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# POSITIVE RECURRENCE OF REFLECTING BROWNIAN MOTION IN THREE DIMENSIONS

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Consider a semimartingale reflecting Brownian motion (SRBM) $Z$ whose state space is the $d$-dimensional nonnegative orthant. The data for such a process are a drift vector $\theta$, a nonsingular $d \times d$ covariance matrix $\Sigma$, and a $d \times d$ reflection matrix $R$ that specifies the boundary behavior of $Z$. We say that $Z$ is positive recurrent, or stable, if the expected time to hit an arbitrary open neighborhood of the origin is finite for every starting state.

In dimension $d = 2$, necessary and sufficient conditions for stability are known, but fundamentally new phenomena arise in higher dimensions. Building on prior work by El Kharroubi, Ben Tahar and Yaacoubi [*Stochastics Stochastics Rep.* **68** (2000) 229–253, *Math. Methods Oper. Res.* **56** (2002) 243–258], we provide necessary and sufficient conditions for stability of SRBMs in three dimensions; to verify or refute these conditions is a simple computational task. As a byproduct, we find that the fluid-based criterion of Dupuis and Williams [*Ann. Probab.* **22** (1994) 680–702] is not only sufficient but also necessary for stability of SRBMs in three dimensions. That is, an SRBM in three dimensions is positive recurrent if and only if every path of the associated fluid model is attracted to the origin. The problem of recurrence classification for SRBMs in four and higher dimensions remains open.

**1. Introduction.** This paper is concerned with the class of $d$-dimensional diffusion processes called *semimartingale reflecting Brownian motions* (SRBMs), which arise as approximations for open $d$-station queueing networks of various kinds; cf. Harrison and Nguyen (1993) and Williams (1995, 1996). The state space for a process $Z = \{Z(t), t \geq 0\}$ in this class is $S = \mathbb{R}^d_+$ (the nonnegative orthant). The data of the process are a drift vector $\theta$, a nonsingular covariance matrix $\Gamma$, and a $d \times d$ "reflection matrix" $R$ that specifies boundary behavior. In the interior of the orthant, $Z$ behaves as an ordinary Brownian motion with parameters $\theta$ and $\Gamma$, and roughly speaking, $Z$ is pushed in direction $R^j$ whenever the boundary surface $\{z \in S : z_j = 0\}$ is hit, where $R^j$ is the $j$th column of $R$, for $j = 1, \ldots, d$. To make this description more precise, one represents $Z$ in the form

$$Z(t) = X(t) + RY(t), \qquad t \geq 0, \tag{1.1}$$

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where $X$ is an unconstrained Brownian motion with drift vector $\theta$, covariance matrix $\Gamma$, and $Z(0) = X(0) \in S$, and $Y$ is a $d$-dimensional process with components $Y_1, \ldots, Y_d$ such that

(1.2) $Y$ is continuous and nondecreasing with $Y(0) = 0$,

(1.3) $Y_j$ only increases at times $t$ for which $Z_j(t) = 0$, $\quad j = 1, \ldots, d$, and

(1.4) $Z(t) \in S, \qquad t \geq 0.$

The complete definition and essential properties of the diffusion process $Z$ will be reviewed in Appendix A, where we also discuss the notion of positive recurrence. As usual in Markov process theory, the complete definition involves a family of probability measures $\{\mathbb{P}_x, x \in S\}$ that specify the distribution of $Z$ for different starting states; informally, one can think of $\mathbb{P}_x(\cdot)$ as a conditional probability given that $Z(0) = x$. Denoting by $\mathbb{E}_x$ the expectation operator associated with $\mathbb{P}_x$ and setting $\tau_A = \inf\{t \geq 0 : Z(t) \in A\}$, we say that $Z$ is positive recurrent if $\mathbb{E}_x(\tau_A) < \infty$ for any $x \in S$ and any open neighborhood $A$ of the origin (see Appendix A for elaboration). For ease of expression, we use the terms "stable" and "stability" as synonyms for "positive recurrent" and "positive recurrence," respectively.

In the foundational theory for SRBMs, the following classes of matrices are of interest. First, a $d \times d$ matrix $R$ is said to be an $\mathcal{S}$*-matrix* if there exists a $d$-vector $w \geq 0$ such that $Rw > 0$ (or equivalently, if there exists $w > 0$ such that $Rw > 0$), and $R$ is said to be *completely-*$\mathcal{S}$ if each of its principal submatrices is an $\mathcal{S}$-matrix. (For a vector $v$, we write $v > 0$ to mean that each component of $v$ is positive, and we write $v \geq 0$ to mean that each component of $v$ is nonnegative.) Second, a square matrix is said to be a $\mathcal{P}$-matrix if all of its principal minors are positive (that is, each principal submatrix of $R$ has a positive determinant). $\mathcal{P}$-matrices are a subclass of completely-$\mathcal{S}$ matrices; the still more restrictive class of $\mathcal{M}$-matrices is defined as in Chapter 6 of Berman and Plemmons (1979). References for the following key results can be found in the survey paper by Williams (1995): there exists a diffusion process $Z$ of the form described above if and only if $R$ is a completely $\mathcal{S}$ matrix; and moreover, $Z$ is unique in distribution whenever it exists.

Hereafter we assume that $R$ is completely-$\mathcal{S}$. Its diagonal elements must then be strictly positive, so we can (and do) assume without loss of generality that

$$R_{ii} = 1 \qquad \text{for all } i = 1, \ldots, d. \tag{1.5}$$

This convention is standard in the SRBM literature; in Sections 5 through 7 of this paper (where our main results are proved) another convenient normalization of problem data will be used. Appendix B explains the scaling procedures that justify both (1.5) and the normalized problem format assumed in Sections 5 through 7.

We are concerned in this paper with conditions that assure the stability of $Z$. An important condition in that regard is the following:

$$R \text{ is nonsingular} \quad \text{and} \quad R^{-1}\theta < 0. \tag{1.6}$$

If $R$ is an $\mathcal{M}$-matrix, then (1.6) is known to be necessary and sufficient for stability of $Z$; Harrison and Williams (1987) prove that result and explain how the $\mathcal{M}$-matrix structure arises naturally in queueing network applications.

El Kharroubi, Tahar and Yaacoubi (2000) further prove the following three results: first, (1.6) is necessary for stability in general; second, when $d = 2$, one has stability if and only if (1.6) holds and $R$ is a $\mathcal{P}$-matrix; and third, (1.6) is not sufficient for stability in three and higher dimensions, even if $R$ is a $\mathcal{P}$-matrix. In Appendix C of this paper, we provide an alternative proof for the first of these results, one that is much simpler than the original proof by El Kharroubi, Tahar and Yaacoubi (2000). Appendix A of Harrison and Hasenbein (2009) contains an alternative proof of the second result. Section 3 of this paper reviews the ingenious example by Bernard and El Kharroubi (1991) that serves to establish the third result, that an SRBM can be unstable, cycling to infinity even if (1.6) holds; Theorem 4 in this paper, together with the examples provided in Section 6, shows that instability can also occur in other ways when (1.6) holds.

A later paper by El Kharroubi, Ben Tahar and Yaacoubi (2002) established sufficient conditions for stability of SRBMs in three dimensions, relying heavily on the foundational theory developed by Dupuis and Williams (1994). In this paper, we show that the conditions identified by El Kharroubi, Ben Tahar and Yaacoubi (2002) are also necessary for stability when $d = 3$; the relevant conditions are easy to verify or refute via simple computations. As a complement to this work, an alternative proof of the sufficiency result by El Kharroubi, Ben Tahar and Yaacoubi (2002) is also being prepared for submission; cf. Dai and Harrison (2009).

The remainder of the paper is structured as follows. First, to allow precise statements of the main results, we introduce in Section 2 the "fluid paths" associated with an SRBM, and the linear complementarity problem that arises in conjunction with linear fluid paths. That section, like the paper's first three appendices, considers a general dimension $d$, whereas all other sections in the body of the paper consider $d = 3$ specifically. Section 3 identifies conditions under which fluid paths spiral on the boundary of the state space $S$. Section 4 states our main conclusions, which are achieved by combining the positive results of El Kharroubi, Ben Tahar and Yaacoubi (2002) with negative results that are new; Figure 2 in Section 4 summarizes succinctly the necessary and sufficient conditions for stability when $d = 3$, and indicates which components of the overall argument are old and which are new. In Sections 5 through 7, we prove the new "negative results" referred to above, dealing first with the case where fluid paths spiral on the boundary, and then with the case where they do not. As stated above, Appendix A reviews the precise definition of SRBM; Appendix B explains the scaling procedures that give rise to normalized problem formats and Appendix C contains a relatively simple proof that (1.6) is necessary for stability. Finally, Appendix D contains several technical lemmas that are used in the probabilistic arguments of Section 7.

## 2. Fluid paths and the linear complementarity problem.

DEFINITION 1. A fluid path associated with the data $(\theta, R)$ is a pair of continuous functions $y, z : [0, \infty) \to \mathbb{R}^d$ that satisfy the following conditions:

$$z(t) = z(0) + \theta t + R y(t) \qquad \text{for all } t \geq 0, \tag{2.1}$$

$$z(t) \in S \qquad \text{for all } t \geq 0, \tag{2.2}$$

$$y(\cdot) \text{ is continuous and nondecreasing with } y(0) = 0, \tag{2.3}$$

$$\begin{gathered} y_j(\cdot) \text{ only increases when } z_j(\cdot) = 0, \text{ i.e.,} \\ \int_0^\infty z_j(t)\, dy_j(t) = 0, \qquad (j = 1, \dots, d). \end{gathered} \tag{2.4}$$

DEFINITION 2. We say that a fluid path $(y, z)$ is *attracted to the origin* if $z(t) \to 0$ as $t \to \infty$.

DEFINITION 3. A fluid path $(y, z)$ is said to be *divergent* if $|z(t)| \to \infty$ as $t \to \infty$, where, for a vector $u = (u_i) \in \mathbb{R}^d$, $|u| = \sum_i |u_i|$.

THEOREM 1 [Dupuis and Williams (1994)]. *Let $Z$ be a $d$-dimensional SRBM with data $(\theta, \Gamma, R)$. If every fluid path associated with $(\theta, R)$ is attracted to the origin, then $Z$ is positive recurrent.*

DEFINITION 4. A fluid path $(y, z)$ is said to be *linear* if it has the form $y(t) = ut$ and $z(t) = vt$, $t \geq 0$, where $u, v \geq 0$.

Linear fluid paths are in one-to-one correspondence with solutions of the following linear complementarity problem (LCP): Find vectors $u = (u_i)$ and $v = (v_i)$ in $\mathbb{R}^d$ such that

$$u, v \geq 0, \tag{2.5}$$

$$v = \theta + Ru, \tag{2.6}$$

$$u \cdot v = 0, \tag{2.7}$$

where $u \cdot v = \sum_i u_i v_i$ is the inner product of $u$ and $v$. [See Cottle, Pang and Stone (1992) for a systematic account of the theory associated with the general problem (2.5)–(2.7).]

DEFINITION 5. A solution $(u, v)$ of the LCP is said to be *stable* if $v = 0$ and to be *divergent* otherwise. It is said to be *nondegenerate* if $u$ and $v$ together have exactly $d$ positive components, and to be *degenerate* otherwise. A stable, nondegenerate solution of the LCP is called *proper*.

LEMMA 1. *Suppose that* (1.6) *holds. Then* $(u^*, 0)$ *is a proper solution of the LCP, where*

$$u^* = -R^{-1}\theta, \tag{2.8}$$

*and any other solution of the LCP must be divergent.*

PROOF. The first statement is obvious. On the other hand, for any stable solution $(u, 0)$ of the LCP, we have from (2.6) that $\theta + Ru = 0$; since (1.6) includes the requirement that $R$ be nonsingular, $u = -R^{-1}\theta = u^*$. That is, there cannot exist a stable solution other than $(u^*, 0)$, which is equivalent to the second statement of the lemma. □

**3. Fluid paths that spiral on the boundary.** Bernard and El Kharroubi (1991) devised the following ingenious example with $d = 3$, referred to hereafter as the B&EK example: let

$$\theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix}.$$

This reflection matrix $R$ is completely $\mathcal{S}$ (moreover, it is a $\mathcal{P}$-matrix), so $Z$ is a well-defined SRBM. (The covariance matrix $\Gamma$ is immaterial to the discussion that follows, provided only that it is nonsingular.) As Bernard and El Kharroubi (1991) observed, the unique fluid path with these process data, starting from $z(0) = (0, 0, \kappa)$ with $\kappa > 0$, is the one pictured in Figure 1; it travels in a counter-clockwise and piecewise linear fashion on the boundary, with the first linear segment ending at $(2\kappa, 0, 0)$, the second one ending at $(0, 4\kappa, 0)$, and so forth. El Kharroubi, Tahar and Yaacoubi (2000) proved that an SRBM with these data is

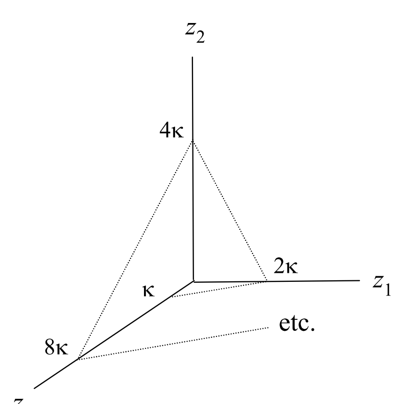


FIG. 1. *Fluid model behavior of the B&EK example.*

not stable, showing that if $\kappa$ is large then $|Z(t) - z(t)|$ remains forever small (in a certain sense) with high probability.

To generalize the B&EK example, let $C_1$ be the set of $(\theta, R)$ pairs that satisfy the following system of inequalities [here $R_{ij}$ denotes the $(i, j)$th element of $R$, or equivalently, the $i$th element of the column vector $R^j$]:

$$\theta < 0, \tag{3.1}$$

$$\theta_1 > \theta_2 R_{12} \quad \text{and} \quad \theta_3 < \theta_2 R_{32}, \tag{3.2}$$

$$\theta_2 > \theta_3 R_{23} \quad \text{and} \quad \theta_1 < \theta_3 R_{13}, \tag{3.3}$$

$$\theta_3 > \theta_1 R_{31} \quad \text{and} \quad \theta_2 < \theta_1 R_{21}. \tag{3.4}$$

[Notation used in this section agrees with that of El Kharroubi, Tahar and Yaacoubi (2000, 2002) in all essential respects, but is different in a few minor respects.]

To explain the meaning of these inequalities, we consider a fluid path associated with $(\theta, R)$ that starts from $z(0) = (0, 0, \kappa)$, where $\kappa > 0$; it is the unique fluid path starting from that state, but that fact will not be used in our formal results. Over an initial time interval $[0, \tau_1]$, the fluid path is linear and adheres to the boundary $\{z_2 = 0\}$, as in Figure 1. During that interval one has $\dot{y}(t) = (0, -\theta_2, 0)'$ and hence the fluid path has the constant velocity vector

$$\dot{z}(t) = \theta + R\dot{y}(t) = \theta - \theta_2 R^2 = \begin{pmatrix} \theta_1 - \theta_2 R_{12} \\ 0 \\ \theta_3 - \theta_2 R_{32} \end{pmatrix}. \tag{3.5}$$

Thus (3.1) and (3.2) together give the following: as in Figure 1, a fluid path starting from state $(0, 0, \kappa)$ has an initial linear segment in which $z_3$ decreases, $z_1$ increases and $z_2$ remains at zero; that initial linear segment terminates at the point $z(\tau_1)$ on the $z_1$ axis that has

$$z_1(\tau_1) = \left(\frac{\theta_1 - \theta_2 R_{12}}{\theta_2 R_{32} - \theta_3}\right)\kappa > 0.$$

Similarly, from (3.1) and (3.3), the fluid path is linear over an ensuing time interval $[\tau_1, \tau_2]$, with $z_1$ decreasing, $z_2$ increasing and $z_3$ remaining at zero; that second linear segment terminates at the point $z(\tau_2)$ on the $z_2$ axis that has

$$z_2(\tau_2) = \left(\frac{\theta_1 - \theta_2 R_{12}}{\theta_2 R_{32} - \theta_3}\right)\left(\frac{\theta_2 - \theta_3 R_{23}}{\theta_3 R_{13} - \theta_1}\right)\kappa > 0.$$

Finally, from (3.1) and (3.4), the fluid path is linear over a next time interval $[\tau_2, \tau_3]$, with $z_2$ decreasing, $z_3$ increasing and $z_1$ remaining at zero; that third linear segment terminates at the point $z(\tau_3)$ on the $z_3$ axis that has $z_3(\tau_3) = \beta_1(\theta, R)\kappa$, where

$$\beta_1(\theta, R) = \left(\frac{\theta_1 - \theta_2 R_{12}}{\theta_2 R_{32} - \theta_3}\right)\left(\frac{\theta_2 - \theta_3 R_{23}}{\theta_3 R_{13} - \theta_1}\right)\left(\frac{\theta_3 - \theta_1 R_{31}}{\theta_1 R_{21} - \theta_2}\right) > 0. \tag{3.6}$$

Thereafter, the piecewise linear fluid path continues its counter-clockwise spiral on the boundary in a self-similar fashion, like the path pictured in Figure 1, except that in the general case defined by (3.1) through (3.4), the spiral may be either inward or outward, depending on whether $\beta_1(\theta, R) < 1$ or $\beta_1(\theta, R) > 1$.

To repeat, $C_1$ consists of all $(\theta, R)$ pairs that satisfy (3.1) through (3.4), and the *single-cycle gain* $\beta_1(\theta, R)$ for such a pair is defined by (3.6). As we have seen, fluid paths associated with problem data in $C_1$ spiral counter-clockwise on the boundary of $S$. Now let $C_2$ consist of all $(\theta, R)$ pairs that satisfy (3.1) and further satisfy (3.2) through (3.4) *with all six of the strict inequalities reversed*. It is more or less obvious that $(\theta, R)$ pairs in $C_2$ are those giving rise to *clockwise* spirals on the boundary, and the appropriate analog of (3.6) is

$$\begin{aligned}\beta_2(\theta, R) &= \frac{1}{\beta_1(\theta, R)} \\ &= \left(\frac{\theta_3 - \theta_2 R_{32}}{\theta_2 R_{12} - \theta_1}\right)\left(\frac{\theta_1 - \theta_3 R_{13}}{\theta_3 R_{23} - \theta_2}\right)\left(\frac{\theta_2 - \theta_1 R_{21}}{\theta_1 R_{31} - \theta_3}\right) > 0.\end{aligned} \tag{3.7}$$

Hereafter we define $C = C_1 \cup C_2$, $\beta(\theta, R) = \beta_1(\theta, R)$ for $(\theta, R) \in C_1$ and $\beta(\theta, R) = \beta_2(\theta, R)$ for $(\theta, R) \in C_2$. Thus $C$ consists of all $(\theta, R)$ pairs whose associated fluid paths spiral on the boundary, and $\beta(\theta, R)$ is the single-cycle gain for such a pair.

**4. Summary of results in three dimensions.** Theorem 2 below is a slightly weakened version of Theorem 1 by El Kharroubi, Ben Tahar and Yaacoubi (2002), which the original authors express in a more elaborate notation; we have deleted one part of their result that is irrelevant for current purposes. The corollary that follows is immediate from Theorem 1 above (the Dupuis–Williams fluid stability criterion) and Theorem 2.

THEOREM 2 [El Kharroubi, Ben Tahar and Yaacoubi (2002)]. *Suppose that* (1.6) *holds and that either of the following additional hypotheses is satisfied*: (a) $(\theta, R) \in C$ *and* $\beta(\theta, R) < 1$; *or* (b) $(\theta, R) \notin C$ *and the linear complementarity problem* (2.5)–(2.7) *has a unique solution*, *which is the proper solution* $(u^*, 0)$ *defined in* (2.8). *Then all fluid paths associated* $(\theta, R)$ *are attracted to the origin*.

COROLLARY. *Suppose that* (1.6) *holds and*, *in addition*, *either* (a) *or* (b) *holds*. *Then* $Z$ *is positive recurrent*.

The proof of Theorem 2 in El Kharroubi, Ben Tahar and Yaacoubi (2002) is not entirely rigorous, containing verbal passages that mask significant technical difficulties; an alternative proof that uses a linear Lyapunov function to prove stability is given in Dai and Harrison (2009).

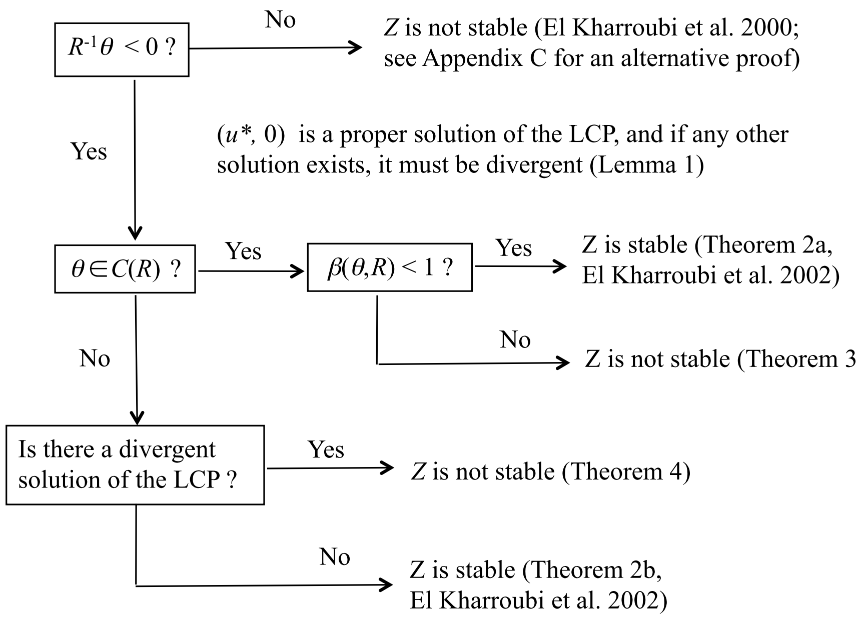


FIG. 2. *Summary of results in three dimensions.*

The new results of this paper are Theorems 3 and 4 below, which will be proved in Sections 5 through 7. Figure 2 summarizes the logic by which these new results combine with previously known results to provide necessary and sufficient conditions for stability (i.e., positive recurrence) of $Z$.

THEOREM 3. *If* $(\theta, R) \in C$ *and* $\beta(\theta, R) \geq 1$, *then* $Z$ *is not positive recurrent.*

THEOREM 4. *Suppose that* (1.6) *is satisfied. If there exists a divergent solution for the linear complementarity problem* (2.5)–(2.7), *then* $Z$ *is not positive recurrent.*

**5. Proof of Theorem 3.** Throughout this section and the next, we assume without loss of generality that our problem data satisfy not only (1.5) but also

$$\theta_i \in \{-1, 0, 1\} \qquad \text{for } i = 1, 2, 3. \tag{5.1}$$

Appendix B explains the scaling procedures that yield this normalized form. To prove Theorem 3 we will assume that $(\theta, R) \in C_1$ and $\beta_1(\theta, R) \geq 1$, then show that $Z$ is not stable; the proof of instability when $(\theta, R) \in C_2$ and $\beta_2(\theta, R) \geq 1$ is identical. Given the normalizations (1.5) and (5.1), the conditions (3.1) through (3.4) that define $C_1$ can be restated as follows:

$$\theta = (-1, -1, -1)', \tag{5.2}$$

$$R_{12}, R_{23}, R_{31} > 1 \quad \text{and} \quad R_{13}, R_{21}, R_{32} < 1. \tag{5.3}$$

Let us now define a $3 \times 3$ matrix $V$ by setting $V_{ij} = R_{ij} - 1$ for $i, j = 1, 2, 3$. Then $V^j$ (the $j$th column of $V$) is the vector $\theta - \theta_j R^j$ for $j = 1, 2, 3$. Note that $V^2$ was identified in (3.5) as the velocity vector on the face $\{Z_2 = 0\}$ for a fluid path corresponding to $(\theta, R)$.

LEMMA 2. *Under the assumption that $\beta_1(\theta, R) \geq 1$, there exists a vector $u > 0$ such that $u'V \geq 0$, or equivalently, $u'V^j \geq 0$ for each $j = 1, 2, 3$.*

PROOF. From (1.5) and (5.3) we have that

$$V = \begin{pmatrix} 0 & a_2 & -b_3 \\ -b_1 & 0 & a_3 \\ a_1 & -b_2 & 0 \end{pmatrix}, \tag{5.4}$$

where $a_i, b_i > 0$ for $i = 1, 2, 3$. In this notation, the definition (3.6) is as follows:

$$\beta_1(\theta, R) = \frac{a_1 a_2 a_3}{b_1 b_2 b_3}. \tag{5.5}$$

Setting

$$u_1 = 1, \qquad u_2 = \frac{a_1 a_2}{b_1 b_2} \quad \text{and} \quad u_3 = \frac{a_2}{b_2},$$

it is easy to verify that $u'V^1 = u'V^2 = 0$, and $u'V^3 = b_3(\frac{a_1 a_2 a_3}{b_1 b_2 b_3} - 1)$. The definition (5.5) and our assumption that $\beta_1(\theta, R) \geq 1$ then give $u'V^3 \geq 0$. $\square$

For the remainder of the proof of Theorem 3, let $e$ denote the three-vector of ones, so (5.2) is equivalently expressed as $\theta = -e$, and we can represent $X$ in (1.1) as

$$X(t) = X(0) + B(t) - et, \qquad t \geq 0, \tag{5.6}$$

where $B$ is a driftless Brownian motion with nonsingular covariance matrix and $B(0) = 0$. Also, we choose a starting state $x = X(0) = Z(0)$ that satisfies

$$Z_1(0) \geq 0, \qquad Z_2(0) = 0 \quad \text{and} \quad Z_3(0) > 0. \tag{5.7}$$

In this section, because the initial state is fixed, we write $\mathbb{E}(\cdot)$ rather than $\mathbb{E}_x(\cdot)$ to signify the expectation operator associated with the probability measure $\mathbb{P}_x$ (see Appendix A). Also, when we speak of stopping times and martingales, the relevant filtration is the one specified in Appendix A.

Let $u > 0$ be chosen to satisfy $u'V \geq 0$, as in Lemma 2, and further normalized so that $u'e = 1$. It is immediate from the definition of $V$ that $u'V = u'R - e'$, and thus one has the following:

$$u'R \geq e'. \tag{5.8}$$

Now define $\xi(t) = u'Z(t)$, $t \geq 0$. From (1.1), (5.6) and (5.8), one has

$$\begin{aligned} \xi(t) - \xi(0) &= u'B(t) - u'et + u'RY(t) \\ &\geq u'B(t) - t + e'Y(t) \qquad \text{for } t \geq 0. \end{aligned} \tag{5.9}$$

Next, let

$$\tau_1 = \inf\{t > 0 : Z_3(t) = 0\}, \qquad \tau_2 = \inf\{t > \tau_1 : Z_1(t) = 0\},$$
$$\tau_3 = \inf\{t > \tau_2 : Z_2(t) = 0\}$$

and so forth. (These stopping times are analogous to the points in time at which the piecewise linear fluid path in Figure 1 changes direction.) The crucial observation is the following: $Z_3(\cdot) > 0$ over the interval $[0, \tau_1)$, $Z_1(\cdot) > 0$ over $[\tau_1, \tau_2)$, $Z_2(\cdot) > 0$ over $[\tau_2, \tau_3)$ and so forth. Thus $Y_3(\cdot)$ does not increase over $[0, \tau_1)$, $Y_1(\cdot)$ does not increase over $[\tau_1, \tau_2)$, $Y_2(\cdot)$ does not increase over $[\tau_2, \tau_3)$ and so forth.

From (1.1) and (5.6), we then have the following relationships:

$$\begin{aligned} Z_2(t) &= B_2(t) - t + Y_2(t) \\ &\quad + R_{21}Y_1(t), \qquad 0 \leq t \leq \tau_1, \end{aligned} \tag{5.10}$$

$$\begin{aligned} Z_3(t) &= [B_3(t) - B_3(\tau_1)] - (t - \tau_1) + [Y_3(t) - Y_3(\tau_1)] \\ &\quad + R_{32}[Y_2(t) - Y_2(\tau_1)], \qquad \tau_1 \leq t \leq \tau_2, \end{aligned} \tag{5.11}$$

$$\begin{aligned} Z_1(t) &= [B_1(t) - B_1(\tau_2)] - (t - \tau_2) + [Y_1(t) - Y_1(\tau_1)] \\ &\quad + R_{13}[Y_3(t) - Y_3(\tau_2)], \qquad \tau_2 \leq t \leq \tau_3. \end{aligned} \tag{5.12}$$

There exist analogous representations for $Z_2$ over the time interval $[\tau_3, \tau_4]$, for $Z_3$ over $[\tau_4, \tau_5]$, for $Z_1$ over $[\tau_5, \tau_6]$, and so on. Now (5.10) gives

$$Y_2(t) = t + Z_2(t) - B_2(t) - R_{21}Y_1(t) \qquad \text{for } 0 \leq t \leq \tau_1. \tag{5.13}$$

Because $Y_3 \equiv 0$ on $[0, \tau_1)$, we can substitute (5.13) into (5.9) to obtain the following:

$$\begin{aligned} \xi(t) - \xi(0) &\geq u'B(t) - t + Y_1(t) \\ &\quad + [t + Z_2(t) - B_2(t) - R_{21}Y_1(t)] \qquad \text{for } 0 \leq t \leq \tau_1. \end{aligned} \tag{5.14}$$

From the definition of $V$ and (5.4), we have $1 - R_{21} = b_1 > 0$, so (5.14) can be rewritten

$$\xi(t) - \xi(0) \geq M(t) + A(t) \qquad \text{for } 0 \leq t \leq \tau_1, \tag{5.15}$$

where

$$M(t) = u'B(t) - B_2(t) \qquad \text{for } 0 \leq t \leq \tau_1, \tag{5.16}$$

$$A(t) = Z_2(t) + b_1Y_1(t) \qquad \text{for } 0 \leq t \leq \tau_1. \tag{5.17}$$

Defining $\tau = \lim \tau_n$, we now extend the definition (5.16) to all $t \in [0, \tau)$ as follows:

$$
\begin{aligned}
M(t) = M(\tau_1) + u'[B(t) - B(\tau_1)] \\
- [B_3(t) - B_3(\tau_1)] \qquad \text{for } \tau_1 \le t \le \tau_2,
\end{aligned} \tag{5.18}
$$

$$
\begin{aligned}
M(t) = M(\tau_2) + u'[B(t) - B(\tau_2)] \\
- [B_1(t) - B_1(\tau_2)] \qquad \text{for } \tau_2 \le t \le \tau_3
\end{aligned} \tag{5.19}
$$

and so forth. Finally, on $\{\tau < \infty\}$, we set $M(t) = M(\tau)$ for all $t \ge \tau$. Then $M = \{M(t), t \ge 0\}$ is a continuous martingale whose quadratic variation $\langle M, M\rangle(\cdot)$ satisfies

$$
\langle M, M\rangle(t) - \langle M, M\rangle(s) \le \gamma(t - s) \qquad \text{for } 0 < s < t < \infty, \tag{5.20}
$$

where $0 < \gamma < \infty$. Also, we extend (5.17) to all $t \in [0, \tau)$ via

$$
A(t) = A(\tau_1) + Z_3(t) + b_2[Y_2(t) - Y_2(\tau_1)] \qquad \text{for } \tau_1 \le t \le \tau_2, \tag{5.21}
$$

$$
A(t) = A(\tau_2) + Z_1(t) + b_3[Y_3(t) - Y_3(\tau_2)] \qquad \text{for } \tau_2 \le t \le \tau_3 \tag{5.22}
$$

and so forth. Thus the process $A = \{A(t), 0 \le t < \tau\}$ is nonnegative and continuous.

LEMMA 3. *$\xi(t) - \xi(0) \ge M(t) + A(t)$ for all $t \in [0, \tau)$.*

PROOF. It has already been shown in (5.15) that this inequality is valid for $0 \le t \le \tau_1$. In exactly the same way, but using (5.11) instead of (5.10), one obtains

$$
\xi(t) - \xi(\tau_1) = [M(t) - M(\tau_1)] + [A(t) - A(\tau_1)] \qquad \text{for } \tau_1 \le t \le \tau_2, \tag{5.23}
$$

so the desired inequality holds for $0 \le t \le \tau_2$. Continuing in this way, the desired inequality is established for $0 \le t < \tau$. □

To complete the proof of Theorem 3, let $T = \inf\{t > 0 : \xi(t) = \epsilon\}$ and let $\sigma = \inf\{t > 0 : \xi(0) + M(t) = \epsilon\}$, where $0 < \epsilon < \xi(0)$. From Lemma 3, the nonnegativity of $A(\cdot)$, and the fact that $\xi(\tau) = 0$ on $\{\tau < \infty\}$, we have the following inequalities: $0 < \sigma \le T \le \tau$. Thus it suffices to prove that $\mathbb{E}(\sigma) = \infty$, which can be shown by essentially the same argument that applies when $M$ is an ordinary (driftless) Brownian motion. That is, we first let $\sigma(b) = \inf\{t > 0 : \xi(0) + M(t) = \epsilon \text{ or } \xi(0) + M(t) = b\}$, where $b > \xi(0)$. Because both $M$ and $M^2 - \langle M, M\rangle$ are martingales and (5.20) holds, one has $0 < \mathbb{E}[\sigma(b)] < \infty$, $\mathbb{E}[M(\sigma(b))] = 0$ and $\mathbb{E}[M^2(\sigma(b))] = \mathbb{E}[\langle M, M\rangle(\sigma(b))]$. It follows by the optional sampling theorem that

$$
\begin{aligned}
\mathbb{E}[\langle M, M\rangle(\sigma(b))] &= \mathbb{E}[M^2(\sigma(b))] \\
&= \bigl(b - \xi(0)\bigr)^2 \frac{\xi(0) - \epsilon}{b - \epsilon} + \bigl(\xi(0) - \epsilon\bigr)^2 \frac{b - \xi(0)}{b - \epsilon} \\
&= \bigl(b - \xi(0)\bigr)\bigl(\xi(0) - \epsilon\bigr).
\end{aligned} \tag{5.24}
$$

The left-hand side of (5.24) is $\le \gamma E[\sigma(b)]$ by (5.20), the right-hand side $\uparrow \infty$ as $b \uparrow \infty$, and obviously $\sigma \ge \sigma(b)$ for all $b > \xi(0)$. Thus $\mathbb{E}(\sigma) = \infty$, and the proof of Theorem 3 is complete.

**6. Categories of divergent LCP solutions.** Our goal in the remainder of the paper is to prove Theorem 4. We continue to assume the canonical problem format in which $R$ satisfies (1.5) and $\theta$ satisfies (5.1). In the following lemma and later, the term "LCP solution" is used to mean a solution $(u, v)$ of the linear complementarity problem (2.5)–(2.7).

LEMMA 4. *If* (1.6) *holds, then* (a) $\theta \ge 0$ *is not possible, and* (b) *there exists no LCP solution* $(u, v)$ *with* $v > 0$.

PROOF. Because $R$ is completely $\mathcal{S}$ by assumption, its transpose is also completely $\mathcal{S}$; cf. Proposition 1.1 of Dai and Williams (1995). Thus there exists a vector $a > 0$ such that $a'R > 0$. Now (1.6) says that $\theta + Ry = 0$ for $y > 0$. Multiplying both sides of the equation by $a'$ and rearranging terms, one has $a'\theta = -a'Ry < 0$, which implies conclusion (a). Also, if $(u, v)$ is a LCP solution with $v > 0$, one has from (2.7) and (2.6) that $u = 0$ and $v = \theta$, which contradicts conclusion (a). This implies conclusion (b). $\square$

We now define five nonoverlapping categories of divergent LCP solutions. Immediately after each category is defined, we shall exhibit a pair $(R, \theta)$ which admits a LCP solution $(u, v)$ in that category, or else direct the reader to a proposition that shows the category to be empty. Readers may verify that the reflection matrix $R$ appearing in each of our examples is completely $\mathcal{S}$. Also, defining $u^* = -R^{-1}\theta$ as in (2.8), we shall display the vector $u^*$ for each example, showing that $u^* > 0$ and hence (1.6) is satisfied.

CATEGORY I. Exactly two components of $v$ are positive, and the complementary component of $u$ is positive. The following is such an example:

$$R = \begin{pmatrix} 1 & 1/3 & 1/3 \\ 2 & 1 & -1/2 \\ 2 & -1/2 & 1 \end{pmatrix}, \qquad \theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},$$

$$u^* = \begin{pmatrix} 1/5 \\ 6/5 \\ 6/5 \end{pmatrix}, \qquad u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

CATEGORY II. Exactly one component of $v$ is positive, $\det(\hat{R}) > 0$, and the two complementary components of $u$ are not both zero, where $\hat{R}$ is the $2 \times 2$

principal submatrix of $R$ corresponding to the two zero components of $v$. Such an example is given by

$$R = \begin{pmatrix} 1 & 1 & 1/2 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \qquad \theta = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix},$$
$$u^* = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \qquad u = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Here, the two complementary components of $u$ are both positive. In the following example, which also falls in Category II, just one of them is positive:

$$R = \begin{pmatrix} 1 & 1/2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}, \qquad \theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},$$
$$u^* = \begin{pmatrix} 1/5 \\ 2/5 \\ 1/5 \end{pmatrix}, \qquad u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

CATEGORY III. Exactly one component of $v$ is positive, $\det(\hat{R}) = 0$, and the two complementary components of $u$ are not both zero. In Lemma 8, it will be shown that no such LCP solutions exist if (1.6) holds.

CATEGORY IV. Exactly one component of $v$ is positive, $\det(\hat{R}) < 0$, and the two complementary components of $u$ are both positive. Such an example is given by

$$R = \begin{pmatrix} 1 & 11/10 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \qquad \theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},$$
$$u^* = \begin{pmatrix} 19/68 \\ 15/34 \\ 2/17 \end{pmatrix}, \qquad u = \begin{pmatrix} 1/12 \\ 5/6 \\ 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ 0 \\ 2/3 \end{pmatrix}.$$

It will be shown in Lemma 7 that if there exists a LCP solution in Category IV, under our restrictions on $R$ and $\theta$, there also exists a solution in Category I or Category II (or both). For the example above, a second LCP solution is $(\hat{u}, \hat{v})$, where $\hat{u} = (0, 1, 0)'$ and $\hat{v} = (1/10, 0, 1)'$; this second solution lies in Category I.

CATEGORY V. Exactly one component of $v$ is positive, $\det(\hat{R}) < 0$, and exactly one of the two complementary components of $u$ is positive. Such an example is given by

$$R = \begin{pmatrix} 1 & 1 & -2/5 \\ 2 & 1 & -6/5 \\ -2 & -1/10 & 1 \end{pmatrix}, \qquad \theta = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$
$$u^* = \begin{pmatrix} 9/8 \\ 5/14 \\ 45/28 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ 0 \\ 0.9 \end{pmatrix}.$$

LEMMA 5. *Suppose that* (1.6) *holds and that* $(u, v)$ *is a divergent LCP solution. Then* $(u, v)$ *belongs to one of the five categories defined immediately above.*

PROOF. Let $m$ and $n$ denote the number of positive components in $u$ and $v$, respectively; the complementarity condition (2.7) implies that $m+n \le 3$. Lemma 4 shows that $n < 3$ (i.e., $v > 0$ cannot hold); also $n > 0$, because $(u, v)$ is a divergent LCP solution by assumption. Thus, either $n = 1$ or $n = 2$. Moreover, it is not possible that $m = 0$, or equivalently $u = 0$, because then (2.5) and (2.6) would imply $\theta = v \ge 0$, which contradicts Lemma 4. So the only remaining possibilities are $(m, n) = (1, 2)$, $(m, n) = (2, 1)$ and $(m, n) = (1, 1)$. Category I is precisely the case where $(m, n) = (1, 2)$, and Categories II through V together cover the cases where $(m, n) = (2, 1)$ and $(m, n) = (1, 1)$. $\square$

It will be shown in Section 7 that $Z$ cannot be positive recurrent if there exists a LCP solution in Category I, Category II or Category V. Lemma 7 in this section will show that the existence of a LCP solution in Category IV implies the existence of a LCP solution in either Category I or Category II. Lemma 8 at the end of this section will show that LCP solutions in Category III cannot occur when (1.6) holds. In combination with Lemma 5 above, these results obviously imply Theorem 4.

We now state and prove Lemma 6, which we need in order to prove Lemma 7. Our scaling convention (1.5) specifies that $R$ has ones on the diagonal, so we can write

$$R = \begin{pmatrix} 1 & a' & c \\ a & 1 & c' \\ b & b' & 1 \end{pmatrix} \tag{6.1}$$

for some constants $a, a', b, b', c$ and $c'$.

LEMMA 6. *Assume that there does not exist a LCP solution in Category* I, *and that there is a divergent LCP solution* $(u, v)$ *with* $u_1 > 0$, $u_2 > 0$, $u_3 = 0$, $v_1 = v_2 = 0$ *and* $v_3 > 0$. *Let* $R$ *be as in* (6.1) *and assume that the principal submatrix* $\hat{R}$ *corresponding to the zero components of* $v$ *satisfies* $\det(\hat{R}) < 0$. *Then* $\theta = (-1, -1, 1)'$ *and* $a, a' > 1$.

PROOF. Because $(u, v)$ is a solution of the LCP (2.5)–(2.7), one has

$$\begin{pmatrix} 1 & a' & c \\ a & 1 & c' \\ b & b' & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\theta_1 \\ -\theta_2 \\ -\theta_3 + v_3 \end{pmatrix}. \tag{6.2}$$

Because $v_1 = v_2 = 0$ and $v_3 > 0$,

$$\hat{R} = \begin{pmatrix} 1 & a' \\ a & 1 \end{pmatrix}. \tag{6.3}$$

Setting $\hat{u} = (u_1, u_2)'$ and $\hat{\theta} = (\theta_1, \theta_2)'$, we have from (6.2) that

$$\hat{R}\hat{u} = -\hat{\theta}. \tag{6.4}$$

Because $\hat{R}$ is an $\mathcal{S}$-matrix with negative determinant,

$$a, a' > 0 \quad \text{and} \quad aa' > 1. \tag{6.5}$$

Because $u_1 > 0$ and $u_2 > 0$ by hypothesis, it is immediate from (6.4) and (6.5) that both components of $\hat{\theta}$ are negative, so our canonical rescaling gives $\hat{\theta} = (-1, -1)'$. Thus, either $\theta = (-1, -1, -1)'$, $(-1, -1, 0)'$ or $(-1, -1, 1)'$ must hold. From (6.5), (6.4) and $\hat{\theta} = (-1, -1)'$, it follows that

$$a, a' > 1. \tag{6.6}$$

We will show that $\theta = (-1, -1, 1)'$ by excluding the other two cases. Suppose first that $\theta = (-1, -1, -1)'$. Then (6.2) becomes

$$\begin{pmatrix} 1 & a' & c \\ a & 1 & c' \\ b & b' & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 + v_3 \end{pmatrix}. \tag{6.7}$$

It must be true that $b, b' \le 1$; otherwise there would be a solution of the LCP that falls into Category I. For example, if $b > 1$ one then has a divergent LCP solution $(\bar{u}, \bar{v})$ with $\bar{u} = (1, 0, 0)'$, $\bar{v}_1 = 0$, $\bar{v}_2 = a - 1 > 0$ and $\bar{v}_3 = b - 1 > 0$. However, one cannot have $a, a' \ge 1$, $b, b' \le 1$ and (6.7) holding simultaneously, which gives a contradiction.

Next suppose that $\theta = (-1, -1, 0)'$. Here, (6.7) holds with $v_3$ in place of $1 + v_3$ on the right-hand side. We must have $b, b' \le 0$, for the same reason as before. This results in a contradiction, and the only remaining possibility under our canonical rescaling is $\theta = (-1, -1, 1)'$. □

LEMMA 7. *If there exists a LCP solution in Category* IV, *then there also exists a solution in Category* I *or Category* II (*or both*).

PROOF. Denoting by $(u, v)$ a solution in Category IV, we assume that no solution in Category I exists. It will then suffice to prove that a solution in Category II exists. By permuting the indices, we can assume that $u_1 > 0$, $u_2 > 0$, $u_3 = 0$, $v_1 = v_2 = 0$ and $v_3 > 0$.

We use the notation in (6.1). By Lemma 6, one has $a, a' > 1$ and $\theta = (-1, -1, 1)'$. We shall assume that $c' \geq c$, then construct a LCP solution $(\tilde{u}, \tilde{v})$ that falls into Category II, with $\tilde{u}_1 > 0$ , $\tilde{u}_3 > 0$, $\tilde{v}_2 > 0$; in exactly the same way, if $c \geq c'$, one can construct a LCP solution $(\tilde{u}, \tilde{v})$ that falls into Category II, with $\tilde{v}_1 > 0$ , $\tilde{u}_2 > 0$ , $\tilde{u}_3 > 0$.

We first observe that $b, b' \leq -1$. Otherwise, contrary to the assumption imposed in the first paragraph of the proof, there would exist a LCP solution in Category I. For example, if $b > -1$, then there is a divergent LCP solution $(\bar{u}, \bar{v})$ with $\bar{u} = (1, 0, 0)'$, $\bar{v}_1 = 0$, $\bar{v}_2 = a - 1 > 0$, and $\bar{v}_3 = b + 1 > 0$.

The $2 \times 2$ submatrix of $R$ that is relevant to our construction is

$$\tilde{R} = \begin{pmatrix} 1 & c \\ b & 1 \end{pmatrix}.$$

Because $\tilde{R}$ is an $\mathcal{S}$-matrix and $b < 0$, we know that $bc < 1$, hence $\det(\tilde{R}) > 0$, and because $b \leq -1$ we also know that $c > -1$. Letting $\gamma = (\gamma_1, \gamma_2)'$ be the two-vector satisfying $\tilde{R}\gamma = (1, -1)'$, one has

$$\gamma = \frac{1}{1 - bc} \begin{pmatrix} 1 & -c \\ -b & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{1 - bc} \begin{pmatrix} 1 + c \\ -1 - b \end{pmatrix}.$$

Defining $\tilde{u} = (\gamma_1, 0, \gamma_2)'$ and $\tilde{v} = \theta + R\tilde{u}$, it follows that $\tilde{v}_1 = \tilde{v}_3 = 0$. Comparing the first and second rows of $R$ term by term, and noting that the first two components of $\theta$ are identical, one sees that

$$\tilde{v}_2 - \tilde{v}_1 = \frac{1}{1 - bc}[(a - 1)(1 + c) - (c' - c)(1 + b)]. \tag{6.8}$$

Because of the inequalities $a > 1$, $c > -1$, $c' \geq c$ and $b \leq -1$, the quantity inside the square brackets in (6.8) is positive. Thus $\tilde{v}_2 > 0$, and hence $(\tilde{u}, \tilde{v})$ is a LCP solution in Category II. $\square$

LEMMA 8. *If* (1.6) *holds, then there cannot exist a LCP solution in Category* III.

PROOF. Arguing as in the proof of Lemma 7, we assume the existence of a LCP solution $(u, v)$ in Category III. By permuting the indices, we can assume that $u_1 > 0$, $u_2 \geq 0$, $u_3 = 0$, $v_1 = v_2 = 0$ and $v_3 > 0$. We use the notation (6.1) and define $\hat{R}$ by (6.3). A minor variation of the first paragraph in the proof of Lemma 6 shows for the current case that both $\theta_1$ and $\theta_2$ are negative, and so $\theta_1 = \theta_2 = -1$ with our scaling convention. One then has $a = a' = 1$ in (6.1), because $\det(\hat{R}) = 0$ by assumption. By (1.6), $\theta + Ru^* = 0$ for some $u^* > 0$, from which it follows that $c = c'$ in (6.1). That is, the first two rows of $R$ are identical, whereas (1.6) includes the requirement that $R$ be nonsingular. $\square$

**7. Proof of Theorem 4.** As we explained immediately after the proof of Lemma 5 in Section 6, the proofs of Lemmas 9, 10 and 12 in this section will complete the proof of Theorem 4. In Lemmas 9 and 10, we actually prove that $Z$ is transient, which is stronger than we require for Theorem 4. The SRBM $Z$ is said to be *transient* if there exists an open ball $C$ centered at the origin such that $\mathbb{P}\{\tau_C = \infty\} > 0$ for some initial state $Z(0) = x \in \mathbb{R}^3_+$ that is outside of the ball, where $\tau_C = \inf\{t \geq 0 : Z(t) \in C\}$. Clearly, when $Z$ is transient, it is not positive recurrent. In this section, we continue to assume the canonical problem format in which $R$ satisfies (1.5) and $\theta$ satisfies (5.1).

LEMMA 9. *If there is a LCP solution $(u, v)$ in Category* I, *then $Z$ is transient.*

PROOF. Without loss of generality, we assume that

$$u_1 > 0, \qquad u_2 = 0, \qquad u_3 = 0, \qquad v_1 = 0, \qquad v_2 > 0, \qquad v_3 > 0. \tag{7.1}$$

Because $v = \theta + Ru$, with $R$ as in (6.1), $\theta_1 < 0$, and so, by our scaling convention, $\theta_1 = -1$. It follows from this that

$$u_1 = 1, \qquad v_2 = \theta_2 + a > 0 \quad \text{and} \quad v_3 = \theta_3 + b > 0. \tag{7.2}$$

One can write (1.1) as

$$Z_1(t) = Z_1(0) + \theta_1 t + B_1(t) + Y_1(t) + a' Y_2(t) + c Y_3(t), \tag{7.3}$$

$$Z_2(t) = Z_2(0) + \theta_2 t + B_2(t) + a Y_1(t) + Y_2(t) + c' Y_3(t), \tag{7.4}$$

$$Z_3(t) = Z_3(0) + \theta_3 t + B_3(t) + b Y_1(t) + b' Y_2(t) + Y_3(t) \tag{7.5}$$

for $t \geq 0$, where $B = \{B(t), t \geq 0\}$ is the three-dimensional driftless Brownian motion with covariance matrix $\Gamma$. Assume $Z(0) = (0, N, N)'$ for some constant $N > 1$ and set $\tau = \inf\{t \geq 0 : Z_2(t) = 1 \text{ or } Z_3(t) = 1\}$. We will show that $\mathbb{P}\{\tau = \infty\} > 0$ for sufficiently large $N$, and thus $Z$ is transient. Because $\theta_1 = -1$ and $Y_2(t) = Y_3(t) = 0$ for $t \in [0, \tau)$, one has, for $t < \tau$,

$$Z_1(t) = -t + B_1(t) + Y_1(t),$$

$$Z_2(t) = N + \theta_2 t + B_2(t) + a Y_1(t),$$

$$Z_3(t) = N + \theta_3 t + B_3(t) + b Y_1(t).$$

For $t \geq 0$, let $\hat{Y}_1(t) = \sup_{0 \leq s \leq t} (-s + B_1(s))^-$, and set

$$\hat{Z}_1(t) = -t + B_1(t) + \hat{Y}_1(t), \tag{7.6}$$

$$\hat{Z}_2(t) = N + \theta_2 t + B_2(t) + a \hat{Y}_1(t), \tag{7.7}$$

$$\hat{Z}_3(t) = N + \theta_3 t + B_3(t) + b \hat{Y}_1(t) \tag{7.8}$$

for $t \geq 0$. Clearly, $Z(t) = \hat{Z}(t)$ for $t \in [0, \tau]$. In particular, $\tau = \hat{\tau}$, where $\hat{\tau} = \inf\{t \geq 0 : \hat{Z}_2(t) = 1 \text{ or } \hat{Z}_3(t) = 1\}$. To show $Z$ is transient, it suffices to prove that, for sufficiently large $N$, $\mathbb{P}\{\hat{\tau} = \infty\} > 0$.

By the functional strong law of large numbers (FSLLN) for a driftless Brownian motion, one has

$$\lim_{t\to\infty} \frac{1}{t} \sup_{0\le s\le 1} |B(ts)| = 0 \qquad \text{almost surely.}$$

This implies that

$$\begin{aligned}\lim_{t\to\infty} t^{-1}\hat{Y}_1(t) &= \lim_{t\to\infty} t^{-1} \sup_{0\le s\le t} \bigl(-s + B_1(s)\bigr)^- \\ &= \lim_{t\to\infty} \sup_{0\le s\le 1} \bigl(-s + t^{-1}B_1(ts)\bigr)^- \\ &= \sup_{0\le s\le 1} (-s)^- = 1 \qquad \text{almost surely.}\end{aligned}$$

Therefore, by (7.2), (7.7) and (7.8), one has $\lim_{t\to\infty} t^{-1}\hat{Z}_2(t) = v_2 > 0$ and $\lim_{t\to\infty} t^{-1}\hat{Z}_3(t) = v_3 > 0$ almost surely. Consequently, there exists a constant $T > 0$ such that $\mathbb{P}(A) \ge 3/4$ for all $N \ge 1$, where

$$A = \{\hat{Z}_2(t) > 1 \text{ and } \hat{Z}_3(t) > 1 \text{ for all } t \ge T\}. \tag{7.9}$$

One can choose $N$ large enough so that $\mathbb{P}(B) \ge 3/4$, where

$$B = \{\hat{Z}_2(t) > 1 \text{ and } \hat{Z}_3(t) > 1 \text{ for all } t \in [0, T]\}. \tag{7.10}$$

Because $A \cap B \subset \{\hat{\tau} = \infty\}$, $\mathbb{P}\{\hat{\tau} = \infty\} \ge 1/2 > 0$, as desired. □

LEMMA 10. *If there is an LCP solution* $(u, v)$ *in Category* II, *then* $Z$ *is transient.*

PROOF. Without loss of generality, we assume that

$$u_1 > 0, \qquad u_2 \ge 0, \qquad u_3 = 0, \qquad v_1 = 0, \qquad v_2 = 0, \qquad v_3 > 0. \tag{7.11}$$

Assume $R$ is given by (6.1), and let $\hat{R}$ be the $2 \times 2$ principal submatrix of $R$ given by (6.3). By assumption, $\det(\hat{R}) > 0$. One can check that conditions (2.5)–(2.7) and (7.11) imply that

$$\hat{R}^{-1}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} \le 0, \tag{7.12}$$

$$v_3 = \theta_3 - (b, b')\hat{R}^{-1}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} > 0. \tag{7.13}$$

Let $Z(0) = (0, 0, N)'$ for some constant $N > 1$ and set $\tau = \inf\{t \ge 0 : Z_3(t) = 1\}$. We will show that for sufficiently large $N$, $\mathbb{P}\{\tau = \infty\} > 0$, and thus $Z$ is transient.

On $\{t < \tau\}$, one has $Z_3(t) > 0$ and thus $Y_3(t) = 0$. Because the SRBM $Z$ satisfies Equation (1.1), on $\{t < \tau\}$,

$$
\begin{aligned}
Z(t) &= Z(0) + \theta t + B(t) + R \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ 0 \end{pmatrix} \\
&= Z(0) + \theta t + B(t) + \tilde{R} \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \end{pmatrix},
\end{aligned} \tag{7.14}
$$

where

$$\tilde{R} = \begin{pmatrix} 1 & a' & 0 \\ a & 1 & 0 \\ b & b' & 1 \end{pmatrix}.$$

One can check that because $R$ is completely $\mathcal{S}$, so is $\tilde{R}$. It therefore follows from Taylor and Williams (1993) that there exists a SRBM $\tilde{Z}$ associated with the data $(\mathbb{R}^3_+, \theta, \Gamma, \tilde{R})$ that starts from $Z(0)$. Following Definition 6 in Appendix A, the three-dimensional process $\tilde{Z}$, together with the corresponding processes $\tilde{B}$ and $\tilde{Y}$, is defined on some filtered probability space $(\tilde{\Omega}, \{\tilde{\mathcal{F}}_t\}, \tilde{\mathbb{P}})$; $\tilde{B}$, $\tilde{Y}$ and $\tilde{Z}$ are adapted to $\{\tilde{\mathcal{F}}_t\}$, and $\tilde{\mathbb{P}}$-almost surely satisfy (1.1)–(1.4); $\tilde{B}$ is a driftless Brownian motion with covariance matrix $\Gamma$, and $\tilde{B}$ is an $\{\tilde{\mathcal{F}}_t\}$-martingale. Furthermore, from Taylor and Williams (1993), the distribution of $\tilde{Z}$ is unique. Because of (7.14), $B$, $Y$ and $Z$ also satisfy (1.1)–(1.4) on $\{t < \tau\}$, with the same data $(\mathbb{R}^3_+, \theta, \Gamma, \tilde{R})$, and so $\tau = \tilde{\tau}$ in distribution, where

$$\tilde{\tau} = \inf\{t \geq 0 : \tilde{Z}_3(t) = 1\}.$$

We now show that for sufficiently large $N$,

$$\tilde{\mathbb{P}}\{\tilde{\tau} = \infty\} > 0, \tag{7.15}$$

which implies that $\mathbb{P}\{\tau = \infty\} > 0$. We note that $(\tilde{Z}_1, \tilde{Z}_2)$ is a two-dimensional SRBM with data $(\mathbb{R}^2_+, \hat{\theta}, \hat{\Gamma}, \hat{R})$, where $\hat{\theta} = (\theta_1, \theta_2)'$ and $\hat{\Gamma}$ is the $2 \times 2$ principal submatrix of $\Gamma$ obtained by deleting the 3rd row and the 3rd column of $\Gamma$. Lemma 14 in Appendix D will show that when $\hat{R}$ is a $\mathcal{P}$-matrix and the condition (7.12) is satisfied, the two-dimensional SRBM $\tilde{Z}$ is "rate stable" in the sense that

$$\lim_{t\to\infty} \frac{1}{t} \tilde{Z}_i(t) = 0 \qquad \text{almost surely, } i = 1, 2. \tag{7.16}$$

Solving for $\tilde{Y}_1$ and $\tilde{Y}_2$ in the first two components of (1.1) and plugging them into the third component yields

$$
\begin{aligned}
\tilde{Z}_3(t) &= N + \theta_3 t + \tilde{B}_3(t) \\
&\quad + (b, b')\tilde{R}^{-1}\left[\begin{pmatrix} \tilde{Z}_1(t) \\ \tilde{Z}_2(t) \end{pmatrix} - \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} t - \begin{pmatrix} \tilde{B}_1(t) \\ \tilde{B}_2(t) \end{pmatrix}\right] \tilde{Y}_3(t), \qquad t \geq 0.
\end{aligned} \tag{7.17}
$$

Equations (7.13), (7.16) and (7.17), together the SLLN for Brownian motion, imply that

$$\liminf_{t\to\infty} \frac{1}{t}\tilde{Z}_3(t) \geq v_3 \qquad \text{almost surely.}$$

Because $v_3 > 0$, one can argue as in (7.9) and (7.10) that for $N$ large enough, $\mathbb{P}\{\tilde{Z}_3(t) > 1 \text{ for all } t \geq 0\} > 0$. This proves (7.15). $\square$

Before stating and proving Lemma 12 for a LCP solution in Category V, we state the following lemma, which is needed in the proof of Lemma 12 and will be proved at the end of this section.

LEMMA 11. *Let $B = (B_1, B_2, B_3)$ be a three-dimensional Brownian motion with zero drift and covariance matrix $\Gamma$, starting from 0. Set*

$$Z_1(t) = -t + B_1(t) + Y_1(t), \tag{7.18}$$

$$Z_2(t) = 2 + B_2(t) - B_1(t) + Z_1(t), \tag{7.19}$$

$$Z_3(t) = 4N + 3\mu t + B_3(t) + aB_1(t) - aZ_1(t) \tag{7.20}$$

*for $t \geq 0$, and given constants $a$, $\mu > 0$ and $N \geq 1$, where*

$$Y_1(t) = \sup_{0\leq s\leq t} \bigl(-s + B_1(s)\bigr)^{-}.$$

*Then for sufficiently large $N$, one has $\mathbb{E}(\sigma) = \infty$, where $\sigma = \inf\{t \geq 0 : Z_2(t) = 1$ or $Z_3(t) = 1\}$.*

LEMMA 12. *If there is a LCP solution $(u, v)$ in Category V, then $Z$ is not positive recurrent.*

PROOF. Without loss of generality, we assume that

$$u_1 > 0, \qquad u_2 = 0, \qquad u_3 = 0, \qquad v_1 = 0, \qquad v_2 = 0, \qquad v_3 > 0. \tag{7.21}$$

Then a minor variation of the first paragraph in the proof of Lemma 6 establishes that both $\theta_1$ and $\theta_2$ are negative, so $\theta_1 = \theta_2 = -1$ with our scaling convention. Assuming $R$ is as in (6.1), it follows as in (6.2) that $u_1 = 1$, $a = 1$ and $v_3 = b + \theta_3 > 0$.

Let $Z(0) = (0, 2, N)'$ for some constant $N > 1$ and let

$$\tau_i = \inf\{t \geq 0 : Z_i(t) = 1\}, \qquad i = 2, 3,$$

with $\tau = \min(\tau_2, \tau_3)$. We will show that $\mathbb{E}_x(\tau) = \infty$ for sufficiently large $N$, which implies that $Z$ is not positive recurrent.

The SRBM $Z$ satisfies equations (7.3)–(7.5). Since $Z_2(t) > 0$ and $Z_3(t) > 0$ for $t < \tau$, one has $Y_2(t) = Y_3(t) = 0$ for $t < \tau$. Because $a = 1$, (7.3)–(7.5) reduce to

$$Z_1(t) = -t + B_1(t) + Y_1(t), \tag{7.22}$$

$$Z_2(t) = 2 - t + B_2(t) + Y_1(t), \tag{7.23}$$

$$Z_3(t) = N + \theta_3 t + B_3(t) + bY_1(t) \tag{7.24}$$

on $t < \tau$. By (7.22), one has $Y_1(t) = Z_1(t) + t - B_1(t)$ for $t < \tau$. Substituting $Y_1(t)$ into (7.23) and (7.24), one has

$$Z_2(t) = 2 + B_2(t) - B_1(t) + Z_1(t),$$

$$Z_3(t) = N + v_3 t + B_3(t) - bB_1(t) + bZ_1(t)$$

on $t < \tau$.

For each $t \geq 0$, let $\hat{Y}_1(t) = \sup_{0\leq s\leq t}(-s + B_1(s))^-$, and set

$$\hat{Z}_1(t) = -t + B_1(t) + \hat{Y}_1(t),$$

$$\hat{Z}_2(t) = 2 + B_2(t) - B_1(t) + \hat{Z}_1(t),$$

$$\hat{Z}_3(t) = N + v_3 t + B_3(t) - bB_1(t) + b\hat{Z}_1(t)$$

for $t \geq 0$. Let $\hat{\tau} = \inf\{t \geq 0 : \hat{Z}_2(t) = 1 \text{ or } \hat{Z}_3(t) = 1\}$; clearly, $\tau = \hat{\tau}$ on every sample path. It follows from Lemma 11 that $\mathbb{E}(\hat{\tau}) = \infty$ for sufficiently large $N$. Therefore, $\mathbb{E}_x(\tau) = \infty$, and so $Z$ is not positive recurrent. $\square$

PROOF OF LEMMA 11. We first prove the case when $a > 0$. When $a \leq 0$, the proof is actually significantly simpler; an outline for this case will be presented at the end of this proof.

Let $X_2(t) = 1 + B_2(t) - B_1(t)$ and $X_3(t) = B_3(t) + aB_2(t)$ for $t \geq 0$. Then $X_2$ is a Brownian motion starting from 1, $X_3$ is a Brownian motion starting from 0, and (7.19)–(7.20) become

$$Z_2(t) = 1 + X_2(t) + Z_1(t) \geq 1 + X_2(t), \tag{7.25}$$

$$\begin{aligned} Z_3(t) = {} & (N + a) + \big(N + \mu t + X_3(t)\big) \\ & + \big(2N + 2\mu t - aX_2(t) - aZ_1(t)\big) \end{aligned} \tag{7.26}$$

for $t \geq 0$. Define

$$\tau_1 = \inf\{t \geq 0 : X_2(t) \leq 0\},$$

$$\tau_2 = \inf\{t \geq 0 : aX_2(t) \geq 2N + 2\mu t - aZ_1(t) \text{ or } aX_2(t) \geq N + \mu t\},$$

$$\tau_3 = \inf\{t \geq 0 : X_3(t) \leq -(N + \mu t)\},$$

$$\tau = \tau_1 \wedge \tau_2 \wedge \tau_3.$$

Assuming $N + a > 1$, it follows from the definition of $\tau$ and (7.25)–(7.26) that for each $t < \tau$, $Z_2(t) > 1$ and $Z_3(t) > 1$, and so $\tau \le \sigma$. To prove the lemma, it therefore suffices to show that

$$\mathbb{E}(\tau) = \infty. \tag{7.27}$$

Since $X_2$ is a driftless Brownian motion, $\mathbb{E}(\tau_1) = \infty$. When $N$ is large, it is intuitively clear that $\tau_1 > \tau_2 \wedge \tau_3$ with only negligible probability, which leads to $\mathbb{E}(\tau) = \infty$. To make the argument rigorous, first note that, because $Z_1$ is adapted to $B_1$, each $\tau_i$ is a stopping time with respect to the filtration generated by the Brownian motion $B$, and hence $\tau$ is a stopping time as well. Because $X_2(0) = 1$ and $X_2$ is a martingale with respect to the filtration generated by $B$, by the optional sampling theorem,

$$\mathbb{E}\big(X_2(\tau \wedge t)\big) = 1$$

for each $t \ge 0$. We will show that, for sufficiently large $N$,

$$\mathbb{E}\big(X_2(\tau)1_{\{\tau<\infty\}}\big) \le \tfrac{1}{2}. \tag{7.28}$$

It follows that

$$\begin{aligned} 1 &= \mathbb{E}\big(X_2(\tau \wedge t)\big) \\ &= \mathbb{E}\big(X_2(\tau)1_{\{\tau<t\}}\big) + \mathbb{E}\big(X_2(t)1_{\{\tau\ge t\}}\big) \\ &\le \mathbb{E}\big(X_2(\tau)1_{\{\tau<\infty\}}\big) + \mathbb{E}\big(((N + \mu t)/a)1_{\{\tau\ge t\}}\big) \\ &= \tfrac{1}{2} + \big((N + \mu t)/a\big)\mathbb{P}\{\tau \ge t\}, \end{aligned}$$

where we have used $X_2(t) \le (N + \mu t)/a$ on $t \le \tau_2$ for the inequality. Consequently,

$$\mathbb{P}\{\tau \ge t\} \ge \frac{a}{2(N + \mu t)} \tag{7.29}$$

for each $t \ge 0$, from which $\mathbb{E}(\tau) = \infty$ follows.

It remains to prove (7.28). Because $X_2(\tau_1) = 0$ when $\tau_1$ is finite,

$$\begin{aligned} \mathbb{E}\big(X_2(\tau)1_{\{\tau<\infty\}}\big) &= \mathbb{E}\big(X_2(\tau_2 \wedge \tau_3)1_{\{\tau_2\wedge\tau_3<\tau_1\}}\big) \\ &\le \mathbb{E}\big(\big(\big(N + \mu(\tau_2 \wedge \tau_3)\big)/a\big)1_{\{\tau_2\wedge\tau_3<\infty\}}\big) \\ &\le \sum_{n=0}^{\infty}\big(\big(N + \mu(n+1)\big)/a\big)\mathbb{P}\{n < \tau_2 \wedge \tau_3 \le n+1\}. \end{aligned} \tag{7.30}$$

To bound the probability $\mathbb{P}\{n < \tau_2 \wedge \tau_3 \le n+1\}$ for each $n \in \mathbb{Z}_+$, we use

$$\{n < \tau_2 \wedge \tau_3 \le n+1\} \subset \{n < \tau_2 \le n+1\} \cup \{n < \tau_3 \le n+1\}.$$

For $\{n < \tau_3 \le n+1\}$, $X_3(t) \le -(N + \mu n)$ first occurs on $(n, n+1]$. By the strong Markov property for Brownian motion and the reflection principle, the probability of the latter event is at most $2\mathbb{P}\{X_3(n+1) < -(N+\mu n)\}$. For $\{n < \tau_2 \le n+1\}$, either $aX_2(t) \ge N + \mu n$ first occurs on $(n, n+1]$, or $aZ_1(t) \ge N + \mu n$ occurs on $(n, n+1]$. One can also apply the strong Markov property and the reflection principle to the first event. One therefore obtains

$$\begin{aligned}\mathbb{P}\{n < \tau_2 \wedge \tau_3 \le n+1\} &\le \mathbb{P}\{n < \tau_2 \le n+1\} + \mathbb{P}\{n < \tau_3 \le n+1\}\\ &\le 2\mathbb{P}\{X_3(n+1) < -(N+\mu n)\}\\ &\quad + 2\mathbb{P}\{aX_2(n+1) > N + \mu n\}\\ &\quad + \mathbb{P}\Big\{\sup_{n<s\le n+1} aZ_1(s) > N + \mu n\Big\}.\end{aligned} \tag{7.31}$$

We proceed to bound each of these three terms and plug these bounds into the last term in (7.30). Note that

$$\begin{aligned}\mathbb{P}\{X_3(n+1) < -(N+\mu n)\} &= \mathbb{P}\Big\{N(0,1) > \frac{N+\mu n}{\gamma\sqrt{n+1}}\Big\}\\ &\le \frac{1}{\sqrt{2\pi}}\frac{\gamma\sqrt{n+1}}{N+\mu n}\exp\Big(-\frac{1}{2}\frac{(N+\mu n)^2}{\gamma^2(n+1)}\Big)\\ &\le \frac{\gamma}{\sqrt{2\pi}\mu}\exp\Big(-\frac{\mu}{2\gamma^2}(N+\mu n)\Big)\end{aligned} \tag{7.32}$$

for $N \ge \mu$, where $N(0,1)$ denotes the standard normal random variable and $\gamma^2$ is the variance of the Brownian motion $X_3$. The first inequality is a standard estimate and is obtained by integrating by parts. Assume $N = N'\mu$, with $N' \in \mathbb{Z}_+$. Then,

$$\begin{aligned}&\sum_{n=0}^{\infty}\big(N + \mu(n+1)\big)2\mathbb{P}\{X_3(n+1) < -(N+\mu n)\}\\ &\qquad \le \sum_{n=N'}^{\infty}\frac{2\gamma}{\sqrt{2\pi}}(n+1)\exp\Big(-\frac{\mu^2}{2\gamma^2}(n+1)\Big),\end{aligned} \tag{7.33}$$

which is less than $1/6$ for sufficiently large $N$. For the same reason,

$$\sum_{n=0}^{\infty}\big(N + \mu(n+1)\big)2\mathbb{P}\{aX_2(n+1) > N + \mu n\} \le \frac{1}{6} \tag{7.34}$$

for sufficiently large $N$. To bound the probability $\mathbb{P}\{\sup_{n<s\le n+1} aZ_1(s) > N + \mu n\}$, we apply Lemma 13 in Appendix D. The lemma states that for appropriate constants $c_1, c_2 > 0$,

$$\mathbb{P}\Big\{\sup_{n\le s\le n+1} Z_1(t) > x\Big\} \le c_1\exp(-c_2 x) \qquad \text{for } x \ge 0.$$

For $N = N'\mu$,

$$\begin{aligned}&\sum_{n=0}^{\infty}\bigl(N+\mu(n+1)\bigr)\mathbb{P}\Bigl\{\sup_{n<s\le n+1} aZ_1(s) > N+\mu n\Bigr\}\\ &\qquad \le \sum_{n=N'}^{\infty}(n+1)\mu c_1\exp(-c_2\mu n/a),\end{aligned}\tag{7.35}$$

which is also less than $1/6$ for large $N$. The bounds obtained for (7.33)–(7.35) together show that the last term in (7.30) is at most $1/2$. This implies (7.28), and hence the lemma for $a > 0$.

When $a \le 0$, the proof is analogous to the case $a > 0$, with the following simplifications. Assume $N + a \ge 1$. The equality (7.26) can be replaced by

$$Z_3(t) \ge (N+a) + \bigl(N+\mu t + X_3(t)\bigr) + \bigl(2N + 2\mu t - aX_2(t)\bigr)$$

for $t \ge 0$, because $a \le 0$. The definition of $\tau_2$ can be replaced by the simpler

$$\tau_2 = \inf\{t \ge 0 : X_2(t) \ge N + \mu t\}.$$

One can again check that for each $t < \tau$, $Z_2(t) > 1$ and $Z_3(t) > 1$, and so $\tau \le \sigma$. To prove the lemma for the case $a \le 0$, it remains to show (7.27). For this, we follow the same procedure as in the case $a > 0$. First, (7.28) still implies (7.29) with $a$ in the right side of (7.29) replaced by 1. From (7.29), (7.27) follows. To demonstrate (7.28), we employ (7.30), with $a$ there replaced by 1. To bound the probability $\mathbb{P}\{n < \tau_2 \wedge \tau_3 \le n+1\}$ for each $n \in \mathbb{Z}_+$, we replace (7.31) with the simpler

$$\begin{aligned}&\mathbb{P}\{n < \tau_2 \wedge \tau_3 \le n+1\}\\ &\qquad \le \mathbb{P}\{n < \tau_2 \le n+1\} + \mathbb{P}\{n < \tau_3 \le n+1\}\\ &\qquad \le 2\mathbb{P}\{X_3(n+1) < -(N+\mu n)\} + 2\mathbb{P}\{X_2(n+1) > N + \mu n\}.\end{aligned}$$

It then follows from bounds (7.32)–(7.34) that (7.28) holds for sufficiently large $N$. This implies the lemma for $a \le 0$. $\square$

## APPENDIX A: SEMIMARTINGALE REFLECTING BROWNIAN MOTIONS

In this section, we present the standard definition of a semimartingale reflecting Brownian motion (SRBM) in the $d$-dimensional orthant $S = \mathbb{R}^d_+$, where $d$ is a positive integer. We also review the standard definition of positive recurrence for an SRBM, connecting it with the alternative definition used in Section 1.

Recall from Section 1 that $\theta$ is a constant vector in $\mathbb{R}^d$, $\Gamma$ is a $d \times d$ symmetric and strictly positive definite matrix, and $R$ is a $d \times d$ matrix. We shall define an SRBM associated with the data $(S, \theta, \Gamma, R)$. For this, a triple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ will be called a *filtered space* if $\Omega$ is a set, $\mathcal{F}$ is a $\sigma$-field of subsets of $\Omega$, and $\{\mathcal{F}_t\} \equiv \{\mathcal{F}_t, t \ge 0\}$ is an increasing family of sub-$\sigma$-fields of $\mathcal{F}$, that is, a filtration.

DEFINITION 6 (Semimartingale reflecting Brownian motion). A SRBM associated with $(S, \theta, \Gamma, R)$ is a continuous $\{\mathcal{F}_t\}$-adapted $d$-dimensional process $Z = \{Z(t), t \geq 0\}$, together with a family of probability measures $\{\mathbb{P}_x, x \in S\}$, defined on some filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ such that, for each $x \in S$, under $\mathbb{P}_x$, (1.1) and (1.4) hold, where, writing $W(t) = X(t) - \theta t$ for $t \geq 0$, $W$ is a $d$-dimensional Brownian motion with covariance matrix $\Gamma$, an $\{\mathcal{F}_t\}$-martingale such that $W(0) = x$ $\mathbb{P}_x$-a.s., and $Y$ is an $\{\mathcal{F}_t\}$-adapted $d$-dimensional process such that $\mathbb{P}_x$-a.s. (1.2) and (1.3) hold. Here (1.2) is interpreted to hold for each component of $Y$, and (1.3) is defined to be

$$\int_0^t 1_{\{Z_i(s) \neq 0\}} \, dY_i(s) = 0 \qquad \text{for all } t \geq 0. \tag{A.1}$$

Definition 6 gives the so-called weak formulation of a SRBM. It is a standard definition adopted in the literature; see, for example, Dupuis and Williams (1994) and Williams (1995). Note that condition (A.1) is equivalent to the condition that, for each $t > 0$, $Z_j(t) > 0$ implies $Y_j(t - \delta) = Y_j(t + \delta)$ for some $\delta > 0$. Reiman and Williams (1988) showed that a necessary condition for a $(S, \theta, \Gamma, R)$-SRBM to exist is that the reflection matrix $R$ is completely $\mathcal{S}$ (this term was defined in Section 1). Taylor and Williams (1993) showed that when $R$ is completely $\mathcal{S}$, a $(S, \theta, \Gamma, R)$-SRBM $Z$ exists and $Z$ is unique in law under $\mathbb{P}_x$ for each $x \in S$. Furthermore, $Z$, together with the family of probability measures $\{\mathbb{P}_x, x \in \mathbb{R}^d_+\}$, is a Feller continuous strong Markov process.

Let $(\theta, \Gamma, R)$ be fixed with $\Gamma$ being a positive definite matrix and $R$ being a completely $\mathcal{S}$ matrix. Dupuis and Williams [(1994), Definition 2.5] and Williams [(1995), Definition 3.1] gave the following definition of positive recurrence.

DEFINITION 7. An SRBM $Z$ is said to be *positive recurrent* if, for each closed set $A$ in $S$ having positive Lebesgue measure, we have $\mathbb{E}_x(\tau_A) < \infty$ for all $x \in S$, where $\tau_A = \inf\{t \geq 0 : Z(t) \in A\}$.

Because each open neighborhood of the origin contains a closed ball that has positive volume, Definition 7 appears to be stronger (i.e., more restrictive) than the definition adopted in Section 1, but one can show that these two notions of positive recurrence are equivalent for a SRBM. Indeed, the last paragraph on page 698 in Dupuis and Williams (1994) provides a sketch of that proof.

## APPENDIX B: CONVENIENT NORMALIZATIONS OF PROBLEM DATA

Let $R$ be a $d \times d$ completely $\mathcal{S}$ matrix and $(X, Y, Z)$ a triple of continuous, $d$-dimensional stochastic processes defined on a common probability space. The diagonal elements of $R$ are necessarily positive. Let $\tilde{D} = \operatorname{diag}(R)$ and $\tilde{R} = R\tilde{D}^{-1}$ (thus $\tilde{R}$ is a $d \times d$ completely-$\mathcal{S}$ matrix that has ones on the diagonal), and define $\tilde{Y}(t) = \tilde{D}Y(t)$ for $t \geq 0$. If $(X, Y, Z)$ satisfy (1.1)–(1.4) with reflection matrix $R$, then $(X, \tilde{Y}, Z)$ satisfy (1.1)–(1.4) with reflection matrix $\tilde{R}$, and vice versa. Thus the distribution of $Z$ is not changed if one substitutes $\tilde{R}$ for $R$, and that substitution assures the standardized problem format (1.5).

Now let $R$ and $(X, Y, Z)$ be as in the previous paragraph, and further suppose that $X$ is a Brownian motion with drift vector $\theta$ and nonsingular covariance matrix $\Gamma$. Define a $d \times d$ diagonal matrix $D$ by setting $D_{ii} = 1$ if $\theta_i = 0$ and $D_{ii} = |\theta_i|^{-1}$ otherwise. Setting $\hat{Z} = DZ$, $\hat{X} = DX$ and $\hat{R} = DR$, one sees that if $(X, Y, Z)$ satisfy (1.1)–(1.4) with reflection matrix $R$, then $(\hat{X}, Y, \hat{Z})$ satisfy (1.1)–(1.4) with reflection matrix $\hat{R}$, and vice versa. Of course, $\hat{X}$ is a Brownian motion whose drift vector $\hat{\theta} = D\theta$ satisfies (5.1); the covariance matrix of $\hat{X}$ is $\hat{\Gamma} = D\Gamma D$. Thus our linear change of variable gives a transformed problem in which (5.1) is satisfied. To achieve a problem format where *both* (1.5) and (5.1) are satisfied, one can first make the linear change of variable described in this paragraph, and then make the substitution described in the previous paragraph.

## APPENDIX C: PROOF THAT (1.6) IS NECESSARY FOR STABILITY OF $Z$

We consider a $d$-dimensional SRBM $Z$ with associated data $(S, \theta, \Gamma, R)$, defined as in Appendix A, assuming throughout that $R$ is completely $\mathcal{S}$. Let us also assume until further notice that $R$ is nonsingular. Because $R$ is an $\mathcal{S}$-matrix, there exist $d$-vectors $w, v > 0$ such that $Rw = v$. That is,

$$R^{-1}v > 0 \qquad \text{where } v > 0. \tag{C.1}$$

Now suppose it is *not* true that $R^{-1}\theta < 0$. That is, defining $\gamma = R^{-1}\theta$, suppose that $\gamma_i \geq 0$ for some $i \in \{1, \dots, d\}$. For future reference let $u$ be the $i$th row of $R^{-1}$. Thus (C.1) implies

$$u \cdot v > 0. \tag{C.2}$$

Our goal is to show that $Z$ cannot be positive recurrent. Toward this end, it will be helpful to represent the underlying Brownian motion $X$ in (1.1) as $X(t) = W(t) + \theta t$, where $W$ is a $d$-dimensional Brownian motion with zero drift and covariance matrix $\Gamma$. Premultiplying both sides of (1.1) by $R^{-1}$ then gives $R^{-1}Z(t) = R^{-1}W(t) + \gamma t + Y(t)$. The $i$th component of the vector equation is

$$\xi(t) \equiv u \cdot Z(t) = u \cdot W(t) + \gamma_i t + Y_i(t), \qquad t \geq 0. \tag{C.3}$$

Let $A = \{z \in S : |z| \leq 1\}$ and $B = \{u \cdot z : z \in A\}$. Then $B \subset \mathbb{R}$ is a compact interval containing the origin, and from (C.2) we know that $B$ contains positive values as well, because $A$ contains $\alpha v$ for sufficiently small constants $\alpha > 0$. Thus $B$ has the form

$$B = [a, b] \qquad \text{where } a \leq 0 \text{ and } b > 0. \tag{C.4}$$

As the initial state $x = Z(0) = W(0)$, we take

$$x = \beta v \qquad \text{where } v \text{ is chosen as in (C.1)} \quad \text{and}$$
$$\beta > \max\bigl(|v|^{-1}, (u \cdot v)^{-1} b\bigr). \tag{C.5}$$

From (C.1), (C.2), (C.4) and (C.5), we have that

$$x \in S, \qquad |x| > 1 \quad \text{and} \quad u \cdot x > b. \tag{C.6}$$

Thus, defining $\tau_A = \inf\{t \geq 0 : Z(t) \in A\}$ and $\sigma = \inf\{t \geq 0 : \xi(t) \in B\}$, it follows from the definitions of $A$, $B$ and $\xi$, plus (C.4) and (C.6), that

$$\tau_A \geq \sigma, \qquad \mathbb{P}_x\text{-a.s.} \tag{C.7}$$

From (C.3), we see that $\xi$ is bounded below by a one-dimensional Brownian motion with nonnegative drift, and $\xi(0) > b$ $\mathbb{P}_x$-a.s. Thus $\mathbb{E}_x(\sigma) = \infty$, implying that $\mathbb{E}_x(\tau_A) = \infty$ as well by (C.7). This establishes that $Z$ is not positive recurrent when $R$ is nonsingular.

We still need to show that $Z$ cannot be positive recurrent when $R$ is singular. In this case, there exists a nontrivial vector $u \in \mathbb{R}^d$ such that $u'R = 0$, and we can assume that $u \cdot \theta \geq 0$ as well (because $-u$ can be exchanged for $u$ if necessary). Premultiplying both sides of (1.1) by $u'$ gives the following analog of (C.3):

$$u \cdot Z(t) = u \cdot W(t) + (u \cdot \theta)t.$$

Because $R$ is an $\mathcal{S}$-matrix, for this given $u$ there exist $w, v \in S$ such that

$$u + Rw = v. \tag{C.8}$$

After premultiplying both sides of (C.8) by $u'$, one obtains

$$u \cdot v = |u|^2 > 0.$$

We choose the initial state $x = Z(0) = W(0)$ exactly as in (C.5), and define the set $A$ as before. The proof that $\mathbb{E}_x(\tau_A) = \infty$, and hence that $Z$ is not positive recurrent, then proceeds exactly as in the case treated above, except that now the process $\xi = u \cdot Z$ is itself a Brownian motion with nonnegative drift, whereas in the case treated earlier $\xi$ was bounded below by such a Brownian motion.

## APPENDIX D: TWO LEMMAS

In this appendix, we demonstrate two lemmas that are used in Section 7. Lemma 13 is employed in the proof of Lemma 11.

LEMMA 13. *Let $X$ be a one-dimensional Brownian motion with drift $\theta < 0$, variance $\sigma^2$, starting from* 0. *Let $Z$ be the corresponding one-dimensional SRBM,*

$$Z(t) = X(t) - \min_{0 \leq s \leq t} X(s) \qquad \text{for } t \geq 0.$$

*There exist constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\mathbb{P}\Big\{\sup_{n-1\le s\le n} Z(s) > x\Big\} \le c_1 \exp(-c_2 x) \tag{D.1}$$

*for all $n \in \mathbb{Z}_+$ and $x > 0$.*

PROOF. One could employ the Lipschitz continuity property of the one-dimensional Skorohod map and the estimate (4.9) of Atar, Budhiraja and Dupuis (2001) to prove the lemma. Here, we provide a direct proof. Since $X$ has negative drift, it is well known [see, e.g., Section 1.9 of Harrison (1985)] that, for each $t > 0$ and $x > 0$,

$$\begin{aligned}\mathbb{P}\{Z(t) > x\} &= \mathbb{P}\Big\{\max_{0\le s\le t}\big(X(t) - X(t-s)\big) > x\Big\}\\ &= \mathbb{P}\Big\{\max_{0\le s\le t} X(s) > x\Big\} \qquad \text{(D.2)}\\ &\le \mathbb{P}\Big\{\sup_{0\le s<\infty} X(s) > x\Big\} = \exp(-2|\theta|x/\sigma^2).\end{aligned}$$

Therefore, for $n \in \mathbb{Z}_+$,

$$\begin{aligned}&\mathbb{P}\Big\{\max_{n-1<s\le n} Z(s) > x\Big\}\\ &\qquad \le \mathbb{P}\{Z(n) > x/2\} + \mathbb{P}\Big\{Z(n) \le x/2, \max_{n-1<s\le n} Z(s) > x\Big\}\\ &\qquad \le \exp(-|\theta|x/\sigma^2) + \mathbb{P}\{n-1 < \tau < n, X(n) - X(\tau) < -x/2\},\end{aligned}$$

where $\tau = \inf\{s \ge n-1 : Z(s) > x\}$.

Note that $\tau$ is a stopping time with respect to the filtration generated by $Z$, which is the filtration generated by the Brownian motion $B$, where $B(t) = X(t) - \theta t$ for $t \ge 0$. By the strong Markov property of $B$,

$$\begin{aligned}&\mathbb{P}\{n-1 < \tau < n, X(n) - X(\tau) < -x/2\}\\ &\qquad \le \mathbb{P}\{n-1 < \tau < n, B(n) - B(\tau) < -x/2 + |\theta|\}\\ &\qquad \le \Phi\big((-x/2 + |\theta|)/\sigma\big)\end{aligned}$$

for $x > 2|\theta|$, where $\Phi$ is the standard normal distribution function. Thus,

$$\mathbb{P}\Big\{\max_{n-1<s\le n} Z(s) > x\Big\} \le \exp(-|\theta|x/\sigma^2) + \Phi\big((-x/2+1)/\sigma\big)$$

for $x > 2|\theta|$, from which (D.1) follows. $\square$

The following lemma is used in the proof of Lemma 10. Let

$$R = \begin{pmatrix} 1 & b\\ a & 1\end{pmatrix} \quad \text{and} \quad \theta = \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix},$$

and assume that $\det(R) > 0$. Then $R$ is a $\mathcal{P}$-matrix and hence is completely $\mathcal{S}$. For any given $2 \times 2$ positive definite matrix $\Gamma$, it therefore follows from Taylor and Williams (1993) that, starting from any fixed state $x \in \mathbb{R}^2_+$, there is a two-dimensional SRBM $Z$ with data $(\theta, \Gamma, R)$, as defined in Definition 6, that is well defined and is unique in law.

LEMMA 14. *Suppose that $R^{-1}\theta \leq 0$. Then each fluid path $(y, z)$ starting from $z(0) = 0$ remains at 0; that is, $z(t) = 0$ for $t \geq 0$. Consequently,*

$$\lim_{t \to \infty} \frac{Z(t)}{t} = 0 \qquad \textit{almost surely}. \tag{D.3}$$

PROOF. It follows from

$$R^{-1} = \frac{1}{1 - ab} \begin{pmatrix} 1 & -b \\ -a & 1 \end{pmatrix},$$

$\det(R) = 1 - ab > 0$ and $R^{-1}\theta \leq 0$ that

$$\theta_1 - b\theta_2 \leq 0, \qquad \theta_2 - a\theta_1 \leq 0.$$

Observe that $R^{-1}$ is a $\mathcal{P}$-matrix because $R$ is a $\mathcal{P}$-matrix. Therefore, $R^{-1}$ is an $\mathcal{S}$-matrix. Consequently, there exists $(d_1, d_2) > 0$ such that $(c_1, c_2) \equiv (d_1, d_2)R^{-1} > 0$. By assumption $R^{-1}\theta \leq 0$, and so one has

$$c_1\theta_1 + c_2\theta_2 = (d_1, d_2)R^{-1}\theta \leq 0.$$

Let $(y, z)$ be a fluid path with $z(0) = 0$. By the oscillation inequality [see, e.g., Lemma 4.3 of Dai and Williams (1995)], $(y, z)$ is Lipschitz continuous. Setting $g(t) = c_1 z_1(t) + c_2 z_2(t)$, we will show that $g(t) = 0$ for $t \geq 0$. From this, it follows that $z(t) = 0$ for $t \geq 0$, which is the first claim in the lemma. It suffices to prove that

$$\dot{g}(t) \leq 0 \text{ at each } t \text{ where } g(t) > 0 \text{ and } (y, z) \text{ is differentiable.}$$

We consider several cases, depending on whether $z_1(t)$ and $z_2(t)$ are strictly positive. When $z_1(t) > 0$ and $z_2(t) > 0$, (2.1)–(2.4) imply that $\dot{z}_1(t) = \theta_1$ and $\dot{z}_2(t) = \theta_2$, and hence $\dot{g}(t) = c_1\dot{z}_1(t) + c_2\dot{z}_2(t) \leq 0$. When $z_1(t) = 0$ and $z_2(t) > 0$, $\dot{z}_1(t) = 0$ and $\dot{y}_2(t) = 0$, from which one has $\dot{y}_1(t) = -\theta_1$ and $\dot{z}_2(t) = \theta_2 - a\theta_1 \leq 0$. Because in this case $\dot{z}_1(t) = 0$ and $\dot{z}_2(t) \leq 0$, $\dot{g}(t) \leq 0$ follows. When $z_2(t) = 0$ and $z_1(t) > 0$, one can similarly argue that $\dot{g}(t) \leq 0$. This shows that $g(t) = 0$ for $t \geq 0$, as desired.

We next demonstrate (D.3). Let $Z$ be a two-dimensional SRBM with data $(\theta, \Gamma, R)$ having a given initial point $Z(0) = x \in \mathbb{R}^2_+$. By Definition 6, $Z$, together with the associated pair $(X, Y)$, satisfies (1.1)–(1.4). For each $r > 0$ and each $t \geq 0$, set

$$\bar{X}^r(t) = \frac{1}{r}X(rt), \qquad \bar{Y}^r(t) = \frac{1}{r}Y(rt), \qquad \bar{Z}^r(t) = \frac{1}{r}Z(rt).$$

By the functional strong law of large numbers (FSLLN) for Brownian motion, almost surely,

$$\lim_{r\to\infty}\sup_{0\le s\le t}|\bar{X}^r(s)-x(s)|=0 \qquad \text{for each } t>0, \tag{D.4}$$

where $x(s)=(\theta_1 s,\theta_2 s)'$ for $s\ge 0$. Fix a sample path that satisfies (D.4) and let $\{r_n\}\subset\mathbb{R}_+$ be a sequence with $r_n\to\infty$. The FSLLN (D.4) implies that $\{\bar{X}^{r_n}\}$ is relatively compact in $C(\mathbb{R}_+,\mathbb{R})$, the space of continuous functions on $\mathbb{R}_+$ endowed with the topology of uniform convergence on compact sets. By the oscillation inequality, $\{(\bar{Y}^{r_n},\bar{Z}^{r_n})\}$ is also relatively compact.

Let $(y,z)$ be a limit point of $\{(\bar{Y}^{r_n},\bar{Z}^{r_n})\}$. It is not difficult to show that $(y,z)$ is a fluid path associated with the data $(\theta,R)$ that satisfies $z(0)=0$. It follows from the first part of the proof that $z(t)=0$ for $t\ge 0$. This is the unique limit point with $z(0)=0$. Therefore, almost surely,

$$\lim_{r\to\infty}\frac{1}{r}Z(r)=\lim_{r\to\infty}\bar{Z}^r(1)=0,$$

which proves (D.3). □

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