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CAPACITIVE FLOWS ON A 2D RANDOM NET

BY OLIVIER GARET

Institut Élie Cartan Nancy

This paper concerns maximal flows on $\mathbb{Z}^2$ traveling from a convex set to infinity, the flows being restricted by a random capacity. For every compact convex set A, we prove that the maximal flow $\Phi(nA)$ between nA and infinity is such that $\Phi(nA)/n$ almost surely converges to the integral of a deterministic function over the boundary of A. The limit can also be interpreted as the optimum of a deterministic continuous max-flow problem. We derive some properties of the infinite cluster in supercritical Bernoulli percolation.

1. Introduction. The problem of finding the maximum flow in a capacitive network is undoubtedly the most known problem in the theory of operational research. We know since Ford and Fulkerson that the search of a maximum capacitive flow and that of the minimal cutset in a graph are two sides of the same coin. In the applications, the problems can come under a form or another. Thus, this duality allows to choose the formulation which is the most adapted to the mathematical treatment.

In the last decade, the min-cut formulation has been shown to be a practical and useful tool for image segmentation (see Xiaodong [16] or Estrada and Jepson [7], for instance). It is not surprising since image segmentation is precisely running the scissors along the line of cut. Let us assume for instance that we have a picture of a person and that we want to cut around the face in such a way that the background is rather white along the break: if η_x represents the blackness of the point x, then one can try to minimize the "cost" $\sum_{x\in C}\eta_x$, where C is a curve, which separates the face of the person (beforehand identified) from the rest of the photograph.

We give in the present article a probabilistic treatment of this kind of cutset problem: the darkness of the points is given here by a collection of identically distributed random variables, and we want to know to what extent the cost of the minimal cutset is determined by the geometry of the form to be encircled. If we reformulate the problem using the max-flow min-cut duality, we have random capacities on the bonds of $\mathbb{Z}^2$ and we study the maximum flow that can be carried from the boundary of a given set to infinity. To be more specific, we fix a compact convex subset $A\subset\mathbb{R}^2$ and study the asymptotic behavior of the maximal flow $\Phi(nA)$ between nA and infinity, which is also the cost of a minimal cutset separating nA from infinity. We will see that the maximal flow $\Phi(nA)$ between

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nA and infinity is such that $\Phi(nA)/n$ almost surely converges to the integral of a deterministic function over the boundary of A.

2. Notation and results.

Flows. Formally, let $\overrightarrow{\mathbb{E}}^2 = \{(x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : \|x - y\|_1 = 1\}$ and $\mathbb{E}^2 = \{\{x, y\} \in \mathbb{Z}^2 \times \mathbb{Z}^2 : \|x - y\|_1 = 1\}$, where $\|\cdot\|_1$ is the ℓ^1-norm: $\|(a, b)\|_1 = |a| + |b|$. As usual, we denote by $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$ the unoriented square lattice.

We say that a map $f : \overrightarrow{\mathbb{E}}^2 \to \mathbb{R}$ is a flow if $f(x, y) = -f(y, x)$ holds for each edge $(x, y) \in \overrightarrow{\mathbb{E}}^2$.

Let $(t_e)_{e\in\mathbb{E}^2}$ be a family of positive numbers.

We say that f is a capacitive flow from A to infinity if it satisfies

$$\begin{cases} |f(x, y)| \le t_{\{x,y\}}, & \text{for each bond } (x, y) \in \overrightarrow{\mathbb{E}}^2, \\ \operatorname{Div} f(x) = 0, & \text{for } x \in \mathbb{Z}^2 \backslash A, \end{cases} \tag{1}$$

where $\operatorname{Div} j(x) = \sum_{y\in\mathbb{Z}^2; \|x-y\|_1=1} j(x, y)$.

We denote by $\operatorname{Capflow}(A, \infty)$ the set of capacitive flows from A to infinity. The aim is to study the maximal flow from a convex set A to infinity, that is,

$$\max\left\{\sum_{x\in A\cap\mathbb{Z}^2} \operatorname{Div} j(x); j \in \operatorname{Capflow}(A, \infty)\right\}, \tag{2}$$

when the $(t_e)_{e\in\mathbb{E}^2}$ are given by some collection of independent identically distributed random variables.

Links with first passage percolation. The efficiency of methods coming from first passage percolation in studying the maximum flow through a randomly capacitated network was initially pointed out by Grimmett and Kesten [12]: precisely, they gave the asymptotic behavior of the maximum flow through the bottom of a rectangle to its top as an application of their advances in first-passage percolation.

As already mentioned, the point is the use of the max-flow min-cut theorem [8]. In the current setting, we can prove that

$$\max\left\{\sum_{x\in A} \operatorname{Div} j(x); j \in \operatorname{Capflow}(A, \infty)\right\} = \operatorname{Mincut}(A, \infty) \qquad \text{a.s.}, \tag{3}$$

where $\operatorname{Mincut}(A, \infty)$ is the minimum of the quantity $\sum_{e\in C} t_e$, where C is taken among the subsets of $\mathbb{E}^2$ that separate A from infinity, or more precisely that are such that every infinite path in $\mathbb{L}^2$ starting from A meets C. Such a set is called a cutset (relative to A). With this definition, we can write

$$\operatorname{Mincut}(A, \infty) = \min\left\{\sum_{e\in C} t_e; C \subset \mathbb{E}^2 \text{ and } C \text{ is a cutset relative to } A\right\}. \tag{4}$$

The cutsets of $\mathbb{L}^2$ can be characterized as follows: Let $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$, $\mathbb{E}^2_* = \{\{a, b\}; a, b \in \mathbb{Z}^2_* \text{ and } \|a - b\|_1 = 1\}$ and $\mathbb{L}^2_* = (\mathbb{Z}^2_*, \mathbb{E}^2_*)$. It is easy to see that $\mathbb{L}^2_*$ is isomorphic to $\mathbb{L}^2$.

For each bond $e = \{a, b\}$ of $\mathbb{L}^2$ (resp. $\mathbb{L}^2_*$), let us denote by $s(e)$ the only subset $\{i, j\}$ of $\mathbb{Z}^2_*$ (resp. $\mathbb{Z}^2$) such that the quadrangle *aibj* is a square in $\mathbb{R}^2$. s is clearly an involution, and it is not difficult to see that s is a one-to-one correspondence between the cutsets in $\mathbb{L}^2$ and the sets in $\mathbb{E}^2_*$ that contain a closed path surrounding A. If C is minimal for inclusion, then $s(C)$ is just a path surrounding A, so the quantity $\sum_{e \in e} t_e$ can be interpreted as the length of the path in a first-passage percolation setting on $\mathbb{Z}^2_*$.

This leads us to recall a basic result in first-passage percolation:

Assume that m is a probability measure on $[0, +\infty)$, such that

$$m(0) < 1/2 \quad \text{and} \quad \exists c > 0, \qquad \int_{[0,\infty)} \exp(cx)\, dm(x) < +\infty. \tag{5}$$

Let $\Omega = [0, +\infty)^{\mathbb{E}^2}$ and consider the probability measure $\mathbb{P} = m^{\otimes \mathbb{E}^2}$ on Ω. For $e \in \mathbb{E}^2$, we define $t_e(\omega) = \omega_e$, thus the variables $(t_e)_{e \in \mathbb{E}^2}$ are independent identically distributed random variables with common law m.

For each $\gamma \subset \mathbb{E}^2$, we define $l(\gamma) = \sum_{e \in \gamma} t_e$. We denote by $d(a, b)$ the length of the shortest path from a to b, that is,

$$d(a, b) = \inf\{l(\gamma); \gamma \text{ contains a path from } a \text{ to } b\}.$$

Then by the Cox–Durrett shape theorem [5], there exists a norm μ on $\mathbb{R}^2$ such that

$$\lim_{\|x\|_1 \to +\infty} \frac{d(0, x)}{\mu(x)} = 1 \qquad \text{a.s.} \tag{6}$$

We can also define l^* by $l^*(A) = l(s(A))$ and a (random) distance d^* on $\mathbb{Z}^2_* \times \mathbb{Z}^2_*$ by

$$d^*(a, b) = \inf\{l^*(\gamma); \gamma \text{ contains a path from } a \text{ to } b\}.$$

Since $\mathbb{L}^2_*$ is isomorphic to $\mathbb{L}^2$, it is easy to see that $d^*(\cdot, \cdot)$ enjoys the same asymptotic properties as $d(\cdot, \cdot)$ does.

Main results. We first recall some common notation: $\mathcal{H}^1$ is the 1-dimensional normalized Hausdorff measure, λ^2 is the 2-dimensional Lebesgue measure, div is the usual divergence operator, and $C^1_c(\mathbb{R}^2, E)$ is the set of compactly supported C^1 vector functions from $\mathbb{R}^2$ to E. Let $A \subset \mathbb{R}^2$ be a Caccioppoli set. We denote by ∂A its boundary and by $\partial^* A$ its reduced boundary, that is constituted by the points $x \in \partial A$, where ∂A admits an unique outer normal, which is denoted by $\nu_A(x)$.

The main goal of the paper is the following theorem.

THEOREM 2.1. *We suppose that* $m(0) < 1/2$ *and that*

$$\int_{[0,\infty)} \exp(cx)\, dm(x) < +\infty$$

for some $c > 0$. *Then for each bounded convex set* $A \subset \mathbb{R}^2$ *with* 0 *in the interior, we have*

$$\lim_{n\to+\infty} \frac{\mathrm{Mincut}(nA, \infty)}{n} = \int_{\partial^* A} \mu(\nu_A(x))\, d\mathcal{H}^1(x).$$

Equivalently,

$$\lim_{n\to+\infty} \frac{1}{n} \max\left\{ \sum_{x\in nA\cap\mathbb{Z}^2} \mathrm{Div}\, j(x);\ j \in \mathrm{Capflow}(nA, \infty) \right\}$$
$$= \sup\left\{ \int_A \mathrm{div}\, f\, d\lambda^2(x);\ f \in C_c^1(\mathbb{R}^2, \mathcal{W}_\mu) \right\},$$

where

$$\mathcal{W}_\mu = \{x \in \mathbb{R}^2 : \langle x, w\rangle \le \mu(w) \textit{ for all } w\}.$$

Note that $\mathcal{W}_\mu$ is sometimes called the Wulff crystal associated to μ.

If we observe the last equality, we can see that the optimal value of a discrete random max-flow problem converges (after a suitable renormalization) to the optimum of a deterministic continuous max-flow problem.

In fact, we even have exponential bounds for the fluctuations around

$$\mathcal{I}(A) = \sup\left\{ \int_A \mathrm{div}\, f\, d\lambda^2(x);\ f \in C_c^1(\mathbb{R}^2, \mathcal{W}_\mu) \right\}. \tag{7}$$

Indeed, we prove the following theorem.

THEOREM 2.2. *Under the assumptions of Theorem* 2.1, *it holds that for each* $\varepsilon > 0$, *there exist constants* $C_1, C_2 > 0$, *depending on* ε *and* m, *such that*

$$\forall n \ge 0 \qquad \mathbb{P}\left(\frac{\mathrm{Mincut}(nA, \infty)}{n\mathcal{I}(A)} \notin (1-\varepsilon, 1+\varepsilon) \right) \le C_1 \exp(-C_2 n). \tag{8}$$

With the help of Menger's theorem, we obtain the following corollaries.

COROLLARY 2.3. *We consider supercritical Bernoulli percolation on the square lattice, where the edges are open with probability* $p > p_c(2) = 1/2$. *Then for each bounded convex set* $A \subset \mathbb{R}^2$ *with* 0 *in the interior, the maximal number* $\mathrm{dis}(A)$ *for a collection of disjoint open paths from A to infinity satisfies*

$$\exists C_1, C_2 > 0\ \forall n \ge 0 \qquad \mathbb{P}\left(\frac{\mathrm{dis}(nA)}{n\mathcal{I}(A)} \notin (1-\varepsilon, 1+\varepsilon) \right) \le C_1 \exp(-C_2 n), \tag{9}$$

where $\mathcal{I}(A)$ *is the quantity defined in* (7), *the law* m *of passage times being the Bernoulli distribution* $(1-p)\delta_0 + p\delta_1$.

This corollary has itself an easy and pleasant consequence.

COROLLARY 2.4. *We consider supercritical Bernoulli percolation on the square lattice. For each integer k, there almost surely exist k disjoint open bi-infinite paths.*

Note, however, that this amusing corollary is not really new; indeed, it can be obtained as a consequence of Grimmett and Marstrand [10]—see also Grimmett [11], page 148, Theorem 7.2.(a).

The paper is organized as follows. In Section 3, we recall some basic properties in first-passage percolation and prove some useful properties of the functional $\mathcal{I}$. Next, the proof of Theorem 2.2 naturally falls into two parts: Section 4 deals with the upper large deviations appearing in the Theorem, whereas Section 5 is about the lower ones. We complete the proof of Theorem 2.1 and establish the corollaries in Section 6. In the final section, we discuss the possibility of an extension to higher dimensions.

3. Preliminary results.

NOTATION. We denote by $\langle \cdot, \cdot \rangle$ the natural scalar product on $\mathbb{R}^2$ and by $\|\cdot\|_2$ the associated norm. $\mathcal{S}$ is the Euclidean unit sphere: $\mathcal{S} = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$.

3.1. *First-passage percolation.* Let us introduce some notation and results related to first passage percolation. As previously, we suppose that (5) is satisfied and write μ for the norm given by (6).

It will be useful to use

$$\mu_{\max} = \sup\{\mu(x); \|x\|_1 = 1\} \quad \text{and} \quad \mu_{\min} = \inf\{\mu(x); \|x\|_1 = 1\}.$$

Of course, $0 < \mu_{\min} \le \mu_{\max} < +\infty$ and we have

$$\forall x \in \mathbb{R}^2 \qquad \mu_{\min}\|x\|_1 \le \mu(x) \le \mu_{\max}\|x\|_1.$$

The speed of convergence in equation (6) can be specified:

PROPOSITION 3.1 (Large deviations, Grimmett–Kesten [12]). *For each $\varepsilon > 0$, there exist $C_3, C_4 > 0$ such that*

$$\forall x \in \mathbb{Z}^2 \qquad \mathbb{P}\big(d(0,x) \in [(1-\varepsilon)\mu(x), (1+\varepsilon)\mu(x)]\big) \ge 1 - C_3 \exp(-C_4\|x\|_1).$$

Note that in [12], the proof of this result is only written in the direction of the first axis, that is, for $x = ne_1$. Nevertheless, it applies in any direction and computations can be followed in order to preserve a uniform control, whatever direction one considers. See, for instance, Garet and Marchand [9] for a detailed proof in an analogous situation. The control of $\mathbb{P}(d(0,x) > (1+\varepsilon)\mu(x))$ could also be obtained as a byproduct of the foregoing Lemma 4.2.

3.2. *Properties of $\mathcal{I}$.* Since μ is a norm, it is obviously a convex function that does not vanish on the Euclidean sphere $\mathcal{S}$. So, it follows from Proposition 14.3 in Cerf [3] that the identity

$$\int_{\partial^* A} \mu(\nu_A(x))\, d\mathcal{H}^1(x) = \mathcal{I}(A) \tag{10}$$

holds for every Cacciopoli set, and particularly for compact convex sets and polygons.

From equation (7), it is easy to see that

$$\mathcal{I}(\lambda A) = \lambda \mathcal{I}(A) \tag{11}$$

holds for each Borel set A and each $\lambda > 0$.

LEMMA 3.2. *$\mathcal{I}(A) > 0$ for each convex set A with nonempty interior.*

PROOF. For each $x \in \partial^* A$, $\mu(\nu_A(x)) \geq \mu_{\min}\|\nu_A(x)\|_1 \geq \mu_{\min}\|\nu_A(x)\|_2 = \mu_{\min}$, so it follows from (10) that $\mathcal{I}(A) \geq \mu_{\min}\mathcal{H}^1(\partial A)$. □

The next lemma clarifies the connection between $\mathcal{I}(A)$ and μ when A is a polygon. Loosely speaking, $\mathcal{I}(A)$ is simply the μ-length of the polygon.

LEMMA 3.3. *Let A be a polygon whose sides are $[s_0, s_1], [s_1, s_2], \ldots, [s_{n_e-1}, s_{n_e}]$, with $s_{n_e} = s_0$. We have*

$$\mathcal{I}(A) = \sum_{i=0}^{n_e-1} \mu(s_i - s_{i+1}).$$

PROOF. For each $x = (a, b) \in \mathbb{Z}^2$, define $x^\perp = (-b, a)$. The map $\mathbb{R}^{\mathbb{Z}^2} \to \mathbb{R}^{\mathbb{Z}^2}$ that maps $(t_{x,y})_{\{x,y\}\in\mathbb{E}^2}$ to $(t_{-y,x})_{\{x,y\}\in\mathbb{E}^2}$ leaves $m^{\otimes\mathbb{E}^2}$ invariant, so it follows from (6) that $\mu(z) = \mu(z^\perp)$ holds for each $z \in \mathbb{Z}^2$. Since μ is homogeneous and continuous, the formula $\mu(z) = \mu(z^\perp)$ also holds for each $z \in \mathbb{R}^2$. We have

$$\begin{aligned}\int_{\partial^* A} \mu(\nu_A(x))\, d\mathcal{H}^1(x) &= \sum_{i=0}^{n_e-1} \|s_i - s_{i+1}\|_2 \mu\left(\left(\frac{s_i - s_{i+1}}{\|s_i - s_{i+1}\|_2}\right)^\perp\right)\\ &= \sum_{i=0}^{n_e-1} \|s_i - s_{i+1}\|_2 \mu\left(\frac{s_i - s_{i+1}}{\|s_i - s_{i+1}\|_2}\right)\\ &= \sum_{i=0}^{n_e-1} \mu(s_i - s_{i+1}).\end{aligned}$$

□

The next property of $\mathcal{I}$ will be decisive in the proof of lower large deviations. Basically, it says that the shortest path surrounding a convex polygon is the frontier of the polygon itself.

LEMMA 3.4. *Let A, B be two polygons with $B \subset A$. We suppose that B is convex. Then $\mathcal{I}(B) \le \mathcal{I}(A)$.*

PROOF. We proceed by induction on the number $n(A, B)$ of vertices of B which do not belong to ∂A. When $n = 0$, we just apply the triangle inequality. When $n > 1$, we build a polygon A' with $B \subset A' \subset A$, $\mathcal{I}(A') \le \mathcal{I}(A)$ and $n(A', B) < n(A, B)$ as follows: let z be a vertex of B which is not in ∂A. Since B is convex, there exists an affine map φ with $\varphi(z) = 0$ and $\varphi(x) < 0$ for x in $B \backslash \{z\}$. Let D be the connected component of z in $A \cap \{x \in \mathbb{R}^2 : \varphi(x) \ge 0\}$. D is a polygon which has a side F in $\{x \in \mathbb{R}^2 : \varphi(x) \ge 0\}$. Note $A' = A \backslash D$. Denote by s_a and s_b the ends of F and define $\mu(F) = \mu(s_b - s_a)$. We have $\mathcal{I}(A) = (\mathcal{I}(A') - \mu(F)) + (\mathcal{I}(D) - \mu(F))$. By the triangle inequality $\mu(F) \le \mathcal{I}(D)/2$, so $\mathcal{I}(A') \le \mathcal{I}(A)$. $\square$

We will also need convenient approximations of a convex set by convex polygons. This is the goal of the next lemma.

LEMMA 3.5. *Let A be a bounded convex set with 0 in the interior of A. For each $\varepsilon > 0$, there exist convex polygons P and Q such that*

$$0 \in P \subset A \subset Q \quad \textit{and} \quad \mathcal{I}(Q) - \varepsilon \le \mathcal{I}(A) \le \mathcal{I}(P) + \varepsilon.$$

PROOF. A proof of the existence of Q can be found in Lachand–Robert and Oudet [14] in a more general setting. The existence of P is simpler: let $(A_p)_{p \ge 1}$ such that:

- for each $p \ge 1$, A_p is a convex polygon,
- for each $p \ge 1$, $A_p \subset A$,
- $0 \in A_p$ for large p,
- $\lim_{p \to +\infty} \lambda^2(A \backslash A_p) = 0$.

(e.g., take A_p as the convex hull of $x_1, \dots, x_p$, where $(x_p)_{p \ge 1}$ is dense in ∂A: this ensures that $\bigcup_{p \ge 1} A_p \supset A \backslash \partial A$.) For fixed $f \in C_c^1(\mathbb{R}^2, \mathcal{W}_\mu)$, $A \mapsto \int_A \operatorname{div} f \, d\lambda^2(x)$ is continuous with respect to the L^1 convergence of Borel sets, so $A \mapsto \mathcal{I}(A)$ is lower semicontinuous. Then $\mathcal{I}(A) \le \underline{\lim}_{p \to +\infty} \mathcal{I}(A_p)$, so there exists $p \ge 1$ with $\mathcal{I}(A) \le \mathcal{I}(A_p) + \varepsilon$ and $0 \in A_p$. $\square$

4. Upper large deviations.

THEOREM 4.1. *For each $\varepsilon > 0$, there exist constants $C_5, C_6 > 0$, such that*

$$\mathbb{P}\big(\operatorname{Mincut}(nA, \infty) \ge n\mathcal{I}(A)(1 + \varepsilon)\big) \le C_5 \exp(-C_6 n). \tag{12}$$

The proof naturally falls into three parts:

1. Approximate nA by a polygon.
2. Parallel outside nA (but close to nA) the boundary of the polygon: it creates a new polygon.
3. Hope that successive vertices of the newly created polygon can be joined by a path which is short enough and does not enter in nA.

Therefore, we need a lemma that would roughly say that one can find a path from x to y that has length smaller than $(1+\varepsilon)\mu(x-y)$ and is not far from a straight line. To this aim, we introduce some definitions:

Let $y, z \in \mathbb{R}^2$, $\hat{x} \in \mathcal{S}$, and $R, h > 0$. We define

$$d(y, \mathbb{R}\hat{x}) = \|y - \langle y, \hat{x}\rangle \hat{x}\|_2$$

(the Euclidean distance from y to the line $\mathbb{R}\hat{x}$),

$$\mathrm{Cyl}_z(\hat{x}, R, h) = \{y \in \mathbb{Z}^2 : d(y-z, \mathbb{R}\hat{x}) \leq R \text{ and } 0 \leq \langle y - z, \hat{x}\rangle \leq h\},$$

For $R > 0$ and $z, z' \in \mathbb{R}^2$ with $z \neq z'$, we also define

$$\widetilde{Cyl}(z, z', R) = \mathrm{Cyl}_z\left(\frac{z'-z}{\|z'-z\|_2}, R, \|z - z'\|_2\right).$$

LEMMA 4.2. *Let $z \in \mathbb{R}^2$, $\hat{x} \in \mathcal{S}$, $h \geq 1$ and $r \geq 1$. We can define s_0 (resp. s_f) to be the integer point in $\mathrm{Cyl}_z(\hat{x}, r, h)$ which is the closest to z (resp. $z + h\hat{x}$). We also define the longitudinal crossing time $t_{\mathrm{long}}(\mathrm{Cyl}_z(\hat{x}, h, r))$ of the cylinder $\mathrm{Cyl}_z(\hat{x}, r, h)$ as the minimal time needed to cross it from s_0 to s_f, using only edges inside the cylinder.*

Then for each $\varepsilon > 0$ and each function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r\to+\infty} f(r) = +\infty$, there exist two strictly positive constants C_7 and C_8 such that

$$\forall z \in \mathbb{R}^2,\ \forall \hat{x} \in \mathcal{S},\ \forall h > 0$$
$$\mathbb{P}\big(t_{\mathrm{long}}(\mathrm{Cyl}_z(\hat{x}, f(h), h)) \geq \mu(\hat{x})(1+\varepsilon)h\big) \leq C_7 \exp(-C_8 h).$$

PROOF. For $x \in \mathbb{Z}^2$ and $t \geq 0$, let

$$\mathcal{B}_x(t) = \{y \in \mathbb{Z}^2 : \mu(x-y) \leq t\}.$$

For $x, y \in \mathbb{Z}^2$ denote by $I_{x,y}$ the length of the shortest path from x to y which is inside $\mathcal{B}_x(1,25\mu(x-y)) \cap \mathcal{B}_y(1,25\mu(x-y))$. Since $I_{x,y}$ as the same law than $I_{0,x-y}$, we simply define $I_x = I_{0,x}$. We begin with an intermediary lemma.

LEMMA 4.3. *For each $\varepsilon \in (0,1]$, there exists $M_0 = M_0(\varepsilon)$ such that for each $M \geq M_0$ there exist $c = c(\varepsilon, M) < 1$ and $t = t(\varepsilon, M) > 0$ with*

$$\|x\| \in [M/2, 2M] \quad \Longrightarrow \quad \mathbb{E}\exp\big(t\big(I_x - (1+\varepsilon)\mu(x)\big)\big) \leq c.$$

PROOF. Let Y be a random variable with law m and let $\gamma > 0$ be such that $\mathbb{E}e^{2\gamma Y} < +\infty$. First, equation (6) easily implies the following almost sure convergence:

$$\lim_{\|x\| \to +\infty} \frac{I_x}{\mu(x)} = 1.$$

By considering a deterministic path from 0 to x with length $\|x\|$, we see that I_x is dominated by a sum of $\|x\|$ independent copies of Y denoted by $Y_1, \dots, Y_{\|x\|}$, and thus $I_x/\|x\|$ is dominated by

$$\frac{1}{\|x\|} \sum_{k=1}^{\|x\|} Y_i.$$

This family is equi-integrable by the law of large numbers. So $(I_x/\|x\|)_{x \in \mathbb{Z}^2 \setminus \{0\}}$ and then $(I_x/\mu(x))_{x \in \mathbb{Z}^2 \setminus \{0\}}$ are also equi-integrable families, which implies that

$$\lim_{\|x\| \to +\infty} \frac{\mathbb{E}I_x}{\mu(x)} = 1. \tag{13}$$

Note that for every $y \in \mathbb{R}$ and $t \in (0, \gamma]$,

$$e^{ty} \le 1 + ty + \frac{t^2}{2} y^2 e^{t|y|} \le 1 + ty + \frac{t^2}{\gamma^2} e^{2\gamma|y|}.$$

Let $\tilde{I}_x = I_x - (1 + \varepsilon)\mu(x)$ and suppose that $t \in (0, \gamma]$. Then since $|\tilde{I}_x| \le I_x + 2\mu(x)$, the previous inequality implies that

$$e^{t\tilde{I}_x} \le 1 + t\tilde{I}_x + \frac{t^2}{\gamma^2} e^{4\gamma\mu(x)} e^{2\gamma I_x}.$$

Since $\mu(x) \le \|x\|\mu_{\max}$ and $I_x \le Y_1 + \cdots + Y_{\|x\|}$, we can define $\rho = e^{4\gamma\mu_{\max}}\mathbb{E}e^{2\gamma Y}$, and thus obtain

$$\mathbb{E}e^{t\tilde{I}_x} \le 1 + t\left[\mathbb{E}\tilde{I}_x + \frac{t}{\gamma^2}\rho^{\|x\|}\right].$$

Considering equation (13), let M_0 be such that $\|x\| \ge M_0/2$ implies $\frac{\mathbb{E}I_x}{\mu(x)} \le 1 + \varepsilon/3$. For x such that $\|x\| \ge M_0$, we have $\mathbb{E}\tilde{I}_x \le -\frac{2}{3}\varepsilon\mu(x)$, so

$$\mathbb{E}e^{t\tilde{I}_x} \le 1 + t\left[-\frac{2}{3}\varepsilon\mu(x) + \frac{t}{\gamma^2}\rho^{\|x\|}\right] \le 1 + t\left[-\frac{2}{3}\varepsilon\mu_{\max} + \frac{t}{\gamma^2}\rho^{\|x\|}\right].$$

Therefore, we can take $t = t(\varepsilon, M) = \min(\gamma, \gamma^2\mu_{\max}\frac{\varepsilon}{3}\rho^{-2M})$ and $c = c(\varepsilon, M) = 1 - \frac{1}{3}\varepsilon\mu_{\max}t(\varepsilon, M)$. $\square$

Let us come back now to the proof of Lemma 4.2. Let $\varepsilon \in (0,1)$ and consider the integer $M_0 = M_0(\varepsilon/3)$ given by the previous lemma. Let $M_1 = M_1(\varepsilon)$ be an integer greater than M_0 and such that

$$(1+\varepsilon/3)\left(1+\frac{\mu_{\max}}{\mu_{\min}}\frac{2}{M_1}\right) \le 1+\varepsilon/2. \tag{14}$$

Let N be the smallest integer which is greater than h/M_1 and, for each $i \in \{0,\dots,N\}$, denote by x_i the integer point in the cylinder which is the closest to $z+\frac{ih\hat{x}}{N}$. Note that

$$\left(1-\frac{1}{N}\right)M_1 \le \frac{h}{N} \le M_1.$$

1. Let i_0 be an integer with $i_0 \ge \max(\frac{1,25(2+M_1)\sqrt{2}}{\mu_{\min}}, 2)$. There exists a deterministic path inside the cylinder from x_0 to x_{i_0} (resp. x_{N-i_0} to x_N) which uses less than $2i_0h/N$ edges: we denote by L_{start} (resp. L_{end}) the random length of this path. Markov's inequality easily gives

$$\mathbb{P}\left(L_{\text{start}} > \frac{\varepsilon}{4}\mu(\hat{x})h\right) + \mathbb{P}\left(L_{\text{end}} > \frac{\varepsilon}{4}\mu(\hat{x})h\right) \tag{15}$$

$$\le 2(\mathbb{E}e^{2\gamma Y})^{2i_0M_1}\exp\left(-\frac{\gamma\varepsilon}{2}h\mu_{\min}\right) \le C'e^{-C_5h}. \tag{16}$$

2. For each $i,j \in \{0,\dots,N-1\}$, we have $|\mu(x_i-x_j) - |j-i|h\mu(\hat{x})/N| \le 2\mu_{\max}$. Thus, if h is larger than some h_0, then $\mathcal{B}_{x_i}(1,25\mu(x_i-x_{i+1})) \cap \mathcal{B}_{x_j}(1,25\mu(x_j-x_{j+1})) = \varnothing$ as soon as $|j-i| \ge 2$.

Let $h_1 = h_1(\varepsilon, f) \ge h_0$ be such that $\forall h \ge h_0,\ f(h) \ge i_0$. If we take h larger than h_1, then the whole set

$$\bigcup_{i=i_0}^{N-i_0-1} \mathcal{B}_{x_i}\big(1,25\mu(x_i-x_{i+1})\big)$$

stays inside the cylinder. So, provided that $h \ge h_1$, we have inside the cylinder a path from x_0 to x_N with length

$$L_{\text{start}} + \sum_{i=i_0}^{N-i_0-1} I_{x_i,x_{i+1}} + L_{\text{end}}.$$

Let

$$S_{\text{odd}} = \sum_{\substack{2\le i\le N-3\\ i \text{ odd}}} I_{x_i,x_{i+1}} \quad\text{and}\quad S_{\text{even}} = \sum_{\substack{2\le i\le N-3\\ i \text{ even}}} I_{x_i,x_{i+1}}.$$

By the definition of $(x_i)_{1\le i\le N}$, we have

$$\begin{aligned}
\sum_{\substack{2\le i\le N-3\\ i \text{ odd}}} \mu(x_{i+1}-x_i) &\le \sum_{\substack{2\le i\le N-3\\ i \text{ odd}}} \frac{h\mu(\hat{x})}{N} + 2\mu_{\max}\\
&\le \frac{N-3}{2}\biggl(\frac{h\mu(\hat{x})}{N} + 2\mu_{\max}\biggr)\\
&\le \frac{N}{2}\frac{h\mu(\hat{x})}{N} + (N-1)\frac{\mu_{\max}}{\mu_{\min}}\mu(\hat{x})\\
&\le \frac{h\mu(\hat{x})}{2}\biggl(1+2\frac{\mu_{\max}}{\mu_{\min}}\frac{1}{M_1}\biggr)
\end{aligned}$$

Then using (14), we can write, for each $t\ge 0$,

$$\begin{aligned}
&\mathbb{P}\biggl(S_{\text{odd}} \ge \frac{h\mu(\hat{x})}{2}(1+\varepsilon/2)\biggr)\\
&\quad\le \mathbb{P}\Biggl(S_{\text{odd}} \ge (1+\varepsilon/3)\sum_{\substack{2\le i\le N-3\\ i \text{ odd}}}\mu(x_{i+1}-x_i)\Biggr)\\
&\quad\le \mathbb{P}\Biggl(\sum_{\substack{2\le i\le N-3\\ i \text{ odd}}} I_{x_i,x_{i+1}} - (1+\varepsilon/3)\mu(x_{i+1}-x_i) \ge 0\Biggr)\\
&\quad\le \mathbb{P}\Biggl(\exp\Biggl(t\sum_{\substack{2\le i\le N-3\\ i \text{ odd}}} I_{x_i,x_{i+1}} - (1+\varepsilon/3)\mu(x_{i+1}-x_i)\Biggr) \ge 1\Biggr)\\
&\quad\le \mathbb{E}\exp\Biggl(t\sum_{\substack{2\le i\le N-3\\ i \text{ odd}}} I_{x_i,x_{i+1}} - (1+\varepsilon/3)\mu(x_{i+1}-x_i)\Biggr)\\
&\quad\le \prod_{\substack{2\le i\le N-3\\ i \text{ odd}}} \mathbb{E}\exp\bigl(tI_{x_i,x_{i+1}} - (1+\varepsilon/3)\mu(x_{i+1}-x_i)\bigr).
\end{aligned}$$

We take now $t = t(\varepsilon/3, M_1(\varepsilon))$ and $\rho = \rho(\varepsilon/3, M_1(\varepsilon))$. For each i, we have $\mu(x_i - x_{i+1}) \in [M_1/2, 2M_1]$, thus we can apply the previous lemma and get

$$\begin{aligned}
\mathbb{P}\biggl(S_{\text{odd}} \ge \frac{h\mu(\hat{x})}{2}(1+\varepsilon/2)\biggr) &\le \rho^{(N-5)/2}\\
&\le \rho^{hM_1(\varepsilon)/2-3/2} = A\exp(-Bh)
\end{aligned}$$

with $A = \rho^{-3/2}$ and $B = -\frac{1}{2M_1(\varepsilon)}\ln\rho$.

Similarly, $\mathbb{P}(S_{\text{even}} \ge \frac{h\mu(\hat{x})}{2}(1+\varepsilon/2)) \le A\exp(-Bh)$, so it suffices to put the pieces together to conclude the proof. $\square$

PROOF OF THEOREM 4.1. We first consider the case where A is a convex polygon. Let us denote by $s_0, s_1, \dots, s_{n_e}$ the vertices of A, with $s_{n_e} = s_0$. We suppose that the vertices are in trigonometric order. For each $i \in \{0, n_e - 1\}$, let v_i be such that $\langle v_i, s_{i+1} - s_i\rangle = 0$ and $\langle v_i, s_i\rangle = 1$. For $x \in \mathbb{R}^2$, define $\varphi_i(x) = \langle v_i, x\rangle$. With our conventions

$$nA = \bigcap_{i=0}^{n_e-1} \{x \in \mathbb{R}^2 : \varphi_i(x) \le n\}.$$

For $z \in \mathbb{R}^2$, we define $\operatorname{Int}(z)$ as the only $x \in \mathbb{Z}^2_*$ such that $z \in x + [-1/2, 1/2) \times [-1/2, 1/2)$. Let $\varepsilon > 0$. For $i \in \{0, \dots, n_e\}$, let $y_i = \operatorname{Int}(n(1+\varepsilon)s_i)$.

Our goal is to build for each i a path from y_i to y_{i+1} which does not enter nA and is short enough. Define $M = \max\{\|v_i\|_2; 0 \le i \le n_e - 1\}$ and $S = \max\{\mu(s_i - s_{i+1}); 0 \le i \le n_e - 1\}$.

It is easy to see that

$$\forall r \ge 0 \qquad \varphi_i \ge n(1+\varepsilon) - M\left(\frac{\sqrt{2}}{2} + r\right) \qquad \textit{on } \widetilde{Cyl}(y_i, y_{i+1}, r).$$

Moreover, for each $i \in \{0, \dots, n_e - 1\}$, we have

$$M\left(\frac{\sqrt{2}}{2} + \frac{\varepsilon}{4MS}\|y_i - y_{i+1}\|_2\right) \le M\left(\frac{\sqrt{2}}{2} + \frac{\varepsilon}{4MS}(nS + \sqrt{2})\right) \le \frac{n\varepsilon}{2},$$

provided that n is large enough. Therefore, it follows that $\varphi_i \ge (1+\varepsilon/2)n$ on $\widetilde{Cyl}(y_i, y_{i+1}, \frac{\varepsilon}{4MS}\|y_i - y_{i+1}\|_2)$, which means that this set is off nA.

Since $\mu(y_i - (1+\varepsilon)ns_i) \le \mu_{\max}$, we know that

$$\sum_{i \in \{0,\dots,n_e\}} \mu(y_i - y_{i+1}) \le n(1+\varepsilon)\mathcal{I}(A) + 2n_e\mu_{\max} \le n(1+\varepsilon)^2\mathcal{I}(A)$$

provided that n is large enough.

Then one can see that for n greater than some (deterministic) integer n_0, the event

$$A_n = \bigcap_{i \in \{0,\dots,n_e-1\}} \left\{ t_{\text{long}}\left(\widetilde{Cyl}\left(y_i, y_{i+1}, \frac{\varepsilon}{4MS}\|y_i - y_{i+1}\|_2\right)\right) < (1+\varepsilon)\mu(y_{i+1} - y_i)\right\}$$

satisfies

$$A_n \subset \{\operatorname{Cut}(nA) \le n(1+\varepsilon)^3\mathcal{I}(A)\}$$

We are now ready to apply Lemma 4.2 with $f(h)=\frac{\varepsilon}{4MS}h$. It comes that

$$
\begin{aligned}
P\big(\mathrm{Cut}(nA)>n(1+\varepsilon)^3\mathcal{I}(A)\big) &\le \mathbb{P}(A_n^c)\\
&\le \sum_{i=0}^{n_e-1} C_7\exp(-C_8\|y_i-y_{i+1}\|_2)\\
&\le \sum_{i=0}^{n_e-1} C_7 e^{C_8\sqrt{2}}\exp(-C_8\|s_i-s_{i+1}\|_2 n),\\
&\le c_1\exp(-c_2 n),
\end{aligned}
$$

with $c_1=n_eC_7e^{C_8\sqrt{2}}$ and $c_2=C_8\min_i\|s_i-s_{i+1}\|_2$.

Since ε is arbitrary, the theorem follows when A is a polygon.

Let us go to the general case: By Lemma 3.5, there exists a convex polygon Q with $Q\supset A$ and $(1+\varepsilon)\mathcal{I}(A)\le(1+\varepsilon/2)\mathcal{I}(Q)$.

By its very definition, $\mathrm{Mincut}(nA,\infty)\le\mathrm{Mincut}(nQ,\infty)$. Then

$$
\begin{aligned}
\mathbb{P}\big(\mathrm{Mincut}(nA,\infty)\ge n\mathcal{I}(A)(1+\varepsilon)\big) &\le \mathbb{P}\big(\mathrm{Mincut}(nQ,\infty)\ge n\mathcal{I}(A)(1+\varepsilon)\big)\\
&\le \mathbb{P}\big(\mathrm{Mincut}(nQ,\infty)\le n\mathcal{I}(Q)(1+\varepsilon/2)\big).
\end{aligned}
$$

Hence, the result follows from the polygonal case. $\square$

5. Lower large deviations.

THEOREM 5.1. *For each $\varepsilon>0$, there exist constants $C_9, C_{10}>0$, such that*

$$
\forall n\ge 1\qquad \mathbb{P}\big(\mathrm{Mincut}(nA,\infty)\le n\mathcal{I}(A)(1-\varepsilon)\big)\le C_9\exp(-C_{10}n). \tag{17}
$$

The choice of a strategy for the proof of lower large deviations is more difficult than for the upper ones. An important point is that it is hopeless to consider the sides of the polygon separately.

Indeed, consider the following picture on Figure 1: the red curve and the green one surround the black triangle. Of course, it is expected that the minimal cutset looks like the green triangle rather than like the red ones. However, the red path from A' to H is shorter than the green one from A' to B'. But this advantage is lost on the next side because the red path from H to C' is much longer than the green one from B' to C'. So, it appears that we must think globally, using the perimeter of surrounding curves. To this aim, Lemma 3.4 will be particularly useful.

PROOF OF THEOREM 5.1. Again, we first deal with the case, where A is a convex polygon whose sides are $[s_0,s_1],[s_1,s_2],\dots,[s_{n_e-1},s_{n_e}]$, with $s_{n_e}=s_0$. We denote by $L_{n,i}$ the points $x\in\mathbb{Z}_*^2$ that touch a bond which intersects $[1,+\infty)ns_i$.

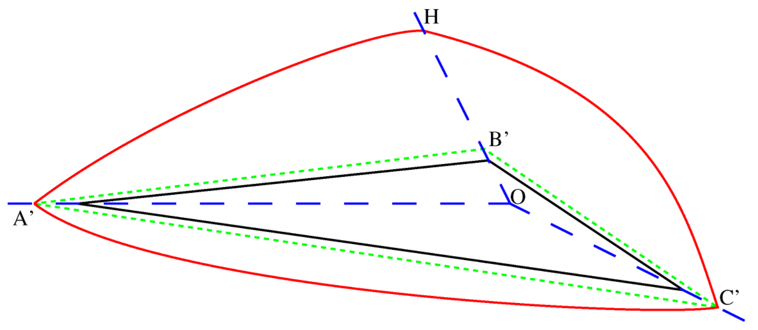

FIG. 1. *Surrounding the polygon.*

LEMMA 5.2. *For each* $\varepsilon > 0$, *there exist* $C_{11} = C_{11}(\varepsilon), C_{12} = C_{12}(\varepsilon)$, *such that*

$$\begin{aligned}&\mathbb{P}\big(\exists i \in \{0, n_e - 1\} \exists (x, y) \in L_{n,i} \times L_{n,i+1}\, d(x, y) \leq (1 - \varepsilon)\mu(x - y)\big)\\&\qquad \leq C_{11} \exp(-C_{12} n).\end{aligned}$$

PROOF. Since $\{0, \dots, n_e - 1\}$ is finite, it is sufficient to prove that for each i, j with $0 \leq i < j < n_e$, there exists $C_{11}(i, j) > 0$ and $C_{12}(i, j) > 0$ with

$$\mathbb{P}\big(\exists (x, y) \in L_{n,i} \times L_{n,j}\, d(x, y) \leq (1 - \varepsilon)\mu(x - y)\big) \leq C_{11}(i, j) \exp(-C_{12}(i, j) n).$$

Thanks to Proposition 3.1, we can write

$$\begin{aligned}&\mathbb{P}\big(\exists (x, y) \in L_{n,i} \times L_{n,j} d(x, y) \leq (1 - \varepsilon)\mu(x - y)\big)\\&\quad \leq \sum_{(x,y)\in L_{n,i}\times L_{n,j}} \mathbb{P}\big(d(x, y) \leq (1 - \varepsilon)\mu(x - y)\big)\\&\quad \leq \sum_{(x,y)\in L_{n,i}\times L_{n,j}} C_3 \exp(-C_4 \|x - y\|_2)\\&\quad \leq C_3 \sum_{p=0}^{+\infty} |A_p| \exp(-C_4 p),\end{aligned}$$

where

$$A_p = \{(x, y) \in L_{n,i} \times L_{n,j};\ \|x - y\|_2 \in [p, p + 1)\}.$$

Let $\alpha = d_2([1, +\infty)s_i, [1, +\infty)s_j)$ and $\theta = \arccos \frac{\langle s_i, s_j\rangle}{\|s_i\|_2 \|s_j\|_2}$. We can see that:

- $|A_p| = 0$ for $p \leq n\alpha - 3$.
- $|A_p| \leq \frac{2000}{\sin^2\theta}(1 + p)^2$ for each $p \geq 0$.

The first point is clear. Let us prove the second point: for each $k \in \{i, j\}$, let $s'_k = s_k/\|s_k\|_2$. Obviously, $A_p \subset B_p \times B'_p$, where $B_p = \{x \in L_{n,i}; d_2(x, \mathbb{R}s'_j) \le p + 3\}$ and $B'_p = \{y \in L_{n,j}; d_2(x, \mathbb{R}s'_i) \le p + 3\}$.

For $r \in \mathbb{R}$, define

$$f(r) = \sum_{x \in B_p} \mathbb{1}_{\{|rs'_i - x| \le \sqrt{2}\}}.$$

Since $d_2(x, \mathbb{R}s_i) \le 1$ for each $x \in L_{n,i}$, it follows that

$$\int_{\mathbb{R}} f(r)\, dr \ge 2|B_p|.$$

For a given r, the sum defining $f(r)$ has at most 9 nonvanishing terms, thus we have

$$f(r) \le 9\mathbb{1}_{\{d_2(rs'_i, \mathbb{R}s_j) \le p+3+\sqrt{2}\}}.$$

Then

$$|B_p| \le \frac{1}{2}\int_{\mathbb{R}} f(r)\, dr \le 9 \times \frac{1}{\beta_{i,j}}\bigl(p + 3 + \sqrt{2}\bigr) \le \frac{9(3+\sqrt{2})}{\beta_{i,j}}(p + 1),$$

where $\beta_{i,j} = |s'_i - \langle s'_i, s'_j\rangle s'_j| = \sqrt{1 - \langle s'_i, s'_j\rangle^2}$.

Similarly, $|B'_p| \le \frac{9(3+\sqrt{2})}{\beta_{i,j}}(p + 1)$. Finally, $|A_p| \le \frac{2000}{\sin^2\theta}(1 + p)^2$.

Let K' be such that $\frac{2000}{\sin^2\theta}C_3(1 + p)^2 \le K' \exp(\frac{C_4}{2}p)$ holds for each $p \ge 0$: we have

$$\begin{aligned}
&\mathbb{P}\bigl(\exists (x, y) \in L_{n,i} \times L_{n,j} : d(x, y) \le (1 - \varepsilon)\mu(x - y)\bigr) \\
&\quad \le \sum_{p=\mathrm{Int}(n\alpha-3)}^{+\infty} K' \exp\left(-\frac{C_4}{2}p\right) \le \frac{K' e^{2C_4}}{1 - \exp(-C_4/2)} \exp\left(-\frac{C_4\alpha}{2}n\right),
\end{aligned}$$

which completes the proof of the lemma. $\square$

We go back to the proof of Theorem 5.1.

Suppose that $\mathrm{Mincut}(nA, \infty) < (1 - \varepsilon)n\mathcal{I}(A)$. Then we can find in the dual lattice a closed path γ that surrounds nA and whose length $l(\gamma)$ is smaller than $(1 - \varepsilon)n\mathcal{I}(A)$. γ necessarily cuts the half-lines $([1, +\infty)ns_i)_{0 \le i \le n_e - 1}$ in some points $y_0, y_1, \dots, y_{n_e-1}$. We also define $y_e = y_0$. The points can be numbered in such a way that γ visits the $(y_i)_{0 \le i \le n_e}$ in the natural order. Let x_i the point in $L_{n,i}$ which is such that $\|y_i - x_i\|_1 \le 1/2$. Obviously,

$$\sum_{i=0}^{n_e-1} d(x_i, x_{i+1}) \le l(\gamma) \le (1 - \varepsilon)n\mathcal{I}(A). \tag{18}$$

Let B be the polygon determined by the y_i: we have

$$\mathcal{I}(B) = \sum_{i=0}^{n_e-1} \mu(y_i - y_{i+1}) \leq \sum_{i=0}^{n_e-1} \big(\mu(x_i - x_{i+1}) + \mu_{\max}\big)$$

nA is convex and contained in B, so by Lemma 3.4, $\mathcal{I}(B) \geq \mathcal{I}(nA)$. It follows that

$$\begin{aligned}\sum_{i=0}^{n_e-1} d(x_i, x_{i+1}) \leq (1-\varepsilon)n\mathcal{I}(A) &\leq (1-\varepsilon)\mathcal{I}(B) \\ &\leq \sum_{i=0}^{n_e-1} (1-\varepsilon)\big(\mu(x_i - x_{i+1}) + \mu_{\max}\big) \\ &\leq \sum_{i=0}^{n_e-1} (1-\varepsilon/2)\mu(x_i - x_{i+1}),\end{aligned}$$

provided that $n \geq \frac{1}{\alpha}(1 + \frac{2}{\varepsilon}\frac{\mu_{\max}}{\mu_{\min}})$.

So, for large n, the event $\{\text{Mincut}(nA, \infty) < (1-\varepsilon)\}$ implies the existence of $i \in \{0, \dots, n_e - 1\}$, $x_i \in L_{n,i}$ and $x_{i+1} \in L_{n,i+1}$ with

$$d(x_i, x_{i+1}) \leq (1-\varepsilon/2)\mu(x_i - x_{i+1}).$$

Then we have

$$\begin{aligned}&\mathbb{P}\big(\text{Mincut}(nA, \infty) < (1-\varepsilon)n\mathcal{I}(A)\big) \\ &\quad\leq \mathbb{P}\big(\exists i \in \{0, \dots, \neq -1\}, \\ &\qquad\quad \exists (x, y) \in L_{n,i} \times L_{n,i+1}\, d(x, y) \leq (1-\varepsilon/2)\mu(x-y)\big) \\ &\quad\leq C_{11}(\varepsilon/2)\exp(-C_{12}(\varepsilon/2)n),\end{aligned}$$

thanks to Lemma 5.2. This ends the proof in the case, where A is a polygon.

Let us go to the general case: By Lemma 3.5, there exists a convex polygon P with $0 \in P$, $P \subset A$ and $(1-\varepsilon)\mathcal{I}(A) \leq (1-\varepsilon/2)\mathcal{I}(P)$.

By its very definition, $\text{Mincut}(nA, \infty) \geq \text{Mincut}(nP)$. Then

$$\begin{aligned}&\mathbb{P}\big(\text{Mincut}(nA, \infty) \leq n\mathcal{I}(A)(1-\varepsilon)\big) \\ &\quad\leq \mathbb{P}\big(\text{Mincut}(nP) \leq n\mathcal{I}(A)(1-\varepsilon)\big) \\ &\quad\leq \mathbb{P}\big(\text{Mincut}(nP) \leq n\mathcal{I}(P)(1-\varepsilon/2)\big),\end{aligned}$$

which has just been proved to decrease exponentially fast with n. $\square$

6. Final proofs.

6.1. *Proof of the theorems.* Obviously, Theorems 4.1 and 5.1 concur to get Theorem 2.2. Since $\mathcal{I}(A) = \int_{\partial^* A} \mu(\nu_A(x))\, d\mathcal{H}^1(x)$, the first equality in Theorem 2.1 directly follows from Theorem 2.2 with the help of the Borel–Cantelli lemma.

It is worth saying a word about equation (3), because the Ford–Fulkerson theorem is initially concerned with finite graphs. Let us recall a version of this theorem.

PROPOSITION 6.1 (Ford–Fulkerson). *For each finite graph $G = (V, E)$ and every disjoint subsets A and B of V, we have*

$$\max\left\{\sum_{x\in A} \operatorname{Div} j(x);\ j \in \operatorname{Capflow}(A, B)\right\} = \operatorname{Mincut}(A, B), \tag{19}$$

where

$$\operatorname{Mincut}(A, B) = \min\left\{\sum_{x\in C} t_x;\ \textit{every path in } G \textit{ from } A \textit{ to } B \textit{ meets } C\right\} \tag{20}$$

and $\operatorname{Capflow}(A, B)$ *is the set of flows* j *that satisfy* $|j(x, y)| \le t_{\{x,y\}}$ *for each* $\{x, y\} \in E$ *and* $\operatorname{Div} j(x) = 0$ *for* $x \in V\backslash(A \cup B)$.

In fact, in the initial paper [8] and in most books, A and B are just singletons. The reduction to this case is easy. Because of the antisymmetry property, the contribution of edges inside A to $\sum_{x\in A} \operatorname{Div} j(x)$ is null; so we neither change the max-flow nor the min-cut if we identify the points that are in A. Obviously, the max-flow and the min-cut are not changed either when we identify the points that are in B.

Now let $G_n = (V_n, E_n)$ be the restriction of $\mathbb{L}^2$ to $V_n = \{x \in \mathbb{Z}^2; \|x\|_1 \le n\}$ and denote by B_n the boundary of V_n.

Let f be a flow from A to infinity; particularly, f is a flow from A to B_n, so $\sum_{x\in A} \operatorname{Div} j(x) \le \operatorname{Mincut}(A, B_n)$. By the definition of a cutset, a minimal cutset from A to infinity is the external boundary of a finite connected set containing A. In particular, a minimal cutset is finite. It follows that $\inf_{n\ge 1} \operatorname{Mincut}(A, B_n) = \operatorname{Mincut}(A, \infty)$. Then $\sup\{\sum_{x\in A} \operatorname{Div} j(x);\ j \in \operatorname{Capflow}(A, \infty)\} \le \operatorname{Mincut}(A, \infty)$. Conversely, let j_n be a flow that realizes $\max\{\sum_{x\in A} \operatorname{Div} j(x);\ j \in \operatorname{Capflow}(A, B_n)\}$. We can extend j_n to $\vec{\mathbb{E}}^d$ by putting $j_n(e) = 0$ outside E_n. Obviously, $j_n \in \prod_{e\in\vec{\mathbb{E}}^d} [-t_e, +t_e]$, thus the sequence $(j_n)_{n\ge 1}$ admits a subsequence $(j_{n_k})_{k\ge 1}$ converging to some $j' \in \prod_{e\in\vec{\mathbb{E}}^d} [-t_e, +t_e]$ in the product topology. Easily, j' is antisymmetric.

$$\begin{aligned}\sum_{x\in A} \operatorname{Div} j'(x) &= \lim_{k\to+\infty} \sum_{x\in A} \operatorname{Div} j_{n_k}(x)\\ &= \lim_{k\to+\infty} \max\left\{\sum_{x\in A} \operatorname{Div} j(x);\ j \in \operatorname{Capflow}(A, B_k)\right\}\end{aligned}$$

$$= \lim_{k\to+\infty} \mathrm{Mincut}(A, B_{n_k})$$
$$= \inf_{n\ge 1} \mathrm{Mincut}(A, B_n) = \mathrm{Mincut}(A, \infty).$$

For each $x \in \mathbb{Z}^2 \backslash A$, there exists k_0 such that $x \in V_n \backslash (B_{n_k} \cup A)$ for $k \ge k_0$; then $\mathrm{Div}\, j_{n_k}(x) = 0$ for $k \ge k_0$, which ensures that $\mathrm{Div}\, j'(x) = 0$. It is now easy to see that j' is a capacitive flow from A to infinity, which completes the proof of equation (3) and, therefore, the proof of Theorem 2.1.

6.2. *Proof of the corollaries.* Let us now recall Menger's theorem (see, for instance, Diestel [6] for a proof).

PROPOSITION 6.2 (Menger's theorem). *Let $G = (V, E)$ be a finite graph and $A, B \subset V$. Then the minimum number of vertices separating A from B is equal to the maximum number of disjoint paths from A to B.*

We can now prove Corollary 2.3.

PROOF OF COROLLARY 2.3. Consider the probability space $(\Omega, \mathcal{B}, \mathbb{P})$, with $\Omega = \{0, 1\}^{\mathbb{E}^2}$ and $\mathbb{P} = \mathrm{Ber}(p)^{\otimes \mathbb{E}^2}$. As usual, e is said to be open if $\omega_e = 1$ and closed otherwise. Let $R = \{e \in \mathbb{E}^2 : \omega_e = 1\}$ and define V_n and E_n as previously. Let $H_n = (V_n, E_n \cap R)$. It is easy to see that the minimum number of vertices separating A from B_n is equal to $\mathrm{Mincut}(A, B_n)$, where the capacity flow is defined by $t_e = 1 - \omega_e$. Then by Menger's theorem, the maximum number of disjoint paths from A to B_n is $\mathrm{Mincut}(A, B_n)$. By a classical compactness argument, the maximum number of disjoint paths from A to infinity is the limit of the maximum number of disjoint paths from A to B_n. Therefore, $\mathrm{dis}(A) = \lim_{n\to+\infty} \mathrm{Mincut}(A, B_n) = \mathrm{Mincut}(A, \infty)$. The variables $(t_e)_{e\in\mathbb{E}^2}$ are independent Bernoulli variables with parameter $1 - p$. Note m for their common distribution. Since $p > p_c(2) = 1/2$, $m(0) = 1 - p < 1/2$, and we can apply Theorem 2.2 to complete the proof of Corollary 2.3. □

We finally prove Corollary 2.4.

PROOF OF COROLLARY 2.4. Let us denote by I_k the event: "there exist k disjoint open biinfinite paths." I_k is obviously translation-invariant, so by the ergodic theorem, its probability is null or full. Let $A = [-1, 1]^2$ and $S_n = \{\mathrm{dis}(An) \ge n\mathcal{I}(A)/2\}$. For large n, we have $n\mathcal{I}(A)/2 > 2k$ and $\mathbb{P}(S_n) > 1/2$. Now consider the event T_n: "all edges inside nA are open." It is not difficult to see that $T_n \cap S_n \subset I_k$ but T_n and S_n are independent, so $P(I_k) \ge P(T_n \cap S_n) = P(T_n)P(S_n) > 0$. Finally, $P(I_k) = 1$. □

7. Perspectives. It is to be expected that these results still hold in higher dimensions. In fact, we make the following conjecture:

CONJECTURE 7.1. We suppose that $m(0) < 1 - p_c(\mathbb{Z}^d)$ and that

$$\int_{[0,\infty)} \exp(cx)\, dm(x) < +\infty$$

for some $c > 0$. Then there exists a map μ on the unit sphere such that for each convex set A with 0 in the interior, we have

$$\lim_{n\to+\infty} \frac{\text{Mincut}(nA, \infty)}{n^{d-1}} = \int_{\partial^* A} \mu(\nu_A(x))\, d\mathcal{H}^{d-1}(x).$$

Equivalently,

$$\lim_{n\to+\infty} \frac{1}{n^{d-1}} \max\left\{ \sum_{x\in nA\cap\mathbb{Z}^d} \text{Div}\, j(x);\ j \in \text{Capflow}(nA, \infty) \right\}$$
$$= \sup\left\{ \int_A \text{div}\, f\, d\lambda^d(x);\ f \in C_c^1(\mathbb{R}^d, \mathcal{W}_\mu) \right\},$$

where

$$\mathcal{W}_\mu = \{x \in \mathbb{R}^d : \langle x, w\rangle \le \mu(w) \text{ for all } w\}.$$

Of course, the situation is more complicated when $d \ge 3$ because cutsets are not paths; therefore, the capacities can not be interpreted in term of first-passage percolation. In a seminal paper [13], Kesten put the basis of a generalization of first-passage percolation which seems to be the appropriate tool for the problem considered here. Basically, he studies the minimal cut between opposite sides of a parallelepiped with (e_1, e_2, e_3) as axes. This allows to define a quantity ν which is a good candidate for $\mu(e_1)$. Later, Boivin [2] extended some of Kesten's results. Particularly, he defined a function on the unit sphere of $\mathbb{R}^3$ which may be convenient for our purpose. The condition $m(0) < 1 - p_c$ is coherent with some previous results; indeed, Zhang [17] proved that $\nu = 0$ for $m(0) \ge 1 - p_c$ whereas Chayes and Chayes [4] had proved (at least in the Bernoulli case) that $\nu > 0$ for $m(0) \ge 1 - p_c$ using a result of Aizenman, Chayes, Chayes, Fröhlich and Russo [1]. Note that Théret [15] recently proved some results that give an independent proof of this fact. So, $m(0) < 1 - p_c$ seems to be a natural assumption for the conjecture. This is also coherent with the expected domain of validity for the d-dimensional version of Corollary 2.3. Of course, this conjecture is at present far from being solved because some of the quantities that are used in the present proof do not have an obvious equivalent in higher dimensions. However, we think that the conjecture presented here is a good motivation to continue the study initiated in Kesten [13].

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UNIVERSITÉ HENRI POINCARÉ NANCY 1
CAMPUS SCIENTIFIQUE, BP 239
54506 VANDOEUVRE-LÈS-NANCY CEDEX
FRANCE
E-MAIL: Olivier.Garet@iecn.u-nancy.fr
URL: http://www.iecn.u-nancy.fr/~garet/