

A functional limit theorem for coin tossing Markov chains

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Abstract. We prove a functional limit theorem for Markov chains that, in each step, move up or down by a possibly state dependent constant with probability $1/2$, respectively. The theorem entails that the law of every one-dimensional regular continuous strong Markov process in natural scale can be approximated with such Markov chains arbitrarily well. The functional limit theorem applies, in particular, to Markov processes that cannot be characterized as solutions to stochastic differential equations. Our results allow to practically approximate such processes with irregular behavior; we illustrate this with Markov processes exhibiting sticky features, e.g., sticky Brownian motion and a Brownian motion slowed down on the Cantor set.

Résumé. Nous prouvons un théorème limite fonctionnelle pour les chaînes de Markov qui, à chaque étape, montent ou descendent avec probabilité $1/2$ d'une constante dépendante de l'état. Le théorème implique que la loi de chaque processus de Markov uni-dimensionnel, fort, continu, régulier et à l'échelle naturelle peut être approximée par de telles chaînes de Markov avec précision quelconque. Le théorème limite fonctionnelle s'applique en particulier aux processus de Markov qui ne peuvent pas être caractérisés comme solutions d'une équation différentielle stochastique. Notamment nos résultats permettent d'approximer de tels processus avec un comportement irrégulier; nous illustrons cela avec des processus de Markov «collants», par exemple, le mouvement brownien «collant» et un mouvement brownien ralenti sur l'ensemble de Cantor.

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Introduction

Let $(\xi_k)_{k \in \mathbb{N}}$ be an iid sequence of random variables, on a probability space with a measure P , satisfying $P(\xi_1 = \pm 1) = \frac{1}{2}$. Given $y \in \mathbb{R}$, $h \in (0, \infty)$ and a function $a_h : \mathbb{R} \rightarrow \mathbb{R}$, we denote by $(X_{kh}^h)_{k \in \mathbb{N}_0}$ the Markov chain defined by

$$X_0^h = y \quad \text{and} \quad X_{(k+1)h}^h = X_{kh}^h + a_h(X_{kh}^h)\xi_{k+1}, \quad \text{for } k \in \mathbb{N}_0. \quad (1)$$

We choose as the Markov chain's index set the set of non-negative multiples of h because we interpret h as the length of a time step. We extend $(X_{kh}^h)_{k \in \mathbb{N}_0}$ to a continuous-time process by linear interpolation, i.e., we set

$$X_t^h = X_{\lfloor t/h \rfloor h}^h + (t/h - \lfloor t/h \rfloor)(X_{(\lfloor t/h \rfloor + 1)h}^h - X_{\lfloor t/h \rfloor h}^h), \quad t \in [0, \infty). \quad (2)$$

Let $\bar{h} \in (0, \infty)$ and let $(a_h)_{h \in (0, \bar{h})}$ be a family of real functions and $(X^h)_{h \in (0, \bar{h})}$ the associated family of extended Markov chains defined as in (2). A fundamental problem of probability theory is to find conditions on $(X^h)_{h \in (0, \bar{h})}$ such that the laws of the processes X^h , $h \in (0, \bar{h})$, converge in some sense as $h \rightarrow 0$. In this article we provide an asymptotic condition on the family $(a_h)_{h \in (0, \bar{h})}$ guaranteeing that the laws of the processes X^h , $h \in (0, \bar{h})$, converge as $h \rightarrow 0$ to the law of a one-dimensional regular continuous strong Markov process (in the sense of Section VII.3 in [39] or Section V.7 in [40]). In what follows we use the term *general diffusions* for the latter class of processes. Recall that a general diffusion $Y = (Y_t)_{t \in [0, \infty)}$ has a state space that is an open, half-open or closed interval $I \subseteq \mathbb{R}$. We denote by $I^\circ = (l, r)$ the interior

of I , where $-\infty \leq l < r \leq \infty$. Moreover, the law of any general diffusion is uniquely characterized by its speed measure m on I , its scale function and its boundary behavior. Throughout the introduction we assume that Y is in natural scale and that every accessible boundary point is absorbing (see the beginning of Section 1 and Section 6 on how to incorporate diffusions in general scale and with reflecting boundary points). This setting covers, in particular, solutions of driftless SDEs with discontinuous and fast growing diffusion coefficient (see Section 2) and also diffusions with sticky features (see Section 7), which cannot be modeled by SDEs whenever a sticky point is located in the interior of the state space.

Our main result, Theorem 1.1, shows that if a family of functions $(a_h)_{h \in (0, \bar{h})}$ satisfies for all $y \in I^\circ$, $h \in (0, \bar{h})$ the equation

$$\frac{1}{2} \int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u - y|) m(du) = h, \tag{3}$$

with a precision of order $o(h)$ uniformly in y over compact subsets of I° (see Condition (A) below for a precise statement), then the associated family $(X^h)_{h \in (0, \bar{h})}$ converges in distribution, as $h \rightarrow 0$, to the general diffusion Y with speed measure m . We show that for every general diffusion a family of functions $(a_h)_{h \in (0, \bar{h})}$ satisfying (3) exists implying that every general diffusion can be approximated by a Markov chain of the form (1). Equation (3) dictates how to compute the functions $(a_h)_{h \in (0, \bar{h})}$ and therefore paves the way to approximate the distribution of a general diffusion numerically (see, e.g., Section 8).

The central idea in the derivation of Equation (3) is to embed for every $h \in (0, \bar{h})$ the Markov chain $(X_{kh}^h)_{k \in \mathbb{N}_0}$ into Y with a sequence of stopping times. To explain this idea assume for the moment that the state space is $I = \mathbb{R}$. For every $h \in (0, \bar{h})$ let $\tau_0^h = 0$ and then recursively define τ_{k+1}^h as the first time Y exits the interval $(Y_{\tau_k^h} - a_h(Y_{\tau_k^h}), Y_{\tau_k^h} + a_h(Y_{\tau_k^h}))$ after τ_k^h . It follows that the discrete-time process $(Y_{\tau_k^h})_{k \in \mathbb{N}_0}$ has the same law as the Markov chain $(X_{kh}^h)_{k \in \mathbb{N}_0}$. Instead of controlling now directly the spatial errors $|Y_{\tau_k^h} - Y_{kh}|$, we first analyze the temporal errors $|\tau_k^h - kh|$, $k \in \mathbb{N}_0$. We show that for every $y \in \mathbb{R}$, $a \in [0, \infty)$ the expected time it takes Y started in y to leave the interval $(y - a, y + a)$ is equal to $\frac{1}{2} \int_{(y-a, y+a)} (a - |u - y|) m(du)$. In particular, if a_h satisfies (3) for all $y \in I$, it follows that for all $k \in \mathbb{N}_0$ the time lag $\tau_{k+1}^h - \tau_k^h$ between two consecutive stopping times is in expectation equal to h . In this case we refer to $(X_{kh}^h)_{k \in \mathbb{N}_0}$ as Embeddable Markov Chain with Expected time Lag h (we write shortly $(X_{kh}^h)_{k \in \mathbb{N}_0} \in \text{EMCEL}(h)$).

For some diffusions Y one can construct EMCEL approximations explicitly (see, e.g., Section 7). For cases where (3) cannot be solved in closed form, we perform a perturbation analysis and show that it suffices to find for all $h \in (0, \bar{h})$, $y \in I^\circ$ a number $a_h(y)$ satisfying (3) with an error of order $o(h)$ uniformly in y belonging to compact subsets of I° . We prove that for the associated stopping times $(\tau_k^h)_{k \in \mathbb{N}_0}$ the temporal errors $|\tau_k^h - kh|$, $k \in \mathbb{N}_0$, converge to 0 as $h \rightarrow 0$ in every L^α -space, $\alpha \in [1, \infty)$. This ultimately implies convergence of $(X^h)_{h \in (0, \bar{h})}$ to Y in distribution as $h \rightarrow 0$.

To illustrate the benefit of the perturbation analysis, we construct in Section 8 approximations for a Brownian motion slowed down on the Cantor set (see Figure 3). Moreover, we note that our main result, Theorem 1.1, is not only applicable to perturbations of the EMCEL approximation but can also be used to derive new convergence results for other approximation methods such as, e.g., weak Euler schemes (see Corollary 2.3).

The idea to use embeddings in order to prove a functional limit theorem goes back to Skorokhod. In the seminal book [42] scaled random walks are embedded into Brownian motion in order to prove Donsker's invariance principle. In [5] we embed Markov chains into the solution process of an SDE and prove a functional limit theorem where the limiting law is that of the SDE. In [42] and [5] the approximating Markov chains have to be embeddable with a sequence of stopping times $(\tau_k)_{k \in \mathbb{N}_0}$ such that the expected distance between two consecutive stopping times is *exactly* equal to h , the time discretization parameter. In contrast, in the present article we require that the expected distance between consecutive embedding stopping times is only approximately equal to h . We show that for the convergence of the laws it is sufficient to require that the difference of the expected distance and h is of the order $o(h)$. Moreover, compared to [5], we allow for a larger class of limiting distributions. Indeed, our setting includes processes that cannot be characterized as the solution of an SDE, e.g., diffusions with sticky points.

There are further articles in the literature using random time grids to approximate a Markov process, under the additional assumption that it solves a one-dimensional SDE. In [17] the authors first fix a finite grid in the state space of the diffusion. Then they construct a Bernoulli random walk on this grid that can be embedded into the diffusion. The authors determine the expected time for attaining one of the neighboring points by solving a PDE.

[37] describes a similar approximation method for the Cox-Ingersoll-Ross (CIR) process. Also here the authors first fix a grid on $[0, \infty)$ and then construct a random walk on the grid that can be embedded into the CIR process. In contrast to [17], the authors in [37] compute the distributions of the embedding stopping times (and not only their expected value) by solving a parabolic PDE. In the numerical implementation of the scheme the authors then draw the random time increments from these distributions and thereby obtain a scheme that is exact along a sequence of stopping times. Note

that in contrast to [17] and [37], in our approach the space grid is not fixed a priori. Instead, we approximately fix the expected time lag between the consecutive embedding stopping times.

Yet a further scheme that uses a random time partition to approximate a diffusion Y with discontinuous coefficients is suggested in [36]. In contrast to our approach the distribution of the time increments is fixed there. More precisely, the authors of [36] use the fact that the distribution of Y sampled at an independent exponential random time is given by the resolvent of the process. Consequently, if it is possible to generate random variables distributed according to the resolvent kernel, one obtains an exact simulation of Y at an exponentially distributed time. Iterating this procedure and letting the parameter of the exponential distribution go to infinity provides an approximation of Y .

We remark that embeddings along random time grids have been recently employed in [20] and [21] in order to obtain convergence rates of (F)BSDE approximations driven by Bernoulli increments.

Recall that, while approximating solutions of SDEs on deterministic time grids usually employs Euler-type schemes (1) with Gaussian increments $(\xi_k)_{k \in \mathbb{N}}$, we use Bernoulli increments in our paper. In this connection, we would like to mention that, from the numerical perspective, convergence results along equidistant time grids, including approximations by the weak Euler schemes with Bernoulli increments, can be found, e.g., in Section 14.1 of [32]. From the more theoretical perspective, we refer to Theorem 7.4.1 in [16] and Theorem IX.4.8 in [28], which are some general functional limit theorems of the Trotter-Kato type for approximating diffusions. A discussion of how to approximate controlled diffusions by Markov chains with Bernoulli increments can be found in [35]. Another perspective on schemes with Bernoulli increments is suggested in [11] and [22], where, on certain machines (like field programmable gate arrays), such schemes are shown to be more efficient for simulation algorithms.

While in our paper a continuous-time Markov process is approximated via (linearly-interpolated) *discrete-time* Markov chains, there is an alternative approach, pioneered in [43], where the approximating processes are themselves *continuous-time* Markov processes. For a recent account, see [10] and references therein. A generalization of the latter approach for variable-speed random walks on trees contained in [6] is worth mentioning as well.

The article is organized as follows. In Section 1 we rigorously formulate and discuss the functional limit theorem. In Section 2 we discuss some of its implications for diffusions that can be described as solution of SDEs. In Sections 3 and 4 we explain, for a given general diffusion, how to embed an approximating coin tossing Markov chain into the diffusion and prove some properties of the embedding stopping times. Section 5 provides the proof of the functional limit theorem, where we, in particular, need the material discussed in Sections 3 and 4. The functional limit theorem is shown under the additional assumption that if a boundary point is attainable, then it is absorbing. In Section 6 we explain how one can extend the functional limit theorem to general diffusions with reflecting boundary points. In the last two sections we illustrate our main result with diffusions exhibiting some stickiness. In Section 7 we construct coin tossing Markov chains approximating sticky Brownian motion, with and without reflection, respectively. In Section 8 we first describe a Brownian motion that is slowed down on the Cantor set, and secondly we explicitly construct coin tossing Markov chains that approximate this process arbitrarily well.

1. Approximating general diffusions with Markov chains

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (P_y)_{y \in I}, (Y_t)_{t \geq 0})$ be a one-dimensional continuous strong Markov process in the sense of Section VII.3 in [39]. We refer to this class of processes as *general diffusions* in the sequel. We assume that the state space is an open, half-open or closed interval $I \subseteq \mathbb{R}$. We denote by $I^\circ = (l, r)$ the interior of I , where $-\infty \leq l < r \leq \infty$, and we set $\bar{I} = [l, r]$. Recall that by the definition we have $P_y[Y_0 = y] = 1$ for all $y \in I$. We further assume that Y is regular. This means that for every $y \in I^\circ$ and $x \in I$ we have that $P_y[H_x(Y) < \infty] > 0$, where $H_x(Y) = \inf\{t \geq 0 : Y_t = x\}$. If there is no ambiguity, we simply write H_x in place of $H_x(Y)$. Moreover, for $a < b$ in \bar{I} we denote by $H_{a,b} = H_{a,b}(Y)$ the first exit time of Y from (a, b) , i.e. $H_{a,b} = H_a \wedge H_b$. Without loss of generality we suppose that the diffusion Y is in natural scale. If Y is not in natural scale, then there exists a strictly increasing continuous function $s : I \rightarrow \mathbb{R}$, the so-called scale function, such that $s(Y_t)$, $t \geq 0$, is in natural scale. Let m be the speed measure of the Markov process Y (see VII.3.7 and VII.3.10 in [39]).¹ Recall that for all $a < b$ in I° we have

$$0 < m([a, b]) < \infty. \quad (4)$$

Finally, we also assume that if a boundary point is accessible, then it is absorbing. We drop this assumption in Section 6, where we extend our approximation method to Markov processes with reflecting boundaries. The extension works for both instantaneous and slow reflection.

¹There are different conventions concerning the normalization of the speed measure. We follow the convention of [39] and [8] and note that our speed measure is thus twice as large as the one for example found in [40].

Let $\bar{h} \in (0, \infty)$ and suppose that for every $h \in (0, \bar{h})$ we are given a measurable function $a_h : \bar{I} \rightarrow [0, \infty)$ such that $a_h(l) = a_h(r) = 0$ and for all $y \in I^\circ$ we have $y \pm a_h(y) \in I$. We refer to each function a_h as a scale factor. We next construct a sequence of Markov chains associated to the family of scale factors $(a_h)_{h \in (0, \bar{h})}$. To this end we fix a starting point $y \in I^\circ$ of Y . Let $(\xi_k)_{k \in \mathbb{N}}$ be an iid sequence of random variables, on a probability space with a measure P , satisfying $P(\xi_k = \pm 1) = \frac{1}{2}$. We denote by $(X_{kh}^h)_{k \in \mathbb{N}_0}$ the Markov chain defined by

$$X_0^h = y \quad \text{and} \quad X_{(k+1)h}^h = X_{kh}^h + a_h(X_{kh}^h)\xi_{k+1}, \quad \text{for } k \in \mathbb{N}_0. \tag{5}$$

We extend $(X_{kh}^h)_{k \in \mathbb{N}_0}$ to a continuous-time process by linear interpolation, i.e., for all $t \in [0, \infty)$, we set

$$X_t^h = X_{\lfloor t/h \rfloor h}^h + (t/h - \lfloor t/h \rfloor)(X_{(\lfloor t/h \rfloor + 1)h}^h - X_{\lfloor t/h \rfloor h}^h). \tag{6}$$

To highlight the dependence of $X^h = (X_t^h)_{t \in [0, \infty)}$ on the starting point $y \in I^\circ$ we also sometimes write $X^{h,y}$.

To formulate our main result we need the following condition.

Condition (A). For all compact subsets K of I° it holds that

$$\sup_{y \in K} \left| \frac{1}{2} \int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u - y|)m(du) - h \right| \in o(h), \quad h \rightarrow 0. \tag{7}$$

Theorem 1.1. Assume that Condition (A) is satisfied. Then, for any $y \in I^\circ$, the distributions of the processes $(X_t^{h,y})_{t \in [0, \infty)}$ under P converge weakly to the distribution of $(Y_t)_{t \in [0, \infty)}$ under P_y , as $h \rightarrow 0$; i.e., for every bounded and continuous functional² $F : C([0, \infty), \mathbb{R}) \rightarrow \mathbb{R}$, it holds that

$$E[F(X^{h,y})] \rightarrow E_y[F(Y)], \quad h \rightarrow 0. \tag{8}$$

Remark 1.2. To better explain Condition (A), for every $\alpha < \beta$ in I , we introduce the Green function $G_{\alpha,\beta} : [\alpha, \beta]^2 \rightarrow \mathbb{R}$ of Y by the formula

$$G_{\alpha,\beta}(u, v) = \frac{(\beta - u \vee v)(u \wedge v - \alpha)}{\beta - \alpha}, \quad u, v \in [\alpha, \beta]$$

(recall that Y is in natural scale) and observe that, for all $y \in [\alpha, \beta]$,

$$E_y[H_{\alpha,\beta}(Y)] = \int_{(\alpha,\beta)} G_{\alpha,\beta}(y, u)m(du) \tag{9}$$

(see, e.g., Section VII.3 in [39]). It follows that, for any $y \in I^\circ$ and $a > 0$ such that $y \pm a \in I$, it holds

$$\begin{aligned} E_y[H_{y-a, y+a}(Y)] &= \int_{(y-a,y)} \frac{1}{2}(u - y + a)m(du) + \int_{\{y\}} \frac{1}{2}am(du) + \int_{(y,y+a)} \frac{1}{2}(y + a - u)m(du) \\ &= \frac{1}{2} \int_{(y-a, y+a)} (a - |u - y|)m(du). \end{aligned} \tag{10}$$

Thus, Condition (A) is an analytic condition that is equivalent to requiring that the scale factors $(a_h)_{h \in (0, \bar{h})}$ satisfy

$$\sup_{y \in K} |E_y[H_{y-a_h(y), y+a_h(y)}(Y)] - h| \in o(h), \quad h \rightarrow 0,$$

for any compact subset K of I° .

²As usual, we equip $C([0, \infty), \mathbb{R})$ with the topology of uniform convergence on compact intervals, which is generated, e.g., by the metric

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} (\|x - y\|_{C[0,n]} \wedge 1), \quad x, y \in C([0, \infty), \mathbb{R}),$$

where $\|\cdot\|_{C[0,n]}$ denotes the sup norm in $C([0, n], \mathbb{R})$.

Remark 1.3. It is worth noting that Condition (A) is, in fact, nearly necessary for weak convergence (8) (see Example 2.1).

Remark 1.4. For all $y \in I^\circ$, $h \in (0, \bar{h})$ it holds that

$$\int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u - y|)m(du) = \int_I (a_h(y) - |u - y|)^+ m(du).$$

This yields an alternative representation of Condition (A) which is occasionally used below.

It is important to note that for every speed measure m there exists a family of scale factors such that Condition (A) is satisfied and hence every general diffusion Y can be approximated by Markov chains of the form (5). Indeed, for all $y \in I^\circ$, $h \in (0, \bar{h})$ let $\hat{a}_h(l) = \hat{a}_h(r) = 0$ and

$$\hat{a}_h(y) = \sup \left\{ a \geq 0 : y \pm a \in I \text{ and } \frac{1}{2} \int_{(y-a, y+a)} (a - |z - y|)m(dz) \leq h \right\} \tag{11}$$

and denote by $(\hat{X}^h)_{h \in (0, \bar{h})}$ the associated family of processes defined in (5) and (6). Then the proof of Corollary 1.5 below shows that for all compact subsets K of I° there exists $h_0 \in (0, \bar{h})$ such that for all $y \in K$, $h \in (0, h_0)$ it holds that

$$\frac{1}{2} \int_{(y-\hat{a}_h(y), y+\hat{a}_h(y))} (\hat{a}_h(y) - |z - y|)m(dz) = h.$$

In particular, the family $(\hat{a}_h)_{h \in (0, \bar{h})}$ satisfies Condition (A) and we show in Section 3 below that the Markov chain $(\hat{X}_{kh}^h)_{k \in \mathbb{N}_0}$ is embeddable into Y with a sequence of stopping times with expected time lag h . We refer to $(\hat{X}_t^h)_{t \in [0, \infty)}$, $h \in (0, \bar{h})$, as EMCEL approximations and write shortly $(\hat{X}_{kh}^h)_{k \in \mathbb{N}_0} \in \text{EMCEL}(h)$.

Corollary 1.5. For every $y \in I^\circ$ the distributions of the EMCEL approximations $(\hat{X}_t^{h,y})_{t \in [0, \infty)}$ under P converge weakly to the distribution of $(Y_t)_{t \in [0, \infty)}$ under P_y as $h \rightarrow 0$.

Proof. Let K be a compact subset of I° . Without loss of generality assume that $K = [l_0, r_0]$ with $l < l_0 < r_0 < r$. Let $a_0 = \frac{r-r_0}{2} \wedge \frac{l_0-l}{2} \wedge 1$. It follows with dominated convergence that the function

$$K \ni y \mapsto \frac{1}{2} \int_I (a_0 - |u - y|)^+ m(du) \in (0, \infty)$$

is continuous. In particular, it is bounded away from zero, i.e., there exists $h_0 \in (0, \bar{h})$ such that for all $y \in K$ it holds that $\frac{1}{2} \int_I (a_0 - |u - y|)^+ m(du) \geq h_0$. Next observe that for all $y \in K$ the function $[0, a_0] \ni a \mapsto \frac{1}{2} \int_I (a - |u - y|)^+ m(du) \in [0, \infty)$ is continuous and strictly increasing. Hence for all $y \in K$, $h \in (0, h_0)$ the supremum in (11) is a maximum and it holds that $\frac{1}{2} \int_{(y-\hat{a}_h(y), y+\hat{a}_h(y))} (\hat{a}_h(y) - |u - y|)m(du) = h$. In particular, Condition (A) is satisfied and the statement of Corollary 1.5 follows from Theorem 1.1. \square

2. Application to SDEs

A particular case of our setting is the case, where Y is a solution to the driftless SDE

$$dY_t = \eta(Y_t) dW_t, \tag{12}$$

where $\eta: I^\circ \rightarrow \mathbb{R}$ is a Borel function satisfying the Engelbert-Schmidt conditions

$$\eta(x) \neq 0 \quad \forall x \in I^\circ, \tag{13}$$

$$\eta^{-2} \in L^1_{\text{loc}}(I^\circ) \tag{14}$$

$(L^1_{\text{loc}}(I^\circ))$ denotes the set of Borel functions locally integrable on I° . Under (13)–(14) SDE (12) has a unique in law weak solution (see [15] or Theorem 5.5.7 in [30]). This means that there exists a pair of processes (Y, W) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, with (\mathcal{F}_t) satisfying the usual conditions, such that W is an (\mathcal{F}_t) -Brownian motion and

(Y, W) satisfies SDE (12). The process Y possibly reaches the endpoints l or r in finite time. By convention we force Y to stay in l (resp., r) in this case. This can be enforced in (12) by extending η to \bar{I} with $\eta(l) = \eta(r) = 0$. In this example Y is a regular continuous strong Markov process with the state space being the interval with the endpoints l and r (whether l and r belong to the state space is determined by the behavior of η near the boundaries). Moreover, Y is in natural scale, and its speed measure on I° is given by the formula

$$m(dx) = \frac{2}{\eta^2(x)} dx.$$

In this situation a change of variables shows that it holds for all $h \in (0, \bar{h})$, $y \in I^\circ$ that

$$\int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u - y|)m(du) = 2a_h(y)^2 \int_{-1}^1 \frac{1 - |z|}{\eta^2(y + a_h(y)z)} dz. \tag{15}$$

Condition (A) hence becomes that for every compact subset K of I° it holds that

$$\lim_{h \rightarrow 0} \left(\sup_{y \in K} \left| \frac{a_h(y)^2}{h} \int_{-1}^1 \frac{1 - |z|}{\eta^2(y + a_h(y)z)} dz - 1 \right| \right) = 0. \tag{16}$$

Example 2.1 (Brownian motion). In the special case where $Y = W$ is a Brownian motion (i.e., $I = \mathbb{R}$, $\eta(x) \equiv 1$), Condition (A) requires that for all compact sets $K \subset \mathbb{R}$ it holds that $\sup_{y \in K} |\frac{a_h(y)^2}{h} - 1| \rightarrow 0$ as $h \rightarrow 0$. In particular, Condition (A) is satisfied for the choice $a_h(y) = \sqrt{h}$, $h \in (0, \infty)$, $y \in \mathbb{R}$, and we recover from Theorem 1.1 Donsker’s functional limit theorem for the scaled simple random walk.

Moreover, in the case of a Brownian motion it is natural to restrict ourselves to space-homogeneous (i.e., constant) scale factors $a_h(y) \equiv a_h$, $h \in (0, \bar{h})$, so that Condition (A) takes the form $\lim_{h \rightarrow 0} \frac{a_h^2}{h} = 1$. It is straightforward to show that the latter condition is also necessary for the weak convergence of approximations (5)–(6) driven by space-homogeneous scale factors to the Brownian motion.

Example 2.2 (Geometric Brownian motion). Let $\sigma > 0$ and assume that η satisfies for all $x \in (0, \infty)$ that $\eta(x) = \sigma x$. Then the solution Y of (12) with positive initial value $Y_0 = y \in (0, \infty)$ is a geometric Brownian motion. Its state space is $I = (0, \infty)$ and both boundary points are inaccessible. Note that for all $y \in (0, \infty)$, $a \in (0, y)$ it holds that

$$\int_{-1}^1 \frac{1 - |z|}{\eta^2(y + az)} dz = \frac{1}{(\sigma y)^2} \int_{-1}^1 \frac{1 - |z|}{(1 + az/y)^2} dz = -\frac{1}{(\sigma a)^2} \log\left(1 - \frac{a^2}{y^2}\right).$$

Hence, Condition (A) requires that for all compact sets $K \subset (0, \infty)$ it holds that

$$\lim_{h \rightarrow 0} \left(\sup_{y \in K} \left| \frac{1}{h\sigma^2} \log\left(1 - \frac{a_h(y)^2}{y^2}\right) + 1 \right| \right) = 0. \tag{17}$$

To obtain the EMCEL approximation of Y we solve for all $y \in (0, \infty)$, $h \in (0, \infty)$ the equation $\frac{1}{h\sigma^2} \log(1 - \frac{a^2}{y^2}) + 1 = 0$ in a and obtain $\hat{a}_h(y) = y\sqrt{1 - e^{-\sigma^2 h}}$. Note that also the usual choice $a_h(y) = \sqrt{h}\sigma y$, $y \in (0, \infty)$, $h \in (0, 1/\sigma^2)$, which corresponds to the weak Euler scheme for geometric Brownian motion, satisfies (17).

Convergence of the weak Euler scheme

Throughout this subsection we assume that $I = \mathbb{R}$. A common method to approximate solutions of SDEs is the Euler scheme. For equations of the form (12) with initial condition $Y_0 = y$ the Euler scheme $(X_{kh}^{Eu,h})_{k \in \mathbb{N}_0}$ with time step $h \in (0, \infty)$ is given by

$$X_0^{Eu,h} = y \quad \text{and} \quad X_{(k+1)h}^{Eu,h} = X_{kh}^{Eu,h} + \eta(X_{kh}^{Eu,h})(W_{(k+1)h} - W_{kh}), \quad \text{for } k \in \mathbb{N}_0.$$

Weak Euler schemes are variations of the Euler scheme, where the normal increments $W_{(k+1)h} - W_{kh}$, $k \in \mathbb{N}_0$, are replaced by an iid sequence of centered random variables with variance h . Therefore, with the choice $a_h(y) = \sqrt{h}\eta(y)$, $h \in (0, \infty)$, $y \in \mathbb{R}$, the Markov chain $(X_{kh}^h)_{k \in \mathbb{N}_0}$ defined in (5) represents a weak Euler scheme with Rademacher increments.

In this subsection we show how Theorem 1.1 can be used to derive new convergence results for weak Euler schemes. To this end let the setting of Section 2 be given and let $a_h(y) = \sqrt{h}\eta(y)$, $h \in (0, \infty)$, $y \in \mathbb{R}$. Then it follows from (16) that Condition (A) is equivalent to assuming that for every compact subset $K \subset \mathbb{R}$ we have

$$\sup_{y \in K} \left| \int_{-1}^1 \frac{\eta^2(y)(1 - |z|)}{\eta^2(y + \sqrt{h}\eta(y)z)} dz - 1 \right| = \sup_{y \in K} \left| \int_{-1}^1 \frac{\eta^2(y) - \eta^2(y + \sqrt{h}\eta(y)z)}{\eta^2(y + \sqrt{h}\eta(y)z)} (1 - |z|) dz \right| \rightarrow 0, \tag{18}$$

as $h \rightarrow 0$.

Suppose that η is continuous, let $K \subset \mathbb{R}$ be compact and let $\varepsilon > 0$. Then η is bounded on K and since every continuous function is uniformly continuous on compact sets, we obtain that there exists $h_0 \in (0, \infty)$ such that for all $h \in (0, h_0]$, $y \in K$, $z \in [-1, 1]$ it holds that

$$|\eta(y) - \eta(y + \sqrt{h}\eta(y)z)| \leq \varepsilon.$$

By (13) and the continuity of η the function η^2 is strictly bounded away from 0 on every compact subset of \mathbb{R} and hence we obtain that there exists $C \in [0, \infty)$ such that for all $h \in (0, h_0]$ it holds that

$$\sup_{y \in K} \left| \int_{-1}^1 \frac{\eta^2(y) - \eta^2(y + \sqrt{h}\eta(y)z)}{\eta^2(y + \sqrt{h}\eta(y)z)} (1 - |z|) dz \right| \leq C\varepsilon.$$

It follows with (18) that Condition (A) is satisfied. Therefore we obtain the following Corollary of Theorem 1.1.

Corollary 2.3. *Assume the setting of Section 2 with $I = \mathbb{R}$ and that η is continuous. Let $a_h : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $a_h(y) = \sqrt{h}\eta(y)$ for all $h \in (0, \infty)$, $y \in \mathbb{R}$. Then for all $y \in \mathbb{R}$ the distributions of the processes $(X_t^{h,y})_{t \in [0, \infty)}$ under P converge weakly to the distribution of $(Y_t)_{t \in [0, \infty)}$ under P_y , as $h \rightarrow 0$.*

Remark 2.4. Corollary 2.3 complements convergence results for the Euler scheme for example obtained in [45] and [25]. Theorem 2.2 in [45] shows weak convergence of the Euler scheme if η has at most linear growth and is discontinuous on a set of Lebesgue measure zero. Theorem 2.3 in [25] establishes almost sure convergence of the Euler scheme if η is locally Lipschitz continuous. Moreover, [25] allows for a multidimensional setting and a drift coefficient. In contrast, Corollary 2.3 above applies to the weak Euler scheme and does not require linear growth or local Lipschitz continuity of η .

Remark 2.5. As stated in Corollary 1.5, EMCEL approximations can be constructed for every general diffusion. In particular, they can be used in cases where η is not continuous and where (weak) Euler schemes do not converge (see, e.g., Section 5.4 in [4]). In Sections 7 and 8 we consider further irregular examples.

3. Embedding the chains into the Markov process

In this section we construct the embedding stopping times. Throughout the section we assume the setting of Section 1.

We need to introduce an auxiliary subset of I° . To this end, if $l > -\infty$, we define, for all $h \in (0, \bar{h})$,

$$l_h := l + \inf \left\{ a \in \left(0, \frac{r-l}{2} \right] : a < \infty \text{ and } \frac{1}{2} \int_{(l, l+2a)} (a - |u - (l+a)|) m(du) \geq h \right\},$$

where we use the convention $\inf \emptyset = \infty$. If $l = -\infty$, we set $l_h = -\infty$. Similarly, if $r < \infty$, then we define, for all $h \in (0, \bar{h})$,

$$r_h := r - \inf \left\{ a \in \left(0, \frac{r-l}{2} \right] : a < \infty \text{ and } \frac{1}{2} \int_{(r-2a, r)} (a - |u - (r-a)|) m(du) \geq h \right\}.$$

If $r = \infty$, we set $r_h = \infty$. It is worth noting that l is inaccessible if and only if $l_h = l$ for all $h \in (0, \bar{h})$; and, similarly, r is inaccessible if and only if $r_h = r$ for all $h \in (0, \bar{h})$. This is verified in Remark 4.2 below. The auxiliary subset is defined by

$$I_h = (l_h, r_h) \cup \{y \in I^\circ : y \pm a_h(y) \in I^\circ\}.$$

Now we have everything at hand to start constructing a sequence of embedding stopping times. Suppose Y starts at a point $y \in I^\circ$ and fix $h \in (0, \bar{h})$. Set $\tau_0^h = 0$. Let $\sigma_1^h = H_{y-a_h(y), y+a_h(y)}$. Recall that we have

$$E_y[\sigma_1^h] = \frac{1}{2} \int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u - y|)m(du)$$

(see Remark 1.2). We now define τ_1^h by distinguishing two cases.

Case 1: $y \in I_h$ (i.e., $y \in (l_h, r_h)$ or $y \pm a_h(y) \in I^\circ$). In this case we set $\tau_1^h = \sigma_1^h$.

Case 2: $y \notin I_h$ (i.e., $y \notin (l_h, r_h)$ and $(y + a_h(y) = r$ or $y - a_h(y) = l)$). In this case we deterministically extend σ_1^h so as to make it have expectation h . Observe that by the definition of l_h and r_h we have in this case $E_y[\sigma_1^h] \leq h$. Moreover, we can assume in this case that it must hold that $P_y(Y_{\sigma_1^h} \in \{l, r\}) = \frac{1}{2}$ (only in the case $\max\{|l|, |r|\} < \infty, y = \frac{l+r}{2}$ and $a_h(y) = \frac{r-l}{2}$ this probability is 1, but we exclude this case by considering a sufficiently small h , so that $a_h(\frac{l+r}{2}) < \frac{r-l}{2}$; notice that Condition (A) implies that, for any $y \in I^\circ, \lim_{h \rightarrow 0} a_h(y) = 0$). We define τ_1^h by

$$\tau_1^h = \sigma_1^h + 2(h - E[\sigma_1^h])1_{\{l, r\}}(Y_{\sigma_1^h}).$$

Observe that the definition implies $E_y[\tau_1^h] = h$ and that the three random variables $Y_{\tau_1^h}, Y_{\sigma_1^h}$ and $X_h^{h,y}$ have all the same law.

We can proceed in a similar way to define the subsequent stopping times. Let $k \in \mathbb{N}$. Suppose that we have already constructed τ_k^h . We first define $\sigma_{k+1}^h = \inf\{t \geq \tau_k^h : |Y_t - Y_{\tau_k^h}| = a_h(Y_{\tau_k^h})\}$. On the event $\{Y_{\tau_k^h} \in I_h\}$ we set $\tau_{k+1}^h = \sigma_{k+1}^h$. On the event $\{Y_{\tau_k^h} \notin I_h\}$ we extend σ_{k+1}^h as follows. Note that $Y_{\tau_k^h}$ takes only finitely many values. Let $v \in I \setminus (l_h, r_h)$ be a possible value of $Y_{\tau_k^h}$ such that $v - a_h(v) = l$ or $v + a_h(v) = r$. Consider the event $A = \{Y_{\tau_k^h} = v\}$. Observe that $c := E_y[\sigma_{k+1}^h - \tau_k^h | A] \leq h$. We extend σ_{k+1}^h on the event A by setting

$$\tau_{k+1}^h = \sigma_{k+1}^h + 2(h - c)1_{\{l, r\}}(Y_{\sigma_{k+1}^h}) \tag{19}$$

(notice that $P_y(Y_{\sigma_{k+1}^h} \in \{l, r\} | A) = \frac{1}{2}$). This implies that $E_y[\tau_{k+1}^h - \tau_k^h | \mathcal{F}_{\tau_k^h}^h] = h$ on the event $\{Y_{\tau_k^h} \notin I_h\}$. Moreover, the processes $(Y_{\tau_j^h})_{j \in \{0, \dots, k+1\}}$ and $(X_{j_h}^{h,y})_{j \in \{0, \dots, k+1\}}$ have the same law. To sum up, we have the following.

Proposition 3.1. *For all $h \in (0, \bar{h})$ and $y \in I^\circ$ the sequence of stopping times $(\tau_k^h)_{k \in \mathbb{N}_0}$ satisfies*

1. $\text{Law}_{P_y}(Y_{\tau_k^h}; k \in \mathbb{N}_0) = \text{Law}_P(X_{k_h}^{h,y}; k \in \mathbb{N}_0)$.
2. For all $k \in \mathbb{N}_0$ we have

$$E_y(\tau_{k+1}^h - \tau_k^h | \mathcal{F}_{\tau_k^h}^h) = \begin{cases} \frac{1}{2} \int_{(Y_{\tau_k^h} - a_h(Y_{\tau_k^h}), Y_{\tau_k^h} + a_h(Y_{\tau_k^h}))} (a_h(Y_{\tau_k^h}) - |u - Y_{\tau_k^h}|)m(du), & \text{if } Y_{\tau_k^h} \in I_h, \\ h, & \text{if } Y_{\tau_k^h} \notin I_h. \end{cases}$$

4. Higher moment estimates for exit times

In this section we provide some moment estimates for the exit times of Y from intervals. We use the estimates in the next section to prove convergence in probability of $\sup_{k \in \{1, \dots, \lfloor T/h \rfloor\}} |\tau_k^h - kh|$ to zero, where $(\tau_k^h)_{k \in \mathbb{N}_0}$ is the sequence of embedding stopping times from Section 3. This is a crucial ingredient in the proof of our main result, Theorem 1.1.

We introduce the function $q : I^\circ \times \bar{I} \rightarrow [0, \infty]$ defined by

$$q(y, x) = \frac{1}{2}m(\{y\})|x - y| + \int_y^x m((y, u)) du, \tag{20}$$

where for $u < y$ we set $m((y, u)) := -m((u, y))$. Notice that, for $y \in I^\circ$, the function $q(y, \cdot)$ is decreasing on $[l, y]$ and increasing on $[y, r]$. For our analysis the key property of q is that it makes the process $q(y, Y_t) - (t \wedge H_{l,r})$, $t \in [0, \infty)$,

a P_y -local martingale (see Lemma 4.1 below). Moreover, q plays a central role in Feller's test for explosions: for any $y \in I^\circ$,

$$l \text{ is accessible (i.e., } l \in I) \iff q(y, l) < \infty, \quad (21)$$

$$r \text{ is accessible (i.e., } r \in I) \iff q(y, r) < \infty \quad (22)$$

(see, e.g., Lemma 2.1 in [2] or Theorem 3.3 in [3]). Consequently, q is finite on $I^\circ \times I$. Notice that for all $y, z \in I^\circ$ and $x \in I$ we have

$$q(z, x) = q(y, x) - q(y, z) - \frac{\partial^0 q}{\partial x}(y, z)(x - z), \quad (23)$$

where $\frac{\partial^0 q}{\partial x}(y, x) = \frac{1}{2}(\frac{\partial^+ q}{\partial x} + \frac{\partial^- q}{\partial x})(y, x)$.

The following lemma identifies a local martingale associated to Y .

Lemma 4.1. *Let $y \in I^\circ$. Then the process $q(y, Y_t) - (t \wedge H_{l,r})$, $t \in [0, \infty)$, is a P_y -local martingale.*

Proof. Let $a, b \in I$ with $a < y < b$. We first show that $q(y, Y_{t \wedge H_{a,b}}) - t \wedge H_{a,b}$, $t \in [0, \infty)$, is a P_y -martingale. It follows from (9) that

$$\begin{aligned} E_y H_{a,b} &= \frac{1}{b-a} \left[\int_{(a,y)} (b-y)(x-a)m(dx) + \int_{(y,b)} (b-x)(y-a)m(dx) + (b-y)(y-a)m(\{y\}) \right] \\ &= \frac{b-y}{b-a} \left[\int_{(a,y)} (x-a)m(dx) + (y-a) \frac{m(\{y\})}{2} \right] + \frac{y-a}{b-a} \left[\int_{(y,b)} (b-x)m(dx) + (b-y) \frac{m(\{y\})}{2} \right] \\ &= \frac{b-y}{b-a} q(y, a) + \frac{y-a}{b-a} q(y, b) \\ &= E_y q(y, Y_{H_{a,b}}), \end{aligned}$$

where, in the third equality, we use the representation $\int_{(a,y)} 1_{\{u < x\}} du$ for $x - a$ (and the similar one for $b - x$) and apply Fubini's theorem. Thus,

$$E_y H_{a,b} = E_y q(y, Y_{H_{a,b}}). \quad (24)$$

Next, we observe that for all $t \in [0, \infty)$ it holds

$$E_y [q(y, Y_{H_{a,b}}) - H_{a,b} | \mathcal{F}_t] = (q(y, Y_{H_{a,b}}) - H_{a,b}) 1_{\{H_{a,b} \leq t\}} + E_y [q(y, Y_{H_{a,b}}) - H_{a,b} | \mathcal{F}_t] 1_{\{H_{a,b} > t\}}. \quad (25)$$

On the event $\{H_{a,b} > t\}$ we have $q(y, Y_{H_{a,b}}) - H_{a,b} = q(y, Y_{H_{a,b}}) \circ \theta_t - H_{a,b} \circ \theta_t - t$, where θ_t denotes the shift operator for Y (see Chapter III in [39]). The Markov property and (24) imply that on the event $\{H_{a,b} > t\}$ we have P_y -a.s.

$$\begin{aligned} E_y [q(y, Y_{H_{a,b}}) - H_{a,b} | \mathcal{F}_t] &= E_z [q(y, Y_{H_{a,b}}) - H_{a,b}] \Big|_{z=Y_t} - t \\ &= E_z [q(y, Y_{H_{a,b}}) - q(z, Y_{H_{a,b}})] \Big|_{z=Y_t} - t. \end{aligned} \quad (26)$$

Formula (23) yields for all $z \in I^\circ$

$$q(y, Y_{H_{a,b}}) - q(z, Y_{H_{a,b}}) = q(y, z) + \frac{\partial^0 q}{\partial x}(y, z)(Y_{H_{a,b}} - z).$$

Since $E_z [Y_{H_{a,b}} - z] = 0$ for all $z \in I^\circ$, equation (26) implies that on the event $\{H_{a,b} > t\}$ we have P_y -a.s.

$$E_y [q(y, Y_{H_{a,b}}) - H_{a,b} | \mathcal{F}_t] = q(y, Y_t) - t.$$

Together with (25) this yields for all $t \in [0, \infty)$

$$E_y [q(y, Y_{H_{a,b}}) - H_{a,b} | \mathcal{F}_t] = q(y, Y_{t \wedge H_{a,b}}) - t \wedge H_{a,b},$$

which shows that $q(y, Y_{t \wedge H_{a,b}}) - t \wedge H_{a,b}$, $t \in [0, \infty)$, is a P_y -martingale.

The statement of the lemma follows via a localization argument. If $l \notin I$, then choose a decreasing sequence $(l_n)_{n \in \mathbb{N}} \subseteq I$ with $l_1 < y$ and $\lim_{n \rightarrow \infty} l_n = l$. If $l \in I$, set $l_n = l$ for all $n \in \mathbb{N}$. Similarly, if $r \notin I$, then choose an increasing sequence $(r_n)_{n \in \mathbb{N}} \subseteq I$ with $r_1 > y$ and $\lim_{n \rightarrow \infty} r_n = r$, and if $r \in I$, then set $r_n = r$ for all $n \in \mathbb{N}$. The sequence of stopping times $\inf\{t \geq 0 : X_t \notin [l_n, r_n]\}$, $n \in \mathbb{N}$, is then a localizing sequence for the process $q(y, Y_t) - (t \wedge H_{l,r})$, $t \in [0, \infty)$. \square

Remark 4.2. We still owe verifying that, for l_h and r_h introduced in Section 3, l is inaccessible if and only if $l_h = l$ for all $h \in (0, \bar{h})$; and r is inaccessible if and only if $r_h = r$ for all $h \in (0, \bar{h})$. Let us prove this equivalence for the left boundary point. To this end, the following identity is helpful: for $y \in I^\circ$ and $a \in (0, \infty)$ such that $y \pm a \in \bar{I}$, it holds

$$\int_{(y-a, y+a)} (a - |u - y|)m(du) = q(y, y + a) + q(y, y - a). \tag{27}$$

Indeed, if $y \pm a \in I$, then (27) follows from (10) and (24). If we only have $y \pm a \in \bar{I}$, then (27) follows by the monotone convergence argument.

Now, if $l = -\infty$, then the equivalence is clear. Let $l > -\infty$. Consider a small enough $a > 0$. For the integral in the definition of l_h , we have due to (27)

$$\int_{(l, l+2a)} (a - |u - (l + a)|)m(du) = q(l + a, l) + q(l + a, l + 2a), \tag{28}$$

where $l + a, l + 2a \in I^\circ$, and hence the second term on the right-hand side is finite, i.e., the integral is infinite if and only if the first term on the right-hans side is infinite. We conclude by applying Feller’s test (21): l is inaccessible if and only if the integral in (28) is infinite for all small $a > 0$, and the latter holds if and only if $l_h = l$ for all $h \in (0, \bar{h})$.

The next result provides conditions guaranteeing that moments of a stopping time τ can be bounded against an integral with respect to the distribution of Y_τ .

Theorem 4.3. Let $\alpha \in [1, \infty)$ and let $y \in I$. Let τ be a stopping time such that $\tau \leq H_{l,r}(Y)$, P_y -a.s. and the process $(q^\alpha(y, Y_{t \wedge \tau}))_{t \in [0, \infty)}$ is of class (D) under P_y . Then $\tau < \infty$, P_y -a.s. and it holds that

$$E_y[\tau^\alpha] \leq \alpha^\alpha E_y[q^\alpha(y, Y_\tau)]. \tag{29}$$

Proof. If $y \in I \setminus I^\circ$, then $\tau = 0$ and (29) is satisfied. For the remainder of the proof we assume that $y \in I^\circ$. We first show by contradiction that $\tau < \infty$ P_y -a.s. So assume that $P_y(\tau = \infty) > 0$. Since on $\{\tau = \infty\}$ we necessarily have $\tau = H_{l,r}(Y)$, we obtain

$$\limsup_{t \rightarrow \infty} q^\alpha(y, Y_{t \wedge \tau}) = \infty \quad P_y\text{-a.s. on } \{\tau = \infty\}.$$

For any $n \in \mathbb{N}$ let $\sigma_n = \inf\{t \in [0, \infty) : q^\alpha(y, Y_{t \wedge \tau}) \geq n\}$. Then we obtain that

$$\lim_{n \rightarrow \infty} q^\alpha(y, Y_{\sigma_n \wedge \tau}) = \infty \quad P_y\text{-a.s. on } \{\tau = \infty\},$$

which contradicts the uniform integrability of $\{q^\alpha(y, Y_{\sigma_n \wedge \tau})\}_{n \in \mathbb{N}}$. Thus, we proved that $\tau < \infty$ P_y -a.s. and, in particular, that Y_τ on the right-hand side of (29) is well-defined.

According to Lemma 4.1 the process $N_t := q(y, Y_t) - (t \wedge H_{l,r}(Y))$, $t \geq 0$, is a P_y -local martingale. The product formula yields for all $t \in [0, H_{l,r}(Y)]$

$$\begin{aligned} t^{\alpha-1}q(y, Y_t) &= t^{\alpha-1}N_t + t^\alpha = (\alpha - 1) \int_0^t s^{\alpha-2}N_s ds + \int_0^t s^{\alpha-1} dN_s + t^\alpha \\ &= (\alpha - 1) \int_0^t s^{\alpha-2}q(y, Y_s) ds + \int_0^t s^{\alpha-1} dN_s + \frac{1}{\alpha}t^\alpha \\ &\geq \int_0^t s^{\alpha-1} dN_s + \frac{1}{\alpha}t^\alpha. \end{aligned} \tag{30}$$

Note that $(\int_0^t s^{\alpha-1} dN_s)_{t \geq 0}$ is a local martingale and let $(\tau'_n)_{n \in \mathbb{N}}$ be a localizing sequence for it. Set $\tau_n := n \wedge \tau'_n$ for all $n \in \mathbb{N}$. In particular, it holds $E_y[\tau_n^\alpha] < \infty$ for all $n \in \mathbb{N}$. With inequality (30) and Hölder’s inequality we obtain for all

$n \in \mathbb{N}$ that

$$E_y[(\tau_n \wedge \tau)^\alpha] \leq \alpha E_y[(\tau_n \wedge \tau)^{\alpha-1} q(y, Y_{\tau_n \wedge \tau})] \leq \alpha (E_y[(\tau_n \wedge \tau)^\alpha])^{\frac{\alpha-1}{\alpha}} (E_y[q^\alpha(y, Y_{\tau_n \wedge \tau})])^{\frac{1}{\alpha}}.$$

This implies for all $n \in \mathbb{N}$

$$E_y[(\tau_n \wedge \tau)^\alpha] \leq \alpha^\alpha E_y[q^\alpha(y, Y_{\tau_n \wedge \tau})]. \quad (31)$$

By monotone convergence the left-hand side converges to $E_y[\tau^\alpha]$ as $n \rightarrow \infty$. Since the process $(q^\alpha(y, Y_{\tau \wedge t}))_{t \in [0, \infty)}$ is of class (D), it follows that the family $(q^\alpha(y, Y_{\tau_n \wedge \tau}))_{n \in \mathbb{N}}$ is uniformly integrable. Vitali's convergence theorem implies that $E_y[q^\alpha(y, Y_{\tau_n \wedge \tau})] \rightarrow E_y[q^\alpha(y, Y_\tau)]$ as $n \rightarrow \infty$. Therefore we obtain

$$E_y[\tau^\alpha] \leq \alpha^\alpha E_y[q^\alpha(y, Y_\tau)],$$

which is precisely (32). \square

Remark 4.4. Under the assumption of Theorem 4.3 we have equality in (29) for $\alpha = 1$, i.e., $E_y[\tau] = E_y[q(y, Y_\tau)]$. Indeed, the inequality \leq is provided by (29), while, for the reverse inequality \geq , use that equality holds in (30) for $\alpha = 1$, localize (30), compute expectations of the both sides and apply Fatou's lemma.

From Theorem 4.3 we obtain the following moment estimate for first exit times.

Corollary 4.5. *Let $\alpha \in [1, \infty)$, let $y \in I$ and let $a \in [0, \infty)$ be such that $[y - a, y + a] \subseteq I$. Then it holds that*

$$E_y[(H_{y-a, y+a}(Y))^\alpha] \leq \frac{\alpha^\alpha}{2} (q^\alpha(y, y-a) + q^\alpha(y, y+a)). \quad (32)$$

Proof. Clearly, $H_{y-a, y+a}(Y) \leq H_{l,r}(Y)$, P_y -a.s. Moreover, under P_y , the process $(q^\alpha(y, Y_{H_{y-a, y+a}(Y) \wedge t}))_{t \in [0, \infty)}$ is bounded (see (21) and (22)) and hence of class (D). Inequality (32) then follows from Theorem 4.3. \square

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1 and show that under Condition (A) the processes $(X^h)_{h \in (0, \bar{h})}$ converge in distribution to Y . We use the embedding stopping times $(\tau_k^h)_{k \in \mathbb{N}_0}$ constructed in Section 3 and control the temporal errors $|\tau_k^h - kh|$, $h \in (0, \bar{h})$, $k \in \mathbb{N}_0$. To this end, for every $h \in (0, \bar{h})$ we apply the Doob decomposition to the process $(\tau_k^h - kh)_{k \in \mathbb{N}_0}$ and write $\tau_k^h - kh = M_k^h + A_k^h$, $k \in \mathbb{N}_0$, for a martingale M^h and a predictable process A^h . Condition (A) guarantees that A^h converges to 0 as $h \rightarrow 0$ (see Proposition 5.3 below). We show that also the martingale part M^h can be nicely controlled (see Proposition 5.2 below).

For all $h \in (0, \bar{h})$ let $(\tau_k^h)_{k \in \mathbb{N}}$ be the sequence of embedding stopping times defined in Section 3. Then we have the following result about the time lags $\rho_k^h = \tau_k^h - \tau_{k-1}^h$, $h \in (0, \bar{h})$, $k \in \mathbb{N}$, between consecutive embedding stopping times.

Lemma 5.1. *Let $\alpha \in [1, \infty)$, $h \in (0, \bar{h})$ and $y \in I^\circ$. Then it holds that*

$$\sup_{k \in \mathbb{N}} \|\rho_k^h\|_{L^\alpha(P_y)} \leq 2h + 2^{1-1/\alpha} \alpha \sup_{z \in I} (E_z[H_{z-a_h(z), z+a_h(z)}(Y)]). \quad (33)$$

Proof. By construction of the sequence $(\tau_k^h)_{k \in \mathbb{N}_0}$ (in particular, recall (19)) it holds for all $k \in \mathbb{N}$ that

$$\rho_k^h = \tau_k^h - \tau_{k-1}^h \leq \inf\{t \geq \tau_{k-1}^h : |Y_t - Y_{\tau_{k-1}^h}| = a_h(Y_{\tau_{k-1}^h})\} + 2h.$$

This and the strong Markov property of Y imply for all $k \in \mathbb{N}$ that

$$\begin{aligned} E_y[(\tau_k^h - \tau_{k-1}^h)^\alpha] &\leq E_y[E_z[(H_{z-a_h(z), z+a_h(z)}(Y) + 2h)^\alpha] \Big|_{z=Y_{\tau_{k-1}^h}}] \\ &\leq \sup_{z \in I} E_z[(H_{z-a_h(z), z+a_h(z)}(Y) + 2h)^\alpha]. \end{aligned} \quad (34)$$

Notice that

$$a^\alpha + b^\alpha \leq (a + b)^\alpha \quad \text{for } a, b \in [0, \infty), \tag{35}$$

because the function $x \mapsto x^\alpha$, $x \in [0, \infty)$, is convex, increasing and starts in zero. It follows from the triangle inequality, Corollary 4.5 and (35) that for all $z \in I$ it holds

$$\begin{aligned} \|H_{z-a_h(z), z+a_h(z)}(Y) + 2h\|_{L^\alpha(P_z)} &\leq (E_z[H_{z-a_h(z), z+a_h(z)}(Y)^\alpha])^{\frac{1}{\alpha}} + 2h \\ &\leq 2^{-1/\alpha} \alpha (q^\alpha(z, z - a_h(z)) + q^\alpha(z, z + a_h(z)))^{\frac{1}{\alpha}} + 2h \\ &\leq 2^{-1/\alpha} \alpha (q(z, z - a_h(z)) + q(z, z + a_h(z))) + 2h \\ &= 2^{1-1/\alpha} \alpha E_z[H_{z-a_h(z), z+a_h(z)}(Y)] + 2h. \end{aligned}$$

Combining this with (34) completes the proof. □

Below, for a random variable ξ , it is convenient to use the notation

$$\|\xi\|_{L^\alpha(P_y)} = (E_y|\xi|^\alpha)^{1/\alpha}$$

for all $\alpha \in (0, \infty)$, even though it is not a norm for $\alpha \in (0, 1)$. Notice that $\|\xi\|_{L^\alpha(P_y)} \leq \|\xi\|_{L^\beta(P_y)}$ for $0 < \alpha < \beta$ by the Jensen inequality.

Proposition 5.2. *Let $\alpha \in (0, \infty)$. Then there exists a constant $C(\alpha) \in (0, \infty)$ such that, for all $T \in (0, \infty)$, $y \in I^\circ$ and $h \in (0, \bar{h})$, it holds that*

$$\left\| \sup_{k \in \{1, \dots, \lfloor T/h \rfloor\}} \left| \tau_k^h - \sum_{n=1}^k E[\rho_n^h | \mathcal{F}_{\tau_{n-1}^h}] \right| \right\|_{L^\alpha(P_y)} \leq C(\alpha) \sqrt{T} \left(\sqrt{h} + \frac{\sup_{z \in I} (E_z[H_{z-a_h(z), z+a_h(z)}(Y)])}{\sqrt{h}} \right). \tag{36}$$

Proof. Without loss of generality we consider $\alpha \in [2, \infty)$. Throughout the proof we fix $T \in (0, \infty)$, $y \in I^\circ$, $h \in (0, \bar{h})$ and let $N = \lfloor T/h \rfloor$. For all $k \in \{0, \dots, N\}$ we define $\mathcal{G}_k = \mathcal{F}_{\tau_k^h}$ and $M_k = \tau_k^h - \sum_{n=1}^k E[\rho_n^h | \mathcal{G}_{n-1}]$. Notice that $(M_k)_{k \in \{0, \dots, N\}}$ is a $(\mathcal{G}_k)_{k \in \{0, \dots, N\}}$ -martingale. The Burkholder–Davis–Gundy inequality ensures that there exists a constant $\tilde{C}(\alpha) \in (0, \infty)$ (only depending on α) such that

$$\begin{aligned} E_y \left[\sup_{k \in \{1, \dots, N\}} \left| \tau_k^h - \sum_{n=1}^k E[\rho_n^h | \mathcal{G}_{n-1}] \right|^\alpha \right] &\leq \tilde{C}(\alpha) E_y \left[\left(\sum_{k=1}^N (M_k - M_{k-1})^2 \right)^{\frac{\alpha}{2}} \right] \\ &= \tilde{C}(\alpha) E_y \left[\left(\sum_{k=1}^N (\rho_k^h - E[\rho_k^h | \mathcal{G}_{k-1}])^2 \right)^{\frac{\alpha}{2}} \right]. \end{aligned}$$

This, together with Jensen’s inequality, proves that

$$\begin{aligned} E_y \left[\sup_{k \in \{1, \dots, N\}} \left| \tau_k^h - \sum_{n=1}^k E[\rho_n^h | \mathcal{G}_{n-1}] \right|^\alpha \right] &\leq \tilde{C}(\alpha) N^{\frac{\alpha}{2}-1} \sum_{k=1}^N E_y [|\rho_k^h - E[\rho_k^h | \mathcal{G}_{k-1}]|^\alpha] \\ &\leq \tilde{C}(\alpha) 2^\alpha N^{\frac{\alpha}{2}-1} \sum_{k=1}^N E_y [|\rho_k^h|^\alpha] \\ &\leq \tilde{C}(\alpha) 2^\alpha N^{\frac{\alpha}{2}} \sup_{k \in \mathbb{N}} E_y [|\rho_k^h|^\alpha] \\ &\leq \tilde{C}(\alpha) 2^\alpha T^{\frac{\alpha}{2}} \left(\frac{\sup_{k \in \mathbb{N}} \|\rho_k^h\|_{L^\alpha(P_y)}}{\sqrt{h}} \right)^\alpha. \end{aligned} \tag{37}$$

Then Lemma 5.1 proves (36). □

Proposition 5.3. *Let $\alpha \in (0, \infty)$. Then, for all $T \in (0, \infty)$, $y \in I^\circ$ and $h \in (0, \bar{h})$, it holds that*

$$\left\| \sup_{k \in \{1, \dots, \lfloor T/h \rfloor\}} \left| \left(\sum_{n=1}^k E[\rho_n^h | \mathcal{F}_{\tau_{n-1}^h}] \right) - kh \right| \right\|_{L^\alpha(P_y)} \leq \frac{T}{h} \sup_{z \in I_h} |E_z[H_{z-a_h(z), z+a_h(z)}(Y)] - h|. \tag{38}$$

Proof. Without loss of generality we consider $\alpha \in [1, \infty)$. Throughout the proof we fix $T \in (0, \infty)$, $y \in I^\circ$, $h \in (0, \bar{h})$ and let $N = \lfloor T/h \rfloor$. For all $k \in \{0, \dots, N\}$ we define $\mathcal{G}_k = \mathcal{F}_{\tau_k^h}$. The triangle inequality ensures that

$$\begin{aligned} \left\| \sup_{k \in \{1, \dots, N\}} \left| \left(\sum_{n=1}^k E[\rho_n^h | \mathcal{G}_{n-1}] \right) - kh \right| \right\|_{L^\alpha(P_y)} &= \left\| \sup_{k \in \{1, \dots, N\}} \left| \left(\sum_{n=1}^k E[\rho_n^h | \mathcal{G}_{n-1}] - h \right) \right| \right\|_{L^\alpha(P_y)} \\ &\leq \sum_{n=1}^N \| (E[\rho_n^h | \mathcal{G}_{n-1}] - h) \|_{L^\alpha(P_y)}. \end{aligned} \tag{39}$$

By Proposition 3.1, on the event $\{Y_{\tau_{n-1}^h} \in I_h\}$ we have

$$\begin{aligned} &|E_y[\rho_n^h | \mathcal{G}_{n-1}] - h| \\ &= \left| \frac{1}{2} \int_{(Y_{\tau_{n-1}^h} - a_h(Y_{\tau_{n-1}^h}), Y_{\tau_{n-1}^h} + a_h(Y_{\tau_{n-1}^h}))} (a_h(Y_{\tau_{n-1}^h}) - |u - Y_{\tau_{n-1}^h}|) m(du) - h \right| \\ &\leq \sup_{z \in I_h} \left| \frac{1}{2} \int_{(z-a_h(z), z+a_h(z))} (a_h(z) - |u - z|) m(du) - h \right| \\ &= \sup_{z \in I_h} |E_z[H_{z-a_h(z), z+a_h(z)}(Y)] - h|. \end{aligned} \tag{40}$$

On the event $\{Y_{\tau_{n-1}^h} \notin I_h\}$ we have $|E_y[\rho_n^h | \mathcal{G}_{n-1}] - h| = 0$. Therefore,

$$\begin{aligned} \left\| \sup_{k \in \{1, \dots, N\}} \left| \left(\sum_{n=1}^k E[\rho_n^h | \mathcal{G}_{n-1}] \right) - kh \right| \right\|_{L^\alpha(P_y)} &\leq N \sup_{z \in I_h} |E_z[H_{z-a_h(z), z+a_h(z)}(Y)] - h| \\ &\leq \frac{T}{h} \sup_{z \in I_h} |E_z[H_{z-a_h(z), z+a_h(z)}(Y)] - h|. \end{aligned} \tag{41}$$

This proves (38). □

By combining the two preceding theorems we obtain a result about uniform in k convergence of the embedding stopping times (τ_k^h) in spaces $L^\alpha(P_y)$, as $h \rightarrow 0$. To this end, we impose a slightly stronger condition than Condition (A), namely,

$$\sup_{y \in I_h} \left| \frac{1}{2} \int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u - y|) m(du) - h \right| \in o(h), \quad h \rightarrow 0. \tag{42}$$

Corollary 5.4. *Assume (42). Let $\alpha \in (0, \infty)$, $T \in (0, \infty)$ and $y \in I^\circ$. Then it holds*

$$\sup_{k \in \{1, \dots, \lfloor T/h \rfloor\}} |\tau_k^h - kh| \xrightarrow{L^\alpha(P_y)} 0, \quad h \rightarrow 0.$$

Proof. The proof is an application of Propositions 5.2 and 5.3. The fact that the right-hand side of (38) converges to zero as $h \rightarrow 0$ is a direct consequence of (42) (also recall Remark 1.2). Similarly, (42) implies

$$\frac{\sup_{z \in I_h} (E_z[H_{z-a_h(z), z+a_h(z)}(Y)])}{\sqrt{h}} \rightarrow 0, \quad h \rightarrow 0.$$

The remaining property

$$\frac{\sup_{z \in I \setminus I_h} (E_z[H_{z-a_h(z), z+a_h(z)}(Y)])}{\sqrt{h}} \rightarrow 0, \quad h \rightarrow 0$$

(cf. (36)), follows from the fact that, by the definition of I_h , we have

$$E_z[H_{z-a_h(z), z+a_h(z)}(Y)] \leq h, \quad z \in I \setminus I_h.$$

This concludes the proof. □

Proof of Theorem 1.1. For any $h \in (0, \bar{h})$, we define the continuous-time process $Y^h = (Y_t^h)_{t \in [0, \infty)}$ by linear interpolation of $(Y_{\tau_k^h})_{k \in \mathbb{N}_0}$. More precisely, we set

$$Y_t^h = Y_{\tau_{\lfloor t/h \rfloor}^h} + (t/h - \lfloor t/h \rfloor)(Y_{\tau_{\lfloor t/h \rfloor + 1}^h} - Y_{\tau_{\lfloor t/h \rfloor}^h}), \quad t \in [0, \infty).$$

Notice that $Y_{kh}^h = Y_{\tau_k^h}$, $k \in \mathbb{N}_0$, and Proposition 3.1 easily extends to

$$\text{Law}_{P_y}(Y_t^h; t \in [0, \infty)) = \text{Law}_P(X_t^{h,y}; t \in [0, \infty)).$$

Therefore, in order to prove Theorem 1.1, it is sufficient to show that the processes $Y^h = (Y_t^h)_{t \in [0, \infty)}$ converge to the process $Y = (Y_t)_{t \in [0, \infty)}$ in probability P_y uniformly on compact intervals, i.e., that, for all $T \in (0, \infty)$, it holds

$$\|Y^h - Y\|_{C[0, T]} \xrightarrow{P_y} 0, \quad h \rightarrow 0,$$

where $\|\cdot\|_{C[0, T]}$ denotes the sup norm in $C([0, T], \mathbb{R})$. In what follows, we use the notation

$$Y^h \xrightarrow{\text{ucp}(P_y)} Y, \quad h \rightarrow 0, \tag{43}$$

for this mode of convergence.

1. In the first step, we prove (43) under assumption (42), which is stronger than Condition (A). To this end, fix $T \in (0, \infty)$ and $\varepsilon > 0$. Take an arbitrary $T' \in (0, \infty)$, $T' > T$, and choose $\delta \in (0, \frac{T'-T}{2})$ such that $P_y(A(\delta)) > 1 - \frac{\varepsilon}{2}$, where

$$A(\delta) = \left\{ |Y_t - Y_s| < \frac{\varepsilon}{2} \text{ for all } s, t \in [0, T'] \text{ such that } |t - s| < 3\delta \right\}.$$

Corollary 5.4 implies

$$\sup_{k \in \{1, \dots, \lfloor T'/h \rfloor\}} |\tau_k^h - kh| \xrightarrow{P_y} 0, \quad h \rightarrow 0,$$

hence, there exists $\gamma \in (0, \bar{h})$ such that $P_y(C(h, \delta)) > 1 - \frac{\varepsilon}{2}$ whenever $h \in (0, \gamma)$, where

$$C(h, \delta) = \left\{ \sup_{k \in \{1, \dots, \lfloor T'/h \rfloor\}} |\tau_k^h - kh| < \delta \right\}.$$

A somewhat tedious check shows that, if $h \in (0, \delta)$,

$$\|Y^h - Y\|_{C[0, T]} < \varepsilon \quad \text{on } A(\delta) \cap C(h, \delta).$$

Thus, we get $P_y(\|Y^h - Y\|_{C[0, T]} > \varepsilon) < \varepsilon$ whenever $h \in (0, \gamma \wedge \delta)$. This completes the proof of the first step.

2. We now prove (43) under Condition (A). Consider strictly monotone sequences $\{l_n\}_{n \in \mathbb{N}}$ and $\{r_n\}_{n \in \mathbb{N}}$ with $l_n \searrow l$ and $r_n \nearrow r$. We define compact subintervals K_n of I° by setting $K_n = [l_n, r_n]$, $n \in \mathbb{N}$, and modified scale factors $\tilde{a}_h^n: \bar{T} \rightarrow [0, \infty)$ by setting

$$\tilde{a}_h^n(y) = \begin{cases} a_h(y), & y \in K_n, \\ \hat{a}_h(y), & y \in \bar{T} \setminus K_n, \end{cases}$$

$n \in \mathbb{N}$, $h \in (0, \bar{h})$, where the scale factors \widehat{a}_h , $h \in (0, \bar{h})$, are the ones from the EMCEL algorithm (recall (11)). Let $(\widetilde{\tau}_k^{n,h})_{k \in \mathbb{N}_0}$ be the associated sequences of the embedding stopping times and $\widetilde{Y}^{n,h} = (\widetilde{Y}_t^{n,h})_{t \in [0, \infty)}$ the analogues of the process $Y^h = (Y_t^h)_{t \in [0, \infty)}$ for the modified scale factors \widetilde{a}_h^n , $n \in \mathbb{N}$, $h \in (0, \bar{h})$. Since the scale factors $(a_h)_{h \in (0, \bar{h})}$ satisfy Condition (A), the modified scale factors $(\widetilde{a}_h^n)_{h \in (0, \bar{h})}$ satisfy (42) for each $n \in \mathbb{N}$. By the first step of the proof,

$$\widetilde{Y}^{n,h} \xrightarrow{\text{ucp}(P_Y)} Y, \quad h \rightarrow 0, \quad (44)$$

for any fixed $n \in \mathbb{N}$.

Fix $T \in (0, \infty)$ and $\varepsilon > 0$. For any $n \in \mathbb{N}$, we define the events

$$A_n = \{H_{l,r}(Y) \leq T + 2 \text{ and } \exists t \in [H_{l_n, r_n}(Y), H_{l,r}(Y)] \text{ such that } |Y_t - Y_{H_{l,r}(Y)}| > \varepsilon\},$$

$$B_n = \{H_{l,r}(Y) > T + 2 \text{ and } H_{l_n, r_n}(Y) \leq T + 1\}.$$

Notice that the expression $Y_{H_{l,r}(Y)}$ in the above formula for A_n is well-defined and finite. Indeed, this is the position of Y at an accessible boundary (because $H_{l,r}(Y) \leq T + 2 < \infty$), while an infinite boundary cannot be accessible (because Y is in natural scale). As $H_{l_n, r_n}(Y) \nearrow H_{l,r}(Y)$ P_Y -a.s., as $n \rightarrow \infty$, and Y is continuous, we can choose a sufficiently big $n_0 \in \mathbb{N}$ such that $P_Y(A_{n_0}) < \frac{\varepsilon}{3}$ and $P_Y(B_{n_0}) < \frac{\varepsilon}{3}$. We also take an arbitrary $T' \in (T, T + 1)$. Corollary 5.4 applied to the modified scale factors $(\widetilde{a}_h^{n_0})_{h \in (0, \bar{h})}$, which satisfy (42), yields that there exists $\gamma > 0$ such that, for any $h \in (0, \gamma)$, we have

$$P_Y(\widetilde{\tau}_{\lfloor T'/h \rfloor}^{n_0, h} \leq T + 1) > 1 - \frac{\varepsilon}{3}.$$

For $h \in (0, \bar{h})$, we define the event

$$C_h = \{\widetilde{\tau}_{\lfloor T'/h \rfloor}^{n_0, h} \leq T + 1\} \cap (A_{n_0} \cup B_{n_0})^c$$

(the notation D^c means the complement of an event D). Notice that $P_Y(C_h) > 1 - \varepsilon$ whenever $h \in (0, \gamma)$. Furthermore, on C_h we have either

$$H_{l,r}(Y) > T + 2, \quad H_{l_{n_0}, r_{n_0}}(Y) > T + 1, \quad \text{hence } Y_t^h = \widetilde{Y}_t^{n_0, h}, \quad t \in [0, T],$$

or

$$H_{l,r}(Y) \leq T + 2, \quad |Y_t - Y_{H_{l,r}(Y)}| \leq \varepsilon \quad \text{for } t \in [H_{l_{n_0}, r_{n_0}}(Y), H_{l,r}(Y)],$$

$$\text{hence } |Y_t^h - Y_t| \leq \varepsilon \quad \text{whenever } Y_t^h \neq \widetilde{Y}_t^{n_0, h}, \quad t \in [0, T].$$

Together with (44), this proves $\|Y^h - Y\|_{C[0, T]} \xrightarrow{P_Y} 0$ as $h \rightarrow 0$. As $T \in (0, \infty)$ is arbitrary, we obtain (43). This concludes the proof. \square

6. Reflecting boundaries

Throughout the preceding sections we assume that if a boundary point is accessible, then it is absorbing. In this section we explain how one can drop this assumption, i.e. how one can extend our functional limit theorem, Theorem 1.1, to Markov processes with reflecting boundaries.

The idea is to reduce the reflecting case to the inaccessible or absorbing case. Indeed, for every Markov process Z with reflecting boundaries one can find a Markov process Y on an extended state space and a Lipschitz function f such that Y has inaccessible or absorbing boundaries and $Z \stackrel{d}{=} f(Y)$.

We illustrate the reduction for a Markov process Z in natural scale with state space $I_Z = [l, \infty)$, where $l > -\infty$ is a reflecting boundary. We denote by m_Z the speed measure of Z . Since l is non-absorbing, it must hold that $m_Z(\{l\}) < \infty$. Notice that $m_Z(\{l\}) = 0$ corresponds to instantaneous reflection, while $m_Z(\{l\}) \in (0, \infty)$ to slow reflection.

To proceed with the construction, we first remark that it holds

$$m_Z(l, l + 1) < \infty. \quad (45)$$

Indeed, in terms of the Feller boundary classification (see Table 15.6.2 in [31]), as the accessible boundary point l is reflecting, it can only be *regular*, which implies (45).

Now let Y be a Markov process in natural scale with state space $I_Y = \mathbb{R}$ and speed measure m_Y satisfying

$$\begin{aligned} m_Y(A) &= m_Z(A), \quad \text{for all } A \in \mathcal{B}(\mathbb{R}), A \subseteq (l, \infty), \\ m_Y(A) &= m_Z(2l - A), \quad \text{for all } A \in \mathcal{B}(\mathbb{R}), A \subseteq (-\infty, l), \\ m_Y(\{l\}) &= 2m_Z(\{l\}), \end{aligned}$$

which is a valid speed measure on $I_Y = \mathbb{R}$ (i.e., (4) holds) due to (45). Then $l + |Y - l|$ has the same distribution as Z (see Proposition VII.3.10 in [39]).

Let $(a_h)_{h \in (0, \bar{h})}$ satisfy the assumptions of Theorem 1.1. Then $(X^h)_{h \in (0, \bar{h})}$ converges in distribution to Y as $h \rightarrow 0$. This implies that the processes $(l + |X^h - l|)_{h \in (0, \bar{h})}$ converge in distribution to Z as $h \rightarrow 0$.

In a similar way, a Markov process Z on a bounded interval I_Z with endpoints l and r ($l < r$), where $l \in I_Z$ is reflecting and r is inaccessible (resp., absorbing), can be reduced to a Markov process Y with state space I_Y , which is the interval with endpoints $2l - r$ and r , where both these endpoints are inaccessible (resp., absorbing).

A Markov process Z with two reflecting boundaries can be reduced to a Markov process with state space \mathbb{R} . To explain this, suppose for simplicity that the state space of Z is $[0, 1]$. Define Y as the Markov process on \mathbb{R} with speed measure m_Y satisfying

$$\begin{aligned} m_Y(A) &= m_Y(-A) = m_Z(A), \quad \text{for all } A \in \mathcal{B}((0, 1)), \\ m_Y(A + 2k) &= m_Z(A), \quad \text{for all } A \in \mathcal{B}((-1, 0) \cup (0, 1)) \text{ and } k \in \mathbb{Z}, \\ m_Y(\{2k\}) &= 2m_Z(\{0\}), \quad \text{for all } k \in \mathbb{Z}, \\ m_Y(\{2k + 1\}) &= 2m_Z(\{1\}), \quad \text{for all } k \in \mathbb{Z}. \end{aligned}$$

Let $f: \mathbb{R} \rightarrow [0, 1]$ be the periodic function with period 2 satisfying $f(x) = |x|$, $x \in [-1, 1]$. Then the process $f(Y)$ has the same distribution as Z (cf. Proposition VII.3.10 in [39]).

7. Examples with sticky points

In this section we apply Theorem 1.1 to sticky Brownian motions on \mathbb{R} and on $[0, \infty)$, where the sticky point is zero. In the latter case one also speaks about slow (or sticky) reflection at 0. Recent years have witnessed an increased interest in the sticky Brownian motion and related processes, see [7,12,14,26,29] and references therein. Newly, diffusions with slow reflection were applied in [13] to provide bounds (via sticky couplings) for the distance between two multidimensional diffusions with different drifts. Stickiness is a convenient concept for modeling repulsive interactions between particles, and, motivated by natural questions from physics, it is discussed in multi- and infinite-dimensional situations in [18,23,24,33,34]. Diffusions with slow reflection also attracted interest in economic theory, where such processes characterize optimal continuation values in dynamic principal-agent problems (see, e.g., [46] and [38]).

On the contrary, the literature on *approximations* of diffusions with atoms in the speed measure is scarce. We remark that [1] provides a sequence of random walks that converges in distribution to the Brownian motion on \mathbb{R} sticky at zero. The random walks considered there are forced to stay in zero for some time whenever they visit zero. In contrast to our approach, the approximating processes are not Markov chains. [19] constructs Markov chains that converge in distribution to the Brownian motion on $[0, \infty)$ with slow reflection at 0. The approximating Markov chains considered there exhibit sticky behavior in zero in the sense that once the Markov chains reach zero they stay there with positive probability also in the next time period. The recent work [9] proposes an approximation of the Brownian motion on $[0, \infty)$ with slow reflection at 0 by continuous-time pure jump Markov processes Y^δ , $\delta \in (0, \infty)$, with uniform grids $\{0, \delta, 2\delta, \dots\}$ as state spaces. The jump times are exponentially distributed and the mean waiting time at the interior points $\{\delta, 2\delta, \dots\}$ is of order δ^2 whereas it is of order δ at the origin 0.

7.1. Brownian motion on \mathbb{R} with sticky point 0

Brownian motion on \mathbb{R} sticky at 0 is a Markov process Y in natural scale with state space $I = \mathbb{R}$ and speed measure

$$m(dx) = \frac{2}{\sigma^2} \lambda(dx) + \frac{2}{\theta} \delta_0(dx), \tag{46}$$

where $\sigma, \theta \in (0, \infty)$ and $\lambda(dx)$ denotes the Lebesgue measure.³ Such a process Y behaves like σ times a Brownian motion outside zero, but spends a positive amount of time at zero having no intervals of zeros. Notice that the bigger θ is, the less time Y spends at zero; $\theta = \infty$ corresponds to a standard Brownian motion (times σ).

It is instructive to compute the function $q(y, x)$, $y, x \in \mathbb{R}$, of (20)

$$q(y, x) = \begin{cases} \frac{(x-y)^2}{\sigma^2} + 2\frac{x^-}{\theta} & \text{if } y > 0, \\ \frac{x^2}{\sigma^2} + \frac{|x|}{\theta} & \text{if } y = 0, \\ \frac{(x-y)^2}{\sigma^2} + 2\frac{x^+}{\theta} & \text{if } y < 0 \end{cases}$$

($x^+ = \max\{x, 0\}$, $x^- = -\min\{x, 0\}$) and to observe that, for any $y \in \mathbb{R}$, the function $q(y, \cdot)$ has a kink at zero.

We now determine, for every $h \in (0, \infty)$, a function $\widehat{a}_h: \mathbb{R} \rightarrow (0, \infty)$ such that the associated Markov chain $(\widehat{X}_{hk}^h)_{k \in \mathbb{N}_0}$, defined in (5), belongs to EMCEL(h). Indeed, one can explicitly determine for all $y \in \mathbb{R}$ the real number $\widehat{a}_h(y)$ satisfying

$$\frac{1}{2} \int_{(y-\widehat{a}_h(y), y+\widehat{a}_h(y))} (\widehat{a}_h(y) - |u - y|) m(du) = h. \tag{47}$$

We state the closed-form representations of \widehat{a}_h in the next Lemma.

Lemma 7.1. *For all $h \in (0, \infty)$ and $y \in \mathbb{R}$ let*

$$\widehat{a}_h(y) = \begin{cases} \sigma\sqrt{h} & \text{if } |y| \geq \sigma\sqrt{h}, \\ \sigma(\sqrt{h + \frac{|y|}{\theta} + (\frac{\sigma}{2\theta})^2} - \frac{\sigma}{2\theta}) & \text{if } |y| < \sigma\sqrt{h}. \end{cases}$$

Then Equation (47) is satisfied for all $h \in (0, \infty)$ and $y \in \mathbb{R}$.

Proof. Throughout the proof fix $h \in (0, \infty)$ and $y \in \mathbb{R}$. For every $a \in [0, \infty)$ it holds that

$$\frac{1}{2} \int_{(y-a, y+a)} (a - |u - y|) m(du) = \frac{a^2}{\sigma^2} + \frac{a - |y|}{\theta} 1_{(y-a, y+a)}(0) = \frac{a^2}{\sigma^2} + \frac{a - |y|}{\theta} 1_{[0, a)}(|y|).$$

Assume first that $|y| \geq \sigma\sqrt{h}$. Then it holds that $|y| \geq \widehat{a}_h(y)$ and hence (47) is satisfied. Next assume that $|y| < \sigma\sqrt{h}$. In this case it holds that $\widehat{a}_h(y) > |y|$. Moreover it holds that

$$\frac{\widehat{a}_h(y)^2}{\sigma^2} + \frac{\widehat{a}_h(y) - |y|}{\theta} = h. \tag{48}$$

This proves (47) in the case $|y| < \sigma\sqrt{h}$. The proof is thus completed. □

Since \widehat{a}_h satisfies Equation (47) exactly, it follows that Condition (A) is satisfied. Theorem 1.1 implies that the processes (X^h) converge in distribution to Y as $h \rightarrow 0$. Figure 1 depicts two realizations of a Brownian motion on \mathbb{R} sticky at 0 with $\sigma = 1$ and different values for θ as well as the empirical distribution function of \widehat{X}_1^h with $h = 10^{-3}$.

7.2. Brownian motion on $[0, \infty)$ with slow reflection at 0

In this section we consider a Brownian motion on $[0, \infty)$ with slow reflection at 0. We first define this process, as in Warren [44], as the solution of SDE (49) below. We subsequently show that its distribution is identical to the distribution of $|Y|$, where Y is the general diffusion analyzed in Section 7.1. From this perspective, the main difference between the processes studied in this section and those in Section 7.1 is the state space.

Let $\sigma, \theta \in (0, \infty)$. According to Theorem IV.7.2 in [27] the stochastic differential equation

$$dZ_t = \theta 1_{\{Z_t=0\}} dt + \sigma 1_{\{Z_t>0\}} dW_t, \quad Z_0 = 0, \tag{49}$$

possesses a weak solution that is unique in law. However, it is worth noting that neither existence of a strong solution nor pathwise uniqueness hold for (49) (see [14] and references therein). The next result shows that Z is a regular diffusion on $[0, \infty)$ and identifies the associated speed measure.

³Since there exist different conventions concerning the normalization of the speed measure (cf. Footnote 1), our representation of m in (46) may differ by a factor of 2 from related representations found in the literature (cf., e.g., [7]).

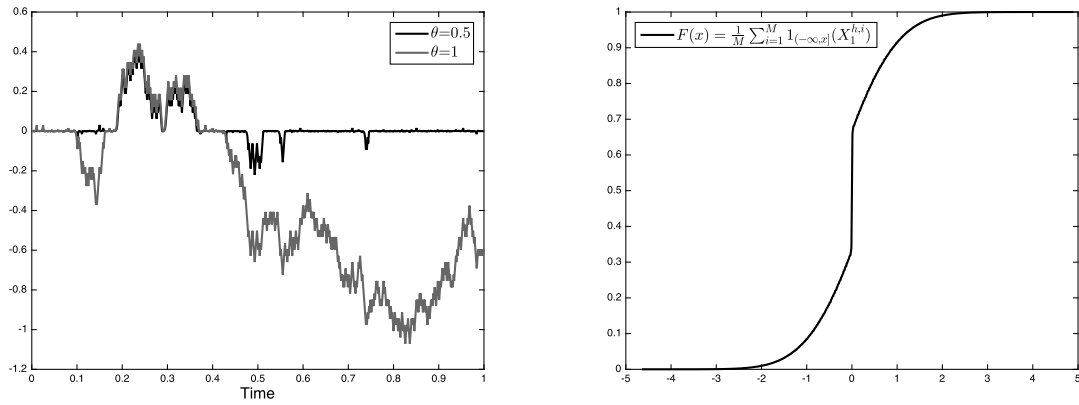


Fig. 1. Left: Two trajectories of the approximation of the sticky Brownian motion. The black line depicts one realization of $(\widehat{X}_t^h)_{t \in [0,1]}$ with $h = 10^{-3}$, $\sigma = 1$ and $\theta = 0.5$. The gray line shows one realization of $(\widehat{X}_t^h)_{t \in [0,1]}$ with $h = 10^{-3}$, $\sigma = 1$ and $\theta = 1$. Both trajectories are generated by the same sample of random increments $(\xi_k)_{k \in \mathbb{N}}$. Observe that the smaller the value of θ is, the more the process sticks to 0. Right: Empirical distribution function of the approximation of the sticky Brownian motion. The figure depicts the function $F: \mathbb{R} \rightarrow [0, 1]$, $F(x) = \frac{1}{M} \sum_{i=1}^M 1_{(-\infty, x]}(\widehat{X}_1^{h,i})$, where $(\widehat{X}_1^{h,i})_{i \in \{1, \dots, M\}}$ are $M = 10^6$ independent realizations of \widehat{X}_1^h with $h = 10^{-3}$, $\sigma = 1$ and $\theta = 1$. Observe that a jump at 0 becomes apparent. This reflects the fact that the (weak) limit Y_1 of \widehat{X}_1^h is with positive probability equal to 0. Notice that the distribution function of Y_1 is known in closed form (cf. Lemma 7.3 below). We refrain from plotting it in the figure on the right-hand side because with the current image scaling it is visually nearly indistinguishable from F .

Lemma 7.2. *The solution Z of (49) is a regular continuous strong Markov process in natural scale with state space $I_Z = [0, \infty)$ and with speed measure*

$$m_Z(dz) = \frac{2}{\sigma^2} \lambda(dz) + \frac{1}{\theta} \delta_0(dz). \tag{50}$$

Proof. Strong Markov property of Z is implied by the uniqueness in law for (49). Clearly, Z is regular with state space $I_Z = [0, \infty)$ and in natural scale. By Itô's formula, we have

$$f(Z_t) = f(Z_0) + \int_0^t \left(\theta f'(0) 1_{\{Z_s=0\}} + \frac{\sigma^2}{2} f''(Z_s) 1_{\{Z_s>0\}} \right) ds + \int_0^t \sigma f'(Z_s) 1_{\{Z_s>0\}} dW_s$$

for C^2 functions $f: [0, \infty) \rightarrow \mathbb{R}$. Therefore, the generator \mathcal{A} of Z takes the form

$$\mathcal{A}f(z) = \begin{cases} \theta f'(0) & \text{if } z = 0, \\ \frac{\sigma^2}{2} f''(z) & \text{if } z > 0 \end{cases} \tag{51}$$

for $f \in C_0^2([0, \infty))$ (this means that the function itself and its first and second derivative vanish at infinity) satisfying the boundary condition

$$\theta f'(0) = \frac{\sigma^2}{2} f''(0).$$

By Theorem VII.3.12 in [39], we have $\mathcal{A}f(z) = \frac{d}{dm_Z} f'(z)$ in the interior of the state space, i.e., for $z > 0$, while, by Proposition VII.3.13 in [39], it holds $f'(0) = m(\{0\}) \mathcal{A}f(0)$ on the boundary. Together with (51), this implies (50) and concludes the proof. \square

It follows from Section 6 that $Z \stackrel{d}{=} |Y|$, where Y is a diffusion in natural scale with state space $I_Y = \mathbb{R}$ and speed measure $m_Y(dz) = \frac{2}{\sigma^2} \lambda(dz) + \frac{2}{\theta} \delta_0(dz)$, i.e., Y is the process studied in Section 7.1 (cf. (46)). In particular, Z can be approximated by $(|\widehat{X}^h|)_{h \in (0, \infty)}$, where each \widehat{X}^h is the EMCEL(h) constructed in Section 7.1.

Warren [44] determines for all $t \in (0, \infty)$ the conditional law of Z_t given the driving Brownian motion W . As a consequence, we obtain for all $t \in (0, \infty)$ closed form representations of the cumulative distribution function and the expected value of Z_t . The precise formulas are provided in Lemma 7.3 below, where we, without loss of generality, consider $\sigma = 1$.

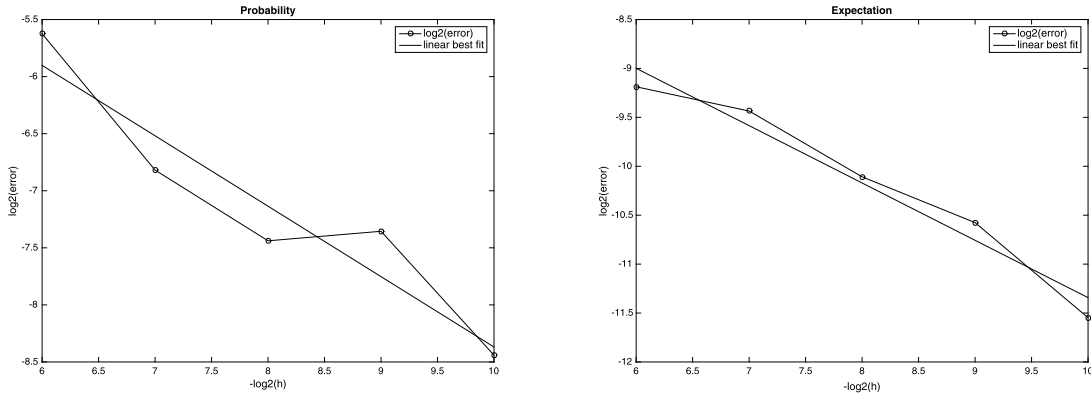


Fig. 2. The parameter values are $\sigma = 1$ and $\theta = 1/2$ for both plots. Left: Empirical rate of convergence of the distribution function. The circles depict the five data points $\{(-\log_2(h), \log_2 |\frac{1}{M} \sum_{i=1}^M 1_{[0,0.1]}(|\widehat{X}_1^{h,i}|) - F_Z(0.1; 1))|, h = 2^{-6}, \dots, 2^{-10}\}$, where $F_Z(0.1; 1) = P_0[Z_1 \leq 0.1] \approx 0.5741$ (see Lemma 7.3) and $(\widehat{X}_1^{h,i})_{i \in \{1, \dots, M\}}$ are $M = 10^8$ independent realizations of \widehat{X}_1^h . The straight line is the linear best fit. Its slope is approximately -0.62 . Right: Empirical rate of convergence of the expected value. The circles depict the five data points $\{(-\log_2(h), \log_2 |\frac{1}{M} \sum_{i=1}^M |\widehat{X}_1^{h,i}| - E_0[Z_1])|, h = 2^{-6}, \dots, 2^{-10}\}$, where $E_0[Z_1] \approx 0.3210$ (see Lemma 7.3) and $(\widehat{X}_1^{h,i})_{i \in \{1, \dots, M\}}$ are $M = 10^8$ independent realizations of X_1^h . The straight line is the linear best fit. Its slope is approximately -0.59 .

The notations P_0 for the probability measure and E_0 for the corresponding expectation operator emphasize that the formulas are given for the case $Z_0 = 0$. We use these formulas to analyze the empirical rate of convergence of EMCEL approximations. The results are presented in Figure 2.

Lemma 7.3. *Let Z be a solution of (49) with $\sigma = 1$. For every $t \in (0, \infty)$ the cumulative distribution function $F(\cdot; t): [0, \infty) \rightarrow [0, 1]$ of Z_t satisfies*

$$F_Z(z; t) := P_0[Z_t \leq z] = 2\Phi\left(\frac{z}{\sqrt{t}}\right) - 1 + 2e^{2\theta(z+\theta t)}\Phi\left(-2\theta\sqrt{t} - \frac{z}{\sqrt{t}}\right), \quad z \in [0, \infty), \tag{52}$$

where $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$ is the cumulative distribution function of the standard normal distribution. Moreover, it holds that

$$E_0[Z_t] = \sqrt{\frac{2t}{\pi}} - \frac{1}{2\theta} + \frac{e^{2\theta^2 t}}{\theta} \Phi(-2\theta\sqrt{t}) = E_0[|W_t|] - \frac{1}{2\theta} + \frac{e^{2\theta^2 t}}{\theta} \Phi(-2\theta\sqrt{t}). \tag{53}$$

Proof. Fix $t \in (0, \infty)$ throughout the proof. Lévy’s distributional theorem implies $W_t + \sup_{s \in [0,t]} (-W_s) \stackrel{d}{=} |W_t|$ (see Theorem VI.2.3 in [39]). Then it follows from Theorem 1 in [44] that for all $z \in [0, \infty)$ it holds

$$\begin{aligned} P_0[Z_t \leq z] &= E_0[e^{-2\theta(W_t + \sup_{s \in [0,t]} (-W_s) - z)^+}] = E_0[1_{[0,z]}(|W_t|) + e^{-2\theta(|W_t| - z)} 1_{[z,\infty)}(|W_t|)] \\ &= 2\Phi\left(\frac{z}{\sqrt{t}}\right) - 1 + \frac{2}{\sqrt{2\pi}} \int_{z/\sqrt{t}}^\infty e^{-2\theta(\sqrt{t}y - z)} e^{-\frac{y^2}{2}} dy \\ &= 2\Phi\left(\frac{z}{\sqrt{t}}\right) - 1 + \frac{2}{\sqrt{2\pi}} e^{2\theta(z+\theta t)} \int_{z/\sqrt{t}}^\infty e^{-\frac{(y+2\theta\sqrt{t})^2}{2}} dy \\ &= 2\Phi\left(\frac{z}{\sqrt{t}}\right) - 1 + 2e^{2\theta(z+\theta t)}\Phi\left(-2\theta\sqrt{t} - \frac{z}{\sqrt{t}}\right). \end{aligned}$$

This proves (52). Moreover, this implies that the density function of Z_t starting in 0 satisfies

$$F'_Z(z; t) = \frac{2}{\sqrt{t}}\Phi'\left(\frac{z}{\sqrt{t}}\right) + 4\theta e^{2\theta(z+\theta t)}\Phi\left(-2\theta\sqrt{t} - \frac{z}{\sqrt{t}}\right) - \frac{2}{\sqrt{t}}e^{2\theta(z+\theta t)}\Phi'\left(-2\theta\sqrt{t} - \frac{z}{\sqrt{t}}\right) \tag{54}$$

for all $z \in (0, \infty)$. Observe that for all $z \in (0, \infty)$ it holds

$$e^{2\theta(z+\theta t)}\Phi'\left(-2\theta\sqrt{t} - \frac{z}{\sqrt{t}}\right) = \frac{1}{\sqrt{2\pi}} e^{2\theta(z+\theta t)} e^{-\frac{(2\theta\sqrt{t} + \frac{z}{\sqrt{t}})^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2t}} = \Phi'\left(\frac{z}{\sqrt{t}}\right). \tag{55}$$

This implies for all $z \in (0, \infty)$ that

$$F'_Z(z; t) = 4\theta e^{2\theta(z+\theta t)} \Phi\left(-2\theta\sqrt{t} - \frac{z}{\sqrt{t}}\right). \tag{56}$$

This and Fubini's theorem prove that

$$\begin{aligned} E_0[Z_t] &= \int_0^\infty z F'_Z(z; t) dz = \frac{4\theta e^{2\theta^2 t}}{\sqrt{2\pi}} \int_0^\infty z e^{2\theta z} \int_{-\infty}^{-2\theta\sqrt{t} - \frac{z}{\sqrt{t}}} e^{-\frac{y^2}{2}} dy dz \\ &= \frac{4\theta e^{2\theta^2 t}}{\sqrt{2\pi}} \int_{-\infty}^{-2\theta\sqrt{t}} e^{-\frac{y^2}{2}} \int_0^{-y\sqrt{t}-2\theta t} z e^{2\theta z} dz dy \\ &= \frac{e^{2\theta^2 t}}{\theta\sqrt{2\pi}} \int_{-\infty}^{-2\theta\sqrt{t}} e^{-\frac{y^2}{2}} [1 - e^{-2\theta(y\sqrt{t}+2\theta t)}(2\theta(y\sqrt{t} + 2\theta t) + 1)] dy \\ &= \frac{e^{2\theta^2 t}}{\theta} \Phi(-2\theta\sqrt{t}) - \frac{1}{\theta\sqrt{2\pi}} \int_{-\infty}^{-2\theta\sqrt{t}} e^{-\frac{(y+2\theta\sqrt{t})^2}{2}} (2\theta\sqrt{t}(y + 2\theta\sqrt{t}) + 1) dy \\ &= \frac{e^{2\theta^2 t}}{\theta} \Phi(-2\theta\sqrt{t}) - \frac{1}{\theta\sqrt{2\pi}} \int_{-\infty}^0 e^{-\frac{y^2}{2}} (2\theta\sqrt{t}y + 1) dy \\ &= \frac{e^{2\theta^2 t}}{\theta} \Phi(-2\theta\sqrt{t}) - \frac{1}{2\theta} - \frac{2\sqrt{t}}{\sqrt{2\pi}} \int_{-\infty}^0 ye^{-\frac{y^2}{2}} dy = \frac{e^{2\theta^2 t}}{\theta} \Phi(-2\theta\sqrt{t}) - \frac{1}{2\theta} + \sqrt{\frac{2t}{\pi}}. \end{aligned} \tag{57}$$

This completes the proof. □

Remark 7.4. Another possibility to get (52) and (53) is as follows. An explicit formula for the transition density of a sticky Brownian motion is provided in Part I, Appendix 1, Section 8 of [8]. This yields formula (56) for the density of Z_t (notice that the factor 4, which is not present in [8], is due to the facts that the mentioned formula in [8] is given for a sticky Brownian motion on \mathbb{R} and the transition density in [8] is given with respect to the speed measure, i.e., twice the Lebesgue measure outside zero). Now (53) follows by the same calculation as above, while the distribution function (52) can be recovered by integrating the density and taking into account the atom at zero.

8. Brownian motion slowed down on the Cantor set

In this section we apply our results to construct a family of Markov chains $(X^h)_{h \in (0,1)}$ that converge in distribution to the general diffusion Y on \mathbb{R} with speed measure $m(dx) = m_C(dx) + 2dx$, where m_C is the Cantor distribution. Such a process Y can be understood as a Brownian motion slowed down on the Cantor set.

For later reference we briefly recall a way to construct the Cantor distribution. To this end let \mathcal{C} be the collection of all subsets of $[0, 1]$ and let $\Psi : \mathcal{C} \rightarrow \mathcal{C}$ be the map given by

$$\Psi(A) = \frac{A}{3} \cup \frac{A+2}{3}, \quad A \subseteq [0, 1]. \tag{58}$$

Next, we define recursively a sequence $(C_n)_{n \in \mathbb{N}_0}$ of subsets of $[0, 1]$. Let $C_0 = [0, 1]$ and for $n \in \mathbb{N}$ let

$$C_n = \Psi(C_{n-1}). \tag{59}$$

The Cantor set is defined as $C = \bigcap_{n \in \mathbb{N}} C_n$.

We define for all $n \in \mathbb{N}$ the probability measure m_n on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by $m_n(dx) = (\frac{3}{2})^n 1_{C_n}(x) dx$. Note that m_n is absolutely continuous with respect to the Lebesgue measure μ_L . It follows from the proof of Theorem 3.1 in [41] that the sequence $(m_n)_{n \in \mathbb{N}}$ converges in distribution to a probability measure m_C on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and that for all $n \in \mathbb{N}$ it holds

$$\sup_{x \in \mathbb{R}} |m_C((-\infty, x]) - m_n((-\infty, x])| \leq 2^{-(n-1)}. \tag{60}$$

Moreover, it holds that $m_C(C) = 1$ (in particular, m_C is concentrated on $[0, 1]$), $\mu_L(C) = 0$ and, for all $x \in \mathbb{R}$, $m_C(\{x\}) = 0$, i.e., m_C is a singular-continuous measure.

Proposition 8.1. *Let m be the measure on \mathbb{R} given by $m(dx) = m_C(dx) + 2dx$ and let Y be the associated diffusion. Let $n: (0, 1) \rightarrow \mathbb{N}$ be a function satisfying $\lim_{h \rightarrow 0} 2^{n(h)}\sqrt{h} = \infty$. Then there exists for all $h \in (0, 1)$, $y \in \mathbb{R}$ a unique solution $a_h(y) \in (0, \sqrt{h}]$ of the equation*

$$\frac{1}{2} \left(\frac{3}{2}\right)^{n(h)} \int_{y-a_h(y)}^{y+a_h(y)} 1_{C_{n(h)}}(u)(a_h(y) - |u - y|) du + a_h^2(y) = h. \quad (61)$$

Let $(X^h)_{h \in (0,1)}$ be the family of Markov chains defined in (5) and (6) (with scale factors a_h , $h \in (0, 1)$, given by the solution of (61)). Then for all $y \in \mathbb{R}$ the distributions of $(X_t^{h,y})_{t \in [0, \infty)}$, $h \in (0, 1)$, under P converge weakly to the distribution of $(Y_t)_{t \in [0, \infty)}$ under P_y , as $h \rightarrow 0$.

Proof. First observe that for all $y \in \mathbb{R}$ the mapping

$$[0, \infty) \ni a \mapsto \frac{1}{2} \left(\frac{3}{2}\right)^{n(h)} \int_{y-a}^{y+a} 1_{C_{n(h)}}(u)(a - |u - y|) du + a^2 \in [0, \infty) \quad (62)$$

is continuous and strictly increasing. This ensures existence of a unique solution $a_h(y) \in [0, \infty)$ of (61). It follows from

$$a_h^2(y) \leq \frac{1}{2} \left(\frac{3}{2}\right)^{n(h)} \int_{y-a_h(y)}^{y+a_h(y)} 1_{C_{n(h)}}(u)(a_h(y) - |u - y|) du + a_h^2(y) = h \quad (63)$$

that $a_h(y) \leq \sqrt{h}$ for all $h \in (0, 1)$, $y \in \mathbb{R}$. Moreover, it follows that for all $h \in (0, 1)$ and $y \in \mathbb{R}$ it holds

$$\begin{aligned} & \frac{1}{2} \int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u - y|)(m_{n(h)} + 2)(du) \\ &= \frac{1}{2} \left(\frac{3}{2}\right)^{n(h)} \int_{y-a_h(y)}^{y+a_h(y)} 1_{C_{n(h)}}(u)(a_h(y) - |u - y|) du + a_h^2(y) = h. \end{aligned} \quad (64)$$

Next, observe that formula (27), definition (20) of q and the fact that m_C and $m_{n(h)}$ do not possess atoms ensure that it holds for all $h \in (0, 1)$ and $y \in \mathbb{R}$ that

$$\begin{aligned} & \int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u - y|)(m_C - m_{n(h)})(du) \\ &= \int_y^{y+a_h(y)} [m_C((y, u)) - m_{n(h)}((y, u))] du + \int_{y-a_h(y)}^y [m_C((u, y)) - m_{n(h)}((u, y))] du \\ &= \int_y^{y+a_h(y)} [m_C((-\infty, u]) - m_{n(h)}((-\infty, u))] du - \int_{y-a_h(y)}^y [m_C((-\infty, u]) - m_{n(h)}((-\infty, u))] du. \end{aligned} \quad (65)$$

This, (60) and the fact that $a_h(y) \leq \sqrt{h}$ show that, for all $h \in (0, 1)$ and $y \in \mathbb{R}$, we have

$$\begin{aligned} & \left| \frac{1}{2} \int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u - y|)(m_C - m_{n(h)})(du) \right| \\ & \leq a_h(y) \sup_{u \in \mathbb{R}} |m_C((-\infty, u]) - m_{n(h)}((-\infty, u])| \leq \sqrt{h} 2^{-(n(h)-1)}. \end{aligned} \quad (66)$$

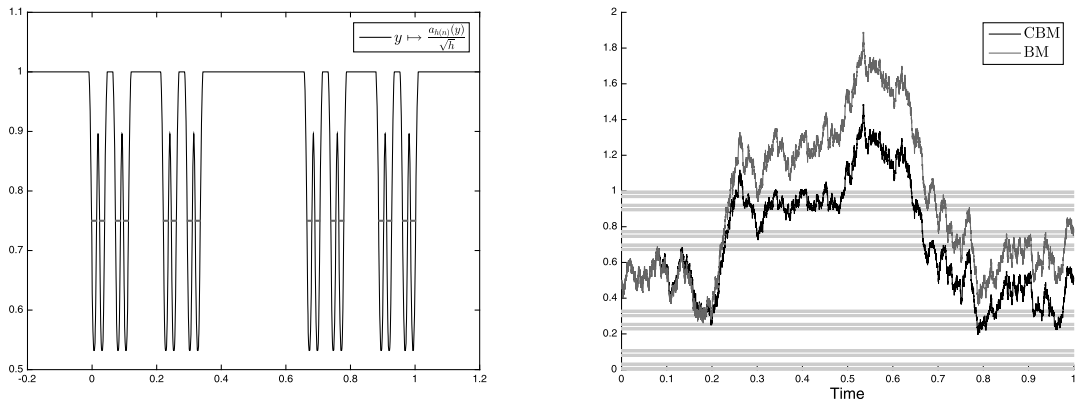


Fig. 3. Left: Normalized scale factor for the approximation of the Brownian motion slowed down on the Cantor set. The black line depicts the function $\mathbb{R} \ni y \mapsto \frac{a_n(h)(y)}{\sqrt{h}} \in \mathbb{R}$, where for all $y \in \mathbb{R}$ the real number $a_n(h)(y)$ is the solution of (61) with $h = 10^{-4}$ and $n(h) = 4$. The gray bars depict the 16 subintervals of the set $C_{n(h)}$. We see that the black line interpolates smoothly between 1 and considerably smaller values around the gray bars. Right: A realization of an approximation of the Brownian motion slowed down on the Cantor set (CBM) and of a Brownian motion (BM). The black line depicts a trajectory of $(X_t^h)_{t \in [0,1]}$ defined in (5) and (6) with scale factor a given by the solution of (61) with $h = 10^{-4}$ and $n(h) = 4$. The gray line shows a realization of $(X_t^h)_{t \in [0,1]}$ defined in (5) and (6) with scale factor a_h given by $a_h \equiv \sqrt{h}$ and $h = 10^{-4}$. Both trajectories are generated from the same sample of random increments $(\xi_k)_{k \in \mathbb{N}}$. The horizontal light gray bars show again the 16 subintervals of the set $C_{n(h)}$. We see that on the set $C_{n(h)}$ the CBM is indeed slowed down (compared with the BM).

Combining (64) and (66) and using the assumption $\lim_{h \rightarrow 0} 2^{n(h)} \sqrt{h} = \infty$ shows that

$$\begin{aligned} & \sup_{y \in \mathbb{R}} \left| \frac{1}{2} \int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u-y|) m(du) - h \right| \\ & \leq \sup_{y \in \mathbb{R}} \left| \frac{1}{2} \int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u-y|) (m_C - m_{n(h)})(du) \right| \\ & \quad + \sup_{y \in \mathbb{R}} \left| \frac{1}{2} \int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u-y|) (m_{n(h)} + 2)(du) - h \right| \\ & = \sup_{y \in \mathbb{R}} \left| \frac{1}{2} \int_{(y-a_h(y), y+a_h(y))} (a_h(y) - |u-y|) (m_C - m_{n(h)})(du) \right| \in o(h). \end{aligned} \tag{67}$$

Hence, Condition (A) is satisfied and weak convergence of X^h to Y follows from Theorem 1.1. □

Proposition 8.1 provides the way to simulate approximations of the Brownian trajectories slowed down on the Cantor set (see Figure 3).

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