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WALKS IN THE QUARTER PLANE: KREWERAS' ALGEBRAIC MODEL

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We consider planar lattice walks that start from $(0, 0)$, remain in the first quadrant $i, j \geq 0$, and are made of three types of steps: North-East, West and South. These walks are known to have remarkable enumerative and probabilistic properties:

- they are counted by nice numbers [Kreweras, *Cahiers du B.U.R.O* **6** (1965) 5–105],
- the generating function of these numbers is algebraic [Gessel, *J. Statist. Plann. Inference* **14** (1986) 49–58],
- the stationary distribution of the corresponding Markov chain in the quadrant has an algebraic probability generating function [Flatto and Hahn, *SIAM J. Appl. Math.* **44** (1984) 1041–1053].

These results are not well understood, and have been established via complicated proofs. Here we give a uniform derivation of all of them, which is more elementary that those previously published. We then go further by computing the full law of the Markov chain. This helps to delimit the border of algebraicity: the associated probability generating function is no longer algebraic, unless a diagonal symmetry holds.

Our proofs are based on the solution of certain functional equations, which are very simple to establish. Finding purely combinatorial proofs remains an open problem.

1. Introduction. Let us begin with a very simple combinatorial statement: the number of planar lattice walks that start and end at $(0, 0)$, consist of $3n$ steps that can be North-East, South or West, and always remain in the nonnegative quadrant $i, j \geq 0$ is

$$a(3n) = \frac{4^n}{(n+1)(2n+1)} \binom{3n}{n}.$$

An example of such a walk is given in Figure 1. This result, first proved by Kreweras in 1965 [23], is rather intriguing, for at least two reasons.

First, this simple looking formula has no simple proof. If we consider instead the more traditional square lattice walks (consisting of North, South, East and West

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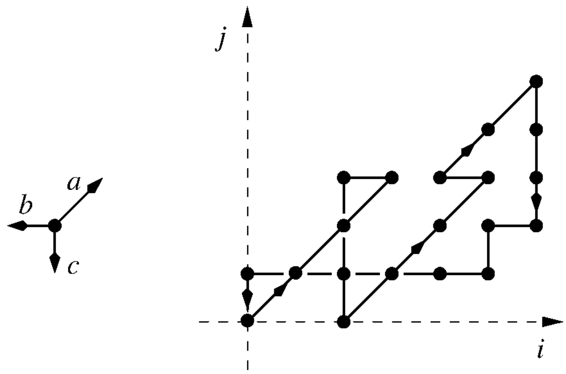

FIG. 1. *Kreweras' walks in a quadrant.*

steps), then there exists a nice formula too: the number of $2n$-step walks, starting and ending at the origin and confined in the first quadrant, is

$$b(2n) = \frac{1}{(2n+1)(2n+2)} \binom{2n+2}{n+1}^2.$$

But the latter formula can be proved in a few lines (count first the number of such walks having $2m$ horizontal steps, and then sum over all values of m) and admits even a direct combinatorial explanation [20]. No similar derivation exists for the numbers $a(3n)$.

The second fact that makes the numbers $a(3n)$ intriguing is that their generating function, that is, the power series $A(t) = \sum_n a(3n)t^n$, is *algebraic*. This means that it satisfies a polynomial equation $P(t, A(t)) = 0$, where P is a nontrivial bivariate polynomial with rational coefficients. For combinatorialists, objects that have an algebraic generating function are really special: this property suggests that one *should* be able to factor them into smaller objects of the same type, and then translate this factorization into a polynomial equation (or a system of polynomial equations) defining the generating function. Let us take an example: it is known that for any (finite) set of steps, the walks confined in the upper half-plane have an algebraic generating function. There is a clear combinatorial understanding of this property: the key idea is to factor the walk at the first time it returns to the x-axis. It is still an open problem to find an explanation of this type for the algebraicity of the series $A(t)$. Let us underline that not all walks in the quadrant have an algebraic generating function: the generating function for the numbers $b(2n)$ is transcendental (see [8] for a stronger result).

A natural question—at least for a computer scientist—is whether the set of words on the alphabet $\{a, b, c\}$ that naturally encode Kreweras' walks forms an *algebraic* (or *context-free*) language [21]. These words contain as many a's as b's, as many a's as c's, and each of their prefixes contains no more b's than a's, and no more c's than a's. Using the *pumping lemma* ([21], Theorem 4.7), one can prove that this language is not algebraic. Moreover, the words satisfying only the second condition above, which encode walks ending anywhere in the quadrant, do not form an algebraic language either [3]. However, we shall see that their generating function is algebraic.

Then how does one prove Kreweras' formula? In his original paper, Kreweras considered n-step walks in the quadrant going from $(0, 0)$ to (i, j). A step by step construction of these walks gives an obvious recurrence relation for their number, denoted below $a_{i,j}(n)$. Kreweras solved this recursion. His proof involves *guessing* a substantial part of the solution, and then proving several hypergeometric identities. The latter part was then simplified by Niederhausen [28, 29]. A different proof, due to Gessel, also requires *guessing* the bivariate generating function of walks ending on the x-axis, and then verifying that it satisfies a certain functional equation [19].

On the probabilistic side, in the early 70's Malyshev began to address the very general problem of computing the stationary distribution of discrete homogeneous Markov chains in the quadrant [26]. Several instances of this question actually correspond to finding the equilibrium behavior of double-queue processes [11, 12, 17, 33]. This work culminated in 1999 with a book that is entirely devoted to solving this problem in the case of unit increments [13]. The techniques used in this book are far from elementary, involving sophisticated complex analysis, Riemann surfaces and boundary value problems. Solutions are often expressed in terms of elliptic functions. The book lists a number of cases in which the stationary distribution has a rational generating function, and mentions exactly *one* case (actually due to Flatto and Hahn [17]) where this generating function is algebraic. Not surprisingly, the set of increments of this random walk is the same as in Kreweras' problem. (A (partial) algebraicity criterion is actually given in [13], Theorems 4.3.1 and 4.3.6, but it is only illustrated by Kreweras' example.)

Hence, the following question: what is so special with this set of three steps? Could one find a single argument that proves both the algebraicity of the generating function that counts these walks and the algebraicity of the generating function for the stationary distribution of the corresponding Markov chain?

This is the question we answer—positively—in this paper. For both the combinatorial problem and the probabilistic one, it is very easy to establish a functional equation defining the generating function. We solve both equations using the same approach. The only difference is that we are dealing with formal power series in the first problem, but with analytic functions in the second one. Our solution is constructive (we do not have to guess anything) and more elementary than the previously published ones. In particular, we always remain in the (small) world of algebraic functions, and do not need to introduce elliptic functions. The key to our approach is the combination of the *kernel method* (which is also central in [17] or [13]) with a special property of the kernel of the equations we consider. Moreover, after having solved the counting problem (Section 2) and the probabilistic one (Section 3), we combine both viewpoints and compute explicitly the full law of the Markov chain (Section 4). This actually marks the end of algebraicity: the probability generating function is transcendental, unless the transition probabilities satisfy a diagonal symmetry. Still, this generating function belongs to the nice class of *D-finite* (*or holonomic*) *series*, which is defined below.

Obviously, since this paper aims at explaining why a specific set of steps has such special properties, it cannot compete in generality with the strength of the machinery developed in [13]. Still, it is natural to ask how far our approach could be generalized. It is actually while fighting with Kreweras' walks that I discovered it. But it turns out that other applications of this approach were published before I was able to complete the present paper. In particular, most of the results in Section 2 are already reported in some conference proceedings, together with a general holonomy criterion for the enumeration of walks in the quadrant [5]. More recently, the same ideas were applied to certain counting problems on permutations [6]. Four new equations were thus solved in a uniform, elementary way. Their solutions are usually transcendental, but holonomic, and can be expressed as integrals of algebraic (quadratic) functions. I have not tried to attack other stationary distribution examples. But it is likely that the method presented here and in [6] can be applied to solve explicitly (and in an elementary way) at least *certain* specific examples.

Let us conclude this section by giving some definitions and notation on formal power series. Given a ring $\mathbb{L}$ and k indeterminates $x_1, \dots, x_k$, we denote by $\mathbb{L}[x_1, \dots, x_k]$ the ring of polynomials in $x_1, \dots, x_k$ with coefficients in $\mathbb{L}$. We denote by $\mathbb{L}[[x_1, \dots, x_k]]$ the ring of formal power series in the x_i, that is, of formal sums

$$\sum_{n_1 \geq 0, \dots, n_k \geq 0} a(n_1, \dots, n_k) x_1^{n_1} \cdots x_k^{n_k}, \tag{1}$$

where $a(n_1, \dots, n_k) \in \mathbb{L}$. A *Laurent polynomial* in the x_i is a polynomial in both the x_i and the $\bar{x}_i = 1/x_i$. A *Laurent series* in the x_i is a series of the form (1) in which the summation runs over $n_i \geq m_i$ for all i, with m_i in $\mathbb{Z}$. For $F \in \mathbb{L}[[t]]$, we denote by $[t^n]F$ the coefficient of t^n in F. If F is a formal series in t whose coefficients are Laurent series in x, we denote by F^+ the *positive part of F in x*, that is,

$$F = \sum_{n \geq 0} t^n \sum_{i \in \mathbb{Z}} f_i(n) x^i \quad \Longrightarrow \quad F^+ = \sum_{n \geq 0} t^n \sum_{i > 0} f_i(n) x^i. \tag{2}$$

We define similarly the negative, nonnegative and nonpositive parts of F.

Assume, from now on, that $\mathbb{L}$ is a field. We denote by $\mathbb{L}(x_1, \dots, x_k)$ the field of rational functions in $x_1, \dots, x_k$ with coefficients in $\mathbb{L}$. A series F in $\mathbb{L}[[x_1, \dots, x_k]]$ is *algebraic* if there exists a nontrivial polynomial P with coefficients in $\mathbb{L}$ such that $P(F, x_1, \dots, x_k) = 0$. The sum and product of algebraic series is algebraic. The series F is *D-finite* if the partial derivatives of F span a finite-dimensional vector space over the field $\mathbb{L}(x_1, \dots, x_k)$; see [31] for the one-variable case, and [24, 25] otherwise. In other words, for $1 \le i \le k$, the series F satisfies a nontrivial partial differential equation of the form

$$\sum_{\ell=0}^{d_i} P_{\ell,i} \frac{\partial^\ell F}{\partial x_i^\ell} = 0,$$

where $P_{\ell,i}$ is a polynomial in the x_j. Any algebraic series is D-finite. The sum and product of D-finite series are D-finite. The specializations of a D-finite series (obtained by giving values from $\mathbb{L}$ to some of the variables) are D-finite, if well defined. Finally, if F is D-finite, then any *diagonal* of F is also D-finite [24] (the diagonal of F in x_1 and x_2 is obtained by keeping only those monomials for which the exponents of x_1 and x_2 are equal). We shall use the following consequence of the proof of this result: if $F(t, x) \in \mathbb{L}[x, \bar{x}][[t]]$ is algebraic (with $\bar{x} = 1/x$), then the positive part of F in x is D-finite, as well as the coefficient of x^i in this series, for all i.

2. Enumeration: the number of walks. Consider walks that start from $(0, 0)$, consist of South, West and North-East steps, and always stay in the first quadrant (Figure 1). Let $a_{i,j}(n)$ be the number of n-step walks of this type ending at (i, j). We denote by $Q(x, y; t)$ the *complete generating function* of these walks:

$$Q(x, y; t) := \sum_{i,j,n \ge 0} a_{i,j}(n) x^i y^j t^n.$$

We can construct these walks recursively, by starting from $(0, 0)$ and adding a step at each time. This gives the equation

$$Q(x, y; t) = 1 + t\left(\frac{1}{x} + \frac{1}{y} + xy\right) Q(x, y; t) - \frac{t}{y} Q(x, 0; t) - \frac{t}{x} Q(0, y; t).$$

The first term in the right-hand side encodes the empty walk, reduced to the point $(0, 0)$. The next term shows the three possible ways one can add a step at the end of a walk. However, one should not add a South step to a walk that ends on the x-axis: the third term subtracts the contribution of this forbidden move, and the last term takes care of the symmetric case. Equivalently,

$$(3) \quad \big(xy - t(x + y + x^2y^2)\big) Q(x, y; t) = xy - xtQ(x, 0; t) - ytQ(0, y; t).$$

We shall often denote $Q(x, y; t)$ by $Q(x, y)$ for short. Let us also denote the series $xtQ(x, 0; t)$ by $R(x; t)$ or even $R(x)$. Using the symmetry of the problem in x and y, the above equation becomes

$$(4) \qquad \big(xy - t(x + y + x^2y^2)\big) Q(x, y) = xy - R(x) - R(y).$$

Equation (3) is equivalent to a recurrence relation defining the numbers $a_{i,j}(n)$ inductively with respect to n. Hence, it defines completely the series $Q(x, y; t)$. Still, the characterization we have in mind is of a different nature:

THEOREM 1 (The number of walks). *Let $W \equiv W(t)$ be the power series in t defined by*

$$W = t(2 + W^3).$$

Then the generating function of Kreweras' walks ending on the x-axis is

$$Q(x,0;t) = \frac{1}{tx}\left(\frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{W} - \frac{1}{x}\right)\sqrt{1 - xW^2}\right).$$

Consequently, the length generating function of walks ending at $(i,0)$ is

$$[x^i]Q(x,0;t) = \frac{W^{2i+1}}{2.4^i\ t}\left(C_i - \frac{C_{i+1}W^3}{4}\right),$$

where $C_i = \binom{2i}{i}/(i+1)$ is the ith Catalan number. The Lagrange inversion formula gives the number of such walks of length $3n + 2i$ as

$$a_{i,0}(3n+2i) = \frac{4^n(2i+1)}{(n+i+1)(2n+2i+1)}\binom{2i}{i}\binom{3n+2i}{n}.$$

The aim of this section is to derive Theorem 1 from the functional equation (3). Note that the complete generating function $Q(x,y)$ can be recovered using (3):

$$Q(x,y;t) = \frac{(1/W - \bar{x})\sqrt{1 - xW^2} + (1/W - \bar{y})\sqrt{1 - yW^2}}{xy - t(x + y + x^2y^2)} - \frac{1}{xyt},$$

with $\bar{x} = 1/x$ and $\bar{y} = 1/y$. For walks ending on the diagonal, we shall also obtain a nice generating function:

THEOREM 2 (Walks ending on the diagonal). *Let $W \equiv W(t)$ be defined as above. Then the generating function of Kreweras' walks ending on the diagonal, defined by*

$$Q_d(x;t) := \sum_{i,n \geq 0} a_{i,i}(n) x^i t^n,$$

satisfies

$$tQ_d(x;t) = \frac{W - \bar{x}}{\sqrt{1 - xW(1 + W^3/4) + x^2W^2/4}} + \bar{x}.$$

The expression of Q_d becomes a bit simpler if we express it in terms of the unique power series $Z \equiv Z(t)$ satisfying $Z = 1 + 4t^3Z^3$. Then $W = 2tZ$ and

$$tQ_d(x;t) = \frac{2tZ - \bar{x}}{\sqrt{1 - xtZ(1 + Z) + x^2t^2Z^2}} + \bar{x}.$$

The last formula of Theorem 1 is due to Kreweras [23]. He also gave a closed form expression for the number of walks containing exactly p West steps, q South steps, and r North-East steps, that is, for walks of length $n = p + q + r$ ending at $(i,j) = (r - p, r - q)$. This expression is a double summation, with alternating signs. We have not found anything simpler.

2.1. *The obstinate kernel method.* The kernel method is basically the only tool we have to attack (4). This method has been around since, at least, the 1970s, and is currently the subject of a certain revival (see [22], Exercises 2.2.1.4 and 2.2.1.11 and [11] for early uses of the method, and [1, 2, 7] for more recent combinatorial applications). It consists in coupling the variables x and y so as to cancel the *kernel* $K(x,y) = xy - t(x + y + x^2y^2)$ [which is the coefficient of $Q(x,y)$ in (4)]. This should give the "missing" information about the series $R(x)$.

As a polynomial in y, this kernel has two roots,

$$Y_0(x) = \frac{1 - t\bar{x} - \sqrt{(1 - t\bar{x})^2 - 4t^2x}}{2tx} = t + \bar{x}t^2 + O(t^3),$$

$$Y_1(x) = \frac{1 - t\bar{x} + \sqrt{(1 - t\bar{x})^2 - 4t^2x}}{2tx} = \frac{\bar{x}}{t} - \bar{x}^2 - t - \bar{x}t^2 + O(t^3).$$

The elementary symmetric functions of the Y_i are

$$Y_0 + Y_1 = \frac{\bar{x}}{t} - \bar{x}^2 \quad \text{and} \quad Y_0Y_1 = \bar{x}. \tag{5}$$

The fact that they are polynomials in $\bar{x} = 1/x$ will play a very important role below.

Only the first root can be substituted for y in (4) [the term $Q(x, Y_1; t)$ is not a well-defined power series in t, because of the negative power of t that occurs in Y_1]. We thus obtain a functional equation for $R(x)$:

$$R(x) + R(Y_0) = xY_0. \tag{6}$$

It is not hard to see that this equation—once restated in terms of $Q(x,0)$—defines uniquely $Q(x,0;t)$ as a formal power series in t with polynomial coefficients in x. Equation (6) is the standard result of the kernel method.

Still, we want to apply here the *obstinate* kernel method. That is, we shall not content ourselves with (6), but we shall go on producing pairs (X,Y) that cancel the kernel and use the information they provide on the series $R(x)$. This obstinacy was inspired by the book [13] by Fayolle, Iasnogorodski and Malyshev and, more precisely, by Section 2.4 of this book, where one possible way to obtain such pairs is described (even though the analytic context is different). We give here an alternative construction.

Let $(X,Y) \neq (0,0)$ be a pair of Laurent series in t with coefficients in some field such that $K(X,Y) = 0$. Recall that, as a function of y, the polynomial $K(x,y)$ is quadratic. Thus, let Y' be *the other solution* of the equation $K(X,y) = 0$. We define the function Ψ by $\Psi(X,Y) = (X,Y')$. For instance, if (X,Y) is the pair (x, Y_0), then $\Psi(X,Y) = (x, Y_1)$. Similarly, we define $\Phi(X,Y) = (X', Y)$, where X' is the other solution of $K(x,Y) = 0$. Note that Φ and Ψ are involutions and that, in view of (5), $X' = Y' = (XY)^{-1}$. In particular, $\Phi(x, Y_0) = (Y_1, Y_0)$. Let us examine the iterated action of Φ and Ψ on the pair (x, Y_0): We obtain the diagram of Figure 2.

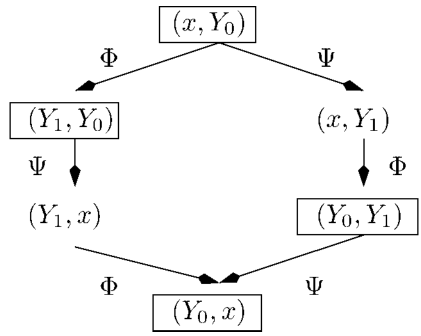

FIG. 2. *The orbit of* (x, Y_0) *under the action of* Φ *and* Ψ. *The framed pairs can be substituted for* (x, y) *in the functional equation.*

All these pairs of power series cancel the kernel. We have already seen that the pair (x, Y_0) can be substituted for (x, y) in (4). This is also true, but less obvious, for the pair (Y_0, Y_1): indeed, if we write

$$\begin{aligned} Q(x, y; t) &= \sum_{k \geq \max(\ell, m)} t^{k+\ell+m} x^{k-\ell} y^{k-m} a_{k-\ell, k-m}(k+\ell+m), \\ &= \sum_{k \geq \max(\ell, m)} t^{k+2m} (xy)^{k-\ell} (ty)^{\ell-m} a_{k-\ell, k-m}(k+\ell+m), \end{aligned}$$

and note that $Y_0 Y_1 = \bar{x}$, while $tY_1 = \bar{x} + O(t)$, then we see that $Q(Y_0, Y_1; t)$ is a well-defined power series in t, with coefficients in $\mathbb{Q}[x, \bar{x}]$. The same argument shows that $R(Y_1) = tY_1 Q(0, Y_1; t)$ is also well defined. Thus, the two pairs than can be substituted for (x, y) in the functional equation give us *two* equations for the unknown series $R(x)$:

$$\begin{aligned} R(x) + R(Y_0) &= xY_0, \\ R(Y_0) + R(Y_1) &= Y_0 Y_1 = \bar{x}. \end{aligned} \tag{7}$$

REMARK. Let p, q, r be three nonnegative numbers such that $p + q + r = 1$. Take $x = (pr)^{1/3} q^{-2/3}$, $y = (qr)^{1/3} p^{-2/3}$, and $t = (pqr)^{1/3}$. Then $K(x, y; t) = 0$, so that $R(x) + R(y) = xy$. This equation can be given a probabilistic interpretation by considering random walks that make a North-East step with (small) probability r and a West (resp. South) step with probability p (resp. q). This probabilistic argument, and the equation it implies, is the starting point in Gessel's solution of Kreweras problem ([19], equation (21)).

2.2. *Symmetric functions of* Y_0 *and* Y_1. After the kernel method, the next tool in our approach is the extraction of the positive part of a power series, defined by (2). This is where the values of the symmetric functions of Y_0 and Y_1 become crucial: the fact that they only involve negative powers of x [see (5)] will simplify the extraction of the positive part of certain equations.

LEMMA 3. *Let $F(u, v; t)$ be a Laurent series in t with coefficients in $\mathbb{C}[u, v]$, symmetric in u and v. That is, $F(u, v; t) = F(v, u; t)$. Then the series $F(Y_0, Y_1; t)$, if well defined, is a Laurent series in t with polynomial coefficients in $\bar{x}$. Moreover, the constant term of this series, taken with respect to $\bar{x}$, is $F(0, 0; t)$.*

PROOF. By linearity, it suffices to check this when F is simply a symmetric polynomial in u and v. But then it is a polynomial in $u + v$ and uv with complex coefficients. The result follows thanks to (5). $\square$

We now want to form a symmetric function of Y_0 and Y_1, starting from the equations of (7). The first one reads

$$R(Y_0) - xY_0 = -R(x).$$

By combining both equations, we obtain the companion expression

$$R(Y_1) - xY_1 = R(x) + 2\bar{x} - 1/t.$$

Taking the difference (an alternative derivation of Kreweras' result, obtained by considering the product $(R(Y_0) - xY_0)(R(Y_1) - xY_1)$, is presented in [5]) and dividing by $Y_0 - Y_1$ gives

$$\frac{R(Y_0) - R(Y_1)}{Y_0 - Y_1} - x = tx \frac{2R(x) + 2\bar{x} - 1/t}{\sqrt{\Delta(x)}}, \tag{8}$$

where $\Delta(x) = (1 - t\bar{x})^2 - 4t^2 x$ is the discriminant that occurs in both Y_0 and Y_1.

As a Laurent polynomial in x, $\Delta(x)$ has three roots. Two of them, say X_0 and X_1, are formal power series in $\sqrt{t}$; the other is a Laurent series in t (for generalities on the roots of a polynomial over $\mathbb{C}(t)$, see [32], Chapter 6). The coefficients of these series can be computed inductively:

$$X_0 = t + 2t^2\sqrt{t} + 6t^4 + 21t^5\sqrt{t} + 80t^7 + \tfrac{1287}{4}t^8\sqrt{t} + \cdots,$$
$$X_1 = t - 2t^2\sqrt{t} + 6t^4 - 21t^5\sqrt{t} + 80t^7 - \tfrac{1287}{4}t^8\sqrt{t} + \cdots,$$
$$X_2 = \frac{1}{4t^2} - 2t - 12t^4 - 160t^7 - 2688t^{10} - 50688t^{13} + \cdots.$$

Hence, $\Delta(x)$ factors as

$$\Delta(x) = \Delta_0 \Delta_+(x) \Delta_-(\bar{x})$$

with

$$\Delta_0 = 4t^2 X_2, \qquad \Delta_+(x) = 1 - x/X_2, \qquad \Delta_-(\bar{x}) = (1 - \bar{x}X_0)(1 - \bar{x}X_1).$$

Note that $\Delta_0, \Delta_+(x)$ and $\Delta_-(\bar{x})$ are power series in t with constant term 1. Moreover, Δ_0 has its coefficients in $\mathbb{Q}$, while $\Delta_+(x)$ has its coefficients in $\mathbb{Q}[x]$, and $\Delta_-(\bar{x})$ has its coefficients in $\mathbb{Q}[\bar{x}]$. This is an instance of the "canonical factorization" of power series of $\mathbb{Q}[x, \bar{x}][[t]]$, which has already proved useful in several path enumeration problems [4, 9, 18]. Going back to (8), and multiplying through by $\sqrt{\Delta_-(\bar{x})}$, one obtains

$$\sqrt{\Delta_-(\bar{x})}\left(\frac{R(Y_0)-R(Y_1)}{Y_0-Y_1}-x\right)=t\,\frac{2xR(x)+2-x/t}{\sqrt{\Delta_0\Delta_+(x)}}.$$

Both sides of this identity are power series in t with coefficients in $\mathbb{Q}[x, \bar{x}]$. But the right-hand side only contains nonnegative powers of x, while the left-hand side, except for a term $-x$, only contains nonpositive powers of x (in view of Lemma 3). Extracting the positive part of the above equation thus gives

$$-x=\frac{t}{\sqrt{\Delta_0}}\left(\frac{2xR(x)+2-x/t}{\sqrt{\Delta_+(x)}}-2\right).$$

The expression of $Q(x, 0)$ announced in Theorem 1 follows, given that $X_2 = 1/W^2$ and $R(x) = xtQ(x, 0)$. The expansion of $Q(x, 0)$ in x is straightforward, using $1 - \sqrt{1-4t} = 2t\sum_{n\ge 0} C_n t^n$. The value of $a_{i,0}(3n + 2i)$ follows using the Lagrange inversion formula ([32], page 38).

2.3. *The algebraic kernel method.* We present in this section another proof of Theorem 1 based on a variation of the kernel method. This variation does not require us to cancel the kernel, but, instead, builds on one of its algebraic properties. This variant has some drawbacks—since the kernel is not zero, we are handling bigger equations—but it also has some advantages. In particular, we obtain at some point an equation that is the counterpart of (8), but in which it is obvious that the left-hand side is nonpositive in x. This will be helpful in the next section, where we handle analytic functions rather than power series. Finally, this variant of the kernel method provides a proof of Theorem 2.

Let us return to the original equation (4), or, equivalently, to

$$xyK_r(x, y)Q(x, y) = xy - R(x) - R(y),$$

where $K_r(x, y) = 1 - t(\bar{x} + \bar{y} + xy)$ is the *rational version* of the kernel K. The fact that the diagram of Figure 2 is nice actually stems from an invariance property of K_r:

$$K_r(x, y) = K_r(\bar{x}\bar{y}, y) = K_r(x, \bar{x}\bar{y}) \equiv K_r.$$

Applying iteratively the (involutive) transformations $\Phi : (x, y) \mapsto (\bar{x}\bar{y}, y)$ and $\Psi : (x, y) \mapsto (x, \bar{x}\bar{y})$ gives the set of pairs of Figure 2A, on which K_r takes the same value. Note that Figure 2A specializes to Figure 2 when $y = Y_0$.

Now, *all* pairs of the above diagram can be substituted for (x, y) in the functional equation: the resulting series are power series in t with coefficients

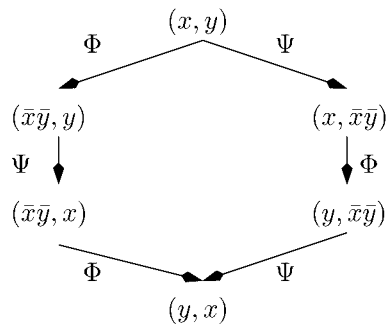

FIG. 2A.

in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$. This gives no less than *three* equations:

$$\begin{aligned}
xyK_r Q(x, y) &= xy - R(x) - R(y),\\
\bar{x}K_r Q(\bar{x}\bar{y}, y) &= \bar{x} - R(\bar{x}\bar{y}) - R(y),\\
\bar{y}K_r Q(x, \bar{x}\bar{y}) &= \bar{y} - R(x) - R(\bar{x}\bar{y}).
\end{aligned}$$

We sum the first and third equations, and subtract the second one, so as to keep $R(x)$ as the only unknown function on the right-hand side:

$$\begin{aligned}
K_r\big(xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + \bar{y}Q(x, \bar{x}\bar{y})\big) &= xy - \bar{x} + \bar{y} - 2R(x)\\
&= \frac{1 - K_r}{t} - 2\bar{x} - 2R(x).
\end{aligned}$$

Equivalently,

$$xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + \bar{y}Q(x, \bar{x}\bar{y}) + \frac{1}{t} = \frac{1}{K_r}\left(\frac{1}{t} - 2\bar{x} - 2R(x)\right). \tag{9}$$

The kernel $K(x, y)$ factors as $-tx^2(y - Y_0)(y - Y_1)$. Converting $1/K$ into partial fractions of y yields the following expression for the reciprocal of the (rational) kernel K_r:

$$\begin{aligned}
\frac{1}{K_r} &= \frac{1}{\sqrt{\Delta(x)}}\left(\frac{1}{1 - \bar{y}Y_0} + \frac{1}{1 - y/Y_1} - 1\right)\\
&= \frac{1}{\sqrt{\Delta(x)}}\left(\sum_{n\geq 0} \bar{y}^n Y_0^n + \sum_{n\geq 1} y^n Y_1^{-n}\right).
\end{aligned}$$

Note that this expansion is valid in the set of formal power series in t with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$. Let us extract in (9) the constant term in y: the series $xyQ(x, y)$ and $\bar{y}Q(x, \bar{x}\bar{y})$ do not contribute, and we obtain

$$-\bar{x}Q_d(\bar{x}) + \frac{1}{t} = \frac{1/t - 2\bar{x} - 2R(x)}{\sqrt{\Delta(x)}},$$

where the series Q_d is the diagonal of $Q(x, y)$, and counts walks ending on the diagonal. The above equation should be compared to (8): basically, both equations are equivalent, but their negative parts on the left-hand side are written in two different ways. We now proceed as above, using the canonical factorization of $\Delta(x)$, which gives

$$\sqrt{\Delta_-(\bar{x})}\left(\frac{1}{t} - \bar{x}Q_d(\bar{x})\right) = \frac{1/t - 2\bar{x} - 2R(x)}{\sqrt{\Delta_0\Delta_+(x)}}.$$

Extracting the nonnegative part gives, as before, the value of $R(x)$, and Theorem 1. Extracting the negative part gives

$$\sqrt{\Delta_-(\bar{x})}\left(\frac{1}{t} - \bar{x}Q_d(\bar{x})\right) - \frac{1}{t} = -\frac{2\bar{x}}{\sqrt{\Delta_0}}.$$

Recall that $\Delta_0 = 4t^2X_2 = 4t^2/W^2$ and $\Delta_-(\bar{x}) = (1 - \bar{x}X_0)(1 - \bar{x}X_1)$, where X_0 and X_1 are the two "small" roots of $\Delta(x)$. We can express their elementary symmetric functions in terms of the third root, $X_2 = 1/W^2$. This gives

$$\Delta_-(\bar{x}) = 1 - \bar{x}W(1 + W^3/4) + \bar{x}^2W^2/4,$$

and this provides the expression of $Q_d(x)$ given in Theorem 2.

3. Probability: a Markov chain and its stationary distribution. We consider a Markov chain on the quadrant, whose transition probabilities $T(i, j; k, \ell)$ are schematized in Figure 3. More precisely, for $i > 0$ and $j > 0$, the probability

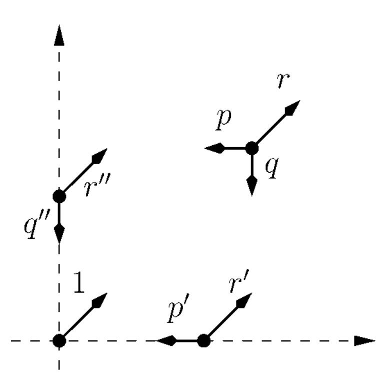

FIG. 3. *The transition probabilities.*

of going from (i, j) to (k, ℓ) is

$$T(i, j; k, \ell) = \begin{cases} p, & \text{if } k = i-1 \text{ and } \ell = j, \\ q, & \text{if } k = i \text{ and } \ell = j-1, \\ r, & \text{if } k = i+1 \text{ and } \ell = j+1, \end{cases}$$

where p, q, r are three positive real numbers summing to 1. When the point (i, j) lies on the border of the quadrant, the transition probabilities are modified as follows: for $i > 0$,

$$T(i, 0; k, \ell) = \begin{cases} p', & \text{if } k = i-1 \text{ and } \ell = 0, \\ r', & \text{if } k = i+1 \text{ and } \ell = 1, \end{cases}$$

and for $j > 0$,

$$T(0, j; k, \ell) = \begin{cases} q'', & \text{if } k = 0 \text{ and } \ell = j-1, \\ r'', & \text{if } k = 1 \text{ and } \ell = j+1, \end{cases}$$

where p', r', q'', r'' are positive numbers such that $p' + r' = q'' + r'' = 1$. Finally, we take $T(0, 0; 1, 1) = 1$. Note that this chain is irreducible (all states communicate) and has period 3.

A probability distribution $(p_{i,j})_{i,j \geq 0}$ is *stationary* for the above transition if for all $k, \ell \geq 0$,

$$p_{k,\ell} = \sum_{i,j} p_{i,j} T(i, j; k, \ell).$$

Our objective is to find the stationary distribution of the above transition, when it exists. It is customary to encode a distribution by its probability generating function

$$\Pi(x, y) = \sum_{i,j \geq 0} p_{i,j} x^i y^j,$$

but is is more convenient here to split $\Pi(x, y)$ into four parts: first, $p_{0,0}$, and then the three following generating functions:

$$P(x, y) = \sum_{i,j \geq 1} p_{i,j} x^i y^j, \qquad P_1(x) = \sum_{i \geq 1} p_{i,0} x^i, \qquad P_2(y) = \sum_{j \geq 1} p_{0,j} y^j.$$

Then the distribution $(p_{i,j})_{i,j \geq 0}$ is stationary if and only if

$$\begin{aligned} &(1 - p\bar{x} - q\bar{y} - rxy) P(x, y) + (1 - p'\bar{x} - r'xy) P_1(x) \\ &\quad + (1 - q''\bar{y} - r''xy) P_2(y) + (1 - xy) p_{0,0} = 0. \end{aligned} \tag{10}$$

Note that the numbers $p_{i,j}$ have to sum to 1: hence, the above series are absolutely convergent for $|x| \leq 1$ and $|y| \leq 1$, and define analytic functions for $|x| < 1$,

$|y| < 1$. Moreover,

$$p_{0,0} + P_1(1) + P_2(1) + P(1,1) = 1. \tag{11}$$

3.1. *The main results.* The stationary distribution of this Markov chain was computed in [13] in the case where the transition probabilities are related by

$$\frac{p}{r} = \frac{p'}{r'} := \alpha \quad \text{and} \quad \frac{q}{r} = \frac{q''}{r''} := \beta. \tag{12}$$

Equivalently,

$$p' = \frac{p}{p+r}, \qquad r' = \frac{r}{p+r}, \qquad q'' = \frac{q}{q+r}, \qquad r'' = \frac{r}{q+r}.$$

It is known that this chain has a stationary distribution if and only if $r < \min(p, q)$ (see [14, 27] for general results on Markov chains in the quadrant). It will be shown in Lemma 6 that the condition is necessary, and it will follow from the results of Section 4, where we compute the law of the chain, that it is sufficient. For the moment, we rely on the general results of [14, 27].

Under the conditions of (12), which we assume to hold in this section, we have

$$\begin{aligned} 1 - p'\bar{x} - r'xy &= \frac{r'}{r}\big(1 - p\bar{x} - q\bar{y} - rxy + q(\bar{y} - 1)\big), \\ 1 - q''\bar{y} - r''xy &= \frac{r''}{r}\big(1 - p\bar{x} - q\bar{y} - rxy + p(\bar{x} - 1)\big), \\ 1 - xy &= \frac{1}{r}\big(1 - p\bar{x} - q\bar{y} - rxy + p(\bar{x} - 1) + q(\bar{y} - 1)\big) \end{aligned}$$

so that the functional equation (10) can be nicely rewritten as

$$(1 - p\bar{x} - q\bar{y} - rxy)Q(x, y) = q(1 - \bar{y})Q(x, 0) + p(1 - \bar{x})Q(0, y) \tag{13}$$

with

$$Q(x, y) = p_{0,0} + r'P_1(x) + r''P_2(y) + rP(x, y). \tag{14}$$

This equation was first met by Flatto and Hahn [17] in their study of a system of two parallel queues with two demands (with continuous time). They solved this equation using nontrivial complex analysis, multivalued analytic functions and a parametrization of the kernel by elliptic functions—to end up with an *algebraic* solution $Q(x, y)$. We shall rederive their result in a more elementary way, and state it in a more symmetric fashion.

THEOREM 4 (Solution of Flatto and Hahn's equation). *Assume* $r < \min(p, q)$. *There exists*, *up to a multiplicative constant*, *a unique solution of* (13) *that is* *analytic in* $\{|x|, |y| < 1\}$ *and whose series expansion converges for* $|x|, |y| \le 1$. *This solution satisfies*

$$\begin{aligned} Q(x,0) &= \frac{Q(0,0)}{(1-qx/p)(1-rx/p)} \\ &\quad \times \left(\left(1-\frac{x}{pw}\right)\sqrt{1-xqrw^2} - \frac{qx}{p}\left(1-\frac{1}{qw}\right)\sqrt{1-prw^2}\right), \\ Q(0,y) &= \frac{Q(0,0)}{(1-py/q)(1-ry/q)} \\ &\quad \times \left(\left(1-\frac{y}{qw}\right)\sqrt{1-yprw^2} - \frac{py}{q}\left(1-\frac{1}{pw}\right)\sqrt{1-qrw^2}\right), \end{aligned}$$

where w *is the smallest positive solution of* $w = 2 + pqrw^3$. *The complete generating function* $Q(x, y)$ *can be obtained using* (13). *When* $p = q$, *then* $w = 1/p$, *and the above expressions simplify to*

$$Q(x,0) = Q(0,x) = \frac{Q(0,0)}{\sqrt{1-rx/p}}.$$

If $r \ge \min(p, q)$, *no solution of* (13) *converges on* $|x|, |y| \le 1$.

Observe that exchanging p and q, and x and y, leaves the solution unchanged, in conformity with the diagonal symmetry of the model. From the algebraic equation defining w, it is not difficult to see that

$$(pw-1)^2(1-qrw^2) = (qw-1)^2(1-prw^2) = (rw-1)^2(1-pqw^2).$$

Moreover, an elementary study of the function $f(z) = pqrz^3 - z + 2$ gives bounds for w:

$$\frac{1}{M} \le w \le \frac{1}{\sqrt{pq}} \le \frac{1}{m} < \frac{1}{r}, \tag{15}$$

where $m = \min(p, q)$ and $M = \max(p, q)$, with equalities holding if and only if $p = q$. Hence,

$$\begin{aligned} 0 &\le (Mw-1)\sqrt{1-mrw^2} \\ &= -(mw-1)\sqrt{1-Mrw^2} = -(rw-1)\sqrt{1-pqw^2}, \end{aligned} \tag{16}$$

and this allows us to rewrite the second part of the expressions of $Q(x, 0)$ and $Q(0, y)$ in various ways. This will be useful in Section 3.3, where we make further comments on this solution, and relate it to Flatto and Hahn's formulation. We shall prove Theorem 4 in Section 3.2. For the moment, let us derive from it the stationary distribution of the Markov chain of Figure 3. This result is actually not given explicitly in [13].

COROLLARY 5 (The stationary distribution). *Assume* $r < \min(p, q)$. *Let* w *be the smallest positive solution of* $w = 2 + pqrw^3$. *The Markov chain schematized in Figure* 3, *with the additional condition* (12), *has a unique stationary distribution* $(p_{i,j})$, *given by*

$$p_{0,0} = \frac{w(p-r)(q-r)|p-q|}{6pq(1-rw)\sqrt{1-pqw^2}},$$

$$\sum_{i>0} p_{i,0}x^i = \frac{1}{r'}\big(Q(x,0) - Q(0,0)\big),$$

$$\sum_{j>0} p_{0,j}y^j = \frac{1}{r''}\big(Q(0,y) - Q(0,0)\big),$$

$$\sum_{i>0,j>0} p_{i,j}x^i y^j = \frac{1}{r}\big(Q(x,y) - Q(x,0) - Q(0,y) + Q(0,0)\big),$$

where $Q(x, y)$ *is the function of Theorem* 4, *taken with* $Q(0, 0) = p_{0,0}$. *When* $p = q$, *then the expression of* $p_{0,0}$ *should be taken to be*

$$p_{0,0} = \tfrac{1}{3}(1 - r/p)^{3/2}.$$

If $r \geq \min(p, q)$, *then the Markov chain has no stationary distribution.*

PROOF. The stationary distribution is related to the series $Q(x, y)$ satisfying (13) by (14), so that the expressions of the three series above are obvious. We only have to determine which value of $Q(0, 0)$ guarantees the normalizing condition (11). This condition reads

$$Q(0,0) + \frac{1}{r'}\big(Q(1,0) - Q(0,0)\big) + \frac{1}{r''}\big(Q(0,1) - Q(0,0)\big)$$
$$+ \frac{1}{r}\big(Q(1,1) - Q(1,0) - Q(0,1) + Q(0,0)\big) = 1,$$

that is,

$$Q(1,1) - qQ(1,0) - pQ(0,1) = r. \tag{17}$$

When $y = 1$, a factor $(1 - \bar{x})$ comes out of (13), leaving

$$(p - rx)Q(x,1) = pQ(0,1). \tag{18}$$

Setting $x = 1$ gives

$$Q(1,1) = \frac{p}{p-r}Q(0,1),$$

and, of course, a symmetric argument yields

$$Q(1,1) = \frac{q}{q-r} Q(1,0).$$

Hence, the normalizing condition (17) reads

$$Q(1,0) = \frac{q-r}{3q}. \tag{19}$$

Now, from the expression of $Q(x,0)$ given in Theorem 4, we obtain, if $p \neq q$,

$$Q(1,0) = \frac{pQ(0,0)}{w(p-q)(p-r)}\big((pw-1)\sqrt{1-qrw^2} - (qw-1)\sqrt{1-prw^2}\big).$$

Using (16), the above expression for $Q(1,0)$ can be rewritten as

$$Q(1,0) = \frac{2pQ(0,0)(1-rw)\sqrt{1-pqw^2}}{w|p-q|(p-r)},$$

and the condition (19) gives the value of $Q(0,0) = p_{0,0}$.

When $p = q$, the simplified expression of $Q(x,0)$, given in Theorem 4, gives $Q(1,0) = Q(0,0)/\sqrt{1-r/p}$, and the result follows. □

3.2. *Proof of Theorem* 4. Our solution of (13) follows the same idea as Section 2.3: we shall exploit an invariance property of the kernel. However, we do not have the length variable t any more, which means that we are no longer in a power series context, but rather in the world of functions of two complex variables x and y. Consequently, certain operations that were performed formally in Section 2.3 (e.g., the extraction of coefficients) now need to be justified analytically. The analytic lemmas we need are gathered in Section 3.2.1. In Section 3.2.2, our main functional equation (13) is transformed into *two* functional equations defining two functions $U(x,y)$ and $F(x,y)$, which are respectively symmetric and anti-symmetric in x and y. Section 3.2.3 is the heart of the proof: there we solve these two equations, using the algebraic kernel method. The reader may skip directly to the latter section in order to recognize the logic of the kernel method, transferred to an analytic context.

3.2.1. *Preliminary results.* Our first lemma tells us that the domain of convergence of $Q(x,y)$ is actually larger than the unit polydisc $|x| \leq 1$, $|y| \leq 1$.

LEMMA 6. *Let $Q(x,y)$ be a power series solution of* (13) *that has nonnegative coefficients and converges for $|x| \leq 1$ and $|y| \leq 1$. Then this series also converges in the following domain*:

$$\{|x| < p/r, |y| \leq 1\} \cup \{|x| \leq 1, |y| < q/r\}.$$

Moreover,

$$Q(x,1)=\frac{Q(0,1)}{1-rx/p} \quad \textit{and} \quad Q(1,y)=\frac{Q(1,0)}{1-ry/q}.$$

In particular, such a solution $Q(x,y)$ can only exist if $r<\min(p,q)$: if this inequality does not hold, the Markov chain has no stationary distribution.

PROOF. For $|x|\le 1$, the expression of $Q(x,1)$ follows from (18). The expression of $Q(1,y)$ is, of course, symmetric. These two series must converge when $x=1$ and $y=1$: this forces r to be smaller than p and q. Given the nonnegativity of the coefficients, the values of $Q(x,1)$ and $Q(1,y)$ imply the convergence of $Q(x,y)$ in the desired domain. $\square$

The extraction of coefficients will be based on the following result ([30], Chapter 10, Exercise 25).

PROPOSITION 7. *Let $f(z)$ be an analytic function in the annulus $\mathcal{A}=\{r<|z|<R\}$. There exists a unique bi-infinite sequence $(a_n)_{n\in\mathbb{Z}}$ such that for all $z\in\mathcal{A}$,*

$$f(z)=\sum_{n\in\mathbb{Z}} a_n z^n.$$

Moreover, the convergence is absolute. In other words, f is the sum of a function analytic in the disk $\{|z|<R\}$ and a function analytic for $|z|>r$.

It will be convenient to work with an equation whose kernel is symmetric in x and y. In (13), let us replace x by px and y by qy. Multiplying through by xy gives

$$(20)\qquad \begin{aligned}&(xy-x-y-pqrx^2y^2)Q(px,qy)\\&\qquad +x(1-qy)Q(px,0)+y(1-px)Q(0,qy)=0.\end{aligned}$$

Let $K(x,y)=xy-x-y-pqrx^2y^2$ denote the kernel of this equation. The discriminant of K, taken as a polynomial in y, is $(x-1)^2-4pqrx^3$. Let us denote

$$\Delta(x)=(1-\bar{x})^2-4pqrx.$$

LEMMA 8. *Assume $r<\min(p,q)$. Let $m=\min(p,q)$ and $M=\max(p,q)$. The three roots of $\Delta(x)$, denoted x_i, $i=0,1,2$, are real and satisfy*

$$0<x_0<1<x_1<\frac{1}{M}\le\frac{1}{m}<\frac{1}{r}\le x_2.$$

PROOF. The variations of $\Delta(x)$ are easy to study. Note that $\Delta(x)$ is a square when $x = 1/p, 1/q$ or $1/r$. $\square$

Recall that w is defined as the smallest positive solution of $w = 2 + pqrw^3$. This implies that $1/(pqrw^2) = x_2$. The lemma above gives the following factorization of Δ, which will play the role of the *canonical factorization* of Section 2:

$$\Delta(x) = \Delta_0 \Delta_+(x) \Delta_-(\bar{x}) \tag{21}$$

with

$$\begin{aligned} \Delta_0 &= 4pqrx_2 = 4/w^2, \\ \Delta_+(x) &= 1 - x/x_2 = 1 - pqrw^2 x, \\ \Delta_-(\bar{x}) &= (1 - \bar{x}x_0)(1 - \bar{x}x_1). \end{aligned} \tag{22}$$

As a polynomial in y, the kernel $K(x, y)$ of (20) has two roots:

$$Y_0(x) = \frac{1 - \bar{x} - \sqrt{\Delta(x)}}{2pqrx}, \qquad Y_1(x) = \frac{1 - \bar{x} + \sqrt{\Delta(x)}}{2pqrx}.$$

The elementary symmetric functions of the Y_i are polynomials in $\bar{x} = 1/x$:

$$Y_0 + Y_1 = \frac{\bar{x}(1 - \bar{x})}{pqr} \quad \text{and} \quad Y_0 Y_1 = \frac{\bar{x}}{pqr}.$$

Using the canonical factorization of Δ, we see that Y_0 and Y_1 are at least analytic in the annulus $x_1 < |x| < x_2$. Let us study these functions a bit more precisely when x is real. The following lemma is illustrated by Figure 4.

LEMMA 9. *We still assume that $r < \min(p, q)$. The functions Y_0 and Y_1 are well defined and real for $x \in (-\infty, x_0] \cup [x_1, x_2]$. In particular,*

$$\begin{aligned} Y_0(1/M) &= 1/m, \qquad Y_0(1/m) = Y_0(1/r) = 1/M, \\ Y_1(1/M) &= Y_1(1/m) = 1/r, \qquad Y_1(1/r) = 1/m. \end{aligned}$$

Each of the derivatives $Y_0'(x)$ and $Y_1'(x)$ admits a unique zero on the interval $[x_1, x_2]$, respectively denoted by v_2 and v_1. Moreover,

$$x_1 < \frac{1}{M} \le v_1 = w \le \frac{1}{m} < v_2 < \frac{1}{r} \le x_2.$$

The function Y_0 decreases between x_1 and v_2, and increases between v_2 and x_2, while the function Y_1 increases between x_1 and v_1, and then decreases up to x_2.

Finally, for $x \in (1/m, x_2)$, one has

$$\frac{1}{pqx} < Y_0(x).$$

PROOF. The proof is a bit tedious, but elementary. We merely sketch the different steps.

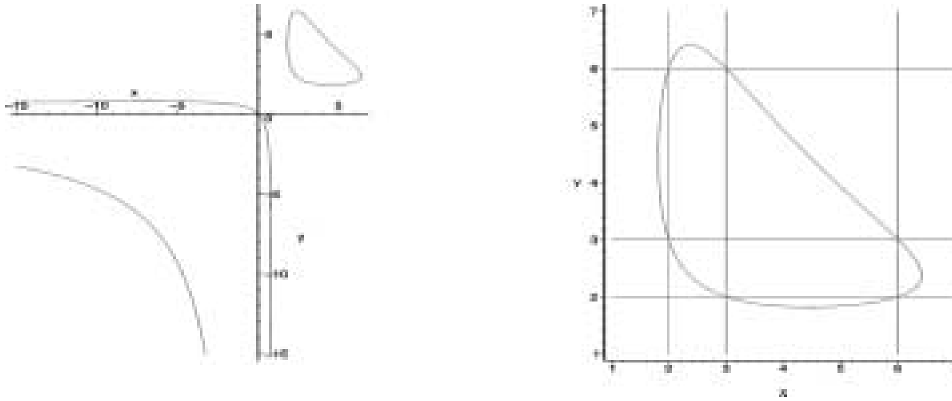

FIG. 4. *The real branches of the functions Y_i, for $p = 1/3, q = 1/2$ and $r = 1/6$.*

The first assertion comes from the study of the discriminant $\Delta(x)$ (Lemma 8). The values of Y_i at the points $1/M, 1/m$ and $1/r$ are obtained by a direct calculation.

Let us now focus on the interval $[x_1, x_2]$. Given that $(1-\bar{x})Y_i = 1 + pqrxY_i^2$ and $x \geq x_1 > 1$, we have $0 < Y_0(x) \leq Y_1(x)$. The derivatives of Y_0 and Y_1 with respect to x can be written

$$Y_0'(x) = \frac{1}{x\sqrt{\Delta}}\left((x-2)\frac{Y_0}{x} - 1\right), \qquad Y_1'(x) = -\frac{1}{x\sqrt{\Delta}}\left((x-2)\frac{Y_1}{x} - 1\right).$$

Given that Y_0 and Y_1 are positive on $[x_1, x_2]$, any root of these derivatives will be larger than 2. The equation satisfied by the Y_i implies that these roots are also solutions of $x - 2 = pqrx^3$. The polynomial $pqrz^3 - z + 2$ has two roots larger than 2. Let us denote them v_1 and v_2, with $v_1 < v_2$. Note that v_1 is actually the number w defined in Theorem 4. If $x = v_i$, then $\Delta(x) = (x-3)^2$. Hence, v_i belongs to the interval $[x_1, x_2]$. Evaluating the numerators of Y_0' and Y_1' at v_1 and v_2 shows that v_1 cancels Y_1', while v_2 cancels Y_0'. Finally, we compute the numerators of Y_0' and Y_1' at x_1 and x_2. This determines the sign of these derivatives and completes the study of the variations of Y_0 and Y_1.

The last assertion is proved by studying the function $y \mapsto K(x, y)$, for x fixed. $\square$

Let $x \in (x_1, x_2)$ and let y belong to the annulus $\{Y_0(x) < |y| < Y_1(x)\}$. Let $K_r = K/(xy) = 1 - \bar{x} - \bar{y} - pqrxy$ be the rational version of the kernel. Then the following expansion is convergent:

$$\begin{aligned} \frac{1}{K_r} &= \frac{1}{\sqrt{\Delta(x)}}\left(\frac{1}{1-\bar{y}Y_0} + \frac{1}{1-y/Y_1} - 1\right) \\ &= \frac{1}{\sqrt{\Delta(x)}}\left(\sum_{n\geq 0}\bar{y}^n Y_0^n + \sum_{n\geq 1} y^n Y_1^{-n}\right). \end{aligned} \tag{23}$$

Let $F(y)$ be analytic in the same annulus. Let us write

$$F(y) = \sum_{i \in \mathbb{Z}} f_i y^i = F^-(\bar{y}) + f_0 + F^+(y),$$

where F^+ and F^- are the positive and negative parts of F. Then

$$\begin{aligned} [y^0]\frac{F(y)}{K_r} &= \frac{1}{\sqrt{\Delta(x)}}\big(F^-(1/Y_1) + f_0 + F^+(Y_0)\big) \\ &= \frac{1}{\sqrt{\Delta(x)}}\big(F^-(pqrxY_0) + f_0 + F^+(Y_0)\big). \end{aligned} \tag{24}$$

In particular, if $F(y) = G(y) - G(\bar{x}\bar{y}/(pqr))$, then

$$[y^0]\frac{F(y)}{K_r} = 0. \tag{25}$$

3.2.2. *Simplification of the functional equation.* Let us go back to (20). The function $\bar{Q}(x, y) = Q(px, qy)/((1 - px)(1 - qy))$ satisfies an equation that is symmetric in x and y:

$$(xy - x - y - pqrx^2y^2)\bar{Q}(x, y) + x\bar{Q}(x, 0) + y\bar{Q}(0, y) = 0. \tag{26}$$

Yet we shall see that this function is, in general, not symmetric in x and y (Section 3.3). We shall symmetrize it by considering $\bar{Q}(x, y) + \bar{Q}(y, x)$. More precisely, we shall study separately the functions $S(x, y)$ and $D(x, y)$ (as in Sum and Difference) defined by

$$\begin{aligned} S(x, y) &= (1 - qx)(1 - py)Q(px, qy) + (1 - px)(1 - qy)Q(py, qx), \\ D(x, y) &= (1 - qx)(1 - py)Q(px, qy) - (1 - px)(1 - qy)Q(py, qx). \end{aligned} \tag{27}$$

By Lemma 6, these functions are analytic in

$$\mathcal{D} = \left\{|x| < \frac{1}{r}, |y| < \frac{1}{M}\right\} \cup \left\{|x| < \frac{1}{M}, |y| < \frac{1}{r}\right\} \tag{28}$$

with $M = \max(p, q)$. Note that $S(x, y)$ and $D(x, y)$ satisfy *the same equation*:

$$\begin{aligned} &S(x, y)(xy - x - y - pqrx^2y^2) \\ &\qquad + x(1 - py)(1 - qy)S(x, 0) + y(1 - px)(1 - qx)S(0, y) = 0, \\ &D(x, y)(xy - x - y - pqrx^2y^2) \\ &\qquad + x(1 - py)(1 - qy)D(x, 0) + y(1 - px)(1 - qx)D(0, y) = 0. \end{aligned} \tag{29}$$

However, $S(x, y)$ is symmetric in x and y, while $D(x, y) = -D(y, x)$. We shall solve separately the two equations, taking into account the respective symmetry or anti-symmetry condition. For each equation, we will obtain a unique solution, up to a multiplicative factor. A relation between the two factors will be found by noticing that, by definition of S and D,

$$S\left(\frac{1}{q},0\right)+D\left(\frac{1}{q},0\right)=0. \tag{30}$$

First we rewrite the above equations on S and D, by expressing $x(1-py)\times(1-qy)$ in terms of the kernel: indeed,

$$rx(1-py)(1-qy)=(r-\bar{x})\big(x+y-xy(p+q)\big)-\bar{x}K(x,y).$$

Let $K_r=K/(xy)=1-\bar{x}-\bar{y}-pqrxy$. The equation satisfied by S can be rewritten as

$$\begin{aligned}K_r\frac{xyrS(x,y)-yS(x,0)-xS(0,y)}{x+y-xy(p+q)}&=(\bar{x}-r)S(x,0)+(\bar{y}-r)S(0,y)\\&=\bar{x}T(x)+\bar{y}T(y)\end{aligned} \tag{31}$$

with

$$T(x)=(1-rx)S(x,0)=(1-rx)S(0,x). \tag{32}$$

Similarly,

$$\begin{aligned}K_r\frac{xyrD(x,y)-yD(x,0)-xD(0,y)}{x+y-xy(p+q)}&=(\bar{x}-r)D(x,0)+(\bar{y}-r)D(0,y)\\&=E(x)-E(y),\end{aligned}$$

where

$$E(x)=(\bar{x}-r)D(x,0)=-(\bar{x}-r)D(0,x). \tag{33}$$

We have taken into account the fact that $D(x,y)$ is anti-symmetric, so that, in particular, $D(0,0)=0$. The functions T and E are analytic for $|x|<1/r$.

From the fact that the curve $x+y=xy(p+q)$ intersects the domain of convergence of $S(x,y)$ (near the origin), we derive from (31) the existence of a function $U(x,y)$, analytic in $\mathcal{D}$, such that

$$xyrS(x,y)-yS(x,0)-xS(0,y)=\big(x+y-xy(p+q)\big)U(x,y).$$

A similar statement holds for the function $xyrD(x,y)-yD(x,0)-xD(0,y)$. But this function vanishes as soon as $x=0$ or $y=0$, so that we can actually write

$$xyrD(x,y)-yD(x,0)-xD(0,y)=xy\big(x+y-xy(p+q)\big)F(x,y),$$

for a function F that is analytic is $\mathcal{D}$. Finally, we shall need the following initial conditions:

$$U(x,0)=-S(0,0)=-2Q(0,0),\qquad T(0)=2Q(0,0). \tag{34}$$

Equations (29) have thus been replaced by the following simpler equations:

$$K_r U(x, y) = \bar{x}T(x) + \bar{y}T(y), \tag{35}$$

$$K_r xyF(x, y) = E(x) - E(y), \tag{36}$$

where $K_r = 1 - \bar{x} - \bar{y} - pqrxy$ and U, T, F, E are analytic in the domain $\mathcal{D}$ defined by (28).

3.2.3. *The algebraic kernel method.* We apply the algebraic kernel method of Section 2.3 to (35)–(36). We observe that K_r satisfies the following invariance condition:

$$K_r(x, y) = K_r\left(\frac{\bar{x}\bar{y}}{pqr}, y\right) = K_r\left(x, \frac{\bar{x}\bar{y}}{pqr}\right) \equiv K_r.$$

Recall that $m = \min(p, q)$ and $M = \max(p, q)$. Let us fix x in the interval $(1/m, 1/r)$ and restrict y to the annulus

$$Y_0(x) < |y| < \frac{1}{M}. \tag{37}$$

By Lemma 9, this annulus is nonempty, and, moreover,

$$\frac{1}{pqx} < |y| < Y_1(x). \tag{38}$$

The pairs (x, y) and $(\bar{x}\bar{y}/(pqr), y)$ both belong to the domain of convergence $\mathcal{D}$, and we thus have, in addition to (35)–(36),

$$K_r U\left(\frac{\bar{x}\bar{y}}{pqr}, y\right) = pqrxyT\left(\frac{\bar{x}\bar{y}}{pqr}\right) + \bar{y}T(y), \tag{39}$$

$$K_r \frac{\bar{x}}{pqr} F\left(\frac{\bar{x}\bar{y}}{pqr}, y\right) = E\left(\frac{\bar{x}\bar{y}}{pqr}\right) - E(y). \tag{40}$$

A linear combination of (35) and (39) gives

$$2U(x, y) - U\left(\frac{\bar{x}\bar{y}}{pqr}, y\right) = \frac{1}{K_r}\left(2\bar{x}T(x) + \bar{y}T(y) - pqrxyT\left(\frac{\bar{x}\bar{y}}{pqr}\right)\right).$$

In view of (37)–(38), the expansion of $1/K_r$ given by (23) is convergent. Recall that x is fixed; we can now use (25) to extract from the above equation the coefficient of y^0. We obtain

$$2U(x, 0) - U_d\left(\frac{\bar{x}}{pqr}\right) = \frac{2\bar{x}T(x)}{\sqrt{\Delta(x)}}, \tag{41}$$

where $U_d(x)$ denotes the diagonal of the series $U(x, y)$:

$$U(x, y) = \sum_{i,j\geq 0} u_{i,j}x^i y^j \quad \Longrightarrow \quad U_d(x) = \sum_{i\geq 0} u_{i,i}x^i.$$

Given that $U(x,y)$ converges absolutely in the domain $\mathcal{D}$ given by (28), the subseries $U_d(x)$ is convergent for $|x| < 1/(rM)$. By analytic continuation, (41) holds in the annulus $\{1/m < |x| < 1/r\}$. Recall that $U(x,0)$ is actually a constant [see (34)]. We now use the canonical factorization of $\Delta(x)$, given by (21). We multiply (41) by $\sqrt{\Delta_-(\bar{x})}$:

$$-\sqrt{\Delta_-(\bar{x})}\left(4Q(0,0) + U_d\left(\frac{\bar{x}}{pqr}\right)\right) = \frac{2\bar{x}T(x)}{\sqrt{\Delta_0\Delta_+(x)}}.$$

Using Proposition 7, we can extract the nonnegative part of this function. Given that $U_d(0) = U(0,0) = -2Q(0,0)$, we obtain, using (22),

$$T(x) = 2Q(0,0)\left(1 - \frac{x}{w}\right)\sqrt{\Delta_+(x)}. \tag{42}$$

In view of (31)–(32), we have completed the determination of the Sum function $S(x,y)$.

Let us now work with (36) and (40). Let us divide (36) by K_r and extract the coefficient of y^0. We obtain

$$E(x) = E(Y_0).$$

If we do the same with (40), we simply find $0 = 0$ [recall that $F(x,y)$ is antisymmetric]. However, if we extract instead the coefficient of y, we obtain, using (24),

$$\frac{\bar{x}}{pqr}F_1\left(\frac{\bar{x}}{pqr}\right) = E(0) - E(Y_0),$$

where

$$F(x,y) = \sum_{i,j\geq 0} f_{i,j}x^i y^j \quad \Longrightarrow \quad F_1(x) = \sum_{i\geq 0} f_{i,i+1}x^i.$$

By combining both equations, and extracting the nonnegative part, one sees that $E(x)$ is actually a constant $E(0)$. In view of (32)–(33) and (42), one has

$$S(x,0) = S(0,x) = 2\frac{Q(0,0)}{1-rx}\left(1 - \frac{x}{w}\right)\sqrt{\Delta_+(x)}$$

and

$$D(x,0) = -D(0,x) = \frac{xE(0)}{1-rx}.$$

The identity (30) completes the determination of E:

$$E(0) = -2Q(0,0)\left(q - \frac{1}{w}\right)\sqrt{\Delta_+(1/q)}.$$

We can now express $Q(x,0)$ and $Q(0,x)$ explicitly, using (27). Thanks to (16), this gives exactly Theorem 4.

3.3. *Comments on the solution.*

3.3.1. *Asymptotics.* For a good understanding of the solution of Theorem 4, or, equivalently, of the stationary distribution of Corollary 5, it is useful to determine the dominant singularities of the functions $Q(x,0)$ and $Q(0,y)$, and, hence, the asymptotic behavior of the numbers $p_{i,0}$ and $p_{0,j}$. This is why we briefly rederive below a result already proven in [17].

PROPOSITION 10. *Assume $r < \min(p,q)$. The asymptotic decay of the stationary probabilities $p_{i,0}$ depends on the relative values of p and q:*

- *If $p = q$, then $Q(x,0)$ is the reciprocal of a square root, and, as i goes to infinity,*

$$p_{i,0} \sim c(r/p)^i i^{-1/2}$$

for some positive constant c.

- *If $p < q$, then $Q(x,0)$ has a simple pole at p/r as its unique dominant singularity. The decay of the numbers $p_{i,0}$ is given by*

$$p_{i,0} \sim c(r/p)^i.$$

- *If $p > q$, then $Q(x,0)$ has a square root singularity at $1/(qrw^2)$ as its unique dominant singularity, and*

$$p_{i,0} \sim c(qrw^2)^i i^{-3/2}.$$

PROOF. We use standard results that relate the singularities of a series to the asymptotic behavior of its coefficients (see, e.g., [16]).

When $p = q$, the result is clear in view of Theorem 4. Otherwise, the three possible singularities of $Q(x,0)$ are p/q, p/r and $1/(qrw^2)$. The inequalities of (15) imply

$$\frac{p}{q} < \frac{p}{r} < \frac{1}{qrw^2}.$$

Hence, our first candidate for the radius of $Q(x,0)$ is p/q. However, the numerator of $Q(x,0)$ vanishes at this point, so that there is no pole at p/q. Our next candidate is p/r. For this value of x, the numerator of $Q(x,0)$ is

$$\left(1 - \frac{1}{rw}\right)\sqrt{1 - pqw^2} - \frac{q}{r}\left(1 - \frac{1}{qw}\right)\sqrt{1 - prw^2}.$$

According to (16), the first term in this difference is negative. If $p < q$, then the second term is positive. Hence, the difference is negative, and $Q(x,0)$ has, indeed, a simple pole at p/r.

However, if $p > q$, then (16) shows that the numerator of $Q(x,0)$ cancels at $x = p/r$, so that the only singularity of $Q(x,0)$ is a square root singularity at $1/(qrw^2)$.

Note that one can compute explicitly, in the same way, the multiplicative constants denoted c in the proposition. □

3.3.2. *An asymmetry of the solution.* As observed above, the function $\bar{Q}(x, y) = Q(px, qy)/((1-px)(1-qy))$ satisfies an equation that is symmetric in x and y [see (26)]. Hence, we could expect $\bar{Q}(x, y)$ to be a symmetric function of x and y. This is equivalent to the condition

$$\bar{Q}(x,0) - \bar{Q}(0,x) = \frac{Q(px,0)}{1-px} - \frac{Q(0,qx)}{1-qx} = 0.$$

However, we derive from Theorem 4 and (16) that

$$\bar{Q}(x,0) - \bar{Q}(0,x) = \frac{2Q(0,0)x(pw-1)\sqrt{1-qrw^2}}{w(1-px)(1-qx)(1-rx)}.$$

By (15), this quantity differs from 0, unless $pw = 1$, which forces $p = q$. If $p = q$, the solution satisfies $Q(x, 0) = Q(0, x)$, so that the symmetry property naturally holds. Otherwise, the asymmetry of the result comes from the asymmetric conditions we have required: $\bar{Q}(x, y)$ must converge when $|px| < 1$ and $|qy| < 1$.

3.3.3. *Flatto and Hahn's expression.* Assume $p < q$. Equation (16) shows that the numerator of $Q(0, y)$ vanishes when $y = q/p$ and $y = q/r$. Let us denote $\delta(y) = \sqrt{1 - yprw^2}$. Then the numerator of $Q(0, y)$ is a polynomial in $\delta(y)$, of degree 3, and two of its roots are $\delta(q/p) = \sqrt{1-qrw^2}$ and $\delta(q/r) = \sqrt{1-pqw^2}$. The third root is then easily determined to be $-\sqrt{1-prw^2}$. Hence, up to a multiplicative constant independent of y, the numerator of $Q(0, y)$ factors as

$$\begin{aligned}&\left(\sqrt{1-yprw^2} - \sqrt{1-qrw^2}\right)\left(\sqrt{1-yprw^2} - \sqrt{1-pqw^2}\right)\\ &\quad\times\left(\sqrt{1-yprw^2} + \sqrt{1-prw^2}\right).\end{aligned}$$

The denominator of $Q(0, y)$ is already factored in y, and also vanishes at $y = q/p$ and $y = q/r$. Up to a multiplicative constant, it factors as

$$\begin{aligned}&\left(\sqrt{1-yprw^2} + \sqrt{1-qrw^2}\right)\left(\sqrt{1-yprw^2} - \sqrt{1-qrw^2}\right)\\ &\quad\times\left(\sqrt{1-yprw^2} + \sqrt{1-pqw^2}\right)\left(\sqrt{1-yprw^2} - \sqrt{1-pqw^2}\right).\end{aligned}$$

Two simplifications occur, and, finally,

$$Q(0,y) = Q(0,0)\frac{\Psi(y)}{\Psi(0)}, \tag{43}$$

where

$$\Psi(y) = \frac{\sqrt{1-yprw^2} + \sqrt{1-prw^2}}{\left(\sqrt{1-yprw^2} + \sqrt{1-qrw^2}\right)\left(\sqrt{1-yprw^2} + \sqrt{1-pqw^2}\right)}.$$

Now, using (16), the function $Q(x,0)$ can be rewritten as

$$Q(x,0) = \frac{Q(0,0)}{(1-qx/p)(1-rx/p)} \times \left(\left(1-\frac{x}{pw}\right)\sqrt{1-xqrw^2} + x\left(1-\frac{1}{pw}\right)\sqrt{1-qrw^2}\right).$$

In this form, the numerator of $Q(x,0)$ now looks more like the numerator of $Q(0,y)$. More precisely, denoting the latter numerator by $P(\delta(y))$, the former numerator is exactly $-P(-\sqrt{1-xqrw^2})$, and, hence, factors as

$$(\sqrt{1-xqrw^2}+\sqrt{1-qrw^2})(\sqrt{1-xqrw^2}+\sqrt{1-pqw^2}) \times (\sqrt{1-xqrw^2}-\sqrt{1-prw^2}).$$

The denominator of $Q(x,0)$ is also easily factored; two simplifications occur again, and we end up with

$$Q(x,0) = Q(0,0)\frac{\Phi(x)}{\Phi(0)}, \tag{44}$$

where

$$\Phi(x) = \frac{\sqrt{1-xqrw^2}+\sqrt{1-qrw^2}}{(\sqrt{1-xqrw^2}+\sqrt{1-prw^2})(\sqrt{1-xqrw^2}-\sqrt{1-pqw^2})}.$$

Expressions (43) and (44) are, with our notation, the forms given in Flatto and Hahn's paper [17]. They are nicely factored, and it is easy to derive from them the singularities of $Q(x,0)$ and $Q(0,y)$. However, they have two drawbacks: first, they are only valid when $p \le q$, and hide the symmetry of the result in p and q, which is clear from the expressions of Theorem 4. Second, they somehow contain "two many" radicals, and suggest that $Q(x,0)$ and $Q(0,y)$ will be algebraic of degree 3×2^4 over the field $\mathbb{Q}(p,q,x,y)$, whereas, as suggested by Theorem 4, they have only degree $3 \times 2^2 = 12$. This can be checked using a computer algebra package, like MAPLE.

4. Enumeration and probability: the law of the chain. In this section we consider again the Markov chain illustrated in Figure 3. We start this chain at time 0 at the origin of the lattice, and address the question of computing the probability $p_{i,j}(n)$ that the walk reaches the point (i,j) at time n. This question is, in essence, close to Section 2: we are again *enumerating* paths according to a certain weight. This weight is the probability that the trajectory begins with this path. But this question is also related to Section 3, since we expect the probability $p_{i,j}(3n-i-j)$ to converge to $3p_{i,j}$ as n goes to infinity, when $r < \min(p,q)$, where $p_{i,j}$ is the stationary distribution of the chain (the factor 3 accounts for the periodicity of the chain).

The notation we adopt is similar to that of Section 3: we introduce the following four generating functions for the probabilities $p_{i,j}(n)$:

$$P_{0,0} = \sum_{n\geq 0} p_{0,0}(n)t^n,$$

$$P_1(x) = \sum_{n,i>0} p_{i,0}(n)x^i t^n,$$

$$P_2(y) = \sum_{n,j>0} p_{0,j}(n)y^j t^n,$$

$$P(x,y) = \sum_{n,i,j>0} p_{i,j}(n)x^i y^j t^n.$$

The step by step construction of the walks gives the following functional equation:

$$(1 - p\bar{x}t - q\bar{y}t - rxyt)P(x,y) + (1 - p'\bar{x}t - r'xyt)P_1(x) \\ + (1 - q''\bar{y}t - r''xyt)P_2(y) + (1 - xyt)P_{0,0} = 1.$$

Again, we assume that the transition probabilities on the border of the quadrant are related to those inside the quadrant by the conditions of (12). This allows us to rewrite the above functional as

$$(1 - p\bar{x}t - q\bar{y}t - rxyt)Q(x,y) + q(\bar{y}t - 1)Q(x,0) + p(\bar{x}t - 1)Q(0,y) = r$$

with

$$Q(x,y) = P_{0,0} + r'P_1(x) + r''P_2(y) + rP(x,y). \tag{45}$$

It will be convenient to have a kernel symmetric in x and y, and our starting point will actually be

$$\big(xy - t(x + y + pqrx^2y^2)\big)Q(px,qy) \\ + (t - qy)xQ(px,0) + (t - px)yQ(0,qy) = rxy. \tag{46}$$

We are back to the (safe) world of formal power series in t with coefficients in $\mathbb{Q}(x,y)$, and we will mimic the obstinate kernel method of Sections 2.1 and 2.2. The only new difficulty arises from the absence of symmetry, since $Q(x,0) \neq Q(0,x)$ when $p \neq q$.

The kernel of the above equation, considered as a polynomial in y, has two roots,

$$Y_0(x) = \frac{1 - t\bar{x} - \sqrt{(1 - t\bar{x})^2 - 4pqrt^2x}}{2pqrtx} = t + \bar{x}t^2 + O(t^3),$$

$$Y_1(x) = \frac{1 - t\bar{x} + \sqrt{(1 - t\bar{x})^2 - 4pqrt^2x}}{2pqrtx} = \frac{\bar{x}}{pqrt} - \frac{\bar{x}^2}{pqr} - t - \bar{x}t^2 + O(t^3).$$

The elementary symmetric functions of the Y_i are again polynomials in $1/x$:

$$Y_0 + Y_1 = \frac{\bar{x}(1 - t\bar{x})}{pqrt} \quad \text{and} \quad Y_0 Y_1 = \frac{\bar{x}}{pqr}.$$

The discriminant $\Delta(x) = (1 - t\bar{x})^2 - 4pqrt^2 x$ vanishes for three values of x: two of them, say X_0 and X_1, are power series in $\sqrt{t}$, while the third one, X_2, is a Laurent series in t that starts with a term in t^{-2}. Let us define $Z \equiv Z(t)$ to be the unique power series in t such that

$$Z = 1 + 4pqrt^3 Z^3.$$

Then $4pqrt^2 X_2 Z^2 = 1$, and the canonical factorization of $\Delta(x)$ reads

$$\Delta(x) = \Delta_0 \Delta_+(x) \Delta_-(\bar{x})$$

with

$$\Delta_0 = 4pqrt^2 X_2 = \frac{1}{Z^2}, \qquad \Delta_+(x) = 1 - x/X_2 = 1 - 4pqrt^2 Z^2 x, \tag{47}$$

$$\Delta_-(\bar{x}) = (1 - \bar{x}X_0)(1 - \bar{x}X_1) = 1 - tZ(1+Z)\bar{x} + t^2 Z^2 \bar{x}^2. \tag{48}$$

As in Section 3, it will be convenient to handle two functions $S(x, y)$ and $D(x, y)$, which are, respectively, symmetric and antisymmetric in x and y. We define them by

$$\begin{aligned} S(x, y) &= (t - qx)(t - py)Q(px, qy) + (t - px)(t - qy)Q(py, qx), \\ D(x, y) &= (t - qx)(t - py)Q(px, qy) - (t - px)(t - qy)Q(py, qx). \end{aligned} \tag{49}$$

Then

$$\begin{aligned} &t\big(xy - t(x + y + pqrx^2y^2)\big)S(x, y) + (t - py)(t - qy)xS(x, 0) \\ &\qquad + (t - px)(t - qx)yS(0, y) = G(x, y) + G(y, x), \end{aligned} \tag{50}$$

$$\begin{aligned} &t\big(xy - t(x + y + pqrx^2y^2)\big)D(x, y) + (t - py)(t - qy)xD(x, 0) \\ &\qquad + (t - px)(t - qx)yD(0, y) = G(x, y) - G(y, x), \end{aligned} \tag{51}$$

where

$$G(x, y) = rxyt(t - qx)(t - py).$$

4.1. *Statement of the results.* After all the algebraic series we have met, one might expect the probability generating function of the law of the chain to be algebraic again. This is, however, only true if $p = q$.

THEOREM 11 (The symmetric case). *Assume $p = q$. The three-variate generating function for the probabilities $p_{i,j}(n)$ is algebraic, and can be expressed explicitly in terms of the unique power series $Z \equiv Z(t)$ satisfying $Z = 1 + 4pqrt^3Z^3$. In particular, the generating function of walks ending at the origin is algebraic of degree* 6:

$$\begin{aligned} P_{0,0} &= \sum_{n \geq 0} p_{0,0}(3n)t^{3n} \\ &= \frac{r}{p}\left(\frac{\sqrt{1 - pZ(1+Z) + p^2Z^2}}{1 - 2pZ} - 1\right) = \frac{r}{p}\left(\frac{\sqrt{\Delta_-(p/t)}}{1 - 2pZ} - 1\right), \end{aligned}$$

where $\Delta_-(\bar{x})$ is given by (48). *More generally, the series $Q(px, 0) = P_{0,0} + r'P_1(px)$ is given by*

$$\begin{aligned} &\big(t - x(1-p) + prx^2t^2\big)Q(px, 0) \\ &\quad = \frac{r}{2p}\left(\frac{(2tZ - x)\sqrt{\Delta_-(p/t)}\sqrt{\Delta_+(x)}}{Z(1 - 2pZ)} - 2t + x(1-p)\right), \end{aligned}$$

where $\Delta_+(x)$ is given by (47). *The expression of $P_{0,0}$ can be recovered from the value of $Q(px, 0)$ by setting $x = 0$.*

This theorem, and all the results of this section, will be proved in Section 4.2.

What happens in the general case? We have expressed the series $Q(px, qy)$ in terms of two series $S(x, y)$ and $D(x, y)$, which are, respectively, symmetric and antisymmetric in x and y. It turns out that the Sum series $S(x, y)$ is always algebraic, while the Difference series $D(x, y)$ is transcendental (unless $p = q$). The algebraicity of $S(x, y)$ has an interesting consequence: The generating function $P_{0,0}$ that counts walks ending at the origin is *always algebraic*, even when $p \neq q$.

THEOREM 12 (The general case: algebraic part). *The series $S(x, y)$ defined by* (49) *is algebraic and can be expressed explicitly in terms of the unique power series $Z \equiv Z(t)$ satisfying $Z = 1 + 4pqrt^3Z^3$. In particular, the coefficient of x^0y^0 in $S(x, y)$ is an algebraic series in t. It is equal to $2t^2P_{0,0}$, where $P_{0,0}$ counts walks ending at the origin, and we have*

$$2pqP_{0,0} + r(1-r) = A_{p,q} + A_{q,p},$$

where $A_{p,q}$ is the following algebraic series in t:

$$A_{p,q} = \frac{(p(1-2p) - qrt^3)\sqrt{\Delta_-(p/t)}}{(1-t^3)(1-2pZ)}$$

and $\Delta_-(\bar{x})$ is given by (48). *The algebraic series $P_{0,0}$ has degree* 6 *if $p = q$, and* *degree* 12 *otherwise. More generally, the series* $S(x, y)$ *satisfies*

$$\big(t-(1-p)x+t^2qrx^2\big)\big(t-(1-q)x+t^2prx^2\big)\frac{S(x,0)}{t}+\frac{rH(x)}{2pq}$$
$$=\frac{(2tZ-x)\sqrt{\Delta_+(x)}}{2pqZ}\big(A_{p,q}F_{p,q}(x)+A_{q,p}F_{q,p}(x)\big),$$

where $\Delta_+(x)$ *is given by* (47), *and* $F_{p,q}(x)$ *and* $H(x)$ *denote the following polynomials in* t *and* x:

$$\begin{aligned} F_{p,q}(x) &= (t-xq)\big(t-(1-q)x+t^2prx^2\big),\\ H(x) &= q(2t-x+px)F_{p,q}(x)+p(2t-x+qx)F_{q,p}(x)\\ &\quad -(p-q)^2x^2\big(2rt-(1-p)(1-q)x+2pqrx^2t^2\big). \end{aligned} \tag{52}$$

The expression of $P_{0,0}$ *can be recovered from the value of* $S(x,0)$ *by setting* $x=0$. *An expression for* $S(x, y)$ *can be obtained using* (50).

This theorem will allow us to complete the proof of the following result, announced in Section 3.

COROLLARY 13. *The Markov chain schematized in Figure* 3, *with the border conditions of* (12), *is ergodic* (*i.e., has a stationary distribution*) *if and only if* $r < \min(p, q)$.

Theorem 12 specializes to Theorem 11 when $p = q$. It states that walks ending at the origin have an algebraic generating function. What about the generating functions $P_1(x)$ and $P_2(y)$ that count walks ending on the x- or y-axis? By symmetry of the model, $P_1(x)$ is algebraic if and only if $P_2(y)$ is algebraic too. In view of (45), (46) and (49), this holds if and only if $S(x,0)$ and $D(x,0)$ are algebraic. If $p = q$, then $D(x, y)$ is obviously zero, and Theorem 11 tells us that all the generating functions under consideration are algebraic. If $p \neq q$, we shall prove that $D(x,0)$ is transcendental (but D-finite), and give an explicit expression of it.

So far, we have expressed many of our series in terms of the canonical factorization of the discriminant $\Delta(x)$. This is the case, for instance, in Theorem 12, where the expression of $S(x,0)$ involves $\sqrt{\Delta_+(x)}$, which we could call the positive *multiplicative* part of $\sqrt{\Delta(x)}$. In order to express $D(x,0)$, we need to introduce the positive *additive* part of $\sqrt{\Delta(x)}$, as defined by (2). More precisely, the expression of $D(x,0)$ will involve the positive (additive) part of

$$B(x) := (Y_0-Y_1)(2t-x+pqrtx^3)=\frac{\sqrt{\Delta(x)}}{pqrt}(1-2t\bar{x}-pqrtx^2). \tag{53}$$

We shall compute below the expansion of B in t and x, using the Lagrange inversion formula. In particular, we will see that the positive (additive) part of $B(x)$ reads

$$B^+(x) = -xt - x^2 + 2C^+(x), \tag{54}$$

where all terms in $C^+(x)$ are multiples of x^3. We then define the series $C^-(\bar{x})$ by

$$B(x) = \frac{1-2\bar{x}t}{pqt} + 2C^-(\bar{x}) - xt - x^2 + 2C^+(x). \tag{55}$$

Observe that $\Delta(t/p)$, and, hence, $B(t/p)$, is a well-defined Laurent series in t. Clearly, $C^+(t/p)$ is well defined too: consequently, by difference, we can define $C^-(p/t)$ as a Laurent series in t, even though it would be meaningless to replace x by p/t in the expansion of $C^-(x)$.

THEOREM 14 (The general case: transcendental part). *When $p \neq q$, the series $D(x, y)$ defined by* (49) *is D-finite but transcendental. The same holds for its specialization $D(x,0)$. Consequently, the series $P_1(x)$ and $P_2(y)$ which count walks ending on the x- or y-axis are transcendental. The series $D(x,0)$ satisfies*

$$\begin{aligned}
&\big(t-(1-p)x+t^2qrx^2\big)\big(t-(1-q)x+t^2prx^2\big)\frac{D(x,0)}{rxt}\\
&\quad + x(p-q)\big(t^2(1-r)rx^2 - x/2 + t\big)\\
&\qquad = rx(p-q)t^2C^+(x) - \frac{t}{1-t^3}\big(pC^-(p/t)F_{p,q}(x) - qC^-(q/t)F_{q,p}(x)\big),
\end{aligned}$$

where $C^+(x)$ and $C^-(\bar{x})$ are defined by (54)–(55) *and, as in Theorem* 12,

$$F_{p,q}(x) = (t-xq)\big(t-(1-q)x+t^2prx^2\big).$$

An expression of $D(x, y)$ can then be obtained using (51).

Note the similarities between the expressions of $S(x,0)$ (Theorem 12) and $D(x,0)$ (Theorem 14). One could take the sum and difference of these expressions to recover the series $Q(px,0)$ and $Q(0,qx)$, but, as no significant simplification arises, we shall not do this.

There is still one natural question that is not answered by the combination of the above two theorems: we have seen that the generating function $P_{0,0}$ of walks ending at the origin is algebraic, but that the series $P_1(x)$ that counts walks ending on the x-axis is transcendental. Yet, for $i > 0$, the coefficient of x^i in $P_1(x)$, being

$$P_{i,0} := \sum_{n\geq 0} p_{i,0}(n)t^n,$$

counts walks ending at $(i,0)$ and might be algebraic. The following corollary tells us that this is not (systematically) the case.

COROLLARY 15. *Some of the series $P_{i,0}$ are transcendental.*

NOTE. We can obtain an explicit expression of the series $C^+(x)$ and $C^-(\bar{x})$ by expanding $B(x)$ in x and t. Let us write

$$Y_0 - Y_1 = 2Y_0 - (Y_0 + Y_1) = 2Y_0 - \frac{\bar{x}(1-\bar{x}t)}{pqrt},$$

and observe that the series Y_0, which cancels the kernel, is Lagrangian in t:

$$Y_0 = t(1 + \bar{x}Y_0 + pqrxY_0^2).$$

The Lagrange inversion formula yields

$$B(x) = \frac{(1-\bar{x}t)(1-2\bar{x}t)}{pqrt} - xt - x^2 + 2\sum_{n\geq 2} t^n \sum_{k=0}^{\lfloor n/2\rfloor} x^{3k-n+2}(pqr)^k \frac{(3k-n+1)(3k-n)(n-2)!}{k!(k+1)!(n-2k)!}.$$

Consequently,

$$C^+(x) = \sum_{n\geq 2} t^n \sum_{k=\lceil (n-1)/3\rceil}^{\lfloor n/2\rfloor} x^{3k-n+2}(pqr)^k \frac{(3k-n+1)(3k-n)(n-2)!}{k!(k+1)!(n-2k)!}.$$

One may also write an explicit expansion of $C^-(\bar{x})$.

4.2. *Proofs.*

PROOF OF THEOREM 12. Let us start from (50) defining $S(x,y)$. As in Section 2.1, the pairs (x, Y_0) and (Y_0, Y_1) cancel the kernel and can be substituted for (x,y) in this equation. We thus obtain two equations:

$$\begin{aligned}
&(t-pY_0)(t-qY_0)T(x) + (t-px)(t-qx)T(Y_0)\\
&\qquad = G(x,Y_0) + G(Y_0,x),\\
&(t-pY_1)(t-qY_1)T(Y_0) + (t-pY_0)(t-qY_0)T(Y_1)\\
&\qquad = G(Y_0,Y_1) + G(Y_1,Y_0),
\end{aligned} \tag{56}$$

with $T(x) = xS(x,0)$. Let us form a symmetric function of Y_0 and Y_1 based on a divided difference: We multiply the first equation by $2(t-pY_1)(t-qY_1)$ and the second one by $(t-px)(t-qx)$, and take the difference of the resulting equations,

$$\begin{aligned}
&2(t-pY_0)(t-qY_0)(t-pY_1)(t-qY_1)T(x)\\
&\quad + (t-px)(t-qx)\big((t-pY_1)(t-qY_1)T(Y_0) - (t-pY_0)(t-qY_0)T(Y_1)\big)\\
&\quad = 2(t-pY_1)(t-qY_1)\big(G(x,Y_0) + G(Y_0,x)\big)\\
&\qquad - (t-px)(t-qx)\big(G(Y_0,Y_1) + G(Y_1,Y_0)\big).
\end{aligned}$$

Recall that

$$K(x,y)=xy-t(x+y+pqrx^2y^2)=-pqrtx^2(y-Y_0)(y-Y_1).$$

We use this identity to express the coefficient of $T(x)$ as a Laurent polynomial in x and t. Then, we separate the symmetric and anti-symmetric parts of the right-hand side using

$$\Psi(Y_0,Y_1)=\tfrac{1}{2}\big(\Psi(Y_0,Y_1)+\Psi(Y_1,Y_0)\big)+\tfrac{1}{2}\big(\Psi(Y_0,Y_1)-\Psi(Y_1,Y_0)\big).$$

This gives

$$\begin{aligned}
&2\frac{(t-(1-p)x+t^2qrx^2)(t-(1-q)x+t^2prx^2)}{pqr^2x^4}T(x)+\frac{tH(x)}{p^2q^2rx^3}\\
&\quad=-(t-px)(t-qx)\\
&\qquad\times\big((t-pY_1)(t-qY_1)T(Y_0)-(t-pY_0)(t-qY_0)T(Y_1)\big)\\
&\qquad+(Y_0-Y_1)\frac{t^2J(x)}{pqx},
\end{aligned}\tag{57}$$

where $H(x)$ is given by (52) and $J(x)$ is also a polynomial in x and t:

$$J(x)=qF_{p,q}(x)+pF_{q,p}(x)+x(p-q)^2(t-rx).$$

As we are getting used to the method, let us merge the next two steps: instead of first dividing by (Y_0-Y_1) and then multiplying by $\sqrt{\Delta_-(\bar{x})}$, let us divide (57) by $2\sqrt{\Delta_+(x)}/(pqrx)=-2t(Y_0-Y_1)/\sqrt{\Delta_0\Delta_-(\bar{x})}$. We obtain, in view of (47),

$$\begin{aligned}
&\frac{(t-(1-p)x+t^2qrx^2)(t-(1-q)x+t^2prx^2)}{rx^3\sqrt{\Delta_+(x)}}T(x)+\frac{tH(x)}{2pqx^2\sqrt{\Delta_+(x)}}\\
&\quad=\frac{\sqrt{\Delta_-(\bar{x})}}{2Z}\\
&\qquad\times\Bigg((t-px)(t-qx)\\
&\qquad\quad\times\frac{(t-pY_1)(t-qY_1)T(Y_0)-(t-pY_0)(t-qY_0)T(Y_1)}{t(Y_0-Y_1)}-\frac{tJ(x)}{pqx}\Bigg).
\end{aligned}$$

The left-hand side of this equation, as a Laurent series in x, has valuation -2, while the right-hand side only involves powers of x smaller than or equal to 2. Extracting the positive part in x, and mutiplying by x^2 gives

$$\begin{aligned}
&\frac{(t-(1-p)x+t^2qrx^2)(t-(1-q)x+t^2prx^2)}{rx\sqrt{\Delta_+(x)}}T(x)\\
&\qquad+\frac{tH(x)}{2pq\sqrt{\Delta_+(x)}}=L(x),
\end{aligned}\tag{58}$$

where $L(x)$ is a polynomial in x, of degree 4, with coefficients in $\mathbb{Q}[t, T_1, T_2, T_3]$, where $T_i \equiv T_i(t)$ denotes the coefficient of x^i in $T(x)$. We do not give the explicit expression of $L(x)$, but refer the reader to his/her favorite computer algebra system.

We now have to determine the three unknown functions T_1, T_2 and T_3. Fortunately, we can compute $T(x)$ at three values of x using (56): first at $x = t/p$, then at $x = t/q$, and finally at $x = W$, where W is the unique power series in t that satisfies $K(W, W) = 0$ [so that $W = Y_0(W)$]. Remarkably, W is simply related to the parameter Z by $W = 2tZ$. The three values of $T(x)$ that we obtain are

$$T(t/p) = \frac{t^3(p-q)(q-r+\sqrt{(1-p)^2-4t^3qr})}{2p^2q(1-t^3)}, \tag{59}$$

$$T(t/q) = \frac{t^3(q-p)(p-r+\sqrt{(1-q)^2-4t^3pr})}{2pq^2(1-t^3)}, \tag{60}$$

$$T(W) = rtW^2.$$

Setting $x = W$ in (58), we find that the left-hand side vanishes. Hence, $L(W) = 0$, and this gives an expression of T_2 in terms of T_1:

$$T_2 = -rt + \frac{2r-1}{2t} T_1.$$

The polynomial $L(x)$ now takes the following form:

$$L(x) = \frac{(W-x)(t(t-x-tpqrx^3)M_0 + M_1x^2)}{pqrW}, \tag{61}$$

where

$$M_0 = pqT_1 + r(p+q)t^2,$$

and M_1 involves both T_1 and T_3. It remains to evaluate (58) at $x = t/p$ and $x = t/q$, using the expressions of $T(t/p)$ and $T(t/q)$ given by (59) and (60), to obtain

$$\begin{aligned} M_0 = \frac{t^3}{1-t^3}\biggl(& \frac{(p(1-2p)-t^3qr)\sqrt{\Delta_-(p/t)}}{t-pW} \\ & + \frac{(q(1-2q)-t^3pr)\sqrt{\Delta_-(q/t)}}{t-qW}\biggr), \\ M_1 = \frac{t^3}{1-t^3}\biggl(& \frac{(p(1-2p)-t^3qr)(q(1-q)+prt^3)\sqrt{\Delta_-(p/t)}}{t-pW} \\ & + \frac{(q(1-2q)-t^3pr)(p(1-p)+qrt^3)\sqrt{\Delta_-(q/t)}}{t-qW}\biggr). \end{aligned}$$

Theorem 12 follows, using (61) and (58). □

PROOF OF COROLLARY 13. We have already seen that the condition $r < \min(p, q)$ is necessary for the chain to have a stationary distribution (Lemma 6). Assume this condition holds. As the chain is irreducible, it suffices to prove that the point $(0, 0)$ is positive recurrent, that is, that the probability $p_{0,0}(3n)$ converges to a positive constant as n goes to infinity [10]. The generating function of these numbers, denoted $P_{0,0}$, is given explicitly in Theorem 12.

This leads us to determine the smallest singularity of the series $A_{p,q}$. The technique is standard for algebraic functions, and we only sketch the main steps. The series Z becomes singular at $t^3 = 1/(27pqr) > 1$. Then, we note that

$$p(1-2p) - qrt^3 = \frac{(1-2pZ)(1+(2p-1)Z(1+2pZ))}{4pZ^3},$$

so that there is actually no pole in $A_{p,q}$ if Z reaches $1/(2p)$. Moreover, if $\Delta_-(p/t) = 0$, then $\Delta(t/p) = 0$. But, as t increases from 0 to 1, $\Delta(t/p) = (1-p)^2 - 4qrt^3$ decreases from $(1-p)^2$ to $(q-r)^2$ and thus does not vanish. Hence, $A_{p,q}$ (and $A_{q,p}$) has its smallest singularity at $t^3 = 1$, and this singularity is a simple pole. Consequently, the coefficient of t^{3n} in $A_{p,q}$ tends to a constant as $n \to \infty$. The same holds for $A_{q,p}$.

It remains to show that the sum of these two constants is not zero. We are actually going to compute them explicitly: this will not only conclude the proof of the corollary, but also allow us to recover the value of $p_{0,0}$ given in Corollary 5. First, we note that $Z(1) = 1 + 4pqrZ(1)^3$, and conclude that $Z(1) = w/2$, where w is the real number defined in Theorem 4. Then, the definition of the canonical factorization gives, when $t = 1$,

$$\Delta(1/p) = (q-r)^2 = \Delta_0 \Delta_+(1/p) \Delta_-(p),$$

and so by (47),

$$\sqrt{\Delta_-(p)} = \frac{(q-r)w}{2\sqrt{1-qrw^2}}.$$

Thus, as $t \to 1$,

$$2pq P_{0,0} \sim \frac{w}{2(1-t^3)} \left(\frac{(p(1-2p)-qr)(q-r)}{(1-pw)\sqrt{1-qrw^2}} + \frac{(q(1-2q)-pr)(p-r)}{(1-qw)\sqrt{1-prw^2}} \right).$$

Note that $p(1-2p) - qr = (q-p)(p-r)$. Using (16), we rewrite the above identity as

$$2pq P_{0,0} \sim \frac{1}{1-t^3} \frac{w(p-r)(q-r)|p-q|}{(1-rw)\sqrt{1-pqw^2}}.$$

It follows that, as $n \to \infty$,

$$p_{0,0}(3n) \to \frac{w(p-r)(q-r)|p-q|}{2pq(1-rw)\sqrt{1-pqw^2}}.$$

Given that the chain has period 3, this agrees with Corollary 5. $\square$

PROOF OF THEOREM 14. Let us start from (51) defining $D(x, y)$. The pairs (x, Y_0) and (Y_0, Y_1) cancel the kernel and can be substituted for (x, y) in this equation. We thus obtain

$$(t-pY_0)(t-qY_0)E(x) - (t-px)(t-qx)E(Y_0) = G(x, Y_0) - G(Y_0, x),$$
$$(t-pY_1)(t-qY_1)E(Y_0) - (t-pY_0)(t-qY_0)E(Y_1) = G(Y_0, Y_1) - G(Y_1, Y_0),$$

with $E(x) = xD(x, 0) = -xD(0, x)$. We now want to form a symmetric function of Y_0 and Y_1 based on a sum. We multiply the first equation by $2(t-pY_1)(t-qY_1)$ and the second one by $(t-px)(t-qx)$, and take the sum of the resulting equations:

$$\begin{aligned}
&2(t-pY_0)(t-qY_0)(t-pY_1)(t-qY_1)E(x)\\
&\quad - (t-px)(t-qx)\big((t-pY_1)(t-qY_1)E(Y_0) + (t-pY_0)(t-qY_0)E(Y_1)\big)\\
&\quad = 2(t-pY_1)(t-qY_1)\big(G(x, Y_0) - G(Y_0, x)\big)\\
&\qquad + (t-px)(t-qx)\big(G(Y_0, Y_1) - G(Y_1, Y_0)\big).
\end{aligned}$$

As above, we use the expression of the kernel to express the coefficient of $E(x)$ as a Laurent polynomial in x and t, and split the right-hand side into a symmetric and an anti-symmetric part. After multiplying by x, we obtain

$$\begin{aligned}
&2\frac{(t-(1-p)x+t^2qrx^2)(t-(1-q)x+t^2prx^2)}{pqr^2x^3}E(x) + \frac{t(p-q)I(x)}{p^2q^2rx^2}\\
&\qquad = x(t-px)(t-qx)\\
&\qquad\quad \times \big((t-pY_1)(t-qY_1)E(Y_0) + (t-pY_0)(t-qY_0)E(Y_1)\big)\\
&\qquad\quad + (p-q)\frac{t^3B(x)}{pq},
\end{aligned} \tag{62}$$

where $B(x)$ is given by (53) and

$$I(x) = -x^3pq - t^3pqrx^3 + pqr(1-2r)t^2x^4 + (r+3pq)x^2t - (1+r)xt^2 + t^3.$$

As a Laurent series in x, the left-hand side of the above identity has valuation -2. The term involving $E(Y_0)$ and $E(Y_1)$ only involves powers of x smaller than or equal to 2. But $B(x)$ is a series in t with coefficients in $\mathbb{Q}[x, \bar{x}]$, containing arbitrarily large positive and negative of x, and this is where the transcendence of the solution stems from.

Note that the coefficients of x and x^2 in $B(x)$ are especially simple, being, respectively, $-t$ and -1. Let $E_i \equiv E_i(t)$ denote the coefficient of x^i in $E(x)$. Given that $E(x) = xD(x,0)$ and $D(x,y)$ is antisymmetric, we have $E_0 = E_1 = 0$. Let us first extract from (62) the coefficient of x: we obtain a relation between E_2 and E_3,

$$E_3 = \frac{r}{t}E_2 + r(q-p).$$

Now, extracting the positive part of (62) and multiplying by x gives

$$\begin{aligned}
&2\frac{(t-(1-p)x+t^2qrx^2)(t-(1-q)x+t^2prx^2)}{pqr^2x^2}E(x) \\
&\qquad = x(p-q)t^3\frac{B^+(x)}{pq} + L(x),
\end{aligned} \tag{63}$$

where $B^+(x)$ is the positive part of $B(x)$ and $L(x)$ is a polynomial in x (of degree 3) with coefficients in $\mathbb{Q}[p,q,t,E_2,E_4]$:

$$\begin{aligned}
L(x) = {}& -2\frac{pqrx^3t^2 + tx - t^2 - r(1-r)x^2t^3 + x^2(r^2-pq)}{pqr^2}E_2 \\
& + 2\frac{x^2t^2}{pqr^2}E_4 - (p-q)\frac{(1-2r)t^3}{pq}x^3 \\
& + (p-q)t\frac{2+2r+t^3r}{pqr}x^2 - 2(p-q)\frac{t^2}{pqr}x.
\end{aligned} \tag{64}$$

We have to determine two unknown functions E_2 and E_4. From (49) and the fact that $T(x) = xS(x,0)$ and $E(x) = xD(x,0)$, we derive that $E(t/p) = T(t/p)$ and $E(t/q) = -T(t/q)$. We evaluate (63) at $x = t/p$ and $x = t/q$, using the expressions of $T(t/p)$ and $T(t/q)$ given by (59) and (60). One thus obtains expressions of E_2 and E_4 in terms of $\sqrt{\Delta(t/p)}$, $\sqrt{\Delta(t/q)}$, $B^+(t/p)$ and $B^+(t/q)$. They become much simpler using

$$B^+(x) = \frac{\sqrt{\Delta(x)}}{pqrt}(1-2t\bar{x}-pqrtx^2) - \frac{1-2t\bar{x}}{pqt} - 2C^-(\bar{x}).$$

One finds

$$E_2 = \frac{t^2r}{1-t^3}\big(qC^-(q/t) - pC^-(p/t)\big)$$

and

$$\begin{aligned}
E_4 = {}& \frac{r^2}{1-t^3}\big(q(1-q-pt^3)C^-(q/t) - p(1-p-qt^3)C^-(p/t)\big) \\
& - \frac{r(p-q)(2r+1)}{2t}.
\end{aligned}$$

The expression of $D(x,0) = E(x)/x$ follows from these values, using (63), (64) and (54).

Let us now discuss the algebraic nature of $D(x,0)$ [equivalently, of $E(x)$]. If $E(x)$ were algebraic, then so would be all the coefficients E_i. In particular, the series $L(x)$ given by (64) and occurring in the right-hand side of (63) would be algebraic too. By difference, the positive part of $B(x)$, denoted above by $B^+(x)$ would be algebraic, and so would be all its coefficients. But the coefficient of x^3 in $B(x)$ is

$$4\sum_{k\geq 0} t^{3k+2}(pqr)^{k+1}\frac{(3k)!}{k!(k+1)!(k+2)!}.$$

As $k \to \infty$, the coefficient of t^{3k+2} in this series is asymptotic to $(27pqr)^k/k^4$, up to a positive multiplicative constant. Because of the factor k^{-4}, this cannot be the asymptotic behavior of the coefficients of an algebraic series [15], so that our initial hypothesis is false: The series $D(x,0)$ is not algebraic. However, the general results on D-finite series recalled at the end of Section 1 imply that it is D-finite. $\square$

PROOF OF COROLLARY 15. Assume all the series $P_{i,0}$ are algebraic. By symmetry, all the series $P_{0,j}$, which count walks ending on the y-axis, are algebraic too. In other words, the coefficient of x^i in $Q(x,0)$ and $Q(0,x)$ is algebraic. In view of (49), this holds for $S(x,0)$ and $D(x,0)$ as well.

We already know, by Theorem 12, that $S(x,0)$ is algebraic. Let us work with $D(x,0)$ to obtain a contradiction. We thus assume that the coefficient of x^i in the series $E(x) = xD(x,0)$ is algebraic. By (63), this implies that the coefficient of x^i in $B^+(x)$ is algebraic too, for all i. The same asymptotic argument as above proves that this is wrong. $\square$

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