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SOME MONOTONICITY AND DEPENDENCE PROPERTIES OF SELF-EXCITING POINT PROCESSES

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Point processes on the positive real axis which are positively self-exciting in a sense expressed by their martingale dynamics are studied in this paper. It is shown that such processes can be realized as increasing mappings of Poisson processes and are therefore associated in appropriate manners. Some examples are presented, including Hawkes, renewal, Pólya–Lundberg, Markov dependent, semi-Markov, in addition to other point processes. As corollaries an extension of the Burton–Waymire association result and a solution of the Glasserman conjecture are obtained. Some results on dependence in stochastic processes of interest in queueing are given as a by product.

1. Introduction. Let N be a simple point process (p.p.) with no fixed atoms on the real line. As the corresponding jump points we take $P = \{\cdots < T_{-1} < T_0 \le 0 < T_1 < \cdots\}$ and we assume that this set has no limit points. The counting measure $N(\cdot)$ is defined for each bounded Borel set B to be the cardinality of the set $B \cap P$. Here $N(\cdot)$ is a random measure determined by its finite-dimensional distributions, that is, the joint distributions of $(N(I_1), \dots, N(I_k))$, for every integer $k > 0$, and all bounded intervals $I_1, \dots, I_k$ in the real line. Other characterizations of point processes are often useful in discussing particular situations. For example, we may consider $X_i = T_i - T_{i-1}$, $i > 1$, $i < 1$ and $Y_0 = -T_0$, $Y_1 = T_1$, which is called the interval sequence. The joint distribution of $(Y_0, Y_1, X_i,\ i \in Z \setminus \{1\})$ determines the distribution of N. Another approach is through the *stochastic intensity* of the point process N. Heuristically, the stochastic intensity $\lambda(t; \mathscr{F}_t^N)$ is the conditional intensity of having a point just after time instant t, provided that the history $\mathscr{F}_t^N = \sigma(N((-\infty, s]),\ s \le t)$ of the point process N up to t is known. In general, stochastic intensity does not necessarily characterize a point process and it is usually difficult to demonstrate the existence of stationary point processes which have a particular stochastic intensity. However, point processes on the positive real half axis that admit stochastic intensity are uniquely determined by predictable versions of their intensities [Jacod (1975), Theorem 3.4].

In this paper we shall consider a natural class of point processes, for which stochastic intensities will be increasing functions of the past evolution of the

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process; that is, for such processes larger (in some sense) realizations before t will imply that the corresponding intensities of having a point just after t will also be larger. We shall call point processes with this property positively self-exciting. This is not precise without explaining what is meant by the term "larger" point process. Therefore, we shall introduce three possible orderings for point processes and we shall always relate the property of being positively self-exciting to a given ordering. We shall show that a number of natural point processes have such a property, for example, Hawkes population point processes, Pólya–Lundberg point processes and some renewal processes. We prove a general theorem that positively self-exciting point processes are "increasing" transformations of Poisson processes, where monotonicity will be related to the underlying ordering of point processes. As a consequence we shall study association properties of positively self-exciting point processes. Among other results we generalize the existing results on association for renewal processes due to Burton and Waymire (1986) (and at the same time we give a rather different approach to the problem), and we also confirm a conjecture on point processes with associated increments stated by Glasserman (1992). Finally, as a byproduct we obtain some association properties of stochastic processes related to point processes of interest, for example, in queueing theory.

The paper is organized as follows. In Section 2 definitions of stochastic intensities of point processes, stochastic orderings for point processes and some constructions of point processes are introduced. In Section 3 we shall discuss association of random measures and point processes, giving some relations between finite- and infinite-dimensional settings. In Section 4 positively self-exciting point processes along with examples, monotonicity and association properties of positively self-exciting point processes are studied. In Section 5 we shall establish the dependency properties of some stochastic processes related to point processes.

2. Preliminaries. Let $\mathscr{M} = \mathscr{M}(E)$ be the space of Radon (i.e., locally finite) measures on a locally compact second countable Hausdorff space E. Let $\mathscr{B} = \mathscr{B}(E)$ be the class of Borel sets generated by the topology of E, and let $\mathscr{F}_c = \mathscr{F}_c(E)$ be the class of nonnegative continuous functions $E \to R$ with compact supports. The space $\mathscr{M}$ can be endowed with the *vague* topology, for which the class of all finite intersections of sets of the form $\{\mu \in \mathscr{M}\colon s < \int_E f\, d\mu < t\}$ for $s, t \in R$ and $f \in \mathscr{F}_c$, may serve as a base. The space $\mathscr{M}$ with the vague topology is metrizable as a Polish space [Kallenberg (1983), 15.7.7]. We call any $\mathscr{M}$-valued random element M a random measure on E. The distributions of vectors $(M(B_1), \dots, M(B_n))$, $n \geq 1$, for arbitrary bounded sets $B_1, \dots, B_n \in \mathscr{B}$, entirely determine the distribution of a random measure M. We define a point process N on the space E as a random measure confined with probability 1 to the subset $\mathscr{N} = \mathscr{N}(E)$ of $\mathscr{M}$ consisting of all integer Radon measures on the space E. Elements of $\mathscr{M}$ or $\mathscr{N}$ will be denoted by μ, ν, π, with indices if necessary.

In the remainder of the paper we shall discuss mainly point processes on R_+. Thus $E = R_+$. In this case the simplest description of a point process is perhaps the one by a sequence of random variables $0 = T_0 < T_1 < \cdots$ on a probability space $(\Omega, \mathscr{F}, P)$, corresponding to the jump points of N. We assume that $T_n \to \infty$ as $n \to \infty$ (the process is nonexplosive) and that $\{T_n\}$ is strictly increasing (the process is simple or without multiple points).

Depending on the context, it is convenient to view realizations of a point process either as elements of $\mathscr{N}(R_+)$ (i.e., measures) or as increasing sequences of jump points. To make this duality formally acceptable, we introduce a pair of transformations between the corresponding subsets of $\mathscr{N}(R_+)$ and $\bar{R}_+^\infty$ and prove their measurability. These transformations can be easily extended to the whole spaces $\mathscr{N}(R_+)$ and $\bar{R}_+^\infty$. To this end let $\mathscr{N}_\infty(R_+) = \{\mu \in \mathscr{N}(R_+)\colon \mu(R_+) = \infty\}$, and

$$\mathscr{J}_+ = \{\mathbf{t} = (t_1, t_2, \dots) \in R_+^\infty\colon t_1 < t_2 < \cdots,\ t_n \to +\infty\}.$$

Consider $\mathscr{J}_+$ to be endowed with the usual product topology. For $\mu \in \mathscr{N}(R_+)$ let $\tau_0(\mu) = 0$ and for $n \geq 1$ let

$$\tau_n(\mu) = \sup\{u > 0\colon \mu(0, u] \leq n\}.$$

Define a mapping $\boldsymbol{\tau}\colon \mathscr{N}_\infty(R_+) \to \mathscr{J}_+$ as

$$\boldsymbol{\tau}(\mu) = (\tau_1(\mu), \tau_2(\mu), \dots).$$

Notice that $\boldsymbol{\tau}$ is invertible and that $\boldsymbol{\tau}^{-1}(\mathbf{t})$ is defined as a measure μ such that $\mu(C) = \sum_{n=1}^\infty \mathbf{1}_C(t_n)$ for all measurable sets $C \subset R_+$. Here $\mathbf{1}_C(\cdot)$ is the indicator function of a set C. The mappings $\boldsymbol{\tau}$ and $\boldsymbol{\tau}^{-1}$ are measurable. Indeed, let $A \subset \mathscr{N}(R_+)$ be a set of the form

$$A = \left\{\mu\colon \int_{R_+} f\, d\mu < s\right\},$$

where $s \in R_+$ and $f \in \mathscr{F}_c(R_+)$. For measurability of $\boldsymbol{\tau}^{-1}$ it is sufficient to show that $\boldsymbol{\tau}(A)$ is a measurable subset of $\mathscr{J}_+$. We have

$$\boldsymbol{\tau}(A) = \left\{\mathbf{t} \in \mathscr{J}_+\colon \sum_{i=1}^\infty f(t_i) < s\right\}. \tag{1}$$

Since the support of f is bounded, for each $\mathbf{t}$ only a finite number of terms in the sum on the right-hand side of (1) are nonzero. By the continuity of f, the same terms (and none other) are nonzero in a neighborhood of $\mathbf{t}$. Again, by the continuity of f, it follows that $\boldsymbol{\tau}(A)$ is open in $\mathscr{J}_+$. Hence $\boldsymbol{\tau}^{-1}$ is measurable.

Let $a, b \in R_+$. For a fixed $i \geq 1$ let

$$B = \{\mathbf{t} \in \mathscr{J}_+\colon t_{i-1} < a < t_i < b < t_{i+1}\}.$$

For measurability of $\boldsymbol{\tau}$ it is sufficient to prove that $\boldsymbol{\tau}^{-1}(B)$ is a measurable subset of $\mathscr{N}(R_+)$. Let $\{f_n\}, \{g_n\} \subset \mathscr{F}_c(R_+)$ and $\{F_n\}, \{G_n\} \subset \mathscr{B}(R_+)$ be such that

$$\mathbf{1}_{(a,b)} \geq f_n \geq \mathbf{1}_{F_n} \nearrow \mathbf{1}_{(a,b)} \quad \text{and} \quad \mathbf{1}_{(0,b)} \geq g_n \geq \mathbf{1}_{G_n} \nearrow \mathbf{1}_{(0,b)}$$

[Kallenberg (1983), 15.6.1]. Then

$$\boldsymbol{\tau}^{-1}(B) = \bigcup_{n=1}^{\infty}\left\{\mu \in \mathscr{N}(R_+)\colon \int_E f_n\,d\mu = 1\right\} \cap \bigcup_{n=1}^{\infty}\left\{\mu \in \mathscr{N}(R_+)\colon \int_E g_n\,d\mu = i\right\}$$

is measurable in $\mathscr{N}(R_+)$. Hence $\boldsymbol{\tau}$ is measurable.

2.1. *Compensator and stochastic intensity.* Consider the canonical p.p. N on $\mathscr{N}(R_+)$. Assume that N is simple and nonexplosive and $0 = T_0 < T_1 < \cdots$ are the jump points of N on R_+. That is, $T_n = \tau_n(N)$. For $n \geq 0$ let

$$F_{n+1}(x;\, t_1, \ldots, t_n) = P\{T_{n+1} - T_n \leq x \mid T_1 = t_1, \ldots, T_n = t_n\},$$
$$R_{n+1}(x;\, t_1, \ldots, t_n) = -\log(1 - F_{n+1}(x;\, t_1, \ldots, t_n))$$

be regular versions of conditional probability distributions of the interpoint distances of N and their cumulative hazard functions, respectively. Since $\mathscr{N}(R_+)$ is a Polish space, the regular versions do exist. Fix $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n$. For $t \geq t_n$ define (regarding empty sums as zeros)

$$a_{n+1}(t;\, t_1, \ldots, t_n) = \sum_{i=1}^{n} R_i(t_i - t_{i-1};\, t_1, \ldots, t_{i-1}) + R_{n+1}(t - t_n;\, t_1, \ldots, t_n),$$

and $a_1(t) = R_1(t)$. The definition of a_{n+1} can be made "consistent" by setting $a_{n+1}(t;\, t_1, \ldots, t_n) = a_{k+1}(t;\, t_1, \ldots, t_k)$ for $t \in (t_k, t_{k+1}]$ and $k < n$. The family of functions $\{a_n(\cdot)\}$ is called *the compensator function family* associated with the point process N. The compensator of the process N is a process $\Lambda(t, N)$ such that, for $t \in R_+$ and $\mu \in \mathscr{N}(R_+)$,

$$\Lambda(t, \mu) = \sum_{n=0}^{\infty} a_{n+1}(t;\, \tau_1(\mu), \ldots, \tau_n(\mu))\mathbf{1}_{(\tau_n(\mu),\, \tau_{n+1}(\mu)]}(t). \tag{2}$$

Suppose now that the conditional distributions $F_{n+1}(x;\, t_1, \ldots, t_n)$ are absolutely continuous with respect to the Lebesgue measure on R_+. Denote the corresponding probability density functions by $f_{n+1}(x;\, t_1, \ldots, t_n)$. Let $r_{n+1}(x;\, t_1, \ldots, t_n)$ be the conditional failure rate of $(T_{n+1} - T_n)$. That is, for $x > 0$,

$$r_{n+1}(x;\, t_1, \ldots, t_n) = \frac{f_{n+1}(x;\, t_1, \ldots, t_n)}{1 - F_{n+1}(x;\, t_1 \ldots, t_n)}.$$

Note that F_{n+1}, as a regular version of a probability distribution, is measurable in $(x, t_1, t_2, \ldots)$, and this yields that r_{n+1} is measurable as well. The stochastic intensity of the process N is a process $\lambda(t, N)$ such that

$$\lambda(t, \mu) = \sum_{n=0}^{\infty} r_{n+1}(t - \tau_n(\mu);\, \tau_1(\mu), \ldots, \tau_n(\mu))\mathbf{1}_{(\tau_n(\mu),\, \tau_{n+1}(\mu)]}(t), \tag{3}$$

where $t \in R_+$ and $\mu \in \mathscr{N}(R_+)$. That is, $\lambda(t, N)$ equals $r_{n+1}(t - T_n;\, T_1, \ldots, T_n)$ for $t \in (T_n, T_{n+1}]$.

Note that since $R_{n+1}(x;\ t_1,\dots,t_n) = \int_0^x r_{n+1}(u;\ t_1,\dots,t_n)\,du$, then $\Lambda(t,N) = \int_0^t \lambda(u,N)\,du$. For $t > 0$, let $\mathscr{F}_t^N$ be the σ-field generated by the family of mappings $N(0,u]$: $(\Omega,\mathscr{F},P) \to R$, where $u \le t$. Then $\lambda(t,N)$ is a $(P,\mathscr{F}_t^N)$-predictable stochastic intensity of the process N [Brémaud (1981), Theorem 3.7], and $\Lambda(t,N)$ is the compensator of N.

2.2. *Orderings.* Define the following partial order $\prec_{\mathscr{M}}$ in $\mathscr{M}(E)$. For $\mu,\nu \in \mathscr{M}$ let $\mu \prec_{\mathscr{M}} \nu$ if and only if $\mu(B) \le \nu(B)$ for all (topologically) bounded sets $B \in \mathscr{B}(E)$. The order $\prec_{\mathscr{M}}$ is closed in $\mathscr{M}$. That is, $\{(\mu,\nu): \mu \prec_{\mathscr{M}} \nu\}$ is closed in $\mathscr{M} \times \mathscr{M}$ [Kallenberg (1983), 15.7]. The ordering $\prec_{\mathscr{M}}$ restricted to $\mathscr{N}$ is often called *thinning* and will be denoted here by $\mu \prec_{\mathscr{N}} \nu$ for $\mu,\nu \in \mathscr{N}$. Since $\prec_{\mathscr{M}}$ is closed in $\mathscr{M}$, $\prec_{\mathscr{N}}$ is closed in $\mathscr{N}$.

When one considers a p.p. on R_+, other orderings corresponding to alternative descriptions of the p.p. also seem natural. The jump point description is often restated in terms of a *counting process* N_t defined for $t > 0$ by

$$N_t = N((0,t]) = \sum_{n=1}^{\infty} \mathbf{1}_{(0,t]}(T_n),$$

where $\mathbf{1}$ denotes the indicator function. Also, the distributions of vectors $(N_{t_1},\dots,N_{t_n})$ for $t_1 < \cdots < t_n$ and $n \ge 1$ completely define the distribution of the p.p. N. A trajectory $\mu_t = \mu((0,t])$ of N_t, where $\mu \in \mathscr{N}(R_+)$, is often regarded as an element of $\mathscr{D}(R_+)$, the space of real-valued functions on R_+, which are right-continuous with left-hand limits. Note that $\mathscr{D}(R_+)$ is usually equipped with the Skorohod topology to be a Polish space. The natural ordering in $\mathscr{D}(R_+)$ is $(\mu_t) \prec_{\mathscr{D}} (\nu_t)$ iff $\mu_t \le \nu_t$ for all $t > 0$. We will, however, always regard a p.p. as a random element of $\mathscr{N}(R_+)$ and for $\mu,\nu \in \mathscr{N}(R_+)$ write $\mu \prec_{\mathscr{D}} \nu$ iff $(\mu_t) \prec_{\mathscr{D}} (\nu_t)$ for the corresponding functions $\mu_t,\nu_t \in \mathscr{D}(R_+)$. Note that $\mu \prec_{\mathscr{D}} \nu$ is equivalent to $\boldsymbol{\tau}(\mu) \ge \boldsymbol{\tau}(\nu)$ coordinatewise. Convergence $\mu_n \to \mu$ in $\mathscr{N}(R_+)$ implies that $\boldsymbol{\tau}(\mu_n) \to \boldsymbol{\tau}(\mu)$ in $\mathscr{J}_+$; hence, $\prec_{\mathscr{D}}$ is closed in $\mathscr{N}(R_+)$ [also, it is closed in $\mathscr{D}(R_+)$].

Another description of a point process is provided by its interpoint distances: the distributions of $(T_1, T_2 - T_1, \dots, T_n - T_{n-1})$, $n \ge 1$, wholly determine the distribution of a point process on R_+. In order to compare interpoint distances of point processes, we introduce the following ordering. For $\mu,\nu \in \mathscr{N}(R_+)$ let $\mu \prec_{\infty} \nu$ iff $\tau_{n+1}(\mu) - \tau_n(\mu) \ge \tau_{n+1}(\nu) - \tau_n(\nu)$ for $n \ge 0$.

We now introduce various concepts of stochastic comparisons of point processes. Depending on which of the descriptions of point processes we adopt, we obtain different kinds of stochastic comparisons.

Suppose that N, N' are two point processes on E. We write

$$N \prec_{\text{st}-\mathscr{N}} N' \quad \text{iff } E(\phi(N)) \le E(\phi(N'))$$

for all functionals ϕ: $\mathscr{N}(E) \to R_+$ which are $\prec_{\mathscr{N}}$-increasing. Suppose now that N, N' are two point processes on R_+. Define

$$N \prec_{\text{st}-\mathscr{D}} N' \quad \text{iff } E(\varphi(N)) \le E(\varphi(N'))$$

for all functionals $\varphi\colon \mathscr{N}(R_+) \to R_+$ which are $\prec_{\mathscr{D}}$-increasing. Finally, define

$$N \prec_{\text{st}-\infty} N' \quad \text{iff } E\psi(N) \le E\psi(N')$$

for all functionals $\psi\colon \mathscr{N}(R_+) \to R_+$ which are $\prec_\infty$-increasing.

The following lemma characterizes the above stochastic orderings in terms of finite-dimensional vectors. For part (i) see, for example, Rolski and Szekli (1991), Theorem 1; for other cases see, for example, Stoyan (1983). In the lemma $\le_{\text{st}}$ denotes the usual stochastic ordering of random vectors.

LEMMA 2.1. (i) *$N \prec_{\text{st}-\mathscr{N}} N'$ iff $(N(B_1),\dots,N(B_n)) \le_{\text{st}} (N'(B_1),\dots, N'(B_n))$;*

(ii) *$N \prec_{\text{st}-\mathscr{D}} N'$ iff $(N_{t_1},\dots,N_{t_n}) \le_{\text{st}} (N'_{t_1},\dots,N'_{t_n})$;*

(iii) *$N \prec_{\text{st}-\infty} N'$ iff $(X'_1,\dots,X'_n) \le_{\text{st}} (X_1,\dots,X_n)$*

for each finite collection of sets $B_1,\dots,B_n \in \mathscr{B}(E)$, $t_1 < \cdots < t_n \in R_+$ and the corresponding interpoint distances $X_n = T_n - T_{n-1}$, $X'_n = T'_n - T'_{n-1}$, $n \ge 1$.

Since both $\mu \prec_{\mathscr{N}} \nu$ and $\mu \prec_\infty \nu$ yield $\mu \prec_{\mathscr{D}} \nu$, then also both $N \prec_{\text{st}-\mathscr{N}} N'$ and $N \prec_{\text{st}-\infty} N'$ imply $N \prec_{\text{st}-\mathscr{D}} N'$.

2.3. *Constructions.* In order to motivate the discussion of this section, consider first a single nonnegative random variable X. Let F_X be the distribution function of X and assume that it is absolutely continuous with density function f_X. Let $R_X = -\log(1 - F_X)$ and $r_X = f_X/(1 - F_X)$. It is easy to verify that $R_X(x) = \int_0^x r_X(u)\,du$, $x \ge 0$.

Denote the left-continuous pseudo-inverses of F_X and of R_X by $F_X^{-1}(u) = \inf\{x\colon F_X(x) \ge u\}$, $u \in (0,1)$, and $R_X^{-1}(p) = \inf\{x\colon R_X(x) \ge p\}$, $p \ge 0$, respectively. Then X has the representation

$$X =_{\text{st}} F_X^{-1}(U),$$

where U is a uniform $(0,1)$ random variable, and $=_{\text{st}}$ denotes equality in law. From this it follows that X also has the representation

$$X =_{\text{st}} R_X^{-1}(E), \tag{4}$$

where E is a standard (i.e., mean 1) exponential random variable.

The idea of the representation (4) can be used for the purpose of representing the interpoint distances $X_1, X_2,\dots$ of a point process N. Let $E_1, E_2,\dots$ be a sequence of independent standard exponential random variables. Then we have the representation

$$T_1 = X_1 =_{\text{st}} R_1^{-1}(E_1) \qquad (= \hat{T}_1, \text{ say}),$$
$$T_2 = T_1 + X_2 =_{\text{st}} \hat{T}_1 + R_2^{-1}(E_2;\, \hat{T}_1) \qquad (= \hat{T}_2, \text{ say}).$$

And, in general, for $n \ge 1$,

$$T_n =_{\text{st}} \hat{T}_{n-1} + R_n^{-1}(E_n;\, \hat{T}_1,\dots,\hat{T}_{n-1}) \qquad (= \hat{T}_n, \text{ say}), \tag{5}$$

where $R_n^{-1}(\cdot\,; t_1, \ldots, t_{n-1})$ is the left-continuous pseudo-inverse of the cumulative hazard function $R_n(\cdot\,; t_1, \ldots, t_{n-1})$ defined in Section 2.1. It follows that

$$\mathbf{T} = (T_1, T_2, \ldots) =_{\mathrm{st}} (\hat{T}_1, \hat{T}_2, \ldots) = \hat{\mathbf{T}}.$$

It is sometimes convenient to represent $\hat{\mathbf{T}}$ as a function of the cumulative sums of the E_n's. Denote $P_n = \sum_{i=1}^{n} E_i$, $n = 1, 2, \ldots$. Then, for $n \geq 1$, (5) can be rewritten as

$$\hat{T}_n =_{\mathrm{st}} a_n^{-1}(P_n; \hat{T}_1, \ldots, \hat{T}_{n-1}), \tag{6}$$

where $a_n^{-1}(\cdot\,; t_1, \ldots, t_{n-1})$ is the left-continuous pseudo-inverse of $a_n(\cdot\,; t_1, \ldots, t_{n-1})$ defined in Section 2.1.

Let $\Pi_{(1)}$ be a Poisson process on R_+ with intensity 1 (the standard Poisson process on R_+), and for $n \geq 1$ let $P_n = \tau_n(\Pi_{(1)})$. Formula (6) assigns a realization of a p.p. $\hat{N} = \boldsymbol{\tau}^{-1}(\hat{T}_1, \hat{T}_2, \ldots)$ to each possible realization of $\Pi_{(1)}$. We will write that in an abbreviated form $\hat{N} = \Gamma(\Pi_{(1)})$, where $\Gamma\colon \mathscr{N}(R_+) \to \mathscr{N}(R_+)$. We summarize this representation (construction) of a p.p. in a lemma. For its point process proof, see Kwieciński and Szekli (1991), Proposition 3.2; in a multidimensional setting, see Daduna and Szekli (1995), Lemma 4.2.

LEMMA 2.2. (i) *The mapping* $\Gamma\colon \mathscr{N}(R_+) \to \mathscr{N}(R_+)$ *is measurable.*

(ii) *Let* $\Pi_{(1)}$ *be a Poisson process on* R_+ *with intensity* 1 (*the standard Poisson process on* R_+). *Then* $N =_{\mathrm{st}} \Gamma(\Pi_{(1)})$.

Another construction of a p.p. is called the *Poisson embedding* [Lindvall (1988), page 127] and is given by the following description. Suppose that N is a simple point process with an intensity process $\lambda(t, N)$ defined in (3). Define a mapping $\boldsymbol{\kappa}\colon N(R_+^2) \to \mathscr{J}_+$ as follows. Let $\pi_{(2)} \in N(R_+^2)$. Set $\kappa_0 = 0$. For $n \geq 1$ let

$$A_{n,t} = \{(u, y) \in R_+^2\colon \kappa_{n-1} < u \leq t,\; y \leq r_n(u - \kappa_{n-1}; \kappa_1, \ldots, \kappa_{n-1})\},$$
$$\kappa_n = \kappa_n(\pi_{(2)}) = \sup\{t > \kappa_{n-1}\colon \pi_{(2)}(A_{n,t}) = 0\}.$$

Now set $\boldsymbol{\kappa}(\pi_{(2)}) = (\kappa_1(\pi_{(2)}), \kappa_2(\pi_{(2)}), \ldots)$. Let $\Upsilon\colon \mathscr{N}(R_+^2) \to \mathscr{N}(R_+)$ be given by $\Upsilon(\pi_{(2)}) = \boldsymbol{\tau}^{-1}\boldsymbol{\kappa}(\pi_{(2)})$. Note that Υ depends on N via the intensity function $\lambda(t, \mu)$. The transformation $\pi_{(2)} \mapsto \Upsilon(\pi_{(2)})$ has an intuitive graphical interpretation. Let $\mu_0 \in \mathscr{N}(R_+)$ be a counting measure that has no points. Let (t_1, y_1) be the point of $\pi_{(2)}$ with the smallest t_1 lying below the line $y = \lambda(t, \mu_0)$. Set $\kappa_1 = t_1$. Continue by defining μ_1 as having an atom in κ_1 and finding the point (t_2, y_2) of $\pi_{(2)}$ with the smallest $t_2 > t_1$ lying below the line $y = \lambda(t, \mu_1)$. Set $\kappa_2 = t_2$, and so forth.

LEMMA 2.3 (Poisson embedding). (i) *The mapping* $\Upsilon\colon \mathscr{N}(R_+^2) \to \mathscr{N}(R_+)$ *is measurable.*

(ii) *Let* $\Pi_{(2)}$ *be a Poisson process on* R_+^2 *with intensity* 1 (*the standard Poisson process on* R_+^2). *Then* $N =_{\mathrm{st}} \Upsilon(\Pi_{(2)})$.

PROOF. Part (i) follows from the fact that the mapping $\pi_{(2)} \mapsto \kappa_n(\pi_{(2)})$ is measurable for each $n \geq 1$, which is a consequence of the measurability of the r_n's. For part (ii) assume that $(T_1, \dots, T_{n-1}) =_{\text{st}} (\kappa_1, \dots, \kappa_{n-1})$ and note that

$$\begin{aligned} P\{\kappa_n - \kappa_{n-1} \leq x \mid \kappa_1 = t_1, \dots, \kappa_{n-1} = t_{n-1}\} \\ = P\{\Pi_{(2)}(A_{n,\, \kappa_{n-1}+x}) > 0 \mid \kappa_1 = t_1, \dots, \kappa_{n-1} = t_{n-1}\} \\ = 1 - \exp(-|A_{n,\, \kappa_{n-1}+x}|) = 1 - \exp(-R_n(x;\, t_1, \dots, t_{n-1})) \\ = F_n(x;\, t_1, \dots, t_{n-1}). \end{aligned}$$

Hence $\boldsymbol{\kappa}(\Pi_{(2)}) =_{\text{st}} (T_1, T_2, \dots)$ and $\Upsilon(\Pi_{(2)}) = \boldsymbol{\tau}^{-1}\boldsymbol{\kappa}(\Pi_{(2)}) =_{\text{st}} \boldsymbol{\tau}^{-1}(T_1, T_2, \dots) = N$. □

REMARK 2.1. Suppose that we have two point processes N and N' with intensity processes λ, λ' and corresponding mappings Υ, Υ'. Suppose also that if $\mu \prec_{\mathscr{N}} \nu$, then $\lambda(\tau_n(\nu), \mu) \leq \lambda'(\tau_n(\nu), \nu)$ for all $n \geq 1$. Then the "simultaneous" Poisson embedding $\Upsilon(\Pi_{(2)})$ and $\Upsilon'(\Pi_{(2)})$ yields versions of N and N' satisfying $\Upsilon(\Pi_{(2)}) \prec_{\mathscr{N}} \Upsilon'(\Pi_{(2)})$ a.s., and is equivalent to the construction of *thinning* formalized differently in Rolski and Szekli (1991). Similarly, if the intensity function of say the process N is bounded, that is, for all t and μ we have $\lambda(t, \mu) \leq a$ where $a > 0$, then $\Upsilon(\Pi_{(2)})$ is the result of thinning of a Poisson process on R_+ with intensity a.

3. Association. Let $(S, \mathscr{S})$ be a Polish space with a closed partial ordering $\prec$. The classical notion of association of (the probability distribution of) a random vector [Esary, Proschan and Walkup (1967)] can be extended in the following way.

DEFINITION 3.1. A probability measure on $(S, \mathscr{S})$ is associated $(\prec)$ if

$$P(C_1 \cap C_2) \geq P(C_1)P(C_2)$$

for all increasing sets $C_1, C_2 \in \mathscr{S}$ (a set C is increasing if $x \in C$ and $x \prec y$ implies $y \in C$).

The following theorem [Lindqvist (1988), Theorem 3.1] gives several conditions equivalent to association of a probability measure on $(S, \mathscr{S}, \prec)$.

THEOREM 3.1. *Let P be a probability measure on $(S, \mathscr{S}, \prec)$. The following conditions are equivalent*:

(i) *P is associated $(\prec)$*;
(ii) *$\int fg\, dP \geq \int f\, dP \int g\, dP$ for all f, g increasing, bounded, real-valued functions*;
(iii) *$P(C_1 \cap C_2) \geq P(C_1)P(C_2)$ for all increasing closed sets C_1, C_2.*

In many cases association of a probability measure on a partially ordered Polish space follows from a useful technical lemma of Lindqvist (1988), Theorem 3.2.

LEMMA 3.1. *Suppose that* $(S_1, \mathscr{S}_1, \prec_1)$ *is a partially ordered Polish space with P associated* $(\prec_1)$, *and* $\varphi: S_1 \to S_2$ *is a measurable mapping to a partially ordered Polish space* $(S_2, \mathscr{S}_2, \prec_2)$. *If* φ *is such that* $\varphi(x) \prec_2 \varphi(x')$ *for all* $x \prec_1 x'$ (*i.e.,* φ *is increasing*), *then* $P \circ \varphi^{-1}$ *is associated* $(\prec_2)$ *on* $(S_2, \mathscr{S}_2)$.

3.1. *Association of random measures.* There are two ways to define association of random measures, and in particular point processes, each based on a different theoretical background. The first approach is based on the classical notion of association of random variables introduced by Esary, Proschan and Walkup (1967). From this viewpoint, a random measure or a point process M on a space E is a collection of random variables $\{M(B): B \in \mathscr{B}(E)\}$. The random measure M is then said to be associated if and only if $(M(B_1), \dots, M(B_n))$ is a vector of associated random variables for any $n \geq 1$ and $B_1, \dots, B_n \in \mathscr{C}$, where $\mathscr{C}$ is a certain subset of $\mathscr{B}$. For a review of this approach, see Burton and Franzosa (1990).

The other approach is to think of M as a random element assuming its values in some functional space $\mathscr{S} = \mathscr{S}(E)$. Suppose that $\mathscr{S}$ has the structure of a Polish space and is endowed with a closed partial order $\prec$. A random measure M is then said to be associated if and only if $\mathrm{Cov}(f(M), g(M)) \geq 0$ for any pair of real-valued, square-integrable functions f, g on $\mathscr{S}$, increasing w.r.t. the order $\prec$. That is, the distribution of M is associated $(\prec)$ in the sense of Definition 3.1.

Sometimes the two approaches can be made equivalent. Suppose, for example, that $E = [0, 1]$ with the topology of open intervals, and $\mathscr{S}$ is the space of nondecreasing and right-continuous functions from E to R, with the Skorohod topology. Such functions represent finite measures on E. The so-called time association of an $\mathscr{S}$-valued random element M is usually defined as the association of all finite-dimensional projections $(M(B_1), \dots, M(B_n))$ such that $B_i = [0, t_i]$ for some $t_i \in [0, 1]$ (the first approach). The time association is equivalent to the association implied by a natural pointwise partial order on S (the second approach); see Lindqvist (1988), Theorem 6.1.

We prove now that a random measure $M: (\Omega, \mathscr{F}, P) \to \mathscr{M}(E)$ is associated w.r.t. $\prec_{\mathscr{M}}$ if and only if all its finite-dimensional projections are associated random vectors. Rolski and Szekli (1991) considered a mapping $\gamma: \mathscr{M} \to R_+^\infty$ defined as follows. Let $\mathscr{I} = \{I_1, I_2, \dots\}$ be a denumerable DC semi-ring generating the ring of all bounded Borel sets in E [Kallenberg (1983)]. For $\mu \in \mathscr{M}$ let

$$\gamma(\mu) = (\mu(I_1), \mu(I_2), \dots).$$

Let $\mathscr{G} = \gamma(\mathscr{M})$. Lemma 2 in Rolski and Szekli (1991) yields the following lemma.

LEMMA 3.2 [Rolski and Szekli (1991)]. *The set* $\mathscr{G}$ *is a closed subset of* R_+^∞, *and* $\gamma: \mathscr{M} \to \mathscr{G}$ *is a homeomorphism, where the topology of* $\mathscr{G}$ *is the one generated by the topology in* R_+^∞.

The following theorem is a consequence of Lemma 3.2.

THEOREM 3.2. *A random measure M is associated if and only if random vectors $(M(B_1)), \ldots, M(B_n))$ are associated for all $n \geq 1$ and bounded sets $B_1, \ldots, B_n \in \mathscr{B}(E)$.*

PROOF. Consider the natural coordinate order $\leq$ in R_+^∞. Both γ and γ^{-1} are increasing with respect to $\prec_{\mathscr{M}}$ and $\leq$, respectively. Therefore, M is associated if and only if $\gamma(M)$ is associated as a random element of R_+^∞ (Lemma 3.1). However, $\gamma(M)$ is associated if and only if $(M(I_1), \ldots, M(I_n))$ is a vector of associated random variables for all $n \geq 1$ [Lindqvist (1988), Theorem 5.1]. Since this is true for any DC semi-ring $\mathscr{I} = \{I_1, I_2, \ldots\}$ as specified above, the statement of the theorem follows. □

REMARK 3.1. By an approximation argument, bounded Borel sets in the statement of Theorem 3.2 can be replaced with (a) all Borel sets for which $M(B_i) < +\infty$ a.s. and (b) all Borel sets, if one allows random variables to take values in the extended real line (with $-\infty$ and $+\infty$ as the extreme points) and defines association of random vectors so as to take that into account.

3.2. *Associated point processes.* It is natural to introduce association of a p.p. N with respect to $\prec_{\mathscr{D}}$, $\prec_{\mathscr{N}}$ and $\prec_\infty$-orderings, which is association, in the sense of Definition 3.1, of the distribution of N on $\mathscr{N}(R_+)$ with the given three closed partial orderings.

To summarize, the above introduced notions of association are related to the association of finite-dimensional random vectors by the following theorem.

THEOREM 3.3. (i) *A p.p. N on R_+ is associated ($\prec_{\mathscr{N}}$) iff the vector $(N(B_1), \ldots, N(B_n))$ is associated for all bounded sets $B_1, \ldots, B_n \in \mathscr{B}(E)$.*

(ii) *A p.p. N on R_+ is associated ($\prec_{\mathscr{D}}$) iff the vector $(N_{t_1}, \ldots, N_{t_n})$ is associated for all $t_1 < \cdots < t_n \in R_+$.*

(iii) *A p.p. N on R_+ is associated ($\prec_\infty$) iff the vector $(-X_1, \ldots, -X_n)$ is associated for all $n \geq 1$, where $X_n = T_n - T_{n-1}$.*

PROOF. Part (i) follows from Theorem 3.2. For part (ii) see Theorem 6.1 of Lindqvist (1988). Part (iii) follows from an obvious observation that the transformation $N \mapsto (-X_1, \ldots, -X_n)$ is $\prec_{\mathscr{N}}$-$\leq$ increasing ($\leq$ being the coordinatewise ordering in R^n). □

It is worth mentioning here that association ($\prec_{\mathscr{N}}$) is closely related to the property of "associated increments" studied by Glasserman (1992).

4. Positively self-exciting point processes. In Brémaud (1981), self-exciting p.p. refers to a description in terms of the internal history, in the sense that the potential of a p.p. to generate a point is dictated by its past evolution. In Hawkes (1971) the phrase self-exciting is used for point processes which, intuitively speaking, would have larger intensities of generating a point if in their past evolutions the number of points would increase. We shall formally describe the above-mentioned intuitive property of point processes by stochastic orderings and call such processes positively self-exciting.

DEFINITION 4.1. Let N be a simple point process with a compensator Λ and a stochastic intensity λ given by, respectively, (2) and (3).

(i) We say that N is positively self-exciting w.r.t. $\prec_{\mathcal{N}}$ if, for all $\mu, \nu \in \mathcal{N}(R_+)$, $n \geq 1$,

$$\mu \prec_{\mathcal{N}} \nu \Rightarrow \Big(\forall_{t \in R_+} \quad \lambda(t, \mu) \leq \lambda(t, \nu) \Big)$$

[it is sufficient to verify this condition for $t = \tau_n(\nu)$, $n \geq 1$].

(ii) We say that N is positively self-exciting w.r.t. $\prec_{\mathcal{D}}$ if, for all $\mu, \nu \in \mathcal{N}(R_+)$,

$$\mu \prec_{\mathcal{D}} \nu \Rightarrow \Big(\forall_{t \in R_+} \quad \Lambda(t, \mu) \leq \Lambda(t, \nu) \Big),$$

or, equivalently,

$$\forall_{n \geq 0,\, t > 0} \quad a_{n+1}(t; \tau_1(\mu), \dots, \tau_n(\mu)) \leq a_{n+1}(t; \tau_1(\nu), \dots, \tau_n(\nu)).$$

(iii) We say that N is positively self-exciting w.r.t. $\prec_\infty$ if, for all $\mu, \nu \in \mathcal{N}(R_+)$,

$$\mu \prec_\infty \nu \Rightarrow \Big(\forall_{n \geq 0,\, x \in R_+} \quad R_{n+1}(x; \tau_1(\mu), \dots, \tau_n(\mu)) \leq R_{n+1}(x; \tau_1(\nu), \dots, \tau_n(\nu)) \Big).$$

Relations between orderings $\prec_{\mathcal{N}}$, $\prec_\infty$ and $\prec_{\mathcal{D}}$ and the fact that $\Lambda(t) = \int_0^t \lambda(u)\, du$ suggest possible relations between classes of point processes which are positively self-exciting (s.e.) w.r.t. the three orderings. However, out of six possible inclusions none holds. Counterexamples are:

($\prec_{\mathcal{N}}$-s.e.$\not\Rightarrow\prec_{\mathcal{D}}$-s.e. and $\prec_\infty$-s.e.$\not\Rightarrow\prec_{\mathcal{D}}$-s.e.). A renewal p.p. with DFR interpoint distances; see Example 4.1.

($\prec_{\mathcal{N}}$-s.e.$\not\Rightarrow\prec_\infty$-s.e.). The process of failures in a system with block replacement policy based on a DFR renewal process; see Example 4.5.

($\prec_\infty$-s.e.$\not\Rightarrow\prec_{\mathcal{N}}$-s.e.). Any renewal process with non-DFR interpoint distances; see Example 4.1.

($\prec_{\mathcal{D}}$-s.e.$\not\Rightarrow\prec_{\mathcal{N}}$-s.e. and $\prec_{\mathcal{D}}$-s.e.$\not\Rightarrow\prec_\infty$-s.e.). A point process with a compensator of the following form. For $a > b > 0$ let $\ell_{a,b}(t)$ and $\ell_{b,a}(t)$ be two increasing, continuous functions $R_+ \to R_+$ piecewise linear on intervals $(kT, kT+T]$, $k \geq 0$. Let $\ell_{a,b}(t) = \ell_{b,a}(t) = ct$ for $t \in (0, T]$ and $c > 0$. Then on successive intervals of the form $(kT, kT+T]$, $k \geq 1$, let the slopes of $\ell_{a,b}$ be cyclically $a, b, a, b, \dots$ and the slopes of $\ell_{b,a}$ cyclically $b, a, b, a, \dots$. Set $\Lambda(t, \mu) = \ell_{a,b}(t)$ iff $\tau_1(\mu) < T$, otherwise set $\Lambda(t, \mu) = \ell_{b,a}(t)$.

EXAMPLE 4.1 (Renewal processes). The following lemma characterizes renewal processes which are positively self-exciting w.r.t. $\prec_{\mathscr{N}}$.

LEMMA 4.1. *Let N be a renewal process on R_+. Suppose that the lifetime distribution is continuous and has a failure rate r. Let the first renewal time also have a continuous distribution, with a failure rate r_0 possibly different from r. Then the following statements are equivalent*:

(i) *N is positively self-exciting w.r.t. $\prec_{\mathscr{N}}$*;
(ii) *the failure rate r is decreasing, and $r_0(t) \le r(t)$ for $t \ge 0$.*

PROOF. The intensity process λ of the process N is given by

$$\lambda(t,\mu) = r_0(t)\ \mathbf{1}_{(0,\,\tau_1(\mu)]}(t) + \sum_{i=1}^{\infty} r(t-\tau_n(\mu))\ \mathbf{1}_{(\tau_n(\mu),\,\tau_{n+1}(\mu)]}(t).$$

Suppose that (ii) holds. Let $\mu \prec_{\mathscr{N}} \nu$ be elements of $\mathscr{N}(R_+)$. Fix $n \ge 0$ and suppose that $\lambda(t,\mu) \le \lambda(t,\nu)$ for all $t \le \tau_n(\nu)$. Let $k_n \le n$ be such that $\tau_{k_n}(\mu) \le \tau_n(\nu) < \tau_{n+1}(\nu) \le \tau_{k_n+1}(\mu)$. Then for $t \in (\tau_n(\nu), \tau_{n+1}(\nu)]$ we have $\lambda(t,\mu) = r^{(1)}(t-\tau_{k_n}(\mu))$ and $\lambda(t,\nu) = r^{(2)}(t-\tau_n(\nu))$, where $r^{(1)} = r^{(2)} = r_0$ if $n = k_n = 0$, $r^{(1)} = r_0$ and $r^{(2)} = r$ if $k_n = 0$ and $n \ge 1$ and $r^{(1)} = r^{(2)} = r$ if $n \ge k_n \ge 0$. In all cases $\lambda(t,\mu) \le \lambda(t,\nu)$ for $t \le \tau_n(\nu)$, hence (i) holds.

Now suppose that (i) holds. To see that r is decreasing, consider $\mu \prec_{\mathscr{N}} \nu$ such that $\tau_1(\mu) = \tau_2(\nu) = t_1$, $\tau_2(\nu) = t_2$ and $\tau_2(\mu) = \tau_3(\nu) = t_3$ for some $0 < t_1 < t_2 < t_3$. Then $r(t_3 - t_1) = \lambda(t_3,\mu) \le \lambda(t_3,\nu) = r(t_3 - t_2)$, hence r is decreasing. To see that $r_0 \le r$, consider $\mu \prec_{\mathscr{N}} \nu$ such that $\tau_1(\nu) = t_1$ and $\tau_1(\mu) = \tau_2(\nu) = t_2$. Then $r_0(t_2) = \lambda(t_2,\mu) \le \lambda(t_2,\nu) = r(t_2 - t_1)$. Since r is decreasing, it is almost everywhere continuous. Taking $t_1 \to 0$, we have $r_0(t_2) \le r(t_2)$ for $t_2 > 0$.

We state the following result without proof.

LEMMA 4.2. *A renewal point process is positively self-exciting w.r.t. $\prec_{\mathscr{D}}$ iff it is a time-homogeneous Poisson point process.*

Since interpoint distances for renewal processes are independent, renewal processes are always positively self-exciting w.r.t. $\prec_{\infty}$.

EXAMPLE 4.2. Hawkes (1971) introduced a class of point processes which he called self-exciting, without referring, however, to any particular order. These processes are formally defined as point processes with a stochastic intensity of the form

$$\lambda(t,\mu) = c + \int_{-\infty}^{t} d(t-u)\,d\mu(u),$$

where $c > 0$, $d(u) \ge 0$ for $u \ge 0$ and

$$0 < m = \int_0^{\infty} d(u)\,du < 1.$$

Hawkes and Oakes (1974) represented such processes as Poisson cluster processes and gave some counting and interval properties of them. Since their intensity can be written as

$$\lambda(t, \mu) = c + \sum_{t_i < t} d(t - t_i),$$

it is immediate that such processes are positively self-exciting w.r.t. $\prec_{\mathscr{N}}$ and positively self-exciting w.r.t. $\prec_{\mathscr{D}}$. Interpretations of c, $d(u)$ are as follows. Kendall (1949) introduced an age-dependent birth-and-death process such that for any individual of age x alive at time t there are probabilities $\lambda(x)\,dt$ of birth and $\gamma(x)dt$ of death [plus $o(dt)$] for the next interval $(t, t+dt)$, independently for each individual. If we take $\lambda(x) = d(x)$, $\gamma \equiv 0$ and we allow immigration at rate c, then $N(t)$ is the counting measure corresponding to the times of birth or immigration.

A nonlinear Hawkes process is a point process N driven by an intensity of the form

$$\lambda(t, \mu) = \phi\left(\int_{-\infty}^{t} d(t-u)\,d\mu(u)\right),$$

where d: $R_+ \to R$ and ϕ: $R \to R_+$; see Hawkes (1971), Brémaud and Massoulié (1994) and Daley and Vere-Jones (1988), page 367. If d is nonnegative and ϕ nondecreasing, then N is positively self-exciting w.r.t. $\prec_{\mathscr{N}}$. If d is nonnegative and nondecreasing and ϕ is nondecreasing, then N is positively self-exciting w.r.t. $\prec_{\mathscr{D}}$.

EXAMPLE 4.3 (Pólya process). Approximate a Poisson process by a discrete-time process which can jump only at the time points $1/n, 2/n, \dots$. Let T be the waiting time to the first jump point in the Poisson process. Let $Y_i = 0$ or 1 according to whether the approximating process does or does not jump at $t = i/n$, $i = 1, 2, \dots$. If Y_i are independent and the probability of a jump is a/n for all i, then $P(T > t)$ can be approximated by $(1 - a/n)^{nt}$ for $t \geq 0$, and, taking limits as $n \to \infty$, it follows $P(T > t) = \exp(-at)$. If we replace independent Y_i's by a sequence generated by Pólya's urn model, the resulting continuous-time point process N is called a Pólya process; see Lundberg (1964) and Marshall and Olkin (1993). It is known from these references that N in this case is a doubly stochastic Poisson process with constant intensity A having a gamma distribution with a density of the form

$$P(A \in dx) = \frac{x^{k-1}}{\alpha^k \Gamma(k)} \exp\left(\frac{-x}{\alpha}\right).$$

The corresponding stochastic intensity then has the following form [see Brémaud (1981), page 173]:

$$\lambda(t, \mu) = \frac{E(A^{\mu((0,t])+1} \exp(-At))}{E(A^{\mu((0,t])} \exp(-At))}.$$

To simplify expressions, we assume $\alpha = 1$ and after integration we get

$$\lambda(t, \mu) = \frac{\mu((0, t]) + k}{t + 1}.$$

From this formula we see that the Pólya point process is positively self-exciting w.r.t. $\prec_{\mathscr{N}}$ and w.r.t. $\prec_{\mathscr{D}}$.

EXAMPLE 4.4. A special class of renewal processes are so-called Thorin processes. The interpoint d.f. for such processes is given by $F(t) = 1 - \int_0^\infty \exp(-tx)\, dV(x)$, where V is a proper or defective d.f. satisfying $V(0) = 0$; that is, a d.f. F is Thorin if it has a completely monotone derivative. Thorin point processes fall into the class of these doubly stochastic Poisson point processes which are renewal [Kingman (1964) and Daley (1965)], because the Laplace transform of a Thorin d.f. can be written in the form [Berg and Forst (1975), Theorem 9.8]

$$F^*(s) = \frac{1}{1 + bs + \int_0^\infty (1 - \exp(-sx))\, dB(x)},$$

where $b \geq 0$ and B is a positive measure on $(0, \infty)$. For example, Weibull and gamma distributions with shape parameter smaller than 1 are Thorin distributions. Thorin point processes are positively self-exciting w.r.t. $\prec_{\mathscr{N}}$ because Thorin distributions are DFR as mixtures of exponential distributions and we may apply Lemma 4.1 [see Barlow and Proschan (1981) for the DFR closure property].

EXAMPLE 4.5. Consider a system consisting of a replaceable element, which is subject to failures. Suppose that the lifetime distribution of an element has a failure rate r. Under a *block replacement* policy, elements are replaced with new ones at fixed (deterministic) times $T_1^b, T_2^b, \ldots$ and, of course, at times of actual failures $T_1, T_2, \ldots$.

Let $N^F(B) = \sum_{i=1}^\infty \mathbf{1}_B(T_i)$, where $B \in \mathscr{B}(R_+)$, be a point process counting failures, and let $N^R(B) = \sum_{i=1}^\infty \mathbf{1}_B(T_i) + \sum_{i=1}^\infty \mathbf{1}_B(T_i^b)$ be a process counting all replacements. The intensity λ^F of the process N^F is

$$\lambda^F(t, N^f) = \sum_{i=0}^\infty r(t - \tau_n(N^R))\, \mathbf{1}_{(\tau_n(N^R),\, \tau_{n+1}(N^R)]}(t).$$

Observe that N^F is not a renewal process. The process N^F is a positively self-exciting process w.r.t. the order $\prec_{\mathscr{N}}$ iff the failure rate r is decreasing on the interval $[0, b]$, where $b = \sup\{T_1^b, T_2^b - T_1^b, T_3^b - T_2^b, \ldots\}$. Indeed, if we fix an $i > 0$ and shift the time origin to the point T_i^b, then the shifted N^F evolves as a renewal process until the next point T_{i+1}^b. The argument from Example 4.1 applies.

EXAMPLE 4.6 (Stream of overflows from a finite queue). Consider a service system with a finite number of exponential servers, numbered $1, \ldots, n$, and a

Poisson arrival stream. Arriving customers go to the idle server with the lowest number; if all servers are busy, then the arriving customer leaves the system immediately (overflows). It is an old problem, tracing back to Palm, to study the overflow point process. It is known [see, e.g., Khintchine (1960), page 95] that the overflow p.p. in this system is a renewal process with interpoint distance distribution of the form

$$F(x) = 1 - \sum_{k=0}^{n} c_k \exp(-a_k x), \qquad x \geq 0,$$

for positive constants c_k, a_k. Therefore, the overflow process is Thorin and as such positively self-exciting w.r.t. $\prec_{\mathscr{N}}$. For systems with renewal arrival stream and a waiting room available, see Çinlar and Disney (1967).

EXAMPLE 4.7 (Wold's p.p.). A point process of which successive intervals form a first-order time-homogeneous Markov chain was put forward by Wold as the simplest alternative to a renewal process. The asymptotic behavior of such point processes has been studied by many authors; see Athreya, Tweedie and Vere-Jones (1980) for a more recent treatment. Assume that the chain has one-step transition kernel $P(x, B)$. From the definition of positively self-exciting point processes, we see that in this case N is self-exciting w.r.t. $\prec_\infty$ iff $P(x_1, \cdot) \leq_{\mathrm{st}} P(x_2, \cdot)$ for all $x_1 \geq x_2$, which is simply stochastic monotonicity of the underlying Markov chain.

EXAMPLE 4.8 (Semi-Markov p.p.). Consider a Markov renewal process $(\mathbf{T}, \mathbf{J})$ with a countable state space and the semi-Markov kernel $Q(i, j, t)$. That is,

$$\begin{aligned} P(T_{n+1} - T_n \leq t,\ J_{n+1} = j \mid J_0, \dots, J_n, T_0, \dots, T_n) \\ = P(T_{n+1} - T_n \leq t,\ J_{n+1} = j \mid J_n), \end{aligned}$$

$$P(T_{n+1} - T_n \leq t,\ J_{n+1} = j \mid J_n = i) = Q(i, j, t).$$

Let $G(i, t) = \sum_j Q(i, j, t)$. The p.p. corresponding to $T_0, T_1, \dots$ is called a semi-Markov point process. It is known [see, e.g., Kwieciński and Szekli (1991)] that the compensator of N in this case is given by

$$a_n(t_{n+1}; t_1, \dots, t_n) = \sum_{k=1}^{n} \alpha_k(t_{k+1} - t_k \mid t_1, \dots, t_k)$$

for

$$\alpha_k(t \mid t_1, \dots, t_k) = -\ln\left(1 - \sum_i G(i, t) p_n(i \mid t_1, \dots, t_n)\right),$$

where

$$p_n(i \mid t_1, \dots, t_n) = P(J_n = i \mid T_1 = t_1, \dots, T_n = t_n).$$

For N to be positively self-exciting w.r.t. $\prec_\infty$, it suffices that for all t the probability $P(T_{n+1} - T_n \le t | T_1 = t_1, \ldots, T_n = t_n)$ is $\prec_\infty$-decreasing as a function of the point process in the conditioning event. Note that

$$P(T_{n+1} - T_n \le t \mid T_1 = t_1, \ldots, T_n = t_n) = \sum_i \sum_j p_{ij} F_{ij}(t) p_n(i \mid t_1, \ldots, t_n),$$

where $p_{ij} = Q(i, j, \infty)$ are transition probabilities for the underlying Markov chain and the F_{ij}'s are renewal distribution functions for given transitions. If the underlying Markov chain is stochastically monotone and $F_{ij} \le_{\mathrm{st}} F_{i'j'}$ for $i \le i'$, $j \le j'$, then $\sum_j p_{ij} F_{ij}(t)$ is decreasing in i for all t. It is now enough to check that $p_n(\cdot \mid t_1, \ldots, t_n)$ is stochastically increasing as a function of the point process in the conditioning event with $\prec_\infty$. This can be checked inductively under the additional condition $F_{ij} \le_{\mathrm{lr}} F_{i'j'}$ for $i \le i'$, $j \le j'$, where $\le_{\mathrm{lr}}$ denotes the likelihood ratio ordering. We omit an elementary, but rather lengthy argument.

Interesting properties of positively self-exciting point processes follow from the fact that such processes can be realized as increasing transformations of Poisson processes. The following two theorems contain the main results of this paper.

THEOREM 4.1. (i) *If N is positively self-exciting w.r.t. $\prec_{\mathscr{N}}$, then $N =_{\mathrm{st}} \Upsilon(\Pi_{(2)})$ for an increasing $\prec_{\mathscr{N}}$-$\prec_{\mathscr{N}}$ measurable function $\Upsilon\colon \mathscr{N}(R_+)^2 \to \mathscr{N}(R_+)$, where $\Pi_{(2)}$ is the standard Poisson process in R_+^2.*

(ii) *If N is positively self-exciting w.r.t. $\prec_{\mathscr{D}}$ $(\prec_\infty)$, then $N =_{\mathrm{st}} \Gamma(\Pi_{(1)})$ for an increasing $\prec_{\mathscr{D}}$-$\prec_{\mathscr{D}}$ $(\prec_\infty$-$\prec_\infty)$ measurable function $\Gamma\colon \mathscr{N}(R_+) \to \mathscr{N}(R_+)$, where $\Pi_{(1)}$ is the standard Poisson process in R_+.*

PROOF. (i) Suppose that N is positively self-exciting w.r.t. the order $\prec_{\mathscr{N}}$. Let Υ be the mapping from Lemma 2.3. Then

$$\forall_{\pi_{(2)}, \pi'_{(2)} \in \mathscr{N}(R_+)^2} \quad \pi_{(2)} \prec_{\mathscr{N}} \pi'_{(2)} \Rightarrow \{\kappa_1(\pi_{(2)}), \kappa_2(\pi_{(2)}), \ldots\} \subset \{\kappa_1(\pi'_{(2)}), \kappa_2(\pi'_{(2)}), \ldots\}.$$

We omit an easy proof by induction. We also have

$$\forall_{\substack{(x_1, x_2, \ldots) \in \mathscr{J}_+ \\ (y_1, y_2, \ldots) \in \mathscr{J}_+}} \quad \{x_1, x_2, \ldots\} \subset \{y_1, y_2, \ldots\} \Rightarrow \boldsymbol{\tau}^{-1}(x_1, x_2, \ldots) \prec_{\mathscr{N}} \boldsymbol{\tau}^{-1}(y_1, y_2, \ldots).$$

It follows easily that $\Upsilon = \boldsymbol{\tau}^{-1}\boldsymbol{\kappa}$ is increasing, that is,

$$\forall_{\pi_{(2)}, \pi'_{(2)} \in \mathscr{N}(R_+)^2} \quad \pi_{(2)} \prec_{\mathscr{N}} \pi'_{(2)} \Rightarrow \Upsilon(\pi_{(2)}) \prec_{\mathscr{N}} \Upsilon(\pi'_{(2)}).$$

(ii) Suppose that N is positively self-exciting w.r.t. $\prec_{\mathscr{D}}$. Let Γ be the mapping from Lemma 2.2. Let $\pi_{(1)}, \pi'_{(1)} \in \mathscr{N}(R_+)$ and $\pi_{(1)} \prec_{\mathscr{D}} \pi'_{(1)}$. Consider $\mu = \Gamma(\pi_{(1)})$ and $\mu' = \Gamma(\pi'_{(1)})$. Suppose that $\tau_i(\mu) \geq \tau_i(\mu')$ for $i = 1, \ldots, n$. Then, for $t > 0$,

$$a_{n+1}(t;\, \tau_1(\mu), \ldots, \tau_n(\mu)) \leq a_{n+1}(t;\, \tau_1(\mu'), \ldots, \tau_n(\mu')).$$

Hence

$$\begin{aligned}\tau_{n+1}(\mu) &= a_{n+1}^{-1}(\tau_{n+1}(\pi_{(1)});\, \tau_1(\mu), \ldots, \tau_n(\mu))\\ &\geq a_{n+1}^{-1}(\tau_{n+1}(\pi_{(1)});\, \tau_1(\mu'), \ldots, \tau_n(\mu')) = \tau_{n+1}(\mu).\end{aligned}$$

It follows that Γ is increasing, that is,

$$\forall_{\pi_{(1)}, \pi'_{(1)} \in \mathscr{N}(R_+)} \quad \pi_{(1)} \prec_{\mathscr{D}} \pi'_{(1)} \Rightarrow \Gamma(\pi_{(1)}) \prec_{\mathscr{D}} \Gamma(\pi'_{(1)}).$$

For processes positively self-exciting w.r.t. $\prec_\infty$ the proof is similar. $\square$

THEOREM 4.2. *If N is a positively self-exciting p.p. w.r.t. $\prec$, then N is associated $(\prec)$ whenever $\prec$ denotes one of the three orderings $\prec_{\mathscr{N}}$, $\prec_{\mathscr{D}}$, $\prec_\infty$.*

PROOF. Let $\Pi_{(2)}$ be the standard Poisson process in R_+^2. Then $N =_{\mathrm{st}} \Upsilon(\Pi_{(2)})$. Since $\Pi_{(2)}$ is associated w.r.t. the thinning ordering on the plane (independent increments) and Υ is increasing, $N =_{\mathrm{st}} \Upsilon(\Pi_{(2)})$ is associated $(\prec_{\mathscr{N}})$ from Lemma 3.1.

The proof for other orderings is similar. $\square$

Burton and Waymire (1985) studied scaling limits for associated point random fields and in this context they found that stationary renewal processes with interval distributions having log convex densities are associated. They proved that fact in Burton and Waymire (1986) using so-called absolute product densities. They also mentioned another interesting context connected with this result. Namely, log convexity of the lifetime density $f(t)$, for integer lattice lifetimes, makes the nearest particle system with constant death rate and with birth rates $f(m)f(n)/f(m+n)$ attractive [Liggett (1983)]. If the first moment of f is finite, the stationary renewal process with log convex lifetime density is the unique time-reversible equilibrium state for the particle system concentrated on configurations with infinitely many occupied sites to the left and to the right of the origin [Spitzer (1977)]. A suitable approximation implies that from the Harris inequality the renewal process with f is associated.

If a lifetime density is log convex, then the corresponding distribution function is DFR. From Lemma 4.1 and Theorem 4.2 we have the following extension of the Burton–Waymire (1986) result.

COROLLARY 4.1. *If N is a delayed renewal process as in Lemma* 4.1, *with a DFR lifetime distribution*, *then N is associated $(\prec_{\mathscr{N}})$.*

It is worth mentioning here that the $r_0(t)$ making N stationary satisfies $r_0(t) \le r(t)$, $t \ge 0$, for r decreasing.

Glasserman (1992) investigated processes with associated increments, that is, processes $X = \{X_t,\ t \ge 0\}$ such that for all $n > 0$ and all $0 \le t_0 < t_1 < \cdots < t_n$ the differences $X_0, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ constitute an associated vector.

Glasserman conjectured (page 325) that if a point process N specified by an intensity $\lambda(t, \mu)$ satisfies

$$\forall_{\mu \prec_{\mathscr{D}} \nu} \quad \lambda(\cdot, \mu) \le \lambda(\cdot, \nu), \tag{7}$$

and if additionally $\lambda(t, \cdot)$ is bounded uniformly in t, then N is a process with associated increments. However, he was unable to prove the conjecture using the techniques of time-inhomogeneous Markov processes exploited in this paper.

Clearly, condition (7) implies that N is self-exciting w.r.t. $\prec_{\mathscr{N}}$. Thus it is associated w.r.t. $\prec_{\mathscr{N}}$. It follows that N is indeed a process with associated increments (note that the boundedness of λ is not necessary).

Moreover, if N is positively self-exciting w.r.t. $\prec_{\mathscr{N}}$, then for each $t > 0$ the shifted point process given the past $\{N_{t+u} \mid \mathscr{F}_t^N,\ u \ge 0\}$ can also be constructed as a monotone transformation of a Poisson process. Therefore, it is associated ($\prec_{\mathscr{N}}$) and has associated increments. Now assume that we have the following convergence in distribution as $t \to \infty$:

$$t^{\beta}(t^{-\gamma} N((0, t]) - \alpha) \to \sigma_N^2 \mathbf{N}(0, 1),$$
$$t^{\beta}(t^{-\gamma} \Lambda(t, N) - \alpha) \to \sigma_\Lambda^2 \mathbf{N}(0, 1)$$

for some constants $\alpha, \beta, \gamma > 0$ and $0 < \sigma_N,\ \sigma_\Lambda < \infty$, where $\mathbf{N}(0, 1)$ is the standard normal variable. Then from Theorem 6.1 in Glasserman (1992) we have the following corollary.

COROLLARY 4.2. *If N is a p.p. positively self-exciting w.r.t. $\prec_{\mathscr{N}}$ with bounded stochastic intensity, then $\sigma_\Lambda^2 \le \sigma_N^2$.*

This asymptotic variance reduction is important in the context of Monte Carlo simulation and it is well known for renewal processes with decreasing failure rate [Brown, Solomon and Stephens (1981)].

5. Related results.

5.1. *Virtual waiting time and the number of customers in the system in a single-server queue.* Consider a single-server G/G/1, FIFO queue driven by a marked point process $N = \{(T_1, S_1), (T_2, S_2), \ldots\}$, where T_n is the arrival time of the nth customer and S_n is its service time. We will regard trajectories $\mu = \{(t_1, s_1), (t_2, s_2), \ldots\}$ of process N as counting measures on the plane and order them by thinning.

Let $V(t, N)$ and $Q(t, N)$ be respectively the virtual waiting time and the number of customers in the system at time t. The trajectories $V(\cdot, \mu)$ and $Q(\cdot, \mu)$ are naturally ordered by the pointwise partial ordering $\prec_{\mathscr{D}}$. The proposition stated below gives a sufficient condition for time association of V and Q.

PROPOSITION 5.1. *Suppose that a marked point process N feeding a single-server queue is associated w.r.t. $\prec_{\mathscr{N}}$. Then processes $V(t, N)$ of the virtual waiting time and $Q(t, N)$ of the number of customers in the system are associated w.r.t. $\prec_{\mathscr{D}}$ (time associated).*

PROOF. Note that the mappings $\mu \mapsto V(\cdot, \mu)$ and $\mu \mapsto Q(\cdot, \mu)$ are increasing w.r.t. the appropriate orderings of trajectories of process N and processes V and Q. To see that, it is sufficient to consider realizations $\nu \prec_{\mathscr{N}} \nu'$ of N, such that $\nu = \{(t_1, s_1), (t_2, s_2), \dots\}$ and ν' is obtained from ν by removing one point, say (t_k, s_k).

Then for $t < t_k$ we have $V(t, \nu') = V(t, \nu)$. At t_k the trajectory $V(t, \nu)$ observes an upward jump of s_k, while $V(t, \nu')$ does not have that jump. If $n \geq k$ and $V(t_n, \nu') \leq V(t_n, \nu)$, then $V(t, \nu') \leq V(t, \nu)$ for $t \in [t_n, t_{n+1})$, because each of the two trajectories decreases linearly or reaches 0 and stays there, and $V(t_{n+1}, \nu') \leq V(t_{n+1}, \nu)$, because after t_k both trajectories observe the same jumps. Thus $V(\cdot, \nu') \prec_{\mathscr{D}} V(\cdot, \nu)$. Similar reasoning applies to $Q(t, \cdot)$. From Lemma 3.1 the proof is complete. □

To give a specific example of a queueing system satisfying the assumption of Proposition 5.1, consider a $GI/GI/1$ queue. Suppose that the interarrival times have a common failure rate $r(t)$ and service times a common density $f(s)$.

In such a queue the underlying marked point process N can be expressed as a function of a standard Poisson process in a similar manner as in Lemma 2.3 (Poisson embedding). Define $\lambda(t, s, \mu) = r(t - t_n) f(s)$ for $t \in [t_n, t_{n+1})$, where (t_n, s_n) is the nth point of μ and $t_0 = 0$. Let μ_0 be a counting measure in R_+^2 that has no points. Let $\pi_{(3)}$ be a trajectory of a standard Poisson process in R_+^3. Let (t_1, s_1, z_1) be the point of $\pi_{(3)}$ with the smallest t_1 lying below the surface $z = \lambda(t, s, \mu_0)$. Point (t_1, s_1) is the first point of a trajectory μ. The construction is then continued recursively: let $\mu_1 = \{(t_1, s_1)\}$ and let (t_2, s_2, z_2) be the point of $\pi_{(3)}$ with the smallest $t_2 > t_1$ lying below the surface $z = \lambda(t, s, \mu_1)$. Point (t_2, s_2) is the second point of the trajectory μ and so on. Denote the resulting trajectory as $\mu = \Phi(\pi_{(3)})$. We state the following lemma without proof.

LEMMA 5.1. *Let $\Pi_{(3)}$ be a standard Poisson process in R_+^3. Then $\Phi(\Pi_{(3)})$ and N have the same distribution.*

That Proposition 5.1 can be applied to certain GI/GI/1 queues follows by the next lemma.

LEMMA 5.2. *If $r(\cdot)$ is decreasing (interarrival times are DFR), then the mapping Φ is increasing and N is associated w.r.t. $\prec_{\mathscr{N}}$.*

5.2. *Association of stochastic intensity and compensator.* For a p.p. N we can view its $(\mathscr{F}_t^N)$ stochastic intensity as a transformation $\mu \to \lambda(\cdot, \mu)$ which is from $\mathscr{N}$ to $\mathscr{D}(R_+)$. Then $\prec_{\mathscr{N}}$-$\prec_{\mathscr{D}}$ increasingness of this transformation is equivalent to N being self-exciting w.r.t. $\prec_{\mathscr{N}}$. This in turn is equivalent to the fact that $\mu \to \Lambda(\cdot, \mu)$ is $\prec_{\mathscr{N}}$-$\prec_{\mathscr{M}}$ increasing if we view the compensator as a random measure. To summarize, we have the following result from Lemma 3.1.

COROLLARY 5.1. *If N is a p.p. positively self-exciting w.r.t. $\prec_{\mathscr{N}}$, then its internal stochastic intensity $\{\lambda(t),\ t \geq 0\}$ is associated $(\prec_{\mathscr{D}})$ (time associated) and its compensator random measure given by $\Lambda(B) = \int_B \lambda(s)\,ds$ is associated $(\prec_{\mathscr{M}})$.*

In a similar way for N a renewal p.p. we can see that the age process $A(t) = t - \sum_{i=0}^{N_t} X_i$ and the residual life process $Z(t) = \sum_{i=0}^{N_t+1} X_i - t$ are time associated provided that N is positively self-exciting w.r.t. $\prec_{\mathscr{N}}$ (i.e., $r_0 \leq r$ and r decreasing).

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REFERENCES

ATHREYA, K. B., TWEEDIE, R. L. and VERE-JONES, D. (1980). Asymptotic behaviour of point processes with Markov-dependent intervals. *Math. Nachr.* **99** 301–313.

BARLOW, R. E. and PROSCHAN, F. (1975). *Statistical Theory of Life Testing: Probability Models*. Holt, Rinehart and Winston, New York.

BERG, C. and FORST, G. (1975). *Potential Theory on Locally Compact Abelian Groups*. Springer, New York.

BRÉMAUD, P. (1981). *Point Processes and Queues; Martingale Dynamics*. Springer, New York.

BRÉMAUD, P. and MASSOULIÉ, L. (1994). Stability of non-linear Hawkes process. Unpublished manuscript.

BROWN, M., SOLOMON, H. and STEPHENS, M. A. (1981). Monte Carlo simulation of the renewal function. *J. Appl. Probab.* **18** 426–434.

BURTON, R. M. and FRANZOSA, M. M. (1990). Positive dependence properties of point processes. *Ann. Probab.* **18** 359–377.

BURTON, R. M. and WAYMIRE, E. (1985). Scaling limits for associated random measures. *Ann. Probab.* **13** 1267–1278.

BURTON, R. M. and WAYMIRE, E. (1986). A sufficient condition for association of a renewal process. *Ann. Probab.* **14** 1272–1276.

ÇINLAR, E. and DISNEY, R. L. (1967). Stream of overflows from a finite queue. *Oper. Res.* **15** 131–134.

DADUNA, H. and SZEKLI, R. (1995). Dependencies in Markovian networks. *Adv. in Appl. Probab.* **25** 226–254.

DALEY, D. J. (1965). On a class of renewal functions. *Math. Proc. Cambridge Philos. Soc.* **61** 519–526.

DALEY, D. J. and VERE-JONES, D. (1988). *An Introduction to the Theory of Point Processes*. Springer, New York.

ESARY, J. D., PROSCHAN, F. and WALKUP, D. W. (1967). Association of random variables, with applications. *Ann. Math. Statist.* **38** 1466–1474.

GLASSERMAN, P. (1992). Processes with associated increments. *J. Appl. Probab.* **29** 313–333.

HAWKES, A. G. (1971). Spectra of some self-exciting and mutually exciting point processes. *Biometrika* **58** 83–90.

HAWKES, A. G. and OAKES, D. (1974). A cluster process representation of a self-exciting process. *J. Appl. Probab.* **11** 493–503.

JACOD, J. (1975). Multivariate point processes: predictable projection, Radon–Nikodym derivatives, representation of martingales. *Z. Wahrsch. Verw. Gebiete* **31** 235–253.

KALLENBERG, O. (1983). *Random Measures*. Academic Press, London.

KENDALL, D. G. (1949). Stochastic processes and population growth. *J. Roy. Statist. Soc. Ser. B* **11** 230–264.

KHINTCHINE, A. Y. (1960). *Mathematical Methods in the Theory of Queueing*. Griffin, London.

KINGMAN, J. F. C. (1964). On doubly stochastic Poisson processes. *Math. Proc. Cambridge Philos. Soc.* **60** 923–930.

KWIECIŃSKI, A. and SZEKLI, R. (1991). Compensator conditions for stochastic ordering of point processes. *J. Appl. Probab.* **28** 751–761.

LIGGETT, T. M. (1983). Attractive nearest particle systems. *Ann. Probab.* **11** 16–33.

LINDQVIST, B. H. (1988). Association of probability measures on partially ordered spaces. *J. Multivariate Anal.* **26** 111–132.

LINDVALL, T. (1988). Ergodicity and inequalities in a class of point processes. *Stochastic Process. Appl.* **30** 121–131.

LUNDBERG, O. (1964). *On Random Processes and Their Application to Sickness and Accident Statistics*, 2nd ed. Almqvist and Wiksells, Stockholm.

MARSHALL, A. W. and OLKIN, I. (1993). Bivariate life distributions from Pólya's urn model for contagion. *J. Appl. Probab.* **30** 497–508.

ROLSKI, T. and SZEKLI, R. (1991). Stochastic ordering and thinning of point processes. *Stochastic Process. Appl.* **37** 299–312.

SPITZER, F. (1977). Stochastic time evolution of one-dimensional infinite-particle systems. *Bull. Amer. Math. Soc.* **83** 880–890.

STOYAN, D. (1983). *Comparison Methods for Queues and Other Stochastic Models*. Wiley, Chichester.

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