*The Annals of Applied Probability*
2010, Vol. 20, No. 4, 1179–1204
DOI: 10.1214/09-AAP642


# ON OPTIMAL ARBITRAGE

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In a Markovian model for a financial market, we characterize the best arbitrage with respect to the market portfolio that can be achieved using nonanticipative investment strategies, in terms of the smallest positive solution to a parabolic partial differential inequality; this is determined entirely on the basis of the covariance structure of the model. The solution is intimately related to properties of strict local martingales and is used to generate the investment strategy which realizes the best possible arbitrage. Some extensions to non-Markovian situations are also presented.

**1. Introduction.** In a Markovian model for an equity market with mean rates of return $\mathrm{b}_i(\mathfrak{X}(t))$ and covariance rates $\mathrm{a}_{ij}(\mathfrak{X}(t))$, $1 \le i, j \le n$, for its asset capitalizations $\mathfrak{X}(t) = (X_1(t), \dots, X_n(t))' \in (0, \infty)^n$ at time $t$, what is the highest return on investment [as in (6.3) below] that can be achieved relative to the market on a given time–horizon $[0, T]$, using nonanticipative investment strategies? What are the weights assigned to the different assets by such an investment strategy that accomplishes this?

Answers: under suitable conditions, $1/U(T, \mathfrak{X}(0))$ and

$$X_i(t) D_i \log U\big(T - t, \mathfrak{X}(t)\big) + \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \qquad i = 1, \dots, n, t \in [0, T],$$

respectively. Here $U : [0, \infty) \times (0, \infty)^n \to (0, 1]$ is the smallest nonnegative solution of the linear parabolic partial differential inequality

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{x}) \ge \widehat{\mathcal{L}} U(\tau, \mathbf{x}), \qquad (\tau, \mathbf{x}) \in (0, \infty) \times (0, \infty)^n, \tag{1.1}$$

subject to the initial condition $U(0, \cdot) \equiv 1$, for the linear operator

$$\widehat{\mathcal{L}} f := \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n x_i x_j \mathrm{a}_{ij}(\mathbf{x}) D^2_{ij} f + \sum_{i=1}^n x_i \left( \sum_{j=1}^n \frac{x_j \mathrm{a}_{ij}(\mathbf{x})}{x_1 + \cdots + x_n} \right) D_i f \tag{1.2}$$

with $D_i = \partial / \partial x_i$, $D = (\partial/\partial x_1, \dots, \partial/\partial x_n)'$ and $D^2_{ij} = \partial^2 / \partial x_i \, \partial x_j$. Furthermore, $U(T, \mathfrak{X}(0))$ is the probability that the $([0, \infty)^n \setminus \{\mathbf{0}\})$-valued diffusion process

Received June 2008; revised June 2009.

[1]Author is on leave from the Department of Mathematics at Columbia University, where his research was supported by NSF Grants DMS-06-01774 and DMS-09-05754.

*AMS 2000 subject classifications.* Primary 60H10, 91B28; secondary 60G44, 35B50.

*Key words and phrases.* Portfolios, arbitrage, parabolic operators, maximum principle, strict local martingales, exit measures for supermartingales, diffusions, Fichera drift.

$\mathfrak{Y}(\cdot) = (Y_1(\cdot), \dots, Y_n(\cdot))'$ with infinitesimal generator $\widehat{\mathcal{L}}$ as above and $\mathfrak{Y}(0) = \mathfrak{X}(0) \in (0,\infty)^n$ does not hit the boundary of the orthant $[0,\infty)^n$ by time $t = T$. We note that the answers involve only the covariance structure of the market, not the actual rates of return; the only role these latter play is to ensure that the diffusion $\mathfrak{X}(\cdot)$ lives in $(0,\infty)^n$.

Arbitrage relative to the market exists on $[0, T]$, iff $U(T, \mathfrak{X}(0)) < 1$; this is deeply related to the importance of *strict local martingales* in the present context, and amounts to failure of uniqueness for the Cauchy problem

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{x}) = \widehat{\mathcal{L}} U(\tau, \mathbf{x}), \qquad (\tau, \mathbf{x}) \in (0,\infty) \times (0,\infty)^n \quad \text{and} \quad U(0, \cdot) \equiv 1.$$

Sufficient conditions for such failure of uniqueness are provided.

Consider an "auxiliary market" with capitalizations $\mathfrak{Y}(\cdot) = (Y_1(\cdot), \dots, Y_n(\cdot))'$ as above. The probabilistic significance of the change of drift inherent in the definition of the operator $\widehat{\mathcal{L}}$, from $\mathrm{b}_i(\mathbf{x})$ for $\mathfrak{X}(\cdot)$ to $\sum_{j=1}^n (x_j \mathrm{a}_{ij}(\mathbf{x}))/(x_1 + \cdots + x_n)$ for $\mathfrak{Y}(\cdot)$, is that it corresponds to a change of probability measure which makes the weights $\nu_i(\cdot) := Y_i(\cdot)/(Y_1(\cdot) + \cdots + Y_n(\cdot))$, $i = 1, \dots, n$, of the auxiliary market portfolio martingales. Its financial significance is that it bestows to the auxiliary market portfolio $\nu(\cdot) = (\nu_1(\cdot), \dots, \nu_n(\cdot))'$ the so-called *numéraire property*: any strategy's relative performance in the market with capitalizations $\mathfrak{Y}(\cdot)$ is a supermartingale, so this market cannot be outperformed. This change need not come from a Girsanov-type (absolutely continuous) transformation; rather it corresponds to, and represents, the *exit measure* of Föllmer (1972) for an appropriate supermartingale.

Sections 2 and 3 set up the model, whereas Section 4 introduces the notion and offers examples of relative arbitrage; Section 5 makes the connection with strict local martingales. Section 6 formulates the problem, and Section 7 offers some preliminary results, actually in some modest generality (including non-Markovian cases). Section 8 sets up the Markovian model; the results are presented in earnest in Sections 9–11, Section 12 discusses a couple of examples in detail and a few open questions are raised in Section 13.

*Related literature*: the questions raised in this study are related to the work of Delbaen and Schachermayer (1995b). They bear an even closer connection with issues raised in the Finance literature under the general rubric of "bubbles" [see Definition 5 and Theorem 1 in Ruf (2009) for the precise connection]. The literature on this topic is large, so let us mention the papers by Loewenstein and Willard (2000), Pal and Protter (2007) and, most significantly, Heston, Loewenstein and Willard (2007), as the closest in spirit to our approach here. We note the recent preprint by Hugonnier (2007), which demonstrates that arbitrage opportunities can arise in equilibrium models; this preprint, and Heston, Loewenstein and Willard (2007), can be consulted for an up-to-date survey of the literature on this subject and for some explicit computations of trading strategies that lead to arbitrage. The need to consider state-price-density processes that are only local (as opposed to true) martingales has also been noticed in the context of "stochastic volatility" models [e.g., Sin (1998), Wong and Heyde (2006)] and of pricing with long maturities [e.g., Hulley and Platen (2008)].

**2. The model.** We consider a model consisting of a money-market $dB(t) = B(t)r(t)\,dt$, $B(0) = 1$ and of $n$ stocks with capitalizations,

$$dX_i(t) = X_i(t)\left(\beta_i(t)\,dt + \sum_{k=1}^{K} \sigma_{ik}(t)\,dW_k(t)\right), \qquad X_i(0) = x_i > 0, \tag{2.1}$$

for $i = 1, \dots, n$. These are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and are driven by the Brownian motion $W(\cdot) = (W_1(\cdot), \dots, W_K(\cdot))'$ whose $K \geq n$ independent components are the model's "factors."

We shall assume throughout that the interest rate process of the money-market is $r(\cdot) \equiv 0$, identically equal to zero; and that the vector-valued process $\mathfrak{X}(\cdot) = (X_1(\cdot), \dots, X_n(\cdot))'$ of capitalizations, the vector-valued process $\beta(\cdot) = (\beta_1(\cdot), \dots, \beta_n(\cdot))'$ of *mean rates of return* for the various stocks and the $(n \times K)$-matrix-valued process $\sigma(\cdot) = (\sigma_{ik}(\cdot))_{1 \leq i \leq n, 1 \leq k \leq K}$ of *volatilities* are all progressively measurable with respect to a right-continuous filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ which represents the "flow of information" in the market with $\mathcal{F}(0) = \{\varnothing, \Omega\}$, mod $\mathbb{P}$. Let $\alpha(\cdot) := \sigma(\cdot)\sigma'(\cdot)$ be the *covariance process* of the stocks in the market, and impose for $\mathbb{P}$-a.e. $\omega \in \Omega$ the condition

$$\sum_{i=1}^{n} \int_0^T \big(|\beta_i(t, \omega)| + \alpha_{ii}(t, \omega)\big)\,dt < \infty \qquad \forall\, T \in (0, \infty). \tag{2.2}$$

Under this condition the processes $X_1(\cdot), \dots, X_n(\cdot)$ can be expressed as $X_i(\cdot) = x_i \exp\{\int_0^{\cdot}(\beta_i(t) - \frac{1}{2}\alpha_{ii}(t))\,dt + \sum_{k=1}^{K} \int_0^{\cdot} \sigma_{ik}(t)\,dW_k(t)\} > 0$.

In this setting, the Brownian motion $W(\cdot)$ need not be adapted to the "observations" filtration $\mathbb{F}$. It *is* adapted, though, to the $\mathbb{P}$-augmentation $\mathbb{G} = \{\mathcal{G}(t)\}_{0 \leq t < \infty}$ of the filtration $\mathbb{F}$, provided that $K = n$ and that the matrix-valued process $\sigma(\cdot)$ is invertible—as in Assumption B below.

**3. Strategies and portfolios.** Consider now a *small investor* who decides, at each time $t$, which proportion $\pi_i(t)$ of current wealth $V(t)$ to invest in the $i$th stock, $i = 1, \dots, n$; the proportion $1 - \sum_{i=1}^{n} \pi_i(t) =: \pi_0(t)$ gets invested in the money market. Thus, the wealth $V(\cdot) \equiv V^{v,\pi}(\cdot)$ for an initial capital $v \in (0, \infty)$ and an investment strategy $\pi(\cdot) = (\pi_1(\cdot), \dots, \pi_n(\cdot))'$ satisfies the initial condition $V(0) = v$ and

$$\begin{aligned} \frac{dV(t)}{V(t)} &= \sum_{i=1}^{n} \pi_i(t)\,\frac{dX_i(t)}{X_i(t)} + \pi_0(t)\,\frac{dB(t)}{B(t)} \\ &= \pi'(t)[\beta(t)\,dt + \sigma(t)\,dW(t)]. \end{aligned} \tag{3.1}$$

We shall call *investment strategy* a $\mathbb{G}$-progressively measurable process $\pi : [0, \infty) \times \Omega \to \mathbb{R}^n$ which satisfies for $\mathbb{P}$-a.e. $\omega \in \Omega$ the analogue

$$\int_0^T \big( |\pi'(t,\omega)\beta(t,\omega)| + \pi'(t,\omega)\alpha(t,\omega)\pi(t,\omega) \big)\, dt < \infty, \qquad \forall\, T \in (0,\infty)$$

of (2.2). The collection of investment strategies will be denoted by $\mathcal{H}$.

A strategy $\pi(\cdot) \in \mathcal{H}$ with $\sum_{i=1}^n \pi_i(t,\omega) = 1$ for all $(t,\omega) \in [0,\infty) \times \Omega$ will be called *portfolio*. A portfolio never invests in the money market and never borrows from it. We shall say that a process $\pi(\cdot)$ is *bounded*, if for it there exists a real constant $C_\pi > 0$ such that $\|\pi(t,\omega)\| \le C_\pi$ holds for all $(t,\omega) \in [0,\infty) \times \Omega$. We shall call *long-only portfolio* one that satisfies $\pi_1(t,\omega) \ge 0, \dots, \pi_n(t,\omega) \ge 0, \forall (t,\omega) \in [0,\infty) \times \Omega$, that is, never sells any stock short. Clearly, a long-only portfolio is also bounded.

Corresponding to an investment strategy $\pi(\cdot)$ and initial capital $v > 0$, the associated wealth process, that is, the solution of (3.1), is

$$V^{v,\pi}(\cdot) = v \exp\left\{ \int_0^{\cdot} \pi'(t)\left( \beta(t) - \frac{\alpha(t)}{2}\pi(t) \right) dt + \int_0^{\cdot} \pi'(t)\sigma(t)\, dW(t) \right\} > 0.$$

The strategy $\varrho(\cdot) \equiv 0$ invests only in the money market at all times; it results in $V^{v,\varrho}(\cdot) \equiv v$, that is, in hoarding the initial wealth under the mattress.

3.1. *The market portfolio.* An important long-only portfolio is the market portfolio; this invests in all stocks in proportion to their relative weights,

$$\mu_i(t) := \frac{X_i(t)}{X(t)}, \qquad i = 1, \dots, n, \text{ where } X(t) := X_1(t) + \cdots + X_n(t). \tag{3.2}$$

Clearly $V^{v,\mu}(\cdot) = vX(\cdot)/X(0)$, and the resulting vector process $\mu(\cdot) = (\mu_1(\cdot), \dots, \mu_n(\cdot))'$ of market weights takes values in the positive simplex $\Delta_+^n := \{(m_1, \dots, m_n)' \in (0,1)^n \,|\, \sum_{i=1}^n m_i = 1\}$ of $\mathbb{R}^n$. An application of Itô's rule gives, after some computation, the dynamics of this process as

$$d\mu_i(t) = \mu_i(t)\left[ \gamma_i^\mu(t)\, dt + \sum_{k=1}^K \tau_{ik}^\mu(t)\, dW_k(t) \right], \qquad i = 1, \dots, n. \tag{3.3}$$

Here $\tau^\mu(t)$ is the matrix with entries $\tau_{ik}^\mu(t) := \sigma_{ik}(t) - \sum_{j=1}^n \mu_j(t)\sigma_{jk}(t)$, $\mathfrak{e}_i$ the $i$th unit vector in $\mathbb{R}^n$ and the vector $\gamma^\mu(t) := (\gamma_1^\mu(t), \dots, \gamma_n^\mu(t))'$ has

$$\gamma_i^\mu(t) := \big(\mathfrak{e}_i - \mu(t)\big)'\big(\beta(t) - \alpha(t)\mu(t)\big). \tag{3.4}$$

**4. Relative arbitrage.** The following notion was introduced in Fernholz (2002): given a real number $T > 0$ and any two investment strategies $\pi(\cdot)$ and $\rho(\cdot)$, we call $\pi(\cdot)$ an *arbitrage relative to* $\rho(\cdot)$ over $[0, T]$, if

$$\mathbb{P}\big(V^{1,\pi}(T) \ge V^{1,\rho}(T)\big) = 1 \quad \text{and} \quad \mathbb{P}\big(V^{1,\pi}(T) > V^{1,\rho}(T)\big) > 0. \tag{4.1}$$

We call such relative arbitrage *strong* if $\mathbb{P}(V^{1,\pi}(T) > V^{1,\rho}(T)) = 1$.

Arbitrage (resp., strong arbitrage) relative to $\varrho(\cdot) \equiv 0$ that invests only in the money market, is called just that, without the qualifier "relative."

4.1. *Examples of arbitrage relative to the market.* Here are some examples taken from the survey Fernholz and Karatzas (2009), especially Sections 7 and 8, Remark 11.4, Examples 11.1 and 11.2. Suppose first that

$$\sum_{i=1}^{n} \mu_i(t)\alpha_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i(t)\alpha_{ij}(t)\mu_j(t) \geq h \qquad \forall\, 0 \leq t < \infty \tag{4.2}$$

holds almost surely for some constant $h > 0$. Then the long-only portfolio $\pi_i(t) = \mu_i(t)(c - \log \mu_i(t))/J(t), i = 1, \dots, n,\ J(t) := \sum_{j=1}^{n} \mu_j(t)(c - \log \mu_j(t))$ is, for sufficiently large $c > 0$, a strong arbitrage relative to the market portfolio $\mu(\cdot)$ over any time–horizon $[0, T]$ with $T > (2 \log n)/h$.

Another condition guaranteeing the existence of strong arbitrage relative to the market is that there exists a real constant $h > 0$ with

$$\sqrt[n]{\mu_1(t) \cdots \mu_n(t)}\left[\sum_{i=1}^{n} \alpha_{ii}(t) - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}(t)\right] \geq h \qquad \forall\, 0 \leq t < \infty \tag{4.3}$$

a.s. Then for $c > 0$ sufficiently large, the long-only portfolio $\pi_i(t) = \lambda(t)(1/n) + (1 - \lambda(t))\mu_i(t),\ 1 \leq i \leq n,\ 1/\lambda(t) := 1 + ((\mu_1(t) \cdots \mu_n(t))^{1/n}/c)$, is strong arbitrage relative to the market over any $[0, T]$ with $T > (2n^{1-(1/n)})/h$.

REMARK 1. Suppose that all the eigenvalues of the covariance matrix-valued process $\alpha(\cdot)$ are bounded away from both zero and infinity, uniformly on $[0, \infty) \times \Omega$, and that (4.2) holds. Then, for any given constant $p \in (0, 1)$, the long-only portfolio $\mu_i^{(p)}(t) = (\mu_i(t))^p (\sum_{j=1}^{n} (\mu_j(t))^p)^{-1}, i = 1, \dots, n$, leads again to strong arbitrage relative to the market portfolio over sufficiently long time–horizons. It is also of great interest that appropriate modifications of the portfolio $\mu^{(p)}(\cdot)$ yield such arbitrage over *any* time–horizon $[0, T]$.

**5. Market price of risk and strict local martingales.** We shall assume from now on that there exists a *market price of risk* $\vartheta : [0, \infty) \times \Omega \to \mathbb{R}^K$, an $\mathbb{F}$-progressively measurable process that satisfies

$$\begin{gathered} \sigma(t, \omega)\vartheta(t, \omega) = \beta(t, \omega) \qquad \forall (t, \omega) \in [0, \infty) \times \Omega \quad \text{and} \\ \mathbb{P}\left(\int_0^T \|\vartheta(t, \omega)\|^2\, dt < \infty,\ \forall\, T \in (0, \infty)\right) = 1. \end{gathered} \tag{5.1}$$

The existence of a market-price-of-risk process $\vartheta(\cdot)$ allows us to introduce an associated exponential local martingale,

$$Z(t) := \exp\left\{-\int_0^t \vartheta'(s)\, dW(s) - \frac{1}{2}\int_0^t \|\vartheta(s)\|^2\, ds\right\}, \qquad 0 \leq t < \infty. \tag{5.2}$$

This process is also a supermartingale; it is a martingale, if and only if $\mathbb{E}(Z(T)) = 1$ holds for all $T \in (0, \infty)$. For the purposes of this work it is important to allow such exponential processes to be strict local martingales; that is, not to exclude the possibility $\mathbb{E}(Z(T)) < 1$ for some $T \in (0, \infty)$.

From (5.2) and (3.1), now written in the form

$$dV^{v,\pi}(t) = V^{v,\pi}(t)\pi'(t)\sigma(t)\, d\widehat{W}(t), \qquad \widehat{W}(t) := W(t) + \int_0^t \vartheta(s)\, ds \tag{5.3}$$

on the strength of (5.1), the product rule of Itô's calculus shows that

$$Z(\cdot)V^{v,\pi}(\cdot) = v + \int_0^{\cdot} Z(t)V^{v,\pi}(t)\big(\sigma'(t)\pi(t) - \vartheta(t)\big)'\, dW(t) \tag{5.4}$$

is a positive local martingale and a supermartingale, for every $\pi(\cdot) \in \mathcal{H}$.

If $\alpha(\cdot)$ is invertible, we can take $\vartheta(\cdot) = \sigma'(\cdot)\alpha^{-1}(\cdot)\beta(\cdot)$ as market price of risk in (5.1). If $\beta(\cdot) = \alpha(\cdot)\mu(\cdot)$ holds we can select $\vartheta(\cdot) = \sigma'(\cdot)\mu(\cdot)$ and get $Z(\cdot) \equiv v/V^{v,\mu}(\cdot) \equiv X(0)/X(\cdot)$ from (5.4); there is then no arbitrage relative to the market because $V^{v,\pi}(\cdot)/V^{v,\mu}(\cdot)$ *is a supermartingale for all* $\pi(\cdot) \in \mathcal{H}$; thus $\mathbb{E}[V^{1,\pi}(T)/V^{1,\mu}(T)] \le 1$, a conclusion at odds with (4.1).

5.1. *Strict local martingales*. Suppose the covariance process $\alpha(\cdot)$ is bounded, and (4.1) holds for two bounded portfolios $\pi(\cdot)$ and $\rho(\cdot)$. Then, for *any* market-price-of-risk process $\vartheta(\cdot)$ as in (5.1), the positive local martingales $Z(\cdot)$ and $Z(\cdot)V^{v,\rho}(\cdot)$ of (5.2), (5.4) are strict: $\mathbb{E}[Z(T)V^{v,\rho}(T)] < v$, $\mathbb{E}(Z(T)) < 1$ [Fernholz and Karatzas (2009), Section 6].

In particular, if the matrix $\alpha(\cdot)$ is bounded, and (4.1) holds for some bounded portfolio $\pi(\cdot)$ and for the market portfolio $\rho(\cdot) \equiv \mu(\cdot)$ (these assumptions are satisfied, e.g., under the conditions in Remark 1), then

$$\begin{aligned} &\mathbb{E}(Z(T)) < 1, \qquad \mathbb{E}[Z(T)X(T)] < X(0), \\ &\mathbb{E}[Z(T)X_i(T)] < X_i(0), \qquad i = 1, \dots, n. \end{aligned} \tag{5.5}$$

**6. Optimal arbitrage relative to the market.** The possibility of strong arbitrage relative to the market, defined and exemplified in Section 4, raises an obvious question: *what is the best possible arbitrage of this kind*?

One way to cast this question is as follows: on a given time–horizon $[0, T]$, what is the *smallest* relative amount,

$$\mathfrak{u}(T) := \inf\{w > 0 \,|\, \exists\, \pi(\cdot) \in \mathcal{H} \text{ s.t. } V^{wX(0),\pi}(T) \ge X(T),\ \text{a.s.}\}, \tag{6.1}$$

of initial capital, starting with which one can match or exceed at time $t = T$ the market capitalization $X(T)$? Clearly, $0 < \mathfrak{u}(T) \le 1$; and for $0 < w < \mathfrak{u}(T)$, no strategy starting with initial capital $wX(0)$ can outperform the market almost surely, over the horizon $[0, T]$. That is, for every $\pi(\cdot) \in \mathcal{H}$ and $0 < w < \mathfrak{u}(T)$, we have $\mathbb{P}[V^{wX(0),\pi}(T) \ge X(T)] < 1$.

We shall impose from now on the following structural assumptions on the filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$, the "flow of information" in the market.

ASSUMPTION A. Every local martingale of the filtration $\mathbb{F}$ can be represented as a stochastic integral, with respect to the driving Brownian motion $W(\cdot)$ in (2.1), of some $\mathbb{G}$-progressively measurable integrand.

ASSUMPTION B. We have $K = n$, and $\sigma(t)$ is invertible, $\forall t \in [0, T]$.

Under these two assumptions, general results about hedging in so-called complete markets [e.g., Karatzas and Shreve (1998), Fernholz and Karatzas (2009), Section 10 or Ruf (2009)] based on martingale representation results, show that the quantity of (6.1) given as

$$(6.2) \qquad \mathfrak{u}(T) = \mathbb{E}[Z(T)X(T)]/X(0); \qquad \text{that } V^{\mathfrak{u}(T)X(0),\widehat{\pi}}(T) = X(T)$$

holds a.s. for some $\widehat{\pi}(\cdot) \in \mathcal{H}$; and that $1/\mathfrak{u}(T)$ gives the highest return,

$$(6.3) \qquad \sup\{q \geq 1 \,|\, \exists\, \pi(\cdot) \in \mathcal{H} \text{ s.t. } V^{1,\pi}(T) \geq q V^{1,\mu}(T), \text{ a.s.}\},$$

on investment, that one can achieve relative to the market over $[0, T]$. Arbitrage relative to the market is possible on $[0, T]$, if and only if $\mathfrak{u}(T) < 1$.

The result in (6.2) provides no information about the strategy $\widehat{\pi}(\cdot)$ that implements this "best possible" arbitrage, apart from ascertaining its existence. In Section 8 we shall specialize the model of (2.1) to a Markovian context and describe $\widehat{\pi}(\cdot)$ in terms of partial differential equations (Section 11). We shall also characterize the quantity $\mathfrak{u}(T)$ in terms of the smallest solution to a parabolic partial differential inequality, and as the probability of nonabsorption by time $T$ for a suitable diffusion (Theorems 1, 2).

Assumption A holds when $\mathbb{F}$ is (the augmentation of) $\mathbb{F}^W$, the filtration generated by the Brownian motion $W(\cdot)$; as well as when Assumption B holds, the $\beta_i(\cdot), \sigma_{i\nu}(\cdot)$ are all progressively measurable with respect to $\mathbb{F}^{\mathfrak{X}} = \{\mathcal{F}^{\mathfrak{X}}(t)\}_{0 \leq t < \infty}$, $\mathcal{F}^{\mathfrak{X}}(t) := \sigma(\mathfrak{X}(s), 0 \leq s \leq t)$, and $\mathbb{F} \equiv \mathbb{F}^{\mathfrak{X}}_{+}$ [Jacod (1977)].

6.1. *Generalized likelihood ratios*. The positive local martingale $Z(\cdot)X(\cdot)$, whose expectation appears in (6.2), can be expressed as

$$(6.4) \qquad Z(t)X(t) = X(0) \cdot \exp\left\{ -\int_0^t (\widetilde{\vartheta}(s))' \, dW(s) - \frac{1}{2}\int_0^t \|\widetilde{\vartheta}(s)\|^2 \, ds \right\}$$

for $0 \leq t \leq T$. Here we have solved equation (5.4) for $\pi(\cdot) \equiv \mu(\cdot)$ and set

$$(6.5) \qquad \widetilde{\vartheta}(\cdot) := \vartheta(\cdot) - \sigma'(\cdot)\mu(\cdot), \qquad \widetilde{W}(\cdot) := W(\cdot) + \int_0^{\cdot} \widetilde{\vartheta}(t)\, dt,$$

whence $\sigma(\cdot)\widetilde{\vartheta}(\cdot) = \beta(\cdot) - \alpha(\cdot)\mu(\cdot)$ from (5.1); we thus re-cast (2.1) as

$$(6.6) \qquad dX_i(t) = X_i(t)\left[ \frac{\sum_{j=1}^n \alpha_{ij}(t) X_j(t)}{X_1(t) + \cdots + X_n(t)} \, dt + \sum_{k=1}^n \sigma_{ik}(t)\, d\widetilde{W}_k(t) \right].$$

On the other hand, we note from (6.4), (6.5) that the reciprocal of the exponential local martingale $Z(\cdot)X(\cdot)/X(0)$ can be expressed as

$$\Lambda(\cdot) := \frac{X(0)}{Z(\cdot)X(\cdot)} = \exp\left\{\int_0^\cdot (\widetilde{\vartheta}(t))'\,d\widetilde{W}(t) - \frac{1}{2}\int_0^\cdot \|\widetilde{\vartheta}(t)\|^2\,dt\right\}; \tag{6.7}$$

similarly, the reciprocal of the local martingale $Z(\cdot)X_i(\cdot)/X_i(0)$ is

$$\Lambda_i(\cdot) := \frac{X_i(0)}{Z(\cdot)X_i(\cdot)} = \exp\left\{\int_0^\cdot (\widetilde{\vartheta}^{(i)}(t))'\,d\widetilde{W}^{(i)}(t) - \frac{1}{2}\int_0^\cdot \|\widetilde{\vartheta}^{(i)}(t)\|^2\,dt\right\}, \tag{6.8}$$

where $\widetilde{\vartheta}^{(i)}(\cdot) := \vartheta(\cdot) - \sigma'(\cdot)\mathfrak{e}_i$ and $\widetilde{W}^{(i)}(\cdot) := W(\cdot) + \int_0^\cdot \widetilde{\vartheta}^{(i)}(t)\,dt$.

Comparing (6.7) and (6.8), we observe that $\mu_i(0)\Lambda(\cdot) = \mu_i(\cdot)\Lambda_i(\cdot)$ and cast the dynamics of (3.3) and (3.4) for the market portfolio $\mu(\cdot)$ as

$$d\mu_i(t) = \mu_i(t)\big(\mathfrak{e}_i - \mu(t)\big)'\sigma(t)\,d\widetilde{W}(t), \qquad i = 1, \dots, n. \tag{6.9}$$

If $\mathfrak{u}(T) = 1$, that is, $Z(\cdot)X(\cdot)$ is a martingale on $[0, T]$, no arbitrage relative to the market is possible on this time–horizon; the "reference" measure

$$\widetilde{\mathbb{P}}_T(A) := \mathbb{E}[Z(T)X(T)1_A]/X(0), \qquad A \in \mathcal{F}(T), \tag{6.10}$$

is a probability, that is, $\mathfrak{u}(T) = \widetilde{\mathbb{P}}_T(\Omega) = 1$; and under $\widetilde{\mathbb{P}}_T$, the process $\widetilde{W}(t), 0 \le t \le T$, in (6.5) is a Brownian motion by the Girsanov theorem, so from (6.9) the market weights $\mu_1(t), \dots, \mu_n(t), 0 \le t \le T$ are martingales.

We shall characterize next $\mathfrak{u}(T)$ in terms of the *Föllmer exit measure*, of a "generalized martingale measure" and of a measure $\mathbb{Q}$ with respect to which $\mathbb{P}$ is locally absolutely continuous [equations (7.3), (7.6)] and which plays, to a considerable extent, the rôle of $\widetilde{\mathbb{P}}_T$ when $Z(\cdot)X(\cdot)$ fails to be a $\mathbb{P}$-martingale. The processes of (6.4)–(6.8) are important in this effort.

**7. Exit measure of a positive supermartingale.** We shall assume in this section that the process $Z(\cdot)$ of (5.2) is adapted to $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ and that this filtration is, in turn, the right-continuous version $\mathcal{F}(t) = \bigcap_{\varepsilon>0}\mathcal{F}^o(t+\varepsilon)$ of a *standard system* $\mathbb{F}^o = \{\mathcal{F}^o(t)\}_{0\le t<\infty}$: to wit, each $(\Omega, \mathcal{F}^o(t))$ is isomorphic to the Borel $\sigma$-algebra of some Polish space, and for any decreasing sequence $\{A_j\}_{j\in\mathbb{N}}$ such that $A_j$ is an atom of $\mathcal{F}^o(t_j)$, for some increasing sequence $\{t_j\}_{j\in\mathbb{N}} \subset [0, \infty)$, we have $\bigcap_{j\in\mathbb{N}} A_j \ne \varnothing$.

The canonical example is the space $\Omega$ of right-continuous paths $\omega : [0, \infty) \to \mathbb{R}^n \cup \{\Delta\}$, where $\Delta$ is an additional "absorbing point"; paths stay at $\Delta$ once they get there, that is, after $\mathcal{T}(\omega) = \inf\{t \ge 0 | \omega(t) = \Delta\}$, and are continuous on $(0, \mathcal{T}(\omega))$. If $\mathcal{F}^o(t) = \sigma(\omega(s), 0 \le s \le t)$, then $\mathbb{F}^o = \{\mathcal{F}^o(t)\}_{0\le t<\infty}$ is a standard system [see Föllmer (1972), the Appendix].

Under these conditions, we can associate to the $(\mathbb{P}, \mathbb{F})$-local martingale $Z(\cdot) \cdot X(\cdot)$ a positive measure $\mathfrak{P}$ on the predictable $\sigma$-algebra of $[0, \infty] \times \Omega$,

$$\mathfrak{P}\big((T, \infty] \times A\big) := \mathbb{E}[Z(T)X(T)1_A]/X(0), \qquad A \in \mathcal{F}(T), T \in [0, \infty),$$

by invoking an extension result [Parthasarathy (1967), Theorem V.4.1, whence the assumptions on the nature of the probability space].

This is the "exit measure" of the supermartingale $Z(\cdot)X(\cdot)$, introduced by Föllmer (1972, 1973) [see also Delbaen and Schachermayer (1995a), Föllmer and Gundel (2006)]. Föllmer (1972) obtained a characterization of the (process-theoretic) properties of supermartingales, such as $Z(\cdot)X(\cdot)$ here, in terms of the properties of $\mathfrak{P}$. It follows from his work that $Z(\cdot)X(\cdot)$ is a:

- *martingale*, if and only if $\mathfrak{P}$ in concentrated on $\{\infty\} \times \Omega$;
- *potential* [i.e., $\mathfrak{u}(\infty) = 0$], if and only if $\mathfrak{P}$ in concentrated on $(0, \infty) \times \Omega$.

7.1. *A representation of the Föllmer measure.* From Theorem 4 in Delbaen and Schachermayer (1995a) and Theorem 1 and Lemma 4 of Pal and Protter (2007), the process $\Lambda(\cdot)$ of (6.7) is a continuous martingale under some probability measure $\mathbb{Q}$ on the filtered space $(\Omega, \mathcal{F})$, $\mathbb{F}$ as above. The measure $\mathbb{P}$ is locally absolutely continuous with respect to $\mathbb{Q}$, with $d\mathbb{P} = \Lambda(T)\, d\mathbb{Q}$ on each $\mathcal{F}(T)$; and the process $\widetilde{W}(\cdot)$ of (6.5) is $\mathbb{Q}$-Brownian motion [cf. Ruf (2009), Section 5]. Thus, from (6.9) *the weights* $\mu_1(\cdot), \dots, \mu_n(\cdot)$ *are martingales* and satisfy $\sum_{i=1}^n \mu_i(\cdot) \equiv 1$ a.e., under $\mathbb{Q}$.

We consider the first time the process $\Lambda(\cdot)$ hits the origin,

$$\mathcal{T} := \inf\{t \geq 0 | \Lambda(t) = 0\} = \inf\{t \geq 0 | Z(t)X(t) = \infty\} \tag{7.1}$$

(infinite, if the set is empty). We have $\mathbb{P}(\mathcal{T} < \infty) = 0$, but $\mathbb{Q}(\mathcal{T} < \infty)$ can be positive, so $\mathbb{Q}$ may not be absolutely continuous with respect to $\mathbb{P}$; whereas, $\mathbb{Q}$-a.e. on $\{\mathcal{T} < \infty\}$, we have $Z(\mathcal{T}+h)X(\mathcal{T}+h) = \infty$, $\forall h \geq 0$ and $\int_0^{\mathcal{T}} \|\widetilde{\vartheta}(t)\|^2\, dt = \infty$. Intuitively, the role of the absorbing state $\Delta$ is to account for events that have zero $\mathbb{P}$-measure, but positive $\mathbb{Q}$-measure. We also introduce the first times the processes $\mu_i(\cdot)$ and $\Lambda_i(\cdot)$ hit the origin,

$$\mathcal{T}_i := \inf\{t \geq 0 | \mu_i(t) = 0\}, \qquad \widetilde{\mathcal{T}}_i := \inf\{t \geq 0 | \Lambda_i(t) = 0\}. \tag{7.2}$$

PROPOSITION 1. (i) *The quantity of* (6.1) *can be represented as*

$$\mathfrak{u}(T) = \mathfrak{P}\big((T, \infty] \times \Omega\big) = \mathbb{Q}(\mathcal{T} > T). \tag{7.3}$$

(ii) *Suppose* $n \geq 2$ *and that all capitalizations* $X_1(\cdot), \dots, X_n(\cdot)$ *are real-valued* $\mathbb{Q}$-*a.e. Then we also have the* $\mathbb{Q}$-*a.e. representations*

$$\mathcal{T} = \min_{1 \leq i \leq n} \widetilde{\mathcal{T}}_i; \quad \text{as well as} \quad \mathcal{T} = \min_{1 \leq i \leq n} \mathcal{T}_i \qquad \text{away from the event } E, \tag{7.4}$$

*where* $E := \{\mathcal{T} < \infty\} \cap \{\mu_1(\mathcal{T}) \cdots \mu_n(\mathcal{T}) > 0\}$. *This event has* $\mathbb{Q}$-*measure equal to zero, if for some real constant* $C > 0$ *we have*

$$\|\vartheta(t, \omega)\|^2 \leq C\big(1 + \mathrm{Tr}(\alpha(t, \omega))\big) \qquad \forall (t, \omega) \in [0, \infty) \times \Omega. \tag{7.5}$$

PROOF. We note $\mathfrak{P}((T,\infty] \times A) = \mathbb{E}^{\mathbb{P}}(\Lambda^{-1}(T)1_{A\cap\{\mathcal{T}>T\}}) = \mathbb{E}^{\mathbb{Q}}(\Lambda(T) \cdot \Lambda^{-1}(T)1_{A\cap\{\mathcal{T}>T\}}) = \mathbb{Q}(A\cap\{\mathcal{T}>T\})$, $\forall A \in \mathcal{F}(T)$. With $A=\Omega$, we get (7.3). For $A=\{\mu_1(T)\cdots\mu_n(T)=0\}$, this gives $\mathbb{Q}(A\cap\{\mathcal{T}>T\})=0$: all the weights $\mu_1(\cdot),\dots,\mu_n(\cdot)$ are strictly positive [equivalently, all $X_1(\cdot),\dots,X_n(\cdot)$ take values in $(0,\infty)$] on $[0,\mathcal{T})$, $\mathbb{Q}$-a.e.

Recall $\mu_i(0)\Lambda(\cdot) \equiv \mu_i(\cdot)\Lambda_i(\cdot)$, $\forall i=1,\dots,n$ from (6.7), (6.8); this gives $1/\Lambda(\cdot)=\sum_{i=1}^n(\mu_i(0)/\Lambda_i(\cdot))$ on $[0,\mathcal{T})$, and the first equation in (7.4).

On the event $\{\mathcal{T}<\infty\}\setminus E$, for some $j\in\{1,\dots,n\}$ we shall have $\mu_j(\mathcal{T})=0$, thus also $\mathcal{T}_j=\mathcal{T}$ and the second equation in (7.4). On the other hand, we have seen that $\mathcal{T}_i=\infty$, $\forall i=1,\dots,n$ holds $\mathbb{Q}$-a.e. on $\{\mathcal{T}=\infty\}$, so this equation is valid on $\{\mathcal{T}=\infty\}$.

Finally, from (6.6), (6.7): $\int_0^{\mathcal{T}} \mathrm{Tr}(\alpha(t,\omega))\,dt<\infty$, $\int_0^{\mathcal{T}}\|\widetilde{\vartheta}(t,\omega)\|^2\,dt=\infty$ for $\mathbb{Q}$-a.e. $\omega\in E\subseteq\{\mathcal{T}<\infty\}$. Then (7.5) implies $\int_0^{\mathcal{T}}\|\vartheta(t,\omega)\|^2\,dt<\infty$, and $\widetilde{\vartheta}(\cdot)=\vartheta(\cdot)-\sigma'(\cdot)\mu(\cdot)$ gives $\int_0^{\mathcal{T}}\|\widetilde{\vartheta}(t,\omega)\|^2\,dt<\infty$, thus $\mathbb{Q}(E)=0$. $\square$

Equation (7.3) can be thought of as a "generalized Wald identity" [cf. Problem 3.5.7 in Karatzas and Shreve (1991)]. In Section 9.3 we shall obtain a characterization of the type (7.3) in a Markovian context, in terms of properties of an auxiliary diffusion and with the help of an appropriate partial differential equation. This will enable us to describe the investment strategy that realizes the optimal arbitrage.

7.2. *A generalized martingale measure.* In a similar vein, there exists on the filtered space $(\Omega,\mathcal{F})$, $\mathbb{F}$ a probability measure $\widehat{\mathbb{Q}}$ under which

$$L(t):=1/Z(t)=\exp\biggl\{\int_0^t\vartheta'(s)\,d\widehat{W}(s)-\frac{1}{2}\int_0^t\|\vartheta(s)\|^2\,ds\biggr\},\qquad 0\le t<\infty,$$

is a martingale, and $d\mathbb{P}=L(T)\,d\widehat{\mathbb{Q}}$ on each $\mathcal{F}(T)$, whereas $\widehat{W}(\cdot)$ of (5.3) is $\widehat{\mathbb{Q}}$-Brownian motion. Under $\widehat{\mathbb{Q}}$, the processes $X_i(\cdot)$, $i=1,\dots,n$ are nonnegative local (and super-)martingales, $dX_i(t)=X_i(t)\sum_{k=1}^K\sigma_{ik}(t)\,d\widehat{W}_k(t)$. This justifies the appellation "generalized martingale measure" for $\widehat{\mathbb{Q}}$.

Defining $\mathcal{S}:=\inf\{t\ge 0|L(t)=0\}$, we have $\mathbb{P}(\mathcal{S}<\infty)=0$ and $Z(\cdot)$ *is a strict* $\mathbb{P}$*-local martingale if and only if* $\widehat{\mathbb{Q}}(\mathcal{S}<\infty)>0$ [a potential, if and only if $\widehat{\mathbb{Q}}(\mathcal{S}<\infty)=1$]; and the expression of (6.1), (6.2) is

$$\mathfrak{u}(T)=\mathbb{E}^{\widehat{\mathbb{Q}}}\bigl[\bigl(X(T)/X(0)\bigr)1_{\{\mathcal{S}>T\}}\bigr]. \tag{7.6}$$

This last expression takes the form $\mathfrak{u}(T)=1-\mathbb{E}^{\widehat{\mathbb{Q}}}[(X(\mathcal{S})/X(0))1_{\{\mathcal{S}\le T\}}]$ when $X(\cdot\wedge T)$ is a $\widehat{\mathbb{Q}}$-martingale; from (5.3), this will be the case under the Novikov condition $\mathbb{E}^{\widehat{\mathbb{Q}}}[\exp\{\frac{1}{2}\int_0^T\mu'(t)\alpha(t)\mu(t)\,dt\}]<\infty$. Moreover, $\mathfrak{u}(T)=1$ (no arbitrage relative to the market is possible on $[0,T]$), if and only if: $X(\cdot\wedge T)$ *is a* $\widehat{\mathbb{Q}}$*-martingale, and* $X(\mathcal{S})1_{\{\mathcal{S}\le T\}}=0$ *holds* $\widehat{\mathbb{Q}}$*-a.e.*

**8. A diffusion model.** We shall assume from now on that $K = n$ and that the processes $\beta_i(\cdot), \sigma_{ik}(\cdot)$, $1 \le i, k \le n$ in (2.1) are of the form

$$\beta_i(t) = \mathrm{b}_i(\mathfrak{X}(t)), \qquad \sigma_{ik}(t) = \mathrm{s}_{ik}(\mathfrak{X}(t)), \qquad 0 \le t < \infty. \tag{8.1}$$

Here $\mathfrak{X}(t) = (X_1(t), \dots, X_n(t))'$ is the vector of capitalizations at time $t$, and $\mathrm{b}_i : (0,\infty)^n \to \mathbb{R}$, $\mathrm{s}_{ik} : (0,\infty)^n \to \mathbb{R}$ are continuous functions. We shall denote by $\mathrm{b}(\cdot) = (\mathrm{b}_1(\cdot), \dots, \mathrm{b}_n(\cdot))'$ and $\mathrm{s}(\cdot) = (\mathrm{s}_{ik}(\cdot))_{1 \le i \le n, 1 \le k \le n}$ the vector and matrix, respectively, of these local rate-of-return and local volatility functions. With this setup, the vector process $\mathfrak{X}(t), 0 \le t < \infty$ of capitalizations becomes a diffusion, with values in $(0,\infty)^n$ and dynamics

$$dX_i(t) = \mathfrak{b}_i(\mathfrak{X}(t))\,dt + \sum_{k=1}^{n} \mathfrak{s}_{ik}(\mathfrak{X}(t))\,dW_k(t), \qquad i = 1, \dots, n, \tag{8.2}$$

where for $\mathbf{x} = (x_1, \dots, x_n)' \in (0,\infty)^n$ we set $\mathrm{a}_{ij}(\mathbf{x}) := \sum_{k=1}^n \mathrm{s}_{ik}(\mathbf{x})\mathrm{s}_{jk}(\mathbf{x})$,

$$\mathfrak{b}_i(\mathrm{x}) := x_i \mathrm{b}_i(\mathbf{x}), \qquad \mathfrak{s}_{ik}(\mathbf{x}) := x_i \mathrm{s}_{ik}(\mathbf{x}), \qquad \mathfrak{a}_{ij}(\mathbf{x}) := x_i x_j \mathrm{a}_{ij}(\mathbf{x}). \tag{8.3}$$

This diffusion $\mathfrak{X}(\cdot)$ has infinitesimal generator

$$\mathcal{L}f := \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathfrak{a}_{ij}(\mathbf{x}) D_{ij}^2 f + \sum_{i=1}^{n} \mathfrak{b}_i(\mathbf{x}) D_i f. \tag{8.4}$$

ASSUMPTION C. For every $\mathbf{x} \in (0,\infty)^n$, the matrix $\mathrm{s}(\mathbf{x}) = (\mathrm{s}_{ij}(\mathbf{x}))_{1 \le i,j \le n}$ is invertible; the system (8.2) has a unique-in-distribution weak solution, with $\mathfrak{X}(0) = \mathbf{x}$ and values in $(0,\infty)^n$; and for $\Theta(\mathbf{x}) := \mathrm{s}^{-1}(\mathbf{x})\mathrm{b}(\mathbf{x})$, the following analogue of (2.2), (5.1) holds for each $T \in (0,\infty)$:

$$\sum_{i=1}^{n} \int_0^T \big( |\mathrm{b}_i(\mathfrak{X}(t))| + \mathrm{a}_{ii}(\mathfrak{X}(t)) + \Theta_i^2(\mathfrak{X}(t)) \big)\,dt < \infty \qquad \text{a.s.} \tag{8.5}$$

It follows from this assumption that the Brownian motion $W(\cdot)$ is adapted to the augmentation of the filtration $\mathbb{F}^{\mathfrak{X}}$, and that $\vartheta(\cdot) = \Theta(\mathfrak{X}(\cdot))$ is a market-price of risk process as postulated in (5.1). The following conditions from Bass and Perkins (2003), in particular their Theorem 1.2 and Corollary 1.3, are sufficient for the existence of a weak solution for (8.2) which is unique in distribution: the functions $\mathfrak{s}_{ik}(\cdot)$, $\mathfrak{b}_i(\cdot)$ of (8.3) can be extended by continuity on all of $[0,\infty)^n$; $\mathfrak{b}_i(\cdot)$ and $\mathfrak{h}_{ij}(\mathbf{x}) := \sqrt{x_i x_j}\,\mathrm{a}_{ij}(\mathbf{x}) > 0$ are Hölder continuous on compact subsets of $[0,\infty)^n$; and we have

$$\begin{aligned} &\mathfrak{b}_i(\mathbf{x}) \ge 0 \qquad \text{for } x_i = 0; \\ &\|\mathfrak{b}(\mathbf{x})\| + \|\mathfrak{s}(\mathbf{x})\| \le C(1 + \|\mathbf{x}\|) \qquad \forall\, \mathbf{x} \in [0,\infty)^n, \end{aligned} \tag{8.6}$$

and $\mathfrak{h}_{ij}(\mathbf{x}) = 0$ for $i \ne j$, $\mathbf{x} \in \mathcal{O}^n$, where $\mathcal{O}^n$ is the boundary of $[0,\infty)^n$.

REMARK 2. The diffusion $\mathfrak{X}(\cdot)$ of (8.2) takes values in $(0,\infty)^n$, if and only if the diffusion $\Xi(\cdot) = (\Xi_1(\cdot), \dots, \Xi_n(\cdot))'$, $\Xi_i(\cdot) := 1/X_i(\cdot)$, with dynamics

$$d\Xi_i(t) = \mathfrak{q}_i(\Xi(t))\,dt + \sum_{k=1}^{n} \mathfrak{r}_{ik}(\Xi(t))\,dW_k(t), \qquad i = 1, \dots, n, \tag{8.7}$$

and $\mathfrak{r}_{ik}(\xi) := -\xi_i \mathrm{s}_{ik}(1/\xi_1, \dots, 1/\xi_n)$, $\mathfrak{q}_i(\xi) := \xi_i(\mathrm{a}_{ii} - \mathrm{b}_i)(1/\xi_1, \dots, 1/\xi_n)$, takes values in $(0,\infty)^n$. Thus, any conditions guaranteeing the existence of a nonexplosive solution to the SDEs of (8.7) for all times, such as linear growth for $\mathfrak{q}_i(\cdot)$ and $\mathfrak{r}_{ik}(\cdot)$, also ensure that $\mathfrak{X}(\cdot)$ takes values in $(0,\infty)^n$.

Alternatively, one may invoke results of Friedman (2006), Section 9.4 and Chapter 11, to obtain conditions on $\mathfrak{b}_i(\cdot)$, $\mathfrak{s}_{ik}(\cdot)$ under which the diffusion $\mathfrak{X}(\cdot)$ of (8.2) never attains any of the faces $\{x_1 = 0\}, \dots, \{x_n = 0\}$ of $\mathcal{O}^n$. In particular, if these functions can be extended by continuity on all of $[0,\infty)^n$; the $\mathfrak{s}_{ik}(\cdot)$ are continuously differentiable; the matrix $\mathfrak{a}(\cdot)$ degenerates on the faces of the orthant; and the so-called *Fichera drifts*

$$\mathfrak{f}_i(\mathbf{x}) := \mathfrak{b}_i(\mathbf{x}) - \frac{1}{2}\sum_{j=1}^{n} D_j \mathfrak{a}_{ij}(\mathbf{x}) \tag{8.8}$$

are nonnegative on $\{x_i = 0\}$, for each $i = 1, \dots, n$; then $\mathfrak{X}(\cdot)$ takes values in $(0,\infty)^n$ [see Friedman (2006), Theorem 9.4.1 and Corollary 9.4.2].

ASSUMPTION D. There exists $H : (0,\infty)^n \to \mathbb{R}$ of class $\mathcal{C}^2$, such that

$$\mathfrak{b}(\mathbf{x}) = \mathfrak{a}(\mathbf{x}) DH(\mathbf{x}), \qquad \forall\, \mathbf{x} \in (0,\infty)^n. \tag{8.9}$$

In light of Assumption C, this new requirement amounts essentially to postulating that the vector field $\mathfrak{a}^{-1}(\cdot)\mathfrak{b}(\cdot)$ be conservative; it is imposed here for technical reasons (cf. discussion in Remark 3). Under it, the generator of (8.4) becomes $\mathcal{L}f(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathfrak{a}_{ij}(\mathbf{x})[\frac{1}{2}D^2_{ij} f(\mathbf{x}) + D_i f(\mathbf{x}) D_j H(\mathbf{x})]$, and we have

$$\Theta(\mathbf{x}) = \mathfrak{s}'(\mathbf{x}) DH(\mathbf{x}) \quad \text{and} \quad \mathfrak{s}(\mathbf{x})\Theta(\mathbf{x}) = \mathfrak{b}(\mathbf{x}), \qquad \mathbf{x} \in (0,\infty)^n. \tag{8.10}$$

*Throughout the remainder, Assumptions B, C, D will be in force, and $\mathbb{F} \equiv \mathbb{F}^{\mathfrak{X}}_{+}$; this is a natural choice, and consistent with Assumption A.*

**9. A parabolic PDE for the function $U(\tau, \mathbf{x})$.** The uniqueness in distribution posited in Assumption C implies that $\mathfrak{X}(\cdot)$ is strongly Markovian; we shall denote by $\mathbb{P}^{\mathbf{x}}$ the distribution of this process started at $\mathfrak{X}(0) = \mathbf{x} \in (0,\infty)^n$. Our objective now is to study

$$U(T, \mathbf{x}) := \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}[Z(T)X(T)]/(x_1 + \cdots + x_n), \tag{9.1}$$

the quantity of (6.1), (6.2) in this diffusion context. We start by observing that with $H(\cdot)$ as in Assumption D and the notation of (8.4) and (8.10), Itô's rule gives

$H(\mathfrak{X}(T)) - H(\mathfrak{X}(0)) - \int_0^T \mathcal{L}H(\mathfrak{X}(t))\,dt = \int_0^T \Theta'(\mathfrak{X}(t))\,dW(t)$, and the exponential local martingale $Z(\cdot)$ of (5.2) becomes

$$Z(\cdot) = \exp\left\{H(\mathfrak{X}(0)) - H(\mathfrak{X}(\cdot)) - \int_0^{\cdot} k(\mathfrak{X}(t))\,dt\right\}. \tag{9.2}$$

In particular, $Z(\cdot)$ is $\mathbb{F}^{\mathfrak{X}}$-adapted. We are setting here

$$k(\mathbf{x}) := -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathfrak{a}_{ij}(\mathbf{x})}{2}[D_{ij}^2 H(\mathbf{x}) + D_i H(\mathbf{x}) D_j H(\mathbf{x})], \tag{9.3}$$

$$g(\mathbf{x}) := e^{-H(\mathbf{x})}\sum_{i=1}^{n} x_i, \qquad G(T,\mathbf{x}) := \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\big[g(\mathfrak{X}(T))e^{-\int_0^T k(\mathfrak{X}(t))\,dt}\big]. \tag{9.4}$$

With this notation, the function of (9.1) becomes $U(T,\mathbf{x}) = G(T,\mathbf{x})/g(\mathbf{x})$. A bit more generally, these considerations—coupled with the Markov property of $\mathfrak{X}(\cdot)$—lead for any $0 \le t \le T$ to the a.s. identity

$$\frac{\mathbb{E}^{\mathbb{P}^{\mathbf{x}}}[X(T)Z(T)|\mathcal{F}(t)]}{X(t)Z(t)} = \frac{G(T-t,\mathbf{y})}{g(\mathbf{y})}\bigg|_{\mathbf{y}=\mathfrak{X}(t)} = U\big(T-t,\mathfrak{X}(t)\big). \tag{9.5}$$

The following Assumption E will also be imposed from now onward. *It amounts to assuming that the function $U(\cdot,\cdot)$ of (9.1) is of class $\mathcal{C}^{1,2}$.* Note that (9.6) is satisfied, at least in the support of $\mathfrak{X}(\cdot)$, thanks to the assumption $U(\cdot,\cdot) \in \mathcal{C}^{1,2}((0,\infty)\times(0,\infty)^n)$ and to the $\mathbb{P}^{\mathbf{x}}$-martingale property of the process $G(T-t,\mathfrak{X}(t))e^{-\int_0^t k(\mathfrak{X}(u))\,du}, 0 \le t \le T$.

ASSUMPTION E. The function $G(\cdot,\cdot)$ in (9.4) takes values in $(0,\infty)$, is continuous on $[0,\infty)\times(0,\infty)^n$, of class $\mathcal{C}^{1,2}$ on $(0,\infty)\times(0,\infty)^n$, and solves

$$\frac{\partial G}{\partial \tau}(\tau,\mathbf{x}) = \mathcal{L}G(\tau,\mathbf{x}) - k(\mathbf{x})G(\tau,\mathbf{x}), \qquad \tau \in (0,\infty), \mathbf{x} \in (0,\infty)^n, \tag{9.6}$$

$$G(0,\mathbf{x}) = g(\mathbf{x}), \qquad \mathbf{x} \in (0,\infty)^n. \tag{9.7}$$

This Cauchy problem is exactly the one arising in classical Feynman–Kac theory [see, for instance, Friedman (2006), Sections 5.6, 6.5, Karatzas and Shreve (1991), Section 5.7 and Janson and Tysk (2006)]. From Theorem 1 and the remark following it in Heath and Schweizer (2000), Assumption E holds if: the functions $\mathfrak{b}_i(\cdot)$, $\mathfrak{s}_{ik}(\cdot)$ of (8.3) are continuously differentiable on $(0,\infty)$ and satisfy the growth condition in (8.6); the functions $\mathfrak{a}_{ij}(\cdot)$ of (8.3) satisfy the nondegeneracy condition (9.14) below; the function $g(\cdot)$ in (9.4) is Hölder continuous, uniformly on compact subsets of $(0,\infty)^n$; the continuous function $k(\cdot)$ of (9.3) is bounded from below; and the function $G(\cdot,\cdot)$ in (9.4) is real-valued and continuous on $(0,\infty)\times(0,\infty)^n$. This latter requirement is satisfied, for instance, if the functions $\mathfrak{r}_{ik}(\cdot)$, $\mathfrak{q}_i(\cdot)$ in (8.7) obey linear growth conditions, and the function $\mathfrak{g}(\xi) := g(1/\xi_1, \dots, 1/\xi_n)$ has polynomial growth [see Karatzas and Shreve (1991), Problem 5.3.15, as well as Heath and Schweizer (2000), Lemma 2 (and the paragraph preceding it)].

Sustained computation shows then that the Cauchy problem of (9.6), (9.7) for $G(\cdot,\cdot)$, leads to a corresponding Cauchy problem for $U(\cdot,\cdot)$, namely

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{x}) = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \mathfrak{a}_{ij}(\mathbf{x}) D_{ij}^2 U(\tau, \mathbf{x}) + \sum_{i=1}^n \sum_{j=1}^n \frac{\mathfrak{a}_{ij}(\mathbf{x}) D_i U(\tau, \mathbf{x})}{x_1 + \cdots + x_n} \tag{9.8}$$

for $(\tau, \mathbf{x}) \in (0,\infty) \times (0,\infty)^n$; and $U(0, \mathbf{x}) = 1$ for $\mathbf{x} \in (0,\infty)^n$.

9.1. *An informal derivation of* (9.8). Rather than including the computations which lead from (9.6) to equation (9.8), we present here a rather simple, informal argument that we shall find useful also in the next subsection, in a more formal setting. We start by casting (6.4) as

$$\frac{d(X(t)Z(t))}{X(t)Z(t)} = \sum_{k=1}^n \left( \sum_{i=1}^n \mu_i(t)\sigma_{ik}(t) - \vartheta_k(t) \right) dW_k(t) = -\sum_{k=1}^n \widetilde{\Theta}_k(\mathfrak{X}(t))\, dW_k(t),$$

where, by analogy with (6.5), we have set

$$\widetilde{\Theta}_k(\mathbf{x}) := \Theta_k(\mathbf{x}) - \sum_{i=1}^n \left( \frac{x_i \mathfrak{s}_{ik}(\mathbf{x})}{x_1 + \cdots + x_n} \right), \qquad k = 1, \dots, n. \tag{9.9}$$

On the other hand, assuming that $U(\cdot,\cdot)$ of (9.1) is of class $\mathcal{C}^{1,2}$, we obtain from Itô's rule and with $R_k(\tau, \mathbf{x}) := \sum_{i=1}^n x_i \mathfrak{s}_{ik}(\mathbf{x}) D_i U(\tau, \mathbf{x})$, $k = 1, \dots, n$,

$$dU\big(T-t, \mathfrak{X}(t)\big) = \left( \mathcal{L}U - \frac{\partial U}{\partial \tau} \right)\big(T-t, \mathfrak{X}(t)\big)\, dt + \sum_{k=1}^n R_k\big(T-t, \mathfrak{X}(t)\big)\, dW_k(t).$$

The product rule of the stochastic calculus applied to the process

$$N(t) := X(t)Z(t)U\big(T-t, \mathfrak{X}(t)\big) = \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}[X(T)Z(T) | \mathcal{F}(t)] \tag{9.10}$$

of (9.5), leads then to

$$\begin{aligned}
\frac{dN(t)}{X(t)Z(t)} &= dU\big(T-t, \mathfrak{X}(t)\big) + U\big(T-t, \mathfrak{X}(t)\big) \frac{d(X(t)Z(t))}{X(t)Z(t)} \\
&\qquad - \sum_{k=1}^n R_k\big(T-t, \mathfrak{X}(t)\big) \widetilde{\Theta}_k(\mathfrak{X}(t))\, dt \\
&= C\big(T-t, \mathfrak{X}(t)\big)\, dt \\
&\qquad + \sum_{k=1}^n \big[ R_k\big(T-t, \mathfrak{X}(t)\big) - U\big(T-t, \mathfrak{X}(t)\big) \widetilde{\Theta}_k(\mathfrak{X}(t)) \big]\, dW_k(t).
\end{aligned}$$

We have set

$$
\begin{aligned}
C(\tau,\mathbf{x}) &:= \left(\mathcal{L}U - \frac{\partial U}{\partial \tau}\right)(\tau,\mathbf{x}) - \sum_{k=1}^{n} R_k(\tau,\mathbf{x})\widetilde{\Theta}_k(\mathbf{x}) \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathfrak{a}_{ij}(\mathbf{x}) D_{ij}^2 U(\tau,\mathbf{x}) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathfrak{a}_{ij}(\mathbf{x}) D_i U(\tau,\mathbf{x})}{x_1+\cdots+x_n} - \frac{\partial U}{\partial \tau}(\tau,\mathbf{x}),
\end{aligned}
$$

where the last equality is checked easily from (8.4) and (8.10). But the process $N(\cdot)$ of (9.10) is a martingale, so the term $C(\tau,\mathbf{x})$ should vanish, and

$$
\frac{dN(t)}{N(t)} = \sum_{k=1}^{n}\left[\frac{R_k(T-t,\mathfrak{X}(t))}{U(T-t,\mathfrak{X}(t))} - \widetilde{\Theta}_k(\mathfrak{X}(t))\right] dW_k(t). \tag{9.11}
$$

In other words, the function $U:[0,\infty)\times(0,\infty)^n \to (0,1]$ of (9.3) must satisfy the parabolic partial differential equation (9.8), as postulated earlier.

REMARK 3. This informal derivation suggests that it may be possible to dispense with Assumptions D, E altogether, if it can be shown from first principles that the function $U$ of (9.1) is of class $\mathcal{C}^{1,2}$ on $(0,\infty)\times(0,\infty)^n$. Indeed, under suitable conditions, one can rely on techniques from the Malliavin calculus and the Hörmander hypoëllipticity theorem [Nualart (1995), pages 99–124] to show that the $(n+2)$-dimensional vector $(\mathfrak{X}(T),\Upsilon(T),\Xi(T))$ with $\Upsilon(T) := \int_0^T \Theta(\mathfrak{X}(t))'\,dW(t)$ and $\Xi(T) := \int_0^T \|\Theta(\mathfrak{X}(t))\|^2\,dt$ has an infinitely differentiable probability density function, for any given $T\in(0,\infty)$. This provides the requisite smoothness for the function

$$
U(T,\mathbf{x}) = \frac{1}{x_1+\cdots+x_n}\mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\big[\big(X_1(T)+\cdots+X_n(T)\big)e^{\Upsilon(T)-(\Xi(T)/2)}\big].
$$

The conditions needed for this approach to work are strong; they include the infinite differentiability of the functions $\mathfrak{s}_{ik}(\cdot)$, $\Theta_i(\cdot)$, $1\le i,k\le n$, as well as additional algebraic conditions which, in the present context, are somewhat opaque and not very easy to state or verify. For these reasons we have opted for sticking with Assumptions D, E; these are satisfied in the Examples of Section 12, are easy to test and allow us to represent Föllmer's exit measure via (9.23), (9.24) without involving stochastic integrals.

9.2. *Results and ramifications.* Equation (9.8) is determined *entirely* from the volatility structure of model (2.1). Furthermore, the Cauchy problem of (9.8), $U(0,\cdot)=1$, admits the trivial solution $U(\tau,\mathbf{x})\equiv 1$; thus, the existence of arbitrage relative to the market portfolio over a finite time–horizon $[0,T]$ is tantamount to *failure of uniqueness* for the Cauchy problem of (9.8), $U(0,\cdot)=1$ over the strip $[0,T]\times(0,\infty)^n$.

REMARK 4. Assume there exists some $h > 0$ such that the continuous functions $\mathrm{a}_{ij}(\cdot)$, $1 \le i, j \le n$ satisfy either of the conditions

$$(x_1 + \cdots + x_n) \sum_{i=1}^{n} x_i \mathrm{a}_{ii}(\mathbf{x}) - \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \mathrm{a}_{ij}(\mathbf{x}) \ge h(x_1 + \cdots + x_n)^2, \tag{9.12}$$

$$(x_1 \cdots x_n)^{1/n} \left[ \sum_{i=1}^{n} \mathrm{a}_{ii}(\mathbf{x}) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{a}_{ij}(\mathbf{x}) \right] \ge h(x_1 + \cdots + x_n) \tag{9.13}$$

for all $\mathbf{x} \in (0, \infty)^n$ [we have just re-written (4.2) and (4.3) in the present context]. Then from the results reviewed in Section 4 we deduce that, for all $T > (2 \log n)/h$ under (9.12), and for all $T > (2n^{1-(1/n)})/h$ under (9.13), we have $U(T, \mathbf{x}) < 1, \forall \mathbf{x} \in (0, \infty)^n$. In particular, under either (9.12) or (9.13), *uniqueness fails for the Cauchy problem of* (9.8), $U(0, \cdot) \equiv 1$.

Whenever uniqueness fails for this problem, it is important to know how to pick the "right" solution from among all possible solutions, the one which gives the quantity of (9.1). The next result addresses this issue; it implies that $G(\cdot, \cdot)$ in (9.4) is the *smallest nonnegative*, continuous function, of class $\mathcal{C}^{1,2}((0, \infty) \times (0, \infty)^n)$, which satisfies $(\partial G/\partial \tau) \ge \mathcal{L}G - kG$ and (9.7) [cf. Karatzas and Shreve (1991), Exercise 4.4.7 for a similar situation].

THEOREM 1. *The function* $U : [0, \infty) \times (0, \infty)^n \to (0, 1]$ *of* (9.1) *is the* smallest nonnegative *continuous function*, *of class* $\mathcal{C}^{1,2}$ *on* $(0, \infty) \times (0, \infty)^n$, *that satisfies* $U(0, \cdot) \equiv 1$ *and* (1.1).

PROOF. Consider any continuous function $\widetilde{U} : [0, \infty) \times (0, \infty)^n \to [0, \infty)$ which is of class $\mathcal{C}^{1,2}$ on $(0, \infty) \times (0, \infty)^n$, and satisfies (1.1) and $\widetilde{U}(0, \cdot) \equiv 1$ on $(0, \infty)^n$; we shall denote by $\mathfrak{U}$ the collection of all such functions. We introduce $\widetilde{N}(t) := X(t)Z(t)\widetilde{U}(T - t, \mathfrak{X}(t)), 0 \le t \le T$ as in (9.10).

Repeating verbatim the arguments in Section 9.1, we use (1.1) to conclude that the nonnegative process $\widetilde{N}(\cdot)$ is a local supermartingale. Thus $\widetilde{N}(\cdot)$ is bona-fide supermartingale, $(x_1 + \cdots + x_n)\widetilde{U}(T, \mathbf{x}) = \widetilde{N}(0) \ge \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}(\widetilde{N}(T)) = \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}(X(T)Z(T))$ holds for every $(T, \mathbf{x}) \in (0, \infty) \times (0, \infty)^n$, and $\widetilde{U}(T, \mathbf{x}) \ge U(T, \mathbf{x})$ follows from (9.1). $\square$

PROPOSITION 2. *Assume that the continuous functions* $(\mathfrak{a}_{ij}(\cdot))_{1 \le i, j \le n}$ *of* (8.3) *satisfy the following nondegeneracy condition*: *for every compact subset* $\mathcal{K}$ *of* $(0, \infty)^n$, *there exists a number* $\varepsilon = \varepsilon_{\mathcal{K}} > 0$ *such that*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathfrak{a}_{ij}(\mathbf{z}) \xi_i \xi_j \ge \varepsilon \|\xi\|^2, \qquad \forall \mathbf{z} \in \mathcal{K}, \xi \in \mathbb{R}^n. \tag{9.14}$$

*Then*, *if*

$$U(T, \mathbf{x}) < 1 \qquad \textit{for some } \mathbf{x} \in (0, \infty)^n \tag{9.15}$$

*holds for some* $T \in (0, \infty)$, *we have*

$$U(T, \mathbf{x}) < 1, \qquad \forall (T, \mathbf{x}) \in (0, \infty) \times (0, \infty)^n. \tag{9.16}$$

PROOF. Let us work first under the stronger assumption

$$U(T, \mathbf{x}) < 1, \qquad \forall \mathbf{x} \in (0, \infty)^n, \tag{9.17}$$

for some $T \in (0, \infty)$. For every $\tau > 0$, we consider the set $\mathcal{S}(\tau) := \{\mathbf{x} \in (0, \infty)^n | U(\tau, \mathbf{x}) = 1\}$ and define $\tau_* := \sup\{\tau \in (0, \infty) | \mathcal{S}(\tau) \neq \varnothing\}$ (with $\tau_* = 0$ if the set is empty). Assumption (9.17) amounts to $\tau_* < \infty$, and the claim (9.16) to $\tau_* = 0$; we shall prove this claim by contradiction.

Suppose $\tau_* > 0$; then $U(\tau_* - \delta, \mathbf{x}_*) = 1$ for any given $\delta \in (0, \tau_*/2)$, and some $\mathbf{x}_* \in (0, \infty)^n$. For *any* given $\mathbf{x} \in (0, \infty)^n$, consider an open, connected set $D$ which contains both $\mathbf{x}$ and $\mathbf{x}_*$, and whose closure $\overline{D}$ is a compact subset of $(0, \infty)^n$; in particular, we have $\inf\{\|\mathbf{y} - \mathbf{z}\| | \mathbf{z} \in \overline{D}, \mathbf{y} \in \mathcal{O}^n\} > 0$. The function $U(\cdot, \cdot)$ attains its maximum value over the cylindrical domain $\mathfrak{C} = \{(\tau, \xi) | 0 < \tau < \tau_* + 1, \xi \in D\}$ at the point $(\tau_* - \delta, \mathbf{x}_*)$, which lies in the interior of this domain. By assumption then, the operator $\widehat{\mathcal{L}} f = (1/2) \sum_{i=1}^n \sum_{j=1}^n \mathfrak{a}_{ij}(\mathbf{x}) D_{ij}^2 f + \sum_{i=1}^n \widehat{\mathfrak{b}}_i(\mathbf{x}) D_i f$ of (1.2) with

$$\widehat{\mathfrak{b}}_i(\mathbf{x}) := x_i \widehat{\mathrm{b}}_i(\mathbf{x}), \qquad \widehat{\mathrm{b}}_i(\mathbf{x}) := \sum_{j=1}^n \frac{x_j \mathrm{a}_{ij}(\mathbf{x})}{x_1 + \cdots + x_n}, \qquad i = 1, \ldots, n, \tag{9.18}$$

is uniformly parabolic with bounded, continuous coefficients on $\mathfrak{C}$, so from the maximum principle for parabolic operators [Friedman (2006), Chapter 6],

$$U(\tau, \mathbf{x}) = 1 \qquad \forall (\tau, \mathbf{x}) \in [0, \tau_* - \delta) \times (0, \infty)^n. \tag{9.19}$$

Now let us recall the $\mathbb{P}^{\mathbf{x}}$-a.s. equality $\mathbb{E}^{\mathbb{P}^{\mathbf{x}}}[X(T)Z(T)|\mathcal{F}(t)] = U(T - t, \mathfrak{X}(t)) \cdot X(t)Z(t)$ from (9.5); we apply it with $0 \le t \le \tau_* - \delta$, $0 \le T - t \le \tau_* - \delta$, then take expectations with respect to the probability measure $\mathbb{P}^{\mathbf{x}}$, and use (9.19) along with (9.1), to obtain for every $T \in [0, 2(\tau_* - \delta)]$,

$$U(T, \mathbf{x}) = \frac{\mathbb{E}^{\mathbb{P}^{\mathbf{x}}}[X(T)Z(T)]}{x_1 + \cdots + x_n} = \frac{\mathbb{E}^{\mathbb{P}^{\mathbf{x}}}[X(t)Z(t)]}{x_1 + \cdots + x_n} = U(t, \mathbf{x}) = 1, \qquad \forall \mathbf{x} \in (0, \infty)^n.$$

But since $2(\tau_* - \delta) > \tau_*$, this contradicts the definition of $\tau_*$.

Now we revert to (9.15); as J. Ruf (private communication) observes, yet another application of the maximum principle, as above, leads to (9.17). □

COROLLARY. *Under the nondegeneracy condition* (9.14), *and with either* (9.12) *or* (9.13), *inequality* (9.16) *holds. That is, arbitrage with respect to the market exists then over* any *time–horizon* $[0, T]$ *with* $T \in (0, \infty)$.

9.3. *An auxiliary diffusion.* Let us consider now the diffusion process $\mathfrak{Y}(\cdot)$ with infinitesimal generator $\widehat{\mathcal{L}}$ as in (1.2), (9.18) and dynamics

$$dY_i(t) = \widehat{\mathfrak{b}}_i(\mathfrak{Y}(t))\,dt + \sum_{k=1}^{n} \mathfrak{s}_{ik}(\mathfrak{Y}(t))\,dW_k(t), \qquad i = 1, \dots, n. \tag{9.20}$$

ASSUMPTION F. The system of SDEs (9.20) admits a unique-in-distribution weak solution with values in $[0,\infty)^n \setminus \{\mathbf{0}\}$.

This will be the case, for instance, if the drift functions $\widehat{\mathfrak{b}}_i(\cdot)$, $1 \le i \le n$ of (9.18) can be extended by continuity on all of $[0,\infty)^n$ and satisfy the Bass and Perkins (2003) conditions preceding, following and including (8.6). The resulting process $\mathfrak{Y}(\cdot)$ is then Markovian, and we shall denote by $\mathbb{Q}^{\mathbf{y}}$ its distribution with $\mathfrak{Y}(0) = \mathbf{y} \in [0,\infty)^n$. Unlike the original process $\mathfrak{X}(\cdot)$, which takes values in $(0,\infty)^n$, this new process $\mathfrak{Y}(\cdot)$ is only guaranteed to take values in the nonnegative orthant $[0,\infty)^n \setminus \{\mathbf{0}\}$. In particular, with $\mathbf{x} \in (0,\infty)^n$ the first hitting time

$$\mathfrak{T} := \inf\{t \ge 0 | \mathfrak{Y}(t) \in \mathcal{O}^n\} \tag{9.21}$$

of the boundary $\mathcal{O}^n$ of $[0,\infty)^n$ may be finite with positive $\mathbb{Q}^{\mathbf{x}}$-probability.

Our next result shows that this possibility amounts to the existence of arbitrage relative to the market, and to the lack of uniqueness for the Cauchy problem of (9.8) and $U(0,\cdot) \equiv 1$.

THEOREM 2. *With the above notation and assumptions, including* (9.14), *the function* $U : [0,\infty) \times (0,\infty)^n \to (0,1]$ *of* (9.1) *admits the representation*

$$U(T,\mathbf{x}) = \mathbb{Q}^{\mathbf{x}}[\mathfrak{T} > T], \qquad (T,\mathbf{x}) \in (0,\infty) \times (0,\infty)^n. \tag{9.22}$$

PROOF. The function on the right-hand side of (9.22) is space–time harmonic for the diffusion $\mathfrak{Y}(\cdot)$ on $(0,\infty) \times (0,\infty)^n$, so it solves equation (9.8) there [cf. Janson and Tysk (2006), Theorem 2.7]. Consider any function $V$ in the collection $\mathfrak{U}$ of Theorem 1; then $V(T-t, \mathfrak{Y}(t))1_{\{\mathfrak{T}>t\}}$, $0 \le t \le T$ is a nonnegative local (thus a true) $\mathbb{Q}^{\mathbf{x}}$-supermartingale, and we deduce

$$V(T,\mathbf{x}) \ge \mathbb{E}^{\mathbb{Q}^{\mathbf{x}}}\big[V(0,\mathfrak{Y}(T))1_{\{\mathfrak{T}>T\}}\big] = \mathbb{Q}^{\mathbf{x}}(\mathfrak{T} > T),$$
$$(T,\mathbf{x}) \in (0,\infty) \times (0,\infty)^n.$$

The claim follows now from the proof of Theorem 1. □

COROLLARY. *Under the assumptions of Theorem* 2, *for any given* $\mathbf{x} \in (0,\infty)^n$ *the* $\mathbb{P}^{\mathbf{x}}$*-supermartingale* $Z(\cdot)X(\cdot)$ *is under* $\mathbb{P}^{\mathbf{x}}$ *a*:

- *martingale*, *if and only if* $\mathbb{Q}^{\mathbf{x}}(\mathfrak{T} < \infty) = 0$;
- *potential* [*i.e.*, $\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}(Z(T)X(T)) = 0$], *iff* $\mathbb{Q}^{\mathbf{x}}(\mathfrak{T} < \infty) = 1$;
- *strict local* (*and super-*)*martingale on any time–horizon* $[0,T]$ *with* $T \in (0,\infty)$, *if and only if* $\mathbb{Q}^{\mathbf{x}}(\mathfrak{T} < \infty) > 0$.

We represent by analogy with (7.3) the exit measure $\mathfrak{P}^{\mathbf{x}}$ of the supermartingale $Z(\cdot)X(\cdot)$ with initial configuration $\mathfrak{X}(0)=\mathbf{x}$, in the form

$$\mathfrak{P}^{\mathbf{x}}\big((T,\infty]\times\Omega\big)=U(T,\mathbf{x})=\mathbb{Q}^{\mathbf{x}}[\mathfrak{T}>T], \tag{9.23}$$

and from (9.2)–(9.5) we have for $A\in\mathcal{F}(t)$, $0\le t\le T$,

$$\begin{aligned}&\mathfrak{P}^{\mathbf{x}}\big((T,\infty]\times A\big)\\&\quad=\mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\left[\frac{g(\mathfrak{X}(t))}{g(\mathbf{x})}1_A\big(\mathbb{Q}^{\mathbf{z}}[\mathfrak{T}>T-t]\big)\big|_{\mathbf{z}=\mathfrak{X}(t)}\,e^{-\int_0^t k(\mathfrak{X}(s))\,ds}\right].\end{aligned} \tag{9.24}$$

When $\mathbf{x}\in(0,\infty)^n$ and the quantity of (9.22) is equal to one, the $\mathbb{Q}^{\mathbf{x}}$-distribution of the process $\mathfrak{Y}(t)$, $0\le t\le T$ in (9.20) is the same as the $\widetilde{\mathbb{P}}^{\mathbf{x}}_T$-distribution of the original stock-price process $\mathfrak{X}(t)$, $0\le t\le T$; this follows by comparing (9.20) and (9.18) with (6.6), and denoting by $\widetilde{\mathbb{P}}^{\mathbf{x}}_T$ the probability measure $\widetilde{\mathbb{P}}_T$ of (6.10) with $\mathfrak{X}(0)=\mathbf{x}$. We have in this spirit the following result, by analogy with Remark 2.

PROPOSITION 3. *Under the assumptions of Theorem 2, suppose that the functions* $\mathfrak{s}_{ik}(\cdot)$ *are continuously differentiable on* $(0,\infty)^n$; *that the matrix* $\mathfrak{a}(\cdot)$ *degenerates on* $\mathcal{O}^n$; *and that the analogues of* (8.8), *the Fichera drifts*

$$\widehat{\mathfrak{f}}_i(\mathbf{x}):=\widehat{\mathfrak{b}}_i(\mathbf{x})-\frac{1}{2}\sum_{j=1}^n D_j\mathfrak{a}_{ij}(\mathbf{x})=\sum_{j=1}^n\left(\frac{\mathfrak{a}_{ij}(\mathbf{x})}{x_1+\cdots+x_n}-\frac{1}{2}D_j\mathfrak{a}_{ij}(\mathbf{x})\right) \tag{9.25}$$

*for the process* $\mathfrak{Y}(\cdot)$ *of* (9.20), *can be extended by continuity on* $[0,\infty)^n$. *If* $\widehat{\mathfrak{f}}_i(\cdot)\ge 0$ *holds on each face* $\{x_i=0\}$, $i=1,\dots,n$ *of the orthant, then we have* $U(\cdot,\cdot)\equiv 1$ *in* (9.22), *and no arbitrage with respect to the market portfolio exists on any time–horizon.*

*If, on the other hand, we have* $\widehat{\mathfrak{f}}_i(\cdot)<0$ *on each face* $\{x_i=0\}$ *of the orthant, then* $U(\cdot,\cdot)<1$ *in* (9.22) *and arbitrage with respect to the market portfolio exists, on every time–horizon* $[0,T]$ *with* $T\in(0,\infty)$.

PROOF. In light of Theorem 2, the first claim follows from Theorem 9.4.1, Corollary 9.4.2 of Friedman (2006), and the second is a consequence of the support theorem for diffusions [Ikeda and Watanabe (1989), Section VI.8]. □

REMARK 5. (i) The "relative weights" $\nu_i(t):=Y_i(t)/(Y_1(t)+\cdots+Y_n(t))$, $i=1,\dots,n$ have dynamics similar to (6.9),

$$d\nu_i(t)=\nu_i(t)\big(\mathfrak{e}_i-\nu(t)\big)'\mathrm{s}(\mathfrak{Y}(t))\,dW(t). \tag{9.26}$$

They are thus $\mathbb{Q}^{\mathbf{x}}$-martingales with values in $[0,1]$ (cf. Section 6.1); so, when any one of them hits either boundary point of the unit interval, it gets absorbed there. In terms of them, the first hitting time of (9.21) can be expressed as in (7.4), $\mathfrak{T}=\min_{1\le i\le n}\mathfrak{T}_i$, where $\mathfrak{T}_i:=\inf\{t\ge 0\,|\,\nu_i(t)=0\}$.

(ii) The measure $\mathbb{Q}^{\mathbf{x}}$ corresponds to a *change of drift*, from $\mathfrak{b}(\cdot)$ in (8.2) to $\widehat{\mathfrak{b}}(\cdot)$ in (9.18), (9.20); this ensures that, under $\mathbb{Q}^{\mathbf{x}}$, the components of the new, "fictitious" market portfolio $\nu(\cdot)$ are martingales, that $\nu(\cdot)$ has the numéraire property, and thus that $\nu(\cdot)$ cannot be outperformed.

**10. Markovian market weights.** Let us assume now the form

$$\mathrm{b}_i(\mathbf{x}) = \mathfrak{B}_i(x_1/x, \dots, x_n/x), \qquad \mathrm{s}_{ik}(\mathbf{x}) = \mathfrak{S}_{ik}(x_1/x, \dots, x_n/x)$$

for the functions of (8.1), with $x := \sum_{j=1}^n x_j$ and suitable continuous functions $\mathfrak{B}_i(\cdot)$, $\mathfrak{S}_{ik}(\cdot)$ on $\Delta_+^n$. For $\mathrm{m} = (m_1, \dots, m_n)' \in \Delta_+^n$, we set $\mathcal{A}_{ij}(\mathrm{m}) := \sum_{k=1}^n \mathfrak{S}_{ik}(\mathrm{m})\mathfrak{S}_{jk}(\mathrm{m})$. In words, we consider instantaneous growth rates and volatilities that depend at time $t$ only on the current configuration $\mu(t) = (\mu_1(t), \dots, \mu_n(t))'$ of relative market weights, so the process $\mu(\cdot)$ of (3.3) is now a diffusion with values in the positive simplex $\Delta_+^n$ and

$$d\mu_i(t) = \mu_i(t)\left[\Gamma_i(\mu(t))\,dt + \sum_{k=1}^n \mathcal{T}_{ik}(\mu(t))\,dW_k(t)\right], \qquad i = 1, \dots, n, \tag{10.1}$$

with $\mathcal{T}_{ik}(\mathrm{m}) := \mathfrak{S}_{ik}(\mathrm{m}) - \sum_{j=1}^n m_j \mathfrak{S}_{jk}(\mathrm{m})$, $\mathcal{P}_{ij}(\mathrm{m}) := \sum_{k=1}^n \mathcal{T}_{ik}(\mathrm{m})\mathcal{T}_{jk}(\mathrm{m})$,

$$\Gamma_i(\mathrm{m}) := \mathfrak{B}_i(\mathrm{m}) - \sum_{j=1}^n m_j \mathfrak{B}_j(\mathrm{m}) - \sum_{j=1}^n m_j \mathcal{A}_{ij}(\mathrm{m}) + \sum_{j=1}^n \sum_{k=1}^n m_j m_\ell \mathcal{A}_{j\ell}(\mathrm{m}).$$

In this setup, the function of (9.1) can be expressed in the form $U(T, \mathbf{x}) = Q(T, x_1/x, \dots, x_n/x)$, in terms of a function $Q : (0, \infty) \times \Delta_+^n \to (0, 1]$ that satisfies the initial condition $Q(0, \cdot) \equiv 1$ and the equation

$$\frac{\partial Q}{\partial \tau}(\tau, \mathrm{m}) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n m_i m_j \mathcal{P}_{ij}(\mathrm{m}) D_{ij}^2 Q(\tau, \mathrm{m}), \qquad (\tau, \mathrm{m}) \in (0, \infty) \times \Delta_+^n,$$

which appears on page 56 of Fernholz (2002) and can be derived from (9.8). On the other hand, by analogy with Theorem 2 and (9.26), the quantity $Q(T, \mathrm{m})$ is the probability that the process $\nu(\cdot) = (\nu_1(\cdot), \dots, \nu_n(\cdot))'$ with $\nu(0) = \mathrm{m} \in \Delta_+^n$ and dynamics (10.2) below, does not hit the boundary of the nonnegative simplex $\Delta^n := \{\mathrm{m} \in [0, 1]^n \,|\, \sum_{i=1}^n m_i = 1\}$ before $t = T$:

$$d\nu_i(t) = \nu_i(t) \sum_{k=1}^n \mathcal{T}_{ik}(\nu(t))\,dW_k(t), \qquad i = 1, \dots, n. \tag{10.2}$$

**11. The investment strategy.** Let us substitute now the expressions of (9.9) into (9.11), to obtain the dynamics of the martingale $N(\cdot) \equiv Z(\cdot)X(\cdot)U(T - \cdot,$

$\mathfrak{X}(\cdot)$) in (9.10), with $N(0) = \xi := X(0)U(T, \mathfrak{X}(0))$,

$$N(t) = \xi + \sum_{k=1}^{n} \int_0^t N(s)\Psi_k\big(T - s, \mathfrak{X}(s)\big)\, dW_k(s), \qquad 0 \le t \le T,$$

$$\Psi_k(\tau, \mathbf{x}) := \sum_{i=1}^{n} \mathrm{s}_{ik}(\mathbf{x})\left(x_i D_i \log U(\tau, \mathbf{x}) + \frac{x_i}{x_1 + \cdots + x_n}\right) - \Theta_k(\mathbf{x}).$$

Thus we can identify the "replicating strategy" $\widehat{\pi}(\cdot)$ of (6.2) as

$$\widehat{\pi}_i(t) = X_i(t) D_i \log U\big(T - t, \mathfrak{X}(t)\big) + \big(X_i(t)/X(t)\big), \qquad i = 1, \dots, n, \tag{11.1}$$

and its value as $V^{\xi,\widehat{\pi}}(t) = N(t)/Z(t) = X(t)U(T - t, \mathfrak{X}(t))$, $0 \le t \le T$.

REMARK 6. In the special case of a Markovian model (10.1) for the market weights of $\mu(\cdot) = (\mu_1(\cdot), \dots, \mu_n(\cdot))'$, expression (11.1) takes the form

$$\widehat{\pi}_i(t) = \mu_i(t)\left(1 + D_i \log Q\big(T - t, \mu(t)\big) - \sum_{j=1}^{n} \mu_j(t) D_j \log Q\big(T - t, \mu(t)\big)\right)$$

of a "functionally-generated portfolio" in the terminology of Fernholz (2002), page 56; whereas the value is $V^{\xi,\widehat{\pi}}(t) = X(t)Q(T - t, \mu(t))$, $0 \le t \le T$.

In this case we have $\sum_{i=1}^{n} \widehat{\pi}_i(\cdot) \equiv 1$: the strategy that implements the best possible arbitrage relative to the equity market never borrows or lends.

**12. Examples.** We discuss in this section two illustrative examples. Additional examples, in which the investment strategy $\widehat{\pi}(\cdot)$ of (11.1) that realizes the optimal arbitrage can be computed in closed form in dimension $n = 1$, can be found in Ruf (2009).

For the first of these examples, take $n = 1$, $\beta(t) = 1/X^2(t)$ and $\sigma(t) = 1/X(t)$ in (2.1) where the process $X(\cdot)$ satisfies $dX(t) = (1/X(t))\, dt + dW(t)$ and $X(0) = 1$. This is a Bessel process in dimension three—the radial part of a 3-D Brownian motion started at unit distance from the origin—and takes values in $(0, \infty)$. We have then $\vartheta(t) = 1/X(t)$, $Z(t) = 1/X(t)$ for $0 \le t < \infty$ in (5.1) and (5.2), so $Z(\cdot)X(\cdot)$ is very clearly a martingale. However, $Z(\cdot)$ is the prototypical example of a *strict* local martingale—we have $\mathbb{E}(Z(T)) < 1$ for every $T \in (0, \infty)$ [e.g., Karatzas and Shreve (1991), Exercise 3.36, page 168]. This example is taken from Karatzas and Kardaras [(2007), page 469], where an arbitrage with respect to the money-market is constructed in closed form. It illustrates that it is possible for $Z(\cdot)$ to be a strict local martingale and $Z(\cdot)X(\cdot)$ to be a martingale; in other words, the second and third inequalities in (5.5) fail, while the first stands.

Here we have $\Theta(x) = 1/x$, $H(x) = \log x$ and $k(\cdot) \equiv 0$, $g(\cdot) \equiv 1$, $G(\cdot, \cdot) \equiv 1$ in (9.3), (9.4), thus $U(T, x) \equiv 1$ for all $T \in [0, \infty)$, $x \in (0, \infty)$. Arbitrage relative to $X(\cdot)$ does not exist here, despite the existence of arbitrage relative to the money market and the fact that $Z(\cdot)$ is a strict local martingale. Note that $\widehat{\mathfrak{b}}(x) = 1/x$ in (9.18), so the diffusion of (9.20) is again a Bessel process in dimension three, $dY(t) = (1/Y(t))\,dt + dW(t)$, $Y(0) = y > 0$. This process never hits the origin, so the probability in (9.22) is equal to one, for all $T \in [0, \infty)$.

12.1. *The volatility-stabilized model.* Our second example is the model of "stabilization by volatility" introduced in Fernholz and Karatzas (2005) and studied further by Goia (2009). With $n \geq 2$, $\zeta \in [0, 1]$ this posits

$$\begin{aligned} &\beta_i(t) = (1+\zeta)/(2\mu_i(t)), \\ &\sigma_{ik}(t) = \delta_{ik}(\mu_i(t))^{-1/2}; \qquad 1 \leq i, k \leq n, \end{aligned} \tag{12.1}$$

that is, rates of return and volatilities which are large for the small stocks and small for the large stocks. The conditions of Bass and Perkins (2003) hold for the resulting system of SDEs in the notation of (3.2) with $\kappa := (1+\zeta)/2$,

$$dX_i(t) = \kappa X(t)\,dt + \sqrt{X_i(t)X(t)}\,dW_i(t), \qquad i = 1, \dots, n. \tag{12.2}$$

The unique-in-distribution solution of (12.2) is expressed in terms of independent Bessel processes $\mathfrak{R}_1(\cdot), \dots, \mathfrak{R}_n(\cdot)$ in dimension $4\kappa$ with $X_i(t) = \mathfrak{R}_i^2(A(t)) > 0$ and $A(t) := (1/4)\int_0^t X(s)\,ds$. In particular, $\mathfrak{X}(\cdot)$ takes values in $(0, \infty)^n$; for more details on these Lamperti-like descriptions and their implications, see Fernholz and Karatzas (2005) and Goia (2009). Condition (8.5) is satisfied in this example, so Assumption C also holds.

For the model of (12.1), we have $\Theta_i(\mathbf{x})/\kappa = \mathrm{s}_{ii}(\mathbf{x}) = ((x_1 + \cdots + x_n)/x_i)^{1/2}$,

$$\mathfrak{b}_i(\mathbf{x}) = \kappa(x_1 + \cdots + x_n), \qquad \mathfrak{h}_{ij}(\mathbf{x}) = \delta_{ij}(x_1 + \cdots + x_n), \qquad \mathfrak{a}_{ij}(\mathbf{x}) = x_i \mathfrak{h}_{ij}(\mathbf{x})$$

for $1 \leq i, j \leq n$. The assumptions of Theorem 2 and of Propositions 1 and 3 are all satisfied here, as are (7.5) and (8.9) with $H(\mathbf{x}) = \kappa \sum_{i=1}^n \log x_i$ and $k(\mathbf{x}) = (1 - \zeta^2)(x_1 + \cdots + x_n)\sum_{j=1}^n (1/(8x_j))$. This function $k(\cdot)$ is nonnegative, since we have assumed $0 \leq \zeta \leq 1$, whereas $g(\mathbf{x}) = (x_1 + \cdots + x_n)(x_1 \cdots x_n)^{-\kappa}$. In particular, with $\zeta = 1$ we get

$$U(T, \mathbf{x}) = \frac{x_1 \cdots x_n}{x_1 + \cdots + x_n} \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\left[\frac{X_1(T) + \cdots + X_n(T)}{X_1(T) \cdots X_n(T)}\right] \tag{12.3}$$

[see Goia (2009) and Pal (2009) for a computation of the joint density of $X_1(T), \dots, X_n(T)$ which leads then to an explicit computation of $U(T, \mathbf{x})$ in (12.3) above, and shows that this function is indeed of class $\mathcal{C}^{1,2}$].

With $\zeta = 1$ one computes $Z(t) = \prod_{j=1}^n (X_j(0)/X_j(t))$, therefore $\Lambda(t) = (X(t)/X(0))^{n-1} \prod_{j=1}^n (\mu_j(t)/\mu_j(0))$ as well as $\Lambda_i(t) = (X(t)/X(0))^{n-1} \cdot \prod_{j \neq i} (\mu_j(t)/\mu_j(0))$ for $i = 1, \dots, n$. Both representations in (7.4) hold for the first hitting time of (7.1) in this case; whereas $\mathcal{S} = \mathcal{T} = \min_{1 \leq i \leq n} \mathcal{T}_i$ as in (7.1)–(7.6), since $L(t) = (1/Z(t)) = (X(t)/X(0))^n \prod_{j=1}^n (\mu_j(t)/\mu_j(0))$.

Both (9.12) and (9.13) hold for the example of (12.1) with $h = n - 1$, the first as equality; from the corollary to Proposition 2 and Remark 3, (9.16) holds. We recover the result of Banner and Fernholz (2008) on the existence of arbitrage relative to market of (12.1) over *arbitrary* time–horizons.

The diffusion process $\mathfrak{Y}(\cdot)$ of (9.20) takes now the form

$$dY_i(t) = Y_i(t)\,dt + \sqrt{Y_i(t)\big(Y_1(t) + \cdots + Y_n(t)\big)}\,dW_i(t). \tag{12.4}$$

The conditions of Bass and Perkins (2003) are satisfied again, though one should compare the "weak drift" $\widehat{\mathfrak{b}}_i(\mathbf{x}) = x_i \geq 0$ in (12.4), which vanishes for $x_i = 0$, with the "strong drift" $\mathfrak{b}_i(\mathbf{x}) = \kappa(x_1 + \cdots + x_n)$ for the the original diffusion $\mathfrak{X}(\cdot)$ in (12.2), which is strictly positive on $[0, \infty)^n \setminus \{\mathbf{0}\}$.

The corresponding Fichera drifts in (9.25), (8.8) are given by $2\widehat{\mathfrak{f}}_i(\mathbf{x}) = x_i - (x_1 + \cdots + x_n)$, $2\mathfrak{f}_i(\mathbf{x}) = \zeta(x_1 + \cdots + x_n) - x_i$, and $\mathfrak{f}_i(\mathbf{x}) > 0 > \widehat{\mathfrak{f}}_i(\mathbf{x})$ hold on $\{x_i = 0\} \cap \{\sum_{j \neq i} x_j > 0\}$; from Remark 2 we verify again that the diffusion $\mathfrak{X}(\cdot)$ of (12.2) takes values in $(0, \infty)^n$.

In contrast, the new diffusion $\mathfrak{Y}(\cdot)$ of (12.4) lives in $[0, \infty)^n \setminus \{\mathbf{0}\}$, and hits the boundary $\mathcal{O}^n$ of this nonnegative orthant with positive probability $\mathbb{Q}^{\mathbf{x}}[\mathfrak{T} \leq T] = 1 - U(T, \mathbf{x})$ for every $T \in (0, \infty)$. The positive $\mathbb{P}^{\mathbf{x}}$-supermartingale $Z(\cdot)X(\cdot)$ is a $\mathbb{P}^{\mathbf{x}}$-potential, for every $\mathbf{x} \in (0, \infty)^n$. In this case, the three inequalities of (5.5) hold for every $T \in (0, \infty)$: *the local martingales* $Z(\cdot)$, $Z(\cdot)X(\cdot)$ *and* $Z(\cdot)X_i(\cdot)$, $i = 1, \ldots, n$ *are all strict.*

The model (12.1) can be cast in the form (10.1) for the relative market weights, as a *multivariate Jacobi diffusion process* with dynamics $d\mu_i(t) = (1 + \zeta)(1 - n\mu_i(t))\,dt + \sqrt{\mu_i(t)}\,dW_i(t) - \mu_i(t)\sum_{k=1}^n \sqrt{\mu_k(t)}\,dW_k(t)$, or

$$d\mu_i(t) = (1 + \zeta)\big(1 - n\mu_i(t)\big)\,dt + \sqrt{\mu_i(t)\big(1 - \mu_i(t)\big)}\,dW_i^\sharp(t) \tag{12.5}$$

with appropriate Brownian motions $W_1^\sharp(\cdot), \ldots, W_n^\sharp(\cdot)$. Thus, each component $\mu_i(\cdot)$ is also a diffusion on the unit interval $(0, 1)$ with local drift $(1 + \zeta)(1 - ny)$ and local variance $y(1 - y)$ of Wright–Fisher type. Goia (2009) studies in detail this multivariate diffusion $\mu(\cdot)$ based on an extension of the Warren and Yor (1999), Gouriéroux and Jasiak (2006) study of skew-products involving Bessel and Jacobi processes.

From (12.4), $Y(\cdot) := Y_1(\cdot) + \cdots + Y_n(\cdot)$ satisfies the stochastic equation $dY(t) = Y(t)[dt + dB(t)]$, where $B(\cdot) := \sum_{j=1}^n \int_0^\cdot \sqrt{Y_j(t)/Y(t)}\,dW_j(t)$ is Brownian motion; thus $Y(\cdot)$ a geometric Brownian motion with drift, under $\mathbb{Q}^{\mathbf{x}}$. The process $\nu(\cdot) = (\nu_1(\cdot), \ldots, \nu_n(\cdot))'$ of (10.2) is related to the auxiliary diffusion $\mathfrak{Y}(\cdot)$ of (12.4) via $\nu_i(\cdot) = Y_i(\cdot)/Y(\cdot)$.

The dynamics of these $\nu_i(\cdot)$'s are easy to describe in the manner of (9.26), namely, $d\nu_i(t) = \sqrt{\nu_i(t)}\,dW_i(t) - \nu_i(t)\sum_{k=1}^n \sqrt{\nu_k(t)}\,dW_k(t)$, or in the notation of (12.5): $d\nu_i(t) = \sqrt{\nu_i(t)(1 - \nu_i(t))}\,dW_i^\sharp(t)$. Then the Feller test [e.g., Karatzas and Shreve (1991), pages 348–350] ensures that each $\nu_i(\cdot)$ hits one of the endpoints of $(0, 1)$ in finite expected time. Thus, all but one of the $Y_i(\cdot)$'s eventually get absorbed at zero; from that time $\mathfrak{T}_*$ [with $\mathbb{E}^{\mathbb{Q}^{\mathbf{x}}}(\mathfrak{T}_*) < \infty$] onward, the only surviving nonzero component $Y(\cdot)$ behaves like geometric Brownian motion with drift; in particular, $\mathfrak{Y}(\cdot)$ never hits the origin.

**13. Some open questions.** What conditions, if any, on the Markovian covariance structure of Section 8 will guarantee that $\widehat{\pi}(\cdot)$ of (11.1) never borrows from the money-market, that is, $\sum_{i=1}^n x_i D_i U(T, \mathbf{x}) \le 0$? That it is a portfolio, i.e., that $\sum_{i=1}^n x_i D_i U(T, \mathbf{x}) = 0$ holds? (See Remark 6 for a partial answer.) Or better, that $\widehat{\pi}(\cdot)$ of (11.1) is a long-only portfolio, meaning that both this condition and $D_i(G(T, \mathbf{x})e^{H(\mathbf{x})}) \ge 0$ hold?

Can an iterative method be constructed which converges to the minimal solution of the parabolic differential inequality (1.1), $U(0, \cdot) \equiv 1$ and is numerically implementable [possibly as in Ekström, Von Sydow and Tysk (2008)]? How about a Monte Carlo scheme that computes the quantity $U(T, \mathbf{x})$ of (9.22) by generating the paths of the diffusion process $\mathfrak{Y}(\cdot)$, then simulating the probability $\mathbb{Q}^{\mathbf{x}}[\mathfrak{T} > T]$ that $\mathfrak{Y}(\cdot)$ does not hit the boundary of the nonnegative orthant by time $T$, when started at $\mathfrak{Y}(0) = \mathbf{x} \in (0, \infty)^n$?

How does $U(T, \mathbf{x})$ behave as $T \to \infty$? If it decreases to zero, then at what rate?

**14. Note added in proof.** In the context of Proposition 1, and under the probability measure $\mathbb{Q}$ of Section 7.1, the processes $X_1(\cdot), \dots, X_n(\cdot)$ are real-valued (do not explode) if and only if their sum $X(\cdot)$ as in (3.2) is real-valued. Now it is fairly straightforward to check from (6.6) that this sum satisfies the equation

$$dX(t) = X(t)[d\langle \widetilde{M}\rangle(t) + d\widetilde{M}(t)],$$

where the continuous, $\mathbb{Q}$-local martingale $\widetilde{M}(\cdot)$ and its quadratic variation process $\langle \widetilde{M}\rangle(\cdot)$ are given, respectively, as

$$\widetilde{M}(t) := \sum_{k=1}^n \int_0^t \left(\sum_{i=1}^n \mu_i(s)\sigma_{ik}(s)\right) d\widetilde{W}_k(s), \qquad \langle \widetilde{M}\rangle(t) = \int_0^t \mu'(s)\alpha(s)\mu(s)\, ds.$$

Thus by the Dambis–Dubins–Schwarz result [e.g., Karatzas and Shreve (1991), Theorem 3.4.6], for some real-valued $\mathbb{Q}$-Brownian motion $\widetilde{B}(\cdot)$ we have

$$\log\left(\frac{X(t)}{X(0)}\right) = \left(\widetilde{B}(u) + \frac{1}{2}u\right)\Big|_{u=\langle \widetilde{M}\rangle(t)}, \qquad 0 \le t < \infty.$$

It is fairly clear form this representation that a sufficient condition for the total capitalization process $X(\cdot)$ to be real-valued, $\mathbb{Q}$-a.e., is that this should hold for the quadratic variation process $\langle \widetilde{M}\rangle(\cdot)$:

$$\mathbb{Q}\big(\langle \widetilde{M}\rangle(t) < \infty, \forall t \in [0, \infty)\big) = 1.$$

In the volatility-stabilized model of Section 12.1 we have $\alpha_{ij}(t) = \delta_{ij}/\mu_i(t)$ and thus $\langle \widetilde{M}\rangle(t) = \sum_{i=1}^n \int_0^t \mu_i(s)\, ds = t$, so this condition is clearly satisfied.

**Acknowledgments.** We wish to thank G. Žitković, N. Sesum, E. R. Fernholz, A. Banner, V. Papathanakos, T. Ichiba and most notably J. Ruf for several helpful discussions. We are grateful to M. Sîrbu, M. Soner, F. Delbaen, W. Schachermayer, J. Hugonnier, J. Teichmann and C. Kardaras for their comments. We are also deeply indebted to the referees and Associate Editor for their meticulous readings, and for suggestions that improved this paper greatly.

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