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ON THE PROPERTIES OF r-EXCESSIVE MAPPINGS FOR A CLASS OF DIFFUSIONS

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We consider the convexity and comparative static properties of a class of r-harmonic mappings for a given linear, time-homogeneous and regular diffusion process. We present a set of weak conditions under which the minimal r-excessive mappings for the considered diffusion are convex and under which an arbitrary nontrivial r-excessive mapping is convex on the regions where it is r-harmonic. Consequently, we are able to present a set of usually satisfied conditions under which increased volatility increases the value of r-harmonic mappings. We apply our results to a class of optimal stopping problems arising frequently in studies considering the pricing of perpetual American contingent claims and state a set of conditions under which the value function is convex on the continuation region and, consequently, under which increased volatility unambiguously increases the value function and expands the continuation region, thus postponing the rational exercise of the claim.

1. Introduction. The r-harmonic mappings, especially the minimal r-excessive mappings, play a major role in most studies considering either the optimal stopping problem or the singular stochastic control problem of a linear diffusion (cf. [4] and references therein, [10], Chapter 8, and [14], Chapter 10; see also [2, 5, 6, 15]). Since the value has to typically be r-excessive and r-harmonic on the continuation region (or in the do-nothing-region) and all r-excessive mappings can be expressed as a linear combination of the minimal r-excessive mappings, it is clear that it is essentially the form of these mappings that determines the convexity properties of the value function. Similarly, since all smooth r-harmonic mappings can be expressed as a linear combination of the minimal r-excessive mappings, we again find that the form of the minimal r-excessive mappings is the principal factor determining the curvature of smooth r-harmonic mappings. Convexity and concavity, being second-order properties, affect the quadratic variation of a transformation and, therefore, are helpful properties when comparing the r-harmonic mappings between two diffusions evolving at the same expected rate but subject to a different diffusion coefficient measuring the infinitesimal variance of the stochastic fluctuations. Somewhat surprisingly, not much has been done in characterizing the form of r-harmonic mappings (except for ordinary Brownian motion) even while it is of essential importance in many applications

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(e.g., in the theory of real options and in studies considering the pricing of perpetual American contingent claims; cf. [8]) due to its significance in the comparative static analysis of r-harmonic mappings, especially r-excessive mappings.

Motivated by this argument, we plan to consider the form and comparative static properties of a class of r-harmonic mappings for a linear, time-homogeneous and regular diffusion. Since most economical, financial and ecological applications of diffusions deal with nonnegative processes, we limit our interest in diffusions defined on $\mathbb{R}_+$ even while it is clear that our principal results can be extended to diffusions defined on an arbitrary subinterval of $\mathbb{R}$ or on the entire $\mathbb{R}$. We state a set of easily verifiable conditions under which the convexity of the minimal r-excessive mappings for the considered diffusion is always unambiguously guaranteed and, consequently, under which all r-excessive mappings for the considered diffusion are convex on the regions of r-harmonicity. Since increased volatility increases the value of convex r-harmonic mappings, we are able to establish a set of weak conditions under which this positive result is always valid and, therefore, under which *increased volatility increases the value of optimal stopping problems and extends the continuation region, thus postponing rational exercise*. Since this conclusion is independent of the reward function, we find that *on the continuation region both the curvature of the value function and the sign of the relationship between stochastic fluctuations and the value are inherently process-specific properties and not reward-specific*. This result is of essential interest, since it extends previous results characterizing the comparative static properties of optimal stopping policies and explains *the positivity of sign of the relationship between volatility and the value for pre-exercise states in models subject to concave rewards* (cf. [1]). Another both economically and ecologically important consequence of our results is that the minimal r-excessive mappings for standard mean-reverting diffusions subject to concave drifts are also convex on $\mathbb{R}_+$ as long as the growth rate of the net depreciation rate is positive at the origin. Consequently, our results show that r-excessive mappings may be convex on the set of r-harmonicity also in the presence of mean reversion.

The contents of this study are as follows. In Section 2 we study the convexity of the minimal r-excessive mappings for the considered diffusion. In Section 3 we present a set of convex inequalities for r-harmonic mappings and study the comparative static properties of these mappings. Finally, in Section 4 we apply our principal results to an optimal stopping problem and state a set of usually satisfied conditions under which increased volatility increases the value of the optimal stopping problem and postpones rational exercise by expanding the continuation region.

2. The convexity of the minimal *r*-excessive mappings. Let $X = \{X(t); t \in [0, \tau(0))\}$, where $\tau(0) = \inf\{t \geq 0 : X(t) \leq 0\}$ (which may be infinite), be a linear, time-homogeneous and regular diffusion defined on a complete filtered probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F})$ and evolving on the state space $\mathbb{R}_+$. We assume that X does not die in the interior of $\mathbb{R}_+$ and that the differential operator $\mathcal{A}$ representing the infinitesimal generator of X is given by

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}, \tag{1}$$

where $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}$ are given continuous mappings on $\mathbb{R}_+$. In accordance with most applications, we assume that $\sigma(x) > 0$ on $(0, \infty)$ and that the upper boundary ∞ is natural for the diffusion X. Whenever the lower boundary 0 is regular for X, we assume that it is killing. Moreover, we also assume throughout this study that (a *transversality condition*)

$$\lim_{t\to\infty} E_x[e^{-rt}X(t); t < \tau(0)] = 0 \tag{2}$$

for all $x \in \mathbb{R}_+$. The continuity of $\mu(x)$ and $\sigma(x)$ imply that the basic characteristics of the diffusion X are now absolutely continuous with

$$S'(x) = \exp\left(-\int^x \frac{2\mu(y)}{\sigma^2(y)}\,dy\right)$$

denoting the density of the scale function S of X and

$$m'(x) = \frac{2}{\sigma^2(x)S'(x)}$$

denoting the density of the speed measure m of X. Before beginning our analysis, we present the following definition.

DEFINITION 1 ([7], Chapter 2, [11], Section 4.6, and [13], Section 2.3). The *Green kernel* $G_r : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ of the linear diffusion $\{X(t); t \in [0, \tau(0))\}$ is defined as

$$G_r(x, y) = \int_0^\infty e^{-rt} p(t; x, y)\,dt,$$

where $p(t; x, y)$ is the transition density of X defined with respect to its speed measure m. There are two linearly independent *fundamental solutions*, $\psi(x)$ and $\varphi(x)$, with $\psi(x)$ increasing and $\varphi(x)$ decreasing, spanning the set of solutions of the ordinary second-order differential equation $((\mathcal{A} - r)u)(x) = 0$ within the domain of the generator of $\{X(t); t \in [0, \tau(0))\}$. In terms of these solutions, $G_r(x, y)$ can be rewritten in the alternative form

$$G_r(x, y) = \begin{cases} B^{-1}\psi(x)\varphi(y), & x < y, \\ B^{-1}\psi(y)\varphi(x), & x \geq y, \end{cases}$$

where

$$B = \frac{\psi'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}\psi(x) > 0$$

denotes the constant (with respect to the scale) Wronskian determinant of the fundamental solutions.

DEFINITION 2 ([7], page 31, [9], Chapter 12, [14], pages 196–200, and [15]). A nonnegative and measurable mapping $f:\mathbb{R}_+ \mapsto \mathbb{R}_+ \cup \{\infty\}$ is called r-excessive if it satisfies the following conditions:

1. $E_x[e^{-rt} f(X(t))] \le f(x)$ for all $x \in \mathbb{R}_+$ and $t \ge 0$.
2. $\lim_{t\downarrow 0} E_x[e^{-rt} f(X(t))] = f(x)$ for all $x \in \mathbb{R}_+$.

A measurable mapping $f:\mathbb{R}_+ \mapsto \mathbb{R}_+$ is called r-harmonic if for all $x \in \mathbb{R}_+$ we have

$$E_x\bigl[e^{-r\tau_U} f\bigl(X(\tau_U)\bigr)\bigr] = f(x),$$

where $\tau(U) = \inf\{t \ge 0 : X(t) \notin U\}$ denotes the first exit time of the diffusion X from an arbitrary open set $U \subset \mathbb{R}_+$ with compact closure $\bar{U} \subset \mathbb{R}_+$.

We now plan to consider the form of the fundamental solutions $\psi(x)$ and $\varphi(x)$ constituting the minimal r-excessive mappings for $\{X(t); t \in [0, \tau(0))\}$ (any nontrivial r-excessive mapping for $\{X(t); t \in [0, \tau(0))\}$ can be written as their linear combination; cf. [7], page 32). Our first main result is now summarized in the following theorem.

THEOREM 1. *Assume that the transversality condition* (2) *holds. Then we have, for all* $x \in \mathbb{R}_+$,

$$\sigma^2(x)\frac{\varphi''(x)}{S'(x)} = 2r\int_x^\infty \varphi(y)\bigl(\theta(y) - \theta(x)\bigr)m'(y)\,dy \tag{3}$$

and

$$\sigma^2(x)\frac{\psi''(x)}{S'(x)} = 2r\biggl[\int_0^x \psi(y)\bigl(\theta(x) - \theta(y)\bigr)m'(y)\,dy + \frac{\psi'(0)}{S'(0)}\frac{\theta(x)}{r}\biggr], \tag{4}$$

where $\theta(x) = rx - \mu(x)$. *Especially, if* 0 *is unattainable* (*i.e., either natural or entrance*) *for* X, *then*

$$\sigma^2(x)\frac{\psi''(x)}{S'(x)} = 2r\int_0^x \psi(y)\bigl(\theta(x) - \theta(y)\bigr)m'(y)\,dy. \tag{5}$$

PROOF. A straightforward application of Dynkin's theorem (cf. [12], page 298) implies that, for all $x \in \mathbb{R}_+$,

$$E_x\bigl[e^{-rT_s} X(T_s)\bigr] = x - E_x\int_0^{T_s} e^{-rs}\theta(X(s))\,ds, \tag{6}$$

where $T_s = s \wedge \tau(s) \wedge \tau(0)$ is an almost surely finite $\mathcal{F}_t$-stopping time, $s \in \mathbb{R}_+$ and $\tau(s) = \inf\{t \ge 0 : X(t) \ge s\}$. Letting s tend to ∞, invoking the transversality condition (2) and reordering terms, we have

$$x = E_x\int_0^{\tau(0)} e^{-rs}\theta(X(s))\,ds. \tag{7}$$

Invoking Definition 1, we can rewrite (7) in terms of the Green kernel in the form

$$x = B^{-1}\varphi(x)\int_0^x \psi(y)\theta(y)m'(y)\,dy + B^{-1}\psi(x)\int_x^\infty \varphi(y)\theta(y)m'(y)\,dy. \tag{8}$$

Standard differentiation then yields

$$1 = B^{-1}\varphi'(x)\int_0^x \psi(y)\theta(y)m'(y)\,dy + B^{-1}\psi'(x)\int_x^\infty \varphi(y)\theta(y)m'(y)\,dy, \tag{9}$$

implying that

$$\frac{d}{dx}\left[\frac{1}{\varphi'(x)}\right] = \frac{2rS'(x)}{\sigma^2(x)\varphi'^2(x)}\left[-\frac{\theta(x)}{r}\frac{\varphi'(x)}{S'(x)} - \int_x^\infty \varphi(y)\theta(y)m'(y)\,dy\right].$$

Since ∞ was assumed to be unattainable, we have

$$-\frac{\varphi'(x)}{S'(x)} = r\int_x^\infty \varphi(y)m'(y)\,dy,$$

implying (3). Analogously, we observe that

$$\frac{d}{dx}\left[\frac{1}{\psi'(x)}\right] = \frac{2rS'(x)}{\sigma^2(x)\psi'^2(x)}\left[\int_0^x \psi(y)\theta(y)m'(y)\,dy - \frac{\psi'(x)}{S'(x)}\frac{\theta(x)}{r}\right].$$

Since

$$\frac{\psi'(x)}{S'(x)} - \frac{\psi'(0)}{S'(0)} = r\int_0^x \psi(y)m'(y)\,dy,$$

we find that

$$-\psi''(x) = \frac{2rS'(x)}{\sigma^2(x)}\left[\int_0^x \psi(y)\big(\theta(y)-\theta(x)\big)m'(y)\,dy - \frac{\psi'(0)}{S'(0)}\frac{\theta(x)}{r}\right],$$

from which (4) follows. If 0 is unattainable for X, then $\psi'(0)/S'(0) = 0$, finally implying (5). □

Theorem 1 demonstrates that under the conditions of this study the signs of $\psi''(x)$ and $\varphi''(x)$ are determined by the signs of the functionals $I_1 : \mathbb{R}_+ \mapsto \mathbb{R}$ and $I_2 : \mathbb{R}_+ \mapsto \mathbb{R}$ defined as

$$I_1(x) = \int_x^\infty \varphi(y)\big(\theta(y)-\theta(x)\big)m'(y)\,dy \tag{10}$$

and

$$I_2(x) = \int_0^x \psi(y)\big(\theta(x)-\theta(y)\big)m'(y)\,dy + \frac{\psi'(0)}{S'(0)}\frac{\theta(x)}{r}. \tag{11}$$

Since $\varphi(x)$, $\psi(x)$ and $m'(x)$ are nonnegative, we observe that it is the behavior of the mapping $\theta(x) = rx - \mu(x)$ (which can be interpreted as the *net depreciation rate* of an asset yielding a revenue flow X) that essentially determines the signs of $\psi''(x)$ and $\varphi''(x)$ and, therefore, the convexity properties of the fundamental solutions (in [4, 5, 6] the mapping $-\theta(x)$ is interpreted as the *convenience yield of holding inventories*). If 0 is a natural boundary for the diffusion X, we find that, given the transversality condition (2), we necessarily have that $\lim_{x\downarrow 0}\theta(x) = 0$, since (cf. [13], page 32)

$$\lim_{x\downarrow 0} E_x \int_0^\infty e^{-rs}\theta(X(s))\,ds = \frac{1}{r}\lim_{x\downarrow 0}\theta(x) = 0.$$

Moreover, it is worth observing that (8) can also be derived by observing that $(r - \mathcal{A})x = \theta(x)$ and invoking standard results characterizing the connection between resolvent operators and the second-order differential operator $(\mathcal{A} - r)$ (cf. [13], pages 29–37, and [14], proof of Theorem 8.1.5 on page 134). A set of interesting implications of Theorem 1 are now stated in the following corollary.

COROLLARY 1. *Assume that $\theta(x)$ is nondecreasing on $\mathbb{R}_+$. Then $\varphi(x)$ is convex on $\mathbb{R}_+$. Moreover, $\psi(x)$ is convex on $\mathbb{R}_+$ if either*:

(i) 0 *is unattainable for X*, *or*
(ii) $\lim_{x\downarrow 0}\mu(x) \le 0$ *and* 0 *is attainable for X*.

PROOF. The alleged results are straightforward consequences of Theorem 1. $\square$

Corollary 1 states conditions under which both the increasing $\psi(x)$ and the decreasing $\varphi(x)$ fundamental solutions are convex on $\mathbb{R}_+$. An obvious but important consequence of Corollary 1 affecting models subject to concave drifts (e.g., logistic diffusions) is now stated in the following corollary.

COROLLARY 2. *Assume that $\mu \in C^1(\mathbb{R}_+)$, that $\mu(x)$ is concave and that*

$$\lim_{x\downarrow 0}\theta'(x) = r - \lim_{x\downarrow 0}\mu'(x) \ge 0.$$

Then $\theta(x)$ is nondecreasing and the conclusions of Corollary 1 *are satisfied*.

PROOF. The concavity of $\mu(x)$ implies that $\theta(x)$ is convex. Since the derivative of a continuously differentiable convex mapping is nondecreasing, we find that the condition $\lim_{x\downarrow 0}\theta'(x) \ge 0$ implies that $\theta'(x) \ge \lim_{x\downarrow 0}\theta'(x) \ge 0$ for all $x \in \mathbb{R}_+$. $\square$

It is now worth noticing that the decreasing fundamental solution $\varphi(x)$ cannot be globally concave, since if it were we would have for all $x > a > 0$ that

$$\varphi(x) \le \varphi(a) + \varphi'(a)(x - a),$$

implying that $\lim_{x\to\infty}\varphi(x) = -\infty$ and, thus, violating the nonnegativity of $\varphi(x)$. However, it is clear from Theorem 1 and especially from (4) that it is possible to state conditions under which the global concavity of the increasing fundamental solution $\psi(x)$ can be guaranteed. Unfortunately, such conditions typically lead to the violation of the transversality condition (2) [and the integrability of $\theta(x)$]. More precisely, we have the following result.

LEMMA 1. *Let $\tau(U) = \inf\{t \geq 0 : X(t) \notin U\}$ denote the first exit time of X from an arbitrary open set $U \subset \mathbb{R}_+$ with compact closure $\bar{U} \subset \mathbb{R}_+$ and assume that* 0 *is not an entrance for X. If the increasing fundamental solution $\psi(x)$ is convex on $\mathbb{R}_+$, then*

$$E_x\big[e^{-r\tau(U)}X(\tau(U))\big] \leq x \tag{12}$$

for all $x \in \mathbb{R}_+$. If $\psi(x)$ is concave on $\mathbb{R}_+$, then it is the opposite inequality that holds (*independently of the boundary behavior of X at* 0).

PROOF. It is clear that the fundamental solution $\psi(x)$ is r-harmonic for the diffusion X (cf. [14], page 171). Thus, if U is an arbitrary open set $U \subset \mathbb{R}_+$ with compact closure $\bar{U} \subset \mathbb{R}_+$, we have

$$E_x\big[e^{-r\tau(U)}\psi\big(X(\tau(U))\big)\big] = \psi(x)$$

for all $x \in \mathbb{R}_+$ (cf. [9], page 7). On the other hand, the convexity of the increasing fundamental solution and the boundary condition $\psi(0) = 0$ (which is valid since 0 is not an entrance; cf. [7], page 19) imply that

$$e^{-r\tau(U)}\psi\big(X(\tau(U))\big) \geq \psi\big(e^{-r\tau(U)}X(\tau(U))\big),$$

since $e^{-r\tau(U)} \in [0, 1]$. Taking expectations and invoking Jensen's inequality then yield that

$$\psi(x) = E_x\big[e^{-r\tau(U)}\psi\big(X(\tau(U))\big)\big] \geq \psi\big(E_x\big[e^{-r\tau(U)}X(\tau(U))\big]\big).$$

The monotonicity of $\psi(x)$ implies that, for all $x \in \mathbb{R}_+$ and arbitrary open set $U \subset \mathbb{R}_+$ with compact closure $\bar{U} \subset \mathbb{R}_+$,

$$E_x\big[e^{-r\tau(U)}X(\tau(U))\big] \leq x,$$

completing the proof in the convex case.

Assume now that $\psi(x)$ is concave. Since $\psi(0) = 0$ if 0 is not an entrance for X and $\psi(0) \geq 0$ if 0 is an entrance for X, we find that $e^{-r\tau(U)}\psi(X(\tau(U))) \leq \psi(e^{-r\tau(U)}X(\tau(U)))$, where $U \subset \mathbb{R}_+$ is an open set with compact closure $\bar{U} \subset \mathbb{R}_+$. Jensen's inequality then yields that

$$\begin{aligned}\psi(x) = E_x\big[e^{-r\tau(U)}\psi\big(X(\tau(U))\big)\big] &\leq E_x\big[\psi\big(e^{-r\tau(U)}X(\tau(U))\big)\big]\\ &\leq \psi\big(E_x\big[e^{-r\tau(U)}X(\tau(U))\big]\big),\end{aligned}$$

completing the proof of our lemma. $\square$

REMARK 1. It is worth observing that the proof of Lemma 1 clearly implies that (12) is also satisfied if one can find a monotonically increasing and convex mapping $h:\mathbb{R}_+\mapsto\mathbb{R}$ satisfying the following conditions:

(i) $h(0)\in(-\infty,0]$;
(ii) $E_x[e^{-r\tau(U)}h(X(\tau(U)))]\le h(x)$ for all $x\in\mathbb{R}_+$ and all open sets $U\subset\mathbb{R}_+$ with compact closure $\bar{U}\subset\mathbb{R}_+$.

In other words, to prove that the process X satisfies (12), it is only sufficient to find a monotonically increasing and convex test function satisfying conditions (i) and (ii).

Lemma 1 shows why the convexity of the increasing fundamental solution is typically needed in order to guarantee the validity of the transversality condition (2). An interesting consequence of (12) is that, given the conditions of Lemma 1, the identity mapping $x\mapsto x$ is r-excessive for the process X, since $x\mapsto x$ is continuous and nonnegative on $\mathbb{R}_+$, and it also satisfies the condition (12) (cf. [9], page 7). Consequently, the process $e^{-rt}X(t)$ constitutes an $\mathcal{F}_t$-supermartingale on $\mathbb{R}_+$ and, therefore, converges almost surely as $t\uparrow\infty$. It is worth emphasizing that this result is closely related to the almost-sure convergence of *Doob's minimal r-excessive transformations* (cf. [15], Definition 2.6).

Let $f:\mathbb{R}_+\mapsto\mathbb{R}_+$ be a continuous mapping on $\mathbb{R}_+$ and define the functional $v:\mathbb{R}_+\mapsto\mathbb{R}_+$ as

$$v(x)=E_x\big[e^{-r\tau_{(a,b)}}f\big(X(\tau_{(a,b)})\big)\big], \tag{13}$$

where $\tau_{(a,b)}=\inf\{t\ge 0;\,X(t)\notin(a,b)\}$ denotes the first exit time from the open interval $(a,b)\subset\mathbb{R}_+$. It is well known that $v(x)$ is r-harmonic for X and that $v(x)$ can be rewritten on (a,b) in terms of the fundamental solutions $\psi(x)$ and $\varphi(x)$ in the form

$$v(x)=f(a)\frac{\varphi(x)-(\varphi(b)/\psi(b))\psi(x)}{\varphi(a)-(\varphi(b)/\psi(b))\psi(a)}+f(b)\frac{\psi(x)-(\psi(a)/\varphi(a))\varphi(x)}{\psi(b)-(\psi(a)/\varphi(a))\varphi(b)}. \tag{14}$$

Our main result summarizing the form of the mapping $v(x)$ is now stated in the following result.

THEOREM 2. *Assume that the conditions of Corollary* 1 *are satisfied and that*

$$\frac{\varphi(b)}{\varphi(a)}\le\frac{f(b)}{f(a)}\le\frac{\psi(b)}{\psi(a)}. \tag{15}$$

Then $v(x)$ is convex on (a,b).

PROOF. It is clear that under the conditions of Corollary 1 the fundamental solutions are convex on $\mathbb{R}_+$. Moreover, $v(x)$ can be rewritten as

$$v(x)=\frac{\psi(b)f(a)-f(b)\psi(a)}{\varphi(a)\psi(b)-\varphi(b)\psi(a)}\varphi(x)+\frac{\varphi(a)f(b)-\varphi(b)f(a)}{\varphi(a)\psi(b)-\varphi(b)\psi(a)}\psi(x).$$

Since the sum of two convex mappings is convex and the multipliers of $\psi(x)$ and $\varphi(x)$ are nonnegative when $\psi(b)f(a) \geq f(b)\psi(a)$ and $\varphi(a)f(b) \geq \varphi(b)f(a)$, the result follows from our assumption (15). $\square$

An important consequence of Theorem 2 is now stated in the following result.

THEOREM 3. *Assume that the conditions of Corollary* 1 *are satisfied*, *that* $f(x)$ *is* r-*excessive for* $\{X(t); t \in [0, \tau(0))\}$ *and that* $f(x)$ *is* r-*harmonic on* (a, b). *Then* $f(x)$ *is convex on* (a, b).

PROOF. The r-excessivity of $f(x)$ implies that $f(x)$ is continuous, nonnegative, and satisfies for all $x, y \in \mathbb{R}_+$ the condition (cf. [7], page 32)

$$\frac{f(y)}{f(x)} \geq \begin{cases} \dfrac{\psi(y)}{\psi(x)}, & x \geq y, \\[2ex] \dfrac{\varphi(y)}{\varphi(x)}, & x \leq y. \end{cases}$$

Thus, we find that f satisfies condition (15). Moreover, the r-harmonicity of f on (a, b) implies that

$$f(x) = \frac{\psi(b)f(a) - f(b)\psi(a)}{\varphi(a)\psi(b) - \varphi(b)\psi(a)}\varphi(x) + \frac{\varphi(a)f(b) - \varphi(b)f(a)}{\varphi(a)\psi(b) - \varphi(b)\psi(a)}\psi(x). \tag{16}$$

The convexity of f then follows from the convexity of the fundamental solutions $\psi(x)$ and $\varphi(x)$. $\square$

REMARK 2. There is an alternative way for proving Theorem 2. It is well known that, for any r-excessive mapping $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ for X, there is a probability measure ν that does not charge the set where $f(x)$ is r-harmonic [i.e., if f is r-harmonic on Λ, then $\nu(\Lambda) = 0$] and such that, for all $x \in \mathbb{R}_+$,

$$f(x) = \int_{(0,\infty)} \frac{G_r(x, y)}{G_r(x_0, y)}\nu(dy) + \frac{\varphi(x)}{\varphi(x_0)}\nu(\{0\}) + \frac{\psi(x)}{\psi(x_0)}\nu(\{\infty\}), \tag{17}$$

where $x_0 \in \mathbb{R}_+$ is a given reference point for which $f(x_0) = 1$ (cf. [7], page 32; see also [15], where this representation is applied for solving optimal stopping problems of linear diffusions). Observing that if $f(x)$ is r-harmonic on (a, b) and $x_0 \in (a, b)$, then

$$f(a) = \frac{\varphi(a)}{\varphi(x_0)}\big[\nu((0, a]) + \nu(\{0\})\big] + \frac{\psi(a)}{\psi(x_0)}\big[\nu([b, \infty)) + \nu(\{\infty\})\big]$$

and

$$f(b) = \frac{\varphi(b)}{\varphi(x_0)}\big[\nu((0, a]) + \nu(\{0\})\big] + \frac{\psi(b)}{\psi(x_0)}\big[\nu([b, \infty)) + \nu(\{\infty\})\big],$$

since

$$0 \le \frac{\psi(a)}{\psi(x_0)} \int_{(a,x_0)} \frac{\varphi(y)}{\psi(y)} \nu(dy) \le \frac{\varphi(a)}{\psi(x_0)} \nu((a, x_0)),$$

$$0 \le \frac{\varphi(b)}{\psi(x_0)} \int_{(x_0,b)} \frac{\psi(y)}{\varphi(y)} \nu(dy) \le \frac{\psi(b)}{\psi(x_0)} \nu((x_0, b)),$$

and $\nu((a, b)) = 0$. A straightforward application of Cramér's rule then yields that

$$\nu((0, a]) + \nu(\{0\}) = \varphi(x_0) \frac{\psi(b) f(a) - \psi(a) f(b)}{\psi(b)\varphi(a) - \psi(a)\varphi(b)}$$

and that

$$\nu([b, \infty)) + \nu(\{\infty\}) = \psi(x_0) \frac{\varphi(a) f(b) - \varphi(b) f(a)}{\psi(b)\varphi(a) - \psi(a)\varphi(b)}.$$

It is then clear from (17) that if $x \in (a, b)$, then

$$f(x) = \frac{\varphi(x)}{\varphi(x_0)} \big[\nu((0, a]) + \nu(\{0\})\big] + \frac{\psi(x)}{\psi(x_0)} \big[\nu([b, \infty)) + \nu(\{\infty\})\big],$$

implying (16).

To illustrate our results explicitly, assume now that $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous mapping satisfying the absolute integrability condition

$$E_x \int_0^{\tau(0)} e^{-rs} |f(X(s))|\, ds < \infty. \tag{18}$$

Define now the functional $R_r f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as

$$(R_r f)(x) = E_x \int_0^{\tau(0)} e^{-rs} f(X(s))\, ds. \tag{19}$$

Since $(R_r f)(x)$ satisfies the ordinary second-order differential equation $((\mathcal{A} - r)R_r f)(x) + f(x) = 0$, we observe that $((\mathcal{A} - r)R_r f)(x) = -f(x) \le 0$ for all $x \in \mathbb{R}_+$ and, therefore, that $(R_r f)(x)$ is r-excessive for $\{X(t); t \in [0, \tau(0))\}$. Moreover, it is also clear that $(R_r f)(x)$ is r-harmonic for $\{X(t); t \in [0, \tau(0))\}$ on the regions where $f(x) = 0$. In light of Definition 1, we find that (19) can be rewritten as

$$\begin{aligned}(R_r f)(x) = {} & B^{-1}\varphi(x) \int_0^x \psi(y) f(y) m'(y)\, dy \\ & + B^{-1}\psi(x) \int_x^\infty \varphi(y) f(y) m'(y)\, dy.\end{aligned} \tag{20}$$

Differentiating (20) twice yields

$$\begin{aligned}(R_r f)''(x) = {} & B^{-1}\varphi''(x) \int_0^x \psi(y) f(y) m'(y)\, dy \\ & + B^{-1}\psi''(x) \int_x^\infty \varphi(y) f(y) m'(y)\, dy - \frac{2 f(x)}{\sigma^2(x)}.\end{aligned}$$

Thus, if $f(x) = 0$ on $(a, b) \subset \mathbb{R}_+$ we find that

$$(R_r f)''(x) = B^{-1}\varphi''(x) \int_0^a \psi(y) f(y) m'(y)\, dy + B^{-1}\psi''(x) \int_b^\infty \varphi(y) f(y) m'(y)\, dy,$$

whenever $x \in (a, b)$. Therefore, we find that whenever the conditions of Corollary 1 are met, $(R_r f)''(x) \geq 0$ for all $x \in (a, b)$. In other words, given the conditions of Corollary 1, the r-excessive mapping $(R_r f)(x)$ is convex on (a, b) as was demonstrated in Corollary 3.

REMARK 3. It is worth pointing out that (20) has an important implication, which can be applied to the identification of the regions where the r-excessive mapping $(R_r f)(x)$ is r-harmonic for $\{X(t); t \in [0, \tau(0))\}$. It is easy to demonstrate that (20) implies that

$$\int_0^x \psi(y) f(y) m'(y)\, dy = \frac{\psi'(x)}{S'(x)}(R_r f)(x) - \frac{(R_r f)'(x)}{S'(x)}\psi(x) := I(x)$$

and that

$$\int_x^\infty \varphi(y) f(y) m'(y)\, dy = \frac{(R_r f)'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}(R_r f)(x) := J(x).$$

Thus,

$$I(b) - I(a) = \int_a^b \psi(y) f(y) m'(y)\, dy$$

and

$$J(a) - J(b) = \int_a^b \varphi(y) f(y) m'(y)\, dy,$$

implying that $I(b) = I(a)$ and $J(a) = J(b)$ on the regions $(a, b) \subset \mathbb{R}_+$ where $f(x) = 0$, that is, on the regions where $(R_r f)(x)$ is r-harmonic for $\{X(t); t \in [0, \tau(0))\}$. Similarly, $I(x)$ is increasing and $J(x)$ is decreasing on the set where $f(x) > 0$, that is, on the set where $(R_r f)(x)$ is r-superharmonic for $\{X(t); t \in [0, \tau(0))\}$ (see [15], Theorem 4.7, for a comparison).

3. Convex inequalities and comparative static analysis. Having considered the convexity properties of the fundamental solutions $\psi(x)$ and $\varphi(x)$, it is our purpose in this section to analyze the comparative static properties of r-harmonic mappings and, especially, of the minimal r-excessive mappings. To accomplish this task, let $\hat{X} = \{\hat{X}(t); t \in [0, \hat{\tau}(0))\}$, where $\hat{\tau}(0) = \inf\{t \geq 0 : \hat{X}(t) \leq 0\}$ (which may be infinite), be a linear, time-homogeneous and regular diffusion defined on the state space $\mathbb{R}_+$ and satisfying the condition that $\hat{X}$ does not die in the interior of $\mathbb{R}_+$. In accordance with the notation of the previous section, let

$$\hat{\mathcal{A}} = \frac{1}{2}\hat{\sigma}^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx} \tag{21}$$

denote the differential operator representing the infinitesimal generator of $\hat{X}$ and assume that $\hat{\sigma}:\mathbb{R}_+ \mapsto (0,\infty)$ is a continuous mapping satisfying the inequality $\hat{\sigma}(x) \geq \sigma(x)$ for all $x \in \mathbb{R}_+$. Thus, $\hat{X}$ can be interpreted as a diffusion evolving at the same expected growth rate as X but subject to greater stochastic fluctuations [in most applications $\hat{\sigma}(x) = \gamma\sigma(x)$, where $\gamma > 1$ is an exogenously given constant]. We immediately find the following result.

LEMMA 2. *Assume that the twice continuously differentiable mapping* $u:(a,b) \mapsto \mathbb{R}_+$ *is* r*-harmonic for* X *on* $(a,b) \subset \mathbb{R}_+$. *That is, assume that* $u(x)$ *satisfies the ordinary differential equation* $((\mathcal{A} - r)u)(x) = 0$ *for all* $x \in (a,b) \subset \mathbb{R}_+$.

(i) *If* $u(x)$ *is convex, then* u *is* r*-subharmonic for* $\hat{X}$ *on* (a,b).
(ii) *If* $u(x)$ *is concave, then* u *is* r*-superharmonic for* $\hat{X}$ *on* (a,b).

PROOF. Assume that $u(x)$ is convex and r-harmonic for X on (a,b). Then we find that, for all $x \in (a,b)$,

$$\big((\hat{\mathcal{A}} - r)u\big)(x) = \big((\hat{\mathcal{A}} - \mathcal{A})u\big)(x) = \tfrac{1}{2}\big(\hat{\sigma}^2(x) - \sigma^2(x)\big)u''(x) \geq 0,$$

implying that u is r-subharmonic for $\hat{X}$ on (a,b). The concave case is proved analogously. □

Let $f:\mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous mapping on $\mathbb{R}_+$ and, in accordance with the notation of the previous section, define the functional $\hat{v}:\mathbb{R}_+ \mapsto \mathbb{R}_+$ as

$$\hat{v}(x) = E_x\big[e^{-r\hat{\tau}_{(a,b)}} f\big(\hat{X}(\hat{\tau}_{(a,b)})\big)\big],$$

where $\hat{\tau}_{(a,b)} = \inf\{t \geq 0; \hat{X}(t) \notin (a,b)\}$ denotes the first exit time from the open interval $(a,b) \subset \mathbb{R}_+$. Our first key result summarizing the comparative static properties for a broad class of r-harmonic mappings is now stated as follows.

THEOREM 4. *Assume that the conditions of Theorem* 2 *are satisfied. Then* $v(x) \leq \hat{v}(x)$ *for all* $x \in \mathbb{R}_+$.

PROOF. Theorem 2 implies that $v(x)$ is convex on (a,b). Lemma 2 then implies that $v(x)$ is r-subharmonic for $\hat{X}$ on (a,b). Therefore,

$$E_x\big[e^{-r\hat{\tau}_{(a,b)}} v\big(\hat{X}(\hat{\tau}_{(a,b)})\big)\big] \geq v(x).$$

Invoking the boundary conditions $v(a) = f(a)$ and $v(b) = f(b)$ and the continuity of $v(x)$ then implies that, on (a, b),

$$\hat{v}(x) = E_x\big[e^{-r\hat{\tau}_{(a,b)}} f\big(\hat{X}(\hat{\tau}_{(a,b)})\big)\big] \geq v(x),$$

completing the proof of our theorem. □

Theorem 4 shows that increased stochastic fluctuations (i.e., volatility) increase or leave unchanged the value of r-harmonic mappings of the type (13) whenever the conditions of Theorem 2 are satisfied. Thus, we find that *the sign of the relationship between volatility and the value of an r-harmonic mapping* (13) *is a process-specific property that does not depend on the form of the underlying mapping $f(x)$*. Moreover, another interesting consequence of Theorem 4 is that if the conditions of Theorem 2 are satisfied, then

$$\sup_{a<b} E_x\big[e^{-r\hat{\tau}_{(a,b)}} f\big(\hat{X}(\hat{\tau}_{(a,b)})\big)\big] \geq \sup_{a<b} E_x\big[e^{-r\tau_{(a,b)}} f\big(X(\tau_{(a,b)})\big)\big].$$

Let $\hat{\psi}(x)$ now denote the increasing and $\hat{\varphi}(x)$ the decreasing fundamental solution of the ordinary differential equation $((\hat{\mathcal{A}} - r)u)(x) = 0$. A set of important implications of Theorem 4 is now summarized in our next corollary.

COROLLARY 3. *Assume that the conditions of Theorem* 2 *are satisfied. Then*

$$E_x[e^{-r\hat{\tau}_{(a,b)}}] \geq E_x[e^{-r\tau_{(a,b)}}]. \tag{22}$$

Especially,

$$\frac{\hat{\psi}(x)}{\hat{\psi}(b)} \geq \frac{\psi(x)}{\psi(b)}$$

for all $x < b < \infty$, and

$$\frac{\hat{\varphi}(x)}{\hat{\varphi}(a)} \geq \frac{\varphi(x)}{\varphi(a)}$$

for all $0 < a < x$.

PROOF. Equation (22) follows directly from Theorem 4 by choosing $f \equiv 1$. The other two inequalities are simple consequences of the convexity of the fundamental solutions $\psi(x)$ and $\varphi(x)$ and the results (cf. [7], page 18)

$$E_x[e^{-r\tau_{(0,b)}}] = \frac{\psi(x)}{\psi(b)}$$

when $x \leq b$ and

$$E_x[e^{-r\tau_{(a,\infty)}}] = \frac{\varphi(x)}{\varphi(a)}$$

when $x \geq a$. $\square$

Corollary 3 proves the intuitively clear result that, given the conditions of Theorem 2, increased stochastic fluctuations increase or leave unchanged the value of functionals of the type $E_x[e^{-r\tau_{(a,b)}}]$ by speeding up the rate at which the process exits from an arbitrary open interval (a, b).

4. Optimal stopping. Given the assumptions of Section 2, we now plan to consider the optimal stopping problem

$$V(x) = \sup_{\tau < \tau(0)} E_x\big[e^{-r\tau} g(X(\tau))\big], \tag{23}$$

where τ is an arbitrary $\mathcal{F}_t$-stopping time subject to the constraint $\tau < \tau(0)$ and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous mapping satisfying the integrability condition

$$E_x\big[e^{-rt} g\big(X(t \wedge \tau(0))\big)\big] < \infty \tag{24}$$

for all $(t, x) \in \mathbb{R}_+^2$. Our main result characterizing generally the form of the value function on the continuation region $\{x \in \mathbb{R}_+ : V(x) > g(x)\}$ under a set of usually satisfied conditions is now summarized in the next theorem.

THEOREM 5. *Assume that the conditions of Theorem* 3 *are satisfied. Then* $V(x)$ *is convex on the continuation region* $\{x \in \mathbb{R}_+ : V(x) > g(x)\}$, *and increased stochastic fluctuations increase or leave unchanged the value and expand or leave unchanged the continuation region. That is,* $\hat{V}(x) \geq V(x)$ *for all* $x \in \mathbb{R}_+$ *and* $\{x \in \mathbb{R}_+ : V(x) > g(x)\} \subseteq \{x \in \mathbb{R}_+ : \hat{V}(x) > g(x)\}$, *where*

$$\hat{V}(x) = \sup_{\tau < \hat{\tau}(0)} E_x\big[e^{-r\tau} g\big(\hat{X}(\tau)\big)\big],$$

and $\hat{X}$ *is defined as in Section* 3. *Especially, if the reward* $g(x)$ *is convex on* $\mathbb{R}_+$, *then* $V(x)$ *is convex on* $\mathbb{R}_+$ *as well.*

PROOF. Under the conditions of Theorem 3, all r-excessive mappings for $\{X(t); t \in [0, \tau(0))\}$ are convex on the regions where they are r-harmonic. Since $V(x)$ is the least of the r-excessive majorants of $g(x)$ for $\{X(t); t \in [0, \tau(0))\}$ and $V(x)$ is r-harmonic on the continuation region $C = \{x \in \mathbb{R}_+ : V(x) > g(x)\}$, Theorem 4 implies that, on C,

$$E_x\big[e^{-r\hat{\tau}(C)} V\big(\hat{X}(\hat{\tau}(C))\big)\big] \geq V(x),$$

where $\hat{\tau}(C) = \inf\{t \geq 0 : \hat{X}(t) \notin C\}$. Invoking now the continuity of $V(x)$ across the boundary of the continuation region then yields

$$V(x) \leq E_x\big[e^{-r\hat{\tau}(C)} g\big(\hat{X}(\hat{\tau}(C))\big)\big] \leq \sup_{\tau < \hat{\tau}(0)} E_x\big[e^{-r\tau} g\big(\hat{X}(\tau)\big)\big] = \hat{V}(x),$$

proving that increased stochastic fluctuations increase or leave unchanged $V(x)$ and, therefore, that increased volatility expands or leaves unchanged the continuation region, since clearly $\{x \in \mathbb{R}_+ : V(x) > g(x)\} \subseteq \{x \in \mathbb{R}_+ : \hat{V}(x) > g(x)\}$. Finally, since $V(x) = g(x)$ on the stopping region, we find that if the reward $g(x)$ is convex, then $V(x)$ is convex as well. $\square$

Theorem 5 states a result that is of essential importance in economic and financial applications of optimal stopping since it demonstrates that, given the conditions of Theorem 3, *the value function is always convex on the continuation region independently of the form of the underlying reward*. As a consequence, we find that *the form of the value for pre-exercise states is inherently a process-specific property, not payoff-specific*. This argument clearly emphasizes the role of the form of the fundamental solutions $\psi(x)$ and $\varphi(x)$ as the *determinants of the sign of the relationship between volatility and both the value and the optimal stopping policy of an optimal stopping problem*. Two interesting associated results are now presented in our next two theorems.

THEOREM 6. *Assume that $g(x)$ is nondecreasing and continuous on $\mathbb{R}_+$. Define the mapping $\zeta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as*

$$\zeta(x) = \frac{g(x)}{\psi(x)}$$

and assume that:

(i) $\zeta(x)$ *attains a unique interior global maximum at* $\tilde{x} = \arg\max\{\zeta(x)\}$;
(ii) $g \in C^1((\tilde{x}, \infty)) \cap C^2((\tilde{x}, \infty) \setminus D)$, *where D is a set of measure* 0;
(iii) *the mapping*

$$K_0(x) = \frac{g'(x)}{S'(x)}\psi(x) - \frac{\psi'(x)}{S'(x)}g(x)$$

is decreasing for $x > \tilde{x}$.

Then the optimal stopping time in (23) *is* $\tau = \inf\{t \geq 0 : X(t) \notin (0, \tilde{x})\}$, *and the value reads as*

$$V(x) = \psi(x) \sup_{y \geq x} \zeta(y) = \begin{cases} g(x), & x \geq \tilde{x}, \\ \psi(x)\zeta(\tilde{x}), & x < \tilde{x}. \end{cases} \tag{25}$$

Especially, if the conditions of Corollary 1 *are satisfied, then $V(x)$ is convex on $(0, \tilde{x})$, and increased volatility increases or leaves unchanged the value $V(x)$ and expands or leaves unchanged the continuation region $\{x \in \mathbb{R}_+ : V(x) > g(x)\} = (0, \tilde{x})$ by increasing or leaving unchanged the optimal stopping boundary $\tilde{x}$.*

PROOF. Denote the proposed value function as $V^*(x)$. Since $V^*(x) = E_x[e^{-r\tau^*} g(X(\tau^*))]$, where $\tau^* = \inf\{t \geq 0 : X(t) \notin (0, \tilde{x})\}$, we find immediately that $V(x) \geq V^*(x)$ for all $x \in \mathbb{R}_+$. To prove the opposite inequality, observe first that $V^*(x)$ is continuous, twice continuously differentiable on $\mathbb{R}_+ \setminus D$, nonnegative and dominates the reward $g(x)$ for all $x \in \mathbb{R}_+$. Moreover, since $(\mathcal{A}V)(x) = rV(x)$ on $(0, \tilde{x})$, our monotonicity assumptions on the mapping $K_0(x)$ imply that for all $x \in (\tilde{x}, \infty) \setminus D$ we have $K_0'(x) = ((\mathcal{A}g)(x) - rg(x))\psi(x)m'(x) = ((\mathcal{A}V)(x) - rV(x))\psi(x)m'(x) \leq 0$, implying that $V^*(x)$ is an r-excessive majorant of the reward $g(x)$ (cf. [7], page 32, and [14], pages 214–217). However, since $V(x)$ is the least of such majorants, we find that $V^*(x) \geq V(x)$, completing the proof of the first part of our theorem. The rest of the proof follows directly from Theorem 5. □

THEOREM 7. *Assume that $g(x)$ is nonincreasing and continuous on $\mathbb{R}_+$. Define the mapping $\rho : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as*

$$\rho(x) = \frac{g(x)}{\varphi(x)}$$

and assume that:

(i) *$\rho(x)$ attains a unique interior global maximum at $\hat{x} = \arg\max\{\rho(x)\}$*;
(ii) *$g \in C^1((0, \hat{x})) \cap C^2((0, \hat{x}) \setminus D)$, where D is a set of measure* 0;
(iii) *the mapping*

$$K_\infty(x) = \frac{g'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}g(x)$$

is decreasing for $x < \hat{x}$.

Then the optimal stopping time in (23) *is $\tau = \inf\{t \geq 0 : X(t) \notin (\hat{x}, \infty)\}$, and the value reads as*

$$V(x) = \varphi(x) \sup_{y \leq x} \rho(y) = \begin{cases} \varphi(x)\rho(\hat{x}), & x > \hat{x}, \\ g(x), & x \leq \hat{x}. \end{cases} \tag{26}$$

Especially, if the conditions of Corollary 1 *are satisfied, then $V(x)$ is convex on $(\hat{x}, \infty)$, and increased volatility increases or leaves unchanged $V(x)$ and expands or leaves unchanged the continuation region by decreasing or leaving unchanged the optimal stopping boundary $\hat{x}$.*

PROOF. The proof is completely analogous with the proof of Theorem 6. □

Theorems 6 and 7 state a set of easily verifiable conditions under which the value of the optimal stopping problem (23) can be explicitly determined in terms of the fundamental solutions $\psi(x)$ and $\varphi(x)$, especially, *under which the standard financial conclusion stating that increased volatility should increase the value and postpone the exercise of a real investment opportunity holds true* (cf. [1, 2, 3, 8]).

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