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COMPLEX PRODUCT MANIFOLDS AND BOUNDS OF CURVATURE*

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Abstract. Let $M = X \times Y$ be the product of two complex manifolds of positive dimension. In this paper, we prove that there is no complete Kähler metric g on M such that: either (i) the holomorphic bisectional curvature of g is bounded by a negative constant and the Ricci curvature is bounded below by $-C(1+r^2)$ where r is the distance from a fixed point; or (ii) g has nonpositive sectional curvature and the holomorphic bisectional curvature is bounded above by $-B(1+r^2)^{-\delta}$ and the Ricci curvature is bounded below by $-A(1+r^2)^{\gamma}$ where A, B, γ, δ are positive constants with $\gamma + 2\delta < 1$. These are generalizations of some previous results, in particular the result of Seshadri and Zheng [8].

Key words. Complex products, Kähler manifolds, bisectional curvature, negative curvature.

AMS subject classifications. Primary 53B25; Secondary 53C40

1. Introduction.

In [8], Seshadri and Zheng proved the following result:

THEOREM 1.1. *Let $M = X \times Y$ be the product of two complex manifolds of positive dimension. Then M does not admit any complete Hermitian metric with bounded torsion and holomorphic bisectional curvature bounded between two negative constants.*

In particular, there is no complete Kähler metric on M with holomorphic bisectional curvature bounded between two negative constants. For earlier results in this direction see [11, 15, 16, 7]. The result of Seshadri-Zheng has been generalized by Tosatti [10] to almost-Hermitian manifolds.

On the other hand, there is an open question whether the assumption on the lower bound of the curvature can be removed. In fact, it is an open question raised by N. Mok (see [8]) whether the bidisc admit a complete Kähler metric with holomorphic bisectional curvature bounded from above by -1.

In this work, by using a local version of the generalized Schwartz lemma of Yau [14] and an Omori-Yau type maximum principle of Takegoshi [9], we prove the following:

THEOREM 1.2. *Let $M = X^m \times Y^n$ be the product of two complex manifolds of positive dimension. Then, there is no complete Kähler metric on M with Ricci curvature $\geq -A(1+r)^2$ and holomorphic bisectional curvature $\leq -B$, where A and B are some some positive constants, and $r(x,y) = d(o,(x,y))$ is the distance of (x,y) and a fixed point $o \in M$.*

On the other hand, Seshadri [7] has constructed a complete Kähler metric on $\mathbb{C}^n$ with negative curvature. It seems that the assumption on the upper bound of the curvature in Theorem 1.1 is necessary. However, one can also relax the upper bound of the curvature as follows:

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THEOREM 1.3. *Let $M = X^m \times Y^n$ be the product of two complex manifolds of positive dimension. Then, there is no complete Kähler metric on M with Ricci curvature $\geq -A(1+r^2)^{\gamma}$, holomorphic bisectional curvature $\leq -B(1+r^2)^{-\delta}$, and nonpositive sectional curvature, where $\gamma \geq 0$, $\delta > 0$ such that $\gamma + 2\delta < 1$, A, B are some positive constants, and $r(x,y) = d(o,(x,y))$ is the distance of (x,y) and a fixed point $o \in M$.*

Our method relies on a simple observation. Suppose there is a Hermitian metric g with negative holomorphic bisectional curvature on $M = X \times Y$. Let q be any fixed point in Y. Then the holomorphic vector bundle V over $X_q := X \times \{q\}$, with fibre $V_{(x,q)} = T_q^{1,0}Y \subset T_{(x,q)}^{1,0}M = T_x^{1,0}X \oplus T_q^{1,0}Y$, as a subbundle of $T^{1,0}M|_{X_q}$, is negative. However, $V = T_q^{(1,0)}Y \times X_q$ is a trivial vector bundle, and hence have nonzero global holomorphic sections. Hence the question reduces to the question on the existence of nontrivial global holomorphic sections on the negative vector bundle V . When X is compact, then one can conclude easily that this is impossible (See Kobayashi-Wu [6]). Hence $X \times Y$ does not have a Kähler metric with negative holomorphic bisectional curvature. This result was first proved by Zheng [16] using different method, and was first proved by Yang [12] under the stronger assumption that g is Kähler. Here the metric is not even assumed to be complete.

When X is noncompact and curvature bounds are relaxed, we can only have vanishing theorem with some restriction on the growth of the global section. Controlling the growth of the global section can be done by a local version of Schwartz lemma. Moreover, we need some geometric condition on X_q, for example the validity of Omori-Yau type maximum principle on X_q. This is guaranteed by a volume estimate of X_q and a theorem of Takegoshi [9].

The paper is organized as follows: In section 2, we will prove Theorem 1.2 and in section 3 we will prove Theorem 1.3.

2. Proof of Theorem 1.2. Before we prove the theorem, we need several lemmas. First, we have the following local version of the Schwartz lemma by Yau [14]. See also [2, Theorem 2.1].

LEMMA 2.1. *Let (M^m, g) and (N^n, h) be two complete Kähler manifolds with Kähler forms ω_g and ω_h respectively. Let f be a holomorphic map from M to N. Let $o \in M$ and let $R > 0$. Suppose the Ricci curvature of $B_o(2R)$ is bounded from below by $-K$ and the holomorphic bisectional curvature at every point in $f(B_o(2R))$ is bounded above by $-B$ where K and B are positive constant. Then on $B_o(R)$,*

$$f^*\omega_h \leq C \cdot \frac{K + R^{-2}\left(1 + K^{\frac{1}{2}}R\coth(K^{\frac{1}{2}}R)\right)}{B}\omega_g \tag{2.1}$$

for some positive constant C depending only on m.

Proof. Let $u = \|\partial f\|^2$ which is half the energy density of f. It is clear that $f^*\omega_h \leq \|\partial f\|^2\omega_g$. u satisfies the following inequality on $B_o(2R)$: (See [4], [14])

$$\frac{1}{2}\Delta u \geq -Ku + Bu^2. \tag{2.2}$$

Let $\eta \geq 0$ be a smooth function on $\mathbb{R}$ such that (1) $\eta(t) = 1$ for $t \leq 1$, (2) $\eta(t) = 0$ for $t \geq 2$, (3) $-C_1 \leq \eta'/\eta^{\frac{1}{2}} \leq 0$ for all $t \in \mathbb{R}$, and (4) $|\eta''(t)| \leq C_1$ for all $t \in \mathbb{R}$ for some absolute constant $C_1 > 0$. Let $\phi = \eta(r/R)$.

Suppose ϕu attains maximum at $\bar{x} \in B_o(2R)$. We can assume that $\phi(\bar{x}) > 0$ otherwise $u(\bar{x}) = 0$ for any $x \in B_o(R)$ and we are done. Using an argument of Calabi as in [3], we may assume that ϕu is smooth at $\bar{x}$. Then, we have (1) $\nabla(\phi u)(\bar{x}) = 0$ which implies that at $\bar{x}$, $\nabla u = -u\phi^{-1}\nabla\phi$, (2) $\Delta(\phi u)(\bar{x}) \leq 0$. Hence at $\bar{x}$, we have:

$$\begin{aligned} 0 \geq & \Delta(\phi u) \\ = & \phi\Delta u + 2\langle\nabla\phi, \nabla u\rangle + u\Delta\phi \\ = & \phi\Delta u + 2\langle\nabla\phi, -u\phi^{-1}\nabla\phi\rangle + u\Delta\phi \\ = & \phi\Delta u - 2uR^{-2}\left|\frac{(\eta')^2}{\eta}\right| + u\left(R^{-1}\eta'\Delta r + R^{-2}\eta''\right) \\ \geq & \phi(-2Ku + 2Bu^2) - C_2R^{-2}\left(1 + K^{\frac{1}{2}}R\coth(K^{\frac{1}{2}}R)\right)u, \end{aligned} \tag{2.3}$$

where C_2 is a positive constant depending only on m. Here we have used (2.2), the properties of η and the Laplacian comparison [5]. Hence

$$2B(\phi u)^2(\bar{x}) \leq \left[2K + C_2R^{-2}\left(1 + K^{\frac{1}{2}}R\coth(K^{\frac{1}{2}}R)\right)\right](\phi u)(\bar{x}),$$

and so

$$\sup_{B_o(R)} u \leq \sup_{B_o(2R)} \phi u = (\phi u)(\bar{x}) \leq \frac{2K + C_2R^{-2}\left(1 + K^{\frac{1}{2}}R\coth(K^{\frac{1}{2}}R)\right)}{2B}.$$

From this the lemma follows. ☐

Before we state the next lemma, let us introduce some notations. Let $M = X \times Y$ and let $o = (p, q) \in X \times Y$ be a fixed point. For any $x \in X$, let $Y_x = \{x\} \times Y$ with induced metric denoted as g^x and for any $y \in Y$, let $X_y = X \times \{y\}$ with induced metric denoted as g^y.

Lemma 2.2. *Let M, X, Y as in Theorem 1.2. Suppose there is a complete Kähler metric on M with Ricci curvature bounded from below by $-A(1+r)^2$ and with holomorphic bisectional curvature bounded from above by $-B$ where A and B are positive constants. Let $o = (p, q) \in X \times Y$ be a fixed point. Let $V_p^{X_q}(r)$ be the volume of the geodesic ball of radius r with center at p with respect to the induced metric g^q. Then*

$$V_p^{X_q}(r) \leq C_1 \exp(C_2 r^2)$$

for some constants C_1 and C_2 independent of r.

Proof. Let $x_0 \in X$ be any point. Consider the projection $\pi''_{x_0} : X \times Y \to Y_{x_0}$ such that $\pi''_{x_0}(x, y) = (x_0, y)$. Note that the holomorphic bisectional curvature of Y_{x_0} is still bounded above by $-B$. By Lemma 2.1, there is a constant C_1 independent of x_0 such that

$$(\pi''_{x_0})^*(g^{x_0})|_{(x,y)} \leq C_1\left(1 + r(x, y)\right)^2 g|_{(x,y)} \tag{2.4}$$

for all (x, y). Similarly, if we choose C_1 large enough, we also have:

$$(\pi'_{y_0})^*(g^{y_0})|_{(x,y)} \leq C_1\left(1 + r(x, y)\right)^2 g|_{(x,y)} \tag{2.5}$$

for any $y_0 \in Y$, and $(x, y) \in M$ where π'_{y_0} is the projection from M onto X_{y_0}.

Let γ be any smooth curve in Y_p with length less than 1 from (p,q). Then by (2.4), there exists $C_1' > 0$ such that for any $(x,q) \in X_q$, the length $L(\pi_x'' \circ \gamma)$ satisfies:

$$L(\pi_x'' \circ \gamma) \le C_1' L(\gamma).$$

Hence there is $\rho > 0$ such that for $R > 1$ if $B^p(\rho)$ is the geodesic ball in Y_p with radius ρ and center at (p,q), then $\pi_x(B^p(\rho)) \subset B_o(2R)$ for all $(x,q) \in B_o(R)$.

On the other hand, by (2.5), the Jacobian $J(\pi_q')$ of π_q' at (x,y) satisfies:

$$J(\pi_q')(x,y) \le C_2 \left(1 + r(x,y)\right)^{2m} \tag{2.6}$$

for some constant C_2 for all (x,y).

Now, let $R > 1$ be any constant. Let dV_x be the volume element of Y_x and dV_y be the volume element of X_y. By the co-area formula (see [1]), we have

$$\begin{aligned} V_o(2R) &= \int_M \chi_{B_o(2R)} dV_g \\ &= \int_{X_q} \int_{y \in Y_x} \chi_{B_o(2R)} |J(\pi_q')|^{-1}(x,y) dV_x dV_q \\ &\ge C_2^{-1}(1+R)^{-2m} \int_{(x,q)\in B_o(R)} \int_{\pi_x''(B^p(\rho))} dV_x dV_q \\ &\ge C_3(1+R)^{-2m}(1+R)^{-2n} V^p(\rho) V^q(R) \end{aligned} \tag{2.7}$$

for some constant $C_3 > 0$ for all R by (2.4). Here $V^p(\rho)$ is the volume of $B^p(\rho)$ in Y_p and $V^q(R)$ is the volume of the geodesic ball in X_q with radius R and center at (p,q).

By volume comparison, we have $V_o(2R) \le \exp(C(1+R)^2)$ for some constant C. From this and (2.7), the result follows. ☐

We also need the following result of Takegoshi [9, Theorem 1.1]:

Theorem 2.1. *Let M be a complete noncompact Riemannian manifold. Suppose there is a smooth function f such that $S = \{f > \delta\}$ is nonempty for some $\delta > 0$ and on S*

$$\Delta f \ge \frac{C f^{1+a}}{(1+r)^b}$$

for some positive constants C, a and $0 \le b < 2$ where r is the distance function from some fixed point. Then the volume $V(r)$ of the geodesic ball with radius r satisfies:

$$\liminf_{r\to\infty} \frac{\log V(r)}{r^{2-b}} = \infty.$$

Proof of Theorem 1.2. We proceed by contradiction. Let g be a complete Kähler metric on M satisfying the assumptions.

Suppose $o = (p,q) \in M$. Let u be vector in $T^{1,0}_{p,q}(M) = T^{1,0}_p(X) \times T^{1,0}_q(Y)$ such that $u \in \{0\} \times T^{1,0}_q(Y)$ and such that $g_{u\bar{u}}(p,q) = 1$. Let

$$f(x) = f(x,q) = \|(\pi_x'')_*(u)\|^2.$$

Then f is a function on X_q. By (2.4)

$$f(x) \le C_1(1 + r(p,q))^2 g_{u\bar{u}}(p,q) = C_1 g_{u\bar{u}}(p,q). \tag{2.8}$$

Hence, f is a positive bounded function.

Let $(z^1, z^2, \cdots, z^m, z^{m+1}, \cdots, z^{n+m})$ be a holomorphic coordinate of M at (x, q) such that (1) $(z^1, z^2, \cdots, z^m)$ is a normal coordinate of X_q at x and (2) $g_{a\bar{b}}(x, q) = \delta_{ab}$, $m+1 \leq a, b \leq m+n$. Then, by identifying $(\pi_x)_*(u)$ with u, we have

$$
\begin{aligned}
\Delta_{X_q} f(x) =& 2\sum_{i=1}^{m} \partial_i \partial_{\bar{i}} g_{u\bar{u}}(x, q) \\
=& 2\sum_{i=1}^{m} (-R_{u\bar{u}i\bar{i}} + g^{\bar{b}a} \partial_i g_{u\bar{b}} \partial_{\bar{i}} g_{a\bar{u}})(x, q) \\
=& -2\sum_{i=1}^{m} R_{u\bar{u}i\bar{i}}(x, q) + 2\sum_{i=1}^{m}\sum_{b=1}^{n+m} |\partial_i g_{u\bar{b}}|^2(x, q) \\
\geq& 2mB g_{u\bar{u}}(x, q) \\
=& 2mBf(x).
\end{aligned} \tag{2.9}
$$

Combining this with (2.8), we have

$$\Delta_{X_q} f \geq \frac{2mB}{C_1} f^2.$$

By Lemma 2.2 and Theorem 2.1, we have a contradiction because $f > 0$. ☐

3. Proof of Theorem 1.3. In order to prove the second main result, we need the following lemma. We will use the notations as in the previous section.

Lemma 3.1. *Let $M = X^m \times Y^n$ be the product of two simply connected complex manifolds with positive dimension. Suppose that g is a complete Kähler metric on M with Ricci curvature $\geq -A(1+r^2)^\gamma$, holomorphic bisectional curvature $\leq -B(1+r^2)^{-\delta}$, and nonpositive sectional curvature, where $\gamma \geq 0$, $\delta > 0$ such that $\gamma + 2\delta < 1$, A, B are some positive constants, and $r = r(x, y)$ is the distance of $(x, y) \in X \times Y$ and a fixed point $o \in M$. Then, there is a positive constant C depending only on m, n, γ, δ, A and B, such that for any $x_0 \in X$,*

$$(\pi''_{x_0})^*(g^{x_0})|_{(x,y)} \leq C(1 + r^2(x, y))^\gamma (1 + r^2(x_0, y))^\delta g|_{(x,y)} \tag{3.1}$$

for any $(x, y) \in M$.

Proof. For any point $x_0 \in X$, let $u = \|\partial \pi''_{x_0}\|^2$. Then as before, by [4], [14], we have:

$$\Delta u(x, y) \geq -2A(1 + r^2(x, y))^\gamma u(x, y) + 2B(1 + r^2(x_0, y))^{-\delta} u(x, y)^2. \tag{3.2}$$

Let $v(x, y) = r^2(x_0, y)$. Since M is simply connected and has nonpositive curvature, $r^2(x, y)$ and v are both smooth functions. In the following α, β range from $m+1$ to $m+n$. For $(x, y) \in M$, let $z^1, z^2, \cdots, z^m$ be holomorphic coordinates of x in X and $z^{m+1}, \cdots, z^{m+n}$ be holomorphic coordinates of y in Y such that (1) $g_{\alpha\bar{\beta}}(x, y) = \delta_{\alpha\beta}$, $m+1 \leq \alpha, \beta \leq m+n$ and (2) $g_{\alpha\bar{\beta}}(x_0, y) = \lambda_\alpha \delta_{\alpha\beta}$. Here $(z^{m+1}, \cdots, z^{m+n})$ is also considered as holomorphic coordinates of Y_{x_0} because π''_{x_0} is a biholomorphism between Y_x and Y_{x_0}. Then $u(x,y) = g^{\bar\beta\alpha}(x,y)g_{\alpha\bar\beta}(x_0,y)$. Then

$$\begin{aligned}
&\|\nabla v(x,y)\|^2(x,y)\\
=&2g^{\bar b a}(x,y)v_a(x,y)v_{\bar b}(x,y)\\
=&2g^{\bar\beta\alpha}(x,y)v_\alpha(x,y)v_{\bar\beta}(x,y)\\
=&2u(x,y)g^{\bar\beta\alpha}(x_0,y)v_\alpha(x,y)v_{\bar\beta}(x,y)\\
\le&4u(x,y)v(x,y)
\end{aligned}\tag{3.3}$$

where we have used the fact that the $|\nabla r(x,y)| = 1$ and $r(x_0,y) = r(x,y)|_{Y_{x_0}}$. On the other hand, since r^2 is convex, we have

$$\begin{aligned}
&\Delta v(x,y)\\
=&2g^{\bar b a}(x,y)v_{a\bar b}(x,y)\\
=&2g^{\bar\beta\alpha}(x,y)v_{\alpha\bar\beta}(x_0,y)\\
=&2\sum_\alpha v_{\alpha\bar\alpha}(x_0,y)\\
\le&2u(x,y)\sum_\alpha \lambda_\alpha^{-1}v_{\alpha\bar\alpha}(x_0,y)\quad(\text{since } v_{\alpha\bar\alpha}>0)\\
=&u(x,y)\Delta_{Y_{x_0}}v(x_0,y)\\
\le&u(x,y)(\Delta r^2)(x_0,y)\quad(\text{since } r^2 \text{ is convex})\\
\le&C_1u(x,y)(1+v(x,y))^{\frac{\gamma+1}{2}}
\end{aligned}\tag{3.4}$$

for some constant C_1 by the Laplacian comparison [5] and the assumption on the Ricci curvature of M. Here and below C_i will denote constants depending only on $m, n, \gamma, \delta, A, B$. Let

$$w(x,y) = u(x,y)(C_2 + v(x,y))^{-\delta}$$

where $C_2 > 1$ is a constant to be determined later. Then,

$$\begin{aligned}
\Delta w =&(C_2+v)^{-\delta}\Delta u - 2\delta(C_2+v)^{-1-\delta}\langle\nabla u,\nabla v\rangle\\
&- u\delta(C_2+v)^{-1-\delta}\Delta v + u\delta(\delta+1)(C_2+v)^{-2-\delta}|\nabla v|^2\\
\ge&\left(2B-\delta C_1\,(C_2+v)^{\frac{\gamma}{2}+\delta-\frac12}\right)w^2 - 2A(1+r^2)^\gamma w - 2\delta(C_2+v)^{-1}\langle\nabla w,\nabla v\rangle\\
\ge&\left(2B-\delta C_1 C_2^{\frac{\gamma}{2}+\delta-\frac12}\right)w^2 - 2A(1+r^2)^\gamma w - 4\delta|\nabla w|w^{\frac12}
\end{aligned}\tag{3.5}$$

where we have used (3.3), (3.4), (3.2) and that $C_2 > 1$ and $\gamma + 2\delta < 1$. Hence we may choose $C_2 > 0$ large enough, so that

$$\Delta w \ge C_3w^2 - 2A(1+r^2)^\gamma w - 4\delta|\nabla w|w^{\frac12}\tag{3.6}$$

for some $C_3 > 0$. Then one can proceed as in the proof of Lemma 2.1 to conclude that (3.1) is true. □

Proof of Theorem 1.3. First observe that we may assume M is simply connected because the distance function in the universal cover of M is greater than or equal to the distance function in M. Suppose there is a complete Kähler metric g on M satisfying the curvature assumptions.

Let $o = (p, q)$. As in the proof of Theorem 1.2, let u be a vector in $T^{1,0}_{p,q}(M) = T^{1,0}_p(X) \times T^{1,0}_q(Y)$ such that $u \in \{0\} \times T^{1,0}_q(Y)$ and such that $g_{u\bar{u}}(p, q) = 1$. Let

$$f(x) = f(x, q) = \|(\pi''_x)_*(u)\|^2.$$

Then $f(x)$ is a function on X_q. By the same computation as in (2.9),

$$\Delta_{X_q} f(x) \geq 2mB(1 + r^2(x, q))^{-\delta} f(x). \tag{3.7}$$

By Lemma 3.1, we have

$$\begin{aligned} 0 < f(x) =& \|(\pi''_x)_*(u)\|^2 \\ =& (\pi''_x)^*(g^x)(u, \bar{u}) \\ \leq& C_1(1 + r^2(p, q))^{\gamma}(1 + r^2(x, q))^{\delta} g(u, \bar{u}) \\ =& C_1(1 + r^2(x, q))^{\delta}, \end{aligned} \tag{3.8}$$

where C_1 is a constant independent of x. We may proceed as in the proof of Theorem 1.2 to estimate the volume growth of X_q and use Theorem 2.1 to finish the proof. However, since the curvature of M is nonpositive, we may proceed in a simpler way.

Let $h(x) = \log f(x) - 2\delta \log(C_2 + r^2(x, q))$ where $C_2 > 1$ is some constant to be determined. By (3.8), h achieves its maximum at some point $(\bar{x}, q) \in X_q$. Then at $(\bar{x}, q)$

$$\nabla_{X_q} \log f(\bar{x}) = 2\delta \nabla_{X_q} \log(C_2 + r^2(\bar{x}, q)) \quad \text{and} \quad \Delta_{X_q} h \leq 0. \tag{3.9}$$

Since r^2 is convex, we have $|\nabla_{X_q} r(x, q)| \leq 1$ and $\Delta_{X_q} r^2(x, q) \leq \Delta r^2(x, q) \leq C_3(1 + r^2(x, q))^{\frac{1+\gamma}{2}}$ for some constant C_3 independent of x. Let $r = r(\bar{x}, q)$, then at $(\bar{x}, q)$, using (3.7) and the fact that $\gamma + 2\delta < 1$, we have

$$\begin{aligned} 0 \geq& \Delta_{X_q} h(\bar{x}) \\ =& f^{-1}\Delta_{X_q} f - (2\delta)^2 |\nabla_{X_q} \log(C_2 + r^2(\bar{x}, q))|^2 - 2\delta(C_2 + r^2)^{-1} \Delta_{X_q} r^2(\bar{x}, q) \\ & + 2\delta |\nabla_{X_q} \log(C_2 + r^2(\bar{x}, q))|^2 \\ \geq& 2mB(1 + r^2)^{-\delta} - 2\delta C_3 (C_2 + r^2)^{\frac{-1+\gamma}{2}} \\ >& 2(C_2 + r^2)^{-\delta} \left(mB - \delta C_3 (C_2 + r^2)^{\frac{-1+\gamma+2\delta}{2}} \right) \\ >& 2(C_2 + r^2)^{-\delta} \left(mB - \delta C_3 C_2^{\frac{-1+\gamma+2\delta}{2}} \right) \\ >& 0, \end{aligned} \tag{3.10}$$

if we chose $C_2 > 1$ large enough, such that $\delta C_3 C_2^{-\frac{1-\gamma-2\delta}{2}} < mB$. This can be done because $\gamma + 2\delta < 1$. Hence we have a contradiction. This completes the proof of the theorem. □

Remark 3.1. *Letting $\gamma = 0$ in Theorem 1.3, we know that there is no complete Kähler metric on $X \times Y$ with Ricci curvature bounded from below and holomorphic bisectional curvature $\leq -A(1 + r^2)^{-\delta}$ for any $\delta < \frac{1}{2}$. We may ask the problem if $\frac{1}{2}$ is the optimal power.*

In [5], Greene-Wu proved that if a Hermitian manifold M has holomorphic sectional curvature $\leq -A(1+r^2)^{-1}$, then M is hyperbolic in the sense of Kobayashi-Royden. Note that $\mathbb{C}^n$ is not hyperbolic in the sense of Kobayashi-Royden. So, there is no Hermitian metric on $\mathbb{C}^n$ with holomorphic sectional curvature $\leq -A(1+r^2)^{-1}$. On the other hand, the example given by Seshadri [7] has holomorphic bisectional curvature $\leq -A[(1+r^2)\log(2+r)]^{-1}$. Therefore the optimal power must be in [1/2,1].

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