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PREDICTION OF STATIONARY MAX-STABLE PROCESSES

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We consider prediction of stationary max-stable processes. The usual metric between max-stable variables can be defined in terms of the L_1 distance between spectral functions and in terms of this metric a kind of projection can be defined. It is convenient to project onto *max-stable* spaces; that is, spaces of extreme value distributed random variables that are closed under scalar multiplication and the taking of finite maxima. Some explicit calculations of max-stable spaces generated by processes of interest are given. The concepts of deterministic and purely nondeterministic stationary max-stable processes are defined and illustrated. Differences between linear and nonlinear prediction are highlighted and some characterizations of max-moving averages and max-permutation processes are given.

1. Introduction. Recent studies of telecommunications traffic (for example, Meier-Hellstern, Wirth, Yan and Hoeflin (1991)) reinforce the need for infinite variance models. Attempts to extend classical time series models to the infinite variance case have centered around autoregressive moving average (ARMA) models with infinite variance noise variables. Such processes have the form

$$X_n = \sum_{j=1}^{p} \phi_j X_{n-j} + \sum_{j=0}^{q} \theta_j Z_{n-j},$$

where $1 - \phi_1 z - \cdots - \phi_p z^p \neq 0$ for all $|z| \leq 1$, $\theta_0 = 1$, and $\{Z_n\}$ are iid random variables having either stable, Pareto or regularly varying tails. Progress has been made with such models [Davis and Resnick (1985a,b), (1986); Cline and Brockwell (1985); Knight (1991); Davis, Knight and Liu (1992)], but the distribution theory is frequently very difficult. For instance, limit distributions for sample correlation functions or coefficient estimators involve ratios of stable random variables and sometimes the random variables comprising the numerator and denominator are dependent.

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An alternative class of infinite variance models consists of max-autoregressive moving averages (MARMA) processes of the form

$$X_n = \bigvee_{j=1}^{p} \phi_j X_{n-j} \vee \bigvee_{j=0}^{q} \theta_j Z_{n-j},$$

where $p \geq 0$, $q \geq 0$, $\phi_j > 0$, $\theta_j > 0$, $\theta_0 = 1$ and $\{Z_n\}$ are iid random variables with $Z_n > 0$ and having the heavy-tailed extreme value distribution

$$P[Z_1 \leq x] = \exp\{-x^{-\alpha}\}, \qquad x > 0, \quad \alpha > 0$$

($\alpha < 2$ for infinite variance). Such processes were explored in Davis and Resnick (1989) where it was shown that they have a relatively easy distributional structure and a coherent prediction theory. Furthermore, we discussed how such prediction parallels classical L_2 prediction and how it differs from it.

In thinking about the applicability of MARMA, it is important to realize that the sample paths of a MARMA are very similar to the sample paths of an ARMA of the same order which use the same heavy-tailed noise sequence and have the same coefficients. This is illustrated by the graphs in Davis and Resnick (1989), which compare 250 realizations from the stationary AR(1) model with 250 realizations of the MAR(1) with same coefficients and noise variables and also similarly compare MA(1) versus MMA(1) (max-moving average). Comparable graphs are almost identical. Given the similarities of the sample paths in the two figures, one wonders if a goodness-of-fit statistical test could be devised to distinguish ARMA from MARMA. To add to the difficulty of distinguishing between ARMA and MARMA with identical heavy-tailed noise variables, we note that both models have identical asymptotic point processes of exceedances of high levels [Davis and Resnick (1985a); Hsing (1986)]. In continuous time, furthermore, upcrossing rates of high levels for stable and max-stable processes seem to be the same [Davis and Resnick (1992)].

In fitting a MARMA model to data one may imagine the following procedure: An estimate of the shape parameter α in the formula $P[Z_1 \leq x] = \exp\{-x^{-\alpha}\}$ (or $P[X_1 \leq x] = \exp\{-cx^{-\alpha}\}$, $c > 0$, $x > 0$) is made and the data are transformed by a power transformation $x \mapsto x^{\alpha}$. The process $\{Y_n\} = \{X_n^{\alpha}\}$ is still MARMA, but the shape parameter is 1: $P[Y_1 \leq x] = \exp\{-c'x^{-1}\}$. Thus, without loss of generality, we may suppose henceforth that $\alpha = 1$; we refer to this as the *standard* case.

MARMA processes are a subclass of the class of stationary max-stable processes, and a primary aim of this paper is to consider the problem of prediction for stationary max-stable processes. A standard max-stable process is a stochastic process $\{X_t, -\infty < t < \infty\}$ whose finite-dimensional distributions are of the form ($k \geq 1$, $x_i \geq 0$, $i = 1, \dots, k$)

$$P[X_1 \leq x_1, \dots, X_k \leq x_k] = \exp\left\{-\int_{[0,1]} \bigvee_{i=1}^{k} \frac{f_i(s)}{x_i} \, ds\right\},$$

where $f_i \in L_1$ ([0, 1]). The functions $\{f_n\}$ are called *spectral functions*. Such a stochastic process can be realized by construction from a Poisson process as follows: Let $\sum_k \varepsilon_{(U_k, \Gamma_k)}$ be a Poisson random measure (PRM) on the state space $[0, 1] \times [0, \infty)$ with mean measure $1_{[0,1]}(u)\, du \times dx$. [For background on point processes and their applications to extreme value theory, see Resnick (1987).] Then as pointed out in de Haan (1984) we have

$$\{X_n\} =_d \left\{ \bigvee_k \frac{f_n(U_k)}{\Gamma_k} \right\}.$$

A condition for stationarity of max-stable sequences is provided by de Haan and Pickands (1986): First, there must exist an L_1 ([0, 1] isometry $\gamma = (r, H)$, called a piston, such that H is a bijection of [0, 1]) and for $f \in L_1$ ([0, 1]),

$$\gamma f(s) = r(s) f(H(s)),$$

and

$$\int_{[0,1]} \gamma f(s)\, ds = \int_{[0,1]} f(s)\, ds.$$

Second, there must exist a nonnegative function $f_0 \in L_1$ ([0, 1]) such that $f_n = \gamma^n f_0$.

In order to discuss prediction of a max-stable process, one needs to define a distance between max-stable variables. If (X, Y) are jointly max-stable with spectral functions (f, g), a distance between X and Y can be defined by the L_1 distance between spectral functions:

$$d(X,Y) = \int_{[0,1]} |f(s) - g(s)|\, ds. \tag{1.1}$$

The distance measure $d(X, Y)$ is well defined despite the fact that the joint distribution of X, Y does not uniquely determine the spectral functions [Davis and Resnick (1989)]. This distance measure can be used to define a notion of prediction for jointly max-stable random variables: The best predictor of Y based on a collection of random variables $\mathscr{M}$ is the element of $\mathscr{M}$ closest to Y. This was used for MARMA processes in Davis and Resnick (1989). As discussed in Theorem 3.1 of Davis and Resnick (1989), such a predictor has the optimal property of minimizing the probability of large deviations from Y among elements in $\mathscr{M}$. We continue to explore this notion of prediction for general stationary max-stable processes in this paper in order to see what parallels and contrasts exist between classical prediction theory and prediction of max-stable processes. As will become clearer in later sections, the prediction theory for max-stable processes is distribution-based rather than moment-based as in classical Hilbert space oriented techniques. Whereas classical theory allows linear combinations to be undone (subtraction is the inverse of addition), in the current theory, there is no inverse operation to the operation of taking maxima. This is what produces deviations between the classical theory and the prediction theory of stationary max-stable processes.

In Section 2 we discuss a convenient context for prediction, namely, the max-stable space. This is defined as follows:

DEFINITION 1.1. A collection $\mathscr{M}$ of random variables defined on a common probability space is a standard max-stable space if:

(i) For each $X \in \mathscr{M}$, there exists $\sigma \geq 0$ such that

$$P[X \leq x] = \Phi_{1,\sigma}(x) := \exp\{-\sigma x^{-1}\}, \qquad x \geq 0.$$

(ii) If $X \in \mathscr{M}$, then for $c \geq 0$ we have $cX \in \mathscr{M}$.

(iii) If $X, Y \in \mathscr{M}$, then $X \vee Y \in \mathscr{M}$.

In this paper we will only consider max-stable spaces that are standard and thus the modifier *standard* will be dropped throughout. As will be apparent, it is possible to construct max-stable spaces in which the extreme value distribution $\Phi_{1,1}$ is replaced by a different extreme value distribution. We may define the *closed max-span* of a collection of max-stable random variables as the smallest closed max-stable space containing the collection, where closed means with respect to the metric d given in (1.1). For a max-stable process $\{X_n\}$, the best predictor of X_{n+k} based on information up to time n will be the element of the closed max-span of $\{X_j, j \leq n\}$, which is closest to X_{n+k}. In Section 2 we make some explicit computations of max-spans. Of special interest is the max-span of max-moving average processes and the max-span of a class of processes we call permutation processes. For a general max-stable process, it is impossible to explicitly compute closed max-spans. We give the list of processes where explicit computations are known.

Section 3 considers max-stable prediction. We note the phenomenon, already seen in Davis and Resnick (1989), of the nonuniqueness of predictors. We define the concepts of a deterministic and purely nondeterministic stationary max-stable process and give some examples of each type. We show that the Wold decomposition of a stationary max-stable process into components consisting of a max-moving average and a deterministic max-stable process does not, in general, exist. Some aspects of our theory are in marked contrast to classical L_2 theory. Section 4 briefly remarks on prediction based on richer information and shows that by increasing the information base of a predictor, one may recover aspects of classical Wold decomposition theory.

Section 5 introduces a class of processes called *permutation processes*, which contain as a proper subclass the infinite order max-moving averages. We present some spectral characterizations of such processes and collect some results about their predictors.

2. Max-stable spaces. Max-stable spaces, as defined in Definition 1.1, provide a natural setting for prediction. We begin by presenting some simple properties and constructions of max-stable spaces.

A quick corollary of the Definition 1.1 is that the joint distribution of any $X_1, \dots, X_d \in \mathscr{M}$ is max-stable with one-dimensional marginals of type $\Phi_{1,1}$. To check this fact note that for any positive constants $c_1, \dots, c_d$ we have that $\bigvee_{i=1}^{d} c_i X_i$ has distribution $\Phi_{1,\sigma}$ for some $\sigma \geq 0$, so by a theorem of de Haan (1978) we get that $X_1, \dots, X_d$ are jointly max-stable. Also, if $\mathscr{M}$ is a max-stable space and $\mathscr{M}_i \subset \mathscr{M}$ are max-stable ($i = 1, 2$), then

$$\mathscr{M}_1 \vee \mathscr{M}_2 := \{X_1 \vee X_2 : X_i \in \mathscr{M}_i, \quad i = 1, 2\}$$

is also a max-stable space.

To fix ideas, here are some easy examples of max-stable spaces:

1. Suppose $\{Z_n, \ n \geq 1\}$ are iid with common distribution $\Phi_{1,1}$. Then the collection

$$\mathscr{M} = \left\{ \bigvee_{i=1}^{\infty} c_i Z_i : c_i \geq 0, 0 \leq \sum_{i=1}^{\infty} c_i < \infty \right\}$$

is a max-stable space. The condition $\sum_i c_i < \infty$ ensures that $\bigvee_i c_i Z_i < \infty$ a.s.

2. Suppose $\sum_i \varepsilon_{(U_i, \Gamma_i)}$ is a Poisson random measure (PRM) on $[0, 1] \times [0, \infty)$ with mean measure

$$1_{[0,1]}(u)\, du\, 1_{[0,\ \infty)}(s)\, ds.$$

For $f \in L_1 := L_1[0, 1]$, define

$$X_f = \bigvee_i \frac{f(U_i)}{\Gamma_i}.$$

Then

$$\mathscr{M} = \{X_f : f \in L_1, f \geq 0\}$$

is a max-stable space. We note that

$$X_f \sim \Phi_{1,\sigma} \quad \text{where} \quad \sigma = \int_0^1 f(s)\, ds,$$

$$cX_f = X_{cf},$$

$$X_f \vee X_g = X_{f \vee g}$$

for $f, g \in L_1$.

If $\mathbf{X} = \{X_\lambda, \ \lambda \in \Lambda\}$ is a max-stable process, recall that the *max-span* of $\mathbf{X}$ is the smallest closed max-stable space containing all the variables X_λ, $\lambda \in \Lambda$. Closure is with respect to the L_1-metric d given in (1.1). We will indicate the max-span by

$$\overline{\bigvee\text{-sp}}\ \{X_\lambda, \lambda \in \Lambda\}.$$

Note that a nonempty intersection of closed max-stable spaces yields a closed max-stable space, so to verify that $\overline{\bigvee\text{-sp}}\ \{X_\lambda, \ \lambda \in \Lambda\}$ exists, it suffices to establish the existence of one closed max-stable space containing $\{X_\lambda,\ \lambda \in \Lambda\}$. To do this let

$$\mathscr{M}_1 = \left\{ \bigvee_{\Lambda'} a_\lambda X_\lambda : a_\lambda \geq 0, \Lambda' \subset \Lambda, \Lambda' \text{ finite} \right\}.$$

Then $\{X_\lambda,\ \lambda \in \Lambda\} \subset \mathscr{M}_1$ and one readily checks that $\mathscr{M}_1$ is a max-stable space. Using Lemma 2 of de Haan [(1984), page 1200], one can establish that the closure of a max-stable space is max-stable. Thus the closure of $\mathscr{M}_1$ is the required max-stable space. (See also Proposition 2.2.)

If

$$\mathscr{M}_X = \overline{\vee\text{-sp}}\ \{X_\lambda,\ \lambda \in \Lambda\}$$

is a closed subspace of the max-stable space $\mathscr{M}$, then for $Y \in \mathscr{M}$ we define $P_{\mathscr{M}_X}(Y)$ to be the set of all $\xi \in \mathscr{M}_X$ that achieve

$$\inf\{d(Y, \eta) : \eta \in \mathscr{M}_X\}.$$

We think of $P_{\mathscr{M}_X}(Y)$ as the predictor of Y based on knowing $\mathscr{M}_X$. When we have a stationary max-stable stochastic sequence $\{X_n,\ -\infty < n < \infty\}$, our standard notation will be to set

$$\mathscr{M}_n = \overline{\vee\text{-sp}}\ \{X_j, j \leq n\}.$$

Then a one-step predictor is denoted by

$$\hat{X}_{n+1} = P_{\mathscr{M}_n} X_{n+1}.$$

For the following examples and later work it is convenient to be able to reference the following lemma. The easy proof, based on Proposition 4.1 of Davis and Resnick (1989) is omitted. Define

$$l_1 = \left\{ \mathbf{x} = (x_1, x_2, \dots) \in R^\infty : \sum_i |x_i| < \infty \right\}.$$

LEMMA 2.1. *Suppose* $\{Z_n,\ n \geq 0\}$ *are iid with common distribution* $\Phi_{1,1}$. *Define for* $1 \leq m \leq \infty$,

$$\xi_m = \bigvee_{j=0}^{\infty} c_j^{(m)} Z_j,$$

where

$$c_j^{(m)} \geq 0, \qquad \sum_j c_j^{(m)} < \infty.$$

Then

$$d(\xi_m, \xi_\infty) \to 0$$

iff, as $m \to \infty$,

$$\left\{c_j^{(m)}, j \geq 0\right\} \to \left\{c_j^{(\infty)}, j \geq 0\right\}$$

in l_1.

Furthermore, $\{\xi_m\}$ *converges in the d-metric iff* $\{\{c_j^{(m)},\ j \geq 0\},\ m \geq 1\}$ *converges in* l_1 *to some* $\{c_j^{(\infty)},\ j \geq 0\}$ *in which case the limit of* $\{\xi_m\}$ *is* $\bigvee_{j=0}^{\infty} c_j^{(\infty)} Z_j$.

Here are some explicit calculations of max-spans. These will be used in Section 3 where prediction is more fully discussed.

EXAMPLE 2.1. Let Y be a random variable with distribution $\Phi_{1,1}$. Then

$$\overline{\vee\text{-sp}}\ \{Y\} = \{cY: c \geq 0\}.$$

The right-hand side is a stable space. To see that it is closed, observe that

$$d(cY, c'Y) = |c - c'|$$

[from Proposition 4,1 in Davis and Resnick (1989)]. Therefore, if $c_n Y$ converges we have $\lim_{n\to\infty} c_n = c_\infty$ exists and $d(c_n Y, c_\infty Y) \to 0$.

EXAMPLE 2.2. Suppose $\{Z_n, n \geq 0\}$ are iid with common distribution $\Phi_{1,1}$. Then

$$\overline{\vee\text{-sp}}\ \{Z_n, n \geq 0\} = \left\{ \bigvee_{i=0}^{\infty} c_i Z_i : c_i \geq 0, \sum_i c_i < \infty \right\}. \tag{2.1}$$

As observed in the beginning of this section, the right-hand side of (2.1), call it r.h.s., is a max-stable space. Also, r.h.s. is closed by Lemma 2.1. So r.h.s. is a closed stable space containing $\{Z_n, n \geq 0\}$ and therefore r.h.s. $\supset \overline{\vee\text{-sp}}\ \{Z_n, n \geq 0\}$. For a reverse inclusion note that if $\bigvee_i c_i Z_i \in$ r.h.s., then

$$\lim_{n\to\infty} d\left(\bigvee_{i=0}^{\infty} c_i Z_i, \bigvee_{i=0}^{n} c_i Z_i \right) = 0$$

and because $\bigvee_{i=1}^{n} c_i Z_i \in \overline{\vee\text{-sp}}\ \{Z_j, j \geq 1\}$, we get r.h.s. $\subset \overline{\vee\text{-sp}}\ \{Z_n, n \geq 0\}$.

The next example is a prototype for more complicated examples.

EXAMPLE 2.3. Let $(X_1, \dots, X_n)$ be a max-stable random vector with spectral functions $(f_1, \dots, f_n)$ such that

$$\sigma_* := \bigwedge_{i=1}^{n} \sigma_i = \bigwedge_{i=1}^{n} \int_0^1 f_i(s)\, ds > 0.$$

(Usually in our applications, $X_1, \dots, X_n$ are identically distributed and then $\sigma_* > 0$ is obvious except in degenerate cases.) Then

$$\mathscr{M}_{1,n} = \overline{\vee\text{-sp}}\ \{X_1, \dots, X_n\} = \left\{ \bigvee_{i=1}^{n} v_i X_i : \bigwedge_{i=1}^{n} v_i \geq 0 \right\}. \tag{2.2}$$

One readily checks that the r.h.s. of (2.2) is a max-stable space and, as in the previous example, the main task is to show that r.h.s. is closed. Suppose $\xi_k = \bigvee_{i=1}^{n} v_i^{(k)} X_i \in$ r.h.s. and there exists ξ_∞ such that $d(\xi_k, \xi_\infty) \to 0$. Suppose the spectral function of ξ_∞ is f_∞. We must show $\xi_\infty = \bigvee_{i=1}^{n} v_i X_i$ for some choice of $\{v_i\}$. Observe for all $1 \le i \le n$ and $k \ge 1$,

$$v_i^{(k)} = \sigma_i^{-1} \int_0^1 v_i^{(k)} f_i(s)\, ds \le \sigma_*^{-1} \int_0^1 \bigvee_{j=1}^{n} v_j^{(k)} f_j(s)\, ds,$$

and because $d(\xi_k, \xi_\infty) \to 0$, we get, as $k \to \infty$,

$$\int_0^1 \bigvee_{j=0}^{n} v_j^{(k)} f_j(s)\, ds \to \int_0^1 f_\infty(s)\, ds < \infty.$$

Thus $\{\vee_{i=1}^n v_i^{(k)}\}$ is bounded. Choose a subsequence $\{k'\}$ such that $(v_i^{(k')}, 1 \le i \le n\}$ converges to some limit $(v_i, 1 \le i \le n)$. Then $\vee_{i=1}^n v_i^{(k')} f_i$, the spectral function of $\xi_{k'}$, converges pointwise to $\vee_{i=1}^n v_i f_i$, and because $d(\xi_k, \xi_\infty) \to 0$, we have

$$\bigvee_{i=1}^{n} v_i^{(k')} f_i \to_{L_1} f_\infty,$$

and, therefore, $f_\infty = \vee_{i=1}^n v_i f_i$ a.e. Thus we conclude

$$d\left(\xi_{k'}, \bigvee_{i=1}^{n} v_i X_i\right) \to 0$$

as desired.

From this example emerges the following simple criterion.

PROPOSITION 2.2. *Suppose* $\{X_1, X_2, \dots\}$ *is a max-stable sequence with spectral functions* $(f_i, i \ge 1)$ *with* $\sigma_i = \int_0^1 f_i(s)\,ds > 0$, $i \ge 1$. *Then*

$$\begin{aligned}\overline{\vee\text{-sp}}\ \{X_j, j \ge 1\} &= \text{closure}\left\{\bigvee_{i\in\Lambda} v_i X_i : \bigwedge_{i\in\Lambda} v_i > 0, \Lambda \subset \{0,1,\dots\}, \Lambda \ \text{ finite}\right\}\\ &= \text{closure}\left(\bigcup_{n=1}^{\infty} \overline{\vee\text{-sp}}\ \{X_1,\dots,X_n\}\right).\end{aligned}$$

Before the next set of examples, we recall the following result.

LEMMA 2.3. *Let* $\{X_n, n \ge 1\}$ *be a max-stable sequence with spectral functions* $\{f_n\}$. *Suppose* $c_i \ge 0$ *for* $i \ge 1$. *Then*

$$\bigvee_i c_i X_i < \infty \quad a.s.\ iff \quad \bigvee_i c_i f_i \in L_1[0,1].$$

This follows because, from the form of the joint distribution of $X_1, \dots, X_n$, we have

$$\begin{aligned}P\left[\bigvee_i c_i X_i \le x\right] &= \lim_{N\to\infty} P\left[\bigvee_{i=1}^{N} c_i X_i \le x\right]\\ &= \exp\left\{-x^{-1} \int_0^1 \bigvee_{i=1}^{\infty} c_i f_i(s)\, ds\right\}.\end{aligned}$$

EXAMPLE 2.4. MMA(∞): Let $\{X_n, -\infty < n < \infty\}$ be a MMA(∞) process of the form

$$X_n = \bigvee_{i=0}^{\infty} c_i Z_{n-i}, \tag{2.3}$$

where $c_i \geq 0$, $\Sigma_i c_i < \infty$ and $\{Z_n, -\infty < n < \infty\}$ is iid with common distribution $\Phi_{1,1}$. Then

$$\begin{aligned} M_n &= \overline{\vee\text{-sp}} \ \{X_j, j \leq n\} \\ &= \left\{ \bigvee_{i=0}^{\infty} v_i X_{n-i} \colon \bigvee_{i=0}^{\infty} v_i X_{n-i} < \infty \quad \text{a.s.}, \bigwedge_i v_i \geq 0 \right\}. \end{aligned} \tag{2.4}$$

Note that $\vee_{i=0}^{\infty} v_i X_{n-i} < \infty$ a.s. iff $\Sigma_{i=0}^{\infty} \psi_j < \infty$, where $\psi_j = \vee_{i=0}^{j} v_i c_{j-i}$ [cf. Davis and Resnick (1989)].

In verifying (2.4), we follow the pattern of the verifications of Examples 2.2 and 2.3. The main task is to show that r.h.s. is closed as it is clear that r.h.s. is a max-stable space. So let $\xi_k \in$ r.h.s. and assume

$$\xi_k = \bigvee_{i=0}^{\infty} v_i^{(k)} X_{n-i} \to \xi$$

in the d-metric. We need to show that $\xi \in$ r.h.s. We have that

$$\xi_k = \bigvee_{i=0}^{\infty} v_i^{(k)} X_{n-i} = \bigvee_{i=0}^{\infty} \left(\bigvee_{l=0}^{i} v_l^{(k)} c_{i-l} \right) Z_{n-i} = \bigvee_{i=0}^{\infty} \psi_i^{(k)} Z_{n-i}.$$

From Lemma 2.1 there exists $\{\psi_i\} \in l_1$ such that in l_1 we have $\{\psi_i^{(k)}\} \to \{\psi_i\}$ as $k \to \infty$. This shows that

$$\xi = \bigvee_{i=0}^{\infty} \psi_i Z_{n-i}.$$

But as in Example 2.3, we have $\{\vee_{i=0}^{\infty} v_i^{(k)}, k \geq 1\}$ bounded. So we may find a convergent subsequence $\{k'\}$ such that for all i, $v_i^{(k')} \to v_i$. Then $\psi_i^{(k')} \to \psi_i = \vee_{j=0}^{i} v_j c_{i-j}$ and, therefore,

$$\begin{aligned} \xi &= \bigvee_{i=0}^{\infty} \psi_i Z_{n-i} = \bigvee_{i=0}^{\infty} \left(\bigvee_{j=0}^{i} v_j c_{i-j} \right) Z_{n-i} \\ &= \bigvee_{j=0}^{\infty} v_j \left(\bigvee_{i=j}^{\infty} c_{i-j} Z_{n-i} \right) = \bigvee_{j=0}^{\infty} v_j X_{n-j}. \end{aligned}$$

The previous example may be extended to the two-sided moving average case where

$$X_n = \bigvee_{i=-\infty}^{\infty} c_i Z_{n-i} = \bigvee_{i=-\infty}^{\infty} c_{n-i} Z_i,$$

and $c_i \geq 0$, $0 < \sum_{i=-\infty}^{\infty} c_i < \infty$. In this case

$$\mathscr{M}_n = \overline{\vee\text{-sp}}\ \{X_j, j \leq n\}$$

$$= \left\{ \bigvee_{i=0}^{\infty} v_i X_{n-i} \colon \bigwedge_i v_i \geq 0, \bigvee_{i=0}^{\infty} v_i X_{n-i} < \infty \right\}. \tag{2.5}$$

Only minor modifications of the previous argument are required.

REMARK. For arbitrary max-stable processes, the representation (2.4) of the max-span will not be valid. To see this, let $\{Y, Z_n, n \geq 1\}$ be iid $\Phi_{1,1}$ random variables and define for $n \geq 1$,

$$X_n = nY \vee Z_n.$$

Then

$$d(X_n/n, Y) = d(Y \vee Z_n/n, Y) = 1/n \to 0,$$

so $Y \in \overline{\vee\text{-sp}}\ \{X_n, n \geq 1\}$. However, Y cannot be represented in the form $\bigvee_{i=1}^{\infty} v_i X_i$ because

$$d\left(Y, \bigvee_{i=1}^{\infty} v_i X_i\right) = \left|1 - \bigvee_{i=1}^{\infty} i v_i\right| + \sum_{i=1}^{\infty} v_i \neq 0$$

for any choice of $\{v_i\}$.

We close this section with a calculation that is needed in Examples 3.1 and 3.2.

EXAMPLE 2.5. Suppose $\{X_n\}$ is a max-stable process and suppose

$$M_X = \overline{\vee\text{-sp}}\ \{X_n, -\infty < n < \infty\}$$

$$= \left\{ \bigvee_i v_i X_i \colon \bigwedge_i v_i \geq 0, \bigvee_i v_i X_i < \infty \right\}.$$

Assume $\{X_n\}$ and Y both belong to the same max-stable space and that Y is independent of $\{X_n\}$. Then

$$\overline{\vee\text{-sp}}\ \{X_n \vee Y, -\infty < n < \infty\}$$

$$= \left\{ \bigvee_i v_i (X_i \vee Y) \colon \bigwedge_i v_i \geq 0, \bigvee_i v_i (X_i \vee Y) < \infty \right\}. \tag{2.6}$$

If we let the spectral functions of X_n be f_n and the spectral function of Y be f_Y, we may arrange things so that the support of f_Y is disjoint from S, the union of the supports of the f_ns. As in the previous examples, we must show the right side of (2.6) is closed. Suppose

$$\xi_k = \bigvee_i v_i^{(k)} (X_i \vee Y) \to \xi$$

in the d-metric. As before we have $\{\bigvee_i v_i^{(k)}\}$ is bounded, so extract a convergent subsequence $\{v_i^{(k')}, -\infty < i < \infty\} \to \{v_i, -\infty < i < \infty\}$. Then

$$0 = \lim_{k' \to \infty} \int_0^1 \left| \left(\bigvee_i v_i^{(k')} f_i(s) \right) \vee \left(\bigvee_i v_i^{(k')} f_Y(s) \right) - f_\xi(s) \right| ds$$

$$\geq \lim_{k' \to \infty} \int_S \left| \bigvee_i v_i^{(k')} f_i(s) - f_\xi(s) \right| ds$$

and we conclude

$$\bigvee_i v_i^{(k')} f_i \to_{L_1} f_\xi 1_S.$$

Therefore, $\bigvee_i v_i^{(k')} X_i$ converges in the d-metric and because of the hypothesized form of the max-span, we get

$$\bigvee_i v_i^{(k')} X_i \to \bigvee_i v_i X_i$$

and thus

$$\bigvee_i v_i^{(k')} X_i \vee \left(\bigvee_i v_i^{(k')} Y \right) \to \bigvee_i v_i X_i \vee \left(\bigvee_i v_i Y \right)$$

in the d-metric.

3. Prediction. Suppose $\mathscr{M}$ is a max-stable space and $\mathscr{M}_1 \subset \mathscr{M}$ is a closed max-stable space and that $Y \in \mathscr{M}$. Then the best predictor of Y based on $\mathscr{M}_1$ is $P_{\mathscr{M}_1} Y$, defined as any variable $\xi \in \mathscr{M}_1$ that achieves

$$\inf\{d(\eta, Y) : \eta \in \mathscr{M}_1\}.$$

As remarked in the introduction, one does not expect as complete a prediction theory in the max-stable case as in the classical second order theory. The max-stable predictor is not necessarily unique. In Davis and Resnick (1989) it was shown that prediction for MAR(p) processes yields a unique predictor, but that for MMA(q) processes this is no longer the case. The following example simply illustrates this nonuniqueness.

EXAMPLE 3.1. Suppose $\{Y, Z_n, -\infty < n < \infty\}$ is iid with common distribution $\Phi_{1,1}$ and define the stationary max-stable process

$$X_n = Z_n \vee Y, \qquad -\infty < n < \infty.$$

The explicit representation for $\mathscr{M}_n := \overline{\vee\text{-sp}}\ \{X_j, j \leq n\}$ can be determined from Examples 2.2 and 2.5. Thus to determine X_{n+1} we compute

$$d\left(X_{n+1}, \bigvee_{j=1}^{\infty} v_j X_{n+1-j} \right) = d\left(Z_{n+1} \vee Y, \bigvee_{j=1}^{\infty} v_j Y \vee \bigvee_{j=1}^{\infty} v_j Z_{n+1-j} \right)$$

$$= 1 + \left| 1 - \bigvee_{j=1}^{\infty} v_j \right| + \sum_{j=1}^{\infty} |v_j - 0|,$$

where Proposition 4.1 of Davis and Resnick (1989) makes the calculation easy. If any $v_j > 1$, we have

$$d\left(X_{n+1}, \bigvee_{j=1}^{\infty} v_j X_{n+1-j}\right) = \bigvee_{j=1}^{\infty} v_j - 1 + 1 + \sum_{j=1}^{\infty} v_j$$
$$= \bigvee_{j=1}^{\infty} v_j + \sum_{j=1}^{\infty} v_j > 2$$

whereas, if $\bigvee_j v_j \le 1$,

$$d\left(X_{n+1}, \bigvee_{j=1}^{\infty} v_j X_{n+1-j}\right) = 1 - \bigvee_{j=1}^{\infty} v_j + 1 + \sum_{j=1}^{\infty} v_j$$
$$= 2 + \Sigma' v_j \ge 2,$$

where Σ' stands for the sum with the maximum summand removed. Pick any $j \ge 1$ and any $v \in (0, 1)$ and set $v_j = v$, $v_k = 0$, $k \ne j$. For this choice

$$d(X_{n+1}, vX_{n+1-j}) = 2,$$

the lowest bound possible.

Call a max-stable process $\{X_n, -\infty < n < \infty\}$ *deterministic* if for each n we have $\mathscr{M}_n = \mathscr{M}_{-\infty} := \cap_{j=-\infty}^{\infty} \mathscr{M}_j$ [cf. Brockwell and Davis (1987), page 180]. In this case prediction can be done without error. The process is *purely nondeterministic* if $\mathscr{M}_{-\infty} = \{0\}$. In this case prediction error is nonzero. A max-moving average MMA(∞) process is purely nondeterministic as the next result shows [cf. Doob (1953), page 578].

PROPOSITION 3.1. *Suppose we have the max-stable, stationary MMA*(∞) *process*

$$X_n = \bigvee_{i=0}^{\infty} c_i Z_{n-i}, \qquad -\infty < n < \infty,$$

where $c_i \ge 0$, $\Sigma_i c_i < \infty$ *and* $\{Z_n\}$ *is iid with* $P[Z_1 \le x] = \exp\{-x^{-1}\}$, $x > 0$. *Then*

$$\mathscr{M}_{-\infty} := \bigcap_{j=-\infty}^{\infty} \mathscr{M}_j = \{0\},$$

so the MMA(∞) *is purely nondeterministic.*

PROOF. To check this, suppose that $\xi \in \mathscr{M}_{-\infty}$ so that for all n we have $\xi \in \mathscr{M}_n$ and from Example 2.4 we have for each n the representation

$$\xi = \bigvee_{j=0}^{\infty} v_j^{(n)} X_{n-j} = \bigvee_{j=0}^{\infty} v_j^{(n)} \bigvee_{i=0}^{\infty} c_i Z_{n-j-i}$$
$$= \bigvee_{k=0}^{\infty} \left(\bigvee_{j=0}^{\infty} v_j^{(n)} c_{k-j}\right) Z_{n-k} =: \bigvee_{k=0}^{\infty} \alpha_k^{(n)} Z_{n-k}.$$

Applying a variant of Lemma 2.1 applicable to doubly infinite sequences, we find that

$$\xi = \lim_{n \to -\infty} \bigvee_{k=0}^{\infty} \alpha_k^{(n)} Z_{n-k} = \lim_{n \to -\infty} \bigvee_{k=-\infty}^{n} \alpha_{n-k}^{(n)} Z_k$$

iff

$$\left\{ \alpha_k^{(n)}, -\infty < k < \infty \right\} \to \{ \dots, 0, \dots \}$$

in $l_1^{-\infty,\infty}$, where the limit sequence contains only zeros as entries. Thus $\xi = 0$ and $\mathscr{M}_{-\infty} = \{0\}$. □

Proposition 3.1 parallels classical second order theory. However, the following type of example stands in sharp contrast to classical theory.

EXAMPLE 3.2. Let $\{X_n\}$ be the process defined in Example 3.1. If $\xi \in \mathscr{M}_{-\infty}$, then from Example 2.4, for every n there exist nonnegative $\{v_j^{(n)}\}$ such that

$$\xi = \bigvee_{j=0}^{\infty} v_j^{(n)} X_{n-j} = \bigvee_{j=0}^{\infty} v_j^{(n)} Y \vee \bigvee_{j=0}^{\infty} v_j^{(n)} Z_{n-j}.$$

Letting $n \to -\infty$ and using Lemma 2.1 as in the proof of Proposition 3.1 yields that ξ is of the form (const)Y. However, for $c \neq 0$ we have

$$d\left(cY, \bigvee_{j=0}^{\infty} v_j^{(n)} Y \vee \bigvee_{j=0}^{\infty} v_j^{(n)} Z_{n-j} \right) = \left| c - \bigvee_{j=0}^{\infty} v_j^{(n)} \right| + \sum_{j=0}^{\infty} v_j^{(n)},$$

which cannot be made to converge to 0 as $n \to -\infty$. Thus we conclude $\mathscr{M}_{-\infty} = \{0\}$. Contrast this with Example 5.7.1 of Brockwell and Davis [(1987), page 183], where the analogous process $X_n = Y + Z_n$ (where $\{Z_n\}$ is white noise) is in Wold decomposition form with both deterministic and nondeterministic components present.

To get an example of a deterministic process, we could of course take $X_n = Y$. Here is a somewhat less trivial example.

EXAMPLE 3.3. As a particular case of the permutation processes discussed in more detail in Section 5, consider a process of the form

$$\{X_n, -\infty < n < \infty\} = \left\{ \bigvee_{i=1}^{m} c_{\pi^n(i)} Z_i, -\infty < n < \infty \right\},$$

where $\{Z_1, \dots, Z_m\}$ are iid and π is a permutation of $\{1, \dots, m\}$. If ν is the least common multiple of the cycle lengths of π, then we have for any n, $d > 0$,

$$X_{n+d\nu} = X_n,$$

which implies $\{X_n\}$ is deterministic.

REMARK. Example 3.2 easily can be extended to cover processes of the form

$$X_n = \left(\bigvee_{i=0}^{\infty} c_i Z_{n-i} \right) \vee V_n,$$

where $\{Z_n\}$ are iid $\Phi_{1,1}$ random variables independent of the stationary max-stable deterministic process $\{V_n\}$. The conclusion of Example 3.2, $\mathscr{M}_{-\infty} = \{0\}$, still holds.

Suppose we have a process of the form

$$X_n = U_n \bigvee V_n, \qquad -\infty < n < \infty, \tag{3.1}$$

where $\{U_n\}$ is a stationary, max-stable, purely nondeterministic process and $\{V_n\}$ is an independent, stationary, max-stable, deterministic process. Example 3.2 and the remarks following it suggest that possibly $\mathscr{M}_{-\infty} = \{0\}$. This is unresolved. All we have succeeded in proving is that

$$\mathscr{M}_{-\infty}(X) \subset M_{-\infty}(V), \tag{3.2}$$

where $\mathscr{M}_n(X)$ is the closed max-stable space generated by $\{X_j,\ j \leq n\}$, etc.

To prove (3.2) we need the following lemma.

LEMMA 3.2. *Suppose $\mathscr{M}$ is a closed max-stable space.*

(a) *If $\{\xi_n\} \subset \mathscr{M}$ with spectral functions $\{f_n\}$ and $f_n \to f$ in L_1, then there exists $\xi \in \overline{\vee\text{-sp}\ \{\xi_n\}}$ such that $d(\xi_n, \xi) \to 0$.*

(b) *Suppose $\{\xi_n\}$ and $\{\eta_n\}$ are independent processes in $\mathscr{M}$ with spectral functions $\{f_n\}$ and $\{g_n\}$, respectively such that for some $L \in \mathscr{M}$,*

$$d(\xi_n \vee \eta_n, L) \to 0 \tag{3.3}$$

as $n \to \infty$. Then there exist $\xi_\infty \in \mathscr{M}$ and $\eta_\infty \in \mathscr{M}$ such that

$$d(\xi_n, \xi_\infty) \to 0, \qquad d(\eta_n, \eta_\infty) \to 0$$

as $n \to \infty$.

PROOF. (a) First note that $\{\xi_n\} =_d \{X_{f_n}\}$, where X_{f_n} is constructed from a Poisson process at the beginning of Section 2. By de Haan [(1984), Lemma 2, page 1200], X_{f_n} converges in probability to X_f. It follows that $\{\xi_n\}$, being fundamental in probability, also converges in probability to some limit ξ. It is easy to check that $\{\xi,\ \xi_n,\ n \geq 1\}$ is a max-stable sequence with spectral function sequence $\{f,\ f_n,\ n \geq 1\}$ and hence $d(\xi_n, \xi) \to 0$ and $\xi \in \overline{\vee\text{-sp}\ \{\xi_n\}}$.

(b) Without loss of generality we may suppose there exist subsets A and B of $[0, 1]$ such that $A \cap B = \varnothing$, $A \cup B = [0, 1]$ and for any n the support of f_n is in A and the support of g_n is contained in B. Let f_L be a spectral function corresponding to L. Then (3.3) implies

$$0 = \lim_{n\to\infty} \int_0^1 |f_n(s) \vee g_n(s) - f_L(s)|\, ds$$
$$= \lim_{n\to\infty} \left(\int_A |f_n(s) - f_L(s)|\, ds + \int_B |g_n(s) - f_L(s)|\, ds \right)$$
$$= \lim_{n\to\infty} \left(\int_0^1 |f_n(s) - f_L(s) 1_A(s)|\, ds + \int_0^1 |g_n(s) - f_L(s) 1_B(s)|\, ds \right)$$

and hence

$$f_n \to_{L_1} f_L 1_A := f_\infty, \qquad g_n \to_{L_1} f_L 1_B := g_\infty.$$

The result then follows from (a). □

Now we consider (3.2).

PROPOSITION 3.3. Suppose $\{X_n\}$ is a stationary max-stable process with decomposition (3.1). Then

$$\mathscr{M}_{-\infty}(X) \subset \mathscr{M}_{-\infty}(V).$$

PROOF. Suppose $\xi \in \mathscr{M}_{-\infty}(X)$. Then from Proposition 2.2 there exist integers $k_n \to \infty$ as $n \to -\infty$ and nonnegative constants $v_i^{(n)}$ such that

$$\lim_{n\to-\infty} d\left(\bigvee_{i=n-k_n}^{n} v_i^{(n)} X_i, \xi \right) = 0.$$

Because

$$\bigvee_{i=n-k_n}^{n} v_i^{(n)} X_i = \left(\bigvee_{i=n-k_n}^{n} v_i^{(n)} U_i \right) \vee \left(\bigvee_{i=n-k_n}^{n} v_i^{(n)} V_i \right),$$

we get from Lemma 3.2 that there exist $U_{-\infty}$ and $V_{-\infty}$ such that

$$d\left(\bigvee_{i=n-k_n}^{n} v_i^{(n)} U_i, U_{-\infty} \right) \to 0, \qquad d\left(\bigvee_{i=n-k_n}^{n} v_i^{(n)} V_i, V_{-\infty} \right) \to 0,$$

and since $U_{-\infty} \in \mathscr{M}_{-\infty}(U) = \{0\}$, we get $\xi = V_{-\infty} \in \mathscr{M}_{-\infty}(V)$ as required. □

The Wold decomposition for a stationary max-stable process does not exist. For example, suppose $\{X_n\}$ is a stationary max-stable process with Wold decomposition

$$X_n = U_n \vee V_n,$$

where $\{U_n\}$ and $\{V_n\}$ are independent stationary max-stable processes, $\{U_n\}$ is purely nondeterministic, $\{V_n\}$ is deterministic, $U_n \in \mathscr{M}_n(X)$ [cf. Brockwell and Davis (1987), page 180]. Because $U_n \in \mathscr{M}_n(X)$, there exist sequences $\{v_j^{(m)}\}$

such that as $m \to \infty$,

$$\bigvee_{j=0}^{m} v_j^{(m)} X_{n-j} = \bigvee_{j=0}^{m} v_j^{(m)} (U_{n-j} \vee V_{n-j}) \to U_n$$

in the d-metric. The independence of $\{U_n\}$ and $\{V_n\}$ implies

$$d\left(\bigvee_{j=0}^{m} v_j^{(m)} (U_{n-j} \vee V_{n-j}), U_n\right) = d\left(\bigvee_{j=0}^{m} v_j^{(m)} U_{n-j}, U_n\right) + d\left(\bigvee_{j=0}^{m} v_j^{(m)} V_{n-j}, 0\right) \to 0.$$

Clearly $d(v_0^{(m)} V_n, 0) \le d(\bigvee_{j=0}^{m} v_j^{(m)} V_{n-j}, 0) \to 0$ so that $v_0^{(m)} \to 0$. It follows that

$$d\left(\bigvee_{j=1}^{m} v_j^{(m)} U_{n-j}, U_n\right) \to 0$$

and therefore $U_n \in \overline{\vee\text{-sp}}\ \{U_{n-1}, U_{n-2}, \dots\}$. Thus $\{U_n\}$ is deterministic, which is a contradiction.

4. Prediction based on more information. Some of the properties discussed in the previous section, which deviate markedly from classical theory, change when one bases prediction on larger max-stable spaces containing M_n. The present short section presents contrasts with prediction based on max-spans.

Let $\mathscr{M}^{\#}$ be the max-stable space generated by the Poisson process discussed in the beginning of Section 2: Start with $\sum_k \varepsilon_{(U_k, \Gamma_k)}$, PRM on $[0, 1] \times [0, \infty)$ with mean measure $1_{[0,1]}(u)\, du\, 1_{[0,\infty)}(s)\, ds$, and for nonnegative $f \in L_1 := L_1([0, 1], du)$ define

$$X_f = \bigvee_k \frac{f(U_k)}{\Gamma_k}$$

and the max-stable space

$$\mathscr{M}^{\#} = \{X_f : f \in L_1, f \ge 0\}.$$

Suppose we have a process $\{X_n\} \subset \mathscr{M}^{\#}$ with spectral functions $\{f_n\}$. Define for $-\infty < n < \infty$ the closed max-stable spaces

$$\mathscr{M}_n^{\#} = \{X \in \mathscr{M}^{\#} : X \in \sigma(X_n, X_{n-1}, \dots)\}$$

and

$$\mathscr{M}_{-\infty}^{\#} = \bigcap_n \mathscr{M}_n^{\#}.$$

EXAMPLE 4.1. Consider again the process of Example 3.1:

$$X_n = Y \vee Z_n, \tag{4.1}$$

where Y, Z_n, $-\infty < n < \infty$, are iid $\Phi_{1,1}$ random variables. The spectral functions are f_Y and f_{Z_n}, and without loss of generality we may assume that the supports are all disjoint. As usual, let f_n be the spectral function of X_n so that

$$f_n = f_Y \vee f_{Z_n} = f_Y + f_{Z_n}$$

and

$$f^{\wedge} := \bigwedge_n f_n = f_Y.$$

But $Y = \bigwedge_j X_{n-j}$ for any n so that $Y \in \mathscr{M}^{\#}_{-\infty}$. Recall that in Example 3.2, $Y \notin \mathscr{M}_{-\infty} = \{0\}$.

In order to obtain a Wold decomposition, consider the space of functions

$$L_1^{(n)} = L_1\big(\sigma(f_j, j \le n)\big) \bigcap \left\{ f \in L_1 : f \ge 0, \text{supp } f \subset \text{closure}\left(\bigcup_{j=-\infty}^{n} \text{supp } f_j \right)\right\}$$

where supp f is the support of f and $L_1(\sigma(f_j, j \le n))$ are the L_1 functions measurable with respect to $\{f_j, j \le n\}$. Define for $-\infty < n < \infty$ the closed max-stable spaces

$$\mathscr{H}_n = \left\{ X_f : f \in L_1^{(n)} \right\}$$

and

$$\mathscr{H}_{-\infty} = \bigcap_n \mathscr{H}_n.$$

Prediction based on $\mathscr{H}_n$ yields different results than prediction based on $\mathscr{M}_n$ or $\mathscr{M}_n^{\#}$. For example, let $\hat{f}_{n+1}$ be a function in $L_1^{(n)}$ that minimizes the distance from $L_1^{(n)}$ to f_{n+1}; that is,

$$\| \hat{f}_{n+1} - f_{n+1} \|_{L_1} = \inf_{f \in L_1^{(n)}} \| f - f_{n+1} \|_{L_1}.$$

Then the best predictor of X_{n+1} in $\mathscr{H}_n$ is $\hat{X}_{n+1} := X_{\hat{f}_{n+1}}$ because

$$\inf_{X_f \in \mathscr{H}_n} d(X_f, X_{n+1}) = \inf_{f \in L_1^{(n)}} \| f - f_{n+1} \|_{L_1}.$$

For the process of Example 4.1, we have

$$f_Y = \bigwedge_n f_n$$

and because the f_n's have disjoint support, $f_Y \in L_1^{(n)}$ for each n. This implies that $Y \in \mathscr{H}_{-\infty}$ and because

$$f_n - f_Y = f_{Z_n}$$

and $f_{Z_n} \in L_1^{(n)}$, we have

$$X_{f_{Z_n}} = Z_n \in \mathscr{H}_n.$$

This gives a Wold-type decomposition for $\{X_n\}$: X_n has a component Y in $\mathscr{H}_{-\infty}$ and a component Z_n in $\mathscr{H}_n$.

Before discussing prediction in this framework, we record the following simple result:

PROPOSITION 4.1. *Let $\mathscr{M}'$, $\mathscr{M}$ be closed max-stable spaces with $\mathscr{M}' \subset \mathscr{M}$. Suppose $X \in \mathscr{M}$ and $X = Y \vee Z$, where $Y \in \mathscr{M}'$ and Z is independent of every element in $\mathscr{M}'$. Then*

$$Y \in P_{M'} X.$$

PROOF. For any $\xi \in \mathscr{M}'$ we have

$$d(\xi, X) = d(\xi, Y \vee Z) = d(\xi, Y) + d(\xi, Z)$$

because the support of f_Z can be taken disjoint from supp $f_\xi \cup$ supp f_Y. Thus $d(\xi, X) \geq d(\xi, Y)$ and thus the minimum distance can be achieved by taking $\xi = Y$. □

Continuing with the prediction problem for this example, we know from the previous proposition that a best predictor of X_{n+1} based on either $\mathscr{M}_n^{\#}$ or $\mathscr{H}_n$ is Y, because Z_{n+1} is independent of $\mathscr{M}_n^{\#}$ and $\mathscr{H}_n$.

These results parallel the classical theory rather well, at the expense of a more artificial notion of prediction. The examples can be extended easily to cover processes where Z_n in (4.1) is replaced by a max-moving average process.

A bizarre aspect of prediction based on $\mathscr{H}_n$ is that it is dependent on the choice of spectral functions. For example, if we choose $f_Y(s) = c1_{(0,a)}(s)$ in Example 4.1, then $f \in \bigcap_n L_1^{(n)}$ implies that supp $f \subset$ supp f_Y and that f is a function of $f_Y, f_{Z_n}, f_{Z_{n-1}}, \dots$. Because f_Y and the f_{Z_j} are assumed to have disjoint supports, it follows that f must be a function of f_Y only and hence $f(s) = (\text{const}) f_Y(s)$. From this we conclude that $\mathscr{H}_{-\infty} = \overline{\vee\text{-sp}}\ \{Y\}$. On the other hand, if we choose $f_Y(s) = cs1_{(0,a)}(s)$, then $\mathscr{H}_{-\infty} = \{X_f \in \mathscr{M}^{\#}: f \in \sigma(f_Y),$ supp $f \subset$ supp $f_Y\}$ and $\overline{\vee\text{-sp}}\ \{Y\} \neq \mathscr{H}_{-\infty}$.

5. Permutation processes. We now discuss a class of stationary max-stable processes, called *permutation processes*, that is broader than the class of max-moving averages. A simple example and application of a permutation process was given in Example 3.3. It will be convenient to work with a two-sided permutation process $\{X_n\}$, which has the form

$$X_n = \bigvee_{i=-\infty}^{\infty} c_{\pi^n(i)} Z_i, \qquad -\infty < n < \infty, \tag{5.1}$$

where π is a bijection of $\{\dots, -1, 0, 1, \dots\}$, $\{Z_i\}$ are iid $\Phi_{1,1}$ random variables and $c_i \geq 0$, $\sum_i c_i < \infty$. A simple argument shows that $\{X_n\}$ is stationary.

THEOREM 5.1. *The following are equivalent.*

(a) *The stationary max-stable process $\{X_n\}$ is a permutation process with form* (5.1).

(b) $\{X_n\}$ *is distributionally equivalent to the following process: There exist an atomic probability measure* μ *on* $(0, 1)$,

$$\mu = \sum_{i=-\infty}^{\infty} p_i \varepsilon_{a_i}, \tag{5.2}$$

with atoms $\{a_i\}$ *and weights* $\{p_i\}$ ($p_i > 0$ *for all* i) *and a piston* $\tilde{\gamma}$ *satisfying for some permutation* π *and all* $f \in L_1(\mu)$,

$$\tilde{\gamma} f(s) = \tilde{r}(s) f(\tilde{H}(s)), \qquad \int_{(0,1)} \tilde{\gamma} f(s) \mu(ds) = \int_{(0,1)} f(s) \mu(ds) \tag{5.3}$$

with

$$\tilde{H}(a_i) = a_{\pi(i)}, \qquad \tilde{r}(a_i) = p_{\pi(i)}/p_i. \tag{5.4}$$

The quantities $\tilde{f}_0$ *and* $\{c_i\}$ *are related by*

$$\tilde{f}_0(a_i) = c_i/p_i, \tag{5.5}$$

and then

$$\{X_n\} =_d \left\{ \bigvee_{k=1}^{\infty} \frac{\tilde{f}_n(U_k)}{\Gamma_k} \right\},$$

where $\tilde{f}_n = \tilde{\gamma}^n \tilde{f}_0$ *and*

$$\sum_{k=1}^{\infty} \varepsilon_{(U_k, \Gamma_k)}$$

is Poisson random measure on $(0, 1) \times [0, \infty)$ *with mean measure* $d\mu \times dx$.

(c) *There exist a decomposition of* $(0, 1)$ *into disjoint intervals* $(0, 1) = \sum_{i=-\infty}^{\infty} I_i$, *where* $I_i = (a_i, b_i]$, *and a piston* $\gamma = (r, H)$ *satisfying* $[f \in L_1(dx)]$

$$\gamma f(s) = r(s) f(H(s)), \qquad \int_{(0,1)} \gamma f(s)\, ds = \int_{(0,1)} f(s)\, ds, \tag{5.6}$$

such that for some permutation π *the map* H: $I_i \mapsto I_{\pi(i)}$ *is defined by*

$$H(s) = \sum_{i=-\infty}^{\infty} \left(a_{\pi(i)} + (s - a_i) \frac{|I_{\pi(i)}|}{|I_i|} \right) 1_{I_i}(s) \tag{5.7}$$

and

$$r(s) = \sum_{i=-\infty}^{\infty} \left(\frac{|I_{\pi(i)}|}{|I_i|} \right) 1_{I_i}(s). \tag{5.8}$$

The quantities f_0 *and* $\{c_i\}$ *are related by the fact that for each* n, *the spectral functions* $\{f_n\}$ *of* $\{X_n\}$ *are constant on every* I_i *and*

$$f_0(s) = \sum_{i=-\infty}^{\infty} \left(\frac{c_i}{|I_i|} \right) 1_{I_i}(s). \tag{5.9}$$

Then

$$\{X_n\} =_d \left\{ \bigvee_{k=1}^{\infty} \frac{f_n(\theta_k)}{\Gamma_k} \right\},$$

where $f_n = \gamma^n f_0$ *and*

$$\sum_{k=1}^{\infty} \varepsilon_{(\theta_k, \Gamma_k)}$$

is a homogeneous Poisson random measure on $(0, 1) \times [0, \infty)$ *with mean measure* $du \times dx$.

PROOF. (a) ⇒ (b): Choose probabilities $\{p_i\}$ and atoms $\{a_i\}$ and define μ by (5.2) and $\tilde{H}$, $\tilde{r}$, $\tilde{\gamma}$, $\tilde{f}_0$ by (5.4), (5.3), and (5.5). The definition of $\tilde{H}$, $\tilde{r}$, $\tilde{f}_0$ off $\{a_i\}$ is immaterial. $\tilde{\gamma}$ is a piston because

$$\begin{aligned}
\int \tilde{\gamma} f(s)\mu(ds) &= \sum_i \tilde{\gamma} f(a_i) p_i \\
&= \sum_i \frac{p_{\pi(i)}}{p_i} f\big(\tilde{H}(a_i)\big) p_i \\
&= \sum_i p_{\pi(i)} f(a_{\pi(i)}) = \sum_i p_i f(a_i) \\
&= \int f d\mu.
\end{aligned}$$

Check that for any n,

$$\tilde{f}_n(a_i) := \tilde{\gamma}^n \tilde{f}_0(a_i) = \frac{c_{\pi^n(i)}}{p_i}.$$

If

$$\sum_{k=1}^{\infty} \varepsilon_{\Gamma_k^{(i)}}, \qquad i = 0, \pm 1, \dots,$$

are independent homogeneous Poisson processes in $[0, \infty)$, then

$$Z_i =_d \bigvee_k \frac{1}{\Gamma_k^{(i)}}.$$

If $\{U_k^{(i)}, k \geq 1, i = 0, \pm 1, \dots)$ are iid random variables with distribution μ, then

$$N_i = \sum_{k=1}^{\infty} \varepsilon_{(U_k^{(i)}, \Gamma_k^{(i)}/p_i)}(\{a_i\} \cap \cdot \times \cdot), \qquad i = 0, \pm 1, \dots,$$

are independent PRMs with mean measure

$$\varepsilon_{a_i}(dx) \times p_i\, ds = \big(p_i \varepsilon_{a_i}(dx)\big) \times ds$$

and therefore $\Sigma_i N_i$ is PRM with mean measure

$$\left(\sum_i p_i \varepsilon_{a_i}(dx)\right) \times ds = \mu(dx) \times ds.$$

Letting

$$N := \sum_i N_i = \sum_{k=1}^{\infty} \varepsilon_{(U_k, \Gamma_k)},$$

we see that

$$\begin{aligned}\{X_n\} &= \left\{\bigvee_{i=-\infty}^{\infty} c_{\pi^n(i)} Z_i\right\}\\ &=_d \left\{\bigvee_{i=-\infty}^{\infty} c_{\pi^n(i)} \bigvee_k \frac{1}{\Gamma_k^{(i)}}\right\}\\ &= \left\{\bigvee_{i,k} \frac{\tilde{f}_n(a_i)}{\Gamma_k^{(i)}/p_i}\right\}\\ &= \left\{\bigvee_k \frac{\tilde{f}_n(U_k)}{\Gamma_k}\right\},\end{aligned}$$

as required.

(a) ⇒ (c): Suppose $\{X_n\}$ has form (5.1) and define γ, H, r, f_0 by (5.6)–(5.9). Then γ is a piston because, for $f \in L_1(dx)$,

$$\begin{aligned}\int_{(0,1)} \gamma f(s)\, ds &= \sum_i \int_{I_i} r(s) f(H(s))\, ds\\ &= \sum_i \frac{|I_{\pi(i)}|}{|I_i|} \int_{I_i} f\left(a_{\pi(i)} + (s - a_i)\frac{|I_{\pi(i)}|}{|I_i|}\right) ds\end{aligned}$$

and changing variables via $y = a_{\pi(i)} + (s - a_i)\,|I_{\pi(i)}|/|I_i|$ yields

$$\begin{aligned}&= \sum_i \frac{|I_{\pi(i)}|}{|I_i|} \int_{I_{\pi(i)}} f(y) \frac{|I_i|}{|I_{\pi(i)}|}\, dy\\ &= \sum_i \int_{I_{\pi(i)}} f(y)\, dy = \int_{(0,1)} f(y)\, dy,\end{aligned}$$

from which it follows that γ is a piston. Next one must check that

$$f_n(s) = \sum_i \frac{c_{\pi^n(i)}}{|I_i|} 1_{I_i}(s) \tag{5.10}$$

for any integer n. Let $\{\theta_k^{(i)},\ k \geq 1,\ i = 0, \pm 1, \dots\}$ be iid random variables uniformly distributed on I_i and independent of $\{\Gamma_k^{(i)}\}$ [defined in (b)] so that

$$N_i = \sum_k \varepsilon_{(\theta_k^{(i)}, \Gamma_k^{(i)}/|I_i|)}$$

is PRM on $I_i \times [0, \infty)$ with mean measure

$$\frac{du}{|I_i|} 1_{I_i}(u)\, dx|I_i| = 1_{I_i}(u)\, du\, dx.$$

Therefore, $N = \sum_i N_i$ is homogeneous PRM on $(0, 1) \times [0, \infty)$ with mean measure $du \times dx$ and we can represent N as

$$N = \sum_k \varepsilon_{(\theta_k, \Gamma_k)},$$

where $\{\theta_k\}$ are iid random variables that are uniformly distributed on (0, 1) and independent of $\{\Gamma_k\}$. To conclude the proof note that

$$\begin{aligned}\{X_n\} &= \left\{\bigvee_{i=-\infty}^{\infty} c_{\pi^n(i)} Z_i\right\} \\ &=_d \left\{\bigvee_{i,k} \frac{c_{\pi^n(i)}}{\Gamma_k^{(i)}}\right\} \\ &= \left\{\bigvee_{i,k} \frac{f_n(\theta_k^{(i)})}{\Gamma_k^{(i)}/|I_i|}\right\} \\ &= \left\{\bigvee_{k} \frac{f_n(\theta_k)}{\Gamma_k}\right\},\end{aligned}$$

as desired. The implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) are readily checked. □

Note the flexibility in the choice of atoms $\{a_i\}$ in part (b) and the choice of intervals in part (c). If in part (b) of the theorem, we define $p_i = c_i/\sum_j c_j$ whenever $c_i > 0$, we see by (5.5) that

$$\tilde{f}_0(s) = \left(\sum_j c_j\right) 1_{\tilde{A}}(s),$$

where $\tilde{A} = \{a_i : c_i > 0\}$. In particular if $c_i > 0$ for all i, then $\tilde{f}_0(s) = \sum_j c_j$. Similarly, in part (c) of the theorem, one may choose I_i to have length $c_i/\sum_j c_j$ whenever $c_i > 0$ to get

$$f_0(s) = \left(\sum_j c_j\right) 1_A(s), \qquad 1_A(s) = \sum_{\{j: c_j > 0\}} 1_{I_j}(s).$$

By relabeling the I_j if necessary, we may always take $A = (0, a]$ for some $a \in (0, 1)$. This last statement is really a special case of the remark, stated without proof, in de Haan and Pickands [(1986), page 486]. Because understanding this remark is important for Proposition 5.3, we supply a proof that for any stationary max-stable process, f_0 may be chosen as $c1_{(0,a)}$.

PROPOSITION 5.2. *Let $f \in L_1(dx)$ be a nonnegative function.*

(a) *If $f(s) > 0$ a.e., then there exists a piston $\gamma = (r, H)$ such that*

$$r(s) f(H(s)) = c 1_{(0,1)}(s) \quad a.e.$$

(b) *If* $\int_{(0,1)} 1_{f>0}(s)\,ds \in (0, 1)$, *then for every* $a \in (0, 1)$ *there exists a piston* $\gamma = (r, H)$ *such that*

$$r(s) f(H(s)) = c 1_{(0,a)}(s) \quad a.e.$$

(c) *It is always possible to choose a sequence of spectral functions* $\{f_n\}$ *for a stationary max-stable process* $\{X_n\}$ *such that*

$$f_0(s) = c 1_{(0,a)}(s)$$

for some $a \in (0, 1]$.

PROOF. (a) Without loss of generality assume $\int_{(0,1)} f(s)\,ds = 1$; otherwise divide f by its integral and absorb this constant into c. Set $I(x) = \int_0^x f(s)\,ds$ and put

$$H(s) = I^{-1}(s),$$
$$r(s) = \frac{1}{f(H(s))}.$$

Then $\gamma = (r, H)$ is a piston because, for $g \in L_1(dx)$, we have by the change of variables $s = H^{-1}(u)$,

$$\int_{(0,1)} r(s) g(H(s))\,ds = \int_{(0,1)} \frac{1}{f(H(s))} g(H(s))\,ds = \int_{(0,1)} g(u)\,du.$$

(b) Assume once again that f has integral 1. With I as previously defined, I^{-1} maps [0, 1] onto the support of the probability measure corresponding to I. Because $(\operatorname{supp}(I))^c$ and $(a, 1)$ are open, we can express both as disjoint unions of intervals, say

$$(\operatorname{supp} I)^c = \bigcup_{j=1}^{\infty} (a_j, b_j]$$

and

$$(a, 1) = \bigcup_{j=1}^{\infty} (c_j, d_j].$$

Define

$$H(s) = \begin{cases} I^{-1}\left(\dfrac{s}{a}\right), & \text{for } s \le a, \\ \left(\dfrac{b_j - a_j}{d_j - c_j}\right)(s - c_j) + a_j, & \text{if } s \in (c_j, d_j], \end{cases}$$

$$r(s) = \begin{cases} [af(H(s))]^{-1}, & \text{if } s \le a \text{ and } f(H(s)) > 0, \\ 1, & \text{if } s \le a \text{ and } f(H(s)) = 0, \\ \dfrac{b_j - a_j}{d_j - c_j}, & \text{if } s \in (c_j, d_j], \end{cases}$$

so that

$$H:(0,a] \mapsto \operatorname{supp}(I),$$

$$H:(a,1) \mapsto (\operatorname{supp}(I))^c = \bigcup_{j=1}^{\infty} (a_j, b_j].$$

The pair (r, H) is a piston because, for $g \in L_1(dx)$, we have by change of variables,

$$\begin{aligned}
&\int_{(0,1)} r(s) g(H(s))\, ds \\
&= \int_{(0,a]} \frac{1}{af(H(s))} g(H(s))\, ds + \sum_{j=1}^{\infty} \int_{(c_j, d_j]} r(s) g(H(s))\, ds \\
&= \int_{\operatorname{supp}(I)} g(u)\, du + \sum_{j=1}^{\infty} \int_{(a_j, b_j]} g(u)\, du \\
&= \int_{(0,1)} g(u)\, du.
\end{aligned}$$

Moreover, for $s \le a$,

$$r(s) f(H(s)) = \frac{1}{a} 1_{(0,a]}(s),$$

and for $s > a$,

$$r(s) f(H(s)) = r(s) f(H(s)) 1_{\{H(s) \in (\operatorname{supp}(I))^c\}} = 0 \quad \text{a.e.},$$

because

$$\int_{(0,1)} r(s) f(H(s)) 1_{\{H(s) \in (\operatorname{supp}(I))^c\}}\, ds = \int_{(0,1)} f(u) 1_{(\operatorname{supp}(I))^c}(u)\, du = 0.$$

This completes the proof of (b).

(c) If $f_0(s) \ne c1_{(0,a)}(s)$, then consider the equivalent sequence of spectral functions $\{g_n(s) := r(s) f_n(H(s))\}$, where (r, H) is the piston in either (a) or (b). [As pointed out in de Haan and Pickands (1986), these two sequences of spectral functions are equivalent because

$$\int_{(0,1)} \bigvee_j \frac{f_j(s)}{x_j}\, ds = \int_{(0,1)} \bigvee_j \frac{g_j(s)}{x_j}\, ds$$

for all sequences of nonnegative numbers $\{x_j\}$. This ensures that the joint distributions of $\{X_n\}$ can be specified with either sequence of spectral functions.] □

We now particularize our characterizations to max-moving averages. The two-sided max-moving average process

$$X_n = \bigvee_{i=-\infty}^{\infty} c_i Z_{n+i} = \bigvee_{i=-\infty}^{\infty} c_{i+n} Z_n, \tag{5.11}$$

where $c_i \geq 0$ and $\sum_{i=-\infty}^{\infty} c_i < \infty$, is a special case of a permutation process with

$$\pi(i) = i + 1.$$

(For compatibility with the notation of permutation process, the subscripts of the Z's differ from what was used in Proposition 3.1.)

Given a sequence of spectral functions $\{g_n\}$ from a stationary process, how can we determine whether or not the associated stationary process $\{X_n\}$ is a max-moving average? Because any piston transform of the sequence $\{g_n\}$, that is, $\{f_n := \gamma g_n\}$, produces the same joint distributions as the g_n, we can expect a wide range of behavior of such spectral functions. However, as long as we start with $g_0(s) = c$ (assuming that $c_j > 0$ for all j), which we may do by Proposition 5.2, the piston (r, H) that generates the spectral functions must be *piecewise* linear in the sense that $r(s)$ is piecewise constant on a partition of (0, 1).

PROPOSITION 5.3. *Suppose $\{X_n\}$ is a max-moving average as given in* (5.11) *with $c_j > 0$ for every j and suppose without loss of generality that the spectral functions $\{g_n\}$ are chosen so that $g_0 \equiv c$. Then there exists a partition $\{A_i\}$ of* (0, 1) *and a piston $\gamma = (r, H)$ such that $H(A_i) = A_{i+1}$ and for every i, r is constant on A_i.*

PROOF. Let $\{X_n\}$ be the max-moving average in (5.11) with $c_j > 0$ for all j and let $\{g_n\}$ be a sequence of spectral functions for $\{X_n\}$ with $g_0(s) \equiv c$, $c = \sum_j c_j$. Suppose $\{f_n\}$ is the sequence of spectral functions specified in part (c) of Theorem 5.1 (see 5.10); that is,

$$f_n(s) = \sum_{i=-\infty}^{\infty} \frac{c_{i+n}}{|I_i|} 1_{I_i}(s),$$

constructed so that $f_0(s) = g_0(s)$; this can be arranged by picking I_j to have length c_j/c. Then by Theorem 4.2 in de Haan and Pickands (1986), there exists a piston $\tilde{\gamma} = (\tilde{r}, \tilde{H})$ such that

$$g_n = \tilde{\gamma} f_n$$

for all n. In particular,

$$c = g_0(s) = \tilde{r}(s) f_0(\tilde{H}(s)) = \tilde{r}(s) c$$

so that $\tilde{r}(s) = 1$. It now follows with $A_j = \tilde{H}^{-1}(I_j)$ that

$$\begin{aligned} g_n(s) &= \sum_{j=-\infty}^{\infty} \frac{c_{n+j}}{|I_j|} 1_{I_j}(\tilde{H}(s)) \\ &= \sum_{j=-\infty}^{\infty} \frac{c_{n+j}}{|I_j|} 1_{A_j}(s). \end{aligned}$$

If $\gamma = (r, H)$ denotes the piston that generates the g_n sequence (i.e., $g_n = \gamma^n g_0$), then we conclude from the preceding equations that H: $A_i \mapsto A_{i+1}$ with

$$r(s) = \frac{|A_{i+1}|}{|A_i|} \quad \text{for } s \in A_i.$$

Note that $|A_i|$ is the Lebesgue measure of A_i and that $|A_i| = |I_i|$, because for $f = 1_{I_i}$, we have

$$\int f(H(s))\, ds = \int f(s)\, ds. \qquad \square$$

If one or more of the c_i's is zero, this result breaks down in two respects. First, we no longer can assume that $g_0 \equiv c$, because if $c_j = 0$, then Z_j is independent of X_0 and hence the spectral functions of Z_j and X_0 must have disjoint supports a.e., a contradiction. Second, even if we take g_0 to be piecewise constant, for example,

$$g_0(s) = \frac{c}{a} 1_{(0,\, a]}(s)$$

for some $a \in (0, 1)$, it is still not necessary for $H(s)$ in the generating piston $\gamma = (r, H)$ to have the properties discussed in Proposition 5.3. This is illustrated in the following example.

EXAMPLE 5.1. For a fixed constant $a \in (0, 1)$, choose $c_j > 0$, $j = 0, -1, -2, \dots$, such that $a = \sum_{j=-\infty}^{0} c_j$. Let $\{I_j\}$ be a partition of $(0, 1)$ as described in the statement of Theorem 5.1(c) with $|I_j| = c_j$, $j \le 0$, and

$$(0,a] = \sum_{j=-\infty}^{0} I_j.$$

Let $\gamma = (r, H)$ be the piston defined by

$$H(s) = \sum_{i=-\infty}^{0} \left(a_{i+1} + (s - a_i) \frac{b_{i+1} - a_{i+1}}{b_i - a_i} \right) I_{(a_i,\, b_i]}(s) + \sum_{i=1}^{\infty} \left(a_{i+1} + (s - a_i)^2 \frac{(b_{i+1} - a_{i+1})^2}{(b_i - a_i)^2} \right) I_{(a_i,\, b_i]}(s)$$

and

$$r(s) = \sum_{i=-\infty}^{0} \left(\frac{b_{i+1} - a_{i+1}}{b_i - a_i} \right) I_{(a_i,\, b_i]}(s) + \sum_{i=1}^{\infty} \left(2(s - a_i) \frac{(b_{i+1} - a_{i+1})^2}{(b_i - a_i)^2} \right) I_{(a_i,\, b_i]}(s)$$

[cf. (5.7) and (5.8)]. It follows that the sequence of spectral functions defined by

$$g_0(s) = \sum_{j=-\infty}^{0} c_j \frac{1}{b_j - a_j} 1_{(a_j,\, b_j]}(s) = 1_{(0,\, a]}(s)$$

and

$$g_n(s) = \gamma^n g_0(s)$$

can be expressed as

$$g_n(s) = \sum_{j=-\infty}^{0} c_j f_{n-j},$$

where $f_n(s) = \gamma^n((1/(b_0 - a_0))1_{(a_0, b_0]}(s))$. Because the f_n have disjoint support, we conclude that $\{g_n\}$ is a spectral function sequence for the MMA(∞) process

$$X_n = \bigvee_{j=-\infty}^{0} c_j Z_{n-j},$$

even though r is not piecewise constant.

The permutation process is a generalization of the moving average processes considered in Examples 2.4 and 2.5. Based on our experience with the examples of Section 2, it is natural to speculate that for a permutation process,

$$\overline{\bigvee\text{-sp}}\ \{X_j, j \le n\} = \left\{ \bigvee_{i=0}^{\infty} v_i X_{n-i} \colon \bigwedge_{i=-\infty}^{n} v_i \ge 0,\ \bigvee_{i=0}^{\infty} v_i X_{n-i} < \infty \right\}.$$

However, carefully checking the derivations of Section 2 shows that in order for an explicit representation of the max-span to be possible, one needs some condition on the permutation π such as

$$\lim_{m \to -\infty} \pi^m(i) = \infty, \qquad \forall\, i \ge 1, \tag{5.12}$$

in order to ensure that $\{v_j^{(k')}\} \to \{v_j\}$ as $k' \to \infty$ implies that

$$\bigvee_{j=-\infty}^{n} v_j^{(k')} c_{\pi^j(i)} \to \bigvee_{j=-\infty}^{n} v_j c_{\pi^j(i)}.$$

In contrast to the simplicity of Example 3.3, a permutation process generated by an infinite number of atoms can be difficult to analyze. When (5.12) is satisfied, we show the process is purely nondeterministic.

PROPOSITION 5.4. *Suppose we have the permutation process*

$$X_n = \bigvee_{i=0}^{\infty} c_{\pi^n(i)} Z_i$$

generated by an infinite number of atoms. If (5.12) *is satisfied, then* $\{X_n\}$ *is purely nondeterministic.*

PROOF. Condition (5.12) guarantees that

$$\mathscr{M}_n = \overline{\vee\text{-sp}}\ \{X_j, j \le n\} = \left\{ \bigvee_{j=-\infty}^{n} \alpha_j X_j \colon \bigwedge_j \alpha_j \ge 0, \bigvee_{j=-\infty}^{n} \alpha_j X_j < \infty \right\}.$$

If $\xi \in \mathscr{M}_{-\infty}$ we have for every n a representation

$$\xi = \bigvee_{j=-\infty}^{n} v_j^{(n)} X_j = \bigvee_{j=-\infty}^{n} v_j^{(n)} \bigvee_{i=0}^{\infty} c_{\pi^j(i)} Z_i = \bigvee_{i=0}^{\infty} \left(\bigvee_{j=-\infty}^{n} v_j^{(n)} c_{\pi^j(i)} \right) Z_i.$$

So taking limits as $n \to -\infty$ and using Lemma 2.1 yields the existence of constants

$$\psi_i = \lim_{n \to -\infty} \bigvee_{j=-\infty}^{n} v_j^{(n)} c_{\pi^j(i)},$$

where the limit is in l_1 and thus

$$\xi = \bigvee_{i=0}^{\infty} \psi_i Z_i.$$

It remains to show $\psi_i = 0$. As in the discussions of Example 2.3 we get that $\{\vee_j v_j^{(n)}\}$ is bounded in n and the conclusion $\psi_i = 0$ follows from condition (5.12). □

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