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LARGE DEVIATIONS FOR THE OCCUPATION TIMES OF INDEPENDENT PARTICLE SYSTEMS

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We prove a large deviation principle for the density field of independent particle systems in an infinite volume. We then deduce from the one-dimensional case of this result the large deviations for the occupation times of various sets (from microscopic to macroscopic scales) and we recover the theorem established by Cox and Griffeath. An expression of the rate function is given using the Brownian motion local time as in Deuschel and Wang.

1. Introduction. The study of large deviations for the occupation times of infinite particle systems was initiated by Cox and Griffeath [2] for independent simple random walks on the lattice $\mathbb{Z}^d$ with an initial equilibrium distribution. From the general results on the large deviations for Markov processes, one could have expected the exponential decay speed a_t of deviation probabilities for the occupation times to be equal to t, yet Cox and Griffeath showed it depends on the dimension d. More precisely, the more the particles are recurrent, the slower the deviation probabilities go to 0:

$$a_t = \begin{cases} \sqrt{t}\,, & \text{if } d = 1,\\ t/\log t, & \text{if } d = 2,\\ t, & \text{if } d \geq 3. \end{cases}$$

More recently, Deuschel and Wang [3] have obtained the same results for a Poisson system of independent Brownian particles.

The proof of Cox and Griffeath relies on accurate estimates on random walks so that the explicit rate function can be computed. Cox and Griffeath conjecture that the dependence of the speeds of large deviations on the dimension of the lattice should also occur for some interacting particle systems.

Landim [9] solves the problem for the symmetric simple exclusion process (SSEP). He obtains the same speeds as for the independent system. As expected, if $d \geq 3$, the large deviations for occupation times of the SSEP follow from general results for Markov processes. The rate function is then given by a variational formula, which turns out to be degenerate for $d = 1$ and $d = 2$. But in the one-dimensional case, Landim proves that the occupation time of a site is related to the density field, that is, the empirical

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distribution of particles. Indeed, a large deviation for the occupation time of a microscopic site occurs if and only if there is an abnormal density in a small macroscopic box centered around this site. Notice that such a result is not valid in higher dimensions, because particles are not recurrent enough. Landim then applies a contraction principle to the large deviations for the density field of the SSEP, which has been established by Kipnis, Olla and Varadhan [8], and he obtains the large deviations for occupation times in dimension $d = 1$. Unfortunately, the rate function is still given by a variational formula.

So it is natural to ask if, in the one-dimensional case, the large deviations for occupation times of the independent random walk system have the same origin as for the SSEP. Landim's method would then provide a variational rate function, but would it be possible to recover Cox and Griffeath's explicit formula? This paper answers these questions and our program is as follows.

In the first step, we prove a large deviation principle for the density field of the independent system. Although the method is by now standard (see [4], [8] and [9]), some technical difficulties appear because of the infinite volume and because the number of particles is an unbounded function of the sites. In the second step, we look at the relation between the occupation times of a site and the density field. From the first and second steps, we deduce large deviations for the occupation times of various sets (from microscopic to macroscopic scales), with a variational expression of the rate function. So, in the third step, we have to find an explicit formula. Such a problem is generally difficult, even if there are some examples in the literature (see, e.g., [3], [6] and [7]). In our case, we can reduce the variational problem to a simple linear partial differential equation (PDE) and we have a stochastic interpretation of its solution in terms of the Brownian motion local time (Deuschel and Wang [3] have obtained the same expression for the rate function). Moreover, with our approach, the result can be extended to some nonequilibrium initial distributions.

Notice that the method is not based on the independence of particles, in the sense that this property is not used for computations. Yet, the noninteraction of particles leads to a very simple PDE. In most interacting cases, it will turn out to be a nonlinear equation. For example, if we apply the method to the SSEP, we get a system of nonlinear PDEs with coupled initial and final conditions, and we do not know how to solve such a system. Otherwise, the nonexplicit use of independence should lead to an extension of the method to the general zero-range process, but in the present state of things, the main ingredient, that is, the large deviation principle for the density field, is not obtained in the classical sense (see [1]), mainly because of the nonstandard form of the rate function (which is not convex).

The paper is divided as follows. In Section 2, the notation and main results are stated. Section 3 deals with the last two steps of the method and the first step is established in Section 4.

2. Notation and main results. Throughout this paper, we are interested in an independent particle system on $\mathbb{Z}$: initially, the particles are distributed according to a probability measure on the state space $X = \mathbb{N}^{\mathbb{Z}}$. Let $\xi_0 \in X$ be an initial configuration. Then each of the $\xi_0(k)$ particles located on site k evolves independently of the others, performing a continuous-time random walk on $\mathbb{Z}$ with a transition $p(i,j)$ satisfying the following conditions:

(i) p is shift invariant: $p(i,j) = p(j-i)$,
(ii) p has a finite range: $p(k) = 0$ if $|k| \geq k_0$,
(iii) p has zero mean: $\sum_k kp(k) = 0$

and we set $\sigma^2 = \sum_k k^2 p(k)$.

If we denote by $\xi_t(k)$ the number of particles located on site $k \in \mathbb{Z}$ at time $t \in [0,T]$, the occupation time of the finite subset A of $\mathbb{Z}$, defined as the particle density on this set up to time t, is given by

$$T_t(A) = \frac{1}{t|A|} \int_0^t \sum_{k \in A} \xi_s(k)\, ds,$$

where $|A|$ denotes the cardinality of A.

It is well known (see, e.g., [5]) that the Poisson product measures ν^λ whose marginals are Poisson laws with parameter λ. That is,

$$\nu^\lambda(\eta(k) = n) = \frac{\lambda^n}{n!} e^{-\lambda},$$

are shift invariant and invariant for the process.

To establish a large deviation principle for the occupation time, when the process is initially distributed with these equilibrium measures, it is enough to compute the limit

$$\psi(A,\theta) = \lim_{N\to\infty} \frac{1}{N} \log E^\lambda\left[\exp\{\theta N T_{N^2}(A)\}\right],$$

where E^λ denotes the expectation with respect to the law of the process. Indeed, a result due to Sievers, Plachky-Steinbach and Ellis (stated as Lemma 1 in [2]) asserts that if $\psi(A,\theta)$ satisfies some smoothness conditions, then the large deviation principle follows with the Legendre transform of $\psi(A,\cdot)$ as rate function:

$$\Phi(A,\alpha) = \sup_{\theta \in \mathbb{R}} \{\alpha\theta - \psi(A,\theta)\}.$$

Moreover, with this approach, the result can be extended to some nonequilibrium initial distributions. For any positive integer N and $\gamma \in \Lambda(\lambda)$, where $\Lambda(\lambda)$ is the set of continuous and bounded functions equal to λ outside a compact subset of $\mathbb{R}$, we consider as initial law the Poisson product measure ν_N^γ with smooth parameter γ, whose marginals are given by

$$\nu_N^\gamma(\eta(k) = n) = \frac{(\gamma(k/N))^n}{n!} \exp\left(-\gamma\left(\frac{k}{N}\right)\right).$$

Notice that

$$E^{\nu_N^\gamma}\left[\exp\{\theta N T_{N^2}(A)\}\right] = E^{\nu_N^\gamma}\left[\exp\left\{\frac{\theta N}{|A|} \int_0^1 \sum_{k \in A} \xi_{sN^2}(k)\, ds\right\}\right],$$

so it is natural to consider the accelerated process $(\eta_t)_{t\in[0,T]} = (\xi_{tN^2})_{t\in[0,T]}$ on $\mathbb{N}^{\mathbb{Z}}$. Its infinitesimal generator is defined by

$$(2.1) \qquad \forall\, \eta \in X, \qquad \mathscr{L}_N f(\eta) = N^2 \sum_{i,j} p(j-i)\eta(i)\big(f(\eta^{i,j}) - f(\eta)\big),$$

where f is a cylinder function on X; that is, f depends only on a finite number of coordinates, and where $\eta^{i,j}$ is the configuration obtained after a jump of a particle from site i to site j:

$$\eta^{i,j}(k) = \begin{cases} \eta(i) - 1, & \text{if } k = i \text{ and } \eta(i) > 0,\\ \eta(j) + 1, & \text{if } k = j \text{ and } \eta(i) > 0,\\ \eta(k), & \text{otherwise.}\end{cases}$$

The density field is the empirical measure corresponding to the distribution of the particles:

$$\mu^N(t) = \frac{1}{N}\sum_{i\in\mathbb{Z}} \eta_t(i)\,\delta_{i/N},$$

where $\delta_{i/N}$ stands for the Dirac mass on i/N. We denote by P_N^γ the law of the accelerated independent process when the initial distribution is ν_N^γ.

The independent particle system satisfies a law of large numbers when it is initially distributed with the Poisson product measures. Indeed $\mu^N(0)$ converges to the macroscopic density field $\gamma(x)\,dx$, and this law of large numbers is conserved: $\mu^N(t)$ converges to the unique solution of the heat equation (the so-called hydrodynamical scaling limit of the particle system) with γ as initial condition. To obtain a large deviation principle from this limit for the density field μ^N, we apply the method introduced by Kipnis, Olla and Varadhan [8] and Donsker and Varadhan [4]: we observe the deviations of the process when some perturbations occur in the jump rates.

We now introduce the notation to define the rate function of the large deviation principle. For topological convenience, we work with the Schwartz space $\mathscr{S}'$ of slowly increasing distributions, endowed with the strong dual topology. The path of the density field $(\mu^N(t))_{t\in[0,T]}$ is then considered as an element of $D([0,T],\mathscr{S}')$, the space of cadlag functions from $[0,T]$ into $\mathscr{S}'$, with the Skorohod topology. If $F \in \mathscr{S}'$ is a positive measure and if $f \in \mathscr{S}$, we will write $\langle F, f\rangle = \int_{\mathbb{R}} f(x)F(dx)$. For any $\rho \in D([0,T],\mathscr{S}')$ and $\gamma \in \Lambda(\lambda)$, we define the following linear functional on the space $\mathscr{D}(\mathbb{R}\times[0,T])$ of infinitely differentiable functions on $\mathbb{R}\times[0,T]$, with a compact support [the set of infinitely differentiable functions on $\mathbb{R}$ with a compact support will be denoted by $\mathscr{D}(\mathbb{R})$]

$$l(\rho, G) = \langle \rho(T), G(\cdot,T)\rangle - \langle \rho(0), G(\cdot,0)\rangle - \int_0^T \left\langle \rho(s), \left(\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) G(\cdot,s)\right\rangle ds.$$

Then we introduce the rate functions

$$I_0(\rho) = \sup_{G \in \mathscr{D}(\mathbb{R}\times[0,T])} \left\{ l(\rho, G) - \frac{\sigma^2}{2}\int_0^T \left\langle \rho(s), \left(\frac{\partial G}{\partial x}\right)^2(\cdot, s)\right\rangle ds \right\},$$

$$h(\rho(0), \gamma) = \sup_{g \in \mathscr{D}(\mathbb{R})} \left\{ \langle \rho(0), g\rangle - \int_{\mathbb{R}} (\exp(g(x)) - 1)\gamma(x)\, dx \right\},$$

$$I_\gamma(\rho) = I_0(\rho) + h(\rho(0), \gamma).$$

Given $M > 0$ and $n \in \mathbb{Z}$, let $\mathscr{A}_n^M$ be the closed and convex subset of $D([0,T], \mathscr{S}')$:

$$\mathscr{A}_n^M = \left\{ \rho \in D([0,T], \mathscr{S}'),\ \rho \text{ positive measure: } \int_0^T \rho(s)([n, n+1])\, ds \le M \right\},$$

$\mathscr{A}^M = \bigcap_{n\in\mathbb{Z}} \mathscr{A}_n^M$ and $\mathscr{A} = \bigcup_{M>0} \mathscr{A}^M$. We set

$$J_\gamma(\rho) = \begin{cases} I_\gamma(\rho), & \text{if } \rho \in \mathscr{A}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Now we can state the following result.

THEOREM 1. *For any closed subset C of $D([0,T], \mathscr{S}')$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N^\gamma[\mu^N \in C] \le -\inf_{\rho \in C} J_\gamma(\rho). \tag{2.2}$$

For any open subset O of $D([0,T], \mathscr{S}')$,

$$\liminf_{N\to\infty} \frac{1}{N} \log P_N^\gamma[\mu^N \in O] \ge -\inf_{\rho \in O} J_\gamma(\rho). \tag{2.3}$$

Notice that this result extends to the independent process on the lattice $\mathbb{Z}^d$, with N^d instead of N as the exponential decay speed.

The occupation time of a site is not a continuous function of the density field. Yet, we will see with a superexponential estimate that the occupation time can be replaced, without any change to the large deviation probabilities, by the average number of particles in a small macroscopic box. Indeed, if we consider

$$\eta_t^{\varepsilon N}(0) = \frac{1}{2\varepsilon N + 1} \sum_{|i| \le \varepsilon N} \eta_t(i),$$

then we can prove the following result.

LEMMA 2.1. *For every $\delta > 0$, $t \in [0,T]$,*

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log P_N^\gamma\left[\left| \int_0^t V_{N,\varepsilon}(\eta_s)\, ds \right| \ge \delta \right] = -\infty, \tag{2.4}$$

where

$$V_{N,\varepsilon}(\eta) = \eta(0) - \eta^{\varepsilon N}(0).$$

Such a result was first established for the symmetric simple exclusion process in infinite volume in Lemma 3.1 of [9].

Using Theorem 1, a large deviation principle is then obtained for the random variable

$$\int_0^1 \eta_s^{\varepsilon N}(0)\, ds$$

and it follows from Lemma 2.1 that, for any finite subset A of $\mathbb{Z}$, the function $\psi(A, \theta)$ is given by

$$\psi(A, \theta) = \lim_{\varepsilon \to 0} \psi^{\varepsilon}(\theta),$$

where

$$\psi^{\varepsilon}(\theta) = \sup_{\rho \in D([0,1], \mathscr{S}')} \left\{ \theta \int_0^1 \langle \rho_s, \alpha_\varepsilon \rangle\, ds - J_\lambda(\rho) \right\}$$

and where $(\alpha_\varepsilon)_{\varepsilon > 0}$ is a regularizing family of functions on $\mathbb{R}$; that is, α_ε is an infinitely differentiable nonnegative function with a compact support in $[-\varepsilon, \varepsilon]$ such that $\int_{\mathbb{R}} \alpha_\varepsilon(x)\, dx = 1$.

The last part of the proof consists of the explicit computation of $\psi^{\varepsilon}(\theta)$. The main idea is to differentiate the function of ρ in the supremum and then to search for the critical paths. We show that some particular functions of these paths satisfy a partial differential equation and the Feynman–Kac formula provides a stochastic interpretation of its solution in terms of the Brownian local time L_t. Finally, letting ε go to 0, we get $\psi(A, \theta)$.

In fact, we will apply this method to study the large deviations for the occupation times of various sets (from microscopic to macroscopic scales). More precisely, let us denote by $(A_t)_{t \geq 0}$ a family of finite subsets of $\mathbb{Z}$. We define the occupation time of A_t up to time t as

$$T_t(A_t) = \frac{1}{t|A_t|} \int_0^t \sum_{k \in A_t} \xi_s(k)\, ds.$$

First, we will study the occupation times of sets in the microscopic scale, that is, an increasing family of sets $(A_t)_{t \geq 0}$ such that

$$\lim_{t \to \infty} \frac{\operatorname{diam} A_t}{\sqrt{t}} = 0, \tag{2.5}$$

where, for any finite subset B of $\mathbb{Z}$, $\operatorname{diam} B = \max\{|x - y|,\ x, y \in B\}$. Let A be a bounded Borel subset of $\mathbb{R}$ with a positive Lebesgue measure such that its frontier has a zero measure. We define the discrete approximation of A as

$$\tilde{A}_t = \left\{ i \in \mathbb{Z},\ \frac{i}{\sqrt{t}} \in A \right\}. \tag{2.6}$$

A family $(A_t)_{t \geq 0}$ of finite subsets of $\mathbb{Z}^d$ is said to be in the macroscopic scale if there exists a bounded Borel subset A of $\mathbb{R}$ such that

$$\lim_{t \to \infty} \frac{|A_t\, \Delta\, \tilde{A}_t|}{\sqrt{t}} = 0 \tag{2.7}$$

and we write $A = \lim_{t\to\infty} A_t/\sqrt{t}$. Notice that, if $(A_t)_{t\geq 0}$ is in the microscopic scale, then $\lim_{t\to\infty} A_t/\sqrt{t} = \{0\}$. Now we can state our main result.

THEOREM 2. *Let* $(A_t)_{t\geq 0}$ *be in the microscopic or macroscopic scale and let* $A = \lim_{t\to\infty} A_t/\sqrt{t}$. *Then, for any closed subset* F,

$$\limsup_{t\to\infty} \frac{1}{\sqrt{t}} \log P^{\nu^{\gamma}_{t^{1/2}}}[T_t(A_t) \in F] \leq -\inf_{a\in F} \Phi_A(a)$$

and, for any open subset Ω,

$$\liminf_{t\to\infty} \frac{1}{\sqrt{t}} \log P^{\nu^{\gamma}_{t^{1/2}}}[T_t(A_t) \in \Omega] \geq -\inf_{a\in \Omega} \Phi_A(a),$$

where

$$\Phi_A(a) = \sup_{\theta\in\mathbb{R}} (a\theta - \psi_A(\theta))$$

and

$$\psi_A(\theta) = \begin{cases} \int_{\mathbb{R}} \left(\mathbb{E}^x\left[\exp\{2\theta\sigma^{-2}L_{\sigma^2}(0)\}\right] - 1\right)\gamma(x)\,dx, & \text{if } A = \{0\},\\ \int_{\mathbb{R}} \left(\mathbb{E}^x\left[\exp\left\{\dfrac{2\theta\sigma^{-2}}{m(A)}\int_A L_{\sigma^2}(y)\,dy\right\}\right] - 1\right)\gamma(x)\,dx, & \text{if } m(A) > 0, \end{cases}$$

where m *denotes the Lebesgue measure on* $\mathbb{R}$.

3. Large deviations for the occupation time.

PROOF OF LEMMA 2.1. We begin with the equilibrium case $\gamma = \lambda$, where λ is a fixed positive constant. We denote by $\eta^{\varepsilon N}(0)$ the average number of particles in the small macroscopic box $[-\varepsilon N, \varepsilon N]$:

$$\frac{1}{2\varepsilon N + 1} \sum_{|j|\leq \varepsilon N} \eta(j).$$

To deal with the unboundedness of $\eta^{\varepsilon N}(0)$, let us introduce $\omega(\eta_s^{\varepsilon N}(0))$, where $\omega(k) = k\log(k/\beta)$, for $k \in \mathbb{N}$ and $\beta > 0$. If $\beta > e\lambda$, then, for every $\theta \leq 1$,

$$r(\theta) = \log E_N^{\lambda}[\exp\{\theta\omega(\eta_0(0))\}] < \infty. \tag{3.1}$$

We first observe that for $\alpha \leq 2/T$,

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty} \frac{1}{N}\log E_N^{\lambda}\left[\exp\left\{\alpha\varepsilon N\int_0^T \omega(\eta_s^{\varepsilon N}(0))\,ds\right\}\right] \leq 0. \tag{3.2}$$

Indeed,

$$\begin{aligned} E_N^{\lambda}\left[\exp\left\{\alpha\varepsilon N\int_0^T \omega(\eta_s^{\varepsilon N}(0))\,ds\right\}\right] &\leq E_N^{\lambda}\left[\exp\{T\alpha\varepsilon N\omega(\eta_0^{\varepsilon N}(0))\}\right]\\ &\leq E_N^{\lambda}\left[\exp\left\{\frac{T\alpha\varepsilon N}{2\varepsilon N + 1}\sum_{|j|\leq\varepsilon N}\omega(\eta_0(j))\right\}\right], \end{aligned} \tag{3.3}$$

where we have successively used Jensen's inequality, the invariance of ν^λ and the convexity of ω. From (3.1), (3.3) and the shift invariance of ν^λ, we get

$$\frac{1}{N}\log E_N^\lambda\left[\exp\left\{\alpha\varepsilon N\int_0^T\omega\left(\eta_s^{\varepsilon N}(0)\right)ds\right\}\right]\le\frac{2\varepsilon N+1}{N}r\left(\frac{T\alpha\varepsilon N}{2\varepsilon N+1}\right).$$

Thus, letting N go to ∞ and ε go to 0, we obtain (3.2).

Notice that it is enough to prove (2.4) without the absolute value. For any $a>0$ and $t\in[0,T]$, the probability that we have to estimate is less than or equal to

$$\begin{aligned}&\exp(-a\delta N)E_N^\lambda\left[\exp\left\{aN\int_0^t V_{N,\varepsilon}(\eta_s)\,ds\right\}\right]\\&\quad\le\exp(-a\delta N)\left(E_N^\lambda\left[\exp N\left\{2a\int_0^t V_{N,\varepsilon}(\eta_s)\,ds-\alpha\varepsilon\int_0^t\omega\left(\eta_s^{\varepsilon N}(0)\right)ds\right\}\right]\right)^{1/2}\\&\qquad\times\left(E_N^\lambda\left[\exp\left\{\alpha\varepsilon N\int_0^t\omega\left(\eta_s^{\varepsilon N}(0)\right)ds\right\}\right]\right)^{1/2}.\end{aligned}$$

So, in view of (3.2), we need to show that, for any $a>0$,

$$\begin{aligned}\limsup_{\varepsilon\to0}\limsup_{N\to\infty}\frac{1}{N}\log E_N^\lambda\Bigg[\exp N\Bigg\{&a\int_0^t V_{N,\varepsilon}(\eta_s)\,ds\\&-\alpha\varepsilon\int_0^t\omega\left(\eta_s^{\varepsilon N}(0)\right)ds\Bigg\}\Bigg]\le0.\end{aligned}\tag{3.4}$$

Now, following the proof of Theorem 2.1 in [8], we use the Feynman–Kac formula, and the expectation above is equal to $\int S(t)1\,d\nu$, where $S(t)$ is the semigroup corresponding to the infinitesimal generator

$$\mathscr{L}_N^\varepsilon f(\eta)=\mathscr{L}_N f(\eta)+N\left(aV_{N,\varepsilon}(\eta)-\alpha\varepsilon\omega\left(\eta_s^{\varepsilon N}(0)\right)\right),$$

where $\mathscr{L}_N$ is defined in (2.1). We then apply a spectral decomposition theorem to the symmetric part of L_N^ε (see [1] for details), and (3.4) will be proved if, for any $a>0$,

$$\begin{aligned}\limsup_{\varepsilon\to0}\limsup_{N\to\infty}\sup_{\substack{\int f(\eta)\nu^\lambda(d\eta)=1\\ f\ge0}}\Bigg\{&a\int f(\eta)V_{N,\varepsilon}(\eta)\nu^\lambda(d\eta)-ND(f)\\&-\alpha\varepsilon\int f(\eta)\,\omega\left(\eta^{\varepsilon N}(0)\right)\nu^\lambda(d\eta)\Bigg\}\le0,\end{aligned}\tag{3.5}$$

where $D(f)$ is the Dirichlet form given by

$$D(f)=\tfrac12\sum_{i,j}\tilde p(j-i)\int\left(\sqrt{f(\eta^{i,j})}-\sqrt{f(\eta)}\right)^2\eta(i)\nu^\lambda(d\eta)$$

and $\tilde p(k)=(p(k)+p(-k))/2$. In order to keep the notation simple, we now suppose that $\tilde p(k)=1/2$ if $|k|=1$ and $\tilde p(k)=0$ otherwise. Since

$$\int f(\eta^{0,j})\eta(0)\nu^\lambda(d\eta)=\int f(\eta)\eta(j)\nu^\lambda(d\eta),\tag{3.6}$$

the first term in the supremum (3.5) is equal to

$$\frac{1}{2\varepsilon N+1}\sum_{|j|\le \varepsilon N}\int \big(f(\eta)-f(\eta^{0,j})\big)\eta(0)\,\nu^{\lambda}(d\eta).$$

We begin to deal with positive indices j in the previous sum

$$\begin{aligned}\frac{1}{2\varepsilon N+1}&\sum_{0\le j\le \varepsilon N}\int \big(f(\eta)-f(\eta^{0,j})\big)\eta(0)\,\nu^{\lambda}(d\eta)\\&=\frac{1}{2\varepsilon N+1}\sum_{0\le j\le \varepsilon N}\ \sum_{0\le k\le j-1}\int \big(f(\eta^{0,k})-f(\eta^{0,k+1})\big)\eta(0)\,\nu^{\lambda}(d\eta).\end{aligned}$$

Noting that

$$f(\eta^{0,k})-f(\eta^{0,k+1})=\left(\sqrt{f(\eta^{0,k})}-\sqrt{f(\eta^{0,k+1})}\right)\left(\sqrt{f(\eta^{0,k})}+\sqrt{f(\eta^{0,k+1})}\right)$$

and using the Cauchy–Schwarz inequality, the last expression is bounded above by

$$\begin{aligned}\sum_{0\le j<\varepsilon N}&\left[\int\left(\sqrt{f(\eta^{0,j})}-\sqrt{f(\eta^{0,j+1})}\right)^2\eta(0)\,\nu^{\lambda}(d\eta)\right]^{1/2}\\&\times\left[\int\left(\sqrt{f(\eta^{0,j})}+\sqrt{f(\eta^{0,j+1})}\right)^2\eta(0)\,\nu^{\lambda}(d\eta)\right]^{1/2}.\end{aligned}$$

Similar inequalities hold for negative indices. Then we apply the classical Cauchy–Schwarz inequality for sums and we obtain the upper bound

$$\begin{aligned}&\left[\frac12\sum_{\substack{|i|\vee|j|\le\varepsilon N\\|i-j|=1}}\int\left(\sqrt{f(\eta^{0,i})}-\sqrt{f(\eta^{0,j})}\right)^2\eta(0)\,\nu^{\lambda}(d\eta)\right]^{1/2}\\&\qquad\times\left[\frac12\sum_{\substack{|i|\vee|j|\le\varepsilon N\\|i-j|=1}}\int\left(\sqrt{f(\eta^{0,i})}+\sqrt{f(\eta^{0,j})}\right)^2\eta(0)\,\nu^{\lambda}(d\eta)\right]^{1/2}.\end{aligned}\tag{3.7}$$

From the change of variable formula (3.6), we see that the first term in the product (3.7) is equal to

$$\left[\sum_{\substack{|i|\vee|j|\le\varepsilon N\\|i-j|=1}}\frac12\int\left(\sqrt{f(\eta^{i,j})}-\sqrt{f(\eta)}\right)^2\eta(i)\,\nu^{\lambda}(d\eta)\right]^{1/2}\le\sqrt{2D(f)}\tag{3.8}$$

and the second term in the product (3.7) is less than

$$\begin{aligned}&\left[2\sum_{|j|\le\varepsilon N}\int f(\eta^{0,j})\eta(0)\,\nu^{\lambda}(d\eta)\right]^{1/2}\\&\qquad=\sqrt{2}\,\sqrt{2\varepsilon N+1}\left[\int\eta^{\varepsilon N}(0)f(\eta)\,\nu^{\lambda}(d\eta)\right]^{1/2}.\end{aligned}\tag{3.9}$$

Moreover, since ω is convex,

$$\int \omega(\eta^{\varepsilon N}(0)) f(\eta)\, \nu^{\lambda}(d\eta) \geq \omega\left(\int \eta^{\varepsilon N}(0) f(\eta)\, \nu^{\lambda}(d\eta)\right).$$

Therefore, in view of (3.5), (3.7), (3.8), (3.9) and the inequality above, the estimate (3.4) will be established if, for every $a > 0$,

$$\limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \sup_{x\geq 0,\, y\geq 0} \left\{ a\sqrt{\frac{2\varepsilon N + 1}{N}}\, \sqrt{xy} - x - \varepsilon\alpha\omega(y) \right\} \leq 0.$$

A simple computation shows that this supremum is equal to

$$\varepsilon\alpha\beta \exp\left(\frac{a^2}{4\alpha}\,\frac{2\varepsilon N + 1}{\varepsilon N} - 1\right),$$

which goes to 0 as $N \to \infty$ and $\varepsilon \to 0$.

To obtain the same result in the general case, notice that, if $\gamma \in \Lambda(\lambda)$, then we have

$$\frac{dP_N^{\gamma}}{dP_N^{\lambda}}(\eta_\cdot) = \frac{d\nu_N^{\gamma}}{d\nu_N^{\lambda}}(\eta_0) = \exp \sum_{i\in\mathbb{Z}} \left(\eta_0(i) \log\frac{\gamma(i/N)}{\lambda} - \gamma\left(\frac{i}{N}\right) + \lambda\right).$$

Therefore, for any $k > 0$,

$$\lim_{N\to\infty} \frac{1}{N} \log E_N^{\lambda}\left[\left(\frac{dP_N^{\gamma}}{dP_N^{\lambda}}\right)^k (\eta_\cdot)\right] = \lambda^{1-k}\int_{\mathbb{R}} (\gamma^k(x) - \lambda^k)\, dx - k\int_{\mathbb{R}} (\gamma(x) - \lambda)\, dx \tag{3.10}$$

and this term is finite since $\gamma \in \Lambda(\lambda)$. □

PROOF OF THEOREM 2. (i) We begin with the case $A_t = \{0\}$. We want to study the limit $\psi(\theta)$ defined by

$$\lim_{N\to\infty} \frac{1}{N} \log E_N^{\gamma}\left[\exp\left\{\theta N \int_0^1 \eta_s(0)\, ds\right\}\right]. \tag{3.11}$$

It is enough to compute, for every $\varepsilon > 0$ and $\theta \in \mathbb{R}$, the limit

$$\psi^{\varepsilon}(\theta) = \lim_{N\to\infty} \frac{1}{N} \log E_N^{\gamma}\left[\exp\left\{\theta N \int_0^1 \eta_s^{\varepsilon N}(0)\, ds\right\}\right]$$

and to prove that, for every $\theta \in \mathbb{R}$, $\lim_{\varepsilon\to 0} \psi^{\varepsilon}(\theta)$ exists and is finite, because in this case the limit is equal to $\psi(\theta)$. Indeed,

$$\begin{aligned} &E_N^{\gamma}\left[\exp\left\{\theta N \int_0^1 \eta_s(0)\, ds\right\}\right] \\ &\quad\leq \exp(\delta\theta N) E_N^{\gamma}\left[\exp\left\{\theta N \int_0^1 \eta_s^{\varepsilon N}(0)\, ds\right\}\right] \\ &\qquad + E_N^{\gamma}\left[\exp \theta N\left\{\int_0^1 \eta_s^{\varepsilon N}(0)\, ds + \int_0^1 V_{N,\varepsilon}(\eta_s)\, ds\right\} \mathbb{1}_{\{\int_0^1 V_{N,\varepsilon}(\eta_s)\, ds > \delta\}}\right]. \end{aligned} \tag{3.12}$$

But the second term on the right-hand side of (3.12) does not contribute to the limit (3.11), since we have

$$\begin{aligned}
&\frac{1}{N}\log E_N^{\gamma}\left[\exp \theta N\left\{\int_0^1 \eta_s^{\varepsilon N}(0)\,ds + \int_0^1 V_{N,\varepsilon}(\eta_s)\,ds\right\}\mathbb{1}_{\{\int_0^1 V_{N,\varepsilon}(\eta_s)\,ds > \delta\}}\right]\\
&\quad\le \frac{1}{5N}\log E_N^{\lambda}\left[\exp N\left\{5\theta\int_0^1 V_{N,\varepsilon}(\eta_s)\,ds - \varepsilon\alpha\int_0^1 \omega\big(\eta_s^{\varepsilon N}(0)\big)\,ds\right\}\right]\\
&\qquad+\frac{1}{5N}\log E_N^{\lambda}\left[\exp\left\{\varepsilon\alpha N\int_0^1 \omega\big(\eta_s^{\varepsilon N}(0)\big)\,ds\right\}\right]\\
&\qquad+\frac{1}{5N}\log E_N^{\lambda}\left[\exp 5\theta N\left\{\int_0^1 \eta_s^{\varepsilon N}(0)\,ds\right\}\right]\\
&\qquad+\frac{1}{5N}\log P_N^{\lambda}\left[\int_0^1 V_{N,\varepsilon}(\eta_s)\,ds > \delta\right] + \frac{1}{5N}\log E_N^{\lambda}\left[\left(\frac{dP_N^{\gamma}}{dP_N^{\lambda}}\right)^5\right],
\end{aligned}$$

and, in view of (3.2) and (3.4), the first and second terms of the previous expression are less than 0 as N goes to ∞ and ε goes to 0. Moreover, in the limit, the third term is bounded above by $\limsup_{\varepsilon\to 0}\psi_\varepsilon(5\theta)/5$, the fourth term goes to $-\infty$ because of Lemma 2.1 and the last term is less than a positive constant by (3.10). So, we obtain

$$(3.13)\qquad \limsup_{N\to\infty}\frac{1}{N}\log E_N^{\gamma}\left[\exp\left\{\theta N\int_0^1 \eta_s(0)\,ds\right\}\right] \le \limsup_{\varepsilon\to 0}\psi^{\varepsilon}(\theta).$$

If we reverse $\eta_s(0)$ and $\eta_s^{\varepsilon N}(0)$ in (3.12), we get in the same way

$$(3.14)\qquad \liminf_{N\to\infty}\frac{1}{N}\log E_N^{\gamma}\left[\exp\left\{\theta N\int_0^1 \eta_s(0)\,ds\right\}\right] \ge \liminf_{\varepsilon\to 0}\psi^{\varepsilon}(\theta).$$

Notice that $\eta_s^{\varepsilon N}(0)$ can be replaced by $\langle \mu^N(s), \alpha_\varepsilon\rangle$, where $(\alpha_\varepsilon)_{\varepsilon>0}$ is a regularizing family. Now, we can compute the limit $\psi^{\varepsilon}(\theta)$ by truncating the continuous function on $D([0,1],\mathscr{S}')$, defined by

$$\rho \mapsto \theta\int_0^1 \langle \rho_s, \alpha_\varepsilon\rangle\,ds,$$

and applying Varadhan's theorem to the large deviation principle for the density field (Theorem 1). It leads to the variational formula:

$$\psi^{\varepsilon}(\theta) = \sup_{\rho\in D([0,1],\mathscr{S}')}\left\{\theta\int_0^1 \langle \rho_s, \alpha_\varepsilon\rangle\,ds - J_\gamma(\rho)\right\}.$$

With some arguments to regularize the paths that we will expand in the proof of the lower bound of Theorem 1 [Section 4, (4.20) to (4.24)], the supremum can be taken over bounded and smooth ρ which are greater than a positive constant and satisfy $J_\lambda(\rho) < \infty$ (we denote by $\mathscr{E}$ the subset of these paths). Moreover, we will prove in the same section [see (4.18) and (4.19)] that, in this case, there exists a function $\partial H/\partial x$ such that

$$\frac{\partial \rho}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial}{\partial x}\left(\sigma^2 \frac{\partial H}{\partial x}\rho\right)$$

and

$$J_\gamma(\rho) = \frac{\sigma^2}{2}\int_0^1 \left\langle \rho(s,\cdot), \left(\frac{\partial H}{\partial x}\right)^2(\cdot, s)\right\rangle ds + \int_{\mathbb{R}}\left(\rho(0,x)\log\frac{\rho(0,x)}{\gamma(x)} - \rho(0,x) + \gamma(x)\right) dx.$$

Considering $\bar{\rho}(t,x) = \rho(\sigma^{-2}t, x)$ and $\overline{H}(x,t) = H(x, \sigma^{-2}t)$, we would like to compute

$$\psi_\varepsilon(\theta) = \sup_{\bar{\rho}\in\mathscr{E}} \Psi_{\varepsilon,\theta}(\bar{\rho}),$$

with

$$\Psi_{\varepsilon,\theta}(\bar{\rho}) = \theta\sigma^{-2}\int_0^{\sigma^2}\langle \bar{\rho}(s,\cdot), \alpha_\varepsilon\rangle\, ds - \frac{1}{2}\int_0^{\sigma^2}\left\langle \bar{\rho}(s,\cdot), \left(\frac{\partial \overline{H}}{\partial x}\right)^2(\cdot,s)\right\rangle ds - \int_{\mathbb{R}}\left(\bar{\rho}(0,x)\log\frac{\bar{\rho}(0,x)}{\gamma(x)} - \bar{\rho}(0,x) + \gamma(x)\right) dx.$$

Notice that, in view of the equation satisfied by ρ and H,

$$\frac{\partial \bar{\rho}}{\partial t} = \frac{1}{2}\frac{\partial^2 \bar{\rho}}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{\partial \overline{H}}{\partial x}\bar{\rho}\right), \tag{3.15}$$

so

$$\frac{\partial \overline{H}}{\partial x}(x,t) = \frac{\int_{-\infty}^{x}\left[(1/2)(\partial^2/\partial x^2) - (\partial/\partial t)\right]\bar{\rho}(t,y)\,dy}{\bar{\rho}(t,x)}. \tag{3.16}$$

Thus, $\Psi_{\varepsilon,\theta}$ is a concave and differentiable function on $\mathscr{E}$. Let ρ be a solution of Euler's equation $D\Psi_{\varepsilon,\theta}(\bar{\rho}) = 0$. Applying elementary rules of infinitesimal calculus, we obtain that, for every $G \in \mathscr{D}(\mathbb{R}\times[0,1])$,

$$\begin{aligned} &\theta\sigma^{-2}\int_0^{\sigma^2}\langle G(\cdot,s), \alpha_\varepsilon\rangle\, ds + \int_0^{\sigma^2}\left\langle G(\cdot,s), \left(\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right)\overline{H}(\cdot,s)\right\rangle ds \\ &+ \frac{1}{2}\int_0^{\sigma^2}\left\langle G(\cdot,s), \left(\frac{\partial \overline{H}}{\partial x}\right)^2(\cdot,s)\right\rangle ds \\ &+ \langle G(\cdot,0), \overline{H}(\cdot,0)\rangle - \langle G(\cdot,\sigma^2), \overline{H}(\cdot,\sigma^2)\rangle \\ &- \int_{\mathbb{R}} G(x,0)\log\frac{\bar{\rho}(0,x)}{\gamma(x)}\, dx = 0. \end{aligned} \tag{3.17}$$

For every $g \in \mathscr{D}(\mathbb{R})$, consider the function $G_n \in \mathscr{D}(\mathbb{R}\times[0,1])$, defined by

$$G_n(x,t) = g(x)T_n(t),$$

where T_n is a smooth function on $[0,1]$ such that $\lim_{n\to\infty} T_n(t) = \mathbb{1}_{\{\sigma^2\}}(t)$. Letting n go to ∞, the above equality implies that, for every $g \in \mathscr{D}(\mathbb{R})$, $\langle g, \overline{H}(\cdot,\sigma^2)\rangle = 0$, so that

$$\overline{H}(\cdot,\sigma^2) = 0. \tag{3.18}$$

We get in the same way

$$\overline{H}(\cdot,0) = \log\frac{\bar\rho(0,\cdot)}{\gamma(x)}. \tag{3.19}$$

Then equalities (3.17) and (3.18) mean that $\overline{H}$ is a solution of the partial differential equation

$$\frac{\partial \overline{H}}{\partial t} + \frac{1}{2}\frac{\partial^2 \overline{H}}{\partial x^2} + \frac{1}{2}\left(\frac{\partial \overline{H}}{\partial x}\right)^2 + \theta\sigma^{-2}\alpha_\varepsilon = 0, \qquad \overline{H}(\cdot,\sigma^2) = 0. \tag{3.20}$$

Now, suppose that $\overline{H}$ and $\bar\rho$ satisfy (3.15) and (3.20). Then we have the following expression:

$$\frac{1}{2}\int_0^{\sigma^2}\left\langle \bar\rho(s,\cdot), \left(\frac{\partial \overline{H}}{\partial x}\right)^2(\cdot,s)\right\rangle ds = -\langle \bar\rho(0,\cdot), \overline{H}(\cdot,0)\rangle + \theta\sigma^{-2}\int_0^{\sigma^2}\langle \bar\rho(s,\cdot), \alpha_\varepsilon\rangle\, ds.$$

Indeed,

$$\begin{aligned}\int_0^{\sigma^2}&\left\langle \bar\rho(s,\cdot), \left(\frac{\partial \overline{H}}{\partial x}\right)^2(\cdot,s)\right\rangle ds\\ &= \langle \bar\rho(\sigma^2,\cdot), \overline{H}(\cdot,\sigma^2)\rangle - \langle \bar\rho(0,\cdot), \overline{H}(\cdot,0)\rangle\\ &\quad - \int_0^{\sigma^2}\left\langle \bar\rho(s,\cdot), \left(\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right)\overline{H}(\cdot,s)\right\rangle ds\\ &= -\langle \bar\rho(0,\cdot), \overline{H}(\cdot,0)\rangle + \int_0^{\sigma^2}\left\langle \bar\rho(s,\cdot), \frac{1}{2}\left(\frac{\partial \overline{H}}{\partial x}\right)^2(\cdot,s) + \theta\sigma^{-2}\alpha_\varepsilon\right\rangle ds.\end{aligned}$$

Therefore, in view of (3.19), we obtain

$$\begin{aligned}\Psi_{\varepsilon,\theta}(\bar\rho) &= \langle \bar\rho(0,\cdot), \overline{H}(\cdot,0)\rangle - \int_{\mathbb{R}}\left(\bar\rho(0,x)\log\frac{\bar\rho(0,x)}{\gamma(x)} - \bar\rho(0,x) + \gamma(x)\right) dx\\ &= \int_{\mathbb{R}}(\bar\rho(0,x) - \gamma(x))\, dx\\ &= \int_{\mathbb{R}}(\exp \overline{H}(x,0) - 1)\gamma(x)\, dx.\end{aligned}$$

In order to assert that this term is the supremum of $\Psi_{\varepsilon,\theta}$, that is, that it is equal to $\psi_\varepsilon(\theta)$, we just have to prove the existence and uniqueness of the solutions of (3.20).

To turn the partial differential equation (3.20) into a linear one, consider, for $x \in \mathbb{R}$ and $0 \le t \le 1$,

$$F(x,t) = \exp \overline{H}(x, 1-t).$$

Then F satisfies

$$\frac{\partial F}{\partial t} = \frac{1}{2}\frac{\partial^2 F}{\partial x^2} + \theta\sigma^{-2}\alpha_\varepsilon F,$$
$$F(\cdot, 0) = 1.$$

This new equation has only one solution. Using the Feynman–Kac theorem, it is given by

$$F(x,t) = \mathbb{E}^x\left[\exp\left\{\theta\sigma^{-2}\int_0^{\sigma^{2+}} \alpha_\varepsilon(W_s)\,ds\right\}\right],$$

where W_s is the Brownian motion started at x. Consequently, if $L_s(y)$ denotes the Brownian local time at $y \in \mathbb{R}$, then

$$\psi_\varepsilon(\theta) = \int_{\mathbb{R}} \left(\mathbb{E}^x\left[\exp\{2\theta\sigma^{-2}\langle \alpha_\varepsilon, L_{\sigma^2}(\cdot)\rangle\}\right] - 1\right)\gamma(x)\,dx.$$

Now, we just have to study the limit $\psi(\theta)$ of $\psi_\varepsilon(\theta)$ as ε goes to 0. On the one hand, we use Jensen's inequality to claim that

$$\int_{\mathbb{R}}\left(\mathbb{E}^x\left[\exp\left\{2\theta\sigma^{-2}\int_{\mathbb{R}}\langle\alpha_\varepsilon, L_{\sigma^2}(\cdot)\rangle\right\}\right] - 1\right)\gamma(x)\,dx$$
$$\le \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\mathbb{E}^0\left[\exp\frac{2\theta}{\sigma^2}L_{\sigma^2}(x)\right] - 1\right)\alpha_\varepsilon(y)\gamma(x+y)\,dy\,dx$$

and, since γ is smooth, this term goes to

$$\int_{\mathbb{R}}\left(\mathbb{E}^0\left[\exp\left\{\frac{2\theta}{\sigma^2}L_{\sigma^2}(x)\right\}\right] - 1\right)\gamma(x)\,dx$$

when ε goes to 0.

On the other hand, we apply Fatou's lemma

$$\liminf_{\varepsilon\to 0}\psi_\varepsilon(\theta) \ge \int_{\mathbb{R}}\left(\lim_{\varepsilon\to 0}\mathbb{E}^x\left[\exp\{2\theta\sigma^{-2}\langle\alpha_\varepsilon, L_{\sigma^2}(\cdot)\rangle\}\right] - 1\right)\gamma(x)\,dx$$

and, since the local time paths are smooth enough, the limit in the integral exists and is equal to

$$\mathbb{E}^x\left[\exp\{2\theta\sigma^{-2}L_{\sigma^2}(0)\}\right].$$

As a conclusion, we get

$$\psi(\theta) = \int_{\mathbb{R}}\left(\mathbb{E}^x\left[\exp\{2\theta\sigma^{-2}L_{\sigma^2}(0)\}\right] - 1\right)\gamma(x)\,dx.$$

To complete the proof of (i), let us recall that the law of the Brownian local time $L_t(0)$ is given by

$$\mathbb{P}^x[2L_t(0) \in da] = \frac{2}{\sqrt{2\pi t}} \int_0^{|x|} \exp\left(-\frac{\alpha^2}{2t}\right) dx\, \delta_0(da) + \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(a+|x|)^2}{2t}\right) da.$$

A straightforward computation proves that, if $\sigma^2 = 1$ and $\gamma = \lambda$ (where λ is a positive constant), then

$$\psi(\theta) = 2\lambda \left[\frac{\exp(\theta^2/2) - 1}{\theta} + \sqrt{\frac{2}{\pi}} \frac{\exp(\theta^2/2)\int_0^\theta \exp(-s^2/2)\, ds - \theta}{\theta} \right].$$

and this is the explicit Legendre transform of the rate function obtained by Cox and Griffeath (Theorem 1 of [2]).

(ii) We now prove Theorem 2 for subsets in the microscopic scale, that is, an increasing family $(A_t)_{t \geq 0}$ of finite subsets of $\mathbb{Z}$ satisfying (2.5). We can suppose 0 to be an element of A_t for every $t \geq 0$. As we did previously, it suffices to study the limit

$$\lim_{N\to\infty} \frac{1}{N} \log E_N^\gamma\left[\exp\left\{\frac{\theta N}{|A_{N^2}|}\int_0^1 \sum_{i\in A_{N^2}} \eta_s(i)\, ds\right\}\right] \tag{3.21}$$

to obtain a large deviation principle for the occupation time of A_t. With the superexponential estimate (Lemma 2.1), we will see that, for any $\delta > 0$,

$$\limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log P_N^\gamma\left[\left|\frac{1}{|A_{N^2}|}\int_0^1 \sum_{i\in A_{N^2}} \eta_s(i)\, ds - \int_0^1 \eta_s^{\varepsilon N}(0)\, ds\right| > \delta\right] = -\infty. \tag{3.22}$$

It easily follows that the limit (3.21) is equal to $\psi(\theta)$ defined by (3.11) and computed in part (i).

Because of (3.10), it is enough to prove (3.22) in the equilibrium case $\gamma = \lambda$, where λ is a positive constant. Then we use the basic inequality

$$\log\left(\sum_{i=1}^k a_i\right) \leq \log k + \max_{1\leq i\leq k} \log a_i \tag{3.23}$$

and we get

$$\begin{aligned} &\frac{1}{N}\log P_N^\lambda\left[\left|\frac{1}{|A_{N^2}|}\int_0^1 \sum_{i\in A_{N^2}} \left(\eta_s(i) - \eta_s^{\varepsilon N}(i)\right) ds\right| > \delta\right] \\ &\quad\leq \frac{\log|A_{N^2}|}{N} + \max_{i\in A_{N^2}} \frac{1}{N}\log P_N^\lambda\left[\left|\int_0^1 \left(\eta_s(i) - \eta_s^{\varepsilon N}(i)\right) ds\right| > \delta\right]. \end{aligned}$$

So, in view of Lemma 2.1 and the shift invariance of ν^λ, (3.22) will be established if

$$\text{(3.24)}\qquad \frac{1}{N}\log P_N^\lambda\left[\left|\frac{1}{|A_{N^2}|}\int_0^1\sum_{i\in A_{N^2}}\eta_s^{\varepsilon N}(i)\,ds-\int_0^1\eta_s^{\varepsilon N}(0)\,ds\right|>\delta\right]$$

goes to $-\infty$ when N goes to $+\infty$ and ε goes to 0. Notice that

$$\left|\frac{1}{|A_{N^2}|}\sum_{i\in A_{N^2}}\eta_s^{\varepsilon N}(i)-\eta_s^{\varepsilon N}(0)\right|\le\frac{2}{2\varepsilon N+1}\sum_{|i\pm\varepsilon N|\le\operatorname{diam}A_{N^2}}\eta_s(i).$$

Therefore, taking N large enough so that $\varepsilon N>\operatorname{diam}A_{N^2}$ and applying Chebyshev's exponential inequality, (3.24) is bounded above, for any $a>0$, by

$$-\delta a+\frac{1}{N}\log E_N^\lambda\left[\exp\left\{\frac{4aN}{2\varepsilon N+1}\sum_{|i|\le\operatorname{diam}A_{N^2}}\int_0^1\eta_s(i)\,ds\right\}\right]$$

and this term is less than or equal to

$$-\delta a+\frac{2\operatorname{diam}A_{N^2}+1}{N}\log E_N^\lambda\left[\exp\left\{\frac{4aN}{2\varepsilon N+1}\eta_0(0)\right\}\right].$$

We conclude, letting $N\to\infty$, $\varepsilon\to0$ and, finally, $a\to+\infty$.

(iii) Let $(A_t)_{t\ge0}$ be in the macroscopic scale with $A=\lim_{t\to\infty}A_t/\sqrt{t}$ [see (2.7)], where A is a bounded Borel subset of $\mathbb{R}$ with positive Lebesgue measure such that its frontier has a zero measure, and denote by $\tilde{A}_t$ the discrete approximation of A [see (2.6)]. Then

$$\left|\frac{1}{|A_{N^2}|}\sum_{i\in A_{N^2}}\eta_s(i)-\frac{1}{|\tilde{A}_{N^2}|}\sum_{i\in\tilde{A}_{N^2}}\eta_s(i)\right|$$
$$\le\frac{|A_{N^2}\,\Delta\,\tilde{A}_{N^2}|}{|A_{N^2}|\,|\tilde{A}_{N^2}|}\sum_{i\in A_{N^2}}\eta_s(i)+\frac{1}{|\tilde{A}_{N^2}|}\sum_{i\in A_{N^2}\Delta\tilde{A}_{N^2}}\eta_s(i).$$

Moreover, it is easy to see that $N^{-1}|A_{N^2}|\to m(A)$ and $N^{-1}|\tilde{A}_{N^2}|\to m(A)$. So, following part (ii) and noting that

$$\frac{1}{|\tilde{A}_{N^2}|}\sum_{i\in\tilde{A}_{N^2}}\eta_s(i)=\frac{N}{|\tilde{A}_{N^2}|}\langle\mu^N(s),\mathbb{1}_A\rangle,$$

we obtain that the limit $\psi_A(\theta)$ defined by (3.21) is equal to

$$\lim_{N\to\infty}\frac{1}{N}\log E_N^\gamma\left[\exp\left\{\frac{\theta N}{m(A)}\int_0^1\langle\mu^N(s),\mathbb{1}_A\rangle\,ds\right\}\right].$$

Now, let $(g_k)_{k\ge0}$ be a sequence of uniformly bounded functions with a compact support such that $g_k\to\mathbb{1}_A$ a.e. As a result, with the same arguments as in part (i) if we define

$$\psi_A^k(\theta) = \lim_{N\to\infty} \frac{1}{N} \log E_N^{\gamma}\left[\exp\left\{\frac{\theta N}{m(A)} \int_0^1 \langle \mu^N(s), g_k\rangle \, ds\right\}\right],$$

then

$$\psi_A^k(\theta) = \int_{\mathbb{R}} \left(\mathbb{E}^x\left[\exp\left\{\frac{2\theta\sigma^{-2}}{m(A)} \int_{\mathbb{R}} L_{\sigma^2}(y) g_k(y) \, dy\right\}\right] - 1\right)\gamma(x) \, dx$$

and $\psi_A(\theta) = \lim_{k\to\infty} \psi_A^k(\theta)$. So, finally,

$$\psi_A(\theta) = \int_{\mathbb{R}} \left(\mathbb{E}^x\left[\exp\left\{\frac{2\theta\sigma^{-2}}{m(A)} \int_A L_{\sigma^2}(y) \, dy\right\}\right] - 1\right)\gamma(x) \, dx. \qquad \square$$

4. Large deviations for the density field in an infinite volume. In this section, using the method introduced in [8], we prove a large deviation principle for the density field of the accelerated independent particle system.

For any $H(\cdot,\cdot) \in \mathscr{D}(\mathbb{R} \times [0,T])$, we introduce the weakly asymmetric process. Its generator acts on cylinder functions f in the following way:

$$\mathscr{L}_N^H f(\eta_s) = N^2 \sum_{i,j} p(j-i) C_{H,N}^{i,j}(s) \eta_s(i) \left[f(\eta_s^{i,j}) - f(\eta_s) \right],$$

where

$$C_{H,N}^{i,j}(s) = \exp\left(H\left(\frac{j}{N}, s\right) - H\left(\frac{i}{N}, s\right)\right).$$

We denote by $P_N^{H,\gamma}$ the law of this process when the initial distribution is ν_N^{γ}.

4.1. *Preliminary lemmas.* In the next subsections, we will need some technical lemmas. The first one aims at proving that the perturbed process is close to the initial independent process. Indeed, $P_N^{H,\lambda}$ is absolutely continuous with respect to P_N^{λ}, and, applying Girsanov's formula, its density $Z_N^H = (Z_N^H(t))_{t\in[0,T]}$ is the exponential martingale

$$\begin{aligned} Z_N^H(t) = \exp\Bigg\{ \sum_{i,j} \Bigg[\int_0^t \left(H\left(\frac{j}{N}, s\right) - H\left(\frac{i}{N}, s\right)\right) dJ_s^{i,j} \\ - N^2 \int_0^t \left(p(j-i) C_{H,N}^{i,j}(s) - 1\right) \eta_s(i) \, ds \Bigg] \Bigg\}, \end{aligned} \tag{4.1}$$

where $J_t^{i,j}$ is the number of jumps from site i to site j up to time t.

LEMMA 4.1. *For any positive number p, we can find a positive constant c_p such that*

$$\forall\, t \in [0,T], \qquad E_N^{\lambda}\left[\left(Z_N^H(t)\right)^p \right] \le \exp(c_p N).$$

PROOF. First, we rewrite the martingale Z_N^H as

$$Z_N^H(t) = \exp\left\{ \sum_{i\in\mathbb{Z}} \left[H\left(\frac{i}{N}, t\right)\eta_t(i) - H\left(\frac{i}{N}, 0\right)\eta_0(i) - \int_0^t \frac{\partial H}{\partial t}\left(\frac{i}{N}, s\right)\eta_s(i)\,ds \right.\right.$$
$$\left.\left. -N^2\int_0^t \left(\sum_j p(j-i)C_{H,N}^{i,j}(s) - 1\right)\eta_s(i)\,ds \right]\right\}.$$

We expand the last term in the exponential using Taylor's formula. Then, as H and its derivatives are continuous with a compact support, there exists a positive continuous function f with a compact support, such that

$$Z_N^H(t) \le \exp\left\{ \sum_{i\in\mathbb{Z}} f\left(\frac{i}{N}\right)\left(\eta_t(i) + \eta_0(i) + \int_0^t \eta_s(i)\,ds\right)\right\}.$$

Now using the Cauchy–Schwarz inequality, we get

$$\begin{aligned} E_N^\lambda\left[\left(Z_N^H\right)^p\right] \le{}& \left(E_N^\lambda\left[\exp\left\{3p\sum_{i\in\mathbb{Z}} f\left(\frac{i}{N}\right)\eta_t(i)\right\}\right]\right)^{1/3} \\ & \times\left(E_N^\lambda\left[\exp\left\{3p\sum_{i\in\mathbb{Z}} f\left(\frac{i}{N}\right)\eta_0(i)\right\}\right]\right)^{1/3} \qquad (4.2)\\ & \times\left(E_N^\lambda\left[\exp\left\{3p\int_0^t\sum_{i\in\mathbb{Z}} f\left(\frac{i}{N}\right)\eta_s(i)\,ds\right\}\right]\right)^{1/3} \end{aligned}$$

and the third term in this product is bounded above by

$$\left(\frac{1}{t}\int_0^t E_N^\lambda\left[\exp\left\{3pt\sum_{i\in\mathbb{Z}} f\left(\frac{i}{N}\right)\eta_s(i)\right\}\right]ds\right)^{1/3}.$$

As ν_N^λ is invariant, (4.2) is less than

$$E_N^\lambda\left[\exp\left\{3p(T\vee 1)\sum_{i\in\mathbb{Z}} f\left(\frac{i}{N}\right)\eta_0(i)\right\}\right].$$

Finally, since ν^λ is the Poisson product measure, this term is equal to

$$\prod_i \exp\lambda\left(\exp\left(3p(T\vee 1)f\left(\frac{i}{N}\right)\right) - 1\right),$$

so there exists a positive constant c_p such that (3.2) is less than $\exp(c_pN)$. □

We can extend Lemma 4.1 to the law of the weakly asymmetric process with ν_N^γ as initial distribution [$\gamma\in\Lambda(\lambda)$]. Indeed, we have $P_N^{H,\gamma} = Z_N^{H,\gamma}P_N^\lambda$, where

$$\begin{aligned} Z_N^{H,\gamma}(t) &= \frac{d\nu_N^\gamma}{d\nu^\lambda}Z_N^H(t) \\ &= \exp\left\{\sum_{i\in\mathbb{Z}}\left(\lambda - \gamma\left(\frac{i}{N}\right)\right) + \sum_{i\in\mathbb{Z}}\log\left(\frac{\gamma(i/N)}{\lambda}\right)\eta_0(i)\right\}Z_N^H(t). \end{aligned} \qquad (4.3)$$

Since $\gamma \in \Lambda(\lambda)$, the terms with i/N outside a compact subset of $\mathbb{R}$ do not contribute to the martingale. So, for any $p > 0$, there exists $c_p(\gamma) > 0$ such that

$$E_N^\lambda\left[\left(\frac{d\nu_N^\gamma}{d\nu^\lambda}\right)^p\right] \le \exp\big(c_p(\gamma)N\big).$$

LEMMA 4.2. *If $\gamma \in \Lambda(\lambda)$ is the initial profile and if f is a positive continuous function on $\mathbb{R}$ which satisfies $\limsup_{|x|\to\infty} x^2 f(x) < \infty$, then*

$$\lim_{M\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log P_N^{H,\gamma}\left[\frac{1}{N}\sum_{i\in\mathbb{Z}} f\left(\frac{i}{N}\right)\int_0^T \eta_s(i)\,ds > M\right] = -\infty.$$

PROOF. We use the Chebyshev exponential inequality and the Cauchy–Schwarz inequality to get

$$\begin{aligned} &P_N^{H,\gamma}\left[\frac{1}{N}\sum_{i\in\mathbb{Z}} f\left(\frac{i}{N}\right)\int_0^T \eta_s(i)\,ds > M\right] \\ &\quad\le e^{-MN}\Big(E_N^\lambda\Big[\big(Z_N^{H,\gamma}(T)\big)^2\Big]\Big)^{1/2}\left(E_N^\lambda\left[\exp\left\{2\sum_{i\in\mathbb{Z}} f\left(\frac{i}{N}\right)\int_0^T \eta_s(i)\,ds\right\}\right]\right)^{1/2}. \end{aligned}$$

With Lemma 4.1 and its extension (4.3), the second term is bounded by $\exp(Nc_2(\gamma)/2)$. For the third term, we proceed as we did in (4.2), using the invariance and shift invariance of ν^λ. We obtain that

$$\left(E_N^\lambda\left[\exp\left\{2\sum_{i\in\mathbb{Z}} f\left(\frac{i}{N}\right)\int_0^T \eta_s(i)\,ds\right\}\right]\right)^{1/2} \le \prod_{i\in\mathbb{Z}} \exp\left\{\frac{\lambda}{2}\left(\exp\left[2Tf\left(\frac{i}{N}\right)\right] - 1\right)\right\}.$$

So, as N goes to ∞, we get

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N}\log P_N^{H,\gamma}\left[\frac{1}{N}\sum_{i\in\mathbb{Z}} f\left(\frac{i}{N}\right)\int_0^T \eta_s(i)\,ds > M\right] \\ &\qquad \le -M + \frac{c_2}{2} + \frac{\lambda}{2}\int_{\mathbb{R}} (\exp[2Tf(\theta)] - 1)\,d\theta \end{aligned} \tag{4.4}$$

and this last integral is finite under our assumptions on f. □

4.2. *Hydrodynamical limits in infinite volume.* In this subsection, we prove the hydrodynamical limit for the weakly asymmetric process.

LEMMA 4.3. *If $\gamma \in \Lambda(\lambda)$ and $H \in \mathscr{D}(\mathbb{R} \times [0,T])$, then the empirical measure μ^N converges in $P_N^{H,\gamma}$ probability to the unique weak solution $\rho \in D([0,T], \mathscr{S}')$, with $\sup_{t\in[0,T]} \langle \rho(t), (1+x^2)^{-1}\rangle < \infty$, of the equation*

$$\begin{aligned} \frac{\partial\rho}{\partial t} &= \frac{\sigma^2}{2}\frac{\partial^2\rho}{\partial x^2} - \frac{\partial}{\partial x}\left(\sigma^2\frac{\partial H}{\partial x}\rho\right), \\ \rho(0,dx) &= \gamma(x)\,dx. \end{aligned} \tag{4.5}$$

PROOF. The proof is similar to the proof of Theorem 3.2 in [8]. Let $Q_N^{H,\gamma}$ (respectively, Q_N^{γ}) be the law of the empirical measure under $P_N^{H,\gamma}$ (respectively, P_N^{γ}). In the first step, we prove the tightness of $(Q_N^{H,\gamma})_{N\geq 1}$. In the second step, we show that any limit point of this sequence is concentrated on weak solutions of (4.5) and finally we establish the uniqueness of such solutions. Let us begin with the second step.

For any $G \in \mathscr{D}(\mathbb{R}\times[0,T])$, we consider the following square integrable martingale with respect to the weakly asymmetric independent process:

$$M_N^G(t) = \langle \mu^N(t), G(\cdot,t)\rangle - \langle \mu^N(0), G(\cdot,0)\rangle - \int_0^t \left\langle \mu^N(s), \frac{\partial G}{\partial t}(\cdot,s)\right\rangle ds$$
$$- N\sum_{i,j}\int_0^t p(j-i)\left(G\left(\frac{j}{N},s\right) - G\left(\frac{i}{N},s\right)\right) C_{H,N}^{i,j}(s)\eta_s(i)\,ds.$$

On the one hand, if we denote, for $\rho \in D([0,T],\mathscr{S}')$,

$$l^H(\rho,t,G) = \langle \rho(t), G(\cdot,t)\rangle - \langle \rho(0), G(\cdot,0)\rangle$$
$$- \int_0^t \left\langle \rho(s), \left(\frac{\sigma^2}{2}\frac{\partial^2 G}{\partial x^2} + \frac{\partial G}{\partial t}\right)(\cdot,s)\right\rangle ds$$
$$- \int_0^t \left\langle \rho(s), \left(\sigma^2 \frac{\partial G}{\partial x}\frac{\partial H}{\partial x}\right)(\cdot,s)\right\rangle ds,$$

then, using Taylor's formula to expand the last term of $M_N^G(t)$, there exists a continuous positive function g_1 with a compact support in $\mathbb{R}$ such that

$$|M_N^G(t) - l^H(\mu^N,t,G)| \leq \frac{1}{N^2}\sum_{i\in\mathbb{Z}} g_1\left(\frac{i}{N}\right)\int_0^T \eta_s(i)\,ds. \tag{4.6}$$

On the other hand, the quadratic variation of M_N^G is given by

$$\langle M_N^G\rangle_t = \sum_{i,j}\left\{\int_0^t p(j-i)\left(G\left(\frac{j}{N},s\right) - G\left(\frac{i}{N},s\right)\right)^2 C_{H,N}^{i,j}(s)\eta_s(i)\,ds\right\},$$

and, using again Taylor's formula, we see that there exists a function g_2 with the same properties as g_1 such that

$$\langle M_N^G\rangle_t \leq \frac{1}{N^2}\sum_{i\in\mathbb{Z}} g_2\left(\frac{i}{N}\right)\int_0^t \eta_s(i)\,ds. \tag{4.7}$$

Therefore, using Lemma 4.1,

$$E_N^{H,\gamma}[\langle M_N^G\rangle_t] \leq \frac{1}{N^2}\log E_N^{\lambda}\left[\frac{dP_N^{H,\gamma}}{dP_N^{\lambda}}\exp\left\{\sum_{i\in\mathbb{Z}} g_2\left(\frac{i}{N}\right)\int_0^t \eta_s(i)\,ds\right\}\right] \leq \frac{C}{N}. \tag{4.8}$$

For every $\delta > 0$, it results from Doob's maximal inequality and (4.6) that

$$P_N^{H,\gamma}\left[\sup_{t\in[0,T]} |l^H(\mu^N, t, G)| > \delta\right] \le 4\left(\frac{2}{\delta}\right)^2 E_N^{H,\gamma}\left[\langle M_N^G\rangle_T\right] + P_N^{H,\gamma}\left[\frac{1}{N}\sum_{i\in\mathbb{Z}} g_1\left(\frac{i}{N}\right)\int_0^T \eta_s(i)\,ds > \frac{N\delta}{2}\right],$$

so with Lemma 4.2 and (4.8), we get, for every $\delta > 0$,

$$\lim_{N\to\infty} P_N^{H,\gamma}\left[\sup_{t\in[0,T]} |l^H(\mu^N, t, G)| > \delta\right] = 0. \tag{4.9}$$

Now, we deal with the first step. A tightness criterion on $D([0,T], \mathscr{S}')$ is given by Theorem 4.1 in [10]. We have to prove that, for any $f \in \mathscr{S}$ and $\varepsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{N\to\infty} P_N^{H,\gamma}\left[\sup_{|s-t|\le\delta} |\langle \mu^N(t), f\rangle - \langle \mu^N(s), f\rangle| > \varepsilon\right] = 0.$$

In fact, we establish a stronger result which will be useful for the proof of the upper bound of the large deviation principle:

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \frac{1}{N}\log P_N^{H,\gamma} \times\left[\sup_{|s-t|\le\delta} |\langle \mu^N(t), f\rangle - \langle \mu^N(s), f\rangle| > \varepsilon\right] = -\infty \qquad \forall f\in\mathscr{S}. \tag{4.10}$$

Using Lemma 4.1, we can suppose that $H = 0$. Then, splitting up the interval $[0,T]$ into intervals $[t_k, t_{k+1}]$ of length δ, it is enough to prove that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_k \frac{1}{N}\log P_N^{\lambda} \times\left[\sup_{t_k\le t\le t_{k+1}} |\langle \mu^N(t), f\rangle - \langle \mu^N(t_k), f\rangle| > \frac{\varepsilon}{2}\right] = -\infty \qquad \forall f\in\mathscr{S},$$

and, with the invariance of ν^λ, we just need the following estimate for any $\varepsilon > 0$:

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \frac{1}{N}\log P_N^{\lambda} \times\left[\sup_{0\le t\le\delta} |\langle \mu^N(t), f\rangle - \langle \mu^N(0), f\rangle| > \varepsilon\right] = -\infty \qquad \forall f\in\mathscr{S}.$$

Considering $-f$ instead of f, we can forget the absolute value. For every $f\in\mathscr{S}$ and $\alpha > 0$, define

$$A_{N,\alpha}^f(t) = \langle \mu^N(t), \alpha f\rangle - \langle \mu^N(0), \alpha f\rangle - N\left(\sum_{i,j} p(j-i)\exp\left\{\alpha\left(f\left(\frac{j}{N}\right) - f\left(\frac{i}{N}\right)\right)\right\} - 1\right)\int_0^t \eta_s(i)\,ds.$$

There exists a positive continuous function h_α such that $\limsup_{|x|\to\infty} x^2 h_\alpha(x) < \infty$ and, for every $t \in [0, \delta]$,

$$\langle \mu^N(t), \alpha f \rangle - \langle \mu^N(0), \alpha f \rangle \le A^f_{N,\alpha}(t) + \frac{1}{N} \sum_{i\in\mathbb{Z}} h_\alpha\left(\frac{i}{N}\right) \int_0^t \eta_s(i)\, ds.$$

Then, for any $\alpha > 0$,

$$\begin{aligned} &P^\lambda_N\left[\sup_{0\le t\le\delta} \langle \mu^N(t), f\rangle - \langle \mu^N(0), f\rangle > \varepsilon\right] \\ &\quad\le P^\lambda_N\left[\sup_{0\le t\le\delta} A^f_{N,\alpha}(t) > \frac{\varepsilon\alpha}{2}\right] \\ &\qquad + P^\lambda_N\left[\frac{1}{N}\sum_{i\in\mathbb{Z}} h_\alpha\left(\frac{i}{N}\right)\int_0^\delta \eta_s(i)\, ds > \frac{\varepsilon\alpha}{2}\right]. \end{aligned} \tag{4.11}$$

On the other hand, we apply Doob's inequalities to the exponential martingale $\exp NA^f_{N,\alpha}(\cdot)$ and we get

$$\limsup_{N\to\infty} \frac{1}{N}\log P^\lambda_N\left[\sup_{t\in[0,\delta]} A^f_{N,\alpha}(t) > \frac{\varepsilon\alpha}{2}\right] \le -\frac{\varepsilon\alpha}{2}. \tag{4.12}$$

On the other hand, a similar computation to (4.4) gives

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N}\log P^\lambda_N\left[\frac{1}{N}\sum_{i\in\mathbb{Z}} h_\alpha\left(\frac{i}{N}\right)\int_0^\delta \eta_s(i)\, ds > \frac{\varepsilon\alpha}{2}\right] \\ &\quad\le -\frac{\varepsilon\alpha}{2} + \lambda\int_{\mathbb{R}} \left(\exp(\delta h_\alpha(\theta)) - 1\right) d\theta. \end{aligned} \tag{4.13}$$

Finally, we complete the proof using (4.12) together with (4.13) in (4.11) and letting δ go to 0 and α go to ∞.

We proved with (4.9) that any limit point $Q^{H,\gamma}$ of the sequence $(Q^{H,\gamma}_N)_{N\ge 1}$ is concentrated on the weak solutions of (4.5). To complete the proof of Lemma 4.3, we need the uniqueness of weak solutions. But the space $D([0,T], \mathscr{S}')$ is too large to obtain such a result. Nevertheless, with the following formula, it suffices to look for solutions in the set of positive measures with moderate increments:

$$\lim_{A\to\infty} \limsup_{N\to\infty} \frac{1}{N}\log P^{H,\gamma}_N\left[\sup_{t\in[0,T]} \left\langle \mu^N(t), \frac{1}{1+x^2}\right\rangle \ge A\right] = -\infty. \tag{4.14}$$

Indeed, with Lemma 4.1, it is enough to study the independent case, that is, $H = 0$, and even if $\phi(x) = (1+x^2)^{-1}$ is not in $\mathscr{S}$, we notice that M^ϕ_N is still a square-integrable martingale and that (4.7) holds. So inequalities (4.11), (4.12) and (4.13) hold with $\varepsilon = A$, $\alpha = 1$, and $\delta = T$, and, letting A go to ∞, we obtain (4.14).

Let $\rho \in D([0,T],\mathscr{S}')$ be a weak solution of (4.5) which satisfies

$$\sup_{t\in[0,T]} \left\langle \rho(t), \frac{1}{1+x^2} \right\rangle < \infty.$$

Then $p(t) = (1+x^2)^{-1}\rho(t)$ is a positive finite measure such that

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial x}(Ap) + Bp$$

(where A and B are smooth functions), with $(1+x^2)^{-1}\gamma(x)\,dx$ as initial condition. Following Propositions 3.4 and 3.5 in [11], we can prove the uniqueness of solutions of this partial differential equation. □

4.3. *Proof of Theorem* 1. Notice that, since $J_\gamma(\rho) = \sup_{M>0}\{(I_\gamma(\rho) \wedge M)\mathbb{1}_{\{\rho\in\mathscr{A}^M\}} + M\,\mathbb{1}_{\{\rho\notin\mathscr{A}^M\}}\}$ and since $\mathscr{A}$ is convex, J_γ is lower semicontinuous and convex.

Upper bounds. To establish estimate (2.2), we first consider the case where C is a compact subset of $D([0,T],\mathscr{S}')$. For given $G \in \mathscr{D}(\mathbb{R}\times[0,T])$ and $g \in \mathscr{D}(\mathbb{R})$, Girsanov's formula (4.1) can be written, using Taylor's expansions as we did in (4.6), in the following way:

$$\begin{aligned}
\frac{dP_N^{G,\gamma\exp(g)}}{dP_N^\gamma} &= \exp N\left\{\langle \mu^N(0), g\rangle - \frac{1}{N}\sum_{i\in\mathbb{Z}}\gamma\left(\frac{i}{N}\right)\left(\exp g\left(\frac{i}{N}\right) - 1\right)\right\}\\
&\quad\times \exp N\left\{\langle \mu^N(T), G(\cdot,T)\rangle - \langle \mu^N(0), G(\cdot,0)\rangle\right.\\
&\qquad - \int_0^T\left\langle \mu^N(s), \frac{\partial G}{\partial t}(\cdot,s)\right\rangle ds\\
&\qquad \left. - N\sum_{i\in\mathbb{Z}}\int_0^T\left(\sum_j p(j-i)C^{i,j}_{G,N}(s) - 1\right)\eta_s(i)\,ds\right\}\\
&\le \exp N\left\{\langle \mu^N(0), g\rangle - \int_{\mathbb{R}}\gamma(x)(\exp g(x) - 1)\,dx + o(1)\right\}\\
&\quad\times \exp N\left\{l(\mu^N, G) - \frac{\sigma^2}{2}\int_0^T\left\langle \mu^N(s), \left(\frac{\partial G}{\partial x}(\cdot,s)\right)^2\right\rangle ds\right.\\
&\qquad\qquad \left. + \frac{1}{N^2}\sum_{i\in\mathbb{Z}} h\left(\frac{i}{N}\right)\int_0^T \eta_s(i)\,ds\right\},
\end{aligned}$$

where h is a positive continuous function with a compact support in $\mathbb{R}$. With Lemma 4.2, the remainder in Taylor's formula does not contribute to the

Girsanov density as N goes to ∞. Indeed,

$$\limsup_{N\to\infty} \frac{1}{N}\log P_N^{\gamma}\left[\mu^N \in C\right] \le \limsup_{N\to\infty} \frac{1}{N}\log P_N^{\gamma}\left[\mu^N \in C^{\delta}\right], \tag{4.15}$$

where

$$C^{\delta} = C \cap \left\{ \frac{1}{N^2}\sum_{i\in\mathbb{Z}} h\left(\frac{i}{N}\right)\int_0^T \eta_s(i)\, ds + o(1) \le \delta \right\}.$$

The lower bound (2.3) can be easily established for every open neighborhood of a smooth path ρ (see the next subsection). To deal with the general case, we need to regularize paths, but it can not be done for every element in $D([0,T],\mathscr{S}')$. We now prove that the deviations from the hydrodynamical limit only occur for paths in $\mathscr{A}$. Indeed, inequality (4.4) shows that, for every $M > 0$ and $n \in \mathbb{Z}$, there exists $c > 0$ which depends neither on M nor on n, such that

$$\forall\, N \ge 1, \qquad \frac{1}{N}\log P_N^{\gamma}\left[\mu^N \notin \mathscr{A}_n^M\right] \le -M + c.$$

Thus, for any $k \in \mathbb{N}$, if we write

$$C^{\delta} = \bigcup_{|n|\le k}\left[C^{\delta} \cap \left(\mathscr{A}_n^M\right)^c\right] \bigcup \left[C^{\delta} \cap \left(\bigcap_{|n|\le k}\mathscr{A}_n^M\right)\right],$$

and if we use the basic inequality (3.23), we get

$$\begin{aligned} &\frac{1}{N}\log P_N^{\gamma}\left[\mu^N \in C^{\delta}\right] \\ &\qquad \le \frac{\log(2k+2)}{N} \\ &\qquad\quad + \frac{1}{N}\log P_N^{\gamma}\left[\mu^N \in C^{\delta} \cap \left(\bigcap_{|n|\le k}\mathscr{A}_n^M\right)\right] \vee (-M+c). \end{aligned} \tag{4.16}$$

Now, we proceed as in Section 4 of [8]:

$$\begin{aligned} &\frac{1}{N}\log P_N^{\gamma}\left[\mu^N \in C^{\delta} \cap \left(\bigcap_{|n|\le k}\mathscr{A}_n^M\right)\right] \\ &\quad = \frac{1}{N}\log E_N^{G,\gamma\exp(g)}\left[\left(Z_N^{G,\gamma\exp(g)}\right)^{-1}\mathbb{1}_{\{\mu^N \in C^{\delta}\cap(\bigcap_{|n|\le k}\mathscr{A}_n^M)\}}\right] \\ &\quad \le \delta + \sup_{\rho\in C^{\delta}\cap(\bigcap_{|n|\le k}\mathscr{A}_n^M)}\left[-\left\{\left(l(\rho,G) - \frac{\sigma^2}{2}\int_0^T\left\langle \rho(s), \left(\frac{\partial G}{\partial x}(\cdot,s)\right)^2\right\rangle ds\right)\right.\right. \\ &\qquad\qquad \left.\left. + \left(\langle \rho(0), g\rangle - \int_{\mathbb{R}}(\exp(g(x)) - 1)\gamma(x)\, dx\right)\right\}\right]. \end{aligned}$$

Since $C^{\delta} \cap (\bigcap_{|n|\le k}\mathscr{A}_n^M)$ is compact and since the function of the variable ρ which appears in the supremum is continuous, we can let k go to ∞. Then, optimizing the last inequality over $\delta > 0$, $G \in \mathscr{D}(\mathbb{R} \times [0, T])$ and $g \in \mathscr{D}(\mathbb{R})$, we obtain from (4.15) and (4.16) that

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N^\gamma\big[\mu^N \in C\big]$$

$$\leq \inf_{\substack{G\in\mathscr{D}(\mathbb{R}\times[0,T])\\ g\in\mathscr{D}(\mathbb{R})}} \sup_{\rho\in C\cap\mathscr{A}^M} \Bigg[-\Bigg\{\Bigg(l(\rho, G) - \frac{\sigma^2}{2}\int_0^T \Big\langle \rho(s), \Big(\frac{\partial G}{\partial x}(\cdot, s)\Big)^2\Big\rangle ds\Bigg)$$

$$+\Big(\langle \rho(0), g\rangle - \int_{\mathbb{R}} (\exp(g(x)) - 1)\gamma(x)\, dx\Big)\Bigg\}\Bigg] \vee (-M + c).$$

With the compactness of $C \cap \mathscr{A}^M$, the infimum and supremum can be reversed (see, e.g., [12]); so, optimizing over $M > 0$, inequality (2.2) is proved for compact subsets.

We conclude, in the general case, noting that (4.10) implies the existence of a sequence of compacts C_L such that

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N^\gamma\big[\mu^N \notin C_L\big] \leq -L.$$

Indeed, consider the set of paths

$$A(g, \delta, \varepsilon) = \Big\{\rho\colon \sup_{|t-s|\leq\delta} |\langle \rho(t), g\rangle - \langle \rho(s), g\rangle| \leq \varepsilon\Big\}.$$

Formula (4.10) means that

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N} \log P_N^\gamma\big[\mu_N \notin A(g, \delta, \varepsilon)\big] = -\infty.$$

Now, take a sequence of functions $(g_l)_{l\geq 1}$ which is dense in $\mathscr{S}$. For every $L > 0$, $l \geq 1$ and $m \geq 1$, we can find $\delta(l, m, L) > 0$ such that, for N large enough, we have

$$P_N^\gamma\big[\mu_N \notin A(g_l, \delta(l, m, L), 1/m)\big] \leq \exp(-NLml).$$

Then consider

$$C_L = \bigcap_{l\geq 1,\, m\geq 1} A(g_l, \delta(l, m, L), 1/m)$$

and

$$K_L = C_L \cap B_L,$$

where B_L is the set of paths ρ such that, for every $t \in [0, T]$, $\langle \rho(t), (1 + x^2)^{-1}\rangle \leq L$. We claim that the closure of K_L is suitable. Indeed,

$$P_N^\gamma\big[\mu_N \notin C_L\big] \leq \sum_{l\geq 1,\, m\geq 1} \exp(-NLml)$$

$$\leq \exp(-NL).$$

Then, in view of Lemma 4.2 and (4.14),

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N^\gamma\big[\mu_N \notin B_L\big] \leq C - L.$$

Moreover, from the definition of K_L, for all $l \geq 1$,

$$\lim_{\delta \to 0} \sup_{\rho \in K_L} \sup_{|t-s| \leq \delta} |\langle \rho(t), g_l \rangle - \langle \rho(s), g_l \rangle| = 0.$$

As (g_l) is dense and as all the paths we look at are in B_L, this property remains valid for every function g in $\mathscr{S}$ and so K_L is relatively compact in $D([0,T], \mathscr{S}')$.

Lower bounds. First, we identify the rate functions. Suppose that $\gamma \in \Lambda(\lambda)$ and $h(\rho(0), \gamma) < \infty$. Then we see that $\rho(0)$ is absolutely continuous with respect to Lebesgue measure and if we denote by $\rho(0, x)$ its density,

$$h(\rho(0), \gamma) = \int_{\mathbb{R}} \left(\rho(0, x) \log \frac{\rho(0, x)}{\gamma(x)} - \rho(0, x) + \gamma(x) \right) dx. \tag{4.17}$$

For $\rho \in D([0,T], \mathscr{S}')$, we consider the Hilbert space $\mathscr{H}(\rho)$ defined in the following way. Let $H_2(\rho)$ be the space of functions H on $\mathbb{R} \times [0,T]$ that are almost everywhere differentiable in space and such that

$$\int_0^T \left\langle \left(\frac{\partial H}{\partial x} \right)^2 (\cdot, s), \rho_s \right\rangle ds < \infty.$$

Define the scalar product

$$[G, H] = \sigma^2 \int_0^T \left\langle \frac{\partial G}{\partial x} \frac{\partial H}{\partial x} (\cdot, s), \rho_s \right\rangle ds$$

and define the equivalence relation $G \sim H$ if

$$\int_0^T \left\langle \left(\frac{\partial G}{\partial x} - \frac{\partial H}{\partial x} \right)^2 (\cdot, s), \rho_s \right\rangle ds = 0.$$

Then we obtain $\mathscr{H}(\rho)$ by the completion of the space $H_2(\rho)/\sim$. It is easy to verify that $\mathscr{D}(\mathbb{R} \times [0,T])$ is dense in $\mathscr{H}(\rho)$. The proof of Lemma 5.1 in [8] can be adapted here; that is, if $I_0(\rho) < \infty$, there exists $\partial H/\partial x \in \mathscr{H}(\rho)$ such that

$$I_0(\rho) = \frac{\sigma^2}{2} \int_0^T \left\langle \rho(s), \left(\frac{\partial H}{\partial x}(\cdot, s) \right)^2 \right\rangle ds, \tag{4.18}$$

and in this case ρ is a weak solution of (4.5).

Obviously, the lower bound (2.3) will be proved if, for any open neighborhood V of $\rho \in D([0,T], \mathscr{S}')$, such that $J_\gamma(\rho) < \infty$ [i.e., $\rho \in \mathscr{A}$ and $I_\gamma(\rho) < \infty$],

$$\liminf_{N \to \infty} \frac{1}{N} \log P_N^\gamma[\mu^N \in V] \geq -J_\gamma(\rho). \tag{4.19}$$

Suppose ρ to be such that the function $\partial H/\partial x$ [defined by (4.18)] is the derivative of a smooth function with a compact support. Then we obtain (4.19) with the same arguments as those developed in the proof of Theorem 5.1 in [8].

To reduce the problem to this situation, we will regularize ρ.

Denote by α_ε a regularizing family and $\rho^\varepsilon = \rho * \alpha_\varepsilon$. Since $\rho \in \mathscr{A}$, ρ^ε is spatially smooth and satisfies

$$\sup_{x \in \mathbb{R}} \int_0^T \rho^\varepsilon(s, x)\, ds < \infty. \tag{4.20}$$

Notice that $\rho^\varepsilon \to \rho$ in $D([0,T], \mathscr{S}')$ and that $h(\rho^\varepsilon(0), \gamma) \to h(\rho(0), \gamma)$. Moreover, since I_γ is convex, lower semicontinuous and invariant under spatial shifts, $I_0(\rho^\varepsilon) \to I_0(\rho)$. Now, define

$$\widetilde{\rho^\varepsilon} = (1 - \varepsilon)\rho^\varepsilon + \varepsilon\lambda. \tag{4.21}$$

We observe that $\widetilde{\rho^\varepsilon}$ converges to ρ as ε goes to 0, that $h(\lambda, \gamma) < \infty$ and that $I_0(\lambda) = 0$ since λ is a solution of the heat equation. So, using convexity and lower semicontinuity of the rate function, $I_\gamma(\widetilde{\rho^\varepsilon}) \to I_\gamma(\rho)$. As a consequence, we just have to prove (4.19) for paths ρ which are spatially smooth, bounded below by a positive constant [from (4.21)] and which satisfy (4.20).

Let ρ be such a path and let $\partial H/\partial x$ be the corresponding function. To regularize ρ with respect to time, we proceed in the same way, but time convolutions of ρ over $[0,T]$ require extending ρ to $[0,T']$, where $T' > T$. Denote by χ the solution of the heat equation with $\rho(\cdot, T)$ as initial condition and define

$$\hat{\rho}(t, x) = \begin{cases} \rho(t, x), & \text{if } 0 \le t \le T, \\ \chi(t - T, x), & \text{otherwise.} \end{cases} \tag{4.22}$$

Since ρ is a weak solution of (4.5), $\hat{\rho}$ satisfies

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 \hat{\rho}}{\partial x^2} - \frac{\partial}{\partial x}\left(\sigma^2 \hat{\rho} \frac{\partial \hat{H}}{\partial x}\right), \tag{4.23}$$

where

$$\frac{\partial \hat{H}}{\partial x}(x, t) = \begin{cases} \dfrac{\partial H}{\partial x}(x, t), & \text{if } 0 \le t \le T, \\ 0, & \text{otherwise.} \end{cases}$$

Now, denote by $\widehat{I_0}$ the natural extension of I_0 to paths over $[0, T']$. In view of (4.18) and (4.22), we notice that

$$\widehat{I_0}(\hat{\rho}) = I_0(\rho). \tag{4.24}$$

For a family of smooth functions β_δ with their support in $[0, \delta]$ and such that $\int_0^\delta \beta_\delta(t)\, dt = 1$, we consider the path defined by

$$\rho_\delta(t, x) = \int_0^T \hat{\rho}(t + s, x)\beta_\delta(s)\, ds.$$

It easily follows from (4.20) and (4.22) that $\sup_{(t,x)\in[0,T]\times\mathbb{R}} \rho_\delta(t,x) < \infty$. Since $t \mapsto \rho(t)$ is right continuous, $\rho_\delta \to \rho$ as $\delta \to 0$. Moreover,

$$I_0(\rho_\delta) \le \int_0^T I_0(\rho(s+\cdot))\beta_\delta(s)\,ds \\ \le \widehat{I_0}(\hat{\rho}),$$

so $I_0(\rho^\delta) \to I_0(\rho)$.

Consequently, it is enough to prove (4.19) for bounded smooth paths ρ which are bounded below by a positive constant. Under these assumptions on ρ and in view of (4.5), the function $\partial H/\partial x$ corresponding to ρ in (4.18) is smooth and bounded. Furthermore, it belongs to $L^2(\mathbb{R}\times[0,T])$. Then we conclude, approximating H with functions with a compact support as in the proof of Theorem 3.3 in [9]. □

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