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ASYMPTOTIC BEHAVIOR FOR ITERATED FUNCTIONS OF RANDOM VARIABLES

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Let $\mathscr{D} \subseteq (-\infty, \infty)$ be a closed domain and set $\xi = \inf\{x; x \in \mathscr{D}\}$. Let the sequence $\mathscr{X}^{(n)} = \{X_j^{(n)}; j \geq 1\}$, $n \geq 1$ be associated with the sequence of measurable iterated functions $f_n(x_1, x_2, \ldots, x_{k_n})$: $\mathscr{D}^{k_n} \to \mathscr{D}$ $(k_n \geq 2)$, $n \geq 1$ and some initial sequence $\mathscr{X}^{(0)} = \{X_j^{(0)}; j \geq 1\}$ of stationary and m-dependent random variables such that $P(X_1^{(0)} \in \mathscr{D}) = 1$ and $X_j^{(n)} = f_n(X_{(j-1)k_n+1}^{(n-1)}, \ldots, X_{jk_n}^{(n-1)})$, $j \geq 1$, $n \geq 1$. This paper studies the asymptotic behavior for the hierarchical sequence $\{X_1^{(n)}; n \geq 0\}$. We establish general asymptotic results for such sequences under some surprisingly relaxed conditions. Suppose that, for each $n \geq 1$, there exist k_n nonnegative constants $\alpha_{n,i}$, $1 \leq i \leq k_n$ such that $\sum_{i=1}^{k_n} \alpha_{n,i} = 1$ and $f_n(x_1, \ldots, x_{k_n}) \leq \sum_{i=1}^{k_n} \alpha_{n,i} x_i$, $\forall (x_1, \ldots, x_{k_n}) \in \mathscr{D}^{k_n}$. If $\prod_{j=1}^{n} \max_{1 \leq i \leq k_j} \alpha_{j,i} \to 0$ as $n \to \infty$ and $E(X_1^{(0)} \vee 0) < \infty$, then, for some $\lambda \in \mathscr{D} \cup \{\xi\}$, $E(X_1^{(n)}) \downarrow \lambda$ as $n \to \infty$ and $X_1^{(n)} \to_P \lambda$. We conclude with various examples, comments and open questions and discuss further how our results can be applied to models arising in mathematical physics.

1. Introduction. Let $\mathscr{D} \subseteq (-\infty, \infty)$ be a closed domain and $\mathscr{X}^{(0)} = \{X_j^{(0)}; j \geq 1\}$ a sequence of independent and identically distributed (i.i.d.) real random variables on a complete probability space $(\Omega, \mathscr{F}, P)$ such that $P(X_1 \in \mathscr{D}) = 1$. Let $f(x_1, \ldots, x_k)$: $\mathscr{D}^k \to \mathscr{D}$ $(k \geq 2)$ be a real measurable function of k real variables. A sequence $\{X_1^{(n)}; n \geq 0\}$ of random variables is defined recursively, using the given f, as follows. Define

$$X_j^{(1)} = f\big(X_{(j-1)k+1}^{(0)}, \ldots, X_{jk}^{(0)}\big), \qquad j \geq 1 \tag{1.1}$$

and denote the resulting sequence $\mathscr{X}^{(1)} = \{X_j^{(1)}; j \geq 1\}$ by $\mathscr{R}_f \mathscr{X}^{(0)}$. Iterating the map, we then obtain a sequence $\mathscr{X}^{(0)}, \mathscr{X}^{(1)}, \ldots, \mathscr{X}^{(n)}, \ldots$ of i.i.d. sequences, where $\mathscr{X}^{(n+1)} = \mathscr{R}_f \mathscr{X}^{(n)}$ for $n = 0, 1, 2, \ldots$. We will write

$$\mathscr{X}^{(n)} = \big\{X_j^{(n)}; j \geq 1\big\}. \tag{1.2}$$

We finally obtain the sequence $\{X_1^{(n)}; n \geq 0\}$ of random variables. Arising originally from statistical physics, this is a special type of *hierarchical model*, often encountered in applications [See Blumenfeld (1988), Schlösser and Spohn (1992), Schenkel, Wehr and Wittwer (1998), Shneiberg (1986), Stinchcombe and Watson (1976) and Wehr (1997)]. Hierarchical models have been studied

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extensively in the literature for diverse purposes; see, for example, Bernasconi (1978), Boppana and Narayan (1993), Bovin, Vas'kin and Shneiberg (1983), Collet and Eckmann (1978), Derrida (1986), Griffiths and Kaufman (1982), Koch and Wittwer (1994), Moore and Shannon (1956a, b), Newman, Gabrielove, Durand, Phoenix and Turcotte (1994), and Sinai (1982), among others.

If $\mathscr{D} = [0,1]$ the unit interval and f is homogeneous of degree one under multiplication by positive numbers, convex separately in each variable and satisfies a normalization condition $f(1,\dots,1) = 1$, then, by a theorem of Shneiberg [(1986), page 137], the sequence $X_1^{(n)}$ converges to a constant in probability. Shneiberg (1986) demonstrates that this theorem can be applied to hierarchical resistor networks with bounded conductivities.

Under the assumption that $\mathscr{D} = [a,\infty)$ for some constant $a \in (-\infty,\infty)$,

$$f(x_1,\dots,x_k) \le \frac{x_1+\cdots+x_k}{k} \tag{1.3}$$

for all $(x_1,\dots,x_k) \in [a,\infty)^k$ and

$$E\big(\big|X_1^{(0)}\big|\big) < \infty. \tag{1.4}$$

Using the reversed-time martingale technique, Wehr (1997) claimed the sequence $X_1^{(n)}$ converges almost surely to a constant; see the main result Theorem 1 of Wehr [(1997), page 1376]. Wehr (1997) then gives several applications to models arising in mathematical physics and other areas which we revisit in Section 4. Unfortunately, the proof of Theorem 1 of Wehr (1997) is incorrect as it stands.

At the beginning of the proof, Wehr [(1997), page 1376] constructed a sequence of σ-algebras

$$\mathscr{F}_n = \sigma\big(X_1^{(n)}+\cdots+X_k^{(n)}, X_{k^{n+1}+1}^{(0)}, X_{k^{n+1}+2}^{(0)},\dots\big), \qquad n \ge 0,$$

and claimed, "Clearly, the $\mathscr{F}_n$ form a decreasing sequence of σ-algebras...". This critical statement, however, is false.

In fact, if $\mathscr{F}_n$, $n \ge 0$ form a decreasing sequence of σ-algebras, then Y_{n+1} is $\mathscr{F}_n$-measurable, where $Y_{n+1} = X_1^{(n+1)}+\cdots+X_k^{(n+1)}$, and since $X_2^{(n+1)},\dots,$ $X_k^{(n+1)}$ are $\mathscr{F}_n$-measurable, it follows that $X_1^{(n+1)}$ is also $\mathscr{F}_n$-measurable. Note that $X_1^{(n+1)}$ is independent of $X_{k^{n+1}+1}^{(0)}, X_{k^{n+1}+2}^{(0)},\dots$, so we must have

$$X_1^{(n+1)} = E\big(X_1^{(n+1)} \,\big|\, \mathscr{F}_n\big) = E\big(X_1^{(n+1)} \,\big|\, X_1^{(n)}+\cdots+X_k^{(n)}\big);$$

in other words,

$$f\big(X_1^{(n)},\dots,X_k^{(n)}\big) = E\big(X_1^{(n+1)} \,\big|\, X_1^{(n)}+\cdots+X_k^{(n)}\big),$$

which is true if and only if there exists a real measurable function $g(x)\colon \mathscr{D}^{(k)} \to \mathscr{D}$ such that

$$f\Big(X_1^{(n)},\dots,X_k^{(n)}\Big) = g\big(X_1^{(n)}+\cdots+X_k^{(n)}\big) \quad \text{a.s.},$$

where $\mathscr{D}^{(k)} = \{y = x_1+\cdots+x_k; x_i \in \mathscr{D}, 1 \le i \le k\}$.

One may want to use

$$\mathscr{F}'_n = \sigma\Big(X_1^{(n)}, \dots, X_k^{(n)}, X_{k^{n+1}+1}^{(0)}, X_{k^{n+1}+2}^{(0)}, \dots\Big), \qquad n \geq 0$$

to replace $\mathscr{F}_n$, $n \geq 0$. Obviously, the $\mathscr{F}'_n$, $n \geq 0$ form a decreasing sequence of σ-algebras, but, in general the sequence $\{(X_1^{(n)} + \cdots + X_k^{(n)})/k;\ n \geq 0\}$ is not necessarily a reversed-time submartingale relative to the family $(\mathscr{F}'_n)$ of σ-algebras because $E((X_1^{(n)} + \cdots + X_k^{(n)})/k \mid X_1^{(n+1)})$ is independent of $X_j^{(n+1)}$, $j \geq 2$ and it follows that

$$\begin{aligned} E\bigg(\frac{X_1^{(n)} + \cdots + X_k^{(n)}}{k}\bigg|\, \mathscr{F}'_{n+1}\bigg) &= E\bigg(\frac{X_1^{(n)} + \cdots + X_k^{(n)}}{k}\bigg|\, X_1^{(n+1)}\bigg) \\ &\geq \frac{X_1^{(n+1)} + \cdots + X_k^{(n+1)}}{k} \quad \text{a.s.,} \end{aligned}$$

which does not hold in general.

We have tried to remedy Wehr's proof without success. On the other hand, applying our Theorem 2.1, under the condition $f(x_1, \dots, x_k) \leq (x_1 + \cdots + x_k)/k$ for all $x_i \in \mathscr{D}$, $1 \leq i \leq k$ [i.e., (1.3)], and $E(|X_1^{(0)}|) < \infty$ [i.e., (1.4)], there follows

$$X_1^{(n)} \to_P \lambda \quad \text{and} \quad \limsup_{n\to\infty} X_1^{(n)} = \lambda \quad \text{a.s.} \tag{1.5}$$

for some finite constant and, if further, f is a symmetric function of k variables, then

$$\lim_{n\to\infty} X_1^{(n)} = \lambda \quad \text{a.s.} \tag{1.6}$$

Here and below $\to_P$ denotes convergence in probability.

The present paper is concerned with providing relaxed conditions under which some asymptotic behaviors hold for such type of hierarchical models. The main result, Theorem 2.1, appears in Section 2 and its proof is provided in Section 3. Theorem 2.1 extends previous studies in two directions as follows: (1) The original i.i.d. sequence is replaced by the sequence of (strictly) stationary and m-dependent random variables, and (2) the sequence $\{X_1^{(n)}; n \geq 0\}$ will be obtained by iterative procedures $\mathscr{X}^{(n)} = \mathscr{R}_{f_n}\mathscr{X}^{(n-1)}$, $n \geq 1$ where, for each $n \geq 1$, $f_n(x_1, \dots, x_{k_n})$: $\mathscr{D}^{k_n} \to \mathscr{D}$ is a real measurable function of k_n (≥ 2) variables which satisfies a more general condition than (1.3). Theorem 2.1 is divided into three parts. The proof of Theorem 2.1 is based on the limit theorems for weighted sums of stationary and m-dependent random variables stated in Proposition 3.1 and appears to have some novel features. The proof of Theorem 2.1(iii) relies on certain submartingale techniques (and the conclusion of Theorem 2.1(ii)). In Section 4, we provide examples, comments, a list of open problems and applications. In light of our results and the examples it seems that the further study of such types of hierarchical models will depend on results of order statistics. We explain in Section 4 how the results can be applied to the theory of disordered systems among other applied areas.

2. Main results. We start this section with some notation. Let $\mathscr{X}^{(0)} = \{X_j^{(0)}; j \geq 1\}$ be a sequence of (strictly) stationary and m-dependent random variables such that $P(X_1^{(0)} \in \mathscr{D}) = 1$ for some closed domain $\mathscr{D} \subseteq (-\infty, \infty)$. That is, for each $n \geq 1$, the sequence $\{X_{n+j}^{(0)}; j \geq 1\}$ has the same distribution as $\{X_j^{(0)}; j \geq 1\}$ and, the two collections

$$\{X_1^{(0)}, \dots, X_n^{(0)}\} \quad \text{and} \quad \{X_{n+m+1}^{(0)}, X_{n+m+2}^{(0)}, \dots\}$$

are independent. In this terminology, an i.i.d. sequence is 0-dependent and, for any $m_1 < m_2$, m_1-dependence implies m_2-dependence. For each $n \geq 1$, let $f_n(x_1, \dots, x_{k_n})\colon \mathscr{D}^{k_n} \to \mathscr{D}$ be a real measurable function of k_n variables where $\{k_n \geq 2; n \geq 1\}$ is a sequence of integers. We will use f_n, $n \geq 1$ to define a sequence $\mathscr{X}^{(n)}$, $n \geq 1$ of strictly stationary and m-dependent sequences as follows

$$\begin{aligned} \mathscr{X}^{(n)} &= \mathscr{R}_{f_n} \mathscr{X}^{(n-1)} = \{X_j^{(n)}; j \geq 1\}, \qquad n \geq 1, \\ X_j^{(n)} &= f_n\left(X_{(j-1)k_n+1}^{(n-1)}, \dots, X_{jk_n}^{(n-1)}\right), \qquad j \geq 1. \end{aligned} \tag{2.1}$$

We define $L(t) = \ln \max\{e, t\}$, $t \in (-\infty, \infty)$ and $\xi = \inf\{x; x \in \mathscr{D}\}$. Clearly, $\xi \in \mathscr{D}$ if and only if $\mathscr{D}$ is bounded below.

The major result of this paper, which provides some relaxed conditions under which certain asymptotic behaviors hold for $\{X_1^{(n)}; n \geq 0\}$, follows.

THEOREM 2.1. *Suppose that, for each $n \geq 1$, there exist k_n nonnegative constants $\alpha_{n,i}$, $1 \leq i \leq k_n$ such that $\sum_{i=1}^{k_n} \alpha_{n,i} = 1$ and the f_n satisfies the subadditive constraint*

$$f_n(x_1, \dots, x_{k_n}) \leq \sum_{i=1}^{k_n} \alpha_{n,i} x_i \qquad \forall\, (x_1, \dots, x_{k_n}) \in \mathscr{D}^{k_n}. \tag{2.2}$$

(i) *If*

$$\beta_n \triangleq \prod_{j=1}^{n} \max_{1 \leq i \leq k_j} \alpha_{j,i} \to 0 \quad \text{as } n \to \infty \tag{2.3}$$

and

$$E\left(X_1^{(0)} \vee 0\right) < \infty, \tag{2.4}$$

then for some $\lambda \in \mathscr{D} \cup \{\xi\}$,

$$E(X_1^{(n)}) \downarrow \lambda \quad \text{as } n \to \infty \text{ and } X_1^{(n)} \to_P \lambda. \tag{2.5}$$

(ii) *If $\mathscr{D}$ is bounded below,*

$$\gamma_n^{-1} \triangleq \prod_{j=1}^{n} \sum_{i=1}^{k_j} \alpha_{j,i}^2 \leq c \cdot b^{-n}, \qquad n \geq 1, \tag{2.6}$$

for some constants $b > 1$ *and* $c > 0$, *and*

$$E\big(|X_1^{(0)}|\big(L(|X_1^{(0)}|)\big)^{\delta}\big) < \infty \tag{2.7}$$

for some $\delta > 1$ *then for some* $\lambda \in \mathscr{D}$, *both* (2.5) *and*

$$\limsup_{n\to\infty} X_1^{(n)} = \lambda \quad a.s. \tag{2.8}$$

hold.

(iii) *If* $\mathscr{D}$ *is bounded below, the* f_n, $n \ge 1$ *are symmetric functions of* $k_1 = k_2 = \cdots = k$ (≥ 2) *real variables* [*i.e., for each* $n \ge 1$ *and each permutation* $(i_1, \dots, i_k)$ *of* $(1, \dots, k)$, *we have* $f_n(x_{i_1}, \dots, x_{i_k}) = f(x_1, \dots, x_k)$] *and* $\mathscr{X}^{(0)} = \{X_j^{(0)}; j \ge 1\}$ *is a sequence of i.i.d. random variables such that*

$$E\big(|X_1^{(0)}|\big) < \infty, \tag{2.9}$$

then for some $\lambda \in \mathscr{D}$,

$$\lim_{n\to\infty} X_1^{(n)} = \lambda \quad a.s. \tag{2.10}$$

REMARK 2.1. Theorem 2.1(i) provides a general technique from which we can get a weak law of large numbers for $\{X_1^{(n)}; n \ge 0\}$ by checking conditions (2.2)–(2.4) [which in fact are weaker than corresponding conditions given in Wehr (1997)]. Theorem 2.1(ii) and (iii) are concerned with almost sure convergence or a strong law of large numbers for $\{X_1^{(n)}; n \ge 0\}$. Generally, the conditions in Theorem 2.1 are easy to check. We now demonstrate how conclusions (2.5), (2.8) and (2.10) can be drawn from Theorem 2.1 by considering the following two general examples. Further examples are provided in Section 4.

EXAMPLE 2.1. If, for each $n \ge 1$, f_n: $\mathscr{D}^k \to \mathscr{D}$ is real measurable function of k (≥ 2) real variables such that

$$f_n(x_1, \dots, x_k) \le \sum_{i=1}^{k} \alpha_i x_i \qquad \forall\,(x_1, \dots, x_k) \in \mathscr{D}^k \tag{2.11}$$

for some k nonnegative constants $\{\alpha_1, \dots, \alpha_k\}$ with

$$\sum_{i=1}^{k} \alpha_i = 1 \quad \text{and} \quad \max_{1\le i\le k} \alpha_i < 1, \tag{2.12}$$

it follows easily that

$$\beta_n = \Big(\max_{1\le i\le k} \alpha_i\Big)^n \to 0 \quad \text{as } n \to \infty$$

and

$$\gamma_n^{-1} = \left(\sum_{i=1}^{n} \alpha_i^2\right)^n = b^{-n} \quad \text{where } b = \left(\sum_{i=1}^{n} \alpha_i^2\right)^{-1} > 1;$$

that is, conditions (2.3) and (2.6) hold. Thus (2.5) follows if (2.4) holds, (2.8) follows provided $\mathscr{D}$ is bounded below and (2.7) holds, and (2.10) follows if the conditions in Theorem 2.1(iii) hold.

EXAMPLE 2.2. Let $f(x_1, \ldots, x_k)$: $\mathscr{D}^k \to \mathscr{D}$ be a real measurable function of k (≥ 2) real variables satisfying conditions (2.11) and (2.12). We now let $f_1 = f_2 = \cdots = f$. This is just a subcase of Example 2.1 which also includes the special case when $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1/k$ discussed by Wehr (1997).

REMARK 2.2. From the proof of Theorem 2.1(ii), we will find that, if condition (2.7) is replaced by (2.4) and the bounded-below condition of $\mathscr{D}$ is removed, then conclusion (2.8) still holds for some $\lambda \in \mathscr{D} \cup \{\xi\}$ when $\alpha_{n,i} = 1/k_n$, $1 \leq i \leq k_n$, $n \geq 1$.

REMARK 2.3. Let $\{W_n; n \geq 1\}$ be a sequence of independent random variables such that

$$P\left(W_n = 1 + \frac{1}{n}\right) = 1 - P(W_n = 0) = \frac{n}{n+1}, \qquad n \geq 1.$$

Then it follows that

$$E(W_n) \equiv 1, \qquad W_n \to_P 1$$

and

$$\limsup_{n\to\infty} W_n = 1 \quad \text{a.s.} \quad \text{and} \quad \liminf_{n\to\infty} W_n = 0 \quad \text{a.s.}$$

From this example, we can see that, under conditions of Theorem 2.1(ii), (2.5) and (2.8) do not imply that

$$\lim_{n\to\infty} X_1^{(n)} = \lambda \quad \text{a.s.}$$

REMARK 2.4. Let $f(x_1, \ldots, x_k)$: $\mathscr{D}^k \to \mathscr{D}$ be a real measurable symmetric function of k (≥ 2) real variables satisfying (2.11). We can assume without loss of generality that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k.$$

Given k real numbers $x_1, \ldots, x_k$, we arrange them in order as follows:

$$x_{k:1} \leq x_{k:2} \leq \cdots \leq x_{k:k}.$$

Because of the symmetry of f, we have

$$f(x_1, \ldots, x_k) \leq \sum_{i=1}^{k} \alpha_i x_{k:i} \qquad \forall (x_1, \ldots, x_k) \in \mathscr{D}^k.$$

By induction, it is easy to see that

$$\sum_{i=1}^{k} \alpha_i x_{k:i} \leq \frac{x_1 + \cdots + x_k}{k},$$

from which it follows that

$$f(x_1,\dots,x_k) \le \sum_{i=1}^{k} \alpha_i x_{k:i} \le \frac{x_1+\cdots+x_k}{k} \qquad \forall (x_1,\dots,x_k) \in \mathscr{D}^k. \tag{2.13}$$

REMARK 2.5. There is, of course, a theorem analogous to Theorem 2.1 for the case when, for each $n \ge 1$,

$$f_n(x_1,\dots,x_{k_n}) \ge \sum_{i=1}^{k_n} \alpha_{n,i} x_i \qquad \forall\, (x_1,\dots,x_{k_n}) \in \mathscr{D}^{k_n}. \tag{2.2$'$}$$

For example, if (2.2), (2.3) and

$$E(X_1^{(0)} \wedge 0) > -\infty \tag{2.4$'$}$$

hold, then for some $\lambda \in \mathscr{D} \cup \{\xi'\}$,

$$E\big(X_1^{(n)}\big) \uparrow \lambda \quad \text{as } n \to \infty \quad \text{and} \quad X_1^{(n)} \to_P \lambda, \tag{2.5$'$}$$

where $\xi' = \sup\{x; x \in \mathscr{D}\}$.

3. Proof of Theorem 2.1 We will need the following two general results for weighted sums of stationary and m-dependent random variables in order to prove Theorem 2.1.

PROPOSITION 3.1. *Let $\{X_n; n \ge 1\}$ be a sequence of stationary and m-dependent random variables and $\{a_{n,k};\, k \ge 1,\, n \ge 1\}$ an array of nonnegative real numbers such that*

$$\sum_{k\ge 1} a_{n,k} = 1, \qquad n \ge 1.$$

(a) *If*

$$\sup_{k\ge 1} a_{n,k} \to 0 \quad \textit{as } n \to \infty \tag{3.1}$$

and

$$E(X_1 \vee 0) < \infty, \tag{3.2}$$

then for each $n \ge 1$, $\sum_{k\ge1} a_{n,k}X_k$ is a well-defined $[-\infty,\infty)$-valued random variable with

$$E\Big(\sum_{k\ge1} a_{n,k}X_k\Big) = E(X_1)$$

and

$$\sum_{k\ge1} a_{n,k}X_k \to_P E(X_1). \tag{3.3}$$

(b) *If*

$$\zeta_n^{-1} \triangleq \sum_{k\geq 1} a_{n,k}^2 \leq c \cdot b^{-n}, \qquad n \geq 1 \tag{3.4}$$

for some constants $b > 1$ *and* $c > 0$, *and*

$$E\big(|X_1|\big(L(|X_1|)\big)^{\delta}\big) < \infty \tag{3.5}$$

for some $\delta > 1$, *then*

$$\sum_{n\geq 1} P\left(\left|\sum_{k\geq 1} a_{n,k} X_k - E(X_1)\right| \geq \varepsilon\right) < \infty \qquad \forall\, \varepsilon > 0 \tag{3.6}$$

and it follows that

$$\lim_{n\to\infty} \sum_{k\geq 1} a_{n,k} X_k = E(X_1) \quad a.s. \tag{3.7}$$

PROOF. For Proposition 3.1(a), we only need to prove its second part [i.e., (3.3)] because the first part is trivial. If $E(X_1 \wedge 0) > -\infty$, then, together with (3.2), we have that

$$E(|X_1|) < \infty.$$

Given $\varepsilon > 0$, we choose $\tau > 0$ such that

$$E\big(|X_1| I_{\{|X_1|\geq \tau\}}\big) \leq \frac{\varepsilon^2}{2}.$$

For each $n \geq 1$, set

$$Y_n(\tau) = X_n I_{\{|X_n|\leq \tau\}} - E\big(X_n I_{\{|X_n|\leq \tau\}}\big), \qquad Z_n(\tau) = X_n - E(X_1) - Y_n(\tau).$$

Since $\sum_{k\geq 1} a_{n,k} = 1$, $\forall\, n \geq 1$, there follows

$$\sum_{k\geq 1} a_{n,k} X_k - E(X_1) = \sum_{k\geq 1} a_{n,k} Y_k(\tau) + \sum_{k\geq 1} a_{n,k} Z_k(\tau).$$

Since $\{X_n; n \geq 1\}$ is a stationary sequence, we have that

$$\begin{aligned} P\left(\left|\sum_{k\geq 1} a_{n,k} Z_k(\tau)\right) \geq \varepsilon\right|\right) &\leq \frac{1}{\varepsilon} E\left(\left|\sum_{k\geq 1} a_{n,k} Z_k(\tau)\right|\right) \\ &\leq \frac{1}{\varepsilon} \sum_{k\geq 1} a_{n,k} E\big(|Z_k(\tau)|\big) \\ &\leq \frac{2}{\varepsilon} E\big(|X_1| I_{\{|X_1|>\tau\}}\big) \\ &\leq \varepsilon. \end{aligned} \tag{3.8}$$

Since $\{X_n; n \geq 1\}$ is a stationary and m-dependent sequence, using condition (3.1), we may conclude that

$$
\begin{aligned}
P\left(\left|\sum_{k\geq 1} a_{n,k} Y_k(\tau)\right| \geq \varepsilon\right) &\leq \frac{1}{\varepsilon^2} \operatorname{Var}\left(\sum_{k\geq 1} a_{n,k} Y_k(\tau)\right) \\
&= \frac{1}{\varepsilon^2}\left(\sum_{k\geq 1} a_{n,k}^2 \operatorname{Var}(Y_k(\tau))\right. \\
&\qquad \left. + 2 \sum_{1 \leq i-j \leq m} a_{n,i} a_{n,j} \operatorname{Cov}(Y_i(\tau), Y_j(\tau))\right) \\
&\leq \frac{1}{\varepsilon^2}(m+1) \sum_{k\geq 1} a_{n,k}^2 \operatorname{Var}(Y_k(\tau)) \qquad (3.9) \\
&\leq \frac{2(m+1)\tau^2}{\varepsilon^2} \sup_{k\geq 1} a_{n,k} \sum_{k\geq 1} a_{n,k} \\
&= \frac{2(m+1)\tau^2}{\varepsilon^2} \sup_{k\geq 1} a_{n,k} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Then (3.8) and (3.9) imply

$$
\limsup_{n\to\infty} P\left(\left|\sum_{k\geq 1} a_{n,k} X_k - E(X_1)\right| \geq 2\varepsilon\right) \leq \varepsilon,
$$

and since $\varepsilon > 0$ is arbitrary, (3.3) follows.

Now if $E(X_1 \wedge 0) = -\infty$, then together with (3.2), we have that

$$
\lim_{\tau\to\infty} E(X_1 I_{\{X_1 \geq -\tau\}}) = -\infty.
$$

Note that, for every $\tau > 0$,

$$
\sum_{k\geq 1} a_{n,k} X_k \leq \sum_{k\geq 1} a_{n,k} X_k I_{\{X_k \geq -\tau\}}
$$

and

$$
\sum_{k\geq 1} a_{n,k} X_k \leq \sum_{k\geq 1} a_{n,k} X_k I_{\{X_k \geq -\tau\}} \to_P \ E(X_1 I_{\{X_1 \geq -\tau\}}).
$$

Letting $\tau \to \infty$, (3.3) follows with $E(X_1) = -\infty$. This completes the proof of Proposition 3.1(a).

We now give the proof of Proposition 3.1(b). Note that, by Proposition 3.1(a),

$$
\sum_{k\geq 1} a_{n,k} X_k \to_P E(X_1).
$$

So, applying Lemma 2.1 of Li, Rao, Jiang and Wang (1995), we may assume that X_1 is symmetric and hence $E(X_1) = 0$. For each $n \geq 1$, set

$$
U_n = \sum_{k\geq 1} a_{n,k} X_k I_{\{|X_k| \leq \zeta_n\}}, \qquad V_n = \sum_{k\geq 1} a_{n,k} X_k I_{\{|X_k| > \zeta_n\}}.
$$

Since $\{X_n; n \geq 1\}$ is a stationary sequence, (3.4) and (3.5) imply, for every given $\varepsilon > 0$, that

$$
\begin{aligned}
\textstyle\sum_{n\geq 1} P\big(|V_n| \geq \varepsilon\big) &\leq \frac{1}{\varepsilon} \sum_{n\geq 1} E(|V_n|) \\
&\leq \frac{1}{\varepsilon} \sum_{n\geq 1} E\big(\big|X_1 I_{\{|X_1|>\zeta_n\}}\big|\big) \\
&\leq \frac{1}{\varepsilon} \sum_{n\geq 1} E\big(\big|X_1 I_{\{|X_1|>b^n/c\}}\big|\big) \\
&\leq \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \frac{b^{j+1}}{c} P\big(b^j/c < |X_1| \leq b^{j+1}/c\big) \\
&\leq \frac{1}{\varepsilon} \sum_{j\geq 1} \frac{jb^{j+1}}{c} P\big(b^j < |cX_1| \leq b^{j+1}\big) \\
&\leq \frac{b}{\varepsilon \ln b} E\big(|X_1| L(|cX_1|)\big) < \infty.
\end{aligned}
\tag{3.10}
$$

Using the same argument as in (3.9), conditions (3.4) and (3.5) also imply for every given $\varepsilon > 0$, that

$$
\begin{aligned}
\sum_{n\geq 1} P\big(|U_n| \geq \varepsilon\big) &\leq \frac{1}{\varepsilon^2} \sum_{n\geq 1} E(U_n^2) \\
&\leq \frac{m+1}{\varepsilon^2} \sum_{n\geq 1} \sum_{k\geq 1} a_{n,k}^2 E\big(X_1^2 I_{\{|X_1|\leq \zeta_n\}}\big) \\
&\leq \frac{m+1}{\varepsilon^2} \sum_{n\geq 1} \zeta_n^{-1} E\big(X_1^2 I_{\{|X_1|\leq \zeta_n\}}\big) \\
&= \frac{m+1}{\varepsilon^2} \sum_{n\geq 1} \zeta_n^{-1} \left(\zeta_n O\left(\frac{1}{(L(\zeta_n))^{\delta}} \right) E\big(|X_1| \big(L(|X_1|)\big)^{\delta}\big) \right) \\
&= \sum_{n\geq 1} O\left(\frac{1}{n^{\delta}} \right) < \infty.
\end{aligned}
\tag{3.11}
$$

Thus, from (3.10) and (3.11), (3.6) follows. This completes the proof of the proposition. $\square$

PROOF OF THEOREM 2.1(i). Set

$$
l_0 = 1 \quad \text{and} \quad l_n = \prod_{i=1}^{n} k_i, \qquad n \geq 1
$$

and

$$
g_n(x_1, \ldots, x_{k_n}) = \sum_{i=1}^{k_n} \alpha_{n,i} x_i, \qquad n \geq 1.
$$

For every $n \geq 1$ define

$$h_n(x_1, \dots, x_{l_n}) = g_n\big(h_{n-1}(x_1, \dots, x_{l_{n-1}}), h_{n-1}(x_{l_{n-1}+1}, \dots, x_{2l_{n-1}}), \dots, h_{n-1}(x_{(k_n-1)l_{n-1}+1}, \dots, x_{l_n})\big),$$

where $h_0(x) = x$. Clearly, for every $n \geq 1$,

$$h_n\left(x_1, \dots, x_{l_n}\right) = \sum_{i=1}^{l_n} d_{n,i} x_i, \qquad n \geq 1, \tag{3.12}$$

where $d_{n,j}$, $1 \leq j \leq l_n$ are l_n nonnegative constants such that

$$d_{n,j} = \alpha_{n,i} d_{n-1,r} \quad \text{if } j = (i-1)l_{n-1} + r, 1 \leq i \leq k_n \text{ and } 1 \leq r \leq l_{n-1}, \tag{3.13}$$

where $d_{0,1} = 1$. Using mathematical induction, we may prove that

$$\max_{1 \leq j \leq l_n} d_{n,j} = \prod_{j=1}^{n} \max_{1 \leq i \leq k_j} \alpha_{j,i} = \beta_n, \qquad n \geq 1 \tag{3.14}$$

and for every $s > 0$,

$$\sum_{j=1}^{l_n} d_{n,j}^s = \prod_{j=1}^{n} \sum_{i=1}^{k_j} \alpha_{j,i}^s, \qquad n \geq 1. \tag{3.15}$$

In particular,

$$\sum_{j=1}^{l_n} d_{n,j} = 1 \quad \text{and} \quad \sum_{j=1}^{l_n} d_{n,j}^2 = \prod_{j=1}^{n} \sum_{i=1}^{k_j} \alpha_{j,i}^2 = \gamma_n^{-1}, \qquad n \geq 1. \tag{3.16}$$

Clearly, conditions $\sum_{i=1}^{k_n} \alpha_{n,i} = 1$, $n \geq 1$, (2.2) and (2.4) imply that $E(X_1^{(n)})$, $n \geq 1$ form a nonincreasing sequence of $[-\infty, \infty)$-valued numbers. Thus there exists $\lambda \in [-\infty, \infty)$ such that

$$E(X_1^{(n)}) \downarrow \lambda \quad \text{as } n \to \infty. \tag{3.17}$$

Note that by (2.2) and the definitions of g_n, h_n, $n \geq 1$, it follows that

$$X_1^{(n)} \leq g_n\big(X_1^{(n-1)}, \dots, X_{k_n}^{(n-1)}\big) \leq \cdots \leq h_n\big(X_1^{(0)}, \dots, X_{l_n}^{(0)}\big), \qquad n \geq 1.$$

It follows from (3.12) that

$$X_1^{(n)} \leq \sum_{i=1}^{l_n} d_{n,i} X_i^{(0)}, \qquad n \geq 1. \tag{3.18}$$

Because of (3.14) and (3.16), by Proposition 2.1(a), (2.3) and (2.4) imply that

$$\sum_{i=1}^{l_n} d_{n,i} X_i^{(0)} \to_P E\big(X_1^{(0)}\big). \tag{3.19}$$

Consequently,

$$X_1^{(n)} \to_P -\infty \quad \text{if } E\big(X_1^{(0)}\big) = -\infty, \tag{3.20}$$

and, for $\forall\,\varepsilon > 0$,

$$\lim_{n\to\infty} P\Big(X_1^{(n)} \geq E\big(X_1^{(0)}\big) + \varepsilon\Big) = 0 \quad \text{if } E\big(|X_1^{(0)}|\big) < \infty. \tag{3.21}$$

When $E(|X_1^{(0)}|) < \infty$, it is easy to see that, for $\forall\,\varepsilon > 0$,

$$\lim_{n\to\infty} E\left(X_1^{(n)} I_{\{X_1^{(n)} \geq E(X_1^{(0)})+\varepsilon\}}\right) = 0. \tag{3.22}$$

In fact, note that

$$X_1^{(n)} I_{\{X_1^{(n)} \geq E(X_1^{(0)})+\varepsilon\}} \leq \sum_{i=1}^{l_n} d_{n,i} X_i^{(0)} I_{\{\sum_{i=1}^{l_n} d_{n,i} X_i^{(0)} \geq E(X_1^{(0)})+\varepsilon\}}, \qquad n \geq 1$$

and, for every $M > 0$,

$$\begin{aligned} &E\bigg(\sum_{i=1}^{l_n} d_{n,i} \big|X_i^{(0)}\big| I_{\{\sum_{i=1}^{l_n} d_{n,i} X_i^{(0)} \geq E(X_1^{(0)})+\varepsilon\}}\bigg) \\ &\qquad \leq MP\bigg(\sum_{i=1}^{l_n} d_{n,i} X_i^{(0)} \geq E(X_1^{(0)}) + \varepsilon\bigg) + E\left(\big|X_1^{(0)}\big| I_{\{|X_1^{(0)}| \geq M\}}\right). \end{aligned}$$

Thus (3.19) and $E(|X_1^{(0)}|) < \infty$ imply (3.22).

For any given integer $n_0 \geq 1$, (2.3) and (2.4) imply that

$$\prod_{j=1}^{n} \max_{1\leq i\leq k_{j+n_0}} \alpha_{j+n_0,\,i} \to 0 \quad \text{as } n\to\infty \quad \text{and} \quad E\big(X_1^{(n_0)} \vee 0\big) < \infty.$$

If we repeat the previous procedures and assume $E(X_1^{(n_0)}) = -\infty$, then

$$X_1^{(n)} \to_P -\infty. \tag{3.23}$$

If $E(|X_1^{(n_0)}|) < \infty$ then, for $\forall\,\varepsilon > 0$, there follows

$$\lim_{n\to\infty} P\big(X_1^{(n)} \geq E\big(X_1^{(n_0)}\big) + \varepsilon\big) = 0 \tag{3.24}$$

and

$$\lim_{n\to\infty} E\left(X_1^{(n)} I_{\{X_1^{(n)} \geq E(X_1^{(n_0)})+\varepsilon\}}\right) = 0. \tag{3.25}$$

Thus, by (3.17), if $\lambda = -\infty$, then

$$X_1^{(n)} \to_P -\infty, \tag{3.26}$$

and if $\lambda > -\infty$, then for $\forall\,\varepsilon > 0$,

$$\lim_{n\to\infty} P\big(X_1^{(n)} \geq \lambda + \varepsilon\big) = 0 \tag{3.27}$$

and

$$\lim_{n\to\infty} E\left(X_1^{(n)} I_{\{X_1^{(n)} \geq \lambda+\varepsilon\}}\right) = 0. \tag{3.28}$$

If $\lambda > -\infty$, then it is easy to show (3.17) and (3.28) imply that, for $\forall\, \varepsilon > 0$,

$$\lim_{n\to\infty} E\left(\left(X_1^{(n)} - \lambda\right) I_{\{X_1^{(n)} \le \lambda - \varepsilon\}}\right) = 0, \tag{3.29}$$

and, together with (3.27), there follows

$$X_1^{(n)} \to_P \lambda.$$

Finally, we prove $\lambda \in \mathscr{D} \cup \{\xi\}$. Since $X_1^{(n)} \to_P \lambda$, there exists a subsequence $\{n_j;\, j \ge 1\}$ of $\{1, 2, \dots, n, \dots\}$ such that

$$X_1^{(n_j)} \to \lambda \quad \text{a.s.}$$

Note that $\mathscr{D}$ is a closed subset of $(-\infty, \infty)$ and

$$P\left(\bigcap_{j=1}^{\infty} \left\{X_1^{(n_j)} \in \mathscr{D}\right\}\right) = 1.$$

So $\lambda \in \mathscr{D} \cup \{\xi\}$ follows. Theorem 2.1(i) is proved. $\square$

PROOF OF THEOREM 2.1(ii). Since the closed subset $\mathscr{D}$ of $(-\infty,\ \infty)$ is bounded below, it follows that $\mathscr{D} \cup \{\xi\} = \mathscr{D}$. By Theorem 2.1(i), (2.6) and (2.7) imply (2.5). Therefore there exists a subsequence $\{n_j;\ j \ge 1\}$ of $\{1, 2, \dots, n, \dots\}$ such that

$$\lim_{j\to\infty} X_1^{(n_j)} = \lambda \quad \text{a.s.}$$

so that

$$\limsup_{n\to\infty} X_1^{(n)} \ge \lambda \quad \text{a.s.} \tag{3.30}$$

By (3.18), (3.16) and (2.6), we have, for $\forall\ n \ge 1$,

$$X_1^{(n)} \le \sum_{i=1}^{l_n} d_{n,i} X_i^{(0)} \quad \text{and} \quad \sum_{i=1}^{l_n} d_{n,i}^2 \le c \cdot b^{-n}$$

for some constants $b > 1$ and $c > 0$. Thus by Proposition 3.1(b), (2.7) implies

$$\lim_{n\to\infty} \sum_{i=1}^{l_n} d_{n,i} X_i^{(0)} = E\left(X_1^{(0)}\right) \quad \text{a.s.}$$

and so

$$\limsup_{n\to\infty} X_1^{(n)} \le E\left(X_1^{(0)}\right) \quad \text{a.s.} \tag{3.31}$$

Note that, for any given $n_0 \ge 1$, the fact that $\mathscr{D}$ is bounded below along with (2.7) implies that

$$E\left(\left|X_1^{(n_0)}\right| \left(L\left(\left|X_1^{(n_0)}\right|\right)\right)^{\delta}\right) < \infty$$

and (2.6) implies

$$\prod_{j=1}^{n} \sum_{i=1}^{k_{j+n_0}} \alpha^2_{j+n_0,\,i} \le c_{n_0} b^{-n}, \qquad n \ge 1,$$

where $c_{n_0} = c \cdot b^{n_0} / \prod_{j=1}^{n_0} \sum_{i=1}^{k_j} \alpha^2_{j,\,i} > 0$. Using the same argument as in the proof of Theorem 2.1(i), applying Proposition 3.1(b) again, we conclude

$$\limsup_{n\to\infty} X_1^{(n)} \le E\big(X_1^{(n_0)}\big) \quad \text{a.s.} \tag{3.32}$$

Note that $E(X_1^{(n)}) \downarrow \lambda$. So letting $n_0 \to \infty$, we have finally

$$\limsup_{n\to\infty} X_1^{(n)} \le \lambda \quad \text{a.s.} \tag{3.33}$$

and this, together with (3.30), implies (2.8). □

PROOF OF THEOREM 2.1(iii). By Remark 2.4, (2.13), the sequence $f_n(x_1, \dots, x_k)$: $\mathscr{D}^k \to \mathscr{D}$, $n \ge 1$ of real measurable symmetric functions with (2.2) and $k_n = k$, $n \ge 1$ must satisfy the condition

$$f_n(x_1, \dots, x_k) \le \frac{x_1 + \cdots + x_k}{k} \qquad \forall (x_1, \dots, x_k) \in \mathscr{D}^k.$$

Note that $\mathscr{D}$ is bounded below, so applying Theorem 2.1(i), (2.9) implies that there exists $\lambda \in \mathscr{D}$ such that

$$X_1^{(n)} \to_P \lambda. \tag{3.34}$$

Further, for every given integer $n_0 \ge 0$, by the same argument as in the proof of Theorem 2.1(i), it is easy to see that

$$X_1^{(n+n_0)} \le \frac{\sum_{i=1}^{k^n} X_i^{(n_0)}}{k^n}, \qquad n \ge 1.$$

From the strong law of large numbers, we then have

$$\limsup_{n\to\infty} X_1^{(n)} \le E\big(X_1^{(n_0)}\big) \quad \text{a.s.} \tag{3.35}$$

Letting $n_0 \to \infty$, together with (3.34), it follows that

$$\limsup_{n\to\infty} X_1^{(n)} = \lambda \quad \text{a.s.} \tag{3.36}$$

Similarly, for every $l > 1$, we have

$$X_l^{(n)} \to_P \lambda \quad \text{and} \quad \limsup_{n\to\infty} X_l^{(n)} = \lambda \quad \text{a.s.} \tag{3.37}$$

We may now prove that the sequence $\{X_1^{(n)} + \cdots + X_k^{(n)};\ n \ge 0\}$ converges almost surely to constant $k\lambda$. Define

$$\mathscr{F}_n = \sigma\big(X_1^{(l)} + \cdots + X_k^{(l)}; l \ge n\big), \qquad n \ge 0.$$

Obviously, the $\mathscr{F}_n$, $n \geq 0$ form a decreasing sequence of σ-algebras and for each n, $X_1^{(n)} + \cdots + X_k^{(n)}$ is $\mathscr{F}_n$-measurable. Furthermore, by the symmetry of $f_n(x_1, \ldots, x_k)$, $(x_1, \ldots, x_k) \in \mathscr{D}^k$, $n \geq 1$, we have

$$E\big(X_{(i-1)k+1}^{(n)} + \cdots + X_{ik}^{(n)} \,\big|\, \mathscr{F}_{n+1}\big) = E\big(X_1^{(n)} + \cdots + X_k^{(n)} \,\big|\, \mathscr{F}_{n+1}\big), \qquad 1 \leq i \leq k.$$

It then follows that

$$\begin{aligned} E\big(X_1^{(n)} + \cdots + X_k^{(n)} \,\big|\, \mathscr{F}_{n+1}\big) &= \frac{E(X_1^{(n)} + X_2^{(n)} + \cdots + X_{k^2}^{(n)} \,|\, \mathscr{F}_{n+1})}{k} \\ &\geq E\big(X_1^{(n+1)} + \cdots + X_k^{(n+1)} \,\big|\, \mathscr{F}_{n+1}\big) \\ &\geq X_1^{(n+1)} + \cdots + X_k^{(n+1)} \quad \text{a.s.} \end{aligned} \tag{3.38}$$

We have shown that the sequence $\{X_1^{(n)} + \cdots + X_k^{(n)}; n \geq 1\}$ is a reversed-time submartingale relative to the family $\{\mathscr{F}_n; n \geq 0\}$ of σ-algebras [see Chow and Teicher (1988) for the definition and fundamental theorems about reversed-time submartingales]. Since $X_1^{(n)} + \cdots + X_k^{(n)} - k\xi \geq 0$, the convergence theorem for nonnegative reversed-time submartingales implies that

$$\lim_{n\to\infty} \big(X_1^{(n)} + \cdots + X_k^{(n)} - k\xi\big) = Z \quad \text{a.s.}$$

for some random variable Z. Note that (3.34) and (3.37) imply that

$$X_1^{(n)} + \cdots + X_k^{(n)} \to_P k\lambda.$$

So $Z = k(\lambda - \xi)$ almost surely and it follows that

$$\lim_{n\to\infty} \big(X_1^{(n)} + \cdots + X_k^{(n)}\big) = k\lambda \quad \text{a.s.} \tag{3.39}$$

Hence, combining (3.37) and (3.39), we have that

$$\begin{aligned} k\lambda &= \liminf_{n\to\infty} \big(X_1^{(n)} + \cdots + X_k^{(n)}\big) \\ &\leq \liminf_{n\to\infty} X_1^{(n)} + \limsup_{n\to\infty} X_2^{(n)} + \cdots + \limsup_{n\to\infty} X_k^{(n)} \\ &\leq \liminf_{n\to\infty} X_1^{(n)} + (k-1)\lambda \quad \text{a.s.} \end{aligned} \tag{3.40}$$

and hence that

$$\liminf_{n\to\infty} X_1^{(n)} \geq \lambda \quad \text{a.s.}$$

which, together with (3.36), (2.10) follows and thus Theorem 2.1(iii) has been established. □

As a by-product of the proof of Theorem 2.1(iii), we can strengthen Theorem 2.1(iii) to the extent of providing a convergence rate.

COROLLARY 3.1. *Suppose that all conditions for Theorem* 2.1(iii) *are satisfied. Then* $X_1^{(n)}$, $n \geq 0$ *converges completely to* λ, *that is*,

$$\sum_{n=0}^{\infty} P\big(\big|X_1^{(n)} - \lambda\big| \geq \varepsilon\big) < \infty \qquad \forall\, \varepsilon > 0. \tag{3.41}$$

PROOF. Note that, for each $n \geq 1$, $X_2^{(n)}$ is determined by random variables $\{X_j^{(0)};\ k^n + 1 \leq j \leq 2k^n\}$ and $k^n + 1 > 2k^{n-1}$. So $\{X_j^{(0)};\ k^n + 1 \leq j \leq 2k^n\}$, $n \geq 0$ are independent and it follows that $\{X_2^{(n)};\ n \geq 0\}$ is a sequence of independent random variables. From the proof of Theorem 2.1(iii), we also have that $X_2^{(n)} \to \lambda$ almost surely as $n \to \infty$. Applying the Borel–Cantelli lemma, $X_2^{(n)}$, $n \geq 0$ converges completely to λ. That is,

$$\sum_{n=0}^{\infty} P\big(\big|X_2^{(n)} - \lambda\big| \geq \varepsilon\big) < \infty \qquad \forall\, \varepsilon > 0.$$

Since, for each $n \geq 0$, $X_1^{(n)}$ and $X_2^{(n)}$ are i.i.d. random variables, (3.41) follows. □

4. Examples, comments and applications. Theorem 2.1 can be applied to a very wide class of iterated functions. In this section we would like to provide some examples, comments, open problems and applications related to Theorem 2.1. Obviously, our results can be applied to the several situations discussed in Wehr (1997) such as *random resistor networks, durability of hierarchical fibers, the biased coin problem* and so on. The following subsections which include many interesting situations are just the application of Theorem 2.1 in this connection.

4.1. *Linear situation.* We consider linear functions

$$f_n(x_1, \ldots, x_{k_n}) = \sum_{i=1}^{k_n} \alpha_{n,i} x_i, \qquad n \geq 1, \tag{4.1}$$

where

$$\alpha_{n,i} \geq 0, 1 \leq i \leq k_n \quad \text{and} \quad \sum_{i=1}^{k_n} \alpha_{n,i} = 1, n \geq 1.$$

Then by (3.12) we have

$$X_1^{(n)} = \sum_{i=1}^{l_n} d_{n,i} X_i^{(0)}, \qquad n \geq 1, \tag{4.2}$$

where, for every $n \ge 1$, $l_n = \prod_{i=1}^n k_i$ and $d_{n,j}$, $1 \le j \le l_n$ are l_n nonnegative constants satisfying (3.13)–(3.16). Thus, applying Theorem 2.1, we have:

1. Under (2.3) and (2.4),

$$E(X_1^{(n)}) \equiv E(X_1^{(0)}) \quad \text{and} \quad X_1^{(n)} \to_P E(X_1^{(0)}). \tag{4.3}$$

2. Under (2.6) and (2.7),

$$\lim_{n\to\infty} X_1^{(n)} = E(X_1^{(0)}) \quad \text{a.s.} \tag{4.4}$$

   In particular, if $\alpha_{n,i} = 1/k_n$, $1 \le i \le k_n$, $n \ge 1$, then (4.4) also holds given (2.4).

PROPOSITION 4.1. *Suppose further that* $\mathscr{X}^{(0)} = \{X_j^{(0)};\ j \ge 1\}$ *is a sequence of i.i.d. random variables and* $\{X_1^{(n)};\ n \ge 0\}$ *is obtained by the iterated linear functions given in* (4.1). *Let*

$$\beta_n = \prod_{j=1}^n \max_{1\le i\le k_j} \alpha_{j,i} \quad \text{and} \quad \gamma_n = \prod_{j=1}^n \sum_{i=1}^{k_j} \alpha_{j,i}^2, \qquad n \ge 1. \tag{4.5}$$

If

$$\frac{\beta_n}{\sqrt{\gamma_n}} = \prod_{j=1}^n \frac{\max_{1\le i\le k_j} \alpha_{j,i}}{\sqrt{\sum_{i=1}^{k_j} \alpha_{j,i}^2}} \to 0 \quad \text{as } n\to\infty \tag{4.6}$$

and

$$E\big((X_1^{(0)})^2\big) < \infty, \tag{4.7}$$

then

$$\frac{X_1^{(n)} - E(X_1^{(0)})}{\sigma\sqrt{\gamma_n}} \to_D N(0,1), \tag{4.8}$$

where $\sigma^2 = \operatorname{Var}(X_1^{(0)}) > 0$, $\to_D$ *denotes convergence in distribution, and* $N(0,1)$ *stands for the standard normal distribution.*

PROOF. Using (4.2), we have

$$\frac{X_1^{(n)} - E(X_1^{(0)})}{\sigma\sqrt{\gamma_n}} = \sum_{i=1}^{l_n} b_{n,i} Z_i, \qquad n \ge 1, \tag{4.9}$$

where

$$b_{n,i} = \frac{d_{n,i}}{\sqrt{\gamma_n}} \quad \text{and} \quad Z_n = \frac{X_n^{(0)} - E(X_1^{(0)})}{\sigma}, \qquad 1 \le i \le l_n,\ n \ge 1.$$

Obviously $\{Z_n;\, n \ge 1\}$ is a sequence of i.i.d. random variables with mean zero and variance one and from (3.14)–(3.16), together with (4.6), we have

$$\max_{1\le i\le l_n} b_{n,i} = \frac{\beta_n}{\sqrt{\gamma_n}} \to 0 \quad \text{as} \quad n\to\infty \quad \text{and} \quad \sum_{i=1}^{l_n} b_{n,i}^2 = 1, \qquad n \ge 1. \tag{4.10}$$

Let $\varphi(\cdot)$ be the characteristic function of Z_1. Then, from the Taylor expansion at $t = 0$, there exists a constant $\delta > 0$ such that

$$\varphi(t) = \exp\left\{-\frac{t^2}{2}(1+\epsilon(t))\right\}, \qquad |t| \le \delta, \tag{4.11}$$

where $\epsilon(t)$, $|t| \le \delta$ is a complex-valued function defined on $[-\delta, \delta]$, depending on $\varphi(\cdot)$ with $\lim_{t\to 0}\epsilon(t) = 0$. Let $\varphi_n(\cdot)$ be the characteristic function of $\big(X_1^{(n)} - E(X_1^{(0)})\big)/\sigma\sqrt{\gamma_n}$, $n \ge 1$. Then, for every given t and for all sufficiently large values of n, (4.9)–(4.11) imply

$$\varphi_n(t) = \prod_{i=1}^{l_n} \exp\left\{-\frac{b_{n,i}^2 t^2}{2}\big(1+\epsilon(b_{n,i}t)\big)\right\} = \exp\left\{-\frac{t^2}{2}\big(1+o(1)\big)\right\}.$$

Consequently, we have

$$\lim_{n\to\infty} \varphi_n(t) = \exp\left\{-\frac{t^2}{2}\right\}.$$

Finally applying the inverse limit theorem for characteristic functions [cf., e.g., Chow and Teicher (1988)], (4.8) follows. □

We can also give an analogue of the law of the iterated logarithm for the $\{X_1^{(n)};\, n \ge 0\}$ just discussed in Proposition 4.1. For example, if $k_1 = k_2 = \cdots = k\ (\ge 2)$ and

$$f_n(x_1, \dots, x_k) = \sum_{i=1}^{k} \alpha_i x_i, \qquad n \ge 1$$

for some k nonnegative constants $\{\alpha_1, \dots, \alpha_k\}$ with

$$\sum_{i=1}^{k} \alpha_i = 1 \quad \text{and} \quad \max_{1\le i\le k} \alpha_i < 1,$$

then clearly,

$$\frac{\beta_n}{\sqrt{\gamma_n}} = \left(\frac{\max_{1\le i\le k}\alpha_i}{\sqrt{\sum_{i=1}^{k}\alpha_i^2}}\right)^n \to 0 \quad \text{as } n\to\infty.$$

Using Proposition 4.1, (4.7) implies

$$\frac{X_1^{(n)} - E(X_1^{(0)})}{\sigma(\sum_{i=1}^{k}\alpha_i^2)^{n/2}} \to_D N(0,1). \tag{4.12}$$

A law of the iterated logarithm-type result for $\{X_1^{(n)};\ n \geq 0\}$ is then

$$\limsup_{n\to\infty} \frac{X_1^{(n)} - E(X_1^{(0)})}{\sigma(\sum_{i=1}^k \alpha_i^2)^{n/2}\sqrt{2\ln n}} = -\liminf_{n\to\infty} \frac{X_1^{(n)} - E(X_1^{(0)})}{\sigma(\sum_{i=1}^k \alpha_i^2)^{n/2}\sqrt{2\ln n}} = 1 \quad \text{a.s.} \tag{4.13}$$

The proof of (4.13) is left to the reader. □

4.2. *L-statistics.* From Remark 2.4, it seems that there exists an interesting relationship between the special type of hierarchical models and *linear combinations of order statistics* (in short, L-statistics).

Let $\{k_n \geq 2;\ n \geq 1\}$ be a sequence of positive integers. Given k_n real numbers $x_1, \dots, x_{k_n}$, we arrange them in order,

$$x_{k_n:1} \leq x_{k_n:2} \leq \cdots \leq x_{k_n:k_n}, \qquad n \geq 1.$$

We now consider the following symmetric functions:

$$f_n(x_1, \dots, x_{k_n}) = \sum_{i=1}^{k_n} \alpha_{n,i} x_{k_n:i}, \qquad n \geq 1, \tag{4.14}$$

where

$$\alpha_{n,j} \geq 0, 1 \leq j \leq l_n \quad \text{and} \quad \sum_{i=1}^{k_n} \alpha_{n,i} = 1, \qquad n \geq 1.$$

If further, $\alpha_{n,1} \geq \alpha_{n,2} \geq \cdots \geq \alpha_{n,k_n} \geq 0$, $n \geq 1$ then from (2.13), we have

$$f_n(x_1, \dots, x_{k_n}) \leq \frac{x_1 + x_2 + \cdots + x_{k_n}}{k_n}, \qquad n \geq 1. \tag{4.15}$$

Let the sequence $\mathscr{X}^{(n)} = \{X_j^{(n)};\ j \geq 1\}$, $n \geq 1$ be associated with some initial sequence $\mathscr{X}^{(0)} = \{X_j^{(0)};\ j \geq 1\}$ of stationary and m-dependent random variables and the sequence of iterated functions defined in (4.14). Then clearly for every $n \geq 1$, $X_1^{(n)}$ is an L-statistic of $X_1^{(n-1)}, \dots, X_{k_n}^{(n-1)}$. Applying Theorem 2.1 under condition (2.4), there exists a $\lambda \in [-\infty, \infty)$ such that

$$E(X_1^{(n)}) \downarrow \lambda \text{ as } n \to \infty,\ X_1^{(n)} \to_P \lambda \quad \text{and} \quad \limsup_{n\to\infty} X_1^{(n)} = \lambda \quad \text{a.s.} \tag{4.16}$$

If conditions $\alpha_{n,1} \geq \alpha_{n,2} \geq \cdots \geq \alpha_{n,k_n} \geq 0$, $n \geq 1$ and (2.4) are replaced by conditions $0 \leq \alpha_{n,1} \leq \alpha_{n,2} \leq \cdots \leq \alpha_{n,k_n}$, $n \geq 1$ and $E(X_1^{(0)} \wedge 0) > -\infty$, then for some $\lambda \in (-\infty, \infty]$,

$$E(X_1^{(n)}) \uparrow \lambda \text{ as } n \to \infty,\ X_1^{(n)} \to_P \lambda \quad \text{and} \quad \liminf_{n\to\infty} X_1^{(n)} = \lambda \quad \text{a.s.} \tag{4.17}$$

If $k_n = k$ (≥ 2), $n \geq 1$ and $\mathscr{X}^{(0)}$ is a sequence of i.i.d. random variables, then, also under (2.4), we conclude

$$\lim_{n\to\infty} X_1^{(n)} = \lambda \quad \text{a.s.} \tag{4.18}$$

We now assume that $\mathscr{X}^{(0)} = \{X_j^{(0)};\ j \geq 1\}$ is a sequence of i.i.d. random variables with common distribution function $F(x)$, $x \in (-\infty, \infty)$ and discuss a few special cases as follows.

CASE I. If $\alpha_{n,1} = \cdots = \alpha_{n,k_n} = 1/k_n$, $n \geq 1$, then

$$f_n(x_1, \ldots, x_{k_n}) = \frac{x_1 + \cdots + x_{k_n}}{k_n}, \qquad n \geq 1$$

and so

$$X_1^{(n)} = \frac{\sum_{i=1}^{l_n} X_i^{(0)}}{l_n}, \qquad n \geq 1. \tag{4.19}$$

This is just a special case discussed in Section 4.1 and, from (4.5) and (4.8), we have

$$\frac{\sqrt{l_n}}{\sigma}\big(X_1^{(n)} - E(X_1^{(0)})\big) \to_D N(0,1) \tag{4.20}$$

provided condition (4.7). That is, under (4.7), suitably normalized random variables $X_1^{(n)} - E(X_1^{(0)})$, $n \geq 0$ converge to the standard normal distribution.

CASE II. If $\alpha_{n,1} = \cdots = \alpha_{n,k_n-1} = 0$ and $\alpha_{n,k_n} = 1$, $n \geq 1$, then

$$f_n(x_1, \ldots, x_{k_n}) = x_{k_n : k_n} \geq \frac{x_1 + \cdots + x_{k_n}}{k_n}, \qquad n \geq 1.$$

For this case we have

$$X_1^{(n)} = \max_{1 \leq j \leq l_n} X_j^{(0)}, \qquad n \geq 1, \tag{4.21}$$

which is a subsequence of $\{\max_{1 \leq j \leq n} X_j^{(0)};\ n \geq 1\}$, the sequence of extreme (maximum) values of $\{X_j^{(0)};\ j \geq 1\}$. Thus the asymptotic behavior for $\{X_1^{(n)};\ n \geq 0\}$ is found to be related to extreme value theory.

Extreme value theory is an elegant and mathematically fascinating theory as well as a subject which pervades a wide variety of applications. See in particular the books by Galambos (1978), Leadbetter, Lindgren and Rootzen (1983) and Resnick (1987).

For any sequence $\mathscr{X}^{(0)} = \{X_j^{(0)};\ j \geq 1\}$ of i.i.d. random variables, we have

$$\lim_{n\to\infty} X_1^{(n)} = \lim_{n\to\infty} \max_{1 \leq j \leq l_n} X_j^{(0)} = \sup\Big\{x;\ P\big(X_1^{(0)} \leq x\big) < 1\Big\} \quad \text{a.s.} \tag{4.22}$$

Similarly, if $f_n(x_1, \ldots, x_{k_n}) = x_{k_n:1}$, $n \geq 1$, then

$$\lim_{n\to\infty} X_1^{(n)} = \lim_{n\to\infty} \min_{1 \leq j \leq l_n} X_j^{(0)} = \inf\{x;\ P(X_1^{(0)} \leq x) > 0\} \quad \text{a.s.} \tag{4.23}$$

Thus condition (2.4) in Theorem 2.1 is not necessary for (2.5) to hold.

On the other hand, unlike Case I, the suitably normalized random variables $X_1^{(n)} - E(X_1^{(0)})$, $n \geq 0$ converge to a non-Gaussian distribution.

Let $\tau > 0$. Write

$$\Phi_\tau(x) = \begin{cases} 0, & \text{if } x < 0, \\ \exp\{-x^{-\tau}\}, & \text{if } x \geq 0, \end{cases}$$

$$\Psi_\tau(x) = \begin{cases} \exp\{-(-x)^{\tau}\}, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$$

$$\Lambda(x) = \exp\{-e^{-x}\}, \qquad x \in (-\infty, \infty).$$

The functions $\Phi_\tau(\cdot)$, $\Psi_\tau(\cdot)$ and $\Lambda(\cdot)$ are referred to as the *extreme value distributions* [see, e.g., Resnick (1987), page 9].

Gnedenko (1943), de Haan (1970, 1976) and Weissman (1975) establish the fundamental result for extreme value theory as follows. Suppose there exists $a_n > 0$, $b_n \in (-\infty, \infty)$, $n \geq 1$ such that

$$\frac{\max_{1\leq j\leq n} X_j^{(0)} - b_n}{a_n} \to_D G(\cdot), \tag{4.24}$$

where $G(\cdot)$ is assumed nondegenerate. Then $G(\cdot)$ is of the type of one of the three classes $\Phi_\tau(\cdot)$, $\Psi_\tau(\cdot)$ and $\Lambda(\cdot)$ defined above. We say $F(\cdot)$ is in the *domain of attraction* of $G(\cdot)$ [and write $F(\cdot) \in \mathbf{D}(G)$] if (4.24) holds. Gnedenko (1943), de Haan (1970, 1976), Weissman (1975) also obtain necessary and sufficient conditions for $F(\cdot) \in \mathbf{D}(\Phi_\tau(\cdot))$, $F(\cdot) \in \mathbf{D}(\Psi_\tau(\cdot))$ and $F(\cdot) \in \mathbf{D}(\Lambda(\cdot))$, respectively. These results are summarized in Resnick (1987).

We now can state a result for the asymptotic distribution of sequence of random variables defined by (4.21).

PROPOSITION 4.2. *If* (4.24) *holds*, *then*

$$\frac{X_1^{(n)} - b_{l_n}}{a_{l_n}} = \frac{\max_{1\leq j\leq l_n} X_j^{(0)} - b_{l_n}}{a_{l_n}} \to_D G(\cdot). \tag{4.25}$$

However, *the converse is not true.*

COUNTEREXAMPLE. Consider distribution function

$$F(x) = \begin{cases} 1 - 2^{-[x]}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

and iterated functions $f_n(x_1, x_2) = x_{2:2}$, $n \geq 1$. Then

$$X_1^{(n)} = \max_{1\leq j\leq 2^n} X_j^{(0)}, \qquad n \geq 1.$$

Then we have

$$\begin{aligned}\lim_{n\to\infty} P\big(X_1^{(n)} - n \le x\big) &= P\big(X_{2^n:2^n} - n \le x\big)\\ &= \lim_{n\to\infty}\big(1 - 2^{-[x+n]}\big)^{2^n}\\ &= \lim_{n\to\infty}(1 - 2^{-[x]-n})^{2^n}\\ &= \exp\{-2^{-[x]}\}.\end{aligned}$$

That is,

$$X_1^{(n)} - n \to_D \exp\{-2^{-[x]}\}.$$

Clearly, $\exp\{-2^{-[x]}\}$ does not belong to any of three classes $\Phi_\tau(\cdot)$, $\Psi_\tau(\cdot)$ and $\Lambda(\cdot)$ defined above. Thus (4.24) does not hold for any $a_n, b_n \in (-\infty, \infty)$, $n \ge 1$.

Note that $\min_{1\le j\le l_n} X_j^{(0)} = -\max_{1\le j\le l_n}(-X_j^{(0)})$, $n \ge 1$. We thus can state a result analogous to Proposition 4.2 for the case when, for each $n \ge 1$, $f_n(x_1, \dots, x_{k_n}) = x_{k_n:1}$.

CASE III. Suppose that the sequence $\{X_1^{(n)};\ n \ge 1\}$ of random variables is obtained by iterated functions,

$$f_n(x_1, x_2) = \alpha_{n,1}x_{2:1} + \alpha_{n,2}x_{2:2}, \qquad n \ge 1,$$

where $\alpha_{n,1} \ge \alpha_{n,2}$, $n \ge 1$. Then, under (2.4), (4.18) holds. The special case where $\alpha_{n,1} = 1 - \varepsilon$, $\alpha_{n,2} = \varepsilon$, $n \ge 1$ with $0 \le \varepsilon \le 1/2$ was considered by Boppana and Narayan (1993) in relation to the so-called *biased coin problem*.

CASE IV. Consider the case when, for each $n \ge 1$, $k_n = 3$, $\alpha_{n,1} = \alpha_{n,3} = 0$ and $\alpha_{n,2} = 1$. Then

$$f_n(x_1, x_2, x_3) = x_{3:2}, \qquad n \ge 1.$$

For this case Theorem 2.1 cannot be applied because condition (2.2) is not satisfied with either $\le$ or $\ge$.

Let the sequence $\mathscr{X}^{(n)} = \{X_j^{(n)};\ j \ge 1t\}$, $n \ge 1$ be associated with some initial sequence $\mathscr{X}^{(0)} = \{X_j^{(0)};\ j \ge 1\}$ of i.i.d. random variables and the sequence of iterated functions $f_n(x_1, x_2, x_3) = x_{3:2}$, $n \ge 1$. For each $n \ge 1$, let $F_n(\cdot)$ be the common distribution function of $\mathscr{X}^{(n)} = \{X_j^{(n)};\ j \ge 1\}$ and define

$$\lambda_1 = \sup\{x; F(x) < \tfrac{1}{2}\} \quad \text{and} \quad \lambda_2 = \inf\{x;\ F(x) > \tfrac{1}{2}\}.$$

Then $\lambda_1 \le \lambda_2$, for every $x \in [\lambda_1, \lambda_2]$, x is a median of random variable $X_1^{(0)}$ and

$$P\big(\lambda_1 < X_1^{(0)} < \lambda_2\big) = 0.$$

PROPOSITION 4.3. (a) *If* $\lambda_1 = \lambda_2$, *then*

$$\lim_{n\to\infty} X_1^{(n)} = \lambda_1 \quad a.s. \tag{4.26}$$

(b) *If* $\lambda_1 < \lambda_2$, *then*

$$X_1^{(n)} \to_D H_1(x) = \begin{cases} 0, & \textit{if } x < \lambda_1, \\ \frac{1}{2}, & \textit{if } \lambda_1 \le x < \lambda_2, \\ 1, & \textit{if } x \ge \lambda_2. \end{cases} \tag{4.27}$$

PROOF. For each $n \ge 1$, we arrange random variables $X_1^{(n-1)}$, $X_2^{(n-1)}$ and $X_3^{(n-1)}$ in order as follows:

$$X_{3:1}^{(n-1)} \le X_{3:2}^{(n-1)} \le X_{3:3}^{(n-1)}.$$

Then $X_1^{(n)} = X_{3:2}^{(n-1)}$, $n \ge 1$. Then we have

$$\begin{aligned} F_n(x) &= P\big(X_1^{(n)} \le x\big) = P\big(X_{3:2}^{(n-1)} \le x\big) \\ &= F_{n-1}^3(x) + 3F_{n-1}^2(x)(1 - F_{n-1}(x)) \\ &= F_{n-1}^2(x)(3 - 2F_{n-1}(x)). \end{aligned} \tag{4.28}$$

Similarly,

$$1 - F_n(x) = P\big(X_1^{(n)} > x\big) = \big(1 - F_n(x)\big)^2\big(3 - 2(1 - F_{n-1}(x))\big). \tag{4.29}$$

For every $\varepsilon > 0$ let

$$p(\varepsilon) = F(\lambda_1 - \varepsilon)(3 - 2F(\lambda_1 - \varepsilon))$$

and

$$q(\varepsilon) = (1 - F(\lambda_2 + \varepsilon))\big(3 - 2(1 - F(\lambda_2 + \varepsilon))\big).$$

Clearly, $p(\varepsilon), q(\varepsilon) \in [0, 1)$. Using mathematical induction, (4.28) and (4.29), we can show that, for $\varepsilon > 0$,

$$\begin{aligned} F_n(\lambda_1 - \varepsilon) &\le p^n(\varepsilon)F(\lambda_1 - \varepsilon) \quad \text{and} \\ 1 - F_n(\lambda_2 + \varepsilon) &\le q^n(\varepsilon)(1 - F(\lambda_2 + \varepsilon)), \end{aligned} \tag{4.30}$$

which implies

$$\sum_{n=1}^{\infty} P\big(X_1^{(n)} \le \lambda_1 - \varepsilon\big) \le F(\lambda_1 - \varepsilon) \sum_{n=1}^{\infty} p^n(\varepsilon) < \infty \tag{4.31}$$

and

$$\sum_{n=1}^{\infty} P\big(X_1^{(n)} > \lambda_2 + \varepsilon\big) \le \big(1 - F(\lambda_2 + \varepsilon)\big) \sum_{n=1}^{\infty} q^n(\varepsilon) < \infty. \tag{4.32}$$

Applying the Borel–Cantelli lemma, Proposition 4.3(a) follows from (4.31) and (4.32).

As for Proposition 4.3(b), note that, for each $n \ge 1$,

$$F_n(x) = \tfrac{1}{2} \quad \text{if } \lambda_1 < x < \lambda_2.$$

So (4.31) and (4.32) also imply (4.27). This completes the proof of Proposition 4.3. $\square$

Similarly, one can consider the case when, for each $n \ge 1$, $k_n = 4$, $\alpha_{n,1} = \alpha_{n,3} = \alpha_{n,4} = 0$ and $\alpha_{n,2} = 1$. Then

$$f_n(x_1, x_2, x_3, x_4) = x_{4:2}, \qquad n \ge 1.$$

Let $\{X_1^{(n)};\, n \ge 1\}$ be associated with some initial $\mathscr{X}^{(0)} = \{X_j^{(0)};\, j \ge 1\}$ of i.i.d. random variables and the sequence of iterated functions $f_n(x_1, x_2, x_3, x_4) = x_{4:2}$, $n \ge 1$. Define

$$\lambda_3 = \sup\left\{x;\, F(x) < \frac{5-\sqrt{13}}{6}\right\} \quad \text{and} \quad \lambda_4 = \inf\left\{x;\, F(x) > \frac{1+\sqrt{13}}{6}\right\}.$$

We have (1) if $\lambda_3 = \lambda_4$, then

$$\lim_{n\to\infty} X_1^{(n)} = \lambda_3 \quad \text{a.s.}$$

and (2) if $\lambda_3 \ne \lambda_4$, then

$$X_1^{(n)} \to_D H_2(x) = \begin{cases} 0, & \text{if } x < \lambda_3, \\ \dfrac{5-\sqrt{13}}{6}, & \text{if } \lambda_3 \le x < \lambda_4, \\ 1, & \text{if } x \ge \lambda_4. \end{cases}$$

We leave the proof of this result to the reader. One may compare this result with Proposition 4.3.

4.3. *Random resistor networks.* Similarly to Wehr (1997), our Theorem 2.1 applies to several hierarchical networks of random resistors. We now consider the so-called *weighted diamond network* as follows:

$$\begin{aligned} f_n(x_1, x_2, x_3, x_4) &= \left(\frac{1}{w_{n,1}x_{j_{n,1}}} + \frac{1}{w_{n,2}x_{j_{n,2}}}\right)^{-1} \\ &\quad + \left(\frac{1}{w_{n,3}x_{j_{n,3}}} + \frac{1}{w_{n,4}x_{j_{n,4}}}\right)^{-1}, \qquad n \ge 1, \end{aligned} \tag{4.33}$$

where for each $n \ge 1$, $\{j_{n,1},\ j_{n,2},\ j_{n,3},\ j_{n,4}\}$ is a permutation of $\{1, 2, 3, 4\}$, x_i, $w_{j_{n,i}}$, $1 \le i \le 4$ are nonegative (and $0^{-1} = \infty$, $\infty^{-1} = 0$) with

$$w_{n,1} + w_{n,2} + w_{n,3} + w_{n,4} = 4.$$

The x_i, $1 \le i \le 4$ represent conductivities of a random resistor while $f_n(x_1, x_2, x_3, x_4)$ is the nth step effective conductivity of the system of four resistors arranged in a "weighted diamond." The example considered in Wehr [(1997), page 1378] is just the special case when one chooses $w_{n,i} = 1$, $j_{n,i} = i$, $1 \le i \le 4$, $n \ge 1$. It is easy to see, using the inequality between harmonic and arithmetic means that

$$\begin{aligned} f_n(x_1, x_2, x_3, x_4) &\le \frac{w_{n,1}x_{j_{n,1}} + w_{n,2}x_{j_{n,2}} + w_{n,3}x_{j_{n,3}} + w_{n,4}x_{j_{n,4}}}{4} \\ &= \alpha_{n,1}x_1 + \alpha_{n,2}x_2 + \alpha_{n,3}x_3 + \alpha_{n,4}x_4, \qquad n \ge 1, \end{aligned} \tag{4.34}$$

where the weights $\alpha_{j_n, i} = w_{n, i}/4$, $1 \le i \le 4$, $n \ge 1$ satisfy

$$\alpha_{n, i} \ge 0, 1 \le i \le 4 \quad \text{and} \quad \alpha_{n, 1} + \alpha_{n, 2} + \alpha_{n, 3} + \alpha_{n, 4} = 1, \qquad n \ge 1.$$

Let $\{X_1^{(n)};\ n \ge 1\}$ be associated with some initial $\mathscr{X}^{(0)} = \{X_j^{(0)};\ j \ge 1\}$ of nonnegative i.i.d. random variables and the sequence of iterated functions $f_n(x_1, x_2, x_3, x_4)$, $n \ge 1$ defined by (4.33). Applying Theorem 2.1(i) and (ii), we conclude (1) if

$$\prod_{j=1}^{n} \max_{1 \le i \le 4} \alpha_{j, i} \to 0 \quad \text{as } n \to \infty \quad \text{and} \quad E\big(X_1^{(0)}\big) < \infty, \tag{4.35}$$

then, for some $\lambda \ge 0$,

$$E\big(X_1^{(n)}\big) \downarrow \lambda \quad \text{as } n \to \infty \quad \text{and} \quad X_1^{(n)} \to_P \lambda, \tag{4.36}$$

and (2) if

$$\prod_{j=1}^{n} \sum_{i=1}^{4} \alpha_{j, i}^2 \le c \cdot b^{-n}, n \ge 1 \quad \text{and} \quad E\left(X_1^{(0)}\big(L\big(X_1^{(0)}\big)\big)^{\delta}\right) < \infty \tag{4.37}$$

for some constants $b > 1$, $c > 0$ and $\delta > 1$, then, for some $\lambda \ge 0$ both (4.36) and

$$\limsup_{n \to \infty} X_1^{(n)} = \lambda \quad \text{a.s.} \tag{4.38}$$

hold. As Wehr (1997) notes, the constant λ is interpreted as the effective conductivity of the hierachical weighted diamond lattice in the infinite-volume limit. The special situation where $w_{n, i} = 1$, $j_{n, i} = i$, $1 \le i \le 4$, $n \ge 1$ has also been extensively studied by Blumenfeld (1988), Schlösser and Spohn (1992), Schenkel, Wehr and Wittwer (1998) and Stinchcombe and Watson (1976).

On the other hand, Theorem 2.1(iii) cannot be applied here even for the special situation discussed by Wehr [(1997), page 1378] because the $f_n(x_1, x_2, x_3, x_4)$, $n \ge 1$ defined by (4.33) are not symmetric functions. All claims in Wehr (1997) regarding applications to particular models can be easily proved using the submartingale method (unlike, apparently, his main theorem), since in all these cases one has enough symmetry to imply that $E(X_j^{(n)} \mid Y_{n+1})$ is independent of $j = 1, \dots, k$, where $Y_{n+1} = X_1^{(n+1)} + \cdots + X_k^{(n+1)}$, $n \ge 0$.

4.4. *Future research directions.* The main theme of this paper has been the asymptotic behavior especially the law of large numbers for hierarchical sequence $\{X_1^{(n)};\ n \ge 0\}$. There remain many open questions concerning this topic.

Although Theorem 2.1 is a sufficiently general result that it can be applied to a wide class of iterated functions, it does not include Proposition 4.3(a) as a special case. Our first questions is, *Can a reasonably general result be found which includes such interesting cases as Proposition* 4.3(a)*, etc. as special cases?*

We have mentioned that the proof of Theorem 1 of Wehr (1997) needs to be repaired. On the other hand, motivated by our Theorem 2.1, we conjecture that, under conditions (2.4) and (2.6), both (2.5) and

$$\lim_{n\to\infty} X_1^{(n)} = \lambda \quad \text{a.s.}$$

hold.

We have discussed a few situation of iterated functions defined by (4.14) in Section 4.2. Let $\{X_1^{(n)};\, n \geq 1\}$ be obtained by iterated functions

$$f_n(x_1, x_2, \dots, x_k) = \alpha_1 x_{k:1} + \alpha_2 x_{k:2} + \cdots + \alpha_k x_{k:k}, \qquad n \geq 1$$

and initial sequence $\{X_j^{(0)};\, j \geq 1\}$ of i.i.d. random variables, where

$$\alpha_i \geq 0, 1 \leq i \leq k \quad \text{and} \quad \alpha_1 + \alpha_2 + \cdots + \alpha_k = 1.$$

Based on that analyses, we conjecture that, under suitable conditions of the random variable $X_1^{(0)}$, appropriately normalized random variables $X_1^{(n)}$, $n \geq 1$ should converge in distribution to some nondegenerate random variable.

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REFERENCES

BERNASCONI, J. (1978). Real renormalization of bond disordered conductance lattices. *Phys. Rev. B* **18** 2185–2191.

BLUMENFELD, R. (1988). Probability densities of homogeneous functions: explicit approximations and applications to percolating networks. *J. Phys. A* **21** 815–825.

BOPPANA, R. and NARAYAN, B. (1993). The biased coin problem. In *Proceedings of the 25th ACM Symposium on Theory of Computing* 252–257. ACM Press, New York.

BOVIN, V. P., VAS'KIN, V. V. and SHNEIBERG, I. YA. (1983). On recursive models in flow theory. *Teor. Mat. Fiz.* **54** 268–276.

CHOW, Y. S. and TEICHER, H. (1988). *Probability: Independence, Interchangeability, Martingales*. Springer, New York.

COLLET, P. and ECKMANN, J.-P. (1978). *A Renormalization Group Analysis of the Hierarchical Model in Statistical Mechanics*. Springer, New York.

DE HAAN, L. (1970). *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes*. *Math Centre Tract* **32**. Math Centrum, Amsterdam.

DE HAAN, L. (1976). Sample extremes: an elementary introduction. *Statist. Neerlandica* **30** 161–172.

DERRIDA, B. (1986). Pure and random models of statistical mechanics on hierarchical lattices. In *Critical Phenomena, Random Systems and Gauge Theories* (K. Osterwalder and R. Stora, eds.) 989–999 North-Holland, Amsterdam.

GALAMBOS, J. (1978). *The Asymptotic Theory of Extreme Order Statistics*. Wiley, New York.

GNEDENKO, B. V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math.* **44** 423–453.

GRIFFITHS, R and KAUFMAN, M. (1982). Spin systems on hierarchical lattices. Introduction and thermodynamic limit. *Phys. Rev. B* **26** 5022–5032.

KOCH, H. and WITTWER, P. (1994). A nontrivial renormalization group fixed point for the Dyson–Baker hierarchical model. *Comm. Math. Phys.* **164** 627–647.

LEADBETTER, M. R., LINDGREN, G. and ROOTZEN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, Berlin.

LI, D. L., RAO, M. B., JIANG, T. F. and WANG, X. C. (1995). Complete convergence and almost sure convergence of weighted sums of random variables. *J. Theoret. Probab.* **8** 49–76.

MOORE, E. and SHANNON, C. E. (1956a). Reliable circuits using less reliable relays I. *J. Franklin Inst.* **262** 191–208.

MOORE, E. and SHANNON, C. E. (1956b). Reliable circuits using less reliable relays II. *J. Franklin Inst.* **262** 281–297.

NEWMAN, W. I., GABRIELOVE, A. M., DURAND, T. A., PHOENIX, S. L. and TURCOTTE, D. L. (1994). An exact renormalization model for earthquakes and material failure. Statics and dynamics. *Phys. D* **77** 200–216.

RESNICK, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.

SCHENKEL, A., WEHR, J. and WITTWER, P. (1998). A non-Gaussian renormalization group fixed point in a hierarchical model of random resistors. Unpublished manuscript.

SCHLÖSSER, T. and SPOHN, H. (1992). Sample-to-sample fluctuations in the conductivity of a disordered medium. *J. Statist. Phys.* **69** 955–967.

SHNEIBERG, I. (1986). Hierarchical sequences of random variables. *Theory Probab. Appl.* **31** 137–141.

SINAI, YA. G. (1982). *Theory and Phase Transitions: Rigorous Results*. Pergamon Press, Oxford.

STINCHCOMBE, R. B. and WATSON, B. P. (1976). Renormalization group approach for percolation conductivity. *J. Phys. C* **9** 3221–3232.

WEHR, J. (1997). A strong law of large numbers for iterated functions of independent random variables. *J. Statist. Phys.* **86** 1373–1384.

WEISSMAN, I. (1975). Extremal processes generated by independent nonidentically distributed random variables. *Ann. Probab.* **3** 172–178.

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