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STOCHASTIC ORDER FOR INSPECTION AND REPAIR POLICIES

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Inspection and repair policies for $(n - r + 1)$-out-of-n systems are compared stochastically with respect to two partial orderings $\geq^{b_1}$ and $\geq^{b_2}$ on the set of permutations of $\{1, 2, \dots, n\}$. The partial ordering $\geq^{b_1}$ is finer than the partial ordering $\geq^{b_2}$. A given permutation π of $\{1, 2, \dots, n\}$ determines the order in which components are visited and inspected. We assume that the reliability of the ith independent component is given by P_i, where $P_1 \leq P_2 \leq \cdots \leq P_n$. If π and π' are two permutations such that $\pi \geq^{b_1} \pi'$, then we show that the number of inspections necessary to achieve minimal or complete repair is stochastically smaller with π than with π'. We also consider three policies for minimal repair when the components are each made up of t "parts" assembled in parallel. It is shown that if $\pi \geq^{b_2} \pi'$, then the number of repairs necessary under π is stochastically smaller than the number necessary under π', but that in general this is not true for the finer ordering $\geq^{b_1}$. The results enable one to make interesting comparisons between various inspection and repair policies, as well as to understand better the relationship between the orderings $\geq^{b_1}$ and $\geq^{b_2}$ on the set of permutations.

1. Introduction and summary. In this article we are interested in comparing inspection and repair policies for an $(n - r + 1)$-out-of-n system. We assume that the components function independently with respective probabilities $P_1, \dots, P_n$, where $P_1 \leq P_2 \leq \cdots \leq P_n$. N will be the random variable indicating the number of failed components at a particular point of time. In some cases we will assume very specific knowledge about N; in others we will not. For example, if we know the system has just failed, then we will assume that $N = r$, while if we know that the system has failed at some time in the past, we will assume $N \geq r$. Normally we will be interested in points of time where the system is down $[N \geq r]$, but of course N may take any integer value between 0 and n. Now let π be any permutation of $\{1, \dots, n\}$. Any such π defines an inspection procedure for the system whereby we first inspect component $\pi(1)$, then component $\pi(2), \dots,$ etc., until a specified objective (say, a minimal repair or a complete repair) is achieved. For any inspection policy π and any integer k, where $1 \leq k \leq n$, we define the random variable

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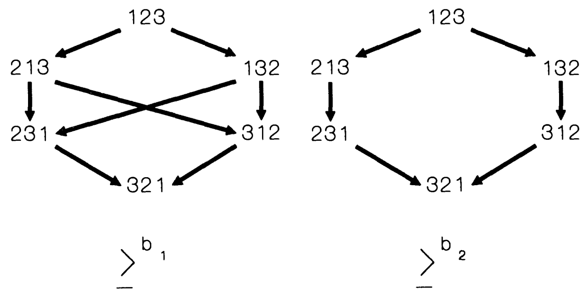

FIG. 1.

$W_{k,\pi}$ (on the space of possible system states w) as follows:

$$W_{k,\pi}(w) = \begin{cases} \text{No. of components inspected (in the order } \pi) & \text{if } N(w) \geq k, \\ \quad \text{until the } k\text{th failure is spotted} & \\ n & \text{if } N(w) < k. \end{cases}$$

We may view $W_{k,\pi}$ as the waiting time under procedure π until the discovery of the kth failed component. For two policies π and π' we will be interested in stochastic comparisons of $W_{k,\pi}$ and $W_{k,\pi'}$ in the situations where (a) nothing is known about N, (b) it is known that $[N = r]$ (the system has just gone down) and (c) it is known that $[N \geq r]$ (the system has been down for some time). Our comparisons will be relative to two partial orderings defined on the permutations of $\{1, \dots, n\}$. For two permutations π and π', we say that $\pi \geq^{b_i} \pi'$ $(i = 1, 2)$ if either $\pi = \pi'$ or π can be obtained from π' by a sequence of permutations, starting with π', whereby each successive permutation is obtained by a pairwise correction of out-of-order elements of the preceding permutation. If the pairwise corrections of elements of the permutations have no restrictions, we say that $\pi \geq^{b_1} \pi'$, while if only adjacent pairs of elements are corrected we say $\pi \geq^{b_2} \pi'$. We note that $\geq^{b_1}$ is a finer partial order on the set σ_n of permutations of $\{1, \dots, n\}$ than is $\geq^{b_2}$, and Figure 1 illustrates the related permutations for $n = 3$.

The partial order $\geq^{b_1}$ was defined by Sobel (1954), Savage (1959) and Lehmann (1966). This ordering is fundamental to the work of Hollander, Proschan and Sethuraman (1977) on arrangement-increasing functions. The ordering $\geq^{b_2}$ is considered by Yanagimoto and Okamoto (1969) in studying monotonicity results for rank correlation coefficients. Both of these partial orders are treated by Block, Chhetry, Fang and Sampson (1988, 1990), who also study other partial orders with applications to concepts of dependence for bivariate distributions.

In Section 2 of this article we show that conditioned on either $[N = r]$, $[N \geq r]$ or unconditionally on N, $\pi \geq^{b_1} \pi' \Rightarrow W_{k,\pi} \leq^{st} W_{k,\pi'}$. Applications are discussed and some stochastic properties of accumulated costs are developed for various inspection procedures.

In Section 3 we assume that each component is in turn composed of t independent parts of similar reliability. Series-parallel systems or, more generally, $(n-r+1)$-out-of-n systems whose components are each enforced by $t-1$ active redundancies, are familiar examples. Such systems occur, for instance, in communication networks and safety design. Assuming independence of parts we have that if p_i is the reliability of a part in component i, then the reliability of component i is $P_i = 1 - q_i^t$. We investigate three different types of inspection and repair policies, where our primary interest is to minimize the number of repairs in order to bring a system which is down back to life. Let π be a given permutation indicating the order in which components are visited (inspected). In policy 1 we inspect only the "first" part of a visited component. If this first part is good, we know the component as a whole is working and we proceed to the next component having made no repair. Otherwise we repair (or replace) that first part (knowing then that the given component will work) and then test the whole system. If the system works, we stop; otherwise, we proceed to the next component (as determined by π). We let R_π^1 be the total number of repairs necessary to bring the system back to life (a form of minimal repair). Policy 2 is similar except that we repair parts in a component sequentially until we meet a good part in the component. If no such part is encountered we make t repairs there, knowing that we have found one of the bad components in the system. We proceed until the system comes back to life, letting R_π^2 be the number of repairs necessary to achieve this. In policy 3, we repair all bad parts in an inspected component, and continue inspection until the system comes back to life. R_π^3 is the random variable indicating the total number of repairs made under this policy in order to obtain a minimal repair. Letting R_π^α be the number of repairs necessary to achieve minimal repair using policy α (assuming either the system is "just down" or has been down for some time), we show that $\pi \geq^{b_2} \pi' \Rightarrow R_\pi^\alpha \leq^{\text{st}} R_{\pi'}^\alpha$, for $\alpha = 1, 2, 3$. We show that in general this stochastic implication is not true for the finer (but more standard) partial order $\geq^{b_1}$.

Ben-Dov (1977) presents some related work where the objective is to determine the bad components in a system. He determines the inspection procedure which is optimal in the sense of minimizing expected costs.

2. Inspection procedures for $(n-r+1)$-out-of-n systems. We assume here that component i has reliability $P_i = 1 - Q_i$ for $i = 1, \ldots, n$, and without loss of generality that $P_1 \leq \cdots \leq P_n$. The components are assumed independent, and the system functions if at least $n-r+1$ of the components function. N is the random variable indicating the number of bad or failed components, and often we will be interested in conditioning on $N = r$ (the system has just failed) or $N \geq r$ (the system is in a failed state). For any given permutation π of $\{1, \ldots, n\}$ which represents the order of inspection of the components, we have defined $W_{k,\pi}$ to be the (waiting time) number of components inspected until the kth failed component is discovered. We now define $W_{N,\pi}$ (respectively, $W_{N-r+1,\pi}$) to be the number of components visited or inspected until all failed components (the failed component whose repair brings the system back to life) are identified and fixed. If the system is working, then of course $W_{N-r+1,\pi} = 0$, and if no components are failed then $W_{N,\pi} = 0$. We may view $W_{N,\pi}$ ($W_{N-r+1,\pi}$) as the waiting time or total number of inspections that are necessary to achieve complete (minimal) repair. We are of course implicitly assuming that every time we encounter a failed component we repair or replace it and test the system to see if it then functions or not. Note that once we perform a repair for a component which results in the system functioning (for the first time), we know that we have just repaired failure number $N - r + 1$ and that there are $r - 1$ failed components left. (Previous to this we might only have known that the system was down without knowing the exact number of failed components.) We are interested in making stochastic comparisons between $W_{N,\pi}$ and $W_{N,\pi'}$ (respectively, $W_{N-r+1,\pi}$ and $W_{N-r+1,\pi'}$) for permutations π and π'.

Definition 2.1. For two permutations π and π' of $\{1,\dots,n\}$, we say $\pi \geq^{b_1} \pi'$ if π can be reached from π' by successive interchanges, each of which corrects an inversion of the natural order. In this ordering we say π immediately precedes π' and write $\pi \geq^{P_1} \pi'$ if there exist indices i and j such that $i < j$, $\pi(i) < \pi(j)$, $\pi(i) = \pi'(j)$, $\pi(j) = \pi'(i)$ and $\pi(l) = \pi'(l)$ for all $l \neq i, j$. Then $\pi \geq^{b_1} \pi'$ if and only if there exists a sequence of permutations $\psi^1,\dots,\psi^k$ such that $\pi = \psi^k \geq^{P_1} \cdots \geq^{P_1} \psi^1 = \pi'$. For example, $(1234) \geq^{b_1} (3421)$. Similarly, we define the partial order $\geq^{b_2}$ on the set σ_n of permutations of $\{1,\dots,n\}$, where $\pi \geq^{b_2} \pi'$ if π can be reached from π' by successive interchanges, each of which corrects an inversion of the natural order of *adjacent* numbers. We say π immediately precedes π' in this ordering, and write $\pi \geq^{P_2} \pi'$ if there exists an index i such that $\pi(i) = \pi'(i+1) < \pi(i+1) = \pi'(i)$ and $\pi(j) = \pi'(j)$ for all $j \neq i, i+1$. Then $\pi \geq^{b_2} \pi'$ if there exists a sequence of permutations $\psi^1,\dots,\psi^k$ such that $\pi = \psi^k \geq^{P_2} \cdots \geq^{P_2} \psi^1 = \pi'$. The partial order $\geq^{b_1}$ is finer than $\geq^{b_2}$ in the sense that $\pi \geq^{b_2} \pi' \Rightarrow \pi \geq^{b_1} \pi'$, but not conversely. For example, $(1243) \not\geq^{b_2} (3421)$, but $(3124) \geq^{b_2} (3421)$. For both partial orderings b_i, $i = 1, 2$, one has that for any permutation $\pi \in \sigma_n$, $(1,2,3,4,\dots,n) \geq^{b_i} \pi = (\pi(1),\dots,\pi(n)) \geq^{b_i} (n, n-1,\dots,1)$.

We now wish to compare the waiting times $W_{k,\pi}$ and $W_{k,\pi'}$ for two different inspection procedures π and π' where $\pi \geq^{b_1} \pi'$.

Lemma 2.2. *Let $W_{k,\pi}$ be the number of inspected components in the system up to and including the discovery of the kth failed component for the inspection procedure π. Let $\pi \geq^{b_1} \pi'$. Then*

$$P[W_{k,\pi} > x,\ N = m] \leq P[W_{k,\pi'} > x,\ N = m] \quad \text{for } x = 0,\dots,n-1.$$

Proof. Without loss of generality we assume $\pi \geq^{P_1} \pi'$. Hence in particular there exists $i < j$ such that $\pi(l) = \pi'(l)$ for $l \neq i, j$ and $P_{\pi(i)} \leq P_{\pi(j)}$. It is easy to see that

$$P[W_{k,\pi} > x,\ N = m] = P[W_{k,\pi'} > x,\ N = m] \quad \text{whenever } x < i \text{ or } x \geq j.$$

We now let $N_{i_1,\dots,i_t}$ indicate the number of failed components among those components with indices $i_1, \dots, i_t$, and define $N^{i_1, \dots, i_t} = N - N_{i_1, \dots, i_t}$ (the number of failed components in $\{1, \dots, n\} - \{i_1, \dots, i_t\}$). Note that $P[W_{k,\pi} > x, N = m] = P[N_{\pi(1), \dots, \pi(x)} \le k - 1, N = m]$.

For $i \le x < j$, we have

$$\begin{aligned} P&[W_{k,\pi} > x, N = m] \\ &= Q_{\pi(i)} Q_{\pi(j)} P\left[N_{\pi(1), \dots, \widehat{\pi(i)}, \dots, \pi(x)} \le k - 2, N^{\pi(i)\pi(j)} = m - 2\right] \\ &\quad + P_{\pi(i)} P_{\pi(j)} P\left[N_{\pi(1), \dots, \widehat{\pi(i)}, \dots, \pi(x)} \le k - 1, N^{\pi(i)\pi(j)} = m\right] \\ &\quad + Q_{\pi(i)} P_{\pi(j)} P\left[N_{\pi(1), \dots, \widehat{\pi(i)}, \dots, \pi(x)} \le k - 2, N^{\pi(i)\pi(j)} = m - 1\right] \\ &\quad + P_{\pi(i)} Q_{\pi(j)} P\left[N_{\pi(1), \dots, \widehat{\pi(i)}, \dots, \pi(x)} \le k - 1, N^{\pi(i)\pi(j)} = m - 1\right]. \end{aligned}$$

Here we use $N_{\pi(1), \dots, \widehat{\pi(i)}, \dots, \pi(x)} = N_{\pi(1), \dots, \pi(i-1), \pi(i+1), \dots, \pi(x)}$, which in the following expression we abbreviate to $N^*_{\pi(i)}$. Then we have

$$\begin{aligned} P&[W_{k,\pi} > x, N = m] - P[W_{k,\pi'} > x, N = m] \\ &= Q_{\pi(i)} P_{\pi(j)} \Big\{ P\left[N^*_{\pi(i)} \le k - 2, N^{\pi(i)\pi(j)} = m - 1\right] \\ &\qquad\qquad - P\left[N^*_{\pi(i)} \le k - 1, N^{\pi(i)\pi(j)} = m - 1\right] \Big\} \\ &\quad - Q_{\pi(j)} P_{\pi(i)} \Big\{ P\left[N^*_{\pi(i)} \le k - 2, N^{\pi(i)\pi(j)} = m - 1\right] \\ &\qquad\qquad - P\left[N^*_{\pi(i)} \le k - 1, N^{\pi(i)\pi(j)} = m - 1\right] \Big\} \\ &= \left(Q_{\pi(i)} P_{\pi(j)} - Q_{\pi(j)} P_{\pi(i)}\right) \Big\{ P\left[N^*_{\pi(i)} \le k - 2, N^{\pi(i)\pi(j)} = m - 1\right] \\ &\qquad\qquad - P\left[N^*_{\pi(i)} \le k - 1, N^{\pi(i)\pi(j)} = m - 1\right] \Big\} \\ &\le 0. \end{aligned}$$

□

As a consequence of this lemma we are able to stochastically compare $W_{N,\pi}$ and $W_{N,\pi'}$ (or $W_{N-r+1,\pi}$ and $W_{N-r+1,\pi'}$) whenever $\pi \ge^{b_1} \pi'$. Such comparisons can be made conditional on some information about N, or unconditionally.

THEOREM 2.3. *Assume $\pi \ge^{b_1} \pi'$. Then for any $x = 0, 1, \dots, n - 1$:*

(i) $$P[W_{N,\pi} > x] \le P[W_{N,\pi'} > x].$$

(ii) $$P[W_{N-r+1,\pi} > x] \le P[W_{N-r+1,\pi'} > x].$$

(iii) $$P[W_{N,\pi} > x \mid N = r] \le P[W_{N,\pi'} > x \mid N = r].$$

(iv) $$P[W_{N-r+1,\pi} > x \mid N = r] \le P[W_{N-r+1,\pi'} > x \mid N = r].$$

(v) $$P[W_{N,\pi} > x \mid N \ge r] \le P[W_{N,\pi'} > x \mid N \ge r].$$

(vi) $$P[W_{N-r+1,\pi} > x \mid N \ge r] \le P[W_{N-r+1,\pi'} > x \mid N \ge r].$$

The proof follows readily from Lemma 2.2 and is therefore omitted.

We now consider the practical implications of parts (iii) and (iv) of Theorem 2.3. If an $(n - r + 1)$-out-of-n system has just gone down, then we know that

$N = r$. We may be interested in the number of component inspections $W_{N-r+1,\pi}$ we must make in order to bring the system back to life (a minimal repair, i.e., in this case find one bad component and repair it), or the number of component inspections $W_{N,\pi}$ we must make in order to make a *complete* repair of the system. In either case we are assuming that π determines the order in which components are inspected. The theorem says that the closer π is to the natural order (inspecting components in the order $1, 2, \dots, n$; remember $P_1 \leq \cdots \leq P_n$ and hence this implies inspecting the "potentially" weakest component first, etc.) with respect to $\geq^{b_1}$, the smaller is (stochastically) the waiting time to minimal or complete repair. Parts (v) and (vi) of the theorem imply this is also true given that you have knowledge that the system has failed or is down (as opposed to the more specific knowledge that it has "just" failed). Note in particular that the inspection procedure $\pi = (1, 2, \dots, n)$ is clearly the best of procedures in all of these circumstances, while $\pi' = (n, n-1, \dots, 1)$ is the worst.

In the rest of this section we deal with comparisons of costs for inspection procedures which are comparable in the $\geq^{b_1}$ ordering. Suppose now that C_i (≥ 0) is the random cost of inspecting component i, for $i = 1, \dots, n$. We assume that $(C_1, \dots, C_n)$ is independent of $(X_1, \dots, X_n)$, the random vector representing the states of the n components where $X_i = 1$ if component i is functioning and 0 otherwise for $i = 1, \dots, n$. In addition we assume $C_1 \leq^{\mathrm{st}} \cdots \leq^{\mathrm{st}} C_n$. Since we already assume that $P_1 \leq \cdots \leq P_n$, this is in line with the belief that the conceptually weaker components are cheaper to inspect. If π is a permutation indicating the order of inspection, then the total cost of inspection in order to make a minimal repair is

$$\sum_{i=1}^{W_{N-r+1,\pi}} C_{\pi(i)}.$$

If a complete repair of the system is to be made, then the total cost of inspection (again assuming the system has failed) is

$$\sum_{i=1}^{W_{N,\pi}} C_{\pi(i)}.$$

We make some stochastic comparisons of total inspection costs for procedures π and π' where $\pi \geq^{b_1} \pi'$. The following lemma is of considerable use.

LEMMA 2.4. *Let $\pi \geq^{b_1} \pi'$. Then*

$$E\left(\sum_{i=1}^{W_{k,\pi}} C_{\pi(i)} | N = m\right) \leq E\left(\sum_{i=1}^{W_{k,\pi'}} C_{\pi'(i)} | N = m\right)$$

for any k and $m = 0, 1, \dots, n$. Moreover, if $C_1, \dots, C_n$ are independent, then conditional on $N = m$,

$$\sum_{i=1}^{W_{k,\pi}} C_{\pi(i)} \leq^{\mathrm{st}} \sum_{i=1}^{W_{k,\pi'}} C_{\pi'(i)}.$$

PROOF. Note that if $k > m$, then $W_{k,\pi} = n$ by definition. Hence without loss of generality we assume $k \leq m$, and in this proof we assume expectations are conditional on $N = m$. It suffices to prove this lemma for the case where $\pi \geq^{P_1} \pi'$. Now

$$E\left(\sum_{i=1}^{W_{k,\pi}} C_{\pi(i)}\right) = \sum_{s=1}^{n}\left[\sum_{i=1}^{s} E(C_{\pi(i)})\right] P[W_{k,\pi} = s] \equiv Eg_1(W_{k,\pi}),$$

and similarly

$$E\left(\sum_{i=1}^{W_{k,\pi'}} C_{\pi'(i)}\right) = \sum_{s=1}^{n}\left[\sum_{i=1}^{s} E(C_{\pi'(i)})\right] P[W_{k,\pi'} = s] \equiv Eg_2(W_{k,\pi'}),$$

where $g_1(s) = \sum_{i=1}^{s} E(C_{\pi(i)})$ and $g_2(s) = \sum_{i=1}^{s} E(C_{\pi'(i)})$.

Now g_1 and g_2 are nondecreasing and $g_1 \leq g_2$. Since $W_{k,\pi} \leq^{\text{st}} W_{k,\pi'}$ (Lemma 2.2), it follows that

$$E\left(\sum_{i=1}^{W_{k,\pi}} C_{\pi(i)} | N = m\right) \leq E\left(\sum_{i=1}^{W_{k,\pi'}} C_{\pi'(i)} | N = m\right).$$

Now we assume that $C_1, \ldots, C_n$ are independent random variables, and let f be any nondecreasing function. We define h_1 and h_2 by $h_1(s) = Ef(\sum_{i=1}^{s} C_{\pi(i)})$ and $h_2(s) = Ef(\sum_{i=1}^{s} C_{\pi'(i)})$. Then h_1 and h_2 are nondecreasing and $h_1 \leq h_2$. Since $E(f(\sum_{i=1}^{W_{k,\pi}} C_{\pi(i)})) = E(h_1(W_{k,\pi}))$ and $E(f(\sum_{i=1}^{W_{k,\pi'}} C_{\pi'(i)})) = E(h_2(W_{k,\pi'}))$, the result follows since $W_{k,\pi} \leq^{\text{st}} W_{k,\pi'}$ (conditional on $N = m$). □

Note that the result of the above lemma holds unconditionally on N as well.

Lemma 2.4 has many applications. If the $(n - r + 1)$-out-of-n system is down, then the total cost of inspection until minimal (complete) repair is

$$\sum_{i=1}^{W_{N-r+1,\pi}} C_{\pi(i)} \qquad \left(\sum_{i=1}^{W_{N,\pi}} C_{\pi(i)}\right),$$

where π is the order of inspection. The above lemma implies that whenever $\pi \geq^{b_1} \pi'$, then for either type of repair, and conditioned on either $[N = r]$ (the system has just failed) or $[N \geq r]$ (the system is failed), π is to be preferred to π' from a cost point of view.

3. Inspection and repair policies for $(n - r + 1)$-out-of-n systems of parallel subsystems. In this section we still assume we have an $(n - r + 1)$-out-of-n system, but now we add the assumption that each component is composed of t independent identically distributed parts in parallel. Letting p_i be the reliability of a part in component (i) (without loss of generality $p_1 \leq p_2 \leq \cdots \leq p_n$), it follows that the reliability of component i is $P_i = 1 - q_i^t$.

Assuming that the system is not functioning, we are interested in stochastic comparisons for three types of minimal repair policies. A given permutation π

will denote the order in which components are visited, but parts within a component are repaired or replaced according to three different policies. The random variables R_π^1, R_π^2 and R_π^3 have been defined previously, and denote, respectively, the number of repairs necessary under the three policies to achieve minimal repair. We show that if $\pi \geq^{b_2} \pi'$, then assuming the system is down, $R_\pi^\alpha \leq_{\mathrm{st}} R_{\pi'}^\alpha$ for $\alpha = 1, 2, 3$.

Policy 1 would be appropriate when it is not possible to test components individually, it is onerous to inspect and/or replace each part, and a very minimal repair is desired. Policy 2 would be appropriate if we feel it advisable to make a somewhat more complete repair within a component, while in policy 3 the idea is that once one has investigated a given component it seems advisable to repair or replace all nonfunctioning parts within the component.

Now let $X_i = 1$ if component i is working and 0 otherwise, and let $Y_{ij} = 1$ if the jth part of component i is working and 0 otherwise. We let G_i be the number of bad parts in component i, up to the first good part (i.e., $G_i = \max\{j\colon Y_{i1} + \cdots + Y_{ij} = 0\}$), and B_i the number of bad parts in component i (i.e., $B_i = t - \Sigma_1^t Y_{i,j}$). G_i is a truncated geometric random variable and B_i is binomially distributed. Note then that

$$R_\pi^1 = \sum_{i=1}^{W_{N-r+1,\pi}} \left[1 - Y_{\pi(i)1}\right],$$

$$R_\pi^2 = \sum_{i=1}^{W_{N-r+1,\pi}} G_{\pi(i)}$$

and

$$R_\pi^3 = \sum_{i=1}^{W_{N-r+1,\pi}} B_{\pi(i)}.$$

THEOREM 3.1. *If* $\pi \geq^{b_2} \pi'$, *then* $P[R_\pi^\alpha \geq s,\ N = m] \leq P[R_{\pi'}^\alpha \geq s,\ N = m]$ *for any* s *and* $\alpha = 1, 2, 3$ *where* $m \geq r$.

PROOF. Without loss of generality we assume $\pi \geq^{P_2} \pi'$, and hence there exists an i such that $\pi(i) < \pi(i+1)$, $\pi'(i) = \pi(i+1)$, $\pi'(i+1) = \pi(i)$ and $\pi(l) = \pi'(l)$ for all other $l \neq i, i+1$.

(a) $\alpha = 1$. Now

$$P\left[R_\pi^1 \geq s, N = m\right] = \sum_{l=s}^{n} P\left[R_\pi^1 \geq s, W_{m-r+1} = l, N = m\right]$$

and for $l \neq i, i+1$,

$$P\left[R_\pi^1 \geq s, W_{m-r+1} = l, N = m\right] = P\left[R_{\pi'}^1 \geq s, W_{m-r+1} = l, N = m\right].$$

Therefore $P[R_\pi^1 \geq s,\ N = m] = P[R_{\pi'}^1 \geq s,\ N = m]$ for $s > i + 1$. Suppose now $s = i + 1$. Then

$$\begin{aligned}
P\big[R_\pi^1 \geq s, W_{m-r+1} = i+1, N = m\big]
&= P\big[R_\pi^1 = i+1, W_{m-r+1} = i+1, N = m\big] \\
&= q_{\pi(i+1)}^t P\big[N_{\pi(1)\cdots\pi(i)} = m - r, R_{\pi(1)\cdots\pi(i)}^1 = i, N_{\pi(i+2)\cdots\pi(n)} = r - 1\big] \\
&= q_{\pi(i+1)}^t P\big[N_{\pi(i+2)\cdots\pi(n)} = r - 1\big] \\
&\quad \times \Big\{q_{\pi(i)}^t P\big[N_{\pi(1)\cdots\pi(i-1)} = m - r - 1, R_{\pi(1)\cdots\pi(i-1)}^1 = i - 1\big] \\
&\qquad + q_{\pi(i)}\big(1 - q_{\pi(i)}^{t-1}\big) P\big[N_{\pi(1)\cdots\pi(i-1)} = m - r, R_{\pi(1)\cdots\pi(i-1)}^1 = i - 1\big]\Big\}.
\end{aligned}$$

[Here $N_{\pi(1)\cdots\pi(i-1)}$ and $R_{\pi(1)\cdots\pi(i-1)}^1$ are, respectively, the number of failed components and number of parts repaired in components $\pi(1), \ldots, \pi(i-1)$. Similarly $N_{\pi(i+2)\cdots\pi(n)}$ is the number of failed components in components $\pi(i+2), \ldots, \pi(n)$.]

It follows therefore that

$$\begin{aligned}
P\big[R_\pi^1 \geq i+1, N = m\big] &- P\big[R_{\pi'}^1 \geq i+1, N = m\big] \\
&= P\big[R_\pi^1 \geq i+1, W_{m-r+1} = i+1, N = m\big] \\
&\quad - P\big[R_{\pi'}^1 \geq i+1, W_{m-r+1} = i+1, N = m\big] \\
&= P\big[N_{\pi(i+2)\cdots\pi(n)} = r - 1\big] P\big[N_{\pi(1)\cdots\pi(i-1)} = m - r, R_{\pi(1)\cdots\pi(i-1)} = i - 1\big] \\
&\quad \times q_{\pi(i+1)} q_{\pi(i)} \big(q_{\pi(i+1)}^{t-1} - q_{\pi(i)}^{t-1}\big) \\
&\leq 0
\end{aligned}$$

since $p_{\pi(i)} \leq p_{\pi(i+1)}$. Similar calculations show that in the case where $s \leq i$,

$$\begin{aligned}
P\big[R_\pi^1 \geq s, N = m\big] - P\big[R_{\pi'}^1 \geq s, N = m\big]
&= q_{\pi(i)} q_{\pi(i+1)} \big(q_{\pi(i)}^{t-1} - q_{\pi(i+1)}^{t-1}\big) P\big[N_{\pi(i+2)\cdots\pi(n)} = r - 1\big] \\
&\quad \times \Big\{P\big[N_{\pi(1)\cdots\pi(i-1)} = m - r, R_{\pi(1)\cdots\pi(i-1)}^1 \geq s - 1\big] \\
&\qquad - P\big[N_{\pi(1)\cdots\pi(i-1)} = m - r, R_{\pi(1)\cdots\pi(i-1)}^1 \geq s - 2\big]\Big\} \\
&\leq 0.
\end{aligned}$$

(b) $\alpha = 2$. Arguing as in (a) and concentrating on terms where $W_{m-r+1} = i$ or $i + 1$, it may be shown that

$$\begin{aligned}
P\big[R_\pi^2 \geq s, N = m\big] &- P\big[R_{\pi'}^2 \geq s, N = m\big] \\
&= P\big[N_{\pi(i+2)\cdots\pi(n)} = r - 1\big] \\
&\quad \times \Bigg\{P\big[N_{\pi(1)\cdots\pi(i-1)} = m - r, R_{\pi(1)\cdots\pi(i-1)}^2 \geq s - t\big] \\
&\qquad \times \Big(q_{\pi(i)}^t\big(1 - q_{\pi(i+1)}^t\big) - q_{\pi(i+1)}^t\big(1 - q_{\pi(i)}^t\big)\Big) \\
&\qquad + \sum_{l=0}^{t-1} a_l P\big[N_{\pi(1)\cdots\pi(i-1)} = m - r, R_{\pi(1)\cdots\pi(i-1)}^2 \geq s - t - l\big]\Bigg\}
\end{aligned}$$

[here $a_l = q^t_{\pi(i+1)} p_{\pi(i)} q^l_{\pi(i)} - q^t_{\pi(i)} p_{\pi(i+1)} q^l_{\pi(i+1)}$ and $R^2_{\pi(1)\cdots\pi(i-1)}$ is the number of repairs performed according to policy 2 in components $\pi(1), \dots, \pi(i-1)$]

$$= P\big[N_{\pi(i+2)\cdots\pi(n)} = r - 1\big]$$
$$\times\left(\sum_{l=1}^{t-1} a_l P\big[N_{\pi(1)\cdots\pi(i-1)} = m - r, s - t > R^2_{\pi(1)\cdots\pi(i-1)} \ge s - t - l\big]\right)$$
$$\le 0$$

[since $a_l \le 0$ for all l and $\sum_{l=0}^{t-1} a_l = q^t_{\pi(i+1)}(1 - q^t_{\pi(i)}) - q^t_{\pi(i)}(1 - q^t_{\pi(i+1)})$].

(c) $\alpha = 3$. In this repair policy, once we decide to inspect a component we then repair or replace all of its faulty parts. We therefore perform a complete repair on each inspected component, and continue doing so until the system as a whole is back in operation and $R^3_\pi = \sum_{i=1}^{W_{N-r+1}} B_{\pi(i)}$. Manipulations similar to those in (b) lead to the expression

$$P\big[R^3_\pi \ge s, N = m\big] - P\big[R^3_{\pi'} \ge s, N = m\big]$$
$$= P\big[N_{\pi(i+2)\cdots\pi(n)} = r - 1\big]$$
$$\times\left(\sum_{l=1}^{t-1} b_l P\big[N_{\pi(1)\cdots\pi(i-1)} = m - r, s - t > R^3_{\pi(1)\cdots\pi(i-1)} \ge s - t - l\big]\right)$$
$$\le 0$$

[where $b_l = q^t_{\pi(i+1)}\binom{t}{l} q^l_{\pi(i)} p^{t-l}_{\pi(i)} - q^t_{\pi(i)}\binom{t}{l} q^l_{\pi(i+1)} p^{t-l}_{\pi(i+1)} \le 0$ for all $l = 0, \dots, t-1$]. □

As a consequence of this theorem, we have the following corollary:

COROLLARY 3.2. *Let $\pi \ge^{b_2} \pi'$. Then for any $\alpha = 1, 2, 3$:*

(i) $$P\big[R^\alpha_\pi \ge s \mid N = r\big] \le P\big[R^\alpha_{\pi'} \ge s \mid N = r\big].$$

(ii) $$P\big[R^\alpha_\pi \ge s \mid N \ge r\big] \le P\big[R^\alpha_{\pi'} \ge s \mid N \ge r\big].$$

Corollary 3.2 implies that for any of the three proposed repair/replacement policies, the number of repairs (replacements) necessary for minimal repair decreases stochastically the closer the inspection procedure π is in the partial ordering $\ge^{b_2}$ to the permutation $(1, 2, \dots, n)$. Hence the permutations $\pi = (1, 2, \dots, n)$ and $\pi^* = (n, n-1, \dots, 2, 1)$ give stochastically the smallest and largest number of repairs, respectively. These results hold either conditional on the knowledge that the system has just failed ($N = r$) or that the system is in a failed state ($N \ge r$).

In the above policies of minimal repair, we have counted the number of repairs made up to and including the component whose functioning brings the system back to life. In policies $\alpha = 2$ and 3 we actually determine the state of an inspected component before repairs are made, and hence we would continue repairing components until all faulty components have been repaired (according to the respective policies 2 or 3). These would be types of complete repair, and it is worth noting that the total number of repairs done under this format would be similarly stochastically ordered according to the partial ordering $\geq^{b_2}$ for both $\alpha = 2$ and 3. The results also go through for the policy $\alpha = 1$, although the mode of inspection for this policy does not allow a practical implementation of the stopping rule $W_{N,\pi}$. Details are not, however, given here.

EXAMPLE 3.3. We now discuss an elementary example to show that our results concerning stochastic order for $\geq^{b_2}$ do not generally hold for the finer (but perhaps more natural) ordering $\geq^{b_1}$. Consider therefore a series system of three components with respective reliabilities $P_i = 1 - q_i^t$, where $p_1 \leq p_2 \leq p_3$ and $t > 1$. We will assume the knowledge that the system has just failed and hence $N = 1$. Consider the inspection permutations $\pi = 213$ and $\pi' = 312$. Then $\pi \geq^{b_1} \pi'$, but of course $\pi \ngeq^{b_2} \pi'$. We show that for certain values of p_1, p_2 and p_3, $R_{213}^1 \nleq^{\text{st}} R_{312}^1$ and in fact $E(R_{213}^1) > E(R_{312}^1)$ when conditioned on $N = 1$. Similar calculations can be done to show similar results for the policies $\alpha = 2$ and 3. Note now that

$$P\big(R_{213}^1 = 3,\, N = 1\big) = q_1\big(1 - q_1^{t-1}\big)q_2\big(1 - q_2^{t-1}\big)q_3^t$$

and

$$\begin{aligned} &P\big(R_{213}^1 \geq 2,\, N = 1\big) \\ &\quad = q_1^t q_2\big(1 - q_2^{t-1}\big)\big(1 - q_3^t\big) + q_3^t\big\{q_1\big(1 - q_1^{t-1}\big)q_2\big(1 - q_2^{t-1}\big) \\ &\qquad\qquad + q_1\big(1 - q_1^{t-1}\big)p_2 + p_1 q_2\big(1 - q_2^{t-1}\big)\big\}. \end{aligned}$$

Letting $p_1 \to 0$, $p_3 \to 1$, it follows therefore that

$$\lim_{\substack{p_1 \to 0 \\ p_3 \to 1}} E\big(R_{213}^1 - R_{312}^1 \mid N = 1\big) = q_2\big(1 - q_2^{t-1}\big)/P(N = 1).$$

Hence for small p_1, large p_3 and, say, $p_2 = \frac{1}{2}$,

$$E\big(R_{213}^1 \mid N = 1\big) > E\big(R_{312}^1 \mid N = 1\big).$$

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