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BARRIER OPTIONS AND TOUCH-AND-OUT OPTIONS UNDER REGULAR LÉVY PROCESSES OF EXPONENTIAL TYPE

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We derive explicit formulas for barrier options of European type and touch-and-out options assuming that under a chosen equivalent martingale measure the stock returns follow a Lévy process from a wide class, which contains Brownian motions (BM), normal inverse Gaussian processes (NIG), hyperbolic processes (HP), normal tilted stable Lévy processes (NTS Lévy), processes of the KoBoL family and any finite mixture of independent BM, NIG, HP, NTS Lévy and KoBoL processes. In contrast to the Gaussian case, for a barrier option, a rebate must be specified not only at the barrier but for all values of the stock on the other side of the barrier. We consider options with a constant or exponentially decaying rebate and options which pay a fixed rebate when the first barrier has been crossed but the second one has not. We obtain pricing formulas by solving boundary problems for the generalized Black–Scholes equation. We use the representation of the q-order harmonic measure of a set relative to a point in terms of the q-potential measure, the Wiener–Hopf factorization method and elements of the theory of pseudodifferential operators.

1. Introduction. Various aspects of pricing of barrier options and touch-and-out options have been considered in a number of papers and books [see, e.g., Rubinstein and Reiner (1991), Wilmott, Dewynne and Howison (1995), Musiela and Rutkowski (1997) and Ingersoll (2000) and the bibliography therein], but to the best of our knowledge only Gaussian processes have been allowed.

In this paper, we consider the case when the returns $X_t = \ln S_t$ on the stock S_t follow a Lévy process from a wide class of processes, which we introduced in Boyarchenko and Levendorskiĭ (1999, 2000) under the name generalized truncated Lévy processes. In a recent paper, in which a generalization of the class for Feller processes is developed, Barndorff-Nielsen and Levendorskiĭ (2001) suggest a new name, "regular Lévy processes of exponential type" (RLPE), and so we will use the new name. (The second author thanks A. N. Shiryaev for pointing out that the old name was noninformative.)

If the Lévy process is neither the Brownian motion nor the Poisson process, the market is incomplete. According to the modern martingale approach to option pricing [see Delbaen and Schachermayer (1994, 1998) and references

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therein], arbitrage-free prices can be obtained as expectations under any equivalent martingale measure (EMM), which is absolutely continuous w.r.t. the historic measure. We assume that the riskless rate $r > 0$ is constant, and an EMM $\mathbf{Q}$ is chosen so that, under $\mathbf{Q}$, X is an RLPE, and we derive explicit formulas for the prices of barrier options on the stock with one fixed barrier and touch-and-out options. In forthcoming papers, we will consider cases of time-dependent barriers and double barrier options [the latter are considered in, e.g., Geman and Yor (1996)].

Notice that, in contrast to the Gaussian case, a rebate (if any) must be specified not only at the barrier but for all values of the stock on the other side of the barrier, the reason being that trajectories of a non-Gaussian Lévy process are discontinuous. In particular, we calculate the value of an option with the constant or exponentially decaying rebate; our general formulas give also explicit formulas for options which pay a fixed rebate when the first barrier has been crossed but the second barrier (situated farther than the first one) has not. We also consider touch-and-out options; they can be considered as barrier options with constant rebate and zero terminal payoff, so the treatment is essentially the same (and even simpler).

The class of regular Lévy processes of exponential type contains, in particular, Brownian motions (BM), normal inverse Gaussian processes (NIG), normal tilted stable Lévy processes (NTS Lévy processes), hyperbolic processes (HP), generalized hyperbolic processes (GHP), truncated Lévy processes (TLP) and any finite mixture of independent processes of these model classes. Not only BM, but the other mentioned processes as well have been widely used to describe the behavior of stock prices in real financial markets.

Hyperbolic processes were constructed and used by Eberlein and co-authors [Eberlein and Keller (1995); Eberlein, Keller and Prause (1998); Eberlein and Prause (1998)]; hyperbolic distributions were constructed by Barndorff-Nielsen (1977).

Normal inverse Gaussian processes were introduced by Barndorff-Nielsen (1998) and used to model German stocks by Barndorff-Nielsen and Jiang (1998); in Eberlein and Prause (1998) and Eberlein (1999), generalized hyperbolic processes were constructed, which contained NIG and HP as subclasses.

The class of NTS Lévy processes was introduced in Barndorff-Nielsen and Levendorskiĭ (2001) and studied in Barndorff-Nielsen and Shephard (2001); it contains NIG as a subclass.

Truncated Lévy processes constructed by Koponen (1995) were used for modeling in real financial markets by Bouchaud and Potters (1997), Cont, Potters and Bouchaud (1997) and Matacz (1997); a simple generalization of this family was constructed in Boyarchenko and Levendorskiĭ (1999, 2000) (the generalization was needed since the probability distributions of Koponen's family have tails of the same rate of exponential decay whereas, in real financial markets, the left tail is usually much fatter). We will call this generalization the KoBoL family. [Later the KoBoL family was used by Carr, Geman, Madan and Yor

(2001) under the name CGMY-model.] Earlier, noninfinitely divisible truncations of stable Lévy distributions had been constructed and used to model the behavior of the Standard & Poor 500 Index by Mantegna and Stanley (1994, 1997).

In the name of the class under consideration, *of exponential type* means that tails of PDF are exponentially decaying, and *regular* indicates that generators of these processes enjoy very favorable features from the point of view of the theory of pseudodifferential operators (PDO); roughly speaking, regular Lévy processes are the best class one can find if the Brownian motion is not available. [We recall the definition of PDO in Section 2; for basic facts of the theory of PDO, see Eskin (1973) and Taylor (1981).] This is important since the PDO technique is very powerful. We applied it in Boyarchenko and Levendorskiĭ (2000, 2002), where we obtained explicit analytical formulas for perpetual American options, showed that the smooth fit principle failed in some cases and suggested a substitute for it. Later, Mordecki (2000) obtained pricing formulas for perpetual American puts and calls by using the probabilistic technique, but without explicit analytic formulas for processes observed in financial markets; his method allows one neither to notice the failure of the smooth fit principle nor to suggest a substitute for it.

By using the Dynkin formula, we reduced the optimal stopping problem to a free boundary problem; to solve the latter, we used the Wiener–Hopf factorization technique in the form of Eskin (1973). In this paper, we use the representation of the q-order harmonic measure of a set relative to a point in terms of the q-potential measure to reduce the pricing problem to the corresponding boundary problem for the generalized Black–Scholes equation; the latter is solved by means of the Wiener–Hopf factorization technique, the Fourier transform and the theory of PDO, and in the end we obtain explicit pricing formulas for barrier options and touch-and-out options.

The pricing formulas are expressed in terms of the factors in the Wiener–Hopf factorization formula, and hence can be expected to hold for variance Gamma processes (VGP) used by Madan and co-authors in a series of papers during the 1990s [see Madan (1999), Madan, Carr and Chang (1998) and the bibliography therein], but the analytical formulas for the factors and some technical details of the proofs are invalid for VGP.

Notice that if X is a process of any of the classes listed above, it belongs to the same class under the Esscher transform of the historic measure, which was used, for example, by Madan and co-authors and Eberlein and co-authors. In Boyarchenko and Levendorskiĭ (1999) we have shown that if X is an RLPE, then it remains a regular Lévy process of exponential type under EMM's from a wide class. This justifies our standing assumption below that X is an RLPE under a chosen EMM.

The reader may decide that our technique is too heavy for the relatively simple case of Lévy processes, when one can obtain many results by using the fluctuation identities and simple tools from complex analysis (we are grateful to the anonymous referee for bringing this fact to our attention), but our main concern is to introduce a general scheme which can be applied not only in the case of one-dimensional Lévy processes but in the multidimensional case and in a much more general situation of Lévy-like Feller processes [cf. Barndorff-Nielsen and Levendorskiĭ (2001), where approximate pricing formulas for European options have been derived]. The Wiener–Hopf method in the form common in the theory of boundary-value problems for PDO [see Eskin (1973)], which we use here, admits straightforward generalization to the multidimensional case, and the methods of Barndorff-Nielsen and Levendorskiĭ (2001) and the present paper taken together will produce approximate pricing formulas for barrier options under Lévy-like Feller processes, with possible applications to interest-rate derivatives. We plan to study these possibilities in the future. In addition, the formulas which our method gives are convenient for developing much simpler approximate formulas for touch-and-out options and some barrier options; the corresponding results will be published elsewhere.

The plan of the paper is as follows. In Section 2, we reduce the pricing problem of a contingent claim to the corresponding boundary problem for the generalized Black–Scholes equation and give the scheme of the solution of these problems for some barrier options and touch-and-out options. Notice that this part admits a generalization to the case of a strong Markov process with absolutely continuous q-potential measure, and constructions in the rest of the paper can be modified and used in the case of Lévy-like Feller processes introduced in Barndorff-Nielsen and Levendorskiĭ (2001), the difference being that here the infinitesimal generator of the (Lévy) process is a PDO with constant symbol (i.e., the symbol depends only on the dual variable), and the infinitesimal generator of a Lévy-like Feller process is a PDO with nonconstant symbol (i.e., with a nontrivial dependence on the state variable). If the symbol depends appropriately on a small (and/or large) parameter, the PDO technique allows one to derive and prove approximate formulas for solutions of boundary value problems for PDO with nonconstant symbols, when complex analysis alone does not suffice.

In Section 3, we give the definition of regular Lévy processes of exponential type and examples, and discuss the main properties of the "generalized Black–Scholes operator" on the real line. In Section 4, we obtain explicit formulas for the factors in the Wiener–Hopf factorization formula and obtain necessary estimates. We also give formulas (in terms of PDO) for the solutions of the boundary problems which are needed in Sections 5 and 6, where we explicitly calculate prices of touch-and-out options and barrier options, respectively. Section 7 concludes, and in the Appendix we prove some auxiliary technical estimates.

2. Pricing of contingent claims and boundary problems for generalized Black–Scholes equation. Consider a model market of a bond yielding the riskless rate of return $r > 0$, and a stock, for which price at time t is denoted by $S_t = \exp X_t$. We assume that $X = \{X_t\}$ is a Lévy process under a chosen equivalent martingale measure $\mathbf{Q}$ on a filtered probability space $(\Omega, \mathcal{F}; (\mathcal{F}_t); \mathbf{P})$

satisfying the usual hypotheses. Let $L := L_X^{\mathbf{Q}}$ be the infinitesimal generator of the transition semigroup of $\{X_t\}$ under $\mathbf{Q}$. Consider a contingent claim; its price at time t we denote by $f(t, X_t)$. Denote by $\mathcal{C}$ the continuing observation region for the claim; for example, for a down-and-out option with expiry date T and barrier $H = e^h$, $\mathcal{C} = (h, +\infty) \times (-\infty, T)$. By analogy with the initial Merton–Black–Scholes approach, we derive the equation (generalized Black–Scholes equation), which the function f obeys on $\mathcal{C}$, and by adding appropriate boundary conditions, which specify a given claim, we obtain a well-posed problem. By solving the problem, we find $f(t, X_t)$, the price of the contingent claim. Though the setup is similar to the initial one, the technique differs significantly at some steps since we no longer live in the Gaussian world; in particular, it is simpler to use not the Itô–Meyer formula but the relation between the generator of the process and the resolvent, and it is necessary to use the Wiener–Hopf factorization method; the representation theorem for analytical semigroups [see Yosida (1964)] can be used to simplify some technical details. At the same time, the technique we use here produces answers in the Gaussian case as well.

2.1. *Reduction to boundary problems for the generalized Black–Scholes equation.* Let X be an n-dimensional Lévy process. Introduce an $n+1$-dimensional process $\hat{X}_t = (X_t, t)$ on the state space $\hat{E} = \mathbf{R}^n \times (-\infty, T]$; its generator is $\hat{L} = \partial_t + L$. Set $B := \hat{E} \setminus \mathcal{C}$, and notice that, for $\hat{X}_t \in B$, the value of the contingent claim is specified by the contract: $f(\hat{X}_t) = g(\hat{X}_t)$, where $g(\hat{X}_t)$ is the payoff (including rebate). Let T'_B be the hitting time of B. If the contingent claim price is a local martingale under $\mathbf{Q}$, we must have

$$f(\hat{x}) = E^{\hat{x}}\big[e^{-rT'_B} g(\hat{X}_{T'_B})\big]. \tag{2.1}$$

[Here $\hat{x} = (x, 0)$.] By comparing (2.1) with Dynkin's formula

$$f(\hat{x}) = E^{\hat{x}}\left[\int_0^{T'_B} e^{-rt}(r - \hat{L}) f(\hat{X}_t)\, dt\right] + E^{\hat{x}}\big[e^{-rT'_B} f(\hat{X}_{T'_B})\big], \tag{2.2}$$

one is tempted to conclude that f solves the following boundary value problem:

$$(r - \hat{L}) f(\hat{x}) = 0, \qquad \hat{x} \in \mathcal{C}, \tag{2.3}$$

$$f(\hat{x}) = g(\hat{x}), \qquad \hat{x} \in B. \tag{2.4}$$

By returning to the variables (x, t), we can rewrite (2.3)–(2.4) as a boundary problem for the generalized Black–Scholes equation:

$$(\partial_t + L - r) f(x, t) = 0, \qquad (x, t) \in \mathcal{C}, \tag{2.5}$$

$$f(x, t) = g(x, t), \qquad (x, t) \in B. \tag{2.6}$$

Usually, one writes (2.6) as a pair: the terminal condition and the boundary condition. For instance, for a down-and-out call option with expiry date T, strike price K, barrier H and without a rebate, (2.6) turns into the following pair:

$$f(x,T) = (e^x - K)_+, \qquad x > \ln H, \tag{2.7}$$

$$f(x,t) = 0, \qquad x \le \ln H,\ t \le T. \tag{2.8}$$

Suppose, one can show that a solution f to the problem (2.3)–(2.4) in an appropriate function class exists and that the f is unique and sufficiently regular that Dynkin's formula applies to the f; then we conclude that the f is the contingent claim price we look for. However, it not easy to realize this program even in some cases when an explicit analytical formula for the f can be obtained, and therefore it is advisable to have a general result which enables one not to be bothered with the verification problem.

We work with RLPE's, and it is easily seen from the definition of RLPE, which we give in Section 3, that the transition density of an RLPE X is absolutely continuous. Then the q-potential measure of the process $\hat{X}$ is absolutely continuous (the verification is straightforward), and therefore, in the case of RLPE, the reduction of the calculation of the price (2.1) to the solution of the boundary problem for the generalized Black–Scholes equation is justified by the theorem in the next subsection.

2.2. *Reduction to the stationary Black–Scholes equation.* In this subsection, X is an n-dimensional Lévy process with absolutely continuous potential measures, which means that its q-potential operators (resolvent operators) U^q,

$$U^q f(x) = E^x\left[\int_0^\infty e^{-qt} f(X_t)\,dt\right],$$

admit the representation

$$U^q f(x) = \int_{\mathbf{R}^n} v^q(y-x) f(y)\,dy,$$

where v^q is nonnegative and measurable. One says that X satisfies the (ACP)-condition [see Sato (1999), page 288].

Let $B \subset \mathbf{R}^n$ be an F_σ-set [for the general definition, see Sato (1999), page 279; for our purposes, it suffices to notice that Borel sets are F_σ-sets], and let T'_B be the hitting time of B by X. For $g \in L_\infty(B)$, define

$$P^q_B g(x) := E^x\left[e^{-qT'_B} g(X_{T'_B})\right]. \tag{2.9}$$

The map $g \mapsto P^q_B g(x)$ defines the measure which is called the q-order harmonic measure of B relative to x.

THEOREM 2.1. *Let X be an n-dimensional Lévy process satisfying the* (*ACP*)-*condition*. *Let $B \subset \mathbf{R}^n$ be a closed set*, *and let B^c be its complement*. *Then the* *function* $f = P_B^q g$ *is a bounded solution to the following boundary value problem*:

$$(q - L) f(x) = 0, \qquad x \in B^c, \tag{2.10}$$

$$f(x) = g(x), \qquad x \in B, \tag{2.11}$$

where (2.10) *is understood in the sense of generalized functions*,

$$\langle f, (q - \tilde{L})w\rangle_{L_2} = 0 \qquad \forall\, w \in C_0^\infty(B^c). \tag{2.12}$$

Here $\tilde{L}$ is the generator of the dual process $\tilde{X} = -X$, and $L_2 = L_2(\mathbf{R}^n; \mathbf{R})$ is the real space.

The proof is based on the following result [see, e.g., Sato (1999), Proposition 42.13, Theorem 42.5 and Definition 42.6].

LEMMA 2.2. *Let B be an F_σ-set. There exists a σ-finite measure $d\mu_B$ supported on $\bar{B}$ such that*

$$P_B^q 1(x) = \int_{\mathbf{R}^n} v^q(y - x)\, d\mu_B(y).$$

If g is constant and B is closed, then (2.12) follows from Lemma 2.2:

$$\langle P_B^q 1, (q - \tilde{L})w\rangle_{L_2} = 0 \qquad \forall\, w \in C_0^\infty(B^c). \tag{2.13}$$

We derive the more general statement

$$\langle P_B^q g, (q - \tilde{L})w\rangle_{L_2} = 0 \qquad \forall\, w \in C_0^\infty(B^c) \tag{2.14}$$

from (2.13) by using an additional assumption which holds for RLPE's: the characteristic exponent ψ of X admits analytic continuation into a tube domain $\mathbf{R}^n + iU$, where U is an open set $U \subset \mathbf{R}^n$ containing 0.

For $\gamma \in \mathbf{R}^n$, define a function u_γ by $u_\gamma(x) = e^{\langle \gamma, x\rangle}$. The first step of the proof is the following lemma.

LEMMA 2.3. *Let $-\gamma \in U$ and $q + \psi(-i\gamma) > 0$. Then $g = u_\gamma$ satisfies* (2.14).

The proof is based on the change of the probability measure: $\mathbf{P}_\gamma = e^{\langle \gamma, x\rangle + t\psi(-i\gamma)}\mathbf{P}$. Under the new measure, X is the Lévy process with characteristic exponent $\psi_\gamma(\xi) = \psi(\xi - i\gamma) - \psi(-i\gamma)$ since

$$E^{\mathbf{P}_\gamma}\big[e^{i\xi X_t}\big] = E^{\mathbf{P}}\big[e^{t\psi(-i\gamma) + i\langle \xi - i\gamma, X_t\rangle}\big] = \exp\big[-t\big(\psi(\xi - i\gamma) - \psi(-i\gamma)\big)\big].$$

Let E_γ, $\{P_{\gamma,t}\}_{t\ge 0}$ and L_γ be the corresponding expectation operator, semigroup and infinitesimal generator. We have

$$\begin{aligned} e^{-\langle \gamma, x\rangle}(P_t u_\gamma g)(x) &= \int_{\mathbf{R}^n} e^{-\langle \gamma, x\rangle} e^{\langle \gamma, x+y\rangle} g(x+y)\mathbf{P}(X_t \in dy) \\ &= e^{-t\psi(-i\gamma)} \int_{\mathbf{R}^n} g(x+y) e^{\langle \gamma, y\rangle + t\psi(-i\gamma)}\mathbf{P}(X_t \in dy) \\ &= e^{-t\psi(-i\gamma)}(P_{\gamma,t} g)(x); \end{aligned} \tag{2.15}$$

therefore

$$e^{-\langle\gamma,x\rangle} P_B^q u_\gamma(x) = E_\gamma^x\big[e^{-(q+\psi(-i\gamma))T'_B}\big] =: P_{\gamma,B}^{q+\psi(-i\gamma)}1(x).$$

Since $q+\psi(-i\gamma)>0$, (2.13) applies to $P_{\gamma,B}^{q+\psi(-i\gamma)}1$ and $\tilde{L}_\gamma$:

$$\big(P_{\gamma,B}^{q+\psi(-i\gamma)}1, (q+\psi(-i\gamma)-\tilde{L}_\gamma)w\big)=0 \qquad \forall\, w\in C_0^\infty(B^c),$$

or, equivalently,

$$\big(u_\gamma^{-1}P_B^q u_\gamma, (q+\psi(-i\gamma)-\tilde{L}_\gamma)u_\gamma w'\big)=0 \qquad \forall\, w'\in C_0^\infty(B^c),$$

and finally,

$$\big(P_B^q u_\gamma, (q-u_\gamma^{-1}(\tilde{L}_\gamma-\psi(-i\gamma))u_\gamma)w'\big)=0 \qquad \forall\, w'\in C_0^\infty(B^c). \tag{2.16}$$

From (2.15), $u_\gamma(L_\gamma-\psi(-i\gamma))u_\gamma^{-1}=L$, hence $u_\gamma^{-1}(\tilde{L}_\gamma-\psi(-i\gamma))u_\gamma=\tilde{L}$, and (2.16) gives (2.14). Lemma 2.3 has been proved.

For the next step, consider a compact B and an arbitrary bounded measurable g. We can approximate g in the L^∞-norm by continuous functions, and each continuous function by polynomials (the Stone–Weierstrass theorem). Fix $\varepsilon>0$ such that $V(\varepsilon):=\{\gamma \mid \|\gamma\|\le\varepsilon\}\subset U$, and $q+\psi(-i\gamma)>0$ for all $\gamma\in V(\varepsilon)$. Since, for any multiindex α,

$$\lim_{\gamma\to 0}\prod_{j=1}^n \gamma_j^{-\alpha_j}(e^{\gamma_j x_j}-1)^{\alpha_j}\to\prod_{j=1}^n x_j^{\alpha_j},$$

uniformly in $x\in B$, we can approximate g in the L^∞-norm by a sequence of functions of the form

$$g_N(x)=\sum_{-\gamma\in\Gamma_N} c_{N,\gamma}e^{\langle\gamma,x\rangle},$$

where $\Gamma_N\subset V(\varepsilon)$ is finite. Since (2.14) is valid for $g=u_\gamma$, provided $-\gamma\in U$ and $q+\psi(-i\gamma)>0$, it holds for $g=g_N$. By passing to the limit $\varepsilon\to 0$, $N\to+\infty$, we conclude that (2.14) holds for any bounded measurable g and compact B.

It remains to drop the assumption that B is bounded. Notice that it suffices to consider nonnegative g. For $R>0$, set $B(R)=B\cap V(R)$, and show first that, for any x,

$$P_{B(R)}^q g(x)\to P_B^q g(x) \qquad \text{as } R\to+\infty. \tag{2.17}$$

We have

$$E^x\big[e^{-qT'_{B(R)}}g\big(X(T'_{B(R)})\big)\big]\le E^x\big[e^{-qT'_B}g\big(X(T'_B)\big)\big]+\|g\|_\infty E^x\big[e^{-qT'_{B\setminus B(R)}}\big]$$

and

$$E^x\big[e^{-qT'_B}g\big(X(T'_B)\big)\big]\le E^x\big[e^{-qT'_{B(R)}}g\big(X(T'_{B(R)})\big)\big]+\|g\|_\infty E^x\big[e^{-qT'_{B\setminus B(R)}}\big].$$

For a Lévy process, $T'(B \setminus B(R)) \to +\infty$ as $R \to +\infty$, a.s.; hence $E^x[e^{-qT'_{B\setminus B(R)}}] \to 0$ as $R \to +\infty$, and (2.17) follows.

By the previous step, (2.14) holds with $B(R)$ instead of B; hence

$$\int_{\mathbf{R}^n} P^q_{B(R)} g(x)(q - \tilde{L})w(x)\,dx = 0 \qquad \forall\, w \in C_0^\infty(B^c), \tag{2.18}$$

and by (2.17), the integrand in (2.18) converges pointwise to $P^q_B g(x)(q - \tilde{L})w(x)$ as $R \to +\infty$. Since $P^q_{B(R)} g(x)$ is bounded uniformly in R and x, and $(q - \tilde{L})w \in L_1(\mathbf{R}^n)$, we can pass to the limit under the integral sign and obtain (2.18) with B instead of $B(R)$. This finishes the proof of (2.14).

REMARK 2.4. (a) In each concrete case, we find bounded solutions of the problem (2.10)–(2.11) by using tools from analysis; in many cases, the bounded solution is unique, and in other cases, assuming that g is continuous and bounded, we single out the solution we need as the unique continuous bounded solution.

(b) In many applications, the datum g may be unbounded (think about calls on the maximum of two stocks). Suppose that g is nonnegative, measurable and finite a.e. Then we use the following scheme:

1. construct a sequence of nonnegative bounded measurable g_n with compact support, which converge pointwise to g a.e., $g_n(x) \uparrow g(x)$;
2. find the unique bounded solution u^n of the problem (2.10)–(2.11) with the datum g_n (or the unique continuous bounded solution);
3. calculate the limit $u := \lim_{n\to+\infty} u^n$; by the monotone convergence theorem, this is the price of the derivative security.

(c) If the process $\hat{X}$ does not satisfy the (ACP)-condition, we can solve the problem (2.10)–(2.11), and after that check that Dynkin's formula is applicable to u^n; that is, $(q - \hat{L})u^n$ is nonnegative and universally measurable. Luckily for an RLPE X, the process $\hat{X}$ does satisfy the (ACP)-condition.

(d) If the characteristic exponent of X does not admit an analytic continuation, we can calculate the stochastic expression (2.9) as follows. Take the Lévy measure $F(dx)$ of X, and define X^ε to be the Lévy process with the generating triplet $(A, b, F_\varepsilon(dx))$, where A and b are the same as for X, and $F_\varepsilon(dx) = e^{-\varepsilon\|x\|}F(dx)$. Define $P^q_{\varepsilon,B} g(x)$ by (2.9) with X^ε instead of X. The $P^q_{\varepsilon,B} g(x)$ can be calculated as described above, and it is possible to show that

$$P^q_{\varepsilon,B} g(x) \to P^q_B g(x) \qquad \text{as } \varepsilon \to 0.$$

By calculating the limit, we obtain the formula for $P^q_B g(x)$.

(e) The constructions above admit a natural generalization to the case of a Markov process having absolutely continuous potential measures.

2.3. *The generalized Black–Scholes equation as a pseudodifferential equation.* Recall that the characteristic exponent $\psi = \psi^{\mathbf{Q}}$ of a Lévy process under a measure $\mathbf{Q}$ is defined by $E[e^{i\xi X_t}] = e^{-t\psi(\xi)}$ [for basic definitions and results of the theory of Lévy processes, see, e.g., Bertoin (1996) and Sato (1999)]. In Boyarchenko and Levendorskiĭ (1999), we used the definition $E[e^{-i\xi X_t}] = e^{-t\psi(\xi)}$, since in the theory of PDO, the standard definition of the Fourier transform $\hat{u}$ of a function u is

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} u(x)\, dx. \tag{2.19}$$

(From now on, we consider the one-dimensional case.) This lead to the uncomfortable appearance of the minus sign in many places, and so we decided to switch to the definition of the characteristic exponent common in probability; but we use (2.19) as the definition of the Fourier transform.

By using the integrodifferential representation of L,

$$Lf(x) = \frac{\sigma^2}{2} f''(x) + bf'(x) + \int_{-\infty}^{+\infty} \big(f(x+y) - f(x) - f'(x)y\mathbf{1}_{[-1,1]}(y)\big)F(dy),$$

where $(\sigma^2, b, F(dy))$ is the characteristic triplet of X_t, and the Lévy–Khintchine formula

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - ib\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y} + i\xi y\mathbf{1}_{|y|\le 1}(y))F(dy),$$

we obtain that L acts on oscillating exponents as follows:

$$(-L)e^{ix\xi} = \psi(\xi)e^{ix\xi}.$$

By using the Fourier inversion formula and this equality, we conclude that, for a sufficiently regular u,

$$(-L)u(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} \psi(\xi)\hat{u}(\xi)\, d\xi.$$

This means that $-L$ is a pseudodifferential operator with symbol $\psi(\xi)$:

$$-L = \psi(D).$$

Recall that a pseudodifferential operator with symbol $a = a(x, \xi)$ is defined by

$$a(x, D)u(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(x, \xi)\hat{u}(\xi)\, d\xi. \tag{2.20}$$

When the symbol is independent of the state variable, x, one writes $a(D)$ and calls a a PDO with constant symbol. Now we can rewrite the generalized Black–Scholes equation in variables $\tau = T - t$ and x as

$$\partial_\tau f + \big(r + \psi(D_x)\big)f = 0. \tag{2.21}$$

REMARK 2.5. If X belongs to the class of Lévy-like Feller processes introduced in Barndorff-Nielsen and Levendorskiĭ (2001), we obtain $-L = \psi(x, D_x)$; it is a PDO with nonconstant symbol $\psi(x, \xi)$.

REMARK 2.6. (a) The problem (2.5)–(2.6) is an analog of the Cauchy problem for a parabolic operator, with Dirichlet boundary condition. In the case of a non-Gaussian Lévy process, the elliptic part, $A := r + \psi(D)$, is not a differential operator but a PDO. The standard technique of the theory of differential operators is no longer applicable, and the adequate technique is to use the Fourier transform and the Wiener–Hopf factorization; to study the problem for the parabolic equation, the representation theorem for analytic semigroups Yosida (1964) can be used to simplify some technical details.

(b) The approximations u^n on step (3) in Remark 2.5(b) will be found with the help of the theory of PDO and the Wiener–Hopf factorization, hence by using the Fourier transform; the resulting formula involves oscillating integrals (which do not converge absolutely), and so the passage to the limit is nontrivial. In the case of Lévy processes, when the resulting formula is given by an explicit analytic expression involving oscillatory integrals, it is fairly straightforward to show that the limit of the sequence u^n, call it U temporarily, exists in the sense of the theory of generalized functions and can be defined by exactly the same expression as u^n, with g substituted for g_n. Moreover, we are capable of proving that U is continuous on $\mathcal{C}$. Since we know that u^n is a nondecreasing sequence of continuous functions, converging pointwise to u, u is its limit in the sense of generalized functions. Hence, $u = U$, and to finish the calculation of the price, it remains to calculate oscillating integrals in formulas involving PDO.

(c) In the multidimensional case, especially in the case of nonflat barriers, and for more general Markov processes, the argument above can be significantly simplified by using the theory of Sobolev spaces. One should construct an approximating sequence so that it converges in an appropriate Sobolev space, and general boundedness theorems on the action of PDO in the scale of Sobolev spaces can be applied to show that the limit of the sequence enjoys necessary properties.

3. Regular Lévy processes of exponential type and main properties of the generalized Black–Scholes equation.

3.1. *Definition of regular Lévy processes of exponential type.* In Boyarchenko and Levendorskiĭ (1999), we have shown that for wide classes of Lévy processes X used in empirical studies of financial markets, characteristic exponents (both under a historic measure and under EMM's from wide classes) satisfy the conditions of the following definition.

DEFINITION 3.1. Let there exist constants $c > 0$, $\nu \in (0, 2]$, $\nu' < \nu$, $\mu \in \mathbf{R}$, $\lambda_- < 0 \leq \lambda_+$ and $C > 0$ such that

$$\psi(\xi) = -i\mu\xi + \phi(\xi), \tag{3.1}$$

where ϕ admits analytic continuation from $\mathbf{R}$ into a strip $\Im\xi \in (\lambda_-, \lambda_+)$, admits continuous extension up to the boundary of the strip and satisfies the following two estimates:

(i) for all ξ in a strip $\Im\xi \in [\lambda_-, \lambda_+]$,

$$|\phi(\xi) - c|\xi|^\nu| \le C\langle\xi\rangle^{\nu'}, \tag{3.2}$$

where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$;

(ii) for any $[\lambda'_-, \lambda'_+] \subset (\lambda_-, \lambda_+)$ and all ξ in a strip $\Im k \in [\lambda'_-, \lambda'_+]$,

$$|\phi'(\xi)| \le C\langle\xi\rangle^{\nu-1}; \tag{3.3}$$

C depends on $[\lambda'_-, \lambda'_+]$ but not on ξ.

We say that X is a regular Lévy process of order ν and exponential type $[\lambda_-, \lambda_+]$.

REMARK 3.2. (a) We have modified the definition from Boyarchenko and Levendorskiĭ (1999, 2000) to allow for a diffusion component, simplify a bound (3.2) and allow for the left tail to decay slower than exponentially. The bound (3.3) is introduced to simplify the proof of uniform estimates in Section 8; it can be significantly relaxed.

(b) In order that the stock itself can be priced under EMM $\mathbf{Q}$, $\psi(-i)$ must be well defined, and hence we must have $\lambda_- \le -1$.

(c) If necessary for applications, one can generalize (3.2):

$$\phi(\xi) \sim c_\pm|\xi|^\nu + O(|\xi|^{\nu'}),$$

as $\Re\xi \to \pm\infty$ in the strip, where $\Re c_\pm \ge 0$. This generalization allows for a significant asymmetry in the central part of the PDF. If $\Re c_\pm > 0$, all the results below hold; only formulas for exponents $\kappa_\pm$ and the factor d in the construction of the factors in the Wiener–Hopf factorization formula in Section 4 change [see the proof of Theorem 6.1 in Eskin (1973)].

EXAMPLE 3.1. A NIG can be described by the characteristic exponent of the form

$$\psi(\xi) = -i\mu\xi + c[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}],$$

where $\alpha > |\beta| > 0$. Clearly, (3.1)–(3.3) hold with $\nu = 1$, $\nu' = 0$ and $\lambda_- = -\alpha + \beta$, $\lambda_+ = \alpha + \beta$. Thus, NIG are processes of order 1. If we use the same formula with $\kappa \in (0, 1)$ instead of $1/2$ in the exponents, we obtain the definition of normal tilted stable Lévy processes; they are RLPEs of order 2κ.

EXAMPLE 3.2. Hyperbolic processes are also processes of order 1. In the symmetric case, a hyperbolic process can be defined by

$$E^{\mathbf{Q}}[e^{i\xi X_1}] = \frac{\alpha}{K_1(\alpha\delta)} \frac{K_1\big(\delta\sqrt{\alpha^2 + \xi^2}\big)}{\sqrt{\alpha^2 + \xi^2}},$$

where K_1 is the modified Bessel function of third kind and order 1, and $\alpha, \delta > 0$.

NIG, NTS Lévy and HP can be obtained from pure diffusions by subordination [see Barndorff-Nielsen (1998) and Eberlein (1999)], which has a natural economic interpretation: the Brownian motion in the random "business time"—see, for example, the general discussion in Geman, Madan and Yor (1998) (for different processes).

EXAMPLE 3.3. Truncated Lévy processes of Koponen's (1995) family provide examples of processes of order $\nu \in (0,2), \nu \neq 1$ with $-\lambda_- = \lambda_+$; a generalization of this family constructed in Boyarchenko and Levendorskiĭ (1999, 2000) provides examples of processes of order $\nu \in [0,2)$ with arbitrary λ_-, λ_+. This is important since, for processes in real financial markets, the left tails are fatter than the right ones, and Koponen's family can contain processes with asymmetric PDF's only when PDF's are asymmetric in the central part as well, whereas PDF's observed in real financial markets are approximately symmetric in the central part. We will call this generalization the KoBoL family. If $\nu \in (0,2)$, $\nu \neq 1$, $c > 0$, then for a KoBoL process X of order ν, ψ is of the form

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)\big[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu\big].$$

Clearly, (3.1)–(3.3) hold; an example satisfying not (3.2) but its modification in Remark 3.1(c) is

$$\psi(\xi) = -i\mu\xi + d_+\Gamma(-\nu)\big[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu\big] + d_-\Gamma(-\nu)\big[(-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu\big],$$

where $d_+ \neq d_-$ are positive.

EXAMPLE 3.4. If in Examples 3.1–3.3 we add a diffusion component or consider a pure diffusion, we obtain a process of order 2.

Clearly, any finite mixture of independent RLPEs is an RLPE.

REMARK 3.3. (a) In Boyarchenko and Levendorskiĭ (1999), we used a definition which regarded variance Gamma processes (VGP) as RLPE of order 0. Some of our constructions below do not apply to VGP, and this is the reason we exclude VGP here.

(b) A convenient feature of a class of HP is its closedness under the Esscher transform, and the same holds for NIG, VGP and TLP.

In the next lemma, two important estimates for the characteristic exponent of an RLPE are derived.

LEMMA 3.4. *Let* (3.1) *and* (3.2) *hold, and let* $r > 0$. *Then the following hold*:

(a) *there exist* $\sigma_- \in [\lambda_-, 0)$, $\sigma_+ \in [0, \lambda_+]$ *and* $\delta > 0$ *such that, for any* ξ *in the strip* $\Im\xi \in [\sigma_-, \sigma_+]$,

$$\Re\psi(\xi) + r \geq \delta; \tag{3.4}$$

(b) *there exist* $\sigma_0 \in \mathbf{R}$ *and* $\delta > 0$ *such that, for all* λ *in the half-plane* $\Im\lambda \le \sigma_0$ *and all* ξ *in the strip* $\Im\xi \in [\lambda_-, \lambda_+]$,

$$\Re\big(i\lambda + r + \psi(\xi)\big) \ge \delta. \tag{3.5}$$

PROOF. (a) Set $M_1(\sigma) = \int_{-\infty}^{+\infty} e^{-\sigma x} p_1(x)\,dx$, where p_1 is the probability density of X_1. By differentiating twice, we conclude that M_1 is convex, and clearly $M_1(0) = 1 < e^r$. Hence, there exist $\sigma_- \in [\lambda_-, 0)$, $\sigma_+ \in [0, \lambda_+]$ and $\delta > 0$ such that, for all $\sigma \in [\sigma_-, \sigma_+]$, $M_1(\sigma) \le e^{r-\delta}$.

Now, for any $\xi \in \mathbf{R}$, and these σ,

$$\begin{aligned}\exp\big(-\Re\psi(\xi + i\sigma)\big) &= |\exp(-\psi(\xi + i\sigma))| \\ &= \left|\int_{-\infty}^{+\infty} e^{i\xi x - \sigma x} p_1(x)\,dx\right| \le \int_{-\infty}^{+\infty} e^{-\sigma x} p_1(x)\,dx;\end{aligned}$$

therefore (3.4) holds, and (a) is proved.

The proof of (b) is similar. □

REMARK 3.5. If $\lambda_+ > 0$, then we can find $\sigma_+ \in (0, \lambda_+)$, and if $\lambda_- < -1$, then we can find $\sigma_- \in (\lambda_-, -1)$.

COROLLARY 3.6. (a) *There exist* $\sigma_- < 0 \le \sigma_+$ *and* $\delta > 0$ *such that, for any* ξ *in the strip* $\Im\xi \in [\sigma_-, \sigma_+]$,

$$\Re\psi(\xi) + r \ge \delta(1 + |\xi|)^\nu. \tag{3.6}$$

(b) *The transition density is of the class* $C^\infty(\mathbf{R})$.

PROOF. (a) Follows from (3.4) and (3.1)–(3.2).

(b) Differentiate under the integral sign in the Fourier inversion formula

$$p_t(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\xi - t\psi(\xi)} d\xi,$$

and use (3.6). □

The main properties (3.1)–(3.6) of the symbol $r + \psi(\xi)$ of the stationary part of the generalized Black–Scholes operator in the LHS of (2.21) having been stated; we can study its action in appropriate scales of normed spaces.

3.2. *Properties of the elliptic part of the generalized Black–Scholes equation as an operator on* $\mathbf{R}$. If $\nu \ge 1$ or $\mu = 0$, then from (3.1), (3.2) and (3.4) we conclude that there exist $C_1, c_1 > 0$ such that $a(\xi) := r + \psi(\xi)$ satisfies both (3.6) and the following estimate:

$$|\Im a(\xi)/\Re a(\xi)| \le C_1, \qquad \xi \in \mathbf{R}. \tag{3.7}$$

Consider $A = r + \psi(D)$ as an unbounded operator in $H^0(\mathbf{R}) = L_2(\mathbf{R})$ with the domain $H^\nu(\mathbf{R})$ [H^s denotes the Sobolev space; see, e.g., Eskin (1973)]. From (3.6), it follows that $\Re A$ is positive definite: for any $u \in H^s(\mathbf{R})$,

$$(\Re Au, u)_0 \geq c_1 \|u\|^2_{\nu/2} \qquad \forall u \in L_2(\mathbf{R}),$$

and from (3.7), $(\Re A)^{-1}\Im A$ is bounded. This means that A is a strongly elliptic PDO, and therefore the generalized Black–Scholes equation (2.21) is an analogue of the parabolic equation. If $\nu \in (0, 1)$ and $\mu \neq 0$, then one can reduce (2.21) to a parabolic equation by changing coordinates $x' = x + \mu\tau$ but this spoils the time-independent boundary for barrier options and touch-and-out options. This observation means, in particular, that in cases of time-dependent barriers, processes of the order $\nu \in [1, 2]$ are more tractable than the ones of the order $\nu \in (0, 1)$.

4. Generalized Black–Scholes equation on a half-axis.

4.1. *The Wiener–Hopf equation.* In this section, we assume that X is an RLPE of order $\nu \in (0, 2]$ and exponential type $[\lambda_-, \lambda_+]$, where $\lambda_- \leq -1 < 0 \leq \lambda_+$. We consider two closely related problems, which arise when we apply the Fourier transform w.r.t. t to a boundary value problem with a time-independent boundary for the generalized Black–Scholes equation (2.21). Let $\lambda \in \mathbf{C}$, and set $a(\lambda, \xi) = i\lambda + r + \psi(\xi)$. The first problem is

$$a(\lambda, D)u(x) = g(x), \qquad x > 0, \tag{4.1}$$

$$u(x) = 0, \qquad x \leq 0, \tag{4.2}$$

and the second is

$$a(\lambda, D)u(x) = 0, \qquad x > 0, \tag{4.3}$$

$$u(x) = g(x), \qquad x \leq 0. \tag{4.4}$$

Notice that each of these two problems can be reduced to the other one and that we will also need similar problems with the inequalities of the opposite signs.

To solve these problems, we apply the Wiener–Hopf factorization method. We modify some constructions and results of Chapters 6 and 7 in Eskin (1973) (see the comment at the end of this paragraph) for the one-dimensional case, which we consider here, though all of them have multidimensional analogs, and analogs for PDO with nonconstant symbols. In particular, this is pertinent to the calculation of the factors in the Wiener–Hopf factorization formula in the next subsection: the formulations and standard proofs of the fluctuation identities in probability are essentially one-dimensional, whereas the constructions in Section 4 are nothing but adaptations of the corresponding multidimensional ones. These observations explain why the method of calculation of prices of barrier options and touch-and-out options, which is used in this paper, admits straightforward generalizations to the multidimensional case, and to the case of Lévy-like Feller processes. [The reader should be aware of the following systematic differences: the monograph Eskin (1973) is chosen as a reference book on PDO since in many respects its exposition is simpler than in later monographs on the subject but it uses the different definition of the Fourier transform, which has become obsolete in the theory of PDO. In the result, to establish the correspondence between the results in Eskin (1973) and their counterparts here, the lower half-plane of the complex plane must be replaced with the upper one and vice versa, etc.]

4.2. *The Wiener–Hopf factorization.* Fix λ, and set $q = i\lambda + r$. If $\Re q > 0$, then the factorization of $a(\lambda, \xi) = q + \psi(\xi)$ can be done for any Lévy process [see, e.g., Theorem 45.2 in Sato (1999)] though without the explicit formulas for the factors; in Boyarchenko and Levendorskiĭ (2000, 2002), explicit formulas are derived for any RLPE and, by using them, one can explicitly solve problems (4.1)–(4.2) and (4.3)–(4.4). We impose the following additional condition on μ in (3.1):

$$\text{if} \quad \nu \in (0, 1), \quad \text{then} \quad \mu = 0. \tag{4.5}$$

This condition enables one to transform the line of integration in the main formulas for barrier options into an appropriate contour and enhance the rate of convergence of the integral. [In our situation, (4.5) is necessary lest the representation theorem for analytical semigroups—see Yosida (1964)—fail.] Until further notice, we add (4.5) to the list of standing assumptions (3.1)–(3.3) on the process X; at the end of the section, we explain which changes are to be made when (4.5) fails.

Let $\sigma_- < 0 \le \sigma_+$ and $\sigma_0 \le 0$ be the same as in Lemma 3.4. For $\theta \in (0, \pi/2)$, set $\Sigma_\theta = \{\lambda \in \mathbf{C} \mid \arg\lambda \in [-\pi - \theta, \theta]\}$ and $\Sigma_{\theta,\sigma_0} = i\sigma_0 + \Sigma_\theta$.

LEMMA 4.1. *There exist $c_1 > 0$ and $\theta \in (0, \pi/2)$ such that the following hold*:

(a) *if $\Im\xi \in [\sigma_-, \sigma_+]$ and $\lambda \in \Sigma_\theta$, then*

$$|i\lambda + r + \psi(\xi)| \ge c_1(1 + |\lambda| + |\xi|^\nu); \tag{4.6}$$

(b) *if $\Im\xi \in [\lambda_-, \lambda_+]$ and $\lambda \in \Sigma_{\theta,\sigma_0}$, then* (4.6) *holds*.

PROOF. We prove (a) with the help of (3.4); in the same way, (b) can be derived from (3.5).

Fix $C_1 > 0$ and $\varepsilon > 0$, and consider domains

$$U^-(C_1, \varepsilon) = \{(\lambda, \xi) \mid |\lambda| \le C_1(1 + |\xi|^\nu),\ \arg\lambda \in (-\pi - \varepsilon, \varepsilon),\ \Im\xi \in [\sigma_-, \sigma_+]\},$$
$$U^+(C_1) = \{(\lambda, \xi) \mid |\lambda| \ge C_1(1 + |\xi|^\nu),\ \lambda \in \mathbf{C},\ \Im\xi \in [\sigma_-, \sigma_+]\}.$$

On $U^+(C_1)$, it suffices to prove (4.6) without $|\xi|$ in the RHS. From (3.1), (3.2), (3.4) and (4.5) it follows that there exists C_0 such that

$$|a(\lambda, \xi)| \le C_0(1 + |\xi|)^\nu,$$

and hence if C_1 is sufficiently large, we obtain $|a(\lambda,\xi)| \ge |\lambda|/2$; thus, any θ fits. On $U^-(C_1,\varepsilon)$, it suffices to prove (4.6) without $|\lambda|$ in the RHS. From (3.1), (3.2), (3.4) and (4.5) it follows that for any C_1 we can find sufficiently small ε and $c_1 = c_1(C_1,\varepsilon)$ such that, for indicated (λ,ξ),

$$\Re a(\lambda,\xi) \ge c_1(1+|\xi|^\nu).$$

Hence, (4.6) holds with $\theta = \varepsilon$. $\square$

Fix a branch of ln by a requirement: $\ln y$ is real for $y > 0$; for $z \in \mathbf{C} \setminus (-\infty, 0]$ and $s \in \mathbf{C}$, set $z^s = \exp[s \ln z]$. Next, fix $\varepsilon_0 > 1 - \lambda_- + \lambda_+$ and $p \ge 1$ such that $p\nu > 1$, and set

$$\Lambda_\pm(\lambda,\xi)^s := \big((i\lambda + r)^{1/p\nu} + (\varepsilon_0 \mp i\xi)^{1/p}\big)^{ps}.$$

(Thus, for processes of order $\nu > 1$, a choice $p = 1$ is admissible.) Then choose $d > 0$ and $\kappa_-, \kappa_+ \in \mathbf{R}$ so that

$$B(\lambda,\xi) := d^{-1}\Lambda_+(\lambda,\xi)^{-\kappa_+}\Lambda_-(\lambda,\xi)^{-\kappa_-}a(\lambda,\xi) \tag{4.7}$$

satisfies, for all $\lambda \in \Sigma_\theta$, $\xi \in \mathbf{R}$ and $\sigma \in [\sigma_-, \sigma_+]$,

$$\lim_{\xi\to\pm\infty} B(\lambda, \xi + i\sigma) = 1, \tag{4.8}$$

and $b(\lambda, \xi + i\sigma) = \ln B(\lambda, \xi + i\sigma)$ is well defined for these λ, ξ, σ.

Choices of d, κ_+ and κ_- depending on properties of B, hence on ν, μ and c in (3.1)–(3.2), we have to consider two cases:

1. if $\nu \in (0, 2]$, $\nu \ne 1$, we set $d = c$, $\kappa_- = \kappa_+ = \nu/2$;
2. if $\nu = 1$, we set $d = (c^2 + \mu^2)^{1/2}$, $\kappa_\pm = 1/2 \pm \pi^{-1}\arctan(\mu/c)$.

In the first case, (4.8) immediately follows from (3.1)–(3.2) and (4.5), and if $\nu = 1$, then the simplest way to prove (4.8) is to check that $\ln B(\lambda, \xi + i\sigma) \to 0$ as $\xi \to \pm\infty$:

$$\begin{aligned}\lim_{\xi\to\pm\infty} \ln B(\lambda, \xi + i\sigma) &= \pm\frac{\pi i}{2}\kappa_+ \mp \frac{\pi i}{2}\kappa_- + \ln\frac{c \mp i\mu}{(c^2+\mu^2)^{1/2}} \\ &= \pm(\kappa_+ - \kappa_-)\frac{\pi i}{2} \mp i\arctan\frac{\mu}{c} = 0.\end{aligned}$$

LEMMA 4.2. *For any $\lambda \in \Sigma_\theta$ and $\sigma \in (\sigma_-, \sigma_+)$, the winding number around the origin of the curve $\{B(\lambda, \xi + i\sigma) \mid -\infty < \xi < +\infty\}$ is zero*:

$$(2\pi)^{-1}\int_{\xi=-\infty}^{\xi=+\infty} d\arg B(\lambda, \xi + i\sigma) = 0. \tag{4.9}$$

PROOF. Due to (4.8) and (4.7), the LHS in (4.9) is an integer. From (4.6), $B(\lambda, \xi) \neq 0$ $\forall \lambda \in \Sigma_\theta$ and ξ in a strip $\Im\xi \in [\sigma_-, \sigma_+]$; hence this integer is independent of $\lambda \in \Sigma_\theta$ and $\sigma \in [\sigma_-, \sigma_+]$. With $\lambda = 0$ and $\Im\xi = \sigma$, the last factor in (4.7) assumes values in the half-plane $\Re z > 0$ by (3.4), and the same is true of the product of the first three factors, since the first one is positive, $\Lambda_-(\lambda, \xi)$ and $\Lambda_+(\lambda, \xi)$ assume values in the same half-plane but in different quadrants, and $0 < \kappa_\pm \le 1$. Hence, for all ξ in a strip $\Im\xi \in [\sigma_-, \sigma_+]$, $-\pi < \arg B(0, \xi) < \pi$, and therefore (4.9) holds. □

Under condition (4.9), $b(\lambda, \xi) := \ln B(\lambda, \xi)$ is well defined on $\Sigma_\theta \times \{\xi \mid \Im\xi \in [\sigma_-, \sigma_+]\}$. Next, for real ξ, $\sigma > \sigma_-$ and $\sigma_1 \in (\sigma_-, \sigma)$, we set

$$b_+(\lambda, \xi + i\sigma) = -(2\pi i)^{-1} \int_{-\infty+i\sigma_1}^{+\infty+i\sigma_1} \frac{b(\lambda, \eta)}{\xi + i\sigma - \eta}\, d\eta, \tag{4.10}$$

and for real ξ, $\sigma < \sigma_+$ and $\sigma_2 \in (\sigma, \sigma_+)$, we set

$$b_-(\lambda, \xi + i\sigma) = (2\pi i)^{-1} \int_{-\infty+i\sigma_2}^{+\infty+i\sigma_2} \frac{b(\lambda, \eta)}{\xi + i\sigma - \eta}\, d\eta. \tag{4.11}$$

By the Cauchy theorem, $b_\pm(\lambda, \xi + i\sigma)$ are independent of choices of σ_1 and σ_2.

It follows from (3.1), (3.2) and (4.8) that there exist $C, \rho > 0$ such that, for all η in a strip $\Im\eta \in [\sigma_-, \sigma_+]$,

$$|b(\lambda, \eta)| \le C(1 + |\eta|)^{-\rho},$$

where C depends on λ but not on η (and $\rho > 0$ is independent of λ and η). Hence, the integrals in (4.10) and (4.11) converge, and $b_\pm(\lambda, \xi)$ is well defined and holomorphic in a half-plane $\pm\Im\xi > \pm\sigma_\mp$. In the Appendix, we will prove the following lemma.

LEMMA 4.3. *For any $[\sigma'_-, \sigma'_+] \subset (\sigma_-, \sigma_+)$, there exists $C > 0$ such that*

$$|b_+(\lambda, \xi)| \le C \qquad \forall \lambda \in \Sigma_\theta, \Im\xi \ge \sigma'_- \tag{4.12}$$

and

$$|b_-(\lambda, \xi)| \le C \qquad \forall \lambda \in \Sigma_\theta, \Im\xi \le \sigma'_+. \tag{4.13}$$

By the residue theorem, for $\sigma_- < \sigma_1 < \sigma < \sigma_2 < \sigma_+$,

$$\begin{aligned} &b_+(\lambda, \xi + i\sigma) + b_-(\lambda, \xi + i\sigma) \\ &\qquad = -(2\pi i)^{-1} \left(\int_{-\infty+i\sigma_1}^{+\infty+i\sigma_1} - \int_{-\infty+i\sigma_2}^{+\infty+i\sigma_2} \right) \frac{b(\lambda, \eta)}{\xi + i\sigma - \eta}\, d\eta = b(\lambda, \xi + i\sigma). \end{aligned}$$

Hence, $B_\pm = \exp b_\pm$ satisfy $B = B_+ B_-$ on $\Sigma_\theta \times \{\xi \mid \Im\xi \in (\sigma_-, \sigma_+)\}$, and if we set

$$a_-(\lambda, \xi) = \Lambda_-(\lambda, \xi)^{\kappa_-} B_-(\lambda, \xi),$$
$$a_+(\lambda, \xi) = d\Lambda_+(\lambda, \xi)^{\kappa_+} B_+(\lambda, \xi),$$

then, for $\lambda \in \Sigma_\theta,\ \Im\xi \in (\sigma_-, \sigma_+)$,

$$a(\lambda, \xi) = a_+(\lambda, \xi)a_-(\lambda, \xi). \tag{4.14}$$

LEMMA 4.4. (a) *For any $\lambda \in \Sigma_\theta$, $a_+(\lambda, \xi)$ is holomorphic in the half-plane $\Im\xi > \sigma_-$, admits continuous extension up to the boundary of the half-plane and satisfies an estimate*

$$c(1 + |\lambda|^{1/\nu} + |\xi|)^{\kappa_+} \le |a_+(\lambda, \xi)| \le C(1 + |\lambda|^{1/\nu} + |\xi|)^{\kappa_+}, \tag{4.15}$$

where $C, c > 0$ are independent of $\lambda \in \Sigma_\theta$ and ξ in the half-plane $\Im\xi \ge \sigma_-$.

(b) *For any $\lambda \in \Sigma_\theta$, $a_-(\lambda, \xi)$ is holomorphic in the half-plane $\Im\xi < \sigma_+$, admits continuous extension up to the boundary of the half-plane and satisfies an estimate*

$$c(1 + |\lambda|^{1/\nu} + |\xi|)^{\kappa_-} \le |a_-(\lambda, \xi)| \le C(1 + |\lambda|^{1/\nu} + |\xi|)^{\kappa_-}, \tag{4.16}$$

where $C, c > 0$ are independent of $\lambda \in \Sigma_\theta$ and ξ in the half-plane $\Im\xi \le \sigma_+$.

(c) *For all $\lambda \in \Sigma_\theta$ and ξ in a strip $\sigma_- \le \Im\xi \le \sigma_+$,* (4.14) *holds.*

(d) *Factors in* (4.14) *are uniquely defined by properties* (a) *and* (b), *up to scalar multiples which may depend on λ.*

PROOF. Fix $[\sigma'_-, \sigma'_+] \subset (\sigma_-, \sigma_+)$, and prove (a)–(c) for $\lambda \in \Sigma_\theta$ and ξ with $\Im\xi \in [\sigma'_-, \sigma'_+]$. Clearly, $\Lambda_\pm(\lambda, \xi)^{\kappa_\pm}$ satisfy (a) and (b), and since $b_\pm$ are holomorphic and bounded on the same set due to (4.12)–(4.13), (a) and (b) are proved; (4.14) has already been proved.

To prove (a)–(c) in full generality, we notice that $a(\lambda, \xi)$ is continuous on the strip $\Im\xi \in [\sigma_-, \sigma_+]$, and hence $a_+(\lambda, \xi)$ admits continuous extension on $\Sigma_\theta \times \{\xi \mid \Im\xi \ge \sigma_-\}$ by $a_+(\lambda, \xi) = a(\lambda, \xi)/a_-(\lambda, \xi)$, and $a_-(\lambda, \xi)$ admits continuous extension on $\Sigma_\theta \times \{\xi \mid \Im\xi \le \sigma_+\}$ by $a_-(\lambda, \xi) = a(\lambda, \xi)/a_+(\lambda, \xi)$; then (4.14) holds for $\lambda \in \Sigma_\theta,\ \Im\xi \in [\sigma_-, \sigma_+]$ by construction. (4.15) and (4.16) for these λ and ξ follow from (3.1)–(3.2) and (3.4) and from the already proved (4.15) and (4.16) for $\lambda \in \Sigma_\theta$, $\Im\xi \in [\sigma'_-, \sigma'_+]$.

To prove (d), fix λ, and let $a(\lambda, \xi) = a'_+(\lambda, \xi)a'_-(\lambda, \xi)$ be another factorization with the same properties. Then $a'_+(\lambda, \xi)/a_+(\lambda, \xi)$ [resp., $a'_-(\lambda, \xi)/a_-(\lambda, \xi)$] is holomorphic in the upper half-plane $\Im\xi > 0$ (resp., the lower half-plane $\Im\xi < 0$) and continuous up to the boundary. Both functions are bounded and nonzero and coincide on $\mathbf{R}$. Hence, the analytic continuation of any of them is a bounded holomorphic function on $\mathbf{C}$. By the Liouville theorem, it must be constant. □

REMARK 4.5. Let $q = i\lambda + r$ be positive, and let

$$\frac{q}{q + \psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi) \tag{4.17}$$

be the Wiener–Hopf factorization standard in probability theory [see Sato (1999), Theorem 45.2]. For an RLPE, $\phi_-(\lambda, \xi) := \phi_q^-(\xi)$, $\phi_+(\lambda, \xi) := \phi_q^+(\xi)$ and their inverses are polynomially bounded w.r.t. ξ in the corresponding half-planes [this is proved in Boyarchenko and Levendorskiĭ (2002)]; therefore by applying the same argument as in the proof of Lemma 4.4(d), and taking into account that $\phi_\pm(\lambda, 0) = 1$, we conclude that, for $q = i\lambda + r > 0$,

$$\phi_\pm(\lambda, \xi) = \frac{a_\pm(\lambda, 0)}{a_\pm(\lambda, \xi)}. \tag{4.18}$$

Since both sides are holomorphic w.r.t. λ in a neighborhood of $i(-\infty, r)$, we conclude that (4.18) holds in the domain of analyticity of $a_\pm(\lambda, \xi)$ w.r.t. $(\lambda, \xi) \in \mathbf{C}^2$.

REMARK 4.6. Since all the constructions and estimates above are based on the estimate (4.6), which is valid not only on $\Sigma_\theta \times \{\xi \mid \Im\xi \in [\sigma_-, \sigma_+]\}$ but on $\Sigma_{\theta,\sigma_0} \times \{\xi \mid \Im\xi \in [\lambda_-, \lambda_+]\}$ as well, we can repeat these constructions and proofs with Σ_{θ,σ_0} and $[\lambda_-, \lambda_+]$ in place of Σ_θ and $[\sigma_-, \sigma_+]$, and use these modifications. When we do it, we refer to the same formulas.

4.3. *Solution of the problem* (4.1)–(4.2). For $\gamma \in \mathbf{R}$ and a function g defined on $\mathbf{R}$ or its subset, set $g_\gamma(x) = e^{\gamma x} g(x)$. We write $g \in L_\infty^\gamma(\mathbf{R})$ iff $g_\gamma \in L_\infty(\mathbf{R})$, and similarly define $L_\infty^\gamma(\mathbf{R}_\pm)$. Let σ_0 be from Lemma 3.4.

THEOREM 4.7. *Let $g \in L_\infty^{-\beta}(\mathbf{R}_+)$, where $\beta < -\lambda_-$. Then, for any λ in the half-plane $\Im\lambda \le \sigma_0$ and any $\beta_1 \in (\beta, -\lambda_-)$, the following hold*:

(i) *a solution to the problem* (4.1)–(4.2) *in the class $L_\infty^{-\beta_1}(\mathbf{R}_+)$ exists*;
(ii) *if $\kappa_- < 1$, it is unique and given by*

$$u = a_-(\lambda, D)^{-1}\theta_+ a_+(\lambda, D)^{-1} e_+ g, \tag{4.19}$$

where $e_+ g$ is the extension of g by 0 *on the negative axis, and θ_+ is the indicator function of the positive axis*;
(iii) *if $\kappa_- = 1$, then the continuous solution of the class $L_\infty^{-\beta_1}(\mathbf{R}_+)$ is unique, and it is given by* (4.19).

REMARK 4.8. When we apply the definition of the action of PDO $a_\pm(\lambda, D)^{-1}$ to (4.19), we assume implicitly that the integration is over the line $\Im\xi = \gamma$, where $\gamma \in (\lambda_-, -\beta)$. Due to the Cauchy theorem, the result is independent of a choice of such a γ.

PROOF OF THEOREM 4.7. Take any $\gamma \in (\lambda_-, -\beta_1)$, set $u_\gamma(x) = e^{\gamma x} u(x)$, insert $u(x) = e^{-\gamma x} u_\gamma(x)$ into (4.1)–(4.2), multiply by $e^{\gamma x}$ and use the equality

$$e^{\gamma x} a(\lambda, D) e^{-\gamma x} = a(\lambda, D + i\gamma). \tag{4.20}$$

The result is

$$a(\lambda, D + i\gamma)u_\gamma(x) = g_\gamma(x), \qquad x > 0, \tag{4.21}$$

$$u_\gamma(x) = 0, \qquad x \le 0. \tag{4.22}$$

Due to our choice of γ, $g_\gamma \in L_2(\mathbf{R}_+) = H^0(\overline{\mathbf{R}_+})$, and since (3.5) holds, the problem (4.21)–(4.22) satisfies conditions of Theorem 7.2 in Eskin (1973). This theorem gives, in particular, that any solution of the class $L_2(\mathbf{R})$ is of the form

$$u_\gamma = a_-(\lambda, D + i\gamma)^{-1}\theta_+ a_+(\lambda, D + i\gamma)^{-1} e_+ g_\gamma + Cw,$$

where $w = a_-(\lambda, D)^{-1}\delta$, δ is the Dirac delta-function and C is a constant. By multiplying by $e^{-\gamma x}$, we obtain (4.19) with an additional term $Ce^{-\gamma x}w$. By integrating by part in the oscillatory integral, which defines w, and using (4.16), one can show that if $\kappa_- < 1$, then w is unbounded in the neighborhood of 0, and if $\kappa_- = 1$, it is discontinuous at 0. Hence, $C = 0$, and we are left with the unique solution (4.19). □

COROLLARY 4.9. *Let σ_0 be as in Lemma* 3.4, *and let $g(x) = e^{\beta x}$, where $\beta < -\lambda_-$.*

Then, for any $\gamma \in (\lambda_-, -\beta)$ and all λ in the half-plane $\Im\lambda \le \sigma_0$,

$$u(\lambda, x) = a_-(\lambda, D)^{-1} a_+(\lambda, -i\beta)^{-1}(\theta_+ e^{\beta\cdot})(x) \tag{4.23}$$

$$= \frac{1}{2\pi a_+(\lambda, -i\beta)} \int_{-\infty+i\gamma}^{+\infty+i\gamma} \frac{e^{ix\xi}}{a_-(\lambda, \xi)(i\xi - \beta)}\, d\xi. \tag{4.24}$$

PROOF. By using (4.15) and the Cauchy theorem, it is straightforward to show that if we evaluate $a_+(\lambda, D)^{-1}(e_+ g(\cdot) - e^{\beta\cdot})$ at $x > 0$, we obtain 0. Hence, in (4.19), we may replace $a_+(\lambda, D)^{-1} e_+ g$ with

$$a_+(\lambda, D)^{-1} e^{\beta\cdot} = e^{\beta\cdot} a_+(\lambda, -i\beta)^{-1}, \tag{4.25}$$

and obtain (4.23)–(4.24). [To see why (4.25) holds, either apply both sides (as functionals) to $f \in C_0^\infty(\mathbf{R})$, or recall (4.18) and the probabilistic meaning of $\phi_-(\lambda, \xi)$—see, e.g., Equation (45.8) in Sato (1999)]. □

4.4. *Solution of the problem* (4.3)–(4.4). We consider only a special case $g(x) = e^{\beta x}$.

LEMMA 4.10. *Let σ_0 be as in Lemma* 3.4, *and let $g(x) = e^{\beta x}$, where $\beta > -\lambda_+$.*

Then, for any $\beta_1 \in (-\lambda_+, \beta)$ and all λ in the half-plane $\Im\lambda \le \sigma_0$, the following hold:

(i) *a solution of the problem* (4.3)–(4.4) *of the class* $L_{\infty}^{-\beta_1}(\mathbf{R}_-)$ *exists*;

(ii) *if* $\kappa_- < 1$, *then the solution is unique*, *and for any* $\gamma \in (-\beta, \lambda_+)$ *it is given by*

$$u(\lambda, x) = \phi_-(\lambda, D)\phi_-(\lambda, -i\beta)^{-1}(\theta_- e^{\beta\cdot})(x) \tag{4.26}$$

$$= (2\pi\phi_-(\lambda, -i\beta))^{-1} \int_{-\infty+i\gamma}^{+\infty+i\gamma} e^{ix\xi}\phi_-(\lambda, \xi)(\beta - i\xi)^{-1}\, d\xi; \tag{4.27}$$

(iii) *if* $\kappa_- = 1$, *then the continuous solution of the class* $L_{\infty}^{-\beta_1}(\mathbf{R}_-)$ *is unique*, *and it is given by* (4.26)–(4.27).

PROOF. Lemma 4.10 is a special case of Theorem 4.2 in Boyarchenko and Levendorskiĭ (2002); for reader's convenience, we give the proof for the case $\beta \in (-\lambda_+, -\lambda_-)$. Introduce a new function by $v(x) = u(x) - e^{\beta x}$. By using (4.20), we see that v solves the problem (4.1)–(4.2) with $g(x) = -a(\lambda, -i\beta)e^{\beta x}$. Then we use (4.23) and (4.18) and arrive at (4.26)–(4.27). $\square$

4.5. *The case* $\nu \in (0, 1)$ *and* $\mu \neq 0$. Let $\nu \in (0, 1)$. If $\mu > 0$, set $\kappa_+ = 1$, $\kappa_- = 0$ and $\Lambda_+(\lambda, \xi) = \varepsilon_0 + i\lambda - i\mu\xi$ (there is no need to introduce Λ_-), and if $\mu < 0$, set $\kappa_+ = 0$, $\kappa_- = 1$ and $\Lambda_-(\lambda, \xi) = \varepsilon_0 + i\lambda - i\mu\xi$ (this time, we do not need Λ_+). After that we can repeat all the constructions, which have been made under assumption (4.5) but this time for λ from a more narrow set: instead of Σ_θ, we can use $\Sigma'(\varepsilon) = \{\lambda \mid \Im\lambda \le \varepsilon\}$, for sufficiently small $\varepsilon > 0$. Remark 4.6, Theorem 4.7, Corollary 4.9 and Lemma 4.10 are valid for all $\nu \in (0, 2]$ and $\mu \in \mathbf{R}$.

5. Touch-and-out options.

5.1. *The setup.* A first-touch digital is a digital contract which pays \$1 when and if a specific event occurs. Consider the first-touch digital (another name is a touch-and-out option) which pays \$1 the first time the stock price S hits or crosses the level H from above. (It would be better to say "hits $(0, H]$;" for simplicity, in the sequel we say "crosses the level H.") If the stock price never crosses the level H before time T, the claim expires worthless. Denote by $V_{\mathrm{d}}(H, T; S, t)$ the no-arbitrage price of such a contract. (It coincides with the price of an American put-like option with a digital payoff. Since the payoff is the same for all levels of the stock price below the barrier, it is optimal to exercise the option the first time the level H is crossed.)

The formulas for the value $V_{\mathrm{u}}(H, T; S, t)$ of a similar contract, which pays \$1 the first time the stock price crosses the level H from below, easily follow by reflection of the real axis w.r.t. the origin. For explicit pricing formulas for the case when the dynamics of the stock price is modelled as the geometric Brownian motion, see, for example, Ingersoll (2000).

The riskless rate $r > 0$ is constant, and EMM $\mathbf{Q}$ is chosen so that, under $\mathbf{Q}$, $\{X_t\} = \{\ln S_t\}$ is an RLPE of order $\nu \in (0, 2]$ and exponential type $[\lambda_-, \lambda_+]$, where $\lambda_- < -1 < 0 < \lambda_+$.

At the end of the section, we obtain similar results for power first-touch contracts; we also consider contracts which pay a nonzero amount when the first barrier has been crossed but the second one (situated farther) has not, and expire worthless if both barriers have been crossed in one jump.

5.2. *Pricing formulas.* Set $x = \ln(S/H)$, $u(x, t) = V_{\mathrm{d}}(H, T; S, t)$. Then, for $t < T$ and $x \in \mathbf{R}$,

$$u(x, t) = E[e^{-rT'}\mathbf{1}_{T' \le T} \mid X_t = x], \tag{5.1}$$

where T' is the hitting time of $(-\infty, 0]$ by X. By applying Theorem 2.1 to the two-dimensional process $\hat{X}_t = (X_t, t)$, we obtain the following theorem.

THEOREM 5.1. *The u is a solution of the following problem*:

$$\big(\partial_t - \big(r + \psi(D_x)\big)\big)u(x, t) = 0, \qquad x > 0,\ t < T, \tag{5.2}$$

$$u(x, t) = 1, \qquad x \le 0,\ t \le T, \tag{5.3}$$

$$u(x, T) = 0, \qquad x > 0, \tag{5.4}$$

in the class of bounded measurable functions.

Set $\tau = T - t$, $v(x, \tau) = u(x, T - \tau)$ and rewrite (5.2)–(5.4) as follows:

$$\big(\partial_\tau + r + \psi(D_x)\big)v(x, \tau) = 0, \qquad x > 0,\ \tau > 0, \tag{5.5}$$

$$v(x, \tau) = 1, \qquad x \le 0,\ \tau \ge 0, \tag{5.6}$$

$$v(x, 0) = 0, \qquad x > 0. \tag{5.7}$$

To solve the problem (5.5)–(5.7), make the Fourier transform w.r.t. τ; since the terminal condition (5.7) is homogeneous, we obtain the following family of the problems on $\mathbf{R}$, parametrized by λ with $\Im\lambda < 0$:

$$\big(i\lambda + r + \psi(D_x)\big)\hat{v}(x, \lambda) = 0, \qquad x > 0, \tag{5.8}$$

$$\hat{v}(x, \lambda) = (i\lambda)^{-1}, \qquad x \le 0. \tag{5.9}$$

Let σ_0 be from Lemma 3.4. By applying Lemma 4.10 and using the equality $\phi_-(\lambda, 0) = 1$, we find, for λ in the half-plane $\Im\lambda \le \sigma_0$,

$$\hat{v}(x, \lambda) = \phi_-(\lambda, D)(i\lambda)^{-1}\theta_-. \tag{5.10}$$

Recall that if $\kappa_- < 1$, the solution is unique in the class of bounded functions, and if $\kappa_- = 1$, it is unique in the class of bounded continuous functions, and notice that the condition $\kappa_- = 1$ holds (or fails) for all λ simultaneously. The equality $\kappa_- = 1$ implies that either the process is of order 2 and hence has a diffusion component, or it is a process of order $\nu \in (0, 1)$ and $\mu < 0$, that is, a process of bounded variation, with negative drift. In both cases, by using (5.1), we obtain that u is continuous.

Explicitly, (5.10) is

$$\hat{v}(x, \lambda) = (2\pi)^{-1} \int_{-\infty+i\omega_+}^{+\infty+i\omega_+} e^{ix\xi} \phi_-(\lambda, \xi)(\lambda\xi)^{-1}\, d\xi,$$

where $\omega_+ \in (0, \lambda_+)$, and v is obtained by the Fourier inversion: for any negative $\sigma \le \sigma_0$,

$$v(x, \tau) = (2\pi)^{-2} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \int_{-\infty+i\omega_+}^{+\infty+i\omega_+} e^{i(\tau\lambda + x\xi)} \phi_-(\lambda, \xi)(\lambda\xi)^{-1}\, d\xi\, d\lambda. \tag{5.11}$$

If $\kappa_- > 0$, then on the strength of (4.16) the integrand admits a bound via

$$C(1 + |\xi|)^{-1-\varepsilon}(1 + |\lambda|)^{-1-\varepsilon},$$

for some $C, \varepsilon > 0$. Hence, v is continuous. If $\kappa_- = 0$, then we integrate by parts w.r.t. x and τ in the open quadrant $(0, +\infty)^2$, and by using the explicit formula for ϕ_-, obtain the integrand, which (locally w.r.t. x and ξ) admits the same bound, and we conclude that v is continuous on $(0, +\infty)^2$. Therefore, (5.11) gives the price we are looking for. By returning to the initial variables, we obtain the following theorem.

THEOREM 5.2. *For $S > H$ and $t < T$,*

$$V_{\mathrm{d}}(H, T; S, t) = (2\pi)^{-2} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \int_{-\infty+i\omega_+}^{+\infty+i\omega_+} e^{i((T-t)\lambda + \ln(S/H)\xi)} \phi_-(\lambda, \xi)(\lambda\xi)^{-1}\, d\xi\, d\lambda, \tag{5.12}$$

for $\sigma \le \sigma_0$ and $\omega_+ \in (0, \lambda_+)$.

By making the inversion of the x-axis w.r.t. the origin, which leads to the change $X \mapsto \tilde{X}$, $S \mapsto -S$, $H \mapsto -H$, $\xi \mapsto -\xi$ and $\phi_-(\lambda, \xi) \mapsto \phi_+(\lambda, -\xi)$, and then making the change of variable $\xi \mapsto -\xi$ in the integral, we obtain the following theorem.

THEOREM 5.3. *For $S < H$ and $t < T$,*

$$V_{\mathrm{u}}(H, T; S, t) = -(2\pi)^{-2} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \int_{-\infty+i\omega_-}^{+\infty+i\omega_-} e^{i((T-t)\lambda + \ln(S/H)\xi)} \phi_+(\lambda, \xi)(\lambda\xi)^{-1}\, d\xi\, d\lambda, \tag{5.13}$$

for $\sigma \le \sigma_0$ and $\omega_- \in (\lambda_-, 0)$.

5.3. *First-touch power options.*

5.3.1. *Down-and-out case.* Let $\beta > -\lambda_+$, $H > 0$, and denote by $V_{\mathrm{d}}^{\beta}(H, T; S, t)$ the price of the contract which pays S^{β} the first time the stock price crosses the level H from above. If this does not happen until the terminal date T, the contract expires worthless. Let $\sigma_0 < 0$ be from Lemma 4.1.

THEOREM 5.4. *For $S > H$ and $t < T$,*

$$V_{\mathrm{d}}^{\beta}(H, T; S, t) = \frac{S^{\beta}}{(2\pi)^2} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \int_{-\infty+i\omega_+}^{+\infty+i\omega_+} \frac{e^{i(T-t)\lambda+i\ln(S/H)\xi}\phi_-(\lambda, \xi - i\beta)}{\lambda\xi\phi_-(\lambda, -i\beta)}\, d\xi\, d\lambda, \tag{5.14}$$

for any $\sigma \le \sigma_0$ and $\omega_+ \in (0, \lambda_+ + \beta)$.

PROOF. Clearly,

$$V_{\mathrm{d}}^{\beta}(H, T; S, t) = H^{\beta} V_{\mathrm{d}}^{\beta}(1, T; S/H, t),$$

and $V_{\mathrm{d}}^{\beta}(1, T; S/H, t)$ can be calculated in essentially the same way as $V_{\mathrm{d}}(H, T; S, t)$. The result is

$$V_{\mathrm{d}}^{\beta}(H, T; S, t) = \frac{H^{\beta}}{(2\pi)^2} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \int_{-\infty+i\omega_+}^{+\infty+i\omega_+} \frac{e^{i(T-t)\lambda+i\ln(S/H)\xi}\phi_-(\lambda, \xi)}{\lambda(\xi + i\beta)\phi_-(\lambda, -i\beta)}\, d\xi\, d\lambda, \tag{5.15}$$

for any $\sigma \le \sigma_0$ and $\omega_+ \in (-\beta, \lambda_+)$. In the case $\beta < 0$, the payoff is unbounded; to be able to use Theorem 2.1, we have to approximate $g(x) = e^{\beta x}$ by a sequence of bounded smooth nonnegative functions g_n, which converges pointwise to g: $g_n(x) \uparrow g(x)$, $\forall x \le 0$. Denote by $V_{\mathrm{d}}(g_n; H, T; S, t)$ the price of the contract with the early exercise payoff g_n. By using Theorem 4.2 in Boyarchenko and Levendorskiĭ (2002) (Lemma 4.10 is a special case of this theorem), we obtain

$$\begin{aligned} V_{\mathrm{d}}(g_n; H, T; S, t) = \frac{H^{\beta}}{2\pi} \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{i(T-t)\lambda}(-i\lambda)^{-1} \\ \times \phi_-(\lambda, D)\theta_-\phi_-(\lambda, D)^{-1} g_n\left(\ln\left(\frac{S}{H}\right)\right) d\lambda, \end{aligned} \tag{5.16}$$

and since $e^{\omega_+ \cdot} g_n \to e^{\omega_+ \cdot} g$ in $L_2(\mathbf{R}_-)$, we have that

$$V_{\mathrm{d}}(g_n; H, T; S, t) \to V_{\mathrm{d}}(g; H, T; S, t)$$

in the sense of generalized functions. By writing the RHS in (5.16) explicitly and integrating by parts, we can prove that the functions involved are continuous in the region $S > H, t < T$. Hence, the limit $V_{\mathrm{d}}(g; H, T; S, t)$ is the price we are looking for. By using the equality $\phi_-(\lambda, D)^{-1}e^{\beta x} = \phi_-(\lambda, -i\beta)^{-1}e^{\beta x}$ and the definition of PDO, we conclude that $V_{\mathrm{d}}(g; H, T; S, t)$ is given by the RHS in (5.15).

Shift the line of integration in (5.15) w.r.t. ξ and integrate over the line $\Im\xi = \omega_+ - \beta$; then change the variable $\xi \mapsto \xi - i\beta$. The result is (5.14). □

5.3.2. *Up-and-out case.* Let $\beta < -\lambda_-$, $H > 0$, and denote by $V_{\mathrm{u}}^{\beta}(H, T; S, t)$ the price of the contract which pays S^β the first time the stock price crosses the level H from below. If this does not happen until the terminal date T, the contract expires worthless.

THEOREM 5.5. *For $S < H$ and $t < T$,*

$$V_{\mathrm{u}}^{\beta}(H, T; S, t) = -\frac{S^\beta}{(2\pi)^2}\int_{-\infty+i\sigma}^{+\infty+i\sigma}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-}\frac{e^{i((T-t)\lambda+\ln(S/H)\xi)}\phi_+(\lambda, \xi - i\beta)}{\lambda\xi\phi_+(\lambda, -i\beta)}\,d\xi\,d\lambda, \tag{5.17}$$

for any $\sigma \le \sigma_0$ and $\omega_- \in (\lambda_- + \beta, 0)$.

5.4. *First-touch power options: the double barrier case.*

5.4.1. *Down-and-out case.* Let $\beta \in \mathbf{R}$, $H_2 > H_1 > 0$, and denote by $V_{\mathrm{d}}^{\beta}((H_1, H_2], T; S, t)$ the price of the contract which pays S^β the first time the stock price crosses the level H_2 from above unless it crosses the level H_1 as well. If both barriers have been crossed in one jump or the first barrier has not been crossed until the terminal date T, the contract expires worthless.

THEOREM 5.6. *For $S > H_2$ and $t < T$,*

$$\begin{aligned} V_{\mathrm{d}}^{\beta}((H_1, H_2], T; S, t) &= (2\pi)^{-1}\int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{i(T-t)\lambda}(-i\lambda)^{-1} \\ &\quad\times \phi_-(\lambda, D)\mathbf{1}_{(-\infty,\ln H_2]}\phi_-(\lambda, D)^{-1}g(\ln S)\,d\lambda, \end{aligned} \tag{5.18}$$

where $g(x) = e^{\beta x}\mathbf{1}_{(\ln H_1, \ln H_2]}(x)$, and $\sigma \le \sigma_0$ is arbitrary.

PROOF. We use Theorem 4.2 in Boyarchenko and Levendorskiĭ (2002) [cf. the derivation of (5.16)]. □

5.4.2. *Up-and-out case.* Let $\beta \in \mathbf{R}$, $0 < H_1 < H_2$, and denote by $V_{\mathrm{u}}^{\beta}([H_1, H_2), T; S, t)$ the price of the contract which pays S^β the first time the stock price crosses the level H_1 from below unless it crosses the level H_2 as well. If both barriers have been crossed in one jump or the first barrier has not been crossed until the terminal date T, the contract expires worthless.

THEOREM 5.7. *For $S < H_1$ and $t < T$,*

$$V_{\mathrm{u}}^{\beta}([H_1, H_2), T; S, t)$$
$$= (2\pi)^{-1} \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{i(T-t)\lambda}(-i\lambda)^{-1} \times \phi_+(\lambda, D)\mathbf{1}_{[H_1,+\infty)}\phi_+(\lambda, D)^{-1} g(\ln S)\, d\lambda, \tag{5.19}$$

where $g(x) = e^{\beta x}\mathbf{1}_{[\ln H_1, \ln H_2)}(x)$, and $\sigma \le \sigma_0$ is arbitrary.

6. Barrier options. Barrier options are reducible to boundary value problems with a nontrivial terminal condition and a fixed early exercise boundary. In the event of the early exercise, the option owner may be entitled to a rebate; that is, the boundary condition can be homogeneous or inhomogeneous. In contrast to the Gaussian case, the rebate must be specified not at the barrier only but everywhere on the other side of the barrier as well.

6.1. *Types of barrier options.*

6.1.1. *Standard barrier options.* Consider a contract which pays the specified amount at the terminal date provided during the lifetime of the contract the price of the underlying asset does not cross a specified barrier $S = H(t)$ from above (down-and-out barrier options) or from below (up-and-out barrier options). When the barrier is crossed, the option expires worthless or the option owner is entitled to some rebate. For simplicity, we consider constant barriers only, though non-constant barriers can also be considered.

The standard variety of barrier options on a stock, without a rebate, comprises four types of down-and-out and up-and-out options, and four types of down-and-in and up-and-in options. The "out" options are as follows:

1. the down-and-out put option with the same terminal payoff as for the European put; denote the price of this contract by $V_{\mathrm{do;put}}(K, H, T; S, t)$;
2. the down-and-out call option with the same terminal payoff as for the European call; the price is denoted by $V_{\mathrm{do;call}}(K, H, T; S, t)$;
3. the up-and-out put option with the same terminal payoff as for the European put; the price is denoted by $V_{\mathrm{uo;put}}(K, H, T; S, t)$;
4. the up-and-out call option with the same terminal payoff as for the European call; the price is denoted by $V_{\mathrm{uo;call}}(K, H, T; S, t)$.

An "in" option becomes the European option when the specified barrier is crossed (before or on the terminal date); otherwise it expires worthless. For instance, the up-and-in put option becomes the European put option when the barrier is crossed from below; we denote its price by $V_{\mathrm{ui;put}}(K, H, T; S, t)$. Similarly, one considers the up-and-in call option, with the price $V_{\mathrm{ui;call}}(K, H, T; S, t)$, and down-and-in put and call options $V_{\mathrm{di;put}}(K, H, T; S, t)$ and $V_{\mathrm{di;call}}(K, H, T; S, t)$.

In all states of nature, a European option pays the same amount as the portfolio of the down-and-out option and up-and-in option with the same barrier (of course, this is pertinent to the portfolio of the up-and-out option and down-and-in option with the same barrier as well); therefore standard no-arbitrage considerations show that the pricing problem for an "in" option reduces to the pricing problem of the corresponding "out" option and the European option, for instance,

$$V_{\mathrm{di,call}}(K, H, T; S, t) = V_{\mathrm{call}}(K, T; S, t) - V_{\mathrm{uo,call}}(K, H, T; S, t), \tag{6.1}$$

where $V_{\mathrm{call}}(K, T; S, t)$ is the price of the European call with the same strike price K and expiry date T. Equalities similar to (6.1) hold for other pairs of barrier options, and in the sequel we will write the explicit formulas for "out" options only.

6.1.2. *Options with a rebate.* Suppose that when the barrier is crossed from above, the European option with terminal payoff $G(S_T)$ expires but the option owner is entitled to some rebate $G^r(S_t, t)$. If the rebate is constant, then standard no-arbitrage consideration shows that the price of the barrier option with a rebate is equal to the price of the portfolio of the same type of barrier option but without a rebate, and the first-touch digital with payoff G^r; the same holds for "up" options. Similarly, the price of an option with a power rebate $G^r(S) = S^\beta$ is equal to the price of the corresponding option without a rebate, and the first-touch power option; for instance, the price of the down-and-out put with a power rebate can be calculated as

$$V^\beta_{\mathrm{do,put}}(K, H, T; S, t) = V_{\mathrm{do,put}}(K, H, T; S, t) + V^\beta_{\mathrm{d}}(H, T; S, t). \tag{6.2}$$

Likewise, consider the contract for the down-and-out put option, which specifies that the rebate S^β is paid provided the barrier H_2 has been crossed from above but the second barrier $H_1 < H_2$ has not, and the option expires worthless, if both barriers have been crossed in one jump. The price of such a contract is

$$\begin{aligned} &V^\beta_{\mathrm{do,put}}(K, (H_1, H_2], T; S, t) \\ &\quad = V_{\mathrm{do,put}}(K, H_2, T; S, t) + V^\beta_{\mathrm{d}}((H_1, H_2], T; S, t). \end{aligned} \tag{6.3}$$

By using (6.2)–(6.3) and their analogue for other types of barrier options, and the formulas for first-touch digitals and power options, we can reduce the pricing problem of barrier options with a constant or power rebate to the case without a rebate.

6.1.3. *More general barrier options.* One can consider barrier options for other terminal payoffs, say, barrier counterparts of power options, and rebates of different types. This increases the variety of barrier options still further.

6.2. *Main results for barrier options without a rebate.* We assume that the riskless rate $r > 0$ is constant and, under a risk-neutral measure $\mathbf{Q}$, X is an RLPE of exponential type $[\lambda_-, \lambda_+]$, where $\lambda_- < -1 < 0 < \lambda_+$.

We consider "out" options with barrier H, without a rebate; the terminal payoffs are of the form

$$G(S_T) = (S_T^\beta - K)_+, \tag{6.4}$$

where $0 \le \beta < -\lambda_-$, which includes payoffs for calls and "power calls" [we denote the price of such a contract by $W^\beta_{*;\mathrm{call}}(K, H, T; S, T)$], or of the form

$$G(S_T) = (K - S_T^\beta)_+, \tag{6.5}$$

where $-\lambda_+ < \beta$, which includes payoffs for puts and "power puts" [the notation used is $W^\beta_{*;\mathrm{put}}(K, H, T; S, T)$]. We also consider barrier contracts with payoffs $G(S_T) = S_T^\beta$; in the down-and-out case, the price is denoted by $W^\beta_{\mathrm{do}}(H, T; S, t)$, and in the up-and-out case, by $W^\beta_{\mathrm{uo}}(H, T; S, t)$. (More general payoffs can also be considered.)

In addition to W^β_{do} and W^β_{uo}, we have to consider separately six cases of standard (power) barrier options:

(i) down-and-out call option $W^\beta_{\mathrm{do;call}}(K, H, T; S, T)$, in the case $K \le H^\beta$;
(ii) up-and-out put option $W^\beta_{\mathrm{uo;put}}(K, H, T; S, T)$, in the case $K \ge H^\beta$;
(iii) down-and-out put option $W^\beta_{\mathrm{do;put}}(K, H, T; S, T)$, in the case $K > H^\beta$;
(iv) up-and-out call option $W^\beta_{\mathrm{uo;call}}(K, H, T; S, T)$, in the case $K < H^\beta$;
(v) down-and-out call option $W^\beta_{\mathrm{do;call}}(K, H, T; S, T)$, in the case $K > H^\beta$;
(vi) down-and-out put option $W^\beta_{\mathrm{do;put}}(K, H, T; S, T)$, in the case $K < H^\beta$.

Of these eight cases, there are only two essentially different: the calculation of W^β_{uo} (resp., $W^\beta_{\mathrm{uo;call}}$) is reducible to the calculation of $W^{-\beta}_{\mathrm{do}}$ (resp., $W^\beta_{\mathrm{do;put}}$) by passing to the dual process, the prices in cases (i) and (ii) can easily be expressed in terms of W^β_{uo} and $W^{-\beta}_{\mathrm{do}}$, and cases (v)–(vi) reduce to (i)–(iv):

THEOREM 6.1. (i) *If $K \le H^\beta$, then*

$$W^\beta_{\mathrm{do;call}}(K, H, T; S, t) = W^\beta_{\mathrm{do}}(H, T; S, t) - K W^0_{\mathrm{do}}(H, T; S, t). \tag{6.6}$$

(ii) *If $K \ge H^\beta$, then*

$$W^\beta_{\mathrm{uo;put}}(K, H, T; S, t) = K W^0_{\mathrm{uo}}(H, T; S, t) - W^\beta_{\mathrm{uo}}(H, T; S, t). \tag{6.7}$$

(v) *If* $K > H^\beta$, *then*

$$W^\beta_{\mathrm{do;call}}(K,H,T;S,t) = W^\beta_{\mathrm{do}}(H,T;S,t) - KW^0_{\mathrm{do}}(H,T;S,t) - W^\beta_{\mathrm{do;put}}(K,H,T;S,t). \tag{6.8}$$

(vi) *If* $K < H^\beta$, *then*

$$W^\beta_{\mathrm{uo;put}}(K,H,T;S,t) = KW^0_{\mathrm{uo}}(H,T;S,t) - W^\beta_{\mathrm{uo}}(H,T;S,t) - W^\beta_{\mathrm{uo;call}}(K,H,T;S,t). \tag{6.9}$$

It remains to write the explicit formulas for functions in the RHS's of (6.6)–(6.9). As in Section 5, the results are formulated in terms of the factors in the Wiener–Hopf factorization formula.

Let σ_0 be from Lemma 3.4. In the formulas below, we may use any negative $\sigma \le \sigma_0$.

THEOREM 6.2. (a) *Let* $\beta < -\lambda_-$. *Then, for* $t < T$ *and* $S > H$,

$$\begin{aligned} &W^\beta_{\mathrm{do}}(H,T;S,t) \\ &\quad = \frac{H^\beta}{(2\pi)^2 a_+(\lambda,-i\beta)} \int_{-\infty+i\sigma}^{+\infty+i\sigma}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-} \\ &\qquad \times \frac{\exp(i[\lambda(T-\tau)+\xi\ln(S/H)])\,d\xi\,d\lambda}{a_-(\lambda,\xi)(i\xi-\beta)}, \end{aligned} \tag{6.10}$$

where $\omega_- \in (\lambda_-, -\beta)$ *is arbitrary.*

(b) *Let* $\beta > -\lambda_+$. *Then, for* $t < T$ *and* $S < H$,

$$\begin{aligned} &W^\beta_{\mathrm{uo}}(H,T;S,t) \\ &\quad = \frac{H^\beta}{(2\pi)^2 a_-(\lambda,-i\beta)} \\ &\qquad \times \int_{-\infty+i\sigma}^{+\infty+i\sigma}\int_{-\infty+i\omega_+}^{+\infty+i\omega_+} \frac{\exp(i[\lambda(T-\tau)+\xi\ln(S/H)])\,d\xi\,d\lambda}{a_+(\lambda,\xi)(\beta-i\xi)}, \end{aligned} \tag{6.11}$$

where $\omega_+ \in (-\beta, \lambda_+)$ *is arbitrary.*

In the last theorem, we need the Fourier transforms of functions

$$g_+(K,H,\beta;x) := \mathbf{1}_{[\ln H, \ln K/\beta]}(x)(K - e^{\beta x}) \qquad \text{when } K > H^\beta$$

and

$$g_-(K,H,\beta;x) := \mathbf{1}_{[\ln K/\beta, \ln H]}(x)(e^{\beta x} - K) \qquad \text{when } K < H^\beta;$$

they are

$$\hat{g}_+(K, H, \beta; \eta) = \frac{K^{1-i\eta/\beta} - KH^{-i\eta}}{-i\eta} - \frac{K^{1-i\eta/\beta} - H^{\beta-i\eta}}{\beta - i\eta}$$

and

$$\hat{g}_-(K, H, \beta; \eta) = \frac{K^{1-i\eta/\beta} - H^{\beta-i\eta}}{\beta - i\eta} - \frac{K^{1-i\eta/\beta} - KH^{-i\eta}}{-i\eta}.$$

THEOREM 6.3. (a) *Let* $K > H^\beta$, $\beta \geq 0$ *and* $\kappa_+ > 0$. *Then, for* $t < T$ *and* $S > H$,

$$\begin{aligned} &W^\beta_{\mathrm{do;put}}(K, H, T; S, t) \\ &\quad = \frac{1}{(2\pi)^3 i} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \int_{-\infty+i\gamma_1}^{+\infty+i\gamma_1} \int_{-\infty+i\gamma}^{+\infty+i\gamma} e^{i(\lambda(T-t)+\ln(S/H)\xi)} \\ &\qquad\qquad \times \frac{\hat{g}_+(K, H, \beta; \eta)\, d\eta\, d\xi\, d\lambda}{a_-(\lambda, \xi)(\xi - \eta)a_+(\lambda, \eta)}, \end{aligned} \tag{6.12}$$

where a negative $\sigma \leq \sigma_0$ *and* $\lambda_- < \gamma_1 < \gamma < \lambda_+$ *are arbitrary.*

(b) *Let* $K < H^\beta$, $0 \leq \beta < -\lambda_+$ *and* $\kappa_- > 0$. *Then, for* $t < T$ *and* $S < H$,

$$\begin{aligned} &W^\beta_{\mathrm{uo,call}}(K, H, T; S, t) \\ &\quad = \frac{1}{(2\pi)^3 i} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \int_{-\infty+i\gamma_1}^{+\infty+i\gamma_1} \int_{-\infty+i\gamma}^{+\infty+i\gamma} e^{i(\lambda(T-t)+\ln(S/H)\xi)} \\ &\qquad\qquad \times \frac{\hat{g}_-(K, H, \beta; \eta)\, d\eta\, d\xi\, d\lambda}{a_+(\lambda, \xi)(\eta - \xi)a_-(\lambda, \eta)}, \end{aligned} \tag{6.13}$$

where a negative $\sigma \leq \sigma_0$ *and* $\lambda_- < \gamma < \gamma_1 < \lambda_+$ *are arbitrary.*

REMARK 6.4. (i) The proof of part (a) [resp., (b)] is valid for $\kappa_+ = 0$ (resp., $\kappa_- = 0$) as well; only the last step needs some modification, and the resulting formulas are more complex.

(ii) If $\nu \in [1, 2)$ or $\nu \in (0, 1)$ and $\mu = 0$, then the factors $a_\pm(\lambda, \xi)$ in the Wiener–Hopf factorization formula can be constructed so that each of them admits analytic continuation w.r.t. λ in the region of the form

$$\Sigma(-C_0, \varepsilon) := -C_0 + \{\lambda \mid \arg\lambda \in (-\pi - \varepsilon, \varepsilon)\},$$

for all $\xi \in (\lambda_-, \lambda_+)$, with appropriate estimates (cf. Lemma 4.1). Denote by $\mathcal{L}(-C_0, \varepsilon)$ the boundary of $\Sigma(-C_0, \varepsilon)$, and notice that in (6.11)–(6.13), we can transform the line $\Im\lambda = \sigma$ into the contour $\mathcal{L}(-C_0, \varepsilon)$. This improves the convergence of the integral.

To prove Theorems 6.2 and 6.3, we first notice that the proof of the (b)-parts can be obtained from the proof of the (a)-parts by using the dual process instead of the initial one and that the (a)-parts are special cases of the following general result. Let g be a nonnegative, continuous function on $[h, +\infty)$, where $h = \ln H$, and let g satisfy an estimate

$$g(x) \le Ce^{\beta x}, \tag{6.14}$$

where $\beta < -\lambda_-$. Below, we identify a function on $[h, +\infty)$ with its extension by zero on $(-\infty, h)$. Denote by $W_{\mathrm{do}}(g; H, T; S, t)$ the price of the down-and-out contract with barrier H, terminal date T and payoff $g(\ln S_T)$, and for any $\gamma \in \mathbf{R}$ set $g_\gamma(x) = e^{\gamma x} g(x)$.

THEOREM 6.5. *For $S > H$ and $t < T$,*

$$\begin{aligned} W_{\mathrm{do}}(g; H, T; S, t) = \frac{e^{-\gamma x}}{2\pi} \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{i\lambda(T-t)} \big[a_-(\lambda, D + i\gamma)^{-1} \\ \times \mathbf{1}_{[h,+\infty)} a_+(\lambda, D + i\gamma)^{-1} g_\gamma\big](\ln S)\, d\lambda, \end{aligned} \tag{6.15}$$

where a negative $\sigma \le \sigma_0$ and $\gamma \in (\lambda_-, -\beta)$ are arbitrary.

PROOF. Construct a sequence $\{g_n\}$ of nonnegative continuous functions with compact supports converging pointwise to g: $g_n(x) \uparrow g(x)$, $n \to \infty$. Then, for any $S > H$ and $t < T$,

$$W_{\mathrm{do}}(g_n; H, T; S, t) \uparrow W_{\mathrm{do}}(g; H, T; S, t). \tag{6.16}$$

By applying Theorem 2.1 to a two-dimensional process $\hat{X}_t = (X_t, t)$, we find that $v_n(x, t) := W_{\mathrm{do}}(g_n; H, T; S, t)$ is a measurable bounded solution to the following problem:

$$\big(\partial_t - \big(r + \psi(D)\big)\big) v(x, t) = 0, \qquad x > h, t < T, \tag{6.17}$$

$$v(x, T) = g_n(x), \qquad x > h, \tag{6.18}$$

$$v(x, t) = 0, \qquad x \le h, t \le T. \tag{6.19}$$

Set $u_n(x, \tau) = v_n(x, T - \tau)$, and for λ on the line $\Im\lambda = \sigma$ consider the problem

$$\big(i\lambda + r + \psi(D)\big) \hat{u}_n(x, \lambda) = g_n(x), \qquad x > h, \tag{6.20}$$

$$\hat{u}_n(x, \lambda) = 0, \qquad x \le h. \tag{6.21}$$

The Wiener–Hopf method gives the unique bounded solution

$$\hat{u}_n(\cdot, \lambda) = a_-(\lambda, D)^{-1} \mathbf{1}_{[h,+\infty)} a_+(\lambda, D)^{-1} g_n. \tag{6.22}$$

Define

$$u_n(x, \tau) = \frac{1}{2\pi} \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{i\lambda\tau} \big(a_-(\lambda, D)^{-1} \mathbf{1}_{[h,+\infty)} a_+(\lambda, D)^{-1} g_n\big)(x)\, d\lambda. \tag{6.23}$$

By invoking the definition of PDO and integrating by parts in the resulting oscillatory integral, it is straightforward to show that u_n is continuous on $(h, +\infty) \times (0, +\infty)$ and, for any $x > 0$, $u_n(x, \tau) \to g_n(x)$, as $\tau \to +0$. Thus, $W_{\mathrm{do}}(g_n; H, T; S, t) = u_n(x, T - t)$, and in view of (6.16), to finish the proof of Theorem 6.5, it suffices to calculate the limit of the RHS in (6.23).

Take $\gamma \in (\lambda_-, -\beta)$, set $g_{n,\gamma}(x) = e^{\gamma x} g_n(x)$, $\hat{u}_{n,\gamma}(x, \lambda) = e^{\gamma x} \hat{u}_n(x, \lambda)$ and rewrite (6.22) as

$$\hat{u}_{n,\gamma}(\cdot, \lambda) = a_-(\lambda, D + i\gamma)^{-1} \mathbf{1}_{[h,+\infty)} a_+(\lambda, D + i\gamma)^{-1} g_{n,\gamma}. \tag{6.24}$$

Define $\hat{u}_\gamma$ and u_γ by (6.24) and (6.23), respectively, with g_γ instead of $g_{n,\gamma}$. Due to the choice of γ, we have $g_{n,\gamma} \to g_\gamma$ in the topology of $L_2(\mathbf{R})$; therefore $u_{n,\gamma} \to u_\gamma$ in the sense of generalized functions. By using the definition of PDO, it is straightforward to show that u_γ is continuous on $(h, +\infty) \times (0, +\infty)$. However, if two continuous functions define the same generalized function, they coincide. Hence, $W_{\mathrm{do}}(g; H, T; S, t) = u(x, T - t)$, and (6.15) has been proved. □

6.3. *Proof of Theorem* 6.2. Now we can deduce (6.10). Without loss of generality, we may assume $H = 1$. Then, for $x > 0$, we have

$$\big(a_+(\lambda, D+i\gamma)^{-1} \mathbf{1}_{[0,+\infty)} e^{(\beta+\gamma)\cdot}\big)(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} \frac{e^{ix\xi}\, d\xi}{a_+(\lambda, \xi + i\gamma)(i\xi - \beta - \gamma)}.$$

In view of (4.15), the integrand is meromorphic in the upper half-plane with the only pole at $\xi = -i(\beta + \gamma)$, which is simple, and in the upper half-plane, outside a vicinity of the pole, the integrand admits an estimate via $C(1 + |\xi|)^{-1-\kappa_+}$. Hence, we can shift the line of integration in the direction $\Im\xi \to +\infty$, and by using the residue theorem we obtain

$$\mathbf{1}_{[0,+\infty)} a_+(\lambda, D + i\gamma)^{-1} \mathbf{1}_{[0,+\infty)} e^{(\beta+\gamma)\cdot} = a_+(\lambda, -i\beta)^{-1} \mathbf{1}_{[0,+\infty)} e^{(\beta+\gamma)\cdot}. \tag{6.25}$$

Once (6.25) has been calculated, we insert the RHS into (6.15), use the definition of PDO and obtain (6.10). □

6.4. *Proof of Theorem* 6.3. Without loss of generality, we may assume that $H = 1$. We write (6.15) explicitly, by using the definition of PDO, and obtain, for any negative $\sigma \le \sigma_0$ and any $\omega \in (\lambda_- - \gamma, 0)$,

$$\begin{aligned} W^{\beta}_{\mathrm{do;put}}(K; H, T; S, t) = \frac{e^{-\gamma x}}{(2\pi)^3} &\int_{-\infty+i\sigma}^{+\infty+i\sigma} d\lambda\, e^{i\lambda\tau} \\ &\times \int_{-\infty+i\omega}^{+\infty+i\omega} d\xi\, e^{ix(\xi+i\gamma)} a_-(\lambda, \xi + i\gamma)^{-1} \int_0^{+\infty} dy\, e^{-iy\xi} \\ &\times \int_{-\infty}^{+\infty} d\eta\, e^{iy\eta} a_+(\lambda, \eta + i\gamma)^{-1} \hat{g}_+(\eta + i\gamma). \end{aligned}$$

We know that $\hat{g}(\eta + i\gamma) = O(|\eta|^{-1})$, as $\eta \to \pm\infty$; therefore in the case $\kappa_+ > 0$, we conclude from (4.15) and the condition $\omega < 0$ that the inner double integral converges absolutely. By applying the Fubini theorem and integrating w.r.t. y first,

$$\int_0^{+\infty} dy\, e^{i(-\xi+\eta)y} = -i(\xi - \eta)^{-1},$$

we obtain

$$\begin{aligned} W^{\beta}_{\mathrm{do;put}}(K; H, T; S, t) = \frac{e^{-\gamma x}}{(2\pi)^3} &\int_{-\infty+i\sigma}^{+\infty+i\sigma} d\lambda\, e^{i\lambda\tau} \\ &\times \int_{-\infty+i\omega}^{+\infty+i\omega} d\xi\, e^{ix(\xi+i\gamma)} a_-(\lambda, \xi + i\gamma)^{-1} \\ &\times \int_{-\infty}^{+\infty} d\eta\, \big(i(\xi - \eta)\big)^{-1} a_+(\lambda, \eta + i\gamma)^{-1} \hat{g}_+(\eta + i\gamma). \end{aligned}$$

Now it remains to change the variables $\xi \mapsto \xi - i\gamma$ and $\eta \mapsto \eta - i\gamma$ to obtain (6.12).

To prove an analogue of (6.12) in the case $\kappa_+ = 0$, we represent a_+ in the form $a_+(\lambda, \xi) = 1 + d_+(\lambda, \xi)$, where $d_+(\lambda, \xi)$ decays at infinity w.r.t. ξ, and work with the resulting two integrals separately. $\square$

7. Conclusion. We have suggested a general procedure of the computation of the price of a contingent claim under a Lévy process and applied it to barrier options and touch-and-out options under regular Lévy processes of exponential type. The first step is reduction of the calculation of the price to the boundary value problem for the generalized Black–Scholes equation, which is a nonlocal pseudodifferential equation. In this step, we use the representation of the q-order harmonic measure of a set relative to a point in terms of the q-potential measure. The reduction procedure generalizes to any Markov process having absolutely continuous potential kernel; the payoff is assumed measurable and bounded.

The next step is solution of the boundary problem by means of the standard tools of the theory of boundary value problems for PDO. Since the infinitesimal generator of a Lévy process is a PDO with constant symbol, the simplest tools—the Fourier transform and the Wiener–Hopf factorization—suffice, but for more general Lévy-like Feller processes considered in Barndorff-Nielsen and Levendorskiĭ (2001), more serious machinery of PDO is necessary.

After the explicit analytical formula for the unique bounded measurable solution is found (in some cases, the solution is singled out as the unique continuous one), we check that it is continuous in the continuation region, and hence this is the solution we are looking for. The verification is necessary since the solution is found as a generalized function, and the analytic expression involves the inverse Fourier transform.

In the case of an unbounded payoff, one more step is needed. In the stochastic representation, which defines the contingent claim with payoff g, we replace g by bounded functions g_n with compact support such that $g_n \uparrow g$ pointwise. This allows one to reduce the problem of the computation of f to the problem which is already solved: we find f_n as the solution of the corresponding boundary problem for the generalized Black–Scholes equation. The formula for f_n being found, we pass to the limit in the sense of generalized functions and obtain the analytic expression for the limit. By inspection, we see that the limit is a continuous function; hence it coincides with f.

We find explicit formulas for touch-and-out options, first-touch power options and power-like barrier options (including standard barrier options); we consider cases of options without rebate, with constant rebate and with exponentially decaying rebate. We also consider the case of the double barrier, when the rebate is paid only when the first barrier has been crossed but the second one, situated farther, has not.

The formulas obtained admit certain simplifications (from the point of view of the numerical calculations, not the length of the resulting formulas) but they are much more involved; we consider them in separate publications.

APPENDIX

PROOF OF LEMMA 4.3. We prove (4.10); (4.11) is proved similarly. By making an appropriate change of variables, we may assume that $\sigma > 0 = \sigma_1 > \sigma_-$.

By using (3.2), (3.3) and (4.6), we easily obtain the estimates

$$C^{-1} \le |B(\lambda, \eta)| \le C, \tag{A.1}$$

$$|B(\lambda, \eta) - 1| \le C_1(1 + |\lambda| + |\eta|^{\nu'})/(1 + |\lambda| + |\eta|^{\nu}), \tag{A.2}$$

$$|\partial_\eta B(\lambda, \eta)/B(\lambda, \eta)| \le C_2(1 + |\lambda|^{1/\nu} + |\eta|)^{-1}, \tag{A.3}$$

where C, C_1 and C_2 are independent of $\lambda \in \Sigma_\theta$ and η in a strip $\Im\eta \in [\sigma_-, \sigma_+]$, as well as all constants below. Set $K = (|\lambda| + 1)^{1/\nu}$, and for each pair (λ, ξ) introduce intervals $J_j \subset \mathbf{R}$:

$$J_1 = \{\eta \mid |\eta - \xi| \le K\}, \qquad J_2 = \{\eta \mid |\eta - \xi| > K,\ |\eta| \le K\},$$
$$J_3 = \{\eta \mid |\eta - \xi| \ge |\eta|,\ |\eta| > K\}, \qquad J_4 = \{\eta \mid K < |\eta - \xi| < |\eta|,\ |\eta| > K\}.$$

By using the mean value theorem and (A.3), we obtain

$$\frac{b(\lambda, \eta)}{\xi + i\sigma - \eta} = \frac{b(\lambda, \xi)}{\xi + i\sigma - \eta} + R(\lambda, \xi, \eta, \sigma), \tag{A.4}$$

where

$$|R(\lambda, \xi, \eta, \sigma)| \le C_3(1 + |\lambda|)^{-1/\nu}. \tag{A.5}$$

Since

$$\left|\int_{-K}^{K} \frac{d\eta}{i\sigma - \eta}\right| = \left|\ln \frac{-K - i\sigma}{K - i\sigma}\right| \le 2\pi,$$

we deduce, from (A.4)–(A.5) and (A.1),

$$\left|\int_{J_1} \frac{b(\lambda, \eta)\, d\eta}{\xi + i\sigma - \eta}\right| \le 2\pi \ln C + C_3 \int_{|\eta - \xi| \le K} (1 + |\lambda|)^{-1/\nu}\, d\eta = C_4. \tag{A.6}$$

To prove the following estimate, only (A.1) is needed:

$$\left|\int_{J_2} \frac{b(\lambda, \eta)\, d\eta}{\xi + i\sigma - \eta}\right| \le C_5 \int_{|\eta| \le K} (1 + |\lambda|)^{-1/\nu}\, d\eta = C_6.$$

Further, we infer from (A.2) that b admits an estimate of the same form as $B - 1$; using this estimate on J_3, we obtain

$$\left|\int_{J_3} \frac{b(\lambda, \eta)\, d\eta}{\xi + i\sigma - \eta}\right| \le C_7 \int_{|\eta| \ge K} \frac{1 + |\lambda| + |\eta|^{\nu'}}{|\eta|(1 + |\lambda| + |\eta|^{\nu})}\, d\eta. \tag{A.7}$$

By changing variables $\eta = K\eta'$, we see that the RHS in (A.7) is bounded uniformly in $\lambda \in \Sigma_\theta$, $\xi \in \mathbf{R}$, $\sigma > 0$.

Since $\nu' \in [0, \nu)$, a function

$$f(s) = (1 + |\lambda| + s^{\nu'})/(1 + |\lambda| + s^{\nu})$$

is decreasing on $[0, +\infty)$, and therefore we deduce from (A.2) an estimate, for $\eta \in J_4$,

$$|b(\lambda, \eta)| \le C_8 (1 + |\lambda| + |\xi - \eta|^{\nu'})/(1 + |\lambda| + |\xi - \eta|^{\nu}). \tag{A.8}$$

From (A.8),

$$\left|\int_{J_4} \frac{b(\lambda, \eta)\, d\eta}{\xi + i\sigma - \eta}\right| \le C_8 \int_{|\xi - \eta| \ge K} \frac{1 + |\lambda| + |\xi - \eta|^{\nu'}}{|\xi - \eta|(1 + |\lambda| + |\xi - \eta|^{\nu})}\, d\eta, \tag{A.9}$$

and the change of variables $\eta = \xi + K\eta'$ shows that the RHS in (A.9) is bounded uniformly in $\lambda \in \Sigma_\theta$.

By gathering bounds (A.6)–(A.7) and (A.9), we obtain (4.10). □

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