

Comparing with octopi

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Abstract. Operator inequalities with a geometric flavour have been successful in studying mixing of random walks and quantum mechanics. We suggest a new way to extract such inequalities using the octopus inequality of Caputo, Liggett and Richthammer.

Résumé. Les inégalités d'opérateurs de nature géométrique ont été très utiles pour étudier le mélange des marches aléatoires et la mécanique quantique. Nous suggérons une nouvelle approche pour exhiber des inégalités de ce type en utilisant l'inégalité « pieuvre » (octopus inequality) de Caputo, Liggett et Richthammer.

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1. Introduction

This note concerns itself with random walks on the symmetric group S_n generated by sets of transpositions, i.e. elements which exchange some $i \neq j$ in $\{1, \dots, n\}$ and keep the others as are. Draw a graph G whose vertices are $\{1, \dots, n\}$ and with an edge between i and j if the transposition (i, j) is in our set of generators. The random walk on S_n with such a set of generators is called “the interchange process on G ” and is an object of interest in interacting particle systems and quantum mechanics. Because of the connection to mathematical physics, the most interesting graphs are those that have a geometric structure. For example, boxes in 2 or 3 dimensional grid.

Geometric graphs are difficult to attack using purely algebraic methods, so some analysis is necessary. One idea that was discovered a number of times independently is an operator inequality approach. Roughly, it goes as follows: Take the standard “multi-commodity flow” estimate for the mixing time [26]. Replace each inequality of numbers in the proof with a corresponding inequality of operators (a very quantum idea!). One gets an inequality comparing the generator of the interchange process on G to the generator of the interchange process on the complete graph K_n . This last object can be attacked purely algebraically. See [11,23,25] – to the best of our knowledge, all three groups of authors developed this idea independently.

From a different starting point, in 1992 David Aldous made a general conjecture about the second eigenvalue of the interchange process. Handjani and Jungrais suggested an inductive approach and used it to prove Aldous’ conjecture when the graph G is a tree [14]. Applying the approach of [14] to general graphs required an inequality, first conjectured in [12] and in a preliminary version of [8], and then proved by Caputo, Liggett and Richthammer [8] who christened it “the octopus inequality” for reasons which remain a mystery (it certainly does not enjoy 8 arms).

Even though the octopus inequality was devised for a specific approach to a specific problem, it turns out to have more applications. The first to use the octopus lemma for something new was Chen [10]. Here we give yet another application. We use it to reprove and strengthen the operator inequality of [11,23,25]. For boxes in \mathbb{Z}^d our estimate is no better than the existing one, but it is better for some graphs, and allows to improve results in the literature [16,20,22].

Is there something especially interesting in the octopus lemma, allowing it to be used for various unintended applications? Or is it the case that any non-trivial inequality for convolution operators on $L^2(S_n)$ is bound to have applications, because they are so hard to get? Only time will tell.

2. The main estimate

For $i \neq j$ we define $\nabla_{ij} = 1 - (ij)$, an element in $S_n[\mathbb{C}]$ (i.e. formal sums of elements in S_n with complex coefficients). Any element $A = \sum c_\sigma \sigma \in S_n[\mathbb{C}]$ can be thought of as a (convolution) operator on $L^2(S_n)$ given by

$$(Af)(\tau) = \sum_{\sigma} c_\sigma f(\sigma\tau).$$

For a general $A \in S_n[\mathbb{C}]$ the corresponding operator is not necessarily self-adjoint, but it is the case for ∇_{ij} .

A function $w : \binom{[1, \dots, n]}{2} \rightarrow [0, \infty)$ will be called a “weight function” and we will denote $w_{ij} = w_{ji} = w(\{i, j\})$. For any such w denote

$$\Delta_w = \sum_{i < j} w_{ij} \nabla_{ij},$$

again, as either an element of $S_n[\mathbb{C}]$ or as an operator on $L^2(S_n)$. Denote

$$w_i = \sum_{j \neq i} w_{ij} \quad w_{\text{tot}} = \sum_i w_i. \tag{1}$$

We will associate a graph G on n vertices with the weight function w with $w_{ij} = 1$ whenever $\{i, j\}$ is an edge and 0 otherwise. For example,

$$\Delta_{K_n} = \sum_{i < j} \nabla_{ij}.$$

The operator Δ_{K_n} has been studied extensively using representation theory and is relatively well-understood. Our main result is a way to compare Δ_w for a general w to Δ_{K_n} . But the comparison uses the notion of the mixing time. Since this has multiple definitions (essentially equivalent, granted), let us state the one we will be using.

Definition. Let w be a weight function. Let R_t be the Markov chain on $\{1, \dots, n\}$ whose transition probabilities $p(i, j)$ are given by

$$p(i, j) = \begin{cases} \frac{w_{ij}}{2w_i} & i \neq j \\ \frac{1}{2} & i = j. \end{cases}$$

In other words, the natural discrete time Markov chain associated with w , with an extra $\frac{1}{2}$ laziness. Let $p_t(i, j)$ be the probability that $R_t = j$, when $R_0 = i$. Let $\pi(j) = w_j/w_{\text{tot}}$ be the stationary measure (it is well-known that, under an assumption of connectivity, it is also the case that $\pi(j) = \lim_{t \rightarrow \infty} p_t(i, j)$ and in particular the limit exists and is independent of i). Denote

$$\text{dmix } w := \min \left\{ t : \forall i, j \ p_t(i, j) > \frac{3}{4} \pi(j) \right\}. \tag{2}$$

(“dmix” standing for discrete time mixing. Unfortunately there will be a few different notions of mixing in this paper, so we need to be careful). Note that the function $\min_{i, j} p_t(i, j)/\pi(j)$ is increasing in t , so in fact the condition of the definition holds for all $t \geq \text{dmix } w$ (this follows from the stationarity of π , we skip the easy calculation). Note also that we only require a lower bound on p_t . This means that our definition of $\text{dmix } w$ is equivalent to the usual (total variation) mixing time and not to the (sometimes larger) l^2 mixing time. We will prove this below in Lemma 5.

Going back to Δ_w , the result we wish to prove is essentially $\Delta_w \geq (c/n \text{dmix } w) \Delta_{K_n}$, but there is an annoying extra term (which we will denote by δ) which is constant in all cases of interest, but not always. Let us therefore state three convenient results which only bound this term, and then define it precisely.

Theorem 1. *Let w be an arbitrary weight function. Then*

$$\Delta_w \geq \frac{c\delta}{\text{dmix } w} \frac{\min_i w_i^2}{w_{\text{tot}}} \Delta_{K_n} \tag{3}$$

where the factor δ has the following estimates:

- (i) $\delta \geq c \left(\frac{\min^* w_{ij}}{\max w_i} \right)^2$ where $\min^* w_{ij} = \min\{w_{ij} : w_{ij} > 0\}$.
- (ii) $\delta \geq c$ when w comes from a regular graph.
- (iii) It always holds that $\delta \geq \frac{1}{2 \text{dmix } w}$.

We note that in the case covered by clause (ii), i.e. when w comes from a regular graph with the degree of all the vertices d , then also $\min w_i^2 / w_{\text{tot}} = d/n$ and we get $\Delta_w \geq cd \Delta_{K_n} / (n \text{dmix } w)$. Here and below c and C stand for universal positive constants. Their value could change from line to line or even within the same line.

Let us now define δ precisely. It is given by

$$\frac{1}{\delta} = \prod_{k=0}^{\lceil \log_2 \text{dmix } w \rceil} \max_i (1 + p_{2^k}(i, i)), \tag{4}$$

where p_t are as in the definition of our lazy mixing time.

Theorem 2. (3) holds with δ given by (4).

Clause (iii) of Theorem 1 is an obvious corollary of Theorem 2 (just bound all probabilities by 1). Clauses (i) and (ii) are not so difficult either, but we will show them later.

The last thing we wish to do before going to the proof of Theorem 2 is to recall the octopus inequality. It claims that for any numbers w_{1i} we have

$$\sum_{2 \leq i \leq n} w_{1i} \nabla_{1i} \geq \sum_{2 \leq i < j \leq n} \frac{w_{1i} w_{1j}}{w_1} \nabla_{ij}$$

where the inequality is as operators on $L^2(S_n)$, and we still to use (1) so $w_1 = \sum_{i=2}^n w_{1i}$. See [8, Theorem 2.3], and perhaps also [9, Theorem 4.2] which has a simplified proof, and a statement identical to ours.

Proof of Theorem 2. For the purpose of the proof, it will be convenient to extend weight functions also to the case $i = j$. Let u be such a weight function, i.e. $u_{ij} \in [0, \infty)$ for $i \leq j$ in $\{1, \dots, n\}$ and $u_{ij} = u_{ji}$. We define $\nabla_{ii} = 0$ so the u_{ii} do not affect Δ_u , but we do redefine $u_i = \sum_j u_{ij}$ so the numbers u_{ij}/u_i still add up to 1 and can be thought of as transition probabilities of a Markov chain, this time one that might stay in the same place for a turn.

For any weight function u we define a new weight function $u^{(2)}$ by

$$u_{ij}^{(2)} = \sum_k \frac{u_{ik} u_{kj}}{u_k}.$$

It is straightforward to check that $u_i^{(2)} = u_i$ and that the corresponding Markov process is the same as doing two steps in the Markov process of u . Our goal is to compare Δ_u and $\Delta_{u^{(2)}}$.

For this purpose write

$$\Delta_u = \frac{1}{2} \sum_i \left(\sum_{j \neq i} u_{ij} \nabla_{ij} \right)$$

and apply the octopus inequality to each term. We get

$$\begin{aligned} \Delta_u &\geq \frac{1}{2} \sum_i \sum_{i \neq j < k \neq i} \frac{u_{ij} u_{ik}}{u_i - u_{ii}} \nabla_{jk} \geq \frac{1}{2} \sum_i \sum_{i \neq j < k \neq i} \frac{u_{ij} u_{ik}}{u_i} \nabla_{jk} \\ &= \frac{1}{2} \sum_{j < k} \nabla_{jk} \sum_{i \notin \{j, k\}} \frac{u_{ji} u_{ik}}{u_i}. \end{aligned}$$

The sum over all i , without the restriction $i \notin \{j, k\}$, is our ‘‘target’’ $\Delta_{u^{(2)}}$ so we get

$$\Delta_u \geq \frac{1}{2} \Delta_{u^{(2)}} - \frac{1}{2} \sum_{j < k} \nabla_{jk} \sum_{i \in \{j, k\}} \frac{u_{ji} u_{ik}}{u_i}. \tag{5}$$

To estimate the remainder, define

$$\epsilon = \max_i \frac{u_{ii}}{u_i}$$

and get

$$\sum_{j < k} \nabla_{jk} \sum_{i \in \{j,k\}} \frac{u_{ji}u_{ik}}{u_i} \leq 2\epsilon \sum_{j < k} \nabla_{jk} u_{jk} = 2\epsilon \Delta_u \tag{6}$$

Summing (5) and (6) we get

$$\Delta_{u^{(2)}} \leq (2 + 2\epsilon) \Delta_u. \tag{7}$$

We now apply (7) inductively. Define

$$u^{(2^k)} = (u^{(2^{k-1})})^{(2)}$$

(so $u^{(2^k)}$ corresponds to the Markov process that does 2^k steps at a time). Define

$$\epsilon_k = \max_i \frac{u_{ii}^{(2^k)}}{u_i^{(2^k)}}.$$

Then applying (7) k times gives

$$\Delta_{u^{(2^k)}} \leq 2^k \Delta_u \prod_{i=0}^{k-1} (1 + \epsilon_i). \tag{8}$$

Recall that in the statement of the theorem we are given a weight function $w_{ij}, i \neq j$. To apply (8) to this w , first define

$$u_{ij} = \begin{cases} w_{ij} & i \neq j \\ w_i & i = j \end{cases}$$

so that the Markov chain corresponding to u is just the Markov chain corresponding to w , with an extra $\frac{1}{2}$ laziness, as in our definition of lazy mixing time. In particular

$$\epsilon_k = \max_i p_{2^k}(i, i).$$

Let k be the minimal such that $2^k \geq \text{dmix } w$. By our definition of the lazy mixing time, after 2^k steps the probability to pass from every i to every j is at least $\frac{3}{4}$ the stationary measure, i.e. $\frac{3}{4} w_j / w_{\text{tot}}$. Using the equality $u_i^{(2^k)} = u_i = 2w_i$ we get

$$u_{ij}^{(2^k)} \geq \frac{3}{2} \frac{w_i w_j}{w_{\text{tot}}} \geq \frac{\min w_i^2}{w_{\text{tot}}} \quad \forall i, j.$$

or

$$\Delta_{u^{(2^k)}} \geq \frac{\min w_i^2}{w_{\text{tot}}} \Delta_{K_n}.$$

With (8) we get

$$\Delta_u \geq 2^{-k} \prod_{i=0}^{k-1} (1 + \epsilon_i)^{-1} \frac{\min w_i^2}{w_{\text{tot}}} \Delta_{K_n}.$$

We have $2^{-k} > 1/(2 \text{dmix } w)$ and $\prod (1 + \epsilon_i)^{-1} \geq \delta$ (in fact they are equal except when $\log_2 \text{dmix } w$ is integer, in which case δ has one more term). The theorem is thus proved. \square

The proof of clauses (i) and (ii) of Theorem 1 uses the well-known connection between isoperimetric inequalities and transition probabilities of random walks. Let us formulate it as a lemma.

Lemma 3. *For any connected weight function w , any $t \leq \text{dmix } w$ and any vertices i and j we have*

$$p_t(i, j) \leq \frac{30}{\sqrt{t}} \cdot \frac{w_j}{\min^* w_{ij}}.$$

(A weight function w is called connected if for any i and j there exists some t such that $p_t(i, j) > 0$. As always, p_t are the transition probabilities after adding to w $\frac{1}{2}$ -laziness.)

Proof. As already mentioned, we use the connection between isoperimetric inequalities and transition probabilities of random walk, [21] will serve as a convenient reference. Here are the necessary definitions: Let $p_t(\cdot, \cdot)$ be the transition probabilities of a finite Markov chain with laziness at least $\frac{1}{2}$, and let π be its stationary measure. For a set S of elements of the chain define

$$\pi(S) = \sum_{i \in S} \pi(i) \quad |\partial S| = \sum_{i \in S, j \notin S} p(i, j)\pi(i).$$

Since w may be scaled without changing the result, let us assume that $w_{\text{tot}} = 1$ which means that $w_{ij} = p(i, j)\pi(i)$, and $w_i = \pi(i)$. Define the isoperimetric profile function $\phi : [0, \frac{1}{2}] \rightarrow [0, \infty)$ by

$$\phi(r) = \inf \left\{ \frac{|\partial S|}{\pi(S)} : S \text{ such that } \pi(S) \leq r \right\}$$

and extend ϕ beyond $\frac{1}{2}$ by defining $\phi(r) = \phi(\frac{1}{2})$ for all $r > \frac{1}{2}$. Then Theorem 1 of [21] states that for every $\lambda > 0$, if

$$t \geq 1 + \int_{4 \min(\pi(i), \pi(j))}^{4/\lambda} \frac{4 dr}{r \phi^2(r)} \tag{9}$$

then

$$\left| \frac{p_t(i, j) - \pi(j)}{\pi(j)} \right| \leq \lambda. \tag{10}$$

(If you try to compare to [21], you will notice that they denote by ϵ what we denote by λ – but have no fear, their results are not restricted to small λ in any way.)

Since our only assumption is connectivity, the only lower bound we can give on ϕ is

$$\phi(r) \geq \frac{\min^* w_{ij}}{r}$$

which allows to bound the integral by

$$\int_{4 \min(\pi(x), \pi(y))}^{4/\lambda} \frac{4 dr}{r \phi^2(r)} \leq \frac{4}{(\min^* w_{ij})^2} \int_0^{4/\lambda} r dr = \frac{32}{(\lambda \min^* w_{ij})^2}.$$

Set therefore

$$\lambda = \sqrt{\frac{32}{(t-2) \min^* w_{ij}^2}} \stackrel{(*)}{\leq} \frac{6}{\sqrt{t} \min^* w_{ij}}$$

where in $(*)$ we assume $t \geq 20$, which we can, as for $t < 20$ the lemma holds trivially (the stated bound for the probability is bigger than one). This choice of λ makes the integral $\leq t - 2$ and hence t (and even $t - 1$) satisfies the requirement (9) and we may conclude (10). Writing (10) with the value of λ above gives

$$p_t(i, j) \leq \left(\frac{6}{\sqrt{t} \min^* w_{ij}} + 1 \right) w_j.$$

To get rid of the annoying +1 factor, argue as follows: if $6/(\sqrt{t} \min^* w_{ij}) < 1/4$ then by (10) we must have $t - 1 \geq \text{dmix } w$ and we assumed this is not the case. Otherwise we have $1 \leq 24/(\sqrt{t} \min^* w_{ij})$ and can write

$$p_t(i, j) \leq \frac{30}{\sqrt{t}} \cdot \frac{w_i}{\min^* w_{ij}}$$

as promised. □

Proof of clause (i) of Theorem 1. Let us first get an uninteresting case off our hands: the case where w is disconnected, i.e. when there are i and j such that there is no paths of positive w between them. In this case $\text{dmix } w = \infty$ and Theorem 1 is trivially true. (Do note that without the assumption of connectedness the δ defined by (4) is ∞ , so this assumption is indeed necessary.)

Since w is connected we may apply Lemma 3. Examine the product

$$\frac{1}{\delta} = \prod_{k=0}^{\lfloor \log_2 \text{dmix } w \rfloor} \max_i (1 + p_{2^k}(i, i)).$$

For the first $10 + 2 \log_2 \frac{\max w_i}{\min^* w_{ij}}$ terms the bound of Lemma 3 is not better than the trivial bound $p \leq 1$, so we use it and get a factor of $1024 \left(\frac{\max w_i}{\min^* w_{ij}}\right)^2$. In the rest of the terms the p_{2^k} decay exponentially, so the product is bounded by a constant. This proves the claim. □

Lemma 4. *Let G be a regular, connected graph, let i and j be vertices and let $t \leq \text{dmix } G$. Then*

$$p_t(i, j) \leq \frac{C}{t^{1/4}}.$$

Proof. Denote the common degree of the vertices of G by d . We divide into two cases, according to whether $d \geq t^{1/4}$ or not. If $d \geq t^{1/4}$ then we note that the function $\max_{i,j} p_t(i, j)$ is decreasing in t (we use here that the graph is regular), and that $p_1(i, j) \leq d^{-1} \leq t^{-1/4}$. If $d < t^{1/4}$ we use Lemma 3 and get that

$$|p_t(i, j)| \leq \frac{30}{\sqrt{t}} d < \frac{30}{t^{1/4}}.$$
□

Proof of clause (ii) of Theorem 1. As in the proof of clause (i), we may assume G is connected. Write, using Lemma 4,

$$\frac{1}{\delta} = \prod_{k=0}^{\lfloor \log_2 \text{dmix } w \rfloor} \max_i (1 + p_{2^k}(i, i)) \leq \prod_{k=0}^{\infty} \left(1 + \frac{C}{2^{k/4}}\right) \leq C.$$
□

We finish this section with a comparison of dmix with the more common notion of mixing time, the total variation mixing time. Denote

$$\text{tvmix } w = \min \left\{ t : \forall i \ \| p_t(i, \cdot) - \pi \|_{\text{TV}} < \frac{1}{4} \right\} \tag{11}$$

where $\| \cdot \|_{\text{TV}}$ is the total variation norm (or $\frac{1}{2}$ of the l^1 norm).

Lemma 5. *For reversible Markov chains, $\frac{1}{8} \text{dmix } w \leq \text{tvmix } w \leq \text{dmix } w$.*

Proof. To show $\text{tvmix} \leq \text{dmix}$, choose some t such that $p_t(i, j) > \frac{3}{4} \pi(j)$ for all i and j and get

$$\| p_t(i, \cdot) - \pi \|_{\text{TV}} = \sum_j (\pi(j) - p_t(i, j))^+ < \sum_j \frac{1}{4} \pi(j) = \frac{1}{4}.$$

(This direction does not require reversibility.)

In the other direction, assume that at some t_0 we have $\max_i \|p_{t_0}(i, \cdot) - \pi\|_{TV} =: d_t < \frac{1}{4}$. It is well known that d_t is monotonically decreasing, and further that $d_{t+s} \leq 2d_t d_s$ [18, lemmas 4.11 and 4.12]. Applying this twice gives for all $t \geq 4t_0$ that $d_t < \frac{1}{32}$. We write

$$p_{2t}(i, j) = \sum_k p_t(i, k)p_t(k, j).$$

Reversibility means that $\pi(j)p_t(j, k) = \pi(k)p_t(k, j)$ so we may continue the calculation to get

$$p_{2t}(i, j) = \pi(j) \sum_k p_t(i, k)p_t(j, k)/\pi(k). \tag{12}$$

Using Cauchy–Schwarz gives

$$\begin{aligned} \sum_k \sqrt{p_t(i, k)p_t(j, k)} &= \sum_k \sqrt{\frac{p_t(i, k)p_t(j, k)}{\pi(k)}} \cdot \sqrt{\pi(k)} \\ &\leq \left(\sum_k \frac{p_t(i, k)p_t(j, k)}{\pi(k)} \right)^{1/2} \left(\sum_k \pi(k) \right)^{1/2} \end{aligned} \tag{13}$$

and the term $\sum \pi(k)$ is of course 1 so we get

$$\begin{aligned} \sqrt{\frac{p_{2t}(i, j)}{\pi(j)}} &\stackrel{(12)}{=} \left(\sum_k p_t(i, k)p_t(j, k)/\pi(k) \right)^{1/2} \\ &\stackrel{(13)}{\geq} \sum p_t(i, k)^{1/2} p_t(j, k)^{1/2} \\ &\geq \sum \min(p_t(i, k), p_t(j, k)) \\ &= \sum \frac{p_t(i, k) + p_t(j, k)}{2} - \frac{|p_t(i, k) - p_t(j, k)|}{2} \\ &= 1 - \|p_t(i, \cdot) - p_t(j, \cdot)\|_{TV} \\ &\geq 1 - \|p_t(i, \cdot) - \pi\|_{TV} - \|\pi - p_t(j, \cdot)\|_{TV} > \frac{15}{16}. \end{aligned}$$

All in all we get $p_t(i, j) > \frac{225}{256}\pi(j)$ for all $t > 8t_0$, proving the lemma. □

For every possible equivalent formulation of the mixing time, including a few quite similar (but not identical) to ours, see [19].

3. Corollaries

3.1. The mixing time

We start with a corollary for the mixing time. Our operator Δ_w is most suitable for studying random walk in *continuous time* because it is the generator of this walk (do not be fooled by the intensive use of discrete time random walk during the proof of Theorem 2, and from the discrete time formulation of δ – this is an artifact of the proof method). So let us recall what is the continuous time interchange process. For a given weight function w we define a stochastic process on S_n in continuous time by

$$p_t(\sigma, \tau) = e^{-t\Delta_w}(\sigma, \tau) \quad \sigma, \tau \in S_n$$

where $e^{-t\Delta}$ is matrix exponentiation (and the matrix is of course $n! \times n!$). It is well-known that this is a Markov chain in continuous time, and further, it is equivalent to the following process: Put marbles on $\{1, \dots, n\}$, all different, and Poisson clocks “on” each couple ij , with the clock on the edge ij having rate w_{ij} , and when the clock rings exchange the two

marbles. This is known as the interchange process corresponding to the weight function w . Mixing time for continuous time random walks is traditionally defined using the total variation distance, i.e. by (11). (The assumption of laziness we had in discrete time is no longer necessary.) Denote by $\text{imix } w$ the mixing time of the interchange process corresponding to w .

Lemma 6. *If $\Delta_w \geq a\Delta_{K_n}$ then*

$$\text{imix } w \leq \frac{C \log n}{a n}.$$

Proof. This result is an easy corollary of a comparison argument which can be found in equation (2.10) in [23], and the determination of $\text{imix } K_n$ by Diaconis and Shahshahani [24]. Let us give some details. First we need to translate the inequality $\Delta_w \geq a\Delta_{K_n}$ to exponentials. Now, in general one cannot conclude from an operator inequality $A \geq B$ that $e^{-A} \leq e^{-B}$ (see [17] for more on this), but in our case the operators Δ_w and Δ_{K_n} commute. Indeed, Δ_{K_n} is in the center of $S_n[\mathbb{C}]$ and commutes with any convolution operator on S_n . For commuting operators, number inequalities translate directly to operator inequalities (simply by taking the joint diagonalisation), so $\Delta_w \geq a\Delta_{K_n}$ implies $e^{-s\Delta_w} \leq e^{-sa\Delta_{K_n}}$ for any $s > 0$.

Let us now explain the notations of [23]. Their setting is that of discrete time random walk on S_n , so we will apply their results for the walk that does steps of $e^{-s\Delta_w}$ for some small $s > 0$. Quoting [23, (2.10)] literally, it states that “if $\tilde{\mathcal{E}} \leq 2\mathcal{E}$ and $\tilde{\mathcal{F}} \leq 2\mathcal{F}$ then for all k , $\|p_k - \pi\|^2 \leq e^{-k/2} + \|\tilde{p}_{k/4} - \pi\|^2$ ”. The notations \mathcal{E} and $\tilde{\mathcal{E}}$ stand for our operators $1 - e^{-s\Delta_w}$ and $1 - e^{-sa\Delta_{K_n}}$ so $\tilde{\mathcal{E}} \leq \mathcal{E}$ without the need for the 2. The notations \mathcal{F} and $\tilde{\mathcal{F}}$ stand for the operators $1 + e^{-s\Delta_w}$ and $1 + e^{-sa\Delta_{K_n}}$. In the setting of [23] one has to care also about periodicity, but in our setting this is automatic: \mathcal{F} and $\tilde{\mathcal{F}}$ are both bounded between 1 and 2, so $\tilde{\mathcal{F}} \leq 2 \leq 2\mathcal{F}$. So the conditions are satisfied. p_k and \tilde{p}_k are the distributions after k steps of the discrete walk, which is just the distributions of our continuous time walk after sk steps. So we get

$$\|p_{sk}^w - \pi\|^2 \leq e^{-k/2} + \|p_{ask/4}^{K_n} - \pi\|^2.$$

Taking $s \rightarrow 0$ and $k \rightarrow \infty$ while preserving $t = sk$ gives

$$\|p_t^w - \pi\|^2 \leq \|p_{at/4}^{K_n} - \pi\|^2$$

which of course implies that

$$\text{imix } w \leq \frac{4}{a} \text{imix } K_n.$$

Finally [24] shows that $\text{imix } K_n \approx \frac{\log n}{n}$, proving the lemma (the number stated in [24], $\frac{1}{2}n \log n$, has to be divided by $\binom{n}{2}$, the number of edges of K_n , because of, yet again, the difference between discrete and continuous time). \square

Corollary 7. *For every weight function w ,*

$$\text{imix } w \leq \frac{\text{dmix } w}{c\delta} \frac{w_{\text{tot}}}{n \min w_i^2} \log n.$$

with δ given by either (4) or its estimates in Theorem 1.

3.2. The chameleon process

Invented by Ben Morris [5], the chameleon process has been used with great success to bound mixing times of interacting particle systems [15,22]. Let us therefore compare the two approaches. This section is somewhat geared towards mixing time aficionados, so some key terms will not be fully explained.

The approach of this paper has two advantages when compared to the chameleon process. The first is that the chameleon process only bounds the mixing time of the exclusion process. The exclusion process is similar to the interchange process, but the marbles have only two colours, so it is a process on a state space of size $\binom{n}{k}$ for some $k \in \{1, \dots, n\}$. The approach works even with multiple colours, but always some proportion of the marbles (i.e. at least cn for some $c > 0$) must be indistinguishable. The second advantage is that, in fact, we have bounded the l^2 mixing time and not just the total variation mixing time (see e.g. [6, Definition 2.2] for the l^2 mixing time). We formulated Corollary 7 for the total variation time because this is how Diaconis and Shahshahani formulate their result for the interchange process on K_n (and, of

course, the total variation time is the most common notion of mixing time in the literature, often called *the* mixing time). But in fact they bound the l^2 mixing time (see their [24, formula (3.1)]), and this was already noted in the literature [6, Theorem 6.9]. Further, an operator inequality implies a comparison of l^2 mixing time [6, Theorem 4.3]. The chameleon process bounds the total variation time, and it does not seem easy to change this fact.

Contrariwise, some of the results using the chameleon process give better bounds than ours. Examine, for example, the results of Oliveira [22]. Denote by $\text{ctvmix } w$ the mixing time of continuous time random walk with respect to the weight function w , and by $\text{etvmix } w$ the same thing but for exclusion process with weight function w (i.e. again in continuous time and with respect to the total variation distance). Then Oliveira showed that

$$\text{etvmix } w \leq C \text{ctvmix } w \log n$$

without all our extra factors. Further, Hermon and Pymar show [15] that this bound is not always tight. For example, for an expander they show $\text{etvmix } w \leq \log n \log \log n$, again using the chameleon process.

We finish this section with a few strengthenings of results of [16], which studied the interchange process using operator comparisons (and not via the chameleon process). Since the results of [16] are formulated in discrete time, one has to scale by the total number edges to compare: in Corollaries 8 and 9 multiply by n and in Corollary 10 by $n \log n$ to get discrete time results. In Corollary 8 our bound matches the lower bound given in [16].

Corollary 8. *For G being a finite regular tree of degree d ,*

$$\text{imix } G \leq C(d)n \log n.$$

Proof. The mixing time of a regular tree is of order n , see [2, Example 5.14]. □

Corollary 9. *For G being a bounded degree expander, $\text{imix } G \leq C \log^2 n$.*

Corollary 10. *For $G = (\mathbb{Z}/2)^d$ (a.k.a. the hypercube),*

$$\text{imix } G \leq C \log n \log \log n.$$

Proof. The discrete-time mixing time of $(\mathbb{Z}/2)^d$ is $d \log d$, see [2, Example 5.15, equation (5.71)]. We bound δ using clause 2 of Theorem 1. □

For the hypercube, Wilson proved a lower bound of $\log n$ and conjectured that it is the correct value [28, Section 9.1]. For the exclusion process Wilson's conjecture was proved recently in [15].

3.3. Appearance of large cycles

We now leave the question of when the interchange process mixes completely and study a different question: when do large cycles first appear? We mean here cycles of permutations: recall that the interchange process is a random walk on the group of permutations, and a permutation may be decomposed into cycles. Here is the corresponding comparison result:

Theorem 11. *If $\Delta_w \geq a \Delta_{K_n}$ then the interchange process at time $\geq C/a n$ satisfies*

$$\mathbb{P}(\exists \text{ cycle of length } > n/2) > c.$$

Proof. The proof uses the representation theory of S_n . As this note is quite short it would not be possible to give all the necessary background, consult [3] for connections specific to the interchange process, or any textbook on the subject, e.g. [13]. In short, an irreducible representation ρ is a subspace of $L^2(S_n)$ which is preserved by any convolution operator, in particular by Δ_w and Δ_{K_n} (and is minimal in that regard). The restriction of Δ_w to the subspace ρ is diagonalisable (recall that Δ_w is self-adjoint). Let us denote its eigenvalues by

$$\lambda_1(w, \rho) \leq \dots \leq \lambda_{\dim \rho}(w, \rho).$$

In the case of the complete graph, Δ_{K_n} restricted to ρ is a scalar matrix (this is an easy corollary of Schur's lemma), and its value was calculated by Diaconis and Shahshahani [24]. In other words $\lambda_i(K_n, \rho)$ are all equal – let us denote the

common value by $\lambda(K_n, \rho)$. The inequality $\Delta_w \geq a\Delta_{K_n}$ restricts, of course, to any irreducible representation, and from it and the scalarity of Δ_{K_n} we get

$$\lambda_i(w, \rho) \geq a\lambda(K_n, \rho). \tag{14}$$

The final element in the proof is a formula connecting $\lambda_i(w, \rho)$ with the probability of large cycles [3]. To state it, let us recall that irreducible representations of S_n are indexed by partitions of n , i.e. numbers $\sigma_1 \geq \dots \geq \sigma_r$ such that $\sum \sigma_i = n$. We denote by $[\sigma_1, \dots, \sigma_r]$ the irreducible representation corresponding to the partition σ , and if a certain number repeats more than once we denote it by a superscript, namely $[3, 1^3]$ is the partition $6 = 3 + 1 + 1 + 1$. We may now state the results of [3]. Fix $k \in \{1, \dots, n\}$ and denote by $s_k(t)$ the number of cycles of length k in the interchange process corresponding to w at time t . Then Lemma 5 of [3] states that

$$\mathbb{E}(s_k(t)) = \frac{1}{k} \sum_{\rho} a_{\rho,k} \sum_{j=1}^{\dim \rho} e^{-t\lambda_j(w, \rho)} \tag{15}$$

where

$$a_{\rho,k} = \begin{cases} 1 & \rho = [n] \\ (-1)^{i+1} & \rho = [k-i-1, n-k+1, 1^i] \text{ for } i \in \{0, \dots, 2k-n-2\} \\ (-1)^i & \rho = [n-k, k-i, 1^i] \text{ for } i \in \{\max\{2k-n, 0\}, \dots, k-1\} \\ 0 & \text{otherwise.} \end{cases}$$

Now, $[n]$ is the one dimensional space of constant functions, so $\lambda_1(w, [n]) = 0$. Our proof strategy will be to show that the corresponding term in the sum, $\frac{1}{k}$, is the main term and all the others are negligible. Hence we need estimates for $\dim \rho$ and for $\lambda(K_n, \rho)$. Both go back to the early 20th century, but a convenient reference would be [7]. From [7, Section 2.3] we get:

$$\lambda(K_n, [k-i-1, n-k+1, 1^i]) = \binom{n}{2} + ik + k - \frac{1}{2}((n-k)^2 + k^2 - n)$$

$$\dim([k-i-1, n-k+1, 1^i]) = \frac{n!(2k-n-i-1)}{i!k(n-k)!(k-i-1)!(n-k+i+1)}$$

while from [7, equations 8 & 9] we get essentially the same formulas for the second family of representations

$$\lambda(K_n, [n-k, k-i, 1^i]) = \binom{n}{2} + ik + k - \frac{1}{2}((n-k)^2 + k^2 - n)$$

$$\dim([n-k, k-i, 1^i]) = \frac{n!(n-2k+i+1)}{i!k(n-k)!(k-i-1)!(n-k+i+1)}$$

(our paper [4] partially explains this similarity). We see that in both cases we have that the dimension is $\binom{n}{k} \binom{k}{i}$ times a rational factor which can be bounded roughly by n^2 . So $\dim \rho \leq 4^n n^2$ for all relevant representations. As for λ , we restrict our attention to $k \in [\frac{1}{2}n, \frac{3}{4}n]$ and get that in all cases $\lambda \geq cn^2$. This means that for any $\rho \neq [n]$ for which $a_{\rho,k} \neq 0$ we have

$$\sum_{j=1}^{\dim \rho} \exp(-t\lambda_j(w, \rho)) \stackrel{(14)}{\leq} \sum_{j=1}^{\dim \rho} \exp(-at\lambda(K_n, \rho)) \leq 4^n n^2 \exp(-actn^2)$$

so

$$\left| \mathbb{E}(s_k(t)) - \frac{1}{k} \right| = \left| \sum_{\rho \neq [n]} a_{\rho,k} \sum_{j=1}^{\dim \rho} e^{-t\lambda_j(w, \rho)} \right| \leq 4^n n^3 \exp(-actn^2).$$

Taking $t > C/an$ for some C sufficiently large makes the right hand side negligible, and we get that $\mathbb{E}(s_k(t)) \geq \frac{1}{2k}$, for all $k \in [\frac{1}{2}n, \frac{3}{4}n]$. Since $k > \frac{1}{2}n$, s_k may only take the values 0 and 1 and $\mathbb{E}(s_k) = \mathbb{P}(s_k = 1)$. Summing over $k \in [\frac{1}{2}n, \frac{3}{4}n]$ gives the theorem. \square

Corollary 12. *For the 2 dimensional Hamming graph, at time $t > C/\sqrt{n}$ we have cycles larger than $n/2$ with positive probability.*

The interchange process on the Hamming graph was studied in [1,20] using different methods.

Proof. The 2 dimensional Hamming graph is the graph given by, for $n = m^2$,

$$w_{((i_1,i_2),(j_1,j_2))} = \begin{cases} 1 & i_1 = j_1 \\ 1 & i_2 = j_2 \\ 0 & \text{otherwise} \end{cases}$$

(recall that w is not defined for $(i_1, i_2) = (j_1, j_2)$). This graph has finite mixing time, $w_i = 2(m - 1)$ and $w_{\text{tot}} = 2(m^3 - m^2)$ so by Theorem 3 $\Delta_w \geq \frac{c}{m} \Delta_{K_n}$. By Theorem 11, we get large cycles at time C/\sqrt{n} , as claimed. \square

Theorem 11 has an analog for the quantum Heisenberg ferromagnet (QHF for short). Rather than defining the QHF, we apply Tóth’s representation [27], and get that the magnetisation of the QHF can be calculated by considering the cycles of the interchange process with a certain weighting. Precisely, let again $s_k(t)$ be the number of cycles of length k at time t and let $s(t) = \sum_k s_k(t)$. Then the magnetisation, m , is defined by

$$Z(t) = \mathbb{E}(2^{s(t)}) \quad m^2(t) = \frac{1}{Z(t)} \mathbb{E}\left(\left(\sum_k k^2 s_k(t)\right) 2^{s(t)}\right).$$

(For the relation to physics we refer to [27], but basically it is a quantum model with spin interactions. The graph over which we perform the interchange process describes the inter-particle interactions, and the time t translates to the inverse of the temperature.)

Theorem 13 below is the analog of Theorem 11 for the quantum Heisenberg ferromagnet. It is proved identically, replacing the results of [7] with the results of [4].

Theorem 13. *If $\Delta_w \geq a \Delta_{K_n}$ then the quantum Heisenberg ferromagnet at $t \geq C/an$ satisfies $m \geq cn$.*

Proof sketch. We keep the notations $\lambda_i(w, \rho)$ and $\lambda(K_n, \rho)$ from the proof of Theorem 11, and recall that $\lambda_i(w, \rho) \geq a\lambda(K_n, \rho)$. We start with the partition function Z , for which we write a simple analog of (15):

$$Z = \sum_{\rho=[a,b]} (a - b + 1) \sum_{j=1}^{\dim \rho} e^{-t\lambda_j(w,\rho)}$$

(the sum includes also $\rho = [n]$ for which we consider $a = n$ and $b = 0$. Ditto below, when we write $[a, b]$ we always entertain the possibility that $b = 0$.) For m we have a more complicated formula, each s_k has

$$\mathbb{E}(s_k(t) 2^{s(t)}) = \sum_{\rho} d_{\rho,k} \sum_{j=1}^{\dim \rho} e^{-t\lambda_j(w,\rho)}$$

with some coefficients $d_{\rho,k}$, which are described by [4, Theorem 3]. Here we will only note a few properties of the $d_{\rho,k}$:

- (i) $d_{\rho,k} = 0$ unless $\rho = [a, b, c, 1^d]$ i.e. has (at most) 3 rows and one column;
- (ii) $d_{[a,b],k} = \frac{2(a-b+1)}{k}$ for all a and b with $a + b = n$, $0 \leq b \leq \lfloor (n - k)/2 \rfloor$;
- (iii) $|d_{\rho,k}| \leq 2n + 2$ for all ρ ; and
- (iv) If $k \in [\frac{1}{2}n, \frac{3}{4}n]$ then for all $\rho \neq [a, b]$ for which $d_{\rho,k} \neq 0$ we have $\lambda(K_n, \rho) > cn^2$.

In particular, the first property allows to bound the dimensions of the relevant representations. It is not difficult to conclude from the hook formula (see e.g. [13]) that $\dim \rho \leq 6^n$ for all ρ of the form $[a, b, c, 1^d]$ (the precise value 6 will play no role).

Assume now $k \in [\frac{1}{2}n, \frac{3}{4}n]$. We make two estimates, the first for Z . We note that for $[a, b]$ with $b > \lfloor (n - k)/2 \rfloor$ we have $\lambda(K_n, [a, b]) > cn^2$ and hence the contribution of these representations to Z is negligible, namely denote

$$Z' = \sum_{\substack{\rho=[a,b] \\ b \leq \lfloor (n-k)/2 \rfloor}} (a - b + 1) \sum_{j=1}^{\dim \rho} e^{-t\lambda_j(w,\rho)}$$

and get

$$|Z - Z'| = \sum_{\substack{\rho=[a,b] \\ b > \lfloor (n-k)/2 \rfloor}} (a - b + 1) \sum_{j=1}^{\dim \rho} e^{-t\lambda_j(w, \rho)} \leq n^2 6^n \exp(-actn^2)$$

(the n^2 term has one n bounding the number of values of b , and one n as a bound for $(a - b + 1)$). Hence for $t > C/an$ we get that $Z - Z'$ is negligible (recall that $Z \geq n + 1$, the contribution of the representation $[n]$).

As for m , we write

$$E'(k) = \sum_{\substack{\rho=[a,b] \\ b \leq \lfloor (n-k)/2 \rfloor}} d_{\rho,k} \sum_{j=1}^{\dim \rho} e^{-t\lambda_j(w, \rho)}$$

and similarly get

$$|\mathbb{E}(s_k 2^{s'}) - E'(k)| \leq \sum_{\rho \neq [a,b]} |d_{\rho,k}| \sum_{j=1}^{\dim \rho} e^{-t\lambda_j(w, \rho)} \leq C \cdot 6^n n^4 \exp(-actn^2).$$

(We used here the third property of the $d_{\rho,k}$ to bound them by Cn , and the fourth property to estimate $\lambda_j(w, \rho) \geq a\lambda(K_n, \rho) \geq acn^2$.) Hence for $t > Ca/n$ this is also negligible. But

$$\frac{E'}{Z'} = \frac{2}{k}. \tag{16}$$

Hence

$$\begin{aligned} m^2 &= \frac{1}{Z} \sum_{k=1}^n k^2 \mathbb{E}(s_k(t) 2^{s(t)}) \geq \frac{1}{Z} \sum_{k=\frac{1}{2}n}^{\frac{3}{4}n} k^2 \mathbb{E}(s_k(t) 2^{s(t)}) \\ &\geq \frac{1}{Z'} \sum_{k=\frac{1}{2}n}^{\frac{3}{4}n} k^2 E'(k) \cdot (1 + O(e^{-cn})) \\ &\stackrel{(16)}{=} \sum_{k=\frac{1}{2}n}^{\frac{3}{4}n} 2k(1 + O(e^{-cn})) \geq cn^2, \end{aligned}$$

as promised. □

Corollary 14. *The quantum Heisenberg ferromagnet on the 2 dimensional Hamming graph, at $t > C/\sqrt{n}$ has $m \geq cn$.*

3.4. A relation with the Caputo–Liggett–Richthammer theorem

Finally, let us close cycles by returning to Aldous’ conjecture. We last mentioned Aldous’ conjecture in the introduction, as its proof was the first application of the octopus inequality. Equipped with the notation $\lambda_i(w, \rho)$ of Section 3.3 we may now state it (which we did not do in the introduction) as follows:

$$\lambda_1(w, \rho) \geq \lambda_1(w, [n - 1, 1]) \quad \forall w, \forall \rho \neq [n]. \tag{17}$$

The notations $[n]$ and $[n - 1, 1]$ are the standard notations for presentations we also used in Section 3.3. We remark, though, that these specific two are very simple: $[n]$ is the trivial, one-dimensional representation while $[n - 1, 1]$ is the $(n - 1)$ -dimensional representation one gets by removing the constants from the standard representation, so $\lambda_1(w, [n - 1, 1])$ is simply the second eigenvalue of the standard representation, i.e. of the continuous time Markov chain corresponding

to w . Operator comparison arguments imply eigenvalue inequalities. Indeed, $\Delta_w \geq a\Delta_{K_n}$ is equivalent to $\lambda_1(w, \rho) \geq a\lambda(K_n, \rho)$ for all ρ . Thus Theorem 1 may be reformulated as

$$\lambda_1(w, \rho) \geq \frac{c\delta}{\text{dmix } w} \frac{\min_i w_i^2}{w_{\text{tot}}} \lambda(K_n, \rho) \quad \forall w, \rho. \quad (18)$$

This begs the question: for which ρ is this estimate better than (17)? Let us restrict our attention to the case that w comes from a d -regular graph. In this case (18) simplifies to

$$\lambda_1(w, \rho) \geq \frac{cd}{n \text{dmix } w} \lambda(K_n, \rho).$$

Further, it is generally true that $\text{dmix } w \leq (Cd \log n)/\lambda_1(w, [n-1, 1])$, see e.g. [2, Lemma 4.23]. (Let us explain the notations of Aldous and Fill, for easier comparison: the relevant clause is the continuous time inequality. Their τ_2 is simply $1/\lambda_1$ and their π^* is the minimum of the stationary measure, which is in this case simply $1/n$. Finally the d comes because our dmix is the discrete mixing time, while their τ_1 is the continuous mixing time.) We get

$$\lambda_1(w, \rho) \geq \frac{c\lambda(K_n, \rho)}{n \log n} \lambda_1(w, [n-1, 1]).$$

Thus we get a better estimate than (17) for any ρ for which $\lambda(K_n, \rho) \geq Cn \log n$. This condition holds for the vast majority of representations, for example for all those with more than $C \log n$ boxes below the first row, see [24, Lemma 7] (again, for easier comparison, our $\lambda(K_n, \rho)$ is $\binom{n}{2}(1-r(\rho))$ where r is the character ratio calculated *ibid.*).

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