

# Long-time limits and occupation times for stable Fleming–Viot processes with decaying sampling rates

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Received 23 May 2017; revised 22 September 2019; accepted 20 February 2020

**Abstract.** A class of Fleming–Viot processes with decaying sampling rates and  $\alpha$ -stable motions that correspond to distributions with growing populations are introduced and analyzed. Almost sure long-time scaling limits for these processes are developed, addressing the question of long-time population distribution for growing populations. Asymptotics in higher orders are investigated. Convergence of particle location occupation and inhabitation time processes are also addressed and related by way of the historical process. The basic results and techniques allow general Feller motion/mutation and may apply to other measure-valued Markov processes.

**Résumé.** Dans cet article, nous introduisons et analysons une classe de processus de Fleming–Viot, avec taux d'échantillonnage décroissant et déplacement  $\alpha$ -stable, correspondant à des distributions de populations croissantes. Les théorèmes limites en temps long presque-sûr pour ces processus sont obtenus, répondant ainsi à la question de la distribution en temps long de la population dans le cas de populations croissantes. Les asymptotiques d'ordres supérieurs sont aussi obtenues. Les convergences des processus de temps d'occupation et d'habitation des particules sont considérées et reliées au moyen du processus historique. Les résultats et techniques autorisent des processus de Feller de déplacement/mutation généraux et peuvent s'appliquer à d'autres processus de Markov à valeurs mesures.

*MSC2020 subject classifications:* Primary 60J80; 60F15; secondary 60B10; 60G57

*Keywords:* Fleming–Viot process;  $\alpha$ -Stable process; Historical process; Occupation times

## 1. Introduction

We consider an  $M_1(\mathbb{R}^d)$ -valued Fleming–Viot process  $X = (X_t, t \geq 0)$  with mutation generator  $-(-\Delta)^{\frac{\alpha}{2}}$  ( $\alpha \in (0, 2]$ ) and sampling rate  $1/\phi(t)$  at time  $t$  for some positive function  $\phi$  defined on  $\mathbb{R}_+ = [0, \infty)$ . Such Fleming–Viot processes can be obtained by normalizing and conditioning the total mass of (possibly non-critical) Dawson–Watanabe processes to have total mass  $\phi(t)$  for all time  $t$ . This was established by Etheridge and March [18] for  $\phi \equiv 1$  and by Perkins [28] for general nonnegative function  $\phi$ . Herein, we investigate the long-time asymptotic of such Fleming–Viot processes when  $1/\phi$  satisfies an integrability condition at infinity. Examples of  $\phi$  satisfying this integrability condition include  $t \rightarrow e^{\beta t}$  for  $\beta > 0$  and  $t \rightarrow 1 + t^N$  for  $N$  sufficiently large.

To be more precise, we let  $W = (W_t, t \geq 0; \mathbb{P}_m)$  be a Dawson–Watanabe process with motion generator  $-(-\Delta)^{\frac{\alpha}{2}}$  on  $\mathbb{R}^d$ , linear growth  $\beta$  and critical branching rate  $\eta > 0$  corresponding to the operator  $-(-\Delta)^{\frac{\alpha}{2}}u + \beta u - \frac{\eta}{2}u^2$  on  $\mathbb{R}^d$ . Then,  $W$  is a measure-valued Markov process starting at a finite measure  $m$  such that

$$M_t^W(f) := W_t(f) - m(f) - \int_0^t W_s((-\Delta)^{\frac{\alpha}{2}} + \beta)f \, ds$$

is a continuous martingale with quadratic variation  $\langle M^W(f) \rangle_t = \int_0^t W_s(\eta f^2) \, ds$  for all  $f \in C_b^2(\mathbb{R}^d)$ . (The reader is referred to [23] for information about stable processes and to [26] as well as [3] for describing measure-valued processes as martingale problems.)  $W$ 's mass growth is subcritical, critical or supercritical if  $\beta < 0$ ,  $\beta = 0$  or  $\beta > 0$  respectively. The expected total mass is found to be  $m(1)e^{\beta t}$  by substituting  $f = 1$  into the above equation and taking expectations.

Now, suppose a population is projected to grow according to a positive continuous function  $\phi$ . Then, following Perkins [28], one finds that the corresponding Fleming–Viot process attained by taking the angular part of  $W$  and conditioning  $W_t(1)$  to have total mass  $\phi(t)$  at every time  $t$  yields an  $M_1(\mathbb{R}^d)$ -valued process  $X = (X_t, t \geq 0)$ , the  $(\alpha, \phi)$  Fleming–Viot superprocess, starting at  $\mu = \frac{m}{m(E)} \in M_1(E)$  and satisfying the martingale problem: For each  $f \in \mathcal{F}((-\Delta)^{\frac{\alpha}{2}})$  (the domain of  $(-\Delta)^{\frac{\alpha}{2}}$ ), the process

$$M_t^X(f) := X_t(f) - \mu(f) - \int_0^t X_s((-\Delta)^{\frac{\alpha}{2}} f) ds \tag{1.1}$$

is a continuous martingale with quadratic variation

$$\langle M^X(f) \rangle_t = \int_0^t \eta \phi(s)^{-1} [X_s(f^2) - X_s^2(f)] ds. \tag{1.2}$$

The law of  $X$  is denoted by  $\mathbb{P}_\mu^\phi$ . Technically, to get  $X$  from  $W$ , we condition on  $W_t(1)$  staying within  $\varepsilon$  of  $\phi(t)$  up to  $T$  and then let  $\varepsilon \rightarrow 0$  and  $T \rightarrow \infty$ . Also, it does not matter whether the original Dawson–Watanabe process is supercritical, critical or even subcritical as the resulting Fleming–Viot after normalizing by  $W_t(1)$  and conditioning so that  $W_t(1) = \phi(t)$  (that is, after considering the angular part) satisfies the same martingale problem. If  $\phi$  is increasing,  $X$  can be considered as a Fleming–Viot process that gives the population distribution for growing populations.

Properly normalized supercritical superprocesses have recently been shown (see e.g. Wang [33], Kouritzin and Ren [22], Liu et al. [25], Eckhoff et al. [16] as well as the more detailed review in Section 2.7 of [17]) to have almost sure long-time scaling limits (often called strong laws of large numbers), generalizing the pioneering branching Markov process work of Watanabe [34] and Asmussen and Hering [1]. Traditionally, superprocesses with ergodic and transient motion models have been handled separately with different scalings in laws of large number results. However, while considering strong laws of large numbers for supercritical, (possibly) non-Markov Gaussian branching processes, Kouritzin et al. [21] showed that these two cases can be considered together. For  $\alpha$ -stable Dawson–Watanabe processes, the result of [22] states that with probability one, as  $t \rightarrow \infty$ ,

$$t^{\frac{d}{\alpha}} \frac{W_t}{W_t(1)} \xrightarrow{d} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} d\theta \lambda_d \quad \text{on } \left\{ \lim_{t \rightarrow \infty} e^{-\beta t} W_t(1) > 0 \right\},$$

where  $\xrightarrow{d}$  denotes shallow convergence of measures and  $\lambda_d$  is the Lebesgue measure on  $\mathbb{R}^d$ . Here and in the sequel, we ease notation by reducing  $\lambda_d(d\theta)$  to just  $d\theta$  when appropriate. *Shallow* convergence is stronger than vague convergence yet still allows convergence to non-finite measures like Lebesgue measure. It is defined in [22] as

$$v_t \xrightarrow{d} v \iff v_t(f) \rightarrow v(f), \quad \forall \text{ continuous } f : \sup_{x \in \mathbb{R}^d} |e^{\epsilon|x|^2} f(x)| < \infty \text{ for some } \epsilon > 0.$$

1.1. *Statement of main results*

For notational simplicity, we will simply call an  $\alpha$ -stable Fleming–Viot process with sampling rate  $1/\phi(t)$  an  $(\alpha, \phi)$ -FV process. As explained previously,  $(\alpha, \phi)$ -FV processes corresponds to  $W_t/W_t(1)$  conditioned so that  $W_t(1) = \phi(t)$  for all  $t$ . For supercritical Dawson–Watanabe processes, the total mass  $W_t(1)$  has expected mean  $m(1)e^{\beta t}$  for some  $\beta > 0$ . This suggests that if  $\phi(t) = e^{\beta t}$ , then we should have the almost-sure, shallow-topology, long-time limit

$$t^{\frac{d}{\alpha}} X_t \xrightarrow{t \rightarrow \infty} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} d\theta \lambda_d. \tag{1.3}$$

In fact, our first main result shows the above almost sure limit for a larger class of sampling functions  $\phi$ .

**Theorem 1.1.** *Assume that  $\mu$  satisfies*

$$\int_{\mathbb{R}^d} |x|^a \mu(dx) < \infty \quad \text{for some } a > 0 \tag{1.4}$$

and  $\phi$  is a positive function on  $\mathbb{R}_+$  such that

$$\int_0^\infty s^{\frac{d}{\alpha} + 1 + \varepsilon_0} \frac{ds}{\phi(s)} < \infty \quad \text{for some } \varepsilon_0 > 0. \tag{1.5}$$

Then, with  $\mathbb{P}_\mu^\phi$ -probability one, the shallow limit (1.3) holds.

In addition, if a test function  $f \in C_c^2(\mathbb{R}^d)$  is fixed, all higher order asymptotics of  $X_t(f)$  can be identified. To state our second main result, we prepare some notation. For each multi-index  $k = (k_1, \dots, k_d) \in \mathbb{N}^d$  and  $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ , we let

$$|k| = k_1 + k_2 + \dots + k_d, \quad k! = k_1!k_2!\dots k_d!, \quad x^k = x_1^{k_1}x_2^{k_2}\dots x_d^{k_d},$$

and define the constant  $\vartheta_{d,\alpha}^k$  and the  $\sigma$ -finite signed measure  $\lambda_d^k$  on  $\mathbb{R}^d$  respectively by

$$\vartheta_{d,\alpha}^k = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} \theta^k \, d\theta \quad \text{and} \quad \lambda_d^k(dy) = \frac{1}{k!} y^k \, dy. \tag{1.6}$$

Obviously,  $\lambda_d^0$  is the Lebesgue measure  $\lambda_d$ .

**Theorem 1.2.** *Let  $N$  be a non-negative integer. Assume that  $\mu$  satisfies (1.4) and  $\phi$  satisfies*

$$\int_0^\infty s^{\frac{2N+d}{\alpha}+1+\varepsilon_0} \frac{ds}{\phi(s)} < \infty \quad \text{for some } \varepsilon_0 > 0. \tag{1.7}$$

*Let  $f$  be a function in  $b\mathcal{E}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$  satisfying*

$$\int_{\mathbb{R}^d} |f(x)| |x|^N \, dx < \infty \tag{1.8}$$

*with its Fourier transform  $\hat{f}$  satisfying*

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)| |\xi|^\alpha \, d\xi < \infty. \tag{1.9}$$

*Then,  $\mathbb{P}_\mu^\phi$ -almost surely*

$$\lim_{t \rightarrow \infty} t^{\frac{N+d}{\alpha}} \left| X_t(f) - \sum_{\substack{k \in \mathbb{N}^d: |k| \leq N \\ |k| \text{ is even}}} \frac{(-1)^{\frac{|k|}{2}} t^{-\frac{d+|k|}{\alpha}}}{(2\pi)^d k!} \int_{\mathbb{R}^d} f(y) y^k \, dy \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} \theta^k \, d\theta \right| = 0. \tag{1.10}$$

*Written another way, we have  $\mathbb{P}_\mu^\phi$ -almost surely*

$$t^{\frac{d}{\alpha}} X_t(f) = \sum_{\substack{k \in \mathbb{N}^d: |k| \leq N \\ |k| \text{ is even}}} (-1)^{\frac{|k|}{2}} t^{-\frac{|k|}{\alpha}} \vartheta_{d,\alpha}^k \lambda_d^k(f) + o(t^{-\frac{N}{\alpha}}) \quad \text{as } t \rightarrow \infty. \tag{1.11}$$

In the above statement,  $b\mathcal{E}(\mathbb{R}^d)$  denotes the space of real-valued bounded measurable functions on  $\mathbb{R}^d$ . Theorem 1.2 gives flexible rate-of-convergence information on the convergence of  $t^{\frac{d}{\alpha}} X_t$  to scaled Lebesgue measure, depending upon the conditions assumed and the test function used. Some of our results herein were later obtained for Dawson–Watanabe processes in [24].

The long-time behaviour of constant rate Fleming–Viot processes has been well studied. When  $\phi \equiv 1$  and  $\alpha = 2$ , the long-time behavior of  $X_t$  is discussed in Dawson and Hochberg [11]. They show that as time gets large, the measure-valued process  $(X_t, t \geq 0)$  concentrates within a random (but stationary) distance from a Brownian motion. The possible long-time distributional limits of even multiple (critical) interacting Fleming–Viot processes are well-known. For example, the stationary distributions were obtained in Shiga [29,30], Shiga and Uchiyama [31] for the two allele case and in Dawson et al. [10], Dawson and Greven [9] for the general case. These results characterize the possible distributional limits of interacting Fleming–Viot processes. However, Fleming–Viot processes that admit almost sure scaling limits do not appear to have been considered.

The proofs of Theorems 1.1 and 1.2 are presented in Section 3.2. Our method, discussed in Section 2.2, improves Asmussen and Hering’s technique in [1] and converts it to the language of martingales and stochastic integration. This formulation provides a clear picture, which may be applicable for superprocesses of both Dawson–Watanabe type and Fleming–Viot type with general Feller motions or mutation processes.

To illustrate our method further, we consider long-time limits of the occupation time  $Y_t = \int_0^t X_s \, ds$  and the inhabitation time, defined for bounded  $f$  as  $Z_t(f) = \mathbb{X}_t(\ell_f)$ . Here,

$$\ell_f(r, y) = \int_0^r f(y_s) \, ds \quad \forall r \geq 0, y \in D(\mathbb{R}^d), f \in b\mathcal{E}(\mathbb{R}^d), \tag{1.12}$$

$D(\mathbb{R}^d)$  is the space of  $\mathbb{R}^d$ -valued cadlag paths equipped with Skorohod  $J_1$  topology and  $\mathbb{X}$  is the  $(\alpha, \phi)$  Fleming–Viot historical process satisfying the martingale problem:

$$\mathbb{M}_t(h) = \mathbb{X}_t(h) - \delta_0 \times \mu^*(h) - \int_0^t \mathbb{X}_s(\mathbb{A}h) \, ds \tag{1.13}$$

is a continuous martingale starting at 0 such that

$$\langle \mathbb{M}(h) \rangle_t = \int_0^t (\mathbb{X}_s(h^2) - \mathbb{X}_s(h)^2) \frac{ds}{\phi(s)} \tag{1.14}$$

for all  $h$  in the domain of bounded functions  $\mathcal{D}(\mathbb{A})$  for the historical generator  $\mathbb{A}$ . (We define  $\mathbb{X}$  precisely and relate it to our  $(\alpha, \phi)$ -FV process below.  $\mu^*$  will be a variant of  $\mu$ , defined on the historical path space.) However,  $\ell_f$  with  $f = 1_{\mathcal{O}}$  (for an open  $\mathcal{O}$ ) is not bounded, hence  $\ell_{1_{\mathcal{O}}} \notin \mathcal{D}(\mathbb{A})$ . Still, the martingale problem (1.13,1.14) does hold for such *natural*  $h = \ell_f$  since

$$\mathbb{A}\ell_f = \text{bp-}\lim_{t \rightarrow \infty} \mathbb{A}\ell_f^t \tag{1.15}$$

exists and

$$\mathbb{X}_t(\ell_f) = \mathbb{X}_t(\ell_f^u) \quad \text{and} \quad \mathbb{X}_t(\mathbb{A}\ell_f) = \mathbb{X}_t(\mathbb{A}\ell_f^u) \quad \forall u \geq t, \tag{1.16}$$

where  $\ell_f^u$  is the bounded variant of  $\ell_f$ . Namely,

$$\ell_f^u(r, y) = \int_0^{r \wedge u} f(y_s) \, ds \quad \forall r \geq 0, y \in D(\mathbb{R}^d), f \in b\mathcal{E}(\mathbb{R}^d)$$

for each fixed  $u$  that we will show is in  $\mathcal{D}(\mathbb{A})$ . (Herein, bp-lim denotes the bounded, pointwise limit.) The definition of  $\mathbb{A}\ell_f$  through the limit (1.15) is established in Lemma 2.13 (to follow). To show (1.16), we let  $y^t = y(\cdot \wedge t) \in D(\mathbb{R}^d)$  for  $t > 0$  and  $y \in D(\mathbb{R}^d)$ , consider  $\mathbb{E} = \{(r, y^r) : r \geq 0, y \in D(\mathbb{R}^d)\}$  as a topological subspace of  $\mathbb{R}_+ \times D(\mathbb{R}^d)$ , and show that  $\mathbb{X}_t$  is supported on

$$\mathbb{E}^t = \{(r, y) \in \mathbb{E} : r = t\}, \tag{1.17}$$

which we do below (see Proposition 2.10 and Remark 2.11). Having observed (1.15) and (1.16), we can define  $\mathbb{M}_t(\ell_f)$  by substituting  $h = \ell_f^u$  in (1.13)

$$\mathbb{M}_t(\ell_f) := \mathbb{X}_t(\ell_f^u) - \delta_0 \times \mu^*(\ell_f^u) - \int_0^t \mathbb{X}_s(\mathbb{A}\ell_f^u) \, ds \quad \forall u \geq t. \tag{1.18}$$

It also follows from (1.16) that

$$\langle \mathbb{M}(\ell_f) \rangle_t = \langle \mathbb{M}(\ell_f^u) \rangle_t \quad \forall u \geq t. \tag{1.19}$$

**Remark 1.3.** The occupation time  $Y_t(1_{\mathcal{O}})$  counts the time in  $\mathcal{O}$  of dead lineages i.e. times of particles that are not ancestors of living particles, while the inhabitation time  $Z_t(1_{\mathcal{O}})$  counts the time of common ancestors multiple times.

Our third main result, connects these two time processes.

**Theorem 1.4.** *Defining  $\ell_f, \mathbb{M}(\ell_f)$  respectively as in (1.12), (1.18) and for all  $t > 0, f \in b\mathcal{E}(\mathbb{R}^d)$ ,*

$$\gamma_d(t) = \begin{cases} t^{1-\frac{d}{\alpha}} & \text{if } d < \alpha, \\ \ln(t \vee 1) & \text{if } d = \alpha, \\ 1 & \text{if } d > \alpha \end{cases} \quad \text{and} \quad \mathcal{N}_d(f) = \begin{cases} \|f\|_{L^1(\mathbb{R}^d)} & \text{if } d < \alpha, \\ \|f\|_{L^1(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |\hat{f}(\theta)| |\theta|^{-\alpha} \, d\theta & \text{if } d = \alpha, \\ \int_{\mathbb{R}^d} |\hat{f}(\theta)| |\theta|^{-\alpha} \, d\theta & \text{if } d > \alpha. \end{cases} \tag{1.20}$$

One has that:

- a:  $\mathbb{M}_t(\ell_f) = Z_t(f) - Y_t(f)$  is a  $\mathcal{F}_{t+}^X$ -martingale for  $f \in b\mathcal{E}(\mathbb{R}^d)$ .
- b:  $\mathbb{M}_\infty(\ell_f) := \lim_{t \rightarrow \infty} \mathbb{M}_t(\ell_f)$  exists a.s. provided that  $\mathcal{N}_d(f) < \infty$  and  $\int_0^\infty \gamma_d^2(s) \frac{ds}{\phi(s)} < \infty$ .
- c:  $Z_\infty(f) := \lim_{t \rightarrow \infty} Z_t(f)$ ,  $Y_\infty(f) := \lim_{t \rightarrow \infty} Y_t(f)$  both exist a.s. in  $\mathbb{R}$  and  $\mathbb{M}_\infty(\ell_f) = Z_\infty(f) - Y_\infty(f)$  a.s. provided that  $\mathbb{M}_\infty(\ell_f)$  exists,  $d > \alpha$  and  $\mathcal{N}_d(f) < \infty$ .

The proof of parts a, b, and c follow respectively from Proposition 2.14, Proposition 4.4(i), and the proof of the high dimensional case of Theorem 1.5 as well as Proposition 4.4.

Notice that the martingale  $\{\mathbb{M}_t(\ell_f), t \geq 0\}$  in Theorem 1.4 is with respect to the right continuous filtration of the (non-historical)  $(\alpha, \phi)$ -FV process, which is possible when the  $(\alpha, \phi)$ -FV process is defined from the  $(\alpha, \phi)$ -historical process through (2.37) below.

Hereafter, the cases  $d < \alpha$ ,  $d = \alpha$  and  $d > \alpha$  are respectively called *low dimension*, *critical dimension* and *high dimension*. One can see from Theorem 1.4 that particle time behaviour is dimensionally dependent as one might expect from the transition from recurrent to transient particle motion.

While in the context of Dawson–Watanabe processes, long-time asymptotics of occupation time processes have been studied extensively starting from the work of Iscoe [20], the corresponding problem for Fleming–Viot processes seems sparse in the literature. As is known in the context of Dawson–Watanabe processes, the limiting behavior of occupation times depends on the relation between  $d$  and  $\alpha$ . Our fourth main result shows the limiting behaviour of occupation and inhabitation times for  $(\alpha, \phi)$ -Fleming–Viot processes will also be dimensionally dependent. Define

$$\kappa_d(\alpha) = \begin{cases} (2\pi)^{-d} \frac{\alpha}{\alpha-d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} d\theta & \text{if } d < \alpha, \\ (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} d\theta & \text{if } d = \alpha. \end{cases} \tag{1.21}$$

**Theorem 1.5.** *Suppose  $\phi$  satisfies*

$$\int_1^\infty \frac{ds}{\phi(s)} < \infty. \tag{1.22}$$

*Then, in low and critical dimensions ( $d \leq \alpha$ ), with  $\mathbb{P}_\mu^\phi$ -probability one, scaled occupation and inhabitation times  $\frac{Y_t}{\gamma_d(t)}$  and  $\frac{Z_t}{\gamma_d(t)}$  both converge shallowly to  $\kappa_d(\alpha)\lambda_d$ . While in high dimensions ( $d > \alpha$ ), with  $\mathbb{P}_\mu^\phi$ -probability one,  $Y_t$  and  $Z_t$  converge shallowly to some random measures.*

This result relies on Proposition 4.1 (to follow) and is proved at the end of Section 4.2.

### 1.2. Explanation of sampling rate assumptions

We would like to thank an anonymous referee for inviting us to speculate around our sampling rate assumptions. Condition (1.5) can be considered heuristically in two ways: Kouritzin and Ren [22] showed the (shallow) a.s. convergence of  $t^{\frac{d}{\alpha}} \frac{W_t}{e^{\beta t}}$  when  $W$  was a superstable process with growth factor  $\beta > 0$ . However, it is well known (see [2]) that  $e^{-\beta t} W_t(1) \rightarrow F$  a.s. for some non-trivial random variable  $F$  in this case and the limits of  $t^{\frac{d}{\alpha}} \frac{W_t}{e^{\beta t}}$  and  $t^{\frac{d}{\alpha}} \frac{W_t}{W_t(1)}$  will only differ by this factor  $F$  (on the set where  $F > 0$ ). Next, conditioning on  $W_t(1)$  to be close to  $e^{\beta t}$  might not have a huge effect since their ratio converges. Finally, Perkins [28]’s argument on  $\frac{W_t}{W_t(1)}$  conditioned so  $W_t(1)$  is  $e^{\beta t}$  has martingale problem (1.1,1.2) with  $\phi(t) = e^{\beta t}$ . In this way, Theorem 1.1 loosely generalizes Kouritzin and Ren [22] from  $\phi(t) = e^{\beta t}$  to any  $\phi$  satisfying (1.5). Secondly, the factor  $t^{\frac{d}{\alpha}}$  on the left of (1.3) is what is needed for a non-trivial limit in Theorem 1.1 but this factor blows up  $X_t$  and its noise  $M_t$ . To have an almost sure limit the noise has to die out fast enough through the  $\phi(s)^{-1}$  factor in (1.2). We can think of the  $s^{\frac{d}{\alpha}}$  factor within the integral of (1.5) as compensation for blowing  $X_t$  up by  $t^{\frac{d}{\alpha}}$  and the integral without this factor as a condition on the noise of  $X$  itself. The full force of (1.5) only comes to bear in Proposition 2.8 through conditions (2.18), (2.23). In the proofs of Theorems 1.1 and 1.2, we will decompose  $X_t(f)$  as

$$X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f) + [X_{t_n}(T_{t_{n+1}-t_n}f) - X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f)] + [X_t(f) - X_{t_n}(T_{t_{n+1}-t_n}f)],$$

where  $T_t$  is the  $\alpha$ -stable semigroup,  $L_t$  is an  $N^{\text{th}}$  order approximation of  $T_t$  and  $\rho$  is a sublinear function. It follows from Proposition 3.2 that  $X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f)$  satisfies the stated scaling limits using Fourier analysis under a lesser condition on  $\phi$  so the other two terms can be thought of as errors. The first error term, handled in (2.28), puts constraints on an

auxiliary sequence  $\{c_n\}$  while the second error term, handled in (2.29), forces a constraint on  $\phi$  depending upon the  $\{c_n\}$ . The two constraints are then solved in (3.20) under (1.5). It would be interesting to know lesser conditions on  $\phi$  under which one has convergence in probability but not necessarily almost sure convergence.

### 1.3. Article outline

Section 2 discusses fundamental results of Fleming–Viot processes. Section 3 focuses on the long-time limit of  $\alpha$ -stable Fleming–Viot processes. In Section 4, long-time asymptotic of the occupation time process as well as the related inhabitation time process of an  $\alpha$ -stable Fleming–Viot process is investigated.

## 2. Fleming–Viot processes

We use  $\nu(f)$  and  $\langle f, \nu \rangle$  to denote  $\int f d\nu$  for a measure  $\nu$  and integrable function  $f$ . Let  $(E, \mathcal{E}(E))$  be a Polish space with its Borel  $\sigma$ -algebra  $\mathcal{E}(E)$  and  $((\xi_t)_{t \geq 0}, (P_x)_{x \in E})$  be an  $E$ -valued Borel strong Markov process with sample paths in  $D(E)$ . Hereafter,  $D(E)$  is the space of cadlag paths from  $\mathbb{R}_+ := [0, \infty)$  to  $E$  equipped with the Skorohod  $J_1$  topology. Define the semigroup on  $b\mathcal{E}(E)$  (the space of real-valued bounded measurable functions on  $E$ ) by

$$T_t f(x) = P_x f(\xi_t),$$

and assume that  $T_t$  maps  $C_b(E)$  (the space of real continuous bounded functions on  $E$ ) to itself. The right-continuity of  $\xi$  implies  $\text{bp-lim}_{t \rightarrow 0} T_t f = f$  for every  $f \in C_b(E)$ . We also assume  $\xi$  is conservative, i.e.  $T_t \mathbf{1} = \mathbf{1}$ . Define

$$A f = \text{bp-lim}_{t \rightarrow 0} \frac{T_t f - f}{t}$$

when the limit exists. The domain  $\mathcal{D}(A)$  of  $A$  contains all functions in  $b\mathcal{E}(E)$  such that the above limit exists, including the constant function  $\mathbf{1}$  for which  $A\mathbf{1} = 0$ .  $(A, \mathcal{D}(A))$  is the so-called weak generator of  $\xi$ . It is known ([27, Corollary II.2.3]) that  $\mathcal{D}(A)$  is bp-dense in  $b\mathcal{E}(E)$ . We adopt the following standard notation:

- $M_F(E), M_1(E)$  denote the spaces of finite, respectively probability measures.
- $(\Omega_F, \mathcal{F}), (\Omega, \mathcal{G})$  are the sample spaces of (compact-open) continuous mappings  $(C([0, \infty), M_F(E)))$  respectively  $(C([0, \infty), M_1(E)))$  with their respective Borel  $\sigma$ -fields.
- $W_t(\omega) = \omega_t, X_t(\omega) = \omega_t$  denote the coordinate mappings on  $\Omega_F$  and  $\Omega$  respectively.
- $\mathcal{F}_t^0 = \sigma(W_s : s \leq t), \mathcal{F}_t = \mathcal{F}_{t+}^0; \mathcal{G}_t^0 = \sigma(X_s : s \leq t), \mathcal{G}_t = \mathcal{G}_{t+}^0$ .

### 2.1. Martingale problems

Let  $E = \mathbb{R}^d$ . For each  $\beta \geq 0, \eta > 0$  and  $m \in M_F(\mathbb{R}^d)$ , there is a unique probability  $\mathbb{P}_m$  on  $(\Omega_F, \mathcal{G})$  such that for all  $f \in \mathcal{D}(A)$

$$M_t^W(f) = W_t(f) - m(f) - \int_0^t W_s(Af + \beta f) ds \tag{2.1}$$

is a continuous  $(\mathcal{F}_t)$ -martingale starting at 0 with quadratic variation

$$\langle M^W(f) \rangle_t = \eta \int_0^t W_s(f^2) ds. \tag{2.2}$$

$\mathbb{P}_m$  is the law of the critical or supercritical  $A$ -Dawson–Watanabe process with drift  $\beta$  and branching variance  $\eta$ .

**Remark 2.1.** There is substantial theory on the existence, uniqueness, path properties and high density limits for Dawson–Watanabe superprocesses under conditions far more general than required here. However, the martingale problem and the connection to finite populations motivate the study of long-time behaviour of our model. Hence, we will expand upon Example 10.1.2.2 in [8] and remind the reader of some basic points in the case  $E = \mathbb{R}^d$  while neglecting details similar to those handled in [8] and [19]. It follows from the proofs of Theorems 9.4.2 and 9.4.3 of [19] that the local martingale problem:

$$\left\{ \begin{array}{l} \exp(-W_t(f)) + \int_0^t \exp(-W_s(f)) W_s(Af + \beta f - \frac{\eta}{2} f^2) ds \\ \text{is a local martingale for all non-negative } f \in \mathcal{D}(A), \end{array} \right. \tag{2.3}$$

is well posed. (Technically, one could first follow [19, Theorems 9.4.2 and 9.4.3] to get well-posedness of the martingale problem. Then, for a potential local martingale solution  $W$ , we realize that  $M_t^\lambda \doteq \exp(-\lambda W_t(1)) + \int_0^t \exp(-\lambda W_s(1))(\lambda\beta - \frac{\lambda^2\eta}{2})W_s(1) ds$  is actually a martingale and use the argument on p. 403 of [19] to show  $\mathbb{P}_m[W_t(1)] \leq m(1)\exp(\beta t)$ . From there, it follows that any solution to the local martingale problem is a solution to the martingale problem.) As part of justifying the use of these proofs, we note that (2.3) is the high density limit of finite branching population models. For example, using the notation of [19] and letting  $c \in (0, \frac{1}{\eta})$ , we find that the population starting with  $n$  individuals, undergoing independent  $A$ -motions with location-independent lifetime rates  $\alpha_n = \frac{\eta}{c}$  and having offspring probability generating function  $\varphi_n(z) = c(\frac{\eta}{2} - \frac{\beta}{n}) + (\frac{c\beta}{n} + 1 - c\eta)z + \frac{c\eta}{2}z^2$  is a well defined model for large enough  $n$  and these populations converge (pathwise) to the solution of (2.3) as  $n \rightarrow \infty$ . Next, it follows from Corollary 2.3.3 of [19] that the local martingale problem in (2.3) is equivalent to the local martingale problem:

$$\left\{ \begin{array}{l} \exp(-W_t(f) + \int_0^t W_s(Af + \beta f - \frac{\eta}{2}f^2) ds) \\ \text{is a local martingale for all non-negative } f \in \mathcal{D}(A). \end{array} \right. \tag{2.4}$$

However, to go further, we must ensure that  $W$  is continuous. This continuity is shown by following Theorem 4.7.2 of [8] for the case  $\beta = 0$  and the case  $\beta \neq 0$  is converted to the case  $\beta = 0$  by Dawson’s Girsanov theorem (Theorem 7.2.2 and Lemma 10.1.2.1 of [8]) with  $r(\mu, y) = \beta$  and  $Q(\mu; dx, dy) = \delta_x(dy)\mu(dx)$ . (This theorem is stated in terms of a larger domain but we already have uniqueness for the smaller domain in (2.3).) Now, by this continuity, the local martingale problem (2.4) is equivalent to the local martingale problem (2.1)–(2.2) by e.g. Theorem 6.2 [7]. Finally, we show that each  $M_t^W(f)$  in (2.1) is a martingale for any continuous local martingale problem solution  $W$  by showing  $\mathbb{P}_m[W_t(1)] \leq m(1)\exp(\beta t)$  through stopping and Gronwall’s inequality.

The process  $\{W_t(1)\}_{t \geq 0}$  describes the evolution of total mass with life time

$$t_W = \inf\{t > 0 : W_t(1) = 0\}.$$

Even in the supercritical regime ( $\beta > 0$ ),  $t_W$  is finite with positive probability. Using the martingale structure of  $W$ , we can describe the evolution of the normalized process  $\bar{W} = \{\frac{W_t}{W_t(1)}, 0 \leq t < t_W\}$  as in the following result.

**Lemma 2.2.** *Assume that  $m \neq 0$ . Let  $\bar{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(W_s(1) : s \geq 0)$  and  $\mu = m/m(1)$ . For every  $f \in \mathcal{D}(A)$*

$$M_t^{\bar{W}}(f) = \bar{W}_t(f) - \mu(f) - \int_0^t \mathbf{1}(s < t_W) \bar{W}_s(Af) ds, \quad t \geq 0$$

*is a continuous  $(\bar{\mathcal{F}}_t)$ -martingale starting at 0 such that*

$$\langle M_t^{\bar{W}}(f) \rangle_t = \eta \int_0^t \mathbf{1}(s < t_W) (\bar{W}_s(f^2) - \bar{W}_s(f)^2) \frac{ds}{\bar{W}_s(1)} \quad \mathbb{P}_m\text{-a.s.}$$

**Proof.** The case when  $\beta = 0$  is proved in Perkins [28] using Itô formula. If  $\beta > 0$ , the proof follows analogously, we omit the details. □

Let  $C_+$  be the space of continuous functions  $\phi : [0, \infty) \rightarrow [0, \infty)$  such that  $\phi(t) > 0$  if  $t \in [0, t_\phi)$  and  $\phi(t) = 0$  if  $t \geq t_\phi$  for some  $t_\phi \in (0, \infty]$ . Let  $Q_{m(1)}$  be the law of  $W.(1)$ , i.e.

$$\mathbb{P}_m(W.(1) \in B) = Q_{m(1)}(B).$$

**Theorem 2.3 (Perkins [28]).** *For every  $\phi \in C_+$  and  $\mu \in M_1(E)$ , there is a unique probability  $\mathbb{P}_\mu^\phi$  on  $(\Omega, \mathcal{G})$  such that under  $\mathbb{P}_\mu^\phi$ , for all  $f \in \mathcal{D}(A)$ ,*

$$M_t^X(f) = X_t(f) - \mu(f) - \int_0^t X_s(Af) ds, \quad t < t_\phi \tag{2.5}$$

*is a continuous  $(\mathcal{G}_t)$ -martingale starting at 0 and such that*

$$\langle M_t^X(f) \rangle_t = \eta \int_0^t (X_s(f^2) - X_s(f)^2) \phi(s)^{-1} ds \quad \forall t < t_\phi \tag{2.6}$$

*and  $X_t = X_{t_\phi}$  for all  $t \geq t_\phi$ .*

**Remark 2.4.** We will use this theorem in Polish spaces  $E = \mathbb{R}^d$  and  $E = \mathbb{E}$ , defined just above (1.17). It is obtained under the assumption that  $E$  is locally compact in Theorem 2 (a) of [28]. The proof in [28] uses detailed arguments, state augmentation and worthy martingale measure representation to change the speed of the sampling martingale relative to the particles motions. This martingale time change argument is then used to infer the existence and uniqueness of  $\mathbb{P}_\mu^\phi$  from that of the law of the classical Fleming–Viot process, i.e.  $\mathbb{P}_\mu^1$ , which was only known on locally compact spaces. This is the only place in [28] where locally compactness was used. The existence and uniqueness of Fleming–Viot processes on Polish spaces have been since obtained by Donnelly and Kurtz in [13,14] based upon earlier ideas of Dawson and Hochberg [11]. Therefore, Perkins’ argument carries through in the setting of Polish spaces.

The connection between Dawson–Watanabe processes and Fleming–Viot processes with time-varying sampling rates  $\phi$  is as follows.

**Theorem 2.5 ([28, Theorem 3]).** *For every  $m \in M_F(E) \setminus \{0\}$ , set  $\mu = m/m(1)$ . For  $Q_{m(1)}$ -a.a.  $\phi$ , we have*

$$\mathbb{P}_m \left( \frac{W}{W(1)} \in A \mid W(1) = \phi \right) = \mathbb{P}_\mu^\phi(A) \quad \forall A \in \mathcal{G}.$$

[28, Theorem 3] is in the setting of locally compact  $E$ , which is fine for our purposes as we only use this theorem in the case of  $E = \mathbb{R}^d$  to motivate our work.

Corollaries 4 and 5 of [28] further establish that for every  $\phi \in C_+$ ,  $\mathbb{P}_\mu^\phi$  is indeed the regular conditional law  $\mathbb{P}_m(\frac{W}{W(1)} \in \cdot \mid W(1) = \phi)$ . Without loss of generality, we assume  $\eta = 1$  hereafter.

2.2. Long term asymptotics

Let  $E = \mathbb{R}^d$ ,  $\mu \in M_1(E)$  and  $\phi \in C_+$  with  $t_\phi = \infty$ . Let  $\mathbb{P}_\mu^\phi$  be the probability law introduced in Theorem 2.3. Recall  $\{T_t\}_{t \geq 0}$  is the semigroup generated by  $A$ . In practice, the semigroup  $T_t$  usually satisfies some asymptotical property. One possibility is the following: for each  $t > 0$ , there exist a deterministic positive scaling  $c(t)$  and an operator  $L_t$  such that

$$\lim_{t \rightarrow \infty} c(t) \|T_t f - L_t f\|_{L^\infty(E)} = 0. \tag{2.7}$$

(2.7) becomes trivial if we choose  $L_t f = T_t f$ . However, we can choose a different  $L_t f$  to our advantage. When  $T_t$  is the symmetric stable semigroup considered in Section 3,  $L_t$  can be chosen as the projection onto a finite dimensional vector space, whose basis are the partial derivatives of the kernel density  $p_t(x)$  (see (3.4) and (3.9) to follow).

In the current section, we present a general procedure to study long term asymptotic for  $X_t(f)$  given a test function  $f \in b\mathcal{E}(E)$ . The method consists of two steps. One first shows that  $X_t(f)$  and  $X_{\rho(t)}(T_{t-\rho(t)}f)$  have the same asymptotic as  $t \rightarrow \infty$ . Hereafter,  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is an increasing sub-linear function, that is  $\rho$  satisfies

$$\lim_{t \rightarrow \infty} \frac{\rho(t)}{t} = 0. \tag{2.8}$$

This step requires a certain integrability condition of the function  $1/\phi$  over  $\mathbb{R}_+$  (see Proposition 2.7 below). Next, by (2.7), one can further deduce the asymptotic of  $X_{\rho(t)}(T_{t-\rho(t)}f)$  from that of  $X_{\rho(t)}(L_{t-\rho(t)}f)$ . In this second step, having chosen  $L_t$  in our favor, we find the asymptotic of  $X_{\rho(t)}(L_{t-\rho(t)}f)$  directly by other tools. In Section 3, we explain how the procedure can be applied to study super stable processes and their occupation times.

In the context of Dawson–Watanabe processes with supercritical branching mechanisms, this method goes back to [1] and has been extended to treat superprocesses with more general Markovian motions (see for instance [6,25]). Until recently, it seemed that Asmussen and Hering’s method required a certain spectral gap assumption on the semigroup  $T_t$ . However, in [21], the same procedure is applied for supercritical branching Gaussian processes. The treatment presented here contains some simplifications and improvements.

Let us now develop a stochastic integration framework which is an essential tool in our approach. Letting  $M_t^X(U) = M_t^X(\mathbf{1}_U)$ , we note that for every  $U, V \in \mathcal{E}(E)$ ,

$$\langle M^X(U), M^X(V) \rangle_t \leq \int_0^t X_s(\mathbf{1}_U \mathbf{1}_V) \frac{ds}{\phi(s)}.$$

In particular,  $(M_t^X)_{t \geq 0}$  is a worthy martingale measure (see [32, Chapter 2]). For every adapted process  $\{g(r, z) = g_r(z) : r \geq 0, z \in E\}$  satisfying

$$\mathbb{P}_\mu^\phi \int_0^\infty X_r(g_r^2) \frac{dr}{\phi(r)} < \infty,$$

one can construct the stochastic integral  $\int_0^\infty \int_E g(r, z) dM^X(r, z)$  such that

$$\mathbb{P}_\mu^\phi \left( \int_0^\infty \int_E g(r, z) dM^X(r, z) \right)^2 = \mathbb{P}_\mu^\phi \int_0^\infty (X_r(g_r^2) - X_r(g_r)^2) \frac{dr}{\phi(r)}. \tag{2.9}$$

We refer to [32, Chapter 2] for a detailed construction.

This worthy martingale measure representation allows us to extend the martingale problem (2.5), (2.6) by an integration by parts argument. In particular, for continuously differentiable  $f_t$  in  $t$  that satisfies  $f_t \in \mathcal{D}(A)$  for all  $t$  and  $\mathbb{P}_\mu^\phi \int_0^\infty X_r(f_r^2) \frac{dr}{\phi(r)} < \infty$ , we have that

$$\int_0^t \int_E f_r(z) dM^X(r, z) = X_t(f_t) - \mu(f_0) - \int_0^t X_r(Af_r) dr - \int_0^t X_r(\partial_r f_r) dr \tag{2.10}$$

is a continuous  $(\mathcal{G}_t)$ -martingale starting at 0 and such that

$$\left\langle \int_0^\cdot \int_E f_r(z) dM^X(r, z) \right\rangle_t = \eta \int_0^t (X_r(f_r^2) - X_r(f_r)^2) \phi(r)^{-1} dr. \tag{2.11}$$

The particular choice  $f_s = \int_0^{t-s} T_r f dr$  for  $t$  fixed and  $f \in b\mathcal{E}(E)$  gives

$$\int_s^t X_r(f) dr = X_s \left( \int_0^{t-s} T_r f dr \right) + \int_s^t \int_E \int_0^{t-r} T_{\bar{r}} f(z) d\bar{r} dM^X(r, z). \tag{2.12}$$

Moreover, it follows from (2.5) (and fact  $t_\phi = \infty$ ) that for every  $f \in b\mathcal{E}(E)$ ,

$$X_t(f) = \mu(T_t f) + \int_0^t \int_E T_{t-r} f(z) dM^X(r, z), \tag{2.13}$$

which is called Green function representation in [27, p. 167]. The representation (2.13) and (2.9) play a central role in our approach. A direct consequence of (2.13) is the following identity

$$X_t(f) - X_s(T_{t-s} f) = \int_s^t \int_E T_{t-r} f(z) dM^X(r, z), \tag{2.14}$$

which holds for every  $0 \leq s \leq t$  and  $f \in b\mathcal{E}(E)$ . Another consequence of (2.13) is

$$\mathbb{P}_\mu^\phi X_t(f) = \mu(T_t f). \tag{2.15}$$

**Lemma 2.6.** *For every  $f \in b\mathcal{E}(E)$  and  $t \geq s \geq 0$ , we have*

$$\mathbb{P}_\mu^\phi [(X_t(f) - X_s(T_{t-s} f))^2] \leq \|T_t(f^2)\|_\infty \int_s^t \frac{dr}{\phi(r)} \tag{2.16}$$

and

$$\mathbb{P}_\mu^\phi \left[ \left( \int_s^t X_r(f) dr - X_s \left( \int_0^{t-s} T_r f dr \right) \right)^2 \right] \leq \int_s^t \left\| \int_0^r T_{\bar{r}} f d\bar{r} \right\|_\infty^2 \frac{dr}{\phi(r)}. \tag{2.17}$$

**Proof.** From (2.14), (2.9) and (2.15)

$$\begin{aligned} \mathbb{P}_\mu^\phi [(X_t(f) - X_s(T_{t-s} f))^2] &\leq \mathbb{P}_\mu^\phi \int_s^t X_r((T_{t-r} f)^2) \frac{dr}{\phi(r)} \\ &\leq \int_s^t \langle T_r(T_{t-r} f)^2, \mu \rangle \frac{dr}{\phi(r)}. \end{aligned}$$

By Jensen inequality,

$$T_r(T_{t-r} f)^2 \leq T_r T_{t-r}(f^2) = T_t(f^2).$$

Hence,  $\langle T_r(T_{t-r}f)^2, \mu \rangle \leq \mu(T_t(f^2)) \leq \|T_t(f^2)\|_\infty$ . The estimate (2.16) follows. Showing (2.17) is similar so we omit the detail.  $\square$

*Convergence along lattice times.* Suppose that  $f$  is a function in  $b\mathcal{E}(E)$ . Let  $\{t_n\}_{n \geq 1}$  be an increasing sequence diverging to infinity such that

$$\sum_n c(t_n) \|T_{t_n} f^2\|_\infty \int_{\rho(t_n)}^{t_n} \frac{ds}{\phi(s)} < \infty \tag{2.18}$$

and

$$\lim_{n \rightarrow \infty} \frac{c(t_n)}{c(t_n - \rho(t_n))} = 1. \tag{2.19}$$

**Proposition 2.7.** *Assuming (2.7), (2.8), (2.18) and (2.19), the following limit holds*

$$\lim_{n \rightarrow \infty} c(t_n) |X_{t_n}(f) - X_{\rho(t_n)}(L_{t_n - \rho(t_n)}f)| = 0 \quad \mathbb{P}_\mu^\phi\text{-a.s.} \tag{2.20}$$

**Proof.** From Lemma 2.6,

$$\sum_n \mathbb{P}_\mu^\phi c(t_n)^2 [|X_{t_n}(f) - X_{\rho(t_n)}(T_{t_n - \rho(t_n)}f)|^2] \leq \sum_n c(t_n) \|T_{t_n} f^2\|_\infty \int_{\rho(t_n)}^{t_n} \frac{ds}{\phi(s)} < \infty$$

by condition (2.18). An application of Borel-Cantelli lemma yields

$$\lim_{n \rightarrow \infty} |c(t_n)X_{t_n}(f) - c(t_n)X_{\rho(t_n)}(T_{t_n - \rho(t_n)}f)| = 0 \quad \mathbb{P}_\mu^\phi\text{-a.s.}$$

Moreover, noting that  $X_s(1) = 1$  for every  $s > 0$ , we have

$$\begin{aligned} & c(t_n) |X_{\rho(t_n)}(T_{t_n - \rho(t_n)}f) - X_{\rho(t_n)}(L_{t_n - \rho(t_n)}f)| \\ & \leq c(t_n) X_{\rho(t_n)}(|T_{t_n - \rho(t_n)}f - L_{t_n - \rho(t_n)}f|) \\ & \leq c(t_n) \|T_{t_n - \rho(t_n)}f - L_{t_n - \rho(t_n)}f\|_\infty, \end{aligned}$$

which converges  $\mathbb{P}_\mu^\phi$ -a.s. to 0 by (2.7), (2.8) and (2.19). The identity (2.20) follows.  $\square$

*From lattice time to continuous time.* If the cost of replacing  $c(t_n)$  by  $c(t)$  for any  $t \in [t_n, t_{n+1}]$  is negligible as  $n \rightarrow \infty$ , then previous result can be transferred to continuous time limit. There are several ways to obtain this. One possibility is the following result while Section 4 provides another way. Hereafter,  $c_n$  denotes  $\sup_{t \in [t_n, t_{n+1}]} c(t)$ .

**Proposition 2.8.** *In addition to the hypothesis in Proposition 2.7, we assume that*

$$\lim_{n \rightarrow \infty} c_n \sup_{t \in [t_n, t_{n+1}]} \|T_{t_{n+1} - t}f - f\|_\infty = 0, \tag{2.21}$$

$$\lim_{n \rightarrow \infty} \frac{c_n}{c(t_n)} = 1, \tag{2.22}$$

and

$$\sum_n c_n \|T_{t_{n+1}}(f^2)\|_\infty \int_{t_n}^{t_{n+1}} \frac{ds}{\phi(s)} < \infty. \tag{2.23}$$

Then

$$\lim_{n \rightarrow \infty} \sup_{t \in [t_n, t_{n+1}]} c(t) |X_t(f) - X_{\rho(t_n)}(L_{t_n - \rho(t_n)}f)| = 0 \quad \mathbb{P}_\mu^\phi\text{-a.s.} \tag{2.24}$$

**Proof.** We adopt an argument from [25], which utilizes the properties of the semigroup  $T_t$  and the martingale  $M_t^X$  at the same time. For every  $t \in [t_n, t_{n+1})$  we have

$$|X_t(f) - X_t(T_{t_{n+1}-t}f)| \leq X_t(|f - T_{t_{n+1}-t}f|) \leq \sup_{t \in [t_n, t_{n+1}]} \|T_{t_{n+1}-t}f - f\|_\infty.$$

It follows from (2.21) that

$$\lim_n \sup_{t \in [t_n, t_{n+1})} c(t) |X_t(f) - X_t(T_{t_{n+1}-t}f)| = 0. \tag{2.25}$$

Hence, to show (2.24), it suffices to prove

$$\lim_{n \rightarrow \infty} \sup_{t \in [t_n, t_{n+1})} c(t) |X_t(T_{t_{n+1}-t}f) - X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f)| = 0 \quad \mathbb{P}_\mu^\phi\text{-a.s.} \tag{2.26}$$

From (2.14) we have

$$\begin{aligned} X_t(T_{t_{n+1}-t}f) &= X_{t_n}(T_{t-t_n}T_{t_{n+1}-t}f) + \int_{t_n}^t \int_E T_{t-s}T_{t_{n+1}-t}f(x) \, dM^X(s, x) \\ &= X_{t_n}(T_{t_{n+1}-t_n}f) + \int_{t_n}^t \int_E T_{t_{n+1}-s}f(x) \, dM^X(s, x). \end{aligned} \tag{2.27}$$

Similar to (2.25), we have

$$\lim_n \sup_{t \in [t_n, t_{n+1})} c(t) |X_{t_n}(T_{t_{n+1}-t_n}f) - X_{t_n}(f)| = 0.$$

Together with Proposition 2.7 and (2.22), this yields

$$\lim_n \sup_{t \in [t_n, t_{n+1})} c(t) |X_{t_n}(T_{t_{n+1}-t_n}f) - X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f)| = 0 \quad \mathbb{P}_\mu^\phi\text{-a.s.} \tag{2.28}$$

Hence, (2.26) follows from (2.27) and (2.28) if we can show that

$$\lim_n c_n \sup_{t \in [t_n, t_{n+1})} \left| \int_{t_n}^t \int_E T_{t_{n+1}-s}f(x) \, dM^X(s, x) \right| = 0 \quad \mathbb{P}_\mu^\phi\text{-a.s.} \tag{2.29}$$

Fixing  $\varepsilon > 0$  and applying the martingale maximal inequality as well as Lemma 2.6 and (2.14), we have

$$\begin{aligned} &\mathbb{P}_\mu^\phi \left( c_n \sup_{t \in [t_n, t_{n+1})} \left| \int_{t_n}^t \int_E T_{t_{n+1}-s}f(x) \, dM^X(s, x) \right| > \varepsilon \right) \\ &\leq \varepsilon^{-2} c_n^2 \mathbb{P}_\mu^\phi \left| \int_{t_n}^{t_{n+1}} \int_E T_{t_{n+1}-s}f(x) \, dM^X(s, x) \right|^2 \\ &\leq \varepsilon^{-2} c_n^2 \|T_{t_{n+1}}f\|_\infty^2 \int_{t_n}^{t_{n+1}} \frac{ds}{\phi(s)}. \end{aligned}$$

Using (2.18), we see that

$$\sum_n \mathbb{P}_\mu^\phi \left( c_n \sup_{t \in [t_n, t_{n+1})} \left| \int_{t_n}^t \int_E T_{t_{n+1}-s}f(x) \, dM^X(s, x) \right| > \varepsilon \right) < \infty.$$

Applying Borel–Cantelli lemma, we find that (2.29) follows and the proof is complete. □

**Remark 2.9.** In view of Proposition 2.7, to study the long-time asymptotic of  $X_t(f)$  for a test function  $f$ , we first study the long-time asymptotic of  $T_t f$  and identify  $c(t)$  and  $L_t$  in (2.7). Then, we establish the long-time limit for  $X_{\rho(t)}(L_{t-\rho(t)}f)$  for a suitable sublinear function  $\rho$ . This procedure will be applied throughout Sections 3 and 4.

### 2.3. Finite particle motivation

The inhabitation time  $Z_t$  discussed in the introduction counts the time spent (in sets) by all ancestors of all particles living at time  $t$ . It counts common ancestors multiple times. It does not count time for particles with no living descendants. As such it requires genealogical information that is not readily available from the Fleming–Viot process  $X$  itself. We need to construct the historical process  $\mathbb{X}$  associated with  $X$ .

To motivate historical processes and the difference between occupation and inhabitation times, we consider a finite particle approximation. Suppose that  $X_t^N = \frac{1}{N} \sum_{\alpha \sim t} \delta_{\xi_t^\alpha}$  is a (Moran particle system empirical measure) pre-high-density limit of  $X$ .  $\{\xi^\alpha\}_{\alpha \in M}$  are particles that undergo independent  $A$ -motions/mutations and are resampled at (time-inhomogeneous) rate proportional to  $N(N - 1)$ . At a resampling time one random particle is selected to move to another random particle’s location. This moved particle disowns her ancestors and adopts those of the particle to which it jumped. (This common convention is consistent with Fleming–Viot superprocesses providing distributional information about Dawson–Watanabe superprocess populations. Sampling is simultaneous deaths and generation of offspring from some of the dying particles.) Here, the set of multi-indices  $\alpha$  keep track of all particles, whether they are living at  $t$  or not, and  $\alpha \sim t$  means particle  $\alpha$  is alive at time  $t$ . Naturally, there are  $N$  particles alive at any time so  $X_t^N$  is a probability measure but the actual particles that are alive is dependent upon which particles are sampled prior to  $t$  and multi-indices  $\alpha$  are used to keep track of ancestors. For example, particle  $(1, 2, 3)$  would be the parent ancestor of  $(1, 2, 3, 1)$  and  $(1, 2, 3, 2)$  for random outcomes where they all exist. Now, let  $\xi_{[0,t]}^\alpha$  denote the ancestral path of particle  $\alpha$  as a  $D(\mathbb{R}^d)$ -path held constant after  $t$  so  $\xi_{[0,t]}^\alpha(u) = \xi_t^\alpha$  for  $u \geq t$ . Then, our times of interest are:

**Occupation:**  $Y_t^N(1_{\mathcal{O}}) = \frac{1}{N} \int_0^t \sum_{\alpha \sim s} 1_{\mathcal{O}}(\xi_s^\alpha) ds$  so  $Y_t^N(f) = \int_0^t X_s^N(f) ds$ .

**Inhabitation:**  $Z_t^N(1_{\mathcal{O}}) = \frac{1}{N} \int_0^t \sum_{\alpha \sim t} 1_{\mathcal{O}}(\xi_s^\alpha) ds$  so  $Z_t^N(f) = \sum_{\alpha \sim t} \int_0^t f(\xi_{[0,t]}^\alpha(s)) ds$ .

for  $\mathcal{O} \subset \mathbb{R}^d$  and  $f \in B(\mathbb{R}^d)$ . Theorem 1.4 in the introduction states that these two times (after high density limits) only differ by a martingale defined in terms of this function  $\ell_f$  i.e. that the multiple counting of common ancestors is similar to the counting of time spent by dead lineages. Whereas  $Y_t^N(f)$  was immediately expressed in terms of the empirical process  $X^N$ , one can only easily express the inhabitation time in terms of the

**Historical Process:**  $\mathbb{X}_t^N = \frac{1}{N} \sum_{\alpha \sim t} \delta_{(t, \xi_{[0,t]}^\alpha)}$  in  $\mathcal{P}(\mathbb{E})$  supported on  $\mathbb{E}^t$ .

In particular,  $Z_t^N(f) = \mathbb{X}_t^N(\ell_f)$ , where  $\ell_f(t, y^t) = \int_0^t f(y_s^t) ds$ . (Here,  $\mathbb{E}$  and  $\mathbb{E}^t$  are defined around (1.17) and since  $\mathbb{X}_t^N$  is supported on  $\mathbb{E}^t$  we also have  $Z_t^N(f) = \mathbb{X}_t^N(\ell_f^t)$ , where  $\ell_f^t(r, y^r) = \int_0^{t \wedge r} f(y_s^r) ds$ .) To relate occupation and inhabitation times, we express  $Y_t^N$  in terms of the historical process as well. For  $f \in B(\mathbb{R}^d)$ , we let  $j^* f(r, y^r) \doteq f(y_r^r) \in B(\mathbb{E})$  and find  $X_t^N(f) = \mathbb{X}_t^N(j^* f)$  so  $Y_t^N(f) = \int_0^t X_s^N(f) ds = \int_0^t \mathbb{X}_s^N(j^* f) ds$ . Notice,  $t$  is included with  $\xi_{[0,t]}^\alpha$  in the definition of the historical process. This is to allow time-inhomogeneous generator and to make support properties obvious as will be seen below. The developments of this motivating subsection survive the process of taking high density limits while martingale problem formulation actually gets easier. We will use the historical martingale problem below to relate the occupation and inhabitation times now that we have expressed them both in terms of the historical process. The first step is to define the historical process when there are infinitely many particles.

### 2.4. Fleming–Viot historical processes

Historical superprocesses were first introduced by Dawson and Perkins [12]. To make our presentation manifest, we assume that  $(\xi_t, P_x)$  is an  $\mathbb{R}^d$ -valued Borel strong Markov process with sample path in  $D := D(\mathbb{R}^d)$ , the Skorohod space defined at the beginning of Section 2. The weak generator of  $\xi$  is still denoted by  $(A, \mathcal{D}(A))$ ,  $\mu \in M_1(E)$  and  $\phi \in C_+$  with  $t_\phi = \infty$ .

For each  $(r, y) \in \mathbb{E}$ , we consider the process  $(\Xi_t)_{t \geq 0}$  in  $\mathbb{E}$  defined by

$$\Xi_t = (r + t, (y \times_r \xi)^{r+t}),$$

where for every  $w, w' \in D(\mathbb{R}^d)$ ,  $w \times_r w'$  is the concatenation path

$$w \times_r w'(s) = \begin{cases} w(s) & \text{for } s \in [0, r) \\ w(r) + w'(s - r) & \text{for } s \in [r, \infty). \end{cases}$$

The law of  $\Xi$  is denoted by  $P_{r,y}$ , namely

$$P_{r,y}(\mathcal{O}) = P_0(\Xi \in \mathcal{O}) \quad \forall \mathcal{O} \in \mathcal{E}(D(\mathbb{E})).$$

$((\Xi_t)_{t \geq 0}, P_{r,y})$  is called the historical process, associated to  $\xi$ , with initial position  $\Xi_0 = (r, y)$ .  $((\Xi_t)_{t \geq 0}, (P_{r,y})_{(r,y) \in \mathbb{E}})$  is a time-homogeneous Borel strong Markov process in  $\mathbb{E}$  with semigroup

$$\begin{aligned} \mathbb{T}_t &: C_b(\mathbb{E}) \rightarrow C_b(\mathbb{E}) \\ \mathbb{T}_t f(r, y) &= P_{r,y} f(\Xi_t). \end{aligned} \tag{2.30}$$

(See [27, Proposition II.2.5] for a more general result.) It is more convenient to express  $\mathbb{T}$  directly through  $\xi$  by

$$\mathbb{T}_t f(r, y) = P_{y,r} f(r + t, (y \times_r \xi)^{r+t}). \tag{2.31}$$

We denote by  $\mathbb{A}$  the (weak) generator of  $\mathbb{T}$ . A function  $f \in b\mathcal{E}(\mathbb{E})$  belongs to the domain of  $\mathbb{A}$ ,  $\mathcal{D}(\mathbb{A})$ , iff the limit

$$\text{bp-}\lim_{h \downarrow 0} \frac{1}{h} (\mathbb{T}_h f(r, y) - f(r, y))$$

exists. In such case, we denote the limit as  $\mathbb{A}f(r, y)$ .

Let  $\tau \geq 0$  and  $\chi$  be a measure in  $M_1(D)$  such that  $\chi(\{y \in D(\mathbb{R}^d) : y^\tau = y\}) = 1$ . Then,  $\delta_\tau \times \chi$  is a probability measure on  $\mathbb{E}$ . By Theorem 2.3 there is a unique solution  $(\mathbb{X}, \mathbb{P}_{\tau,\chi}^\phi (\equiv \mathbb{P}_{\delta_\tau \times \chi}^\phi))$  on  $(\Omega, \mathcal{G})$  (with  $E = \mathbb{E}$ ) to the  $\mathbb{A}$ -martingale problem, meaning

$$\mathbb{M}_t(f) = \mathbb{X}_t(f) - \delta_\tau \times \chi(f) - \int_0^t \mathbb{X}_s(\mathbb{A}f) ds \tag{2.32}$$

is a continuous  $(\mathcal{G}_t)$ -martingale starting at 0 such that

$$\langle \mathbb{M}(f) \rangle_t = \int_0^t (\mathbb{X}_s(f^2) - \mathbb{X}_s(f)^2) \frac{ds}{\phi(s)} \tag{2.33}$$

for all  $f \in \mathcal{D}(\mathbb{A})$ . The process  $(\mathbb{X}_t, \mathbb{P}_{\tau,\chi}^\phi)$  is called the (time-homogeneous) historical Fleming–Viot process. The relations (2.13) and (2.14) in the current context become respectively

$$\mathbb{X}_t(f) = \delta_\tau \times \chi(\mathbb{T}_t f) + \int_0^t \int_{\mathbb{E}} \mathbb{T}_{t-s} f(r, y) d\mathbb{M}(s, (r, y)), \tag{2.34}$$

$$\mathbb{X}_u(f) - \mathbb{X}_t(\mathbb{T}_{u-t} f) = \int_t^u \int_{\mathbb{E}} \mathbb{T}_{u-s} f(r, y) d\mathbb{M}(s, (r, y)), \tag{2.35}$$

which hold for every  $0 \leq t \leq u$  and  $f \in b\mathcal{E}(\mathbb{E})$ . In particular,

$$\mathbb{P}_{\tau,\chi}^\phi \mathbb{X}_t(f) = \delta_\tau \times \chi(\mathbb{T}_t f) \quad \forall t \geq 0, \forall f \in b\mathcal{E}(\mathbb{E}). \tag{2.36}$$

It is possible to recover the Fleming–Viot process  $X$  from  $\mathbb{X}$ . We just define the projection

$$\begin{aligned} J &: \mathbb{E} \rightarrow \mathbb{R}^d \\ J(r, y) &= y_r, \end{aligned}$$

and put  $X_t = \mathbb{X}_t \circ J^{-1}$ ,  $M_t^X = \mathbb{M}_t \circ J^{-1}$ , respectively the pushforward measures of  $\mathbb{X}_t, \mathbb{M}_t$  via  $J$ . Each function  $f$  in  $b\mathcal{E}(\mathbb{R}^d)$  induces the function  $J^* f$  in  $C_b(\mathbb{E})$  by  $J^* f(r, y) = f(y_r)$ . In addition, for each  $f \in b\mathcal{E}(\mathbb{R}^d)$  we have

$$X_t(f) = \mathbb{X}_t(J^* f) \quad \text{and} \quad M_t^X(f) = \mathbb{M}_t(J^* f) \quad \forall t \geq \tau. \tag{2.37}$$

If  $f$  belongs to the domain of  $A$ , then  $J^* f$  belongs to the domain of  $\mathbb{A}$  and  $\mathbb{A}J^* f = Af$ . It follows from (2.32) and (2.33) that  $(X, M^X)$  is a Fleming–Viot process with law  $\mathbb{P}_\mu^\phi$ , where  $\mu = (\delta_\tau \times \chi) \circ J^{-1}$ .

We give a brief investigation on the support of  $\mathbb{X}_t$ . Let  $\Pi : \mathbb{E} \rightarrow D(\mathbb{R}^d)$  be the projection  $\Pi(r, y) = y$  and define an  $M_1(D)$ -valued process  $(H_t, t \geq \tau)$  by

$$H_{\tau+t} = \mathbb{X}_t \circ \Pi^{-1} \quad \forall t \geq 0.$$

Define  $\mathbb{D}^t = \Pi \mathbb{E}^t = \{y \in D : y^t = y\}$  for each  $t \geq 0$  and note  $\mathbb{E} = \bigcup_{t \geq 0} \mathbb{E}^t$ . The following result is an analog of [27, Lemma II.8.1].

**Proposition 2.10.**  $\mathbb{X}_t = \delta_{\tau+t} \times H_{\tau+t}$  and  $\text{supp } H_{\tau+t} \subset \mathbb{D}^{\tau+t}$  for all  $t \geq 0$   $\mathbb{P}_{\tau, \chi}^\phi$ -a.s.

**Proof.** We define

$$\Lambda(t) = \{(r, y) \in \mathbb{E} : r \neq \tau + t\}.$$

Then, by (2.34) and (2.31),

$$\begin{aligned} \mathbb{P}_{\tau, \chi}^\phi \mathbb{X}_t(\mathbf{1}_{\Lambda(t)}) &= \int_D \mathbb{T}_t \mathbf{1}_{\Lambda(t)}(\tau, y) d\chi(y) \\ &= \int_D E_0 \mathbf{1}_{\Lambda(t)}(\tau + t, (y \times_{\tau} \xi)^{\tau+t}) d\chi(y) = 0. \end{aligned}$$

This shows  $\mathbb{X}_t = \delta_{\tau+t} \times H_{\tau+t}$   $\mathbb{P}_{\tau, \chi}^\phi$ -a.s. for each  $t \geq 0$  and hence for all  $t \geq 0$  by the right-continuity of both sides. The later assertion in the proposition statement follows from the former. Indeed, for every  $\mathcal{O} \in \mathcal{E}(D)$ ,

$$H_{\tau+t}(\mathcal{O}) = \mathbb{X}_t(\Pi^{-1}\mathcal{O}) = \delta_{\tau+t} \times H_{\tau+t}(\{(r, y) \in \mathbb{E} : y^r \in \mathcal{O}\}) = H_{\tau+t}(\{y \in \mathcal{O} : y^{\tau+t} = y\}),$$

which implies  $\text{supp } H_{\tau+t} \subset \mathbb{D}^{\tau+t}$ . □

**Remark 2.11.** The process  $(H_t)_{t \geq \tau}$  is time inhomogeneous and is called historical superprocess in literature ([12, 15]). In the current article, we use its time-homogeneous counter part  $(\mathbb{X}_t)_{t \geq 0}$ . It is evident from the previous result that under  $\mathbb{P}_{\tau, \chi}^\phi$ ,  $\text{supp } \mathbb{X}_t \subset \mathbb{E}^{\tau+t}$ . Consequently, for every bounded measurable function  $f$  on  $\mathbb{E}^{\tau+t}$

$$\mathbb{X}_t(f) = \int_{\mathbb{E}} f(r, y) d\mathbb{X}_t(r, y) = \int_{\mathbb{E}} f(r, y) \mathbf{1}_{(r=\tau+t)} d\mathbb{X}_t(r, y) \quad (2.38)$$

and

$$|\mathbb{P}_{\tau, \chi}^\phi \mathbb{X}_t(f)| \leq \|f\|_{L^\infty(\mathbb{E}^{\tau+t})}. \quad (2.39)$$

In addition, it is seen from (2.33) that  $\text{supp } \mathbb{M}_t \subset \text{supp } \mathbb{X}_t \subset \mathbb{E}^{\tau+t}$ .

Our interest is the superprocess  $(X_t)_{t \geq 0}$  starting from a specified initial measure  $X_0 = \mu$ . Hence, it is natural to simply take  $\tau = 0$  for the historical process  $(\mathbb{X}_t)_{t \geq 0}$ . In such case, the measure  $\chi$  can also be constructed (uniquely) from  $\mu$  by

$$\chi(\mathcal{O}) = \mu^*(\mathcal{O}) = \mu(\{y(0) : y \in \mathcal{O}\}) \quad \forall \mathcal{O} \in \mathcal{E}(\mathbb{E}).$$

### 2.5. Occupation times and inhabitation times

The *occupation time* process  $(Y_t)_{t \geq 0}$  associated with  $(X_t)_{t \geq 0}$  is the measure-valued process defined by

$$Y_t(\mathcal{O}) = \int_0^t X_s(\mathcal{O}) ds \quad \forall \mathcal{O} \in \mathcal{E}(\mathbb{R}^d). \quad (2.40)$$

In the context of critical Dawson–Watanabe processes, the occupation time process was introduced and studied by [20] by means of Laplace functionals. Our other time of interest *inhabitation time* is defined through the historical process and the counting function  $\ell_f$ . It is natural to ask whether  $\ell_f$ , defined in (1.12), is measurable when restricted to  $\mathbb{E}$ .

**Proposition 2.12.** For every  $f \in b\mathcal{E}(\mathbb{R}^d)$ ,  $\ell_f : (\mathbb{E}, \mathcal{E}(\mathbb{E})) \rightarrow (\mathbb{R}, \mathcal{E}(\mathbb{R}))$  is measurable.

**Proof.** First, suppose  $f$  is continuous. Then, it follows by Ethier and Kurtz [19, Problems 3.11.13 and 3.11.26] that  $D(\mathbb{E}) \ni y \rightarrow \int_0^r f(y_s) ds \in D(\mathbb{R})$  is continuous and so  $(r, y) \rightarrow \int_0^r f(y_s) ds$  is also continuous. Now, let  $\mathcal{O}$  be a closed set in  $\mathbb{R}^d$ . Then, there exist continuous  $f^n$  such that  $f^n \rightarrow \mathbf{1}_{\mathcal{O}}$  pointwise by Billingsley [4, Theorem 1.2] so for every  $(r, y) \in \mathbb{E}$ ,

$$\lim_{n \rightarrow \infty} \ell_{f^n}(r, y) = \int_0^r \lim_{n \rightarrow \infty} f^n(y_s) ds = \int_0^r \mathbf{1}_{\mathcal{O}}(y_s) ds = \ell_{\mathbf{1}_{\mathcal{O}}}(r, y)$$

by dominated convergence and  $\ell_{1_{\mathcal{O}}}$  is measurable. Finally, the family  $\mathcal{H} = \{f \in b\mathcal{E}(\mathbb{R}^d) : \ell_f \text{ is measurable}\}$  contains  $\mathbf{1}_{\mathcal{O}} \in \mathcal{H}$  for every closed set  $\mathcal{O} \subset \mathbb{R}^d$  and is closed under additions, scalar multiplications and pointwise limits. Hence,  $\mathcal{H} = b\mathcal{E}(\mathbb{R}^d)$  by the monotone class theorem.  $\square$

Let  $(\mathbb{X}_t)_{t \geq 0}$  be the historical Fleming–Viot process constructed in Section 2.4. The inhabitation time process  $(Z_t)_{t \geq 0}$  associated with  $X$  is the measure-valued process defined by

$$Z_t(\mathcal{O}) = \mathbb{X}_t(\ell_{\mathbf{1}_{\mathcal{O}}}) \quad \forall t \geq 0, \mathcal{O} \in \mathcal{E}(\mathbb{R}^d).$$

$\mathbb{X}_t(\ell_f)$  makes sense at least for non-negative  $f$  since  $\ell_f$  is measurable. As mentioned in the introduction  $\mathbb{X}_t(\ell_f)$  satisfies martingale problem (1.13), (1.14) once we know that each  $\ell_f^t \in \mathcal{D}(\mathbb{A})$ .

**Lemma 2.13.** *Let  $f$  be a function in  $b\mathcal{E}(\mathbb{R}^d)$ . Then, for every  $t, h > 0$  and every  $(r, y)$  in  $\mathbb{E}$ ,*

$$\mathbb{T}_h \ell_f^t(r, y) = \ell_f^t(r, y) + \mathbf{1}_{(r < t)} \int_0^{(r+h) \wedge t - r} T_s f(y_r) \, ds. \tag{2.41}$$

In addition,  $\ell_f^t$  belongs to the domain of  $\mathbb{A}$  and

$$\mathbb{A} \ell_f^t(r, y) = f(y_r) \mathbf{1}_{(r < t)} \quad \text{and} \quad \mathbb{A} \ell_f(r, y) = \text{bp-}\lim_{t \rightarrow \infty} \mathbb{A} \ell_f^t(r, y) = f(y_r). \tag{2.42}$$

**Proof.** We observe that for every path  $\omega \in D(\mathbb{R}^d)$

$$\begin{aligned} \ell_f^t(r+h, \omega^{r+h}) &= \int_0^{r \wedge t} f(\omega_s) \, ds + \mathbf{1}_{(r < t)} \int_r^{(r+h) \wedge t} f(\omega_s) \, ds \\ &= \ell_f^t(r, \omega^r) + \mathbf{1}_{(r < t)} \int_0^{(r+h) \wedge t - r} f(\omega_{r+s}) \, ds. \end{aligned}$$

This implies that

$$\begin{aligned} \mathbb{T}_h \ell_f^t(r, y) &= P_{r,y} \ell_f^t(r+h, (y \times_r \xi)^{r+h}) \\ &= \ell_f^t(r, y) + \mathbf{1}_{(r < t)} P_{y_r} \int_0^{(r+h) \wedge t - r} f(\xi_s) \, ds, \end{aligned}$$

which yields (2.41). Equation (2.42) is obtained by differentiating (2.41) at  $h = 0$  and then letting  $t \rightarrow \infty$ .  $\square$

We observe that  $Z_0 \equiv 0$ . In comparison with the occupation time process  $Y$  defined in (2.40), it is easy to derive from (2.36) that for every  $f \in b\mathcal{E}(E)$  and  $t \geq 0$ ,  $Y_t(f)$  and  $Z_t(f)$  have the same mean, that is

$$\mathbb{P}_\mu^\phi Y_t(f) = \mathbb{P}_\mu^\phi Z_t(f) = \mu \left( \int_0^t T_s f \, ds \right).$$

In fact, a deeper relation between  $Z$  and  $Y$  holds.

**Proposition 2.14.** *For every  $f \in b\mathcal{E}(\mathbb{R}^d)$  and  $t \geq 0$*

$$Z_t(f) = \mathbb{M}_t(\ell_f) + Y_t(f), \tag{2.43}$$

where the process  $(\mathbb{M}_t(\ell_f))_{t \geq 0}$  is the continuous  $(\mathcal{G}_t)$ -martingale defined in (1.18), with quadratic variation

$$\langle \mathbb{M}(\ell_f) \rangle_t = \int_0^t (\mathbb{X}_s((\ell_f)^2) - \mathbb{X}_s(\ell_f)^2) \frac{ds}{\phi(s)} \quad \forall t \geq 0. \tag{2.44}$$

**Proof.** Fix  $u > 0$  and recall the bounded variant  $\ell_f^u$  of  $\ell_f$  is in  $\mathcal{D}(\mathbb{A})$  so  $\mathbb{X}_t(\ell_f) = \mathbb{X}_t(\ell_f^u)$ ,  $\mathbb{X}_t(\mathbb{A} \ell_f) = \mathbb{X}_t(\mathbb{A} \ell_f^u)$  and  $\langle \mathbb{M}(\ell_f) \rangle_t = \langle \mathbb{M}(\ell_f^u) \rangle_t$  for  $t \leq u$  by (1.16) and (1.19). In addition, by definition  $Z_t(f) = \mathbb{X}_t(\ell_f)$ , hence we derive from (1.18) that

$$\mathbb{M}_t(\ell_f) = Z_t(f) - \mathbb{X}_0(\ell_f) - \int_0^t \mathbb{X}_s(\mathbb{A} \ell_f) \, ds.$$

We know  $\mathbb{X}_0(\ell_f) = 0$  and have from (2.42), (2.37) that

$$\mathbb{X}_s(\mathbb{A}\ell_f) = \int_E f(y_r) d\mathbb{X}_s(r, y) = \mathbb{X}_s(J^* f) = X_s(f)$$

for every  $s$ . This yields (2.43) and  $(\mathbb{M}_t(\ell_f))_{t \geq 0}$  is a  $(\mathcal{G}_t)$ -martingale with the required quadratic variation by (2.33) by the arbitrariness of  $u$ . □

In relation (2.43), if the long term asymptotics of any two among the three quantities are known, then, this implies the long term asymptotic of the other term. Since  $\mathbb{M}_t(\ell_f)$  is a martingale, its analysis is subjected to martingale limit theorems. Depending on the situation at hand, the asymptotic of one of  $Y_t$  and  $Z_t$  is easier than the other. This is the case for  $\alpha$ -stable Fleming–Viot process considered in Section 4 below.

### 3. Stable Fleming–Viot processes

Hereafter, we consider the specific case when  $A = -(-\Delta)^{\frac{\alpha}{2}}$  on  $\mathbb{R}^d$  for some  $\alpha \in (0, 2]$ . The historical  $\alpha$ -stable generator is still denoted by  $\mathbb{A}$ . The motion of each particle has the law of the  $\alpha$  stable process in  $\mathbb{R}^d$ . The associated superprocess  $(X_t)_{t \geq 0}$  constructed in Theorem 2.3 is called  $\alpha$ -stable Fleming–Viot process. The associated historical superprocess  $(\mathbb{X}_t)_{t \geq 0}$  with law  $\mathbb{P}_{0, \mu^*}^\phi$  constructed in Section 2.4 is called historical  $\alpha$ -stable Fleming–Viot process. The relation (2.37) describes the connection between  $X$  and  $\mathbb{X}$ . In this section, we develop several intermediate results following the guideline described in Remark 2.9. These considerations eventually lead to the proofs of Theorems 1.1 and 1.2 stated in the Introduction.

#### 3.1. The stable semigroup

Let  $T_t$  be the semigroup corresponding to a symmetric  $\alpha$ -stable process. In particular, for each test function  $f$ ,

$$T_t f(x) = \int_{\mathbb{R}^d} p_t(x - y) f(y) dy, \tag{3.1}$$

where

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \theta} e^{-t|\theta|^\alpha} d\theta. \tag{3.2}$$

Let  $\hat{f}$  be the Fourier transform of  $f$ ,  $\hat{f}(\theta) = \int_{\mathbb{R}^d} e^{-i\theta \cdot x} f(x) dx$ . Using Fourier transform,  $T_t f$  takes an alternative form

$$T_t f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \theta - t|\theta|^\alpha} \hat{f}(\theta) d\theta. \tag{3.3}$$

We have seen in Section 2.2 that the long term asymptotic of  $X_t(f)$  depends upon that of  $T_t f$ . It is therefore natural to study  $T_t f$  as  $t \rightarrow \infty$  for a given test function  $f$ . If  $k = (k_1, \dots, k_d) \in \mathbb{N}^d$  is a multi-index, we define  $\partial^k f = \partial_1^{k_1} \partial_2^{k_2} \dots \partial_d^{k_d} f$ .

**Proposition 3.1 (Semigroup expansion).** *Let  $f$  be a bounded measurable function on  $\mathbb{R}^d$  and  $N$  be a non-negative integer such that (1.8) holds. Then, we have*

$$\lim_{t \rightarrow \infty} t^{\frac{N+d}{\alpha}} \sup_{x \in \mathbb{R}^d} \left| T_t f(x) - \sum_{k \in \mathbb{N}^d: |k| \leq N} \frac{(-1)^{|k|}}{k!} \int_{\mathbb{R}^d} f(y) y^k dy \partial^k p_t(x) \right| = 0. \tag{3.4}$$

**Proof.** We begin with a rescaled version of (3.3)

$$t^{d/\alpha} T_t f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it^{-1/\alpha} x \cdot \theta - |\theta|^\alpha} \hat{f}(t^{-1/\alpha} \theta) d\theta. \tag{3.5}$$

The condition (1.8) ensures that the derivative  $\partial^k \hat{f}$  exists and is continuous and bounded for every multi-index  $k$  such that  $|k| \leq N$ . Hence, we have the following Taylor’s expansion for  $\hat{f}(u)$  around  $u = 0$ ,

$$\hat{f}(u) = \sum_{|k| \leq N} \frac{\partial^k \hat{f}(0)}{k!} u^k + R_N(u). \tag{3.6}$$

The remainder term satisfies

$$\lim_{u \rightarrow 0} |u|^{-N} |R_N(u)| = 0 \quad \text{and} \quad \sup_{u \in \mathbb{R}^d \setminus \{0\}} \frac{|R_N(u)|}{|u|^N} = O(1). \tag{3.7}$$

The second estimate in (3.7) comes from the first estimate, (3.6) and the fact that  $\hat{f}$  is bounded. Hence, we can rewrite the right-hand side of (3.5) as follows:

$$\sum_{|k| \leq N} \frac{\partial^k \hat{f}(0)}{k!} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it^{-1/\alpha} x \cdot \theta - |\theta|^\alpha} (t^{-1/\alpha} \theta)^k \, d\theta + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it^{-1/\alpha} x \cdot \theta - |\theta|^\alpha} R_N(t^{-1/\alpha} \theta) \, d\theta.$$

Taking into account the facts that

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it^{-1/\alpha} x \cdot \theta - |\theta|^\alpha} (t^{-1/\alpha} \theta)^k \, d\theta = i^{-|k|} t^{d/\alpha} \partial^k p_t(x)$$

and

$$\partial^k \hat{f}(0) = (-i)^{|k|} \int_{\mathbb{R}^d} f(y) y^k \, dy, \tag{3.8}$$

we obtain

$$\begin{aligned} t^{d/\alpha} T_t f(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it^{-1/\alpha} x \cdot \theta - |\theta|^\alpha} \hat{f}(t^{-1/\alpha} \theta) \, d\theta \\ &= t^{d/\alpha} \sum_{|k| \leq N} \frac{(-1)^{|k|}}{k!} \int_{\mathbb{R}^d} f(y) y^k \, dy \partial^k p_t(x) + \tilde{R}_N(x), \end{aligned}$$

where

$$\tilde{R}_N(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it^{-1/\alpha} x \cdot \theta - |\theta|^\alpha} R_N(t^{-1/\alpha} \theta) \, d\theta.$$

Hence, it remains to show  $\lim_{t \rightarrow \infty} t^{\frac{N}{\alpha}} \|\tilde{R}_N\|_\infty = 0$ . In fact, we have

$$t^{\frac{N}{\alpha}} \sup_{x \in \mathbb{R}^d} |\tilde{R}_N(x)| \lesssim \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} t^{\frac{N}{\alpha}} |R_N(t^{-\frac{1}{\alpha}} \theta)| \, d\theta,$$

which converges to 0 as  $t \rightarrow \infty$  by dominated convergence theorem and (3.7). (Here and below, we use  $\lesssim$  in the standard way:  $A \lesssim B$  means there exists a constant  $C > 0$  such that  $A \leq CB$ .)  $\square$

As an immediate consequence, the stable semigroup  $T_t$  satisfies (2.7) with  $c(t) = t^{(N+d)/\alpha}$  and

$$L_t f = \sum_{|k| \leq N} \frac{(-1)^{|k|}}{k!} \int_{\mathbb{R}^d} f(y) y^k \, dy \partial^k p_t. \tag{3.9}$$

In view of Proposition 2.8 and (3.9), the long term asymptotic of  $X_t(f)$  is reduced to the long term asymptotic along a sequence of

$$X_{\rho(t)}(\partial^k p_{t-\rho(t)}), \quad k \in \mathbb{N}^d, |k| \leq N,$$

which we will describe in Section 3.2.

3.2. Limit theorems for super stable processes

For each  $\theta \in \mathbb{R}^d$ , we denote  $e_\theta(x) = e^{i\theta \cdot x}$ ,  $\cos_\theta(x) = \cos(\theta \cdot x)$  and  $\sin_\theta(x) = \sin(\theta \cdot x)$  and recall the definition of  $\vartheta_{d,\alpha}^k$  in (1.6).

**Proposition 3.2.** *Let  $\rho$  be a sublinear function such that  $\lim_{t \rightarrow \infty} \frac{\rho(t)}{t^{1-\varepsilon_0}} = 0$  for some  $\varepsilon_0 > 0$ . Suppose that  $\phi$  satisfies*

$$\int_0^\infty \frac{ds}{\phi(s)} < \infty \tag{3.10}$$

and  $\mu$  satisfies (1.4). With  $\mathbb{P}_\mu^\phi$ -probability one, we have for every  $k \in \mathbb{N}^d$  that

$$\lim_{t \rightarrow \infty} t^{\frac{d+|k|}{\alpha}} X_{\rho(t)}(\partial^k p_{t-\rho(t)}) = \begin{cases} 0 & \text{if } |k| \text{ is odd,} \\ (-1)^{\frac{|k|}{2}} \vartheta_{d,\alpha}^k & \text{if } |k| \text{ is even.} \end{cases} \tag{3.11}$$

**Proof.** We note that for every function  $f \in L^1(\mathbb{R}^d)$ , by Fubini’s theorem,

$$X_t(f) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} X_t(e_\theta) \hat{f}(\theta) d\theta. \tag{3.12}$$

Hence,

$$X_{\rho(t)}(\partial^k p_{t-\rho(t)}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(t-\rho(t))|\theta|^\alpha} X_{\rho(t)}(e_\theta)(i\theta)^k d\theta.$$

In addition, from (2.5), we obtain

$$X_{\rho(t)}(e_\theta) = \mu(e_\theta) - |\theta|^\alpha \int_0^{\rho(t)} X_s(e_\theta) ds + M_{\rho(t)}^X(e_\theta). \tag{3.13}$$

It follows that

$$X_{\rho(t)}(\partial^k p_{t-\rho(t)}) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(t-\rho(t))|\theta|^\alpha} \mu(e_\theta)(i\theta)^k d\theta, \\ I_2 &= -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(t-\rho(t))|\theta|^\alpha} \int_0^{\rho(t)} X_s(e_\theta) ds |\theta|^\alpha (i\theta)^k d\theta, \\ I_3 &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(t-\rho(t))|\theta|^\alpha} M_{\rho(t)}^X(e_\theta)(i\theta)^k d\theta. \end{aligned}$$

We will show that

$$\lim_{t \rightarrow \infty} t^{\frac{d+|k|}{\alpha}} I_1 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} (i\theta)^k d\theta \quad \text{a.s.}, \tag{3.14}$$

$$\lim_{t \rightarrow \infty} t^{\frac{d+|k|}{\alpha}} I_2 = 0 \quad \text{a.s.} \quad \text{and} \quad \lim_{t \rightarrow \infty} t^{\frac{d+|k|}{\alpha}} I_3 = 0 \quad \text{a.s.} \tag{3.15}$$

By a change of variable, we see that

$$I_1 = t^{-\frac{d+|k|}{\alpha}} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(1-\frac{\rho(t)}{t})|\theta|^\alpha} \mu(e_{t^{-1/\alpha}\theta})(i\theta)^k d\theta.$$

This, together with dominated convergence theorem yields (3.14). For  $I_2$ , we observe that

$$\begin{aligned} |I_2| &\lesssim \rho(t) \int_{\mathbb{R}^d} e^{-(t-\rho(t))|\theta|^\alpha} |\theta|^{|k|+\alpha} d\theta \\ &\lesssim \frac{\rho(t)}{t} t^{-\frac{d+|k|}{\alpha}} \int_{\mathbb{R}^d} e^{-(1-\frac{\rho(t)}{t})|\theta|^\alpha} |\theta|^{|k|+\alpha} d\theta, \end{aligned}$$

which due to sublinearity of  $\rho$  immediately implies the first assertion in (3.15). For  $I_3$ , putting  $a_n = e^n$  and utilizing the Borel-Cantelli lemma, we merely need to show

$$\sum_{n \geq 1} \mathbb{P}_\mu^\phi \left( \sup_{a_n \leq t \leq a_{n+1}} t^{\frac{d+|k|}{\alpha}} |I_3| \right)^2 < \infty. \tag{3.16}$$

Set  $\rho_n = \rho(a_n)$  and note by a change of variables that

$$\int_{\mathbb{R}^d} e^{-(a_n - \rho_{n+1})|\theta|^\alpha} |\theta|^{|k|} d\theta \lesssim a_n^{-\frac{d+|k|}{\alpha}}.$$

By Jensen’s inequality, we have

$$\mathbb{P}_\mu^\phi \left( \sup_{a_n \leq t \leq a_{n+1}} t^{\frac{d+|k|}{\alpha}} |I_3| \right)^2 \lesssim \left( \frac{a_{n+1}^2}{a_n} \right)^{\frac{d+|k|}{\alpha}} \int_{\mathbb{R}^d} e^{-(a_n - \rho_{n+1})|\theta|^\alpha} \mathbb{P}_\mu^\phi \sup_{a_n \leq t \leq a_{n+1}} |M_{\rho(t)}^X(e_\theta)|^2 |\theta|^{|k|} d\theta. \tag{3.17}$$

For each  $\theta \in \mathbb{R}^d$ ,  $(M_t(e_\theta))_{t \geq 0}$  is a complex valued martingale with quadratic variations satisfying

$$\begin{aligned} \langle \Re M(e_\theta) \rangle_t &= \int_0^t [X_s(\cos_\theta^2) - X_s^2(\cos_\theta)] \frac{ds}{\phi(s)} \leq \int_0^t X_s((1 - \cos_\theta)^2) \frac{ds}{\phi(s)}, \\ \langle \Im M(e_\theta) \rangle_t &= \int_0^t [X_s(\sin_\theta^2) - X_s^2(\sin_\theta)] \frac{ds}{\phi(s)} \leq \int_0^t X_s(\sin_\theta^2) \frac{ds}{\phi(s)}. \end{aligned}$$

Hence, using the elementary identity  $1 - \cos_\theta = 2 \sin_{\theta/2}^2$ , we obtain

$$\begin{aligned} \mathbb{P}_\mu^\phi |M_t^X(e_\theta)|^2 &\lesssim \int_0^t \mathbb{P}_\mu^\phi X_s(\sin_{\theta/2}^4 + \sin_\theta^2) \frac{ds}{\phi(s)} \\ &\lesssim \int_0^t \langle T_s(\sin_{\theta/2}^4 + \sin_\theta^2), \mu \rangle \frac{ds}{\phi(s)}. \end{aligned}$$

Note that for every  $x \in \mathbb{R}^d$

$$\begin{aligned} 2T_s \sin_\theta^2(x) &= 1 - \cos_{2\theta}(x) e^{-s|2\theta|^\alpha} = (1 - \cos_{2\theta}(x)) e^{-s|2\theta|^\alpha} + 1 - e^{-s|2\theta|^\alpha} \\ &\lesssim (1 \wedge |\theta||x|)^2 + s|\theta|^\alpha. \end{aligned}$$

Similarly,  $4T_s \sin_{\frac{\theta}{2}}^4(x) = T_s(1 - 2\cos_\theta + \cos_\theta^2) \leq 2T_s(1 - \cos_\theta) \lesssim (1 \wedge |\theta||x|)^2 + s|\theta|^\alpha$ . Using (3.10) and (1.4), it follows that

$$\mathbb{P}_\mu^\phi |M_t^X(e_\theta)|^2 \lesssim |\theta|^{2\wedge a} + t|\theta|^\alpha. \tag{3.18}$$

By martingale maximal inequality

$$\mathbb{P}_\mu^\phi \sup_{a_n \leq t \leq a_{n+1}} |M_{\rho(t)}^X(e_\theta)|^2 \lesssim \mathbb{P}_\mu^\phi |M_{\rho_{n+1}}^X(e_\theta)|^2 \lesssim |\theta|^{2\wedge a} + \rho_{n+1}|\theta|^\alpha. \tag{3.19}$$

Applying the above estimate in (3.17) and a change of variables yields

$$\begin{aligned} \mathbb{P}_\mu^\phi \left( \sup_{a_n \leq t \leq a_{n+1}} t^{\frac{d+|k|}{\alpha}} |I_3| \right)^2 &\lesssim \left( \frac{a_{n+1}^2}{a_n} \right)^{\frac{d+|k|}{\alpha}} \int_{\mathbb{R}^d} e^{-(a_n - \rho_{n+1})|\theta|^\alpha} (|\theta|^{2\wedge a} + \rho_{n+1}|\theta|^\alpha) |\theta|^{|k|} d\theta \\ &\lesssim e^{\frac{d+|k|}{\alpha}} \int_{\mathbb{R}^d} e^{-(e^{-1} - \frac{\rho_{n+1}}{a_{n+1}})|\theta|^\alpha} \left( a_{n+1}^{-(2\wedge a)/\alpha} |\theta|^{2\wedge a} + \frac{\rho_{n+1}}{a_{n+1}} |\theta|^\alpha \right) |\theta|^{|k|} d\theta. \end{aligned}$$

Observing that  $\frac{\rho_n}{a_n} \lesssim a_n^{-\varepsilon_0}$  and  $\sum_n a_n^{-\varepsilon} < \infty$  for any  $\varepsilon > 0$ , the above estimate implies (3.16).

Finally, combining (3.14) and (3.15) yields

$$\lim_{t \rightarrow \infty} t^{\frac{d+|k|}{\alpha}} X_{\rho(t)}(\partial^k p_{t-\rho(t)}) = \frac{i^{|k|}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} \theta^k \, d\theta.$$

The equality (3.11) follows from here, after observing that  $X_{\rho(t)}(\partial^k p_{t-\rho(t)})$  is a real number. □

**Proof of Theorem 1.2.** We are going to verify the hypotheses in Proposition 2.8. As we have seen previously, the identity (3.4) verifies condition (2.7) with  $c(t) = t^{(N+d)/\alpha}$  and  $L_t$  defined in (3.9). We choose  $\rho(t) = t^\kappa$  and  $t_n = n^\delta$  with  $\kappa, \delta \in (0, 1)$  such that

$$\frac{N+d}{\alpha} + 1 + \varepsilon_0 > \frac{1}{\delta} > \frac{N+d}{\alpha} + 1 \quad \text{and} \quad \left( \frac{2N+d}{\alpha} + 1 + \varepsilon_0 \right) \kappa > \frac{N}{\alpha} + \frac{1}{\delta}. \tag{3.20}$$

It is easy to verify conditions (2.8), (2.19) and (2.22). To check the condition (2.18), we note that  $\|T_t f^2\|_\infty \lesssim t^{-d/\alpha} \|f\|_{L^2}^2$ . So we need to verify that

$$\sum_{n=1}^\infty n^{\delta \frac{N}{\alpha}} \int_{n^{\kappa\delta}}^{n^\delta} \frac{ds}{\phi(s)} < \infty.$$

By Tonelli’s theorem, the left-hand side above is at most a constant multiple of

$$\int_1^\infty s^{\frac{1}{\kappa} \frac{N}{\alpha} + \frac{1}{\kappa\delta}} \frac{ds}{\phi(s)}.$$

The ranges of  $\kappa, \delta$  chosen in (3.20) ensures that  $\frac{2N+d}{\alpha} + 1 + \varepsilon_0 > \frac{1}{\kappa} \frac{N}{\alpha} + \frac{1}{\kappa\delta}$ . Hence, the above integral is finite due to (1.7) and we have verified condition (2.18). The condition (2.23) is verified analogously. Finally, we verify (2.21). The assumption (1.9) ensures that  $f \in \mathcal{D}(A)$  and

$$|Af(x)| = \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix \cdot \xi} |\xi|^\alpha \, d\xi \right| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{f}(\xi)| |\xi|^\alpha \, d\xi.$$

It follows that

$$c_n \sup_{t \in [t_n, t_{n+1}]} \|T_{t_{n+1}-t} f - f\|_\infty \lesssim c_n (t_{n+1} - t_n) \lesssim n^{\delta \frac{N+d}{\alpha} + \delta - 1}$$

and, hence, (2.21) is satisfied because of our assumption on the range of  $\delta$  in (3.20). Therefore, applying Proposition 2.8, we find that (2.24) is valid with  $c(t) = t^{\frac{N+d}{\alpha}}$  and  $L_t$  defined by (3.9). In particular, we have

$$\lim_n \sup_{t \in [t_n, t_{n+1}]} t^{\frac{N+d}{\alpha}} \left| X_t(f) - \sum_{|k| \leq N} \frac{(-1)^{|k|}}{k!} \int_{\mathbb{R}^d} f(y) y^k \, dy X_{\rho(t_n)}(\partial^k p_{t_n - \rho(t_n)}) \right| = 0.$$

The long-time limit of  $X_{\rho(t_n)}(\partial^k p_{t_n - \rho(t_n)})$  is given by Proposition 3.2. This implies (1.10). □

**Proof of Theorem 1.1.** The class of functions  $C_c^2(\mathbb{R}^d)$  strongly separates points in the sense of Ethier and Kurtz [19]. From [5, Lemma 2], there exists a countable subset  $\mathcal{M}$  of  $C_c^2(\mathbb{R}^d)$  which strongly separates points and is closed under multiplication. Set  $\tilde{\mathcal{M}} = \{e^{-\varepsilon|\cdot|^2} f : f \in \mathcal{M}, \varepsilon > 0\}$ . By Theorem 1.2, we see that with  $\mathbb{P}_\mu^\phi$ -probability one, (1.10) with  $N = 0$  holds for every  $f \in \tilde{\mathcal{M}}$ . An application of [22, Lemma 7] implies that with  $\mathbb{P}_\mu^\phi$ -probability one, (1.10) with  $N = 0$  holds for every continuous functions  $g$  such that  $e^{\varepsilon|\cdot|^2} g$  is bounded for some  $\varepsilon > 0$ . This yields almost-sure shallow convergence of  $t^{\frac{d}{\alpha}} X_t$  to  $\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} \, d\theta \lambda_d$  as  $t \rightarrow \infty$ . □

#### 4. Occupation times of stable Fleming–Viot processes

Let  $(X_t)_{t \geq 0}$  be the  $(\alpha, \phi)$  Fleming–Viot superprocess and  $(\mathbb{X}_t)_{t \geq 0}$  be the corresponding  $(\alpha, \phi)$  Fleming–Viot historical process with martingale measure  $\mathbb{M}$ . Then, we established the occupation time process  $Y$  and the inhabitation time

process  $Z$  for  $X$  are connected through  $Z_t(f) - Y_t(f) = \mathbb{M}(\ell_f)$ , where  $\ell_f$  is defined in (1.12). (See Theorem 1.4a and Proposition 2.14.) Using this  $Z - Y$  relation and the method described in Section 2.2, we are able to obtain long term asymptotics of both time processes. As we saw earlier at the beginning of Section 3, the  $(\alpha, \phi)$  Fleming–Viot superprocess can be recovered from the  $(\alpha, \phi)$  Fleming–Viot historical process so we need only consider one probability measure,  $\mathbb{P}_{0, \mu^*}$ , which we relabel  $\mathbb{P}_\mu$  to ease notation. Recall that  $\mathcal{N}_d$  is defined in (1.20) and  $\mu$  is a probability measure on  $\mathbb{R}^d$ . The following result, whose proof is presented in Section 4.2, is the key step in showing Theorem 1.5.

**Proposition 4.1.** *Assume that  $\phi$  satisfies (1.22). Let  $f$  be a function in  $b\mathcal{E}(\mathbb{R}^d)$  such that  $\mathcal{N}_d(f) < \infty$ . Then, the following assertions hold  $\mathbb{P}_\mu^\phi$ -a.s.*

(i) *(Low and critical dimensions,  $d \leq \alpha$ )*

$$\lim_{t \rightarrow \infty} \frac{Y_t(f)}{\gamma_d(t)} = \lim_{t \rightarrow \infty} \frac{Z_t(f)}{\gamma_d(t)} = \varkappa_d(\alpha) \int_{\mathbb{R}^d} f(x) \, dx. \tag{4.1}$$

(ii) *(High dimension,  $d > \alpha$ ) The limits  $Y_\infty(f) := \lim_{t \rightarrow \infty} Y_t(f)$ ,  $Z_\infty(f) := \lim_{t \rightarrow \infty} Z_t(f)$  and  $\mathbb{M}_\infty(\ell_f) := \lim_{t \rightarrow \infty} \mathbb{M}_t(\ell_f)$  exist and are finite random variables. In addition, we have the following relation*

$$Z_\infty(f) = \mathbb{M}_\infty(\ell_f) + Y_\infty(f).$$

**Remark 4.2.** The condition  $\mathcal{N}_d(f) < \infty$  ensures that  $\int_0^t T_s f(x) \, ds$  is finite for every  $t > 0$  and  $x \in \mathbb{R}^d$ . This can be seen from the following identity, which is a consequence of (3.3),

$$\int_0^t T_s f(x) \, ds = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\theta \cdot x} \frac{1 - e^{-t|\theta|^\alpha}}{|\theta|^\alpha} \hat{f}(\theta) \, d\theta. \tag{4.2}$$

Indeed, when  $d < \alpha$ ,  $\frac{1 - e^{-t|\theta|^\alpha}}{|\theta|^\alpha}$  is integrable over  $\mathbb{R}^d$ , then the right-hand side above is bounded above by a multiple constant of  $\|f\|_{L^1(\mathbb{R}^d)}$ . When  $d \geq \alpha$ ,  $\frac{1 - e^{-t|\theta|^\alpha}}{|\theta|^\alpha}$  is not integrable as  $|\theta| \rightarrow \infty$ . However, the right-hand side of (4.2) is finite if  $\int_{\mathbb{R}^d} |\hat{f}(\theta)| |\theta|^{-\alpha} \, d\theta$  is finite. The finiteness of  $\int_{\mathbb{R}^d} |\hat{f}(\theta)| |\theta|^{-\alpha} \, d\theta$  is also necessary to control  $\int_0^1 T_s f(x) \, ds$  when  $d = \alpha$ .

The following lemma will be useful later.

**Lemma 4.3.** *Let  $f$  be a function in  $b\mathcal{E}(\mathbb{R}^d)$  with  $\mathcal{N}_d(f) < \infty$ .*

(i) *If  $d \leq \alpha$ , then for every  $x \in \mathbb{R}^d$ ,*

$$\lim_{t \rightarrow \infty} \frac{1}{\gamma_d(t)} \int_0^t T_s f(x) \, ds = \varkappa_d(\alpha) \lambda_d(f), \tag{4.3}$$

where we recall that  $\varkappa_d$  is defined in (1.21) and  $\lambda_d$  is the Lebesgue measure on  $\mathbb{R}^d$ .

(ii) *If  $d > \alpha$ , then*

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \left| \int_0^t T_s f(x) \, ds - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \theta} \hat{f}(\theta) |\theta|^{-\alpha} \, d\theta \right| = 0. \tag{4.4}$$

**Proof.** Consider first the case  $d < \alpha$ . From (4.2), we have

$$\int_0^t T_s f(x) \, ds = t^{1 - \frac{d}{\alpha}} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it^{-1/\alpha} \theta \cdot x} \frac{1 - e^{-|\theta|^\alpha}}{|\theta|^\alpha} \hat{f}(t^{-1/\alpha} \theta) \, d\theta. \tag{4.5}$$

Using the facts that  $\int_{\mathbb{R}^d} \frac{1 - e^{-|\theta|^\alpha}}{|\theta|^\alpha} \, d\theta$  is integrable and  $\lim_{t \rightarrow \infty} \hat{f}(t^{-1/\alpha} \theta) = \hat{f}(0) = \lambda_d(f)$ , we can derive (4.3) from the dominated convergence theorem.

The case  $d = \alpha$  is a bit more subtle. From (3.3), we have

$$\int_1^t T_s f(x) \, ds = \frac{1}{(2\pi)^d} \int_1^t \int_{\mathbb{R}^d} e^{ix \cdot \theta - s|\theta|^d} \hat{f}(\theta) \, d\theta \, ds = \frac{1}{(2\pi)^d} \int_1^t \int_{\mathbb{R}^d} e^{is^{-1/d} x \cdot \theta - |\theta|^d} \hat{f}(s^{-1/d} \theta) \, d\theta \frac{ds}{s},$$

which implies

$$\left\| \int_1^u T_s f \, ds \right\|_\infty \lesssim \ln(u) |\hat{f}(0)| \quad \forall u \geq 1. \tag{4.6}$$

Now, let  $\varepsilon$  be a positive number and choose  $K > 0$  such that

$$\int_{|\theta|>K} e^{-|\theta|^d} \, d\theta \leq \varepsilon$$

and then  $u > 1$  such that

$$\sup_{s \geq u} \sup_{|\theta| \leq K} |e^{is^{-1/d}x \cdot \theta} \hat{f}(s^{-\frac{1}{d}}\theta) - \hat{f}(0)| \leq \varepsilon.$$

Such a choice is always possible because of the continuity of  $\hat{f}$  at 0. It follows that

$$\begin{aligned} & \left\| \int_u^t T_s f(x) \, ds - \hat{f}(0) \int_u^t \int_{\mathbb{R}^d} e^{-|\theta|^d} \, d\theta \frac{ds}{s} \right\|_\infty \\ & \leq \frac{1}{(2\pi)^d} \left( \int_u^t \int_{|\theta| \leq K} + \int_u^t \int_{|\theta| > K} \right) e^{-|\theta|^d} |e^{is^{-1/d}x \cdot \theta} \hat{f}(s^{-\frac{1}{d}}\theta) - \hat{f}(0)| \, d\theta \frac{ds}{s} \\ & \lesssim \varepsilon \int_{\mathbb{R}^d} e^{-|\theta|^d} \, d\theta \ln\left(\frac{t}{u}\right) + \varepsilon |\hat{f}(0)| \ln\left(\frac{t}{u}\right). \end{aligned}$$

Combining with (4.6), one has that

$$\limsup_{t \rightarrow \infty} \frac{1}{\ln t} \left\| \int_1^t T_s f(x) \, ds - \hat{f}(0) \int_1^t \int_{\mathbb{R}^d} e^{-|\theta|^d} \, d\theta \frac{ds}{s} \right\|_\infty \lesssim \varepsilon.$$

Sending  $\varepsilon \rightarrow 0$ , we obtain

$$\lim_{t \rightarrow \infty} \frac{1}{\ln t} \int_1^t T_s f(x) \, ds = \varkappa_d(\alpha) \lambda_d(f).$$

Finally, since  $|\int_0^1 T_s f(x) \, ds| \lesssim \int_{\mathbb{R}^d} |\hat{f}(\theta)| |\theta|^{-\alpha} \, d\theta$ , which is finite, the above implies (4.3).

In case  $d > \alpha$ , from (4.2), we have

$$\int_0^t T_s f(x) \, ds - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \theta} \hat{f}(\theta) |\theta|^{-\alpha} \, d\theta = \frac{-1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \theta - t|\theta|^\alpha} \hat{f}(\theta) |\theta|^{-\alpha} \, d\theta.$$

Hence,

$$\sup_{x \in \mathbb{R}^d} \left| \int_0^t T_s f(x) \, ds - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \theta} \hat{f}(\theta) |\theta|^{-\alpha} \, d\theta \right| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|\theta|^\alpha} |\hat{f}(\theta)| |\theta|^{-\alpha} \, d\theta,$$

which together with the dominated convergence theorem implies (4.4). □

From now on, we assume that  $f$  is a bounded measurable function on  $\mathbb{R}^d$  such that  $\mathcal{N}_d(f)$  is finite. From the proof of Lemma 4.3, it follows that in every dimension,

$$\left\| \int_0^t T_s f \, ds \right\|_\infty \lesssim \mathcal{N}_d(f) (\gamma_d(t) \vee 1) \quad \forall t \geq 0. \tag{4.7}$$

By the homogeneous Markov property of  $\xi$ , we also have

$$\sup_{x \in \mathbb{R}^d} \left| P_x \left( \int_0^t f(\xi_u) \, du \right)^2 \right| = 2 \sup_{x \in \mathbb{R}^d} \left| \int_0^t \int_0^{t-u} T_u [f T_s f](x) \, ds \, du \right| \lesssim \mathcal{N}_d^2(f) (\gamma_d(t) \vee 1)^2 \tag{4.8}$$

for every  $t \geq 0$ .

4.1. Martingale corrector

We investigate the long time limit of the martingale difference  $\mathbb{M}_t(\ell_f)$ . For each  $q > 1$  and  $n \in \mathbb{N}_0$ , define

$$t_n = t_n(q) = \begin{cases} q^{\frac{\alpha}{\alpha-d}n} & \text{if } d < \alpha, \\ e^{qn} & \text{if } d = \alpha \end{cases} \quad \text{so that } \gamma_d(t_n) = q^n. \tag{4.9}$$

**Proposition 4.4.** *Let  $f$  be a bounded measurable function on  $\mathbb{R}^d$  such that  $\mathcal{N}_d(f) < \infty$ . Then, (i)  $\mathbb{M}_t(\ell_f)$  converges  $\mathbb{P}_\mu^\phi$ -a.s. and in  $L^2(\Omega)$  as  $t \rightarrow \infty$  if  $\int_0^\infty \frac{\gamma_d^2(s)}{\phi(s)} ds < \infty$ .*

(ii)  $\lim_{t \rightarrow \infty} \frac{\mathbb{M}_t(\ell_f)}{\gamma_d(t)} = 0$   $\mathbb{P}_\mu^\phi$ -a.s. if (1.22) holds.

**Proof.** (i) By martingale convergence theorem, it suffices to show

$$\sup_{t \geq 0} \mathbb{P}_\mu^\phi[\mathbb{M}_t(\ell_f)^2] < \infty. \tag{4.10}$$

Indeed, from (2.44) and (2.36) we have that

$$\mathbb{P}_\mu^\phi[\mathbb{M}_t(\ell_f)^2] \leq \mathbb{P}_\mu^\phi \int_0^t \mathbb{X}_s((\ell_f^s)^2) \frac{ds}{\phi(s)} = \int_0^t \langle \mathbb{T}_s((\ell_f^s)^2), \delta_0 \times m \rangle \frac{ds}{\phi(s)}.$$

We observe that for every path  $\omega \in D(\mathbb{R}^d)$

$$\ell_f^s(r + s, \omega^{r+s}) = \ell_f^s(r, \omega^r) + \mathbf{1}_{(r < s)} \int_0^{s-r} f(\omega_{r+u}) du.$$

Thus,

$$(\ell_f^s(r + s, \omega^{r+s}))^2 \leq 2(\ell_f^s(r, \omega^r))^2 + \mathbf{1}_{(r < s)} 2 \left( \int_0^{s-r} f(\omega_{r+u}) du \right)^2.$$

Together with (4.8), this implies that

$$\begin{aligned} \mathbb{T}_s(\ell_f^s)^2(r, y) &= P_{y_r} [(\ell_f^s(r + s, (y \times_r \xi)^{r+s}))^2] \\ &\leq 2(\ell_f^s(r, y))^2 + \mathbf{1}_{(r < s)} 2 P_{y_r} \left( \int_0^{s-r} f(\xi_u) du \right)^2 \\ &\lesssim r^2 \|f\|_\infty^2 + \mathbf{1}_{(r < s)} \mathcal{N}_d^2(f) (\gamma_d(s-r) \vee 1)^2. \end{aligned} \tag{4.11}$$

Therefore, we have

$$\int_0^t \langle \mathbb{T}_s(\ell_f^s)^2, \delta_0 \times m \rangle \frac{ds}{\phi(s)} \lesssim \mathcal{N}_d^2(f) \int_0^t (\gamma_d(s) \vee 1)^2 \frac{ds}{\phi(s)},$$

which is uniformly bounded in  $t$  by our assumptions on  $f$  and  $\phi$ . The estimate (4.10) and the convergence of  $\mathbb{M}_t(\ell_f)$  follow.

(ii) Let  $\{t_n\}$  be the sequence defined in (4.9). It suffices to show that

$$\sum_n \frac{1}{\gamma_d^2(t_n)} \mathbb{P}_\mu^\phi \left[ \left( \sup_{t \in [t_{n-1}, t_n]} \mathbb{M}_t(\ell_f) \right)^2 \right] < \infty.$$

By martingale maximal inequality and the computations in the previous case, we see that

$$\mathbb{P}_\mu^\phi \left[ \left( \sup_{t \in [t_{n-1}, t_n]} \mathbb{M}_t(\ell_f) \right)^2 \right] \lesssim \mathbb{P}_\mu^\phi [(\mathbb{M}_{t_n}(\ell_f^{t_n}))^2] \lesssim \int_0^{t_n} (\gamma_d(s) \vee 1)^2 \frac{ds}{\phi(s)}.$$

It remains to show that

$$\sum_n \frac{1}{\gamma_d^2(t_n)} \int_0^{t_n} (\gamma_d(s) \vee 1)^2 \frac{ds}{\phi(s)} < \infty. \tag{4.12}$$

Since  $\gamma_d(t_n) = q^n$ ,  $\sum_n q^{-2n} < \infty$  and  $\int_0^1 (\gamma_d(s) \vee 1)^2 \frac{ds}{\phi(s)} < \infty$ , we can replace 0 in the lower limit of each integral above by 1. Consider the case  $d < \alpha$ . Interchanging the order of summation and integration, we see that

$$\sum_n \frac{1}{\gamma_d^2(t_n)} \int_1^{t_n} (\gamma_d(s) \vee 1)^2 \frac{ds}{\phi(s)} \lesssim \int_1^\infty \sum_{n: q^n > s^{1-\frac{d}{\alpha}}} \frac{1}{q^{2n}} (\gamma_d(s) \vee 1)^2 \frac{ds}{\phi(s)} \lesssim \int_1^\infty \frac{ds}{\phi(s)}.$$

In the second estimate above, we use  $\sum_{n: q^n > s^{1-\frac{d}{\alpha}}} \frac{1}{q^{2n}} \lesssim \frac{1}{\gamma^2(s)}$ . It is straightforward to verify that in the case  $d = \alpha$ , we have the same estimate. That is

$$\sum_n \frac{1}{\gamma_d^2(t_n)} \int_1^{t_n} (\gamma_d(s) \vee 1)^2 \frac{ds}{\phi(s)} \lesssim \int_1^\infty \frac{ds}{\phi(s)}.$$

The integral on the right-hand side above is finite by our assumption. Hence, (4.12) follows and so does the result.  $\square$

### 4.2. Limit theorems for occupation times

We present the proofs of Proposition 4.1 and Theorem 1.5.

**Proof of Proposition 4.1(ii).** Without loss of generality, we assume that  $f$  is non-negative. The process  $Y_t(f)$  is non-negative and increasing. Hence, the limit  $\lim_{t \rightarrow \infty} Y_t(f)$  exists. In addition, using Tonelli’s theorem, (2.15) and (4.4), we have

$$\lim_{t \rightarrow \infty} \mathbb{P}_\mu^\phi Y_t(f) = \lim_{t \rightarrow \infty} \int_0^t \mu(T_s f) ds = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mu(e_\theta) \hat{f}(\theta) |\theta|^{-\alpha} d\theta.$$

Hence, by Fatou’s lemma and the fact that  $\mathcal{N}_d(f) < \infty$ ,

$$\mathbb{P}_\mu^\phi \lim_{t \rightarrow \infty} Y_t(f) \leq \lim_{t \rightarrow \infty} \mathbb{P}_\mu^\phi Y_t(f) < \infty.$$

It follows that  $\lim_{t \rightarrow \infty} Y_t(f)$  is a finite random variable. From Proposition 4.4, the limit  $\lim_{t \rightarrow \infty} \mathbb{M}_t(\ell_f^t)$  exists and is a finite random variable. Together with the relation (2.43), these observations imply Proposition 4.1(ii).  $\square$

**Proof of Proposition 4.1(i).** Without loss of generality, we can assume  $f \geq 0$ . Let  $q$  be at least 1 and  $\{t_n\} = \{t_n(q)\}$  be the sequence defined in (4.9).

*Step 1. Reduce to subsequence convergence:* Suppose that

$$\lim_n \frac{Y_{t_n(q)}(f)}{\gamma_d(t_n(q))} = \varkappa_d(\alpha) \int_{\mathbb{R}^d} f(x) dx \quad \text{a.s.} \tag{4.13}$$

for all  $q > 1$ . For every  $t \in [t_n, t_{n+1})$ , by monotonicity of  $Y_t(f)$ , we see that

$$\frac{1}{q} \lim_n \frac{Y_{t_n}(f)}{\gamma_d(t_n)} \leq \liminf_t \frac{Y_t(f)}{\gamma_d(t)} \leq \limsup_t \frac{Y_t(f)}{\gamma_d(t)} \leq q \lim_n \frac{Y_{t_{n+1}}(f)}{\gamma_d(t_{n+1})}.$$

By sending  $q \downarrow 1$ , one has  $\lim_{t \rightarrow \infty} \frac{Y_t(f)}{\gamma_d(t)} = \varkappa_d(\alpha) \int_{\mathbb{R}^d} f(x) dx$ . Now, Proposition 4.4 (ii) implies (4.1).

*Step 2. Reduce to  $\mu(\int_0^{t_n} T_s f ds)$ :* From Lemma 2.6 and (4.7), we have

$$\mathbb{P}_\mu^\phi \left| \int_0^{t_n} X_s(f) ds - \mu \left( \int_0^{t_n} T_s f ds \right) \right|^2 \lesssim \mathcal{N}_d^2(f) \int_0^{t_n} \gamma_d^2(s) \frac{ds}{\phi(s)}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{\gamma_d^2(t_n)} \mathbb{P}_\mu^\phi \left| \int_0^{t_n} X_s(f) ds - \mu \left( \int_0^{t_n} T_s f ds \right) \right|^2 \lesssim \sum_{n=1}^{\infty} \frac{1}{q^{2n}} \int_0^{t_n} \gamma_d^2(s) \frac{ds}{\phi(s)}.$$

The series on the right-hand side above appeared earlier in (4.12). The same reasoning as in the proof of Proposition 4.4 shows that the above series is finite under condition (1.22). Hence, the Borel-Cantelli lemma implies

$$\lim_n \frac{1}{\gamma_d(t_n)} \left| \int_0^{t_n} X_s(f) ds - \mu \left( \int_0^{t_n} T_s f ds \right) \right| = 0.$$

*Step 3.* From Lemma 4.3, (4.6) and dominated convergence theorem, we deduce that

$$\lim_n \frac{1}{\gamma_d(t_n)} \mu \left( \int_0^{t_n} T_s f ds \right) = \varkappa_d(\alpha) \lambda_d(f).$$

Combining previous steps, one has the result.  $\square$

**Proof of Theorem 1.5.** We note that each function in  $C_c^2(\mathbb{R}^d)$  satisfies the hypotheses of Proposition 4.1. Therefore, by an analogous argument as in the proof of Theorem 1.1 on page 2614, we can easily deduce Theorem 1.5 from Proposition 4.1. We omit the details.  $\square$

## Acknowledgements

Good refereeing is gratefully acknowledged by the authors. KL gratefully acknowledges Pacific Institute for the Mathematical Sciences (PIMS) for its support through the Postdoctoral Training Centre in Stochastics during the preparation of this work.

## References

- [1] S. R. Asmussen and H. Hering. Strong limit theorems for general supercritical branching processes with applications to branching diffusions. *Z. Wahrsch. Verw. Gebiete* **36** (3) (1976) 195–212. MR0420889 <https://doi.org/10.1007/BF00532545>
- [2] K. B. Athreya and P. E. Ney. *Branching Processes*. Dover Publications, Inc., Mineola, NY, 2004. MR2047480
- [3] R. F. Bass and E. A. Perkins. On the martingale problem for super-Brownian motion. In *Séminaire de Probabilités, XXXV* 195–201. *Lecture Notes in Math.* **1755**. Springer, Berlin, 2001. [https://doi.org/10.1007/978-3-540-44671-2\\_14](https://doi.org/10.1007/978-3-540-44671-2_14)
- [4] P. Billingsley. *Convergence of Probability Measures*. Wiley, New York, 1968. MR0233396
- [5] D. Blount and M. A. Kouritzin. On convergence determining and separating classes of functions. *Stochastic Process. Appl.* **120** (10) (2010) 1898–1907. MR2673979 <https://doi.org/10.1016/j.spa.2010.05.018>
- [6] Z.-Q. Chen and Y. Shiozawa. Limit theorems for branching Markov processes. *J. Funct. Anal.* **250** (2) (2007) 374–399. MR2352485 <https://doi.org/10.1016/j.jfa.2007.05.011>
- [7] K. L. Chung and R. J. Williams. *Introduction to Stochastic Integration*, 2nd edition. *Probability and Its Applications*. Birkhäuser Boston, Boston, MA, 1990. MR1102676 <https://doi.org/10.1007/978-1-4612-4480-6>
- [8] D. A. Dawson. Measure-valued Markov processes. In *École d'Été de Probabilités de Saint-Flour XXI—1991* 1–260. *Lecture Notes in Math.* **1541**. Springer, Berlin, 1993. MR1242575 <https://doi.org/10.1007/BFb0084190>
- [9] D. A. Dawson and A. Greven. Hierarchically interacting Fleming–Viot processes with selection and mutation: Multiple space time scale analysis and quasi-equilibria. *Electron. J. Probab.* **4** (1999) 4. MR1670873 <https://doi.org/10.1214/EJP.v4-41>
- [10] D. A. Dawson, A. Greven and J. Vaillancourt. Equilibria and quasiequilibria for infinite collections of interacting Fleming–Viot processes. *Trans. Amer. Math. Soc.* **347** (7) (1995) 2277–2360. MR1297523 <https://doi.org/10.2307/2154827>
- [11] D. A. Dawson and K. J. Hochberg. Wandering random measures in the Fleming–Viot model. *Ann. Probab.* **10** (3) (1982) 554–580. MR0659528
- [12] D. A. Dawson and E. A. Perkins. Historical processes. *Mem. Amer. Math. Soc.* **93** (454) (1991) iv + 179. MR1079034 <https://doi.org/10.1090/memo/0454>
- [13] P. Donnelly and T. G. Kurtz. A countable representation of the Fleming–Viot measure-valued diffusion. *Ann. Probab.* **24** (2) (1996) 698–742. MR1404525 <https://doi.org/10.1214/aop/1039639359>
- [14] P. Donnelly and T. G. Kurtz. Genealogical processes for Fleming–Viot models with selection and recombination. *Ann. Appl. Probab.* **9** (4) (1999) 1091–1148. MR1728556 <https://doi.org/10.1214/aop/1029962866>
- [15] E. B. Dynkin. Path processes and historical superprocesses. *Probab. Theory Related Fields* **90** (1) (1991) 1–36. MR1124827 <https://doi.org/10.1007/BF01321132>
- [16] M. Eckhoff, A. E. Kyprianou and M. Winkel. Spines, skeletons and the strong law of large numbers for superdiffusions. *Ann. Probab.* **43** (5) (2015) 2545–2610. MR3395469 <https://doi.org/10.1214/14-AOP944>
- [17] J. Engländer. *Spatial Branching in Random Environments and with Interaction*. *Advanced Series on Statistical Science & Applied Probability* **20**. World Scientific, Hackensack, NJ, 2015. MR3362353 <https://doi.org/10.1142/8991>

- [18] A. Etheridge and P. March. A note on superprocesses. *Probab. Theory Related Fields* **89** (2) (1991) 141–147. <https://doi.org/10.1007/BF01366902>
- [19] S. N. Ethier and T. G. Kurtz. *Markov Processes: Characterization and Convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. Wiley, New York, 1986. MR0838085 <https://doi.org/10.1002/9780470316658>
- [20] I. Iscoe. A weighted occupation time for a class of measure-valued branching processes. *Probab. Theory Related Fields* **71** (1) (1986) 85–116. MR0814663 <https://doi.org/10.1007/BF00366274>
- [21] M. A. Kouritzin, K. Lê and D. Sezer. Laws of large numbers for supercritical branching Gaussian processes. *Stochastic Process. Appl.* **129** (9) 2018 3463–3498. MR3985570 <https://doi.org/10.1016/j.spa.2018.09.011>
- [22] M. A. Kouritzin and Y.-X. Ren. A strong law of large numbers for super-stable processes. *Stochastic Process. Appl.* **124** (1) (2014) 505–521. MR3131303 <https://doi.org/10.1016/j.spa.2013.08.009>
- [23] A. E. Kyprianou. *Fluctuations of Lévy Processes with Applications: Introductory Lectures*, 2nd edition. Universitext. Springer, Heidelberg, 2014. MR3155252 <https://doi.org/10.1007/978-3-642-37632-0>
- [24] K. Lê. Long-Time Asymptotic of Stable Dawson–Watanabe Processes in Supercritical Regimes. *Acta Math. Sci. Ser. B Engl. Ed.* **39** (1) (2019) 37–45. MR4064232 <https://doi.org/10.1007/s10473-019-0104-y>
- [25] R.-L. Liu, Y.-X. Ren and R. Song. Strong law of large numbers for a class of superdiffusions. *Acta Appl. Math.* **123** (2013) 73–97. MR3010225 <https://doi.org/10.1007/s10440-012-9715-1>
- [26] E. Perkins. On the martingale problem for interactive measure-valued branching diffusions. *Mem. Amer. Math. Soc.* **115** (549) (1995) vi + 89. MR1249422 <https://doi.org/10.1090/memo/0549>
- [27] E. Perkins. Dawson–Watanabe superprocesses and measure-valued diffusions. In *Lectures on Probability Theory and Statistics (Saint-Flour, 1999)* 125–324. *Lecture Notes in Math.* **1781**. Springer, Berlin, 2002. MR1915445
- [28] E. A. Perkins. Conditional Dawson–Watanabe processes and Fleming–Viot processes. In *Seminar on Stochastic Processes, 1991 (Los Angeles, CA, 1991)* 143–156. *Progr. Probab.* **29**. Birkhäuser Boston, Boston, MA, 1992. MR1172149
- [29] T. Shiga. An interacting system in population genetics. *J. Math. Kyoto Univ.* **20** (2) (1980) 213–242. MR0582165 <https://doi.org/10.1215/kjm/1250522276>
- [30] T. Shiga. An interacting system in population genetics. II. *J. Math. Kyoto Univ.* **20** (4) (1980) 723–733. MR0592356 <https://doi.org/10.1215/kjm/1250522168>
- [31] T. Shiga and K. Uchiyama. Stationary states and their stability of the stepping stone model involving mutation and selection. *Probab. Theory Related Fields* **73** (1) (1986) 87–117. MR0849066 <https://doi.org/10.1007/BF01845994>
- [32] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d’été de Probabilités de Saint-Flour XIV—1984* 265–439. *Lecture Notes in Math.* **1180**. Springer, Berlin, 1986. MR0876085 <https://doi.org/10.1007/BFb0074920>
- [33] L. Wang. An almost sure limit theorem for super-Brownian motion. *J. Theoret. Probab.* **23** (2) (2010) 401–416. MR2644866 <https://doi.org/10.1007/s10959-008-0200-8>
- [34] S. Watanabe. Limit theorem for a class of branching processes. In *Markov Processes and Potential Theory (Proc. Sympos. Math. Res. Center, Madison, Wis., 1967)* 205–232. Wiley, New York, 1967. MR0237007