cdot) + (1-p)^{n+1}(1-\pi), \end{aligned}$$

and (A.2) follows. \square

COROLLARY A.1. *Since, each $u_{n,r}(\cdot)$ is continuous, so is $u_{\infty,r}(\cdot)$ thanks to Remark A.3. Then, the identity $u_{\infty,r}(\cdot) = H_r^{(0)}[u_{\infty,r}](\cdot)$ and Lemma 3.2 imply that the function $u_{\infty,r}(\cdot)$ solves*

$$(A.3) \quad (-\lambda + \mathcal{A}_0)u_{\infty,r}(\pi) + \lambda u_{\infty,r}(\mathbb{S}(y)) = 0 \quad \text{on } (0, r),$$

and at $\pi = 0$, we have $u_{\infty,r}(0) = u_{\infty,r}(p)$ [see (3.15)].

PROPOSITION A.1. *The limit function $u_{\infty,r}(\cdot)$ coincides on $[0, 1]$ with the exit time expectation $F_r(\cdot)$ defined in (5.1).*

PROOF. The characterization in (3.13) indicates that the derivative of $u_{\infty,r}$ is bounded on (l, r) , for $0 < l < r$. Then, for $\pi \in (l, r)$, a localization argument and Itô's rule gives

$$(A.4) \quad \begin{aligned} & \mathbb{E}^\pi u_{\infty,r}(\Pi_{\tilde{\tau}_{[l,r]}}) \\ &= u_{\infty,r}(\pi) + \mathbb{E}^\pi \int_0^{\tilde{\tau}_{[l,r]}} [(-\lambda + \mathcal{A}_0)u_{\infty,r}(\Pi_{u-}) + \lambda u_{\infty,r}(\mathbb{S}(\Pi_{u-}))] du \\ &= u_{\infty,r}(\pi), \end{aligned}$$

where $\tilde{\tau}_{[l,r]}$ is the exit time of Π from the interval (l, r) . The boundary $\{0\}$ is natural for the diffusive part of the process Π and its jumps are positive (toward $\{1\}$). This implies that $\tilde{\tau}_{[l,r]} \nearrow \tilde{\tau}_r = \inf\{t \geq 0 : \Pi_t \geq r\}$ as $l \rightarrow 0^+$, \mathbb{P}^π -almost surely [see also (2.12)]. Therefore, when we let $l \rightarrow 0^+$ in (A.4) we obtain

$$\begin{aligned} u_{\infty,r}(\pi) &= \lim_{l \rightarrow 0^+} \mathbb{E}^\pi u_{\infty,r}(\Pi_{\tilde{\tau}_{[l,r]}}) \\ &= \lim_{l \rightarrow 0^+} u_{\infty,r}(l) \mathbb{P}^\pi \{\tilde{\tau}_{[l,r]} < \tilde{\tau}_r\} + \mathbb{E}^\pi 1_{\{\tilde{\tau}_{[l,r]} = \tilde{\tau}_r\}} h(\Pi_{\tilde{\tau}_r}) \\ &= \mathbb{E}^\pi h(\Pi_{\tilde{\tau}_r}). \end{aligned}$$

This shows $u_{\infty,r}(\cdot) = F_r(\cdot)$ on $(0, r)$.

When $\Pi_0 = 0$, the process stays at $\{0\}$ until the first arrival time T_1 of N . It jumps to $\{p\}$ at T_1 . Hence, by strong Markov property, we have $F_r(0) = F_r(p)$, and this shows $u_{\infty,r}(0) = F_r(0)$ [since $u_{\infty,r}(0) = u_{\infty,r}(p)$]. Finally, for $\pi \geq r$, we have $u_{\infty,r}(\pi) = 1 - \pi$ by the construction in (A.1); hence, the equality $u_{\infty,r}(\cdot) = F_r(\cdot)$ is obvious. \square

APPENDIX B: OTHER PROOFS

PROOF OF (2.10). The process \widehat{W} is a (\mathbb{P}, \mathbb{F}) -Brownian motion (this can be verified using Lévy's characterization for Brownian motion) and N is a (\mathbb{P}, \mathbb{F}) -Poisson process. Therefore, it is sufficient to show (2.10) for $s_1 = s_2 = s$.

Note that the process \widehat{W} can be written as

$$(B.1) \quad \widehat{W}_t = W_t + \mu \int_0^t [1_{\{\Theta \leq u\}} - \Pi_u] du.$$

Therefore, if we apply Itô formula to real and imaginary parts of the process $K_t := f(\widehat{W}_t, N_t)$, for $f(x, y) = \exp\{irx + iqy\}$, we obtain

$$(B.2) \quad \begin{aligned} K_s = K_t &+ i \left[\int_t^s r K_u dW_u + \int_t^s r \mu K_u 1_{\{\Theta \leq u\}} du - \int_t^s r \mu K_u \Pi_u du \right] \\ &- \frac{1}{2} r^2 \int_t^s K_u du + \int_t^s (e^{iq} - 1) K_u (dN_u - \lambda du) \\ &+ \int_t^s \lambda (e^{iq} - 1) K_u du \end{aligned}$$

for $t \leq u \leq s$. Clearly, we have

$$\mathbb{E} \left[\int_t^s (e^{iq} - 1) K_u (dN_u - \lambda du) \middle| \mathcal{F}_t \right] = 0 = \mathbb{E} \left[\int_t^s K_u dW_u \middle| \mathcal{F}_t \right].$$

Moreover, for a set $A \in \mathcal{F}_t$ we have $\mathbb{E} 1_A K_u 1_{\{\Theta \leq u\}} = \mathbb{E} 1_A K_u \Pi_u$. Then by multiplying both sides in (B.2) with $1_A/K_t = 1_A \cdot e^{-ir\widehat{W}_t - iqN_t}$ and taking the expectations we get

$$\begin{aligned} &\mathbb{E}[1_A \exp\{ir(\widehat{W}_s - \widehat{W}_t) + iq(N_s - N_t)\}] \\ &= P(A) + \int_t^s \left(-\frac{r^2}{2} + \lambda(e^{iq} - 1) \right) \\ &\quad \times \mathbb{E}[1_A \exp\{ir(\widehat{W}_u - \widehat{W}_t) + iq(N_u - N_t)\}] du. \end{aligned}$$

By solving this integral equation for the (deterministic) function

$$\varrho_t(\cdot) : s \mapsto \mathbb{E}[1_A \exp\{ir(\widehat{W}_s - \widehat{W}_t) + iq(N_s - N_t)\}]$$

we obtain $\varrho_t(s) = P(A) \cdot \exp\{(-\frac{r^2}{2} + \lambda(e^{iq} - 1))(s - t)\}$, and this proves (2.10) for $s_1 = s_2 = s$. \square

PROOF OF (3.20). Let π be a fixed point on $(0, r[w])$. For any $r \geq r[w]$, $H_r[w](\pi)$ is given by (3.13) with

$$\begin{aligned} \frac{\partial H_r[w](\pi)}{\partial r} &= -\frac{\psi(\pi)}{\psi^2(r)} \left\{ h(r)\psi'(r) + \psi(r) - (m_1 - m_2) \int_0^r u_2[w](y) dy \right\} \\ &= -\frac{\psi(\pi)}{\psi^2(r)} B[w](r). \end{aligned}$$

The last expression is strictly positive for $r > r[w]$ since $B[w](r)$ is strictly negative thanks to Lemma 3.3. This implies that $H_{r[w]}[w](\pi) < H_{r_1}(\pi) < H_{r_2}(\pi)$, for

all $r[w] < r_1 < r_2$, and we have $H_r[w](\pi) < \lim_{r \nearrow 1} H_r[w](\pi)$. Since the right boundary is natural, $\tau_r \nearrow \infty$ as $r \nearrow 1$. Then by dominated convergence theorem (see Remark 3.2), we obtain

$$\begin{aligned} H_r[w](\pi) &< \mathbb{E}^\pi \left[\int_0^\infty e^{-\lambda t} (g(Y_t) + \lambda w(\mathbb{S}(Y_t))) dt \right] \\ &\leq \int_0^\infty (g(\pi) + \lambda w(\mathbb{S}(\pi))) dt \\ &= \frac{g(\pi) + \lambda w(\mathbb{S}(\pi))}{\lambda}, \end{aligned}$$

where the second inequality is by Jensen's inequality [recall that Y is a martingale and $w(\cdot)$ is concave], and (3.20) follows. \square

PROOF OF REMARK 5.1. The limits in (5.7) follow easily from (3.19). It is also clear that $c \mapsto \pi_\infty(c)$ is nonincreasing thanks to (1.2) and Remark 3.3. Here, we show that $c \mapsto \pi_\infty(c)$ is continuous on $(0, \infty)$.

Let $V_c(\pi)$ and $B_c[\cdot]$ denote respectively the dependence on c of the value function V and the operator $B[\cdot]$ defined in (3.18).

Since the value function V is a fixed point of the operator J , Lemma 3.3 gives

$$B_{c_1}[V_{c_1}](\pi_\infty(c_1)) = 0 = B_{c_1}[V_{c_1}](\pi_\infty(c_2)) \quad \text{for } 0 < c_1 \leq c_2 < \infty.$$

By using these equalities together with the explicit form of $B[\cdot]$ in (3.18) [and the identity $\mathcal{A}_0\psi(\cdot) = \lambda\psi(\cdot)$], we obtain

$$\begin{aligned} 0 &\leq B_{c_1}[V_{c_1}](\pi_\infty(c_2)) - B_{c_1}[V_{c_1}](\pi_\infty(c_1)) \\ \text{(B.3)} \quad &\equiv \int_{\pi_\infty(c_2)}^{\pi_\infty(c_1)} \frac{\psi''(y)}{\lambda} [c_1 y + \lambda V_{c_1}(\mathbb{S}(y)) - \lambda h(y)] dy \\ &= \int_0^{\pi_\infty(c_2)} \frac{\psi''(y)}{\lambda} [(c_2 - c_1)y + \lambda(V_{c_2}(\mathbb{S}(y)) - V_{c_1}(\mathbb{S}(y)))] dy. \end{aligned}$$

Moreover, we have $V_{c_2}(\pi) \leq \mathbb{P}^\pi(\tilde{\tau}_{\pi_\infty(c_1)} < \Theta) + c_2 \mathbb{E}^\pi(\tilde{\tau}_{\pi_\infty(c_1)} - \Theta)^+$, and this gives the Lipschitz condition

$$\frac{V_{c_2}(\pi) - V_{c_1}(\pi)}{c_2 - c_1} \leq \mathbb{E}^\pi(\tilde{\tau}_{\pi_\infty(c_1)} - \Theta)^+ \leq \frac{V_{c_1}(\pi)}{c_1} \leq \frac{1}{\delta}$$

for any $\delta < c_1$. Using this inequality in (B.3), we obtain

$$\begin{aligned} 0 &\leq \int_{\pi_\infty(c_2)}^{\pi_\infty(c_1)} \frac{\psi''(y)}{\lambda} [c_1 y + \lambda V_{c_1}(\mathbb{S}(y)) - \lambda h(y)] dy \\ \text{(B.4)} \quad &\leq \int_0^{\pi_\infty(c_2)} \frac{\psi''(y)}{\lambda} (c_2 - c_1) \left[y + \frac{\lambda}{\delta} \right] dy \\ &\leq (c_2 - c_1) \left[\frac{1}{\lambda} + \frac{1}{\delta} \right] \psi'(\pi_\infty(c_2)). \end{aligned}$$

This implies that $\pi_\infty(c_2) \nearrow \pi_\infty(c_1)$ as $c_2 \searrow c_1$.

Similarly, it is easy to show that

$$\begin{aligned} 0 &\leq \int_{\pi_\infty(c_2)}^{\pi_\infty(c_1)} \frac{\psi''(y)}{\lambda} [c_2 y + \lambda V_{c_2}(\mathbb{S}(y)) - \lambda h(y)] dy \\ &= \int_0^{\pi_\infty(c_1)} \frac{\psi''(y)}{\lambda} [(c_2 - c_1)y + \lambda(V_{c_2}(\mathbb{S}(y)) - V_{c_1}(\mathbb{S}(y)))] dy \\ &\leq \int_0^{\pi_\infty(c_1)} \frac{\psi''(y)}{\lambda} (c_2 - c_1) \left[1 + \frac{\lambda}{\delta}\right] dy \\ &= (c_2 - c_1) \left[\frac{1}{\lambda} + \frac{1}{\delta}\right] \psi'(\pi_\infty(c_1)) \end{aligned}$$

for some $\delta < c_1$. This shows that $\pi_\infty(c_1) \searrow \pi_\infty(c_2)$ as $c_1 \nearrow c_2$, and the continuity of $c \mapsto \pi_\infty(c)$ follows. \square

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