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Hausdorff dimension of the uniform measure of Galton–Watson trees without the XlogX condition

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Abstract. We consider a Galton–Watson tree with offspring distribution ν of finite mean. The uniform measure on the boundary of the tree is obtained by putting mass 1 on each vertex of the n-th generation and taking the limit $n \to \infty$. In the case $E[\nu \log(\nu)] < \infty$, this measure has been well studied, and it is known that the Hausdorff dimension of the measure is equal to $\log(m)$ (*J. Lond. Math. Soc. (2)* **24** (1981) 373–384; *Ergodic Theory Dynam. Systems* **15** (1995) 593–619). When $E[\nu \log(\nu)] = \infty$, we show that the dimension drops to 0. This answers a question of Lyons, Pemantle and Peres (In *Classical and Modern Branching Processes. Proceedings of the IMA Workshop* (1997) 223–237 Springer).

Résumé. Nous considérons un arbre de Galton–Watson dont le nombre d'enfants ν a une moyenne finie. La mesure uniforme sur la frontière de l'arbre s'obtient en chargeant chaque sommet de la n-ième génération avec une masse 1, puis en prenant la limite $n \to \infty$. Dans le cas $E[\nu \log(\nu)] < \infty$, cette mesure est bien étudiée, et l'on sait que la dimension de Hausdorff de la mesure est égale à $\log(m)$ (*J. Lond. Math. Soc. (2)* **24** (1981) 373–384; *Ergodic Theory Dynam. Systems* **15** (1995) 593–619). Lorsque $E[\nu \log(\nu)] = \infty$, nous montrons que la dimension est 0. Cela répond à une question posée par Lyons, Pemantle et Peres (In *Classical and Modern Branching Processes. Proceedings of the IMA Workshop* (1997) 223–237 Springer).

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1. Introduction

Let $\mathcal{T}$ be a Galton–Watson tree of root e, associated to the offspring distribution $q := (q_k, k \geq 0)$. We denote by GW the distribution of $\mathcal{T}$ on the space of rooted trees, and ν a generic random variable on $\mathbb{N}$ with distribution q. We suppose that $m := \sum_{k\geq 0} kq_k \in (1, \infty)$: the tree has a positive probability of survival, denoted by ρ. We let GW* be the Galton–Watson measure conditionally on $\mathcal{T}$ being infinite. For any vertex u, we write $|u|$ for the height of vertex u ($|e| = 0$), $\nu(u)$ for the number of children of u, and Z_n is the population at height n. We define $\partial\mathcal{T}$ as the set of all infinite self-avoiding paths of $\mathcal{T}$ starting from the root and we define a metric on $\partial\mathcal{T}$ by $d(r, r') := e^{-|r\wedge r'|}$ where $r \wedge r'$ is the highest vertex belonging to r and r'. The space $\partial\mathcal{T}$ is called boundary of the tree, and elements of $\partial\mathcal{T}$ are called rays. The metric space $(\partial\mathcal{T}, d)$ is a random compact ultra-metric space.

When $E[\nu \log(\nu)] < \infty$ (with $0\log(0) := 0$), it is well-known that the martingale $m^{-n} Z_n$ converges in L^1 and almost surely to a limit which is positive GW*-a.s. [6]. Seneta [17] and Heyde [4] proved that in the general case (i.e allowing $E[\nu \log(\nu)]$ to be infinite), there exist constants $(c_n)_{n\geq 0}$ such that

(a) $W_\infty := \lim_{n\to\infty} \frac{Z_n}{c_n}$ exists a.s.
(b) $W_\infty > 0$ GW*-a.s.
(c) $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = m$.

In particular, for each vertex $u \in \mathcal{T}$, if $Z_k(u)$ stands for the number of descendants v of u such that $|v| = |u| + k$, we can define

$$W_\infty(u) := \lim_{k\to\infty} \frac{Z_k(u)}{c_k}$$

and we notice that $m^{-n}\sum_{|u|=n} W_\infty(u) = W_\infty(e)$. Notice that the variables $W_\infty(u)$ depend on the choice of the constants $(c_n)_n$. We refer to Section 2 for our choice of the constants $(c_n)_n$.

Definition. On the event $\partial\mathcal{T} \neq \varnothing$, the uniform measure (also called branching measure) is the unique Borel measure on $\partial\mathcal{T}$ such that

$$\mathbf{UNIF}\big(\{r \in \partial\mathcal{T}, r_n = u\}\big) := \frac{m^{-n} W_\infty(u)}{W_\infty(e)}$$

for any integer n and any vertex u of height n.

We observe that, for any vertex u of height n,

$$\mathbf{UNIF}\big(\{r \in \partial\mathcal{T}, r_n = u\}\big) = \lim_{k\to\infty} \frac{Z_k(u)}{Z_{n+k}}.$$

In particular, it does not depend on the particular choice of the constants $(c_n)_n$. The uniform measure can be seen informally as the probability distribution of a ray taken uniformly in the boundary. This paper is interested in the Hausdorff dimension of **UNIF**, defined by

$$\dim(\mathbf{UNIF}) := \min\big\{\dim(E), \mathbf{UNIF}(E) = 1\big\},$$

where the minimum is taken over all subsets $E \subset \partial\mathcal{T}$ of **UNIF**-measure 1 and $\dim(E)$ is the Hausdorff dimension of set E. The case $E[\nu\log(\nu)] < \infty$ has been well studied. In the seminal paper [5], then in [3] and in [14] for a simple proof, it is shown that $\dim(\mathbf{UNIF}) = \log(m)$ GW*-almost surely. Exact Hausdorff and packing measures are given in [2,7,8,19,20]. A description of the multifractal spectrum is available in [9,16,18]. The case $E[\nu\log(\nu)] = \infty$ presented as Question 3.1 in [15] was left open. This case is proved to display an extreme behaviour.

Theorem 1.1. *If* $E[\nu\log(\nu)] = \infty$, *then* $\dim(\mathbf{UNIF}) = 0$ *for* GW*-*a.e. tree* $\mathcal{T}$.

It is known (see [11]) that the Hausdorff dimension of $\partial\mathcal{T}$ is $\log m$ as soon as $m \in (1,\infty)$. Therefore the theorem tells us that the uniform measure is concentrated on a thin part of the boundary. Indeed we will see that for **UNIF**-a.e. ray r, the number of children of r_n is greater than $(m - o(1))^n$ for infinitely many n, and it is easy to see that the Hausdorff dimension of the set of such rays is zero. The proof relies on the construction of a particular measure Q, under which the distribution of the number of children of a uniformly chosen ray is more tractable. Using a size-biased measure to gain information on the uniform ray already appears in the paper of Duquesne [2], see Lemma 3.1 there.

Section 2 contains the description of the new measure in terms of a spine decomposition. Then we prove Theorem 1.1 in Section 3. Section 4 contains some open questions.

2. A spine decomposition

For $k \geq 1$ and $s \in (0,1)$, we call $\phi_k(s)$ the probability generating function of Z_k

$$\phi_k(s) := E\big[s^{Z_k}\big].$$

We denote by $\phi_k^{-1}(s)$ the inverse map on $(q_0, 1)$. Recall that ρ is the survival probability hence the unique solution in $[0,1)$ of $\phi(s) = s$. Let $s \in (\rho, 1)$. Then $M_n := \phi_n^{-1}(s)^{Z_n}$ defines a martingale and converges in L^1 to some $M_\infty > 0$ a.s. [4]. Therefore we can take for the Seneta–Heyde norming in (a)

$$c_n := \frac{-1}{\log(\phi_n^{-1}(s))},$$

which we will do from now on. Hence we can rewrite equivalently $M_n = e^{-Z_n/c_n}$ and $M_\infty = e^{-W_\infty(e)}$. Notice that M_n depends on our choice of s. In [10], Lynch introduces the so-called derivative martingale (because it is the martingale obtained by differentiating with respect to s)

$$D_n := e^{1/c_n} \frac{Z_n}{\phi_n'(\phi_n^{-1}(s))} M_n. \tag{2.1}$$

The fact that it is a martingale is straightforward by differentiation. It is nonnegative hence converges almost surely to some D_∞. Let us follow the argument of Lynch (restricted to our case $E[\nu] < \infty$) to give some properties of this martingale. Since $(M_n)_n$ is a martingale, $(-M_n \log(M_n))_n$ defines a (nonnegative) supermartingale. Therefore its expectation is nonincreasing. Notice that

$$-M_n \log(M_n) = \phi_n'\big(\phi_n^{-1}(s)\big)\frac{1}{c_n}e^{-1/c_n} D_n. \tag{2.2}$$

Taking the expectation implies that $\phi_n'(\phi_n^{-1}(s))\frac{1}{c_n}e^{-1/c_n}$ is nonincreasing hence converges. We show now that the limit is nonzero. In view of (2.1), since D_n and M_n converge almost surely (and c_n goes to $+\infty$), $\frac{Z_n}{\phi_n'(\phi_n^{-1}(s))}$ converges almost surely. Since Z_n/c_n converges almost surely to a nondegenerate random variable, the limit of $\phi_n'(\phi_n^{-1}(s))/c_n$ is nonzero indeed. From (2.1), we see that D_∞ is positive GW*-almost surely and $(D_n)_n$ is bounded by some constant. In particular, it converges in L^1.

We proved that $\phi_n'(\phi_n^{-1}(s))/c_n$ converges to some positive constant. It follows from (c) that

$$\lim_{n\to\infty} \frac{\phi_{n+1}'(\phi_{n+1}^{-1}(s))}{\phi_n'(\phi_n^{-1}(s))} = m. \tag{2.3}$$

We are interested in the probability measure Q on the space of rooted trees defined by

$$\frac{dQ}{d\mathrm{GW}} := D_\infty.$$

Let us describe this change of measure. We call a marked tree a couple (T, r) where T is a rooted tree and r a ray of the tree T. Let $(\mathbb{T}, \xi)$ be a random variable in the space of all marked trees (equipped with some probability $\mathbb{P}(\cdot)$), whose distribution is given by the following rules. Conditionally on the tree up to level k and on the location of the ray at level k, (which we denote respectively by $\mathbb{T}_k$ and ξ_k),

- the number of children of the vertices at generation k are independent;
- the vertex ξ_k has a number $\nu(\xi_k)$ of children such that for any ℓ

$$\mathbb{P}\big(\nu(\xi_k) = \ell\big) = \tilde{q}_\ell^s := q_\ell \ell \exp\left(-\frac{\ell - 1}{c_{k+1}}\right)\frac{\phi_k'(\phi_k^{-1}(s))}{\phi_{k+1}'(\phi_{k+1}^{-1}(s))}; \tag{2.4}$$

- the number of children of a vertex $u \neq \xi_k$ at generation k verifies for any ℓ

$$\mathbb{P}\big(\nu(u) = \ell\big) = \tilde{q}_\ell := q_\ell e^{1/c_k} \exp\left(-\frac{\ell}{c_{k+1}}\right); \tag{2.5}$$

- the vertex ξ_{k+1} is chosen uniformly among the children of ξ_k.

We call the ray ξ the spine. We refer to [12,13] for motivation on spine decompositions. In our case, we can see $\mathbb{T}$ as a Galton–Watson tree in varying environment and with immigration. The fact that (2.4) and (2.5) define probabilities come from the equations (remember that by definition $e^{-1/c_k} = \phi_k^{-1}(s)$)

$$E\big[\big(\phi_{k+1}^{-1}(s)\big)^\nu\big] = \phi_k^{-1}(s),$$

$$E\big[\nu\big(\phi_{k+1}^{-1}(s)\big)^{\nu-1}\big] = \frac{\phi_{k+1}'(\phi_{k+1}^{-1}(s))}{\phi_k'(\phi_k^{-1}(s))}.$$

When $s = 1$, equations (2.4) and (2.5) become $\mathbb{P}(\nu(\xi_k) = \ell) = \ell q_\ell/m$ and $\mathbb{P}(\nu(u) = \ell) = q_\ell$, which is the size-biased Galton–Watson tree of [13].

Proposition 2.1. *Under Q, the tree $\mathcal{T}$ has the distribution of $\mathbb{T}$. Besides, for $\mathbb{P}$-almost every tree $\mathbb{T}$, the distribution of ξ conditionally on $\mathbb{T}$ is the uniform measure* **UNIF**. *In other words, for any nonnegative measurable function F,*

$$E\left[D_\infty \int_{\partial\mathcal{T}} \mathbf{UNIF}(dr) F(\mathcal{T}, r)\right] = E_{\mathbb{P}}\big[F(\mathbb{T}, \xi)\big].$$

Proof. For any tree T, we define T_n the tree T obtained by keeping only the n first generations. Let T be a tree. We will prove by induction that, for any integer n and any vertex u at generation n,

$$\mathbb{P}(\mathbb{T}_n = T_n, \xi_n = u) = \frac{D_n}{Z_n} \mathrm{GW}(\mathcal{T}_n = T_n). \tag{2.6}$$

For $n = 0$, it is straightforward since $\mathbb{T}_0$ and $\mathcal{T}_0$ are reduced to the root. We suppose that this is true for $n-1$, and we prove it for n. Let $\overleftarrow{u}$ denote the parent of u, and, for any vertex v at height $n-1$, let $k(v)$ denote the number of children of v in the tree T. We have

$$\begin{aligned}
&\mathbb{P}(\mathbb{T}_n = T_n, \xi_n = u \mid \mathbb{T}_{n-1} = T_{n-1}, \xi_{n-1} = \overleftarrow{u}) \\
&\quad = \frac{1}{k(\overleftarrow{u})} \frac{\tilde{q}^s_{k(\overleftarrow{u})}}{\tilde{q}_{k(\overleftarrow{u})}} \prod_{|v|=n-1} \tilde{q}_{k(v)} \\
&\quad = \frac{e^{1/c_n}}{e^{1/c_{n-1}}} \frac{\phi'_{n-1}(\phi^{-1}_{n-1}(s))}{\phi'_n(\phi^{-1}_n(s))} \frac{e^{Z_{n-1}/c_{n-1}}}{e^{Z_n/c_n}} \prod_{|v|=n-1} q_{k(v)} \\
&\quad = \frac{e^{1/c_n}}{e^{1/c_{n-1}}} \frac{\phi'_{n-1}(\phi^{-1}_{n-1}(s))}{\phi'_n(\phi^{-1}_n(s))} \frac{e^{Z_{n-1}/c_{n-1}}}{e^{Z_n/c_n}} \mathrm{GW}(\mathcal{T}_n = T_n \mid \mathcal{T}_{n-1} = T_{n-1}).
\end{aligned}$$

We use the induction assumption to get

$$\mathbb{P}(\mathbb{T}_n = T_n, \xi_n = u) = e^{1/c_n} \frac{1}{\phi'_n(\phi^{-1}_n(s))} e^{\frac{-Z_n}{c_n}} \mathrm{GW}(\mathcal{T}_n = T_n),$$

which proves (2.6). Summing over the n-th generation of T gives

$$\mathbb{P}(\mathbb{T}_n = T_n) = D_n \mathrm{GW}(\mathcal{T}_n = T_n) = Q(\mathcal{T}_n = T_n).$$

This computation also shows that $\mathbb{P}(\xi_n = u \mid \mathbb{T}_n) = 1/Z_n$ which implies that ξ is uniformly distributed on the boundary $\partial\mathbb{T}$. □

3. Proof of Theorem 1.1

Proposition 3.1. *Suppose that $E[\nu \log(\nu)] = \infty$. Then we have $\mathbb{P}$-a.s.*

$$\limsup_{n\to\infty} \frac{1}{n} \log\bigl(\nu(\xi_n)\bigr) = \log(m).$$

Proof. Let $1 < a < b < m$ and $n \ge 0$. We get from (2.4)

$$\begin{aligned}
\mathbb{P}\bigl(\nu(\xi_n) \in (a^n, b^n)\bigr) &= \frac{\phi'_n(\phi^{-1}_n(s))}{\phi'_{n+1}(\phi^{-1}_{n+1}(s))} E\bigl[\nu e^{-(\nu-1)/c_{n+1}}, \nu \in (a^n, b^n)\bigr] \\
&\ge \frac{\phi'_n(\phi^{-1}_n(s))}{\phi'_{n+1}(\phi^{-1}_{n+1}(s))} e^{-b^n/c_n} E\bigl[\nu, \nu \in (a^n, b^n)\bigr].
\end{aligned}$$

From (c) and (2.3), we deduce that for n large enough, we have

$$\mathbb{P}\bigl(\nu(\xi_n) \in (a^n, b^n)\bigr) \ge \frac{1}{2m} E\bigl[\nu, \nu \in (a^n, b^n)\bigr].$$

Therefore, under the condition $E[\nu \log(\nu)] = \infty$, we have

$$\sum_{n\ge 0} \mathbb{P}\bigl(\nu(\xi_n) \in (a^n, b^n)\bigr) = \infty. \tag{3.1}$$

We use the standard converse of the Borel–Cantelli lemma to see that $\nu(\xi_n) > a^n$ infinitely often. Then let a go to m to prove the lower bound of the proposition. The upper bound is easy since $\nu(\xi_n) \le Z_{n+1}$, and we know that $\frac{Z_n}{m^n}$ goes to 0 almost surely. □

We turn to the proof of the theorem.

Proof. Proof of Theorem 1.1 By Proposition 3.1, we have

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{1}{n}\log\bigl(\nu(\xi_n)\bigr) = \log(m)\right) = 1.$$

In particular, for $\mathbb{P}$-a.e. $\mathbb{T}$,

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{1}{n}\log\bigl(\nu(\xi_n)\bigr) = \log(m)\middle|\mathbb{T}\right) = 1.$$

By Proposition 2.1, the distribution of ξ given $\mathbb{T}$ is **UNIF**. Therefore, for $\mathbb{P}$-a.e. $\mathbb{T}$,

$$\textbf{UNIF}\left(r \in \partial\mathbb{T} : \limsup_{n\to\infty} \frac{1}{n}\log\bigl(\nu(r_n)\bigr) = \log(m)\right) = 1. \tag{3.2}$$

Again by Proposition 2.1, the distribution of $\mathbb{T}$ is the one of $\mathcal{T}$ under Q. We deduce that (3.2) holds for Q-a.e. tree $\mathcal{T}$. Since Q and GW* are equivalent, equation (3.2) holds for GW*-a.e. tree $\mathcal{T}$. To finish the proof, we just need to prove that the set $F := \{r \in \partial\mathcal{T} : \limsup_{n\to\infty} \frac{1}{n}\log(\nu(r_n)) = \log(m)\}$ has Hausdorff dimension 0. Let $\alpha > 0$. For a vertex u, let B_u be the set of rays r such that $u \in r$. Then B_u is a ball in the metric space $(\partial\mathcal{T}, d)$ with diameter $\operatorname{diam}(B_u) = e^{-|u|}$. For any integer $k \ge 1$, notice that we can cover F with a countable number of balls B_{u_i}, where $(u_i)_i$ are vertices of height greater than k such that $\nu(u_i) \ge m^{|u_i|}e^{-|u_i|\alpha/2}$. Finally, observe that

$$\sum_i \bigl(\operatorname{diam}(B_{u_i})\bigr)^\alpha = \sum_{\ell\ge 0} e^{-\ell\alpha} \sum_i \mathbf{1}_{\{|u_i|=\ell\}}.$$

For any ℓ, $Z_{\ell+1} \ge \sum_i \mathbf{1}_{\{|u_i|=\ell\}}\nu(u_i) \ge m^\ell e^{-\ell\alpha/2}\sum_i \mathbf{1}_{\{|u_i|=\ell\}}$. Therefore

$$\sum_i \bigl(\operatorname{diam}(B_{u_i})\bigr)^\alpha \le \sum_{\ell\ge 0} e^{-\ell\alpha/2}\frac{Z_{\ell+1}}{m^\ell} < \infty.$$

It yields that the Hausdorff dimension of F is smaller than α, which completes the proof by letting $\alpha \to 0$. □

4. Questions

These questions concern the case $E[\nu\log(\nu)] = \infty$.

Question 1. What is the packing dimension of **UNIF**?

Question 2. Can we give an exact Hausdorff measure for the boundary ∂T? This question is answered in the case $E[\nu\log(\nu)] < \infty$ by Liu [7] and Watanabe [20] in great generality. See also Duquesne [2] for an elementary proof.

Question 3. We know from [11] that the Hausdorff dimension of the boundary is $\log(m)$. What would be a "natural" measure on the boundary ∂T with Hausdorff dimension $\log(m)$?

We saw that the measure **UNIF** was putting its mass on exceptional rays. Therefore, one could thing of removing "bad" rays in order to construct a well spread out measure. This is the goal of the last question.

Question 4. Let $(a_n)_n$ be a sequence of integers going to ∞, such that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = a \in [1, m]$. Consider the Galton–Watson process in varying environment, with offspring distribution at generation n given by $\nu\mathbf{1}_{\{\nu\le a_n\}}$. It is associated with a uniform measure (see Theorem 3 of [1]). What is the Haudorff dimension of this measure?

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