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Superdiffusions with super-exponential growth: Construction, mass and spread

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Abstract. Superdiffusions corresponding to differential operators of the form $Lu + \beta u - \alpha u^2$ with mass creation (potential) terms $\beta(\cdot)$ that are 'large functions' are studied. Our construction for superdiffusions with large mass creations works for the branching mechanism $\beta u - \alpha u^{1+\gamma}$, $0 < \gamma < 1$, as well.

Let $D \subseteq \mathbb{R}^d$ be a domain in $\mathbb{R}^d$. When β is large, the generalized principal eigenvalue λ_c of $L + \beta$ in D is typically infinite. Let $\{T_t, t \geq 0\}$ denote the Schrödinger semigroup of $L + \beta$ in D with zero Dirichlet boundary condition. Under the mild assumption that there exists an $0 < h \in C^2(D)$ so that $T_t h$ is finite-valued for all $t \geq 0$, we show that there is a unique $\mathcal{M}_{\mathrm{loc}}(D)$-valued Markov process that satisfies a log-Laplace equation in terms of the minimal nonnegative solution to a semilinear initial value problem. Although for super-Brownian motion (SBM) this assumption requires β to be less than quadratic, the quadratic case will be treated as well.

When $\lambda_c = \infty$, the usual machinery, including martingale methods and PDE as well as other similar techniques cease to work effectively, both for the construction and for the investigation of the large time behavior of superdiffusions. In this paper, we develop the following two new techniques for the study of the local/global growth of mass and for the spread of superdiffusions:

- a generalization of the Fleischmann–Swart 'Poisson-coupling,' linking superprocesses with branching diffusions;
- the introduction of a new concept: the '*p-generalized principal eigenvalue.*'

The precise growth rate for the total population of SBM with $\alpha(x) = \beta(x) = 1 + |x|^p$ for $p \in [0, 2]$ is given in this paper.

Résumé. Nous étudions des superdiffusions qui correspondent à des opérateurs différentiels de la forme $Lu + \beta u - \alpha u^2$ tels que le terme de création de masse (potentiel) $\beta(\cdot)$ est une « grande fonction ». Notre construction pour les superdiffusions avec un grand terme de création de masse fonctionne aussi pour le mécanisme de branchement $\beta u - \alpha u^{1+\gamma}$, $0 < \gamma < 1$.

Soit $D \subseteq \mathbb{R}^d$ un domaine dans $\mathbb{R}^d$. Lorsque β est grand, la valeur propre principale généralisée λ_c de $L + \beta$ dans D est typiquement infinie. Soit $\{T_t, t \geq 0\}$ le semigroupe de Schrödinger de $L + \beta$ dans D avec condition aux limites de Dirichlet égale à zéro. Sous l'hypothèse légère qu'il existe un $0 < h \in C^2(D)$ tel que $T_t h$ a une valeur finie pour tout $t \geq 0$, nous montrons qu'il existe un unique processus de Markov à valeurs dans $\mathcal{M}_{\mathrm{loc}}(D)$ satisfaisant une équation log-Laplace en fonction de la solution positive minimale d'un problèmeaux valeurs initiales semi-linéaire. Bien que pour le super-mouvement brownien (SMB), cette hypothèse demande que la fonction β soit dominée par une fonction quadratique, nous traitons aussi le cas quadratique.

Quand $\lambda_c = \infty$, les techniques habituelles, y compris les méthodes de martingale et les équations différentielles partielles, ainsi que d'autres techniques similaires, cessent d'être efficaces pour la construction des superdiffusions et pour l'étude de leur comportement en temps grand.

Dans cet article, nous développons les deux nouvelles techniques suivantes pour l'étude de la croissance locale/globale de la masse et pour l'étude de la propagation des superdiffusions :

- une généralisation du « Poisson-coupling » de Fleischmann–Swart, liant les super-processus aux diffusions branchantes ;
- l'introduction d'un nouveau concept : la « *valeur propre principale p-généralisée* ».

Nous identifions aussi le taux de croissance précis de la population totale du SMB pour $\alpha(x) = \beta(x) = 1 + |x|^p$ et $p \in [0, 2]$.

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1. Introduction

1.1. *Superdiffusions*

Like Brownian motion, *super-Brownian motion* is also a building block in stochastic analysis. Just as Brownian motion is a prototype of the more general diffusion processes, super-Brownian motion is a particular *superdiffusion*. Superdiffusions are measure-valued Markov processes, but unlike for branching diffusions, the values of superdiffusions taken for $t > 0$ are no longer discrete measures. Intuitively, such a process describes the evolution of a random cloud in space, or random mass distributed in space, creating more mass at some regions while annihilating mass at some other regions along the way.

The usual way of defining or constructing a superdiffusion X is:

(1) as a measure-valued Markov process via its Laplace functional; or
(2) as a scaling limit of branching diffusions.

The second approach means that X arises as the short lifetime and high density diffusion limit of a *branching particle system with killing at the boundary*, that can be described as follows: in the nth approximation step each particle has mass $1/n$ and lives for a random lifetime which is exponentially distributed with mean $1/n$. While a particle is alive, its motion is described by a diffusion process in D with infinitesimal generator L (where D is a subdomain of $\mathbb{R}^d$ and the diffusion process is killed upon leaving D). If the particle is located at $x \in D$ at the end of its life, it dies and is replaced by a random number of offspring situated at the same location x, if the particle is located at the boundary of D at the end of its life, it dies with no offspring. The law of the number of descendants is spatially varying such that the number of descendants has mean $1 + \frac{\beta(x)}{n}$ and variance $2\alpha(x)$. Different particles undergo branching and migration independently of each other; the branching of a given particle may interact with its motion, as the branching mechanism is spatially dependent. Hence a superdiffusion can be described by the quadruple $(L, \beta, \alpha; D)$, where L is the second order elliptic operator corresponding to the underlying spatial motion, β (the 'mass creation term') describes the growth rate of the superdiffusion,[3] $\alpha > 0$ (sometimes called the 'intensity parameter') is related to the variance of the branching mechanism, and D is the region where the particles live. (A more general branching mechanism, including an integral term, corresponding to infinite variance, was introduced by E. B. Dynkin, but we do not work with those branching mechanisms in this paper.)

The idea behind the notion of superprocesses can be traced back to W. Feller, who observed in his 1951 paper on diffusion processes in genetics, that for large populations one can employ a model obtained from the Galton-Watson process, by rescaling and passing to the limit. The resulting *Feller diffusion* thus describes the scaling limit of the population mass. This is essentially the idea behind the notion of *continuous state branching processes*. They can be characterized as $[0, \infty)$-valued Markov processes, having paths which are right-continuous with left limits, and for which the corresponding probabilities $\{P_x, x \geq 0\}$ satisfy the branching property: the distribution of the process at time $t \geq 0$ under P_{x+y} is the convolution of its distribution under P_x and its distribution under P_y for $x, y \geq 0$. Note that Feller diffusions focus on the evolution of the *total mass* while ignoring the location of the individuals in the population. The first person who studied continuous state branching processes was the Czech mathematician M. Jiřina in 1958 (he called them 'stochastic branching processes with continuous state space').

When the *spatial motion* of the individuals is taken into account as well, one obtains a scaling limit which is now a measure-valued branching process, or superprocess. The latter name was coined by E. B. Dynkin in the 1980's. Dynkin's work (including a long sequence of joint papers with S. E. Kuznetsov) concerning superprocesses and their connection to nonlinear partial differential equations was ground breaking. These processes are also called *Dawson–Watanabe processes* after the fundamental work of S. Watanabe [30] in the late 1960's (see also the independent work by M. L. Silverstein [29] at the same time) and of D. Dawson [7] in the late 1970's. Among the large number of contributions to the superprocess literature we just mention the 'historical calculus' of E. Perkins, the 'Brownian snake representation' of J.-F. LeGall, the 'look down construction' (a countable representation) of P. Donnelly and T. G. Kurtz, and the result of R. Durrett and E. Perkins showing that for $d \geq 2$, rescaled contact processes converge to super-Brownian motion. In addition,

[3] In a region where $\beta < 0$, one actually has mass annihilation.

interacting superprocesses and *superprocesses in random media* have been studied, for example, by D. Dawson, J.-F. Delmas, A. Etheridge, K. Fleischmann, H. Gill, P. Mörters, L. Mytnik, Y. Ren, R. Song, P. Vogt, J. Xiong, and H. Wang, as well as by the authors of this article.

1.2. *Motivation*

A natural and interesting question in the theory of superprocesses is how fast the total mass and local mass grow as time evolves. When β is bounded from above (or more generally, when λ_c, the generalized principal eigenvalue[4] of $L+\beta$ on D is finite), the problem of the local growth has been settled (see e.g. [13] and the references therein) and it is known that the growth rate is at most exponential.

The local and the global growth are not necessarily the same. In fact, another quantity, denoted by λ_∞ is the one that gives the rate of the *global* exponential growth, when it is finite. It may coincide with λ_c or it may be larger. Under the so-called *Kato-class assumption* on β, it is finite. See Section 1.15.5 in [13] for more explanation.

In general, the growth rates of the superprocess can be super-exponential, and up to now, very little is known about the exact growth rates then. It is important to point out that in the general case, even the *existence* of superdiffusions needs to be justified. The difficulty with the construction in such a situation (i.e. when $\lambda_c = \infty$) is compounded by the fact that in the lack of positive harmonic functions (i.e. functions that satisfy $(L+\beta-\lambda)u = 0$ with some λ), all the usual machinery of martingales, Doob's h-transforms, semigroup theory etc. becomes unavailable. (When $\sup_{x\in\mathbb{R}^d}\beta(x) = \infty$ but $\lambda_c < \infty$, one can actually reduce the construction to the case when β is a constant, see p. 88 in [13].) New ideas and approaches are needed for the construction and growth rate estimates.

Obtaining the precise growth rate of superprocesses with 'large' mass creation turns out to be quite a challenging question, and there are many possible scenarios, depending on how large β is; see Corollary 1.4 below for an example. The main part of this paper is devoted to address this question for a class of superprocesses with large mass creation. The effective method of 'lower and upper solutions' for the partial differential equations associated with superprocess through the log-Laplace equation in the study of exponential growth rate for superdiffusions with bounded mass creation term β becomes intimidatingly difficult if not impossible when β is unbounded. (For a beautiful application of lower and upper solutions see [26,27].)

In Section 5 of this article, we are going to introduce the new concept of the 'p-generalized principal eigenvalue,' in an effort to capture the super-exponential growth rate of superprocesses with large mass creation term.

In the last part of this paper, we will employ the 'Poisson-coupling' method (see Section 6) to study super-exponential growth rate for superprocesses with large mass creation, by relating them to discrete branching particle systems. In order to do this, we extend some results of Fleischmann and Swart given in [21] concerning the coupling of superprocesses and discrete branching particle systems, from deterministic times to stopping times. This part may be of independent interest. An advantage of this method over the use of test functions, even in the case of $\lambda_c < \infty$, is that it enables one to transfer results directly from the theory of branching diffusions, where a whole different toolset is available as one is working with a discrete system.

Remark 1.1 (Our method vs. skeleton decomposition). In classical so-called 'skeleton decompositions' the measure-valued process conditioned on survival is decomposed in such a way that a discrete spatial branching process called the 'skeleton' becomes part of it (see e.g. [12]). Therefore, intuitively (and it is not difficult to make that intuition rigorous), one can estimate certain quantities (e.g. hitting probabilities) related to the measure-valued process conditioned on survival by those analogous quantities for the skeleton process. This comparison works in one direction only, as one only knows that, loosely speaking, the measure-valued process is 'larger' than the discrete one.

The proof of Theorem 1.3 below, will be based on the 'Poisson-coupling' method of Section 6. This result exemplifies, that one can in fact get a comparison in both directions by using this method.

The first main result of this paper, Theorem 3.1, gives the construction of superdiffusions with super-exponential growth. Below is the other main result that provides the connection between the growth rate of superdiffusions and that of the corresponding branching processes. The reader should check Definition 3.2 for the rigorous definition of superprocesses. We will use the appellations '$(L,\beta,\alpha;D)$-superdiffusion' and 'the superprocess corresponding to the operator $u\mapsto Lu+\beta u-\alpha u^2$ on D' interchangeably.

Before stating the theorem, a remark is due.

[4] For the definition and properties of λ_c see Chapter 4 in [25].

Remark 1.2 (The $\alpha = \beta$ case). Consider the $(L, \beta, \alpha; D)$-*superdiffusion* when $\alpha = \beta > 0$, in other words, the superprocess corresponding to the operator $u \mapsto Lu + \beta u - \beta u^2$. This case is particularly interesting because this semi-linear operator appears also in the log-Laplace equation (see (5.4), (5.5)) for the branching diffusion on D with spatial motion corresponding to L and with rate β.

Another convenient feature in this case is that the solutions of the steady-state equation $Lu + \beta(u - u^2) = 0$ are the same as those of $\widehat{L}u + (u - u^2) = 0$, where $\widehat{L} := \beta^{-1}L$, and this latter equation corresponds to a superprocess with spatially constant branching mechanism (and to a time changed motion process).

We now present the result.

Theorem 1.3 (General comparison between Z and X). *Let (X, P_0) be the superprocess corresponding to the operator $Lu + \beta u - \beta u^2$ on D starting with Dirac measure at the origin, and (Z, P_0) the branching diffusion on D with branching rate β started at the origin with a* Poisson(1) *number of particles. Let $|X|$ and $|Z|$ denote the total mass processes. Denote by*

$$S := \{|X_t| > 0 \textit{ for every } t \geq 0\}$$

the event of survival for the superdiffusion. Let $f : [0, \infty) \to [0, \infty)$ be a continuous function such that $\lim_{x \to \infty} f(x) = \infty$. *Then*

(i) *the condition*

$$\mathsf{P}_0\Big(\limsup_{t \to \infty} |Z_t| \big/ f(t) \leq 1\Big) = 1 \tag{1.1}$$

implies that $P_0(\limsup_{t \to \infty} |X_t|/f(t) \leq 1) = 1$;

(ii) *the condition*

$$\mathsf{P}_0\Big(\liminf_{t \to \infty} |Z_t| \big/ f(t) \geq 1\Big) = 1 \tag{1.2}$$

implies that $P_0(\liminf_{t \to \infty} |X_t|/f(t) \geq 1 \mid S) = 1$, *provided that one has* $P_0(\lim_{t \to \infty} |X_t| = \infty \mid S) = 1$. *This latter condition is always satisfied if the coefficients of $\frac{1}{\beta}L$ are bounded from above.*

Using Theorem 1.3 and the results from [1,2] on the corresponding branching Brownian motions, we have the following result, which illustrates some possible super-exponential growth rates the total mass of a super-Brownian motion with large mass creation term β may have.

Corollary 1.4 (Total mass of a SBM with large mass creation term). *Let X be a one-dimensional super-Brownian motion corresponding to $(\frac{1}{2}\Delta, \beta, \beta; \mathbb{R})$. Let S be as in Theorem 1.3. Then*:

(1) *If $\beta(x) = 1 + |x|^p$ for $0 \leq p < 2$, then*

$$\lim_{t \to \infty} t^{-(2+p)/(2-p)} \log |X_t| = K_p, \quad P_0\text{-a.s. on } S,$$

where K_p is positive constant, depending on p.

(2) *If $\beta(x) = 1 + C|x|^2$, with $C > 0$, then*

$$\lim_{t \to \infty} \big(\log \log |X_t|\big)/t = 2\sqrt{2C}, \quad P_0\text{-a.s. on } S.$$

Note. it is not difficult to show that the survival set S is not-trivial. In fact, $P_{\delta_x}(S) \geq e^{-1}$ for every $x \in \mathbb{R}$; see Section 6.3.

Notation. For two nonempty sets D_1 and D_2 in $\mathbb{R}^d, d \geq 1$, the notation $D_1 \Subset D_2$ will mean that $\overline{D_1} \subset D_2$ and D_1 is bounded.

Regarding the *local* growth of mass, we will obtain the following upper estimate. (Cf. Example 5.22.)

Theorem 1.5 (Local upper estimate for SBM with $|x|^\ell$-potential when $0 \le \ell < 2$). *For the* $((1/2)\Delta, |x|^\ell, \alpha; \mathbb{R}^d)$-*superdiffusion, with* $0 \le \ell < 2$, *one has that almost surely, as* $t \to \infty$,

$$X_t(B) = \mathcal{O}\big(\exp\big\{\text{const} \cdot t^{(2+\ell)/(2-\ell)}\big\}\big), \quad B \Subset \mathbb{R}^d,$$

provided that α *is such that the compact support property holds.*

Regarding the notion of the compact support property and the assumption on α, see Section 5.2, along with Remark 5.21.

1.3. *Outline*

The rest of the paper is organized as follows. Section 2 gives some preliminaries including notation that will be used later in the paper. The first main result of this paper, regarding the construction of superdiffusions with general large mass creation, is given in Section 3. When the generalized principal eigenvalue λ_c of $L + \beta$ on D is infinite, we show in Section 4 that the local mass of the superprocess can no longer grow at an exponential rate: the growth will be 'super-exponential.' In Section 5 we will focus on super-Brownian motion on $\mathbb{R}^d$ with mass creation $\beta(x) = a|x|^\ell$ for $0 \le \ell \le 2$; construction and some basic properties are discussed, in particular, the growth of the total mass for the case when $d = 1$.

We then introduce a new notion we dubbed the 'p-generalized principal eigenvalue' (a notion more general than λ_c). Some of its properties are investigated in Appendix A, where the proof of Theorem 1.5 is given too (see Section A.1).

Section 6 is devoted to employing a 'Poisson-coupling' method to obtain precise growth rate for the total mass of the superprocess from that of the total mass of the corresponding discrete branching process; see Theorem 1.3. The proof of Corollary 1.4 is given at the end of Section 6 as a corollary to Theorem 1.3.

Finally, Appendix B gives some background material on superprocesses and related functions.

2. Preliminaries

2.1. *Notation*

For convenience, we first recall basic notation. Let $d \ge 1$ and $D \subseteq \mathbb{R}^d$ be a domain and let $\mathcal{B}(D)$ denote the Borel sets of D. We write $\mathcal{M}_f(D)$ and $\mathcal{M}_c(D)$ for the class of finite measures and the class of finite measures with compact support on $\mathcal{B}(D)$, respectively, and $\mathcal{M}_{\text{loc}}(D)$ denotes the space of locally finite measures on $\mathcal{B}(D)$.[5] For $\mu \in \mathcal{M}_f(D)$, denote $|\mu| := \mu(D)$ and let $B_b^+(D)$, $C_b^+(D)$ and $C_c^+(D)$ be the class of non-negative bounded Borel measurable, non-negative bounded continuous and non-negative continuous functions $D \to \mathbb{R}$ having compact support, respectively. For integer $k \ge 0$ and $0 \le \eta < 1$, we use $C^{k,\eta}(D)$ to denote the space of continuous functions on D that have continuous derivatives up to and including the kth order and whose kth order partial derivatives are locally η-Hölder continuous in D. We write $C^k(D)$ for $C^{k,0}(D)$ and $C^\eta(D)$ for $C^{0,\eta}(D)$. For space-time functions defined on $D \times \mathbb{R}^+$, $C^{2,1,\eta}(D \times \mathbb{R}^+)$ will denote the space of functions which belong to $C^{2,\eta}(D)$ in the space variable for $t \ge 0$ fixed, and to $C^{1,\eta}(\mathbb{R}^+)$ in the time variable for $x \in D$ fixed.

The notation $\mu_t \overset{v}{\Rightarrow} \mu$ ($\mu_t \overset{w}{\Rightarrow} \mu$) will be used for the vague (weak) convergence of measures.

Let L be an elliptic operator on D of the form

$$L := \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla,$$

where $a_{i,j}, b_i \in C^{1,\eta}(D)$, $i, j = 1, \dots, d$, for some $\eta \in (0, 1]$, and the matrix $a(x) := (a_{i,j}(x))$ is symmetric and positive definite for all $x \in D$. In addition, let $\alpha, \beta \in C^\eta(D)$, with $\alpha > 0$.

Let $Y = \{Y_t, t \ge 0, \mathbb{P}_x, x \in D\}$ be the *minimal diffusion process* in D having infinitesimal generator L in D; that is, Y is a diffusion process having infinitesimal generator L with killing upon exiting D. Note that typically Y may have finite lifetime ζ and thus $\mathbb{P}_x(Y_t \in D) \le 1$ in general. (In the terminology of [25], Y is the solution of the *generalized martingale problem* for L on D. The world 'generalized' refers to the fact that conservativeness is not assumed.) Finally, let

$$\begin{aligned}\lambda_c &= \lambda_c(L+\beta, D) \\ &:= \inf\big\{\lambda \in \mathbb{R} : \exists u \in C^2 \text{ with } u > 0, (L + \beta - \lambda)u = 0 \text{ in } D\big\}\end{aligned}$$

denote the *generalized principal eigenvalue* for $L + \beta$ on D. See Section 4.3 in [25] for more on this notion, and on its relationship with L^2-theory. (Here the word 'generalized' basically refers to the fact that L is not necessarily self-adjoint.)

[5] I.e. Borel measures on D whose charge on each compact subset of D is finite.

2.2. *The construction of the* $(L,\beta,\alpha;D)$*-superdiffusion*

In [15] the $\mathcal{M}_{\mathrm{loc}}(D)$-valued $(L,\beta,\alpha;D)$-superdiffusion X corresponding to the semilinear elliptic operator $Lu+\beta u-\alpha u^2$ has been constructed, under the assumption that

$$\lambda_c(L+\beta,D)<\infty. \tag{2.1}$$

For the case when β is bounded from above, the construction of an $\mathcal{M}_f(D)$-valued process relied on the method of Dynkin and Fitzsimmons [11,19,20], but instead of the mild equation, the strong equation (PDE) was used in the construction. Then a nonlinear h-transform (producing 'weighted superprocesses') has been introduced in [15], and with the help of this transformation it became possible to replace $\sup_D\beta<\infty$ by (2.1) and get an $\mathcal{M}_{\mathrm{loc}}(D)$-valued process. The condition (2.1) is always satisfied when β is bounded from above, and in many other cases as well (for example on a bounded domain β can be allowed to blow up quite fast at the boundary – see p. 691 in [15]).

Nevertheless, (2.1) is often very restrictive. For example, when L on $\mathbb{R}^d$ has constant coefficients, then even a "slight unboundedness" destroys (2.1), as the following lemma shows.

Lemma 2.1. *Assume that L on $\mathbb{R}^d$ has constant coefficients and that there exists an $\varepsilon>0$ and a sequence $\{x_n\}$ in $\mathbb{R}^d$ such that*

$$\lim_{n\to\infty}\inf_{x\in B(x_n,\varepsilon)}\beta(x)=\infty.$$

Then (2.1) does not hold for $D=\mathbb{R}^d$.

Proof. By the assumption, for every $K>0$ there exists an $n=n(K)\in\mathbb{N}$ such that $\beta\ge K$ on $B_\varepsilon(x_n)$. Let λ^ε denote the principal eigenvalue of L on a ball of radius ε. (Since L has constant coefficients, λ^ε is well defined.) Since

$$\lambda_c=\lambda_c\big(L+\beta,\mathbb{R}^d\big)\ge\lambda_c\big(L+\beta,B_\varepsilon(x_n)\big)\ge\lambda^\varepsilon+K,$$

and $K>0$ was arbitrary, it follows that $\lambda_c=\infty$. □

The first purpose of this paper is to replace (2.1) by a much milder condition. We note that for the discrete setting (branching diffusions), super-exponential growth has been studied in [1,2,23]. In the recent paper [12] the connection between the two types of processes has been studied.

2.3. *Condition replacing* (2.1)

Recalling that Y is the diffusion process corresponding to L on D with lifetime $\tau_D:=\inf\{t\ge 0\mid Y_t\notin D\}\in(0,\infty]$, let us define $\{T_t,t\ge 0\}$, the formal[6] 'Dirichlet–Schrödinger semigroup' of $L+\beta$ in D, by

$$(T_tg)(x):=\mathbb{E}_x\left[\exp\left(\int_0^t\beta(Y_s)\,\mathrm{d}s\right)g(Y_t);t<\tau_D\right]\in[0,\infty],$$

when $g\in C^+(D)$, $t\ge 0$, and $x\in D$.

The following assumption, requiring that $T_t(h)$ is finite for all times for just a single positive function, will be crucial in the construction of the superprocess.

Assumption 2.2 (Existence of $\{T_th,t\ge 0\}$ for a single $h>0$). Assume that there exists a positive function $h\in C^2(D)$ satisfying that $T_th(x)<\infty$ for all $t>0$ and $x\in D$.

Proposition 2.3 (Equivalent formulation). *Assumption 2.2 is equivalent to the following condition: For some (or equivalently, all) non-vanishing $0\le\psi\in C_c^2(D)$ $T_t\psi<\infty$ for all $t>0$.*

Proof. It is sufficient to show that

(a) if for some non-vanishing $0\le\psi\in C_c^2(D)$, $T_t\psi<\infty$ for all $t>0$, then Assumption 2.2 holds;

[6] Finiteness, continuity or the semigroup property are not required, hence the adjective.

(b) if for some non-vanishing $0 \le \psi \in C_c^2(D)$ and some $x_0 \in D$, $t > 0$, we have $T_t(\psi)(x_0) = \infty$, then Assumption 2.2 fails.

Indeed, in the first case, for every $x \in D$,

$$h(x) := T_1\psi(x) = \mathbb{E}_x\big[e^{\int_0^t \beta(Y_s)\,\mathrm{d}s}\psi(Y_t); t < \tau_D\big] > 0,$$

because

$$\mathbb{P}_x\big[e^{\int_0^t \beta(Y_s)\,\mathrm{d}s}\psi(Y_t) > 0 \mid t < \tau_D\big] > 0,$$

as $e^{\int_0^t \beta(Y_s)\,\mathrm{d}s} > 0$ and $\mathbb{P}_x[\psi(Y_t) > 0 \mid t < \tau_D] > 0$. Clearly, $T_t h = T_{t+1}\psi < \infty$ for all $t > 0$. On the other hand, $u(t,x) := T_t\psi(x)$ is the minimal non-negative solution of $\frac{\partial u}{\partial t} = Lu + \beta u$ with $u(0,x) = \psi(x)$ in D, and by the same argument as in the proof of Theorem 1 in [16], $u(t,x) \in C^{1,2}((0,\infty) \times D)$ (in fact, all second spatial derivatives of u are η-Hölder continuous in D). This shows that Assumption 2.2 hold.

In the second case, for any $0 < h \in C^2(D)$, there exists a $C > 0$ such that $Ch > \psi$, implying $T_t(h)(x_0) = \infty$, and thus Assumption 2.2 cannot hold. □

Remark 2.4 (Feynman–Kac representation). Approximating D by an increasing sequence of relatively compact domains and using standard compactness arguments, it is not difficult to show that under Assumption 2.2, the function u defined by $u(x,t) := T_t h(x)$ solves the parabolic equation

$$\frac{\partial u}{\partial t} = (L + \beta)u \quad \text{in } D \times (0,\infty),$$

and in particular, $u \in C([0,\infty) \times D)$.

Finally, it is important to point out that condition (2.1) implies Assumption 2.2. This is because if (2.1) holds, then there is a $C^2(D)$-function $h > 0$ such that $(L + \beta - \lambda_c)h = 0$ in D. (See Section 4.3 in [25].) Clearly, it is enough to show that $T_t(h) \le h$.

Let $\{D_k; k \ge 1\}$ be an increasing sequence of relatively compact smooth subdomains of D with $D_k \Subset D_{k+1} \Subset D$ that increases to D. By the Feynman–Kac representation, for every $k \ge 1$,

$$0 \le u^{(k)}(x,t) := \mathbb{E}_x\big[e^{\int_0^t [\beta(Y_s) - \lambda_c]\,\mathrm{d}s} h(Y_t); t < \tau_{D_k}\big], \quad x \in D_k, t \ge 0,$$

is the unique parabolic solution for $\dot{u} = (L + \beta - \lambda_c)u$ on D_k with zero boundary condition and initial condition h.

By taking $k \to \infty$, and using the above Feynman–Kac representation (or the parabolic maximum principle), $u^{(k)}$ are monotone nondecreasing in k, and are all bounded from above by h (which itself is a nonnegative parabolic solution on each domain D_k with initial condition h restricted on D_k). Therefore, by the Monotone Convergence Theorem, the limiting function u satisfies that

$$h(x) \ge u(x,t) = \mathbb{E}_x\big[e^{\int_0^t [\beta(Y_s) - \lambda_c]\,\mathrm{d}s} h(Y_t); t < \tau_D\big] = T_t(h)(x).$$

2.4. *A useful maximum principle*

In the remaining part of this paper, for convenience, we will use either $\dot{u}$ or $\partial_t u$ to denote $\frac{\partial u}{\partial t}$. We will frequently refer to the following parabolic semilinear maximum principle due to R. Pinsky [15, Proposition 7.2]:

Proposition 2.5 (Parabolic semilinear maximum principle). *Let L, β and α be as in Section 2.1 and let $U \Subset D$ be a non-empty domain. Assume that the functions $0 \le v_1, v_2 \in C^{2,1}(U \times (0,\infty)) \cap C(\overline{U} \times (0,\infty))$ satisfy*

$$Lv_1 + \beta v_1 - \alpha v_1^2 - \dot{v}_1 \le Lv_2 + \beta v_2 - \alpha v_2^2 - \dot{v}_2 \quad \text{in } U \times (0,\infty),$$

$v_1(x,0) \ge v_2(x,0)$ for $x \in U$, and $v_1(x,t) \ge v_2(x,t)$ on $\partial U \times (0,\infty)$. Then $v_1 \ge v_2$ in $U \times [0,\infty)$.

3. Superprocess with general mass creation

The following theorem is one of the main results of this paper, on the construction of the superprocess with large mass creation.

Theorem 3.1 (Superprocess with general mass creation). *Under Assumption 2.2 there exists a unique $\mathcal{M}_{\text{loc}}(D)$-valued Markov process $\{(X, P_\mu); \mu \in \mathcal{M}_c(D)\}$ satisfying the log-Laplace equation*

$$E_\mu \exp\big(\langle -g, X_t\rangle\big) = \exp\big(\langle -S_t(g), \mu\rangle\big), \quad g \in C_c^+(D), \mu \in \mathcal{M}_c(D), \tag{3.1}$$

where $S_t(g)(\cdot) = u(\cdot, t)$ is the minimal nonnegative solution to the semilinear initial value problem ("cumulant equation")

$$\begin{cases} \dot{u} = Lu + \beta u - \alpha u^2 & \text{in } D \times (0, t), \\ \lim_{t\downarrow 0} u(\cdot, t) = g(\cdot). \end{cases} \tag{3.2}$$

Definition 3.2. The process X under the probabilities $\{P_\mu, \mu \in \mathcal{M}_c(D)\}$ in Theorem 3.1 will be called the $(L, \beta, \alpha; D)$-*superdiffusion*, or *the superprocess corresponding to the operator $u \mapsto Lu + \beta u - \beta u^2$ on D*.

Remark 3.3.

(i) Although we only consider the operator $Lu + \beta u - \alpha u^2$ in this paper, the construction of the superprocess goes through for the operator $Lu + \beta u - \alpha u^{1+p}$, $0 < p < 1$, as well.
(ii) The smoothness (Hölder-continuity) assumptions on the coefficients a, b, α, β are a convenience, as one often uses standard PDE machinery for the solutions of (3.2) or to its steady-state version. (For example, this is the case in Proposition 2.5 and several times in [15], the results of which we use in this paper.)

From (3.1) it follows that X possesses the *branching property*.

Corollary 3.4 (Branching property). *If $\mu, \nu \in \mathcal{M}_c(D)$, $t \geq 0$ and $g \in C_c^+(D)$, then the distribution of $\langle g, X_t\rangle$ under $P_{\mu+\nu}$ is the convolution of the distributions of $\langle g, X_t\rangle$ under P_μ and under P_ν.*

Proof of Theorem 3.1. We first recall the definition of the nonlinear space-time H-transform. Consider the backward operator

$$\mathcal{A}(u) := \partial_s u + (L + \beta)u - \alpha u^2,$$

and let $0 < H \in C^{2,1,\eta}(D \times \mathbb{R}^+)$. Analogously to Doob's h-transform for linear operators, introduce the new operator $\mathcal{A}^H(\cdot) := \frac{1}{H}\mathcal{A}(H\cdot)$. Then a direct computation gives that

$$\mathcal{A}^H(u) = \frac{\partial_s H}{H}u + \partial_s u + Lu + a\frac{\nabla H}{H} \cdot \nabla u + \beta u + \frac{LH}{H}u - \alpha H u^2. \tag{3.3}$$

This transformation of operators has the following probabilistic impact. Let X be a $(L, \beta, \alpha; D)$-superdiffusion. We define a new process X^H by

$$X_t^H := H(\cdot, t)X_t \qquad \left(\text{that is, } \frac{\mathrm{d}X_t^H}{\mathrm{d}X_t} = H(\cdot, t)\right), \quad t \geq 0. \tag{3.4}$$

In this way, one obtains a new superdiffusion, which, in general, *is not finite measure-valued* but only $\mathcal{M}_{\text{loc}}(D)$-valued. The connection between X^H and $\mathcal{A}^H$ is given by the following result.

Lemma 3.5 (Lemma 3 in [18]). *The process X^H, defined by (3.4), is a superdiffusion corresponding to $\mathcal{A}^H$ on D.*

Note that the differential operator L is transformed into

$$L_0^H := L + a\frac{\nabla H}{H} \cdot \nabla,$$

while β and α transform into $\beta^H := \beta + \frac{(\partial_s + L)H}{H}$ and $\alpha^H := \alpha H$, respectively.

It is clear that given a superdiffusion, H-transforms can be used to produce new superdiffusions that are weighted versions of the old one. See [18] for more on H-transforms. We now show that, under the assumption of Theorem 3.1, one can always use H-transforms to construct the superdiffusion.

Recall that by Assumption 2.2, there exists an $h > 0$ such that $(T_t h)(x) < \infty$ for all $t \geq 0$ and $x \in D$. Let us fix such an h. We first work with a *fixed finite time horizon*. Fix $t > 0$ and for $x \in D$, $r \in [0, t]$, consider

$$H(x, r; t, h) := (T_{t-r} h)(x) < \infty.$$

Then $0 < H \in C^{2,1,\eta}(D \times \mathbb{R}^+)$ and H is the minimal non-negative solution to the backward equation

$$\begin{cases} -\partial_r H = LH + \beta H & \text{in } D \times (0, t), \\ \lim_{r \uparrow t} H(\cdot, r; t, h) = h(\cdot). \end{cases} \tag{3.5}$$

(One can approximate D by an increasing sequence of compactly embedded domains D_n and consider the Cauchy problem with zero Dirichlet boundary condition. By the parabolic maximum principle, the solutions are growing in n, and, by the assumption on h, the limit is finite; minimality follows again by the parabolic maximum principle. That the limiting function is a solution and it belongs to $C^{2,1,\eta}(D \times \mathbb{R}^+)$, follows by using standard *a priori* estimates and compactness in the second order Hölder norm; see Theorems 5 and 7 in Chapter 3 in [22].)

For the rest of this subsection, fix a measure $\mu \in \mathcal{M}_c(D)$. Keep $t > 0$ still fixed, and define $H(x, s) := (T_{t-s} h)(x)$. Then $\beta^H = 0$ and

$$\left(L_0^H, \beta^H, \alpha^H; D\right) = \left(L + a\frac{\nabla H}{H} \cdot \nabla, 0, \alpha H; D\right).$$

We first show that the (time-inhomogeneous) critical measure-valued process $\widehat{X} = X^H$ corresponding to this quadruple is is well defined on the time interval $[0, t]$. To check this, recall the construction in Appendix A in [15]. That construction goes through for this case too, despite the time-dependence of the drift coefficient of the diffusion and the variance term α. Indeed, the first step in the construction of the measure-valued process is the construction of the minimal nonnegative solution to the semilinear parabolic Cauchy problem (3.2). It is based on the approximation of D with compacts $D_n \Subset D$, $\bigcup_{n=1}^{\infty} D_n = D$, and imposing zero Dirichlet boundary condition on them (see the Appendix A in [15]). By the local boundedness of β, the solution with zero boundary condition for the original operator is well defined on compacts, and therefore it is also well defined for the H-transformed operator on compacts. As $n \to \infty$, the solution to this latter one does not blow up, because the new potential (zeroth order) term is zero and because of Proposition 2.5. Hence, the solution to the original Cauchy problem does not blow up either.

Once we have the minimal nonnegative solution to the H-transformed Cauchy problem we have to check that it defines, via the log-Laplace equation, a finite measure-valued Markov process on the time interval $[0, t]$.

Let $S_s^H(g)(x) := u^{(g)}(x, s)$, where $u^{(g)}$ denotes the minimal nonnegative solution to the H-transformed nonlinear Cauchy problem

$$\dot{u} = L_0^H u - \alpha^H u^2$$

with $\lim_{t \downarrow 0} u(x, t) = g(x) \in C_b^+(D)$. Note that

$$S_s^H(g_n) \downarrow 0 \text{ pointwise, whenever } g_n \in C_b^+(D) \text{, and } g_n \downarrow 0 \text{ pointwise,} \tag{3.6}$$

because, using the semilinear parabolic maximum principle and the fact that β^H,

$$S_s^H(g_n) \leq T_s^H(g_n) \leq \|g_n\|_\infty,$$

where $\{T_s^H; s \geq 0\}$ is the semigroup associated with the infinitesimal generator L^H with zero Dirichlet boundary condition on ∂D. This also shows that the shift S_t^h leaves $C_b^+(D)$ invariant.

Before proceeding further, let us note that, by the minimality of the solution, S^H forms a semigroup on $C_b^+(D)$:

$$S_{s+z}^H = S_s^H \circ S_z^H, \quad \text{for } 0 \leq s, z \text{ and } s + z \leq t. \tag{3.7}$$

(Obviously, S_0 is the unit element of the semigroup.)

The nice properties of S^H then enable one to define a superprocess.

Lemma 3.6. *In order to define the $\mathcal{M}_f(D)$-valued branching Markov process (superprocess) $\widehat{X}$ corresponding to S^H (on $[0,t]$) via the log-Laplace equation*

$$\mathbb{E}_\mu^H \exp\bigl(\langle -g, \widehat{X}_s\rangle\bigr) = \exp\bigl(\bigl\langle -S_s^H(g), \mu\bigr\rangle\bigr), \quad g \in C_b^+(D), \mu \in \mathcal{M}_f(D), \tag{3.8}$$

one only needs that S^H satisfies (3.6) and (3.7).

Proof. Following the method in Appendix A in [15], the fundamental observation is that S^H enjoys the following three properties:

(1) $S_s^H(0) = 0$;
(2) The property under (3.6);
(3) S_s^H is an *N-function* on $C_b^+(D)$; that is,[7]

$$\sum_{i,j=1}^n \lambda_i\lambda_j S_s^H(f_i + f_j) \le 0 \quad \text{if} \sum_i^n \lambda_i = 0, \forall n \ge 2, \forall f_1, \dots, f_n \in C_b^+(D).$$

For the third property, just like in [15], one utilizes [10] (more precisely, the argument on p. 1215).

Then, one defines $\mathcal{L}_s^H(\cdot) := \exp(-S_s^H(\cdot))$, $0 \le s \le t$ on $C^+(D)$, and checks that it satisfies

(1) $\mathcal{L}_s^H(0) = 1$;
(2) $\mathcal{L}_s^H g \in (0,1]$ for $f \in C^+(D)$;
(3) The property under (3.6), if decreasing sequences are replaced by increasing ones;
(4) $\mathcal{L}_s^H$ is a *P-function* on $C_b^+(D)$; that is,

$$\sum_{i,j=1}^n \lambda_i\lambda_j \mathcal{L}_s^H(f_i + f_j) \ge 0, \quad \forall n \in \mathbb{N}, \forall f_1, \dots, f_n \in C_b^+(D), \forall \lambda_1, \dots, \lambda_n \in \mathbb{R}.$$

(For the fourth property, see p. 74 in [3].) As noted in [15], these four properties of $\mathcal{L}^H$ imply that for every $x \in D$ and $0 \le s \le t$ fixed, there exists a unique probability measure $\widehat{P}^{x,s}$ on $\mathcal{M}_f(D)$ satisfying for all $g \in C_b^+(D)$ that

$$\mathcal{L}_s^H(g)(x) = \int_{\mathcal{M}_f(D)} e^{-\langle g,\nu\rangle} \widehat{P}^{x,s}(\mathrm{d}\nu). \tag{3.9}$$

As explained on p. 722 in [15], one can use Corollary A.6 in [19] with a minimal modification. Alternatively, use Theorem 3.1 in [11] instead of [19].

It then follows from the property under (3.7) that the functional $\mathcal{L}^H$ defined by

$$\mathcal{L}^H(s,\mu,g) := \exp\bigl(-\bigl\langle S_s^H g(x)\mu\bigr\rangle\bigr), \quad g \in C_b^+(D), \mu \in \mathcal{M}_f(D)$$

is a *Laplace-transition functional*, that is, there exists a unique $\mathcal{M}_f(D)$-valued Markov process $(\widehat{X}, \widehat{P})$, satisfying that

$$\mathcal{L}^H(s,\mu,g) = \widehat{\mathbb{E}}_\mu\bigl[e^{-\langle g,\widehat{X}_s\rangle}\bigr], \quad s \ge 0, g \in C_b^+(D), \mu \in \mathcal{M}_f(D),$$

finishing the construction of $\widehat{X}$. □

Note. The integral representation (3.9) is essentially a consequence of the Krein–Milman Theorem, which can be found e.g. in Section 2.5 in [3], while the Markov property for the superprocess $\widehat{X}$ is a consequence of the semigroup property (3.7) for S^H, given the log-Laplace equation.

Having constructed the measure-valued process $(\widehat{X}, \widehat{P})$ corresponding to the quadruple $(L + a\frac{\nabla H}{H}\cdot\nabla, 0, \alpha H; D)$, now consider it on the time interval $[0,t]$ starting with initial measure $\widehat{\mu}^{t,h} := H(\cdot, 0; t, h)\mu$. By the properties of the H-transform reviewed above, the measure-valued process $X_r := H^{-1}(\cdot, r; t, h)\widehat{X}_r$ corresponds to the quadruple $(L, \beta, \alpha; D)$ on the same time interval $r \in [0,t]$, with initial measure μ.

[7] An explanation of the terminology 'P-function' and 'N-function' is given on pp. 40-41 in [11]. Note that in [15] we used the names positive semidefinite and negative semidefinite, respectively.

In other words, stressing now the dependence on t in the notation, if $\widehat{\mathbb{P}}^{(t)}$ corresponds to $\widehat{X}^{(t)}$, then the measure valued process

$$X_r^{(t)} := H^{-1}(\cdot, r; t, h)\widehat{X}_r^{(t)}$$

under $\widehat{\mathbb{P}}^{(t)}_{\widehat{\mu}^{t,h}}$ satisfies the log-Laplace equation (3.1), and moreover, clearly, $\widehat{\mathbb{P}}^{(t)}_{\widehat{\mu}^{t,h}}(X_0^{(t)} = \mu) = 1$.

This, in particular, shows that the definition is consistent, that is, if $t < t'$, then $\widehat{P}^{(t)}_{\widehat{\mu}^{t',h}}(X_\cdot^{(t)} \in \cdot)$ and $\widehat{P}^{(t')}_{\widehat{\mu}^{t,h}}(X_\cdot^{(t')} \in \cdot)$ agree on $\mathcal{F}_t$, and thus we can extend the time horizon of the process X to $[0, \infty)$ and define a probability P for paths on $[0, \infty)$. Indeed the finite dimensional distributions up to t are determined by the same log-Laplace equation and $\widehat{P}^{(t')}_{\widehat{\mu}^{t',h}}(X_0^{(t')} = \mu) = 1$ is still true when we work on $[0, t']$.

The semigroup property (or equivalently, the Markov property) is inherited from S^H to S (from $\widehat{X}$ to X) by the definition of the H-transform.

Our conclusion is that the $\mathcal{M}_{\mathrm{loc}}(D)$-valued Markov process $\{(X, P_\mu); \mu \in \mathcal{M}_c(D)\}$ is well defined on $[0, \infty)$ by the log-Laplace equation (3.1) and the cumulant equation (3.2). $\square$

Remark 3.7. There is a similar construction in [28], but instead of our Assumption 2.2, the existence of a function h with far more restrictive conditions is assumed.

Remark 3.8 (Global supersolutions). If there exists an $0 < H \in C^{2,\eta}(D) \times C^{1,\eta}(\mathbb{R}^+)$ which is a *global* super-solution to the backward equation, i.e.

$$\dot{H} + (L + \beta)H \le 0 \quad \text{in } D \times (0, \infty),$$

then there is a shorter way to proceed, since instead of working first with finite time horizons, one can work directly with $[0, \infty)$. Indeed, similarly to what we have done in the general case, now the time-inhomogeneous (sub)critical measure-valued Markov process $\widehat{X}$ corresponding to the quadruple $(L + a\frac{\nabla H}{H} \cdot \nabla, (\dot{H} + (L + \beta)H)/H, \alpha H; D)$ is well defined, because the potential term is non-positive. Just like before, the measure-valued process $X_t := H^{-1}(\cdot, t)\widehat{X}_t$ corresponds to the quadruple $(L, \beta, \alpha; D)$.

When $\lambda_c < \infty$, let $h > 0$ be a C^2-function on D with $(L + \beta)h = \lambda h$ for some $\lambda \ge \lambda_c$. Then $H(x, t) := e^{-\lambda t}h(x)$ is a global *solution* to the backward equation in $D \times (0, \infty)$; when $\lambda_c = \infty$, a global backward super-solution might not exist.

Remark 3.9. In [15], instead of Property (3.6), the continuity on $C_b^+(D)$ with respect to bounded convergence was used. Clearly, if one knows that (3.6) (together with the other properties) guarantees the existence of $\widehat{P}^{x,s}$ for all x, s, then this latter continuity property will guarantee it too: if $0 \le g_n \uparrow g$ and g is bounded, then the convergence is bounded. In [15], in fact, the continuity of the semigroup with respect to bounded convergence was proved.

As far as the path continuity of X is concerned, the reader can find a result in Section 5; see Claim 5.15.

Remark 3.10 (Re-weighting and Assumption 2.2). Recall our basic assumption (Assumption 2.2) for defining a superprocess X: there exists an $0 < h \in C^2(D)$ such that $(T_t h)(x) < \infty$ for all $t \ge 0$ and $x \in D$. Let $0 < \widehat{h} \in C^{2,\eta}(D)$. Then the re-weighted superprocess $X^{\widehat{h}}$ defined by (3.4) with $H(x, t) = \widehat{h}(x)$, satisfies Assumption 2.2 too, since $0 < h^* := h/\widehat{h} \in C^2(D)$ and

$$T_t^{\widehat{h}}\big(h^*\big)(x) = T_t(h)(x) < \infty.$$

It is important to emphasize though that even if Assumption 2.2 is not satisfied but we do know that the superprocess is well defined (such is the case of a super-Brownian motion with quadratic mass creation in Example 5.7), the definition

$$X_t^{\widehat{h}} := \widehat{h}(\cdot)X_t \qquad \left(\text{that is, } \frac{\mathrm{d}X_t^{\widehat{h}}}{\mathrm{d}X_t} = \widehat{h}(\cdot)\right), \quad t \ge 0,$$

clearly yields a a new (re-weighted) superprocess which corresponds to the h-transformed ($h = \widehat{h}$) semilinear operator.

The following result is sometimes called the Many-to-One Principle.

Lemma 3.11 (Expectation formula). *For $\mu \in \mathcal{M}_c(D)$ and $0 \le g \in C^+(D)$,*

$$E_\mu\langle g, X_t\rangle = \langle T_t g, \mu\rangle,$$

in the sense that if one side is infinite, then so is the other.

Proof. Using monotone convergence, it is enough to work with $0 \le g \in C_c^+(D)$. Next, by construction, on $[0,t]$ the superprocess can be H transformed into a critical ($\beta^H = 0$) one, for which the expectation formula is well known (the standard proof is differentiating the log-Laplace equation with respect to ε when it is started with εg). Using an inverse H transform (one with $H' = 1/H$), one recovers our statement. □

4. Super-exponential growth when $\lambda_c = \infty$

When the generalized principal eigenvalue is infinite, the local mass of the superprocess can no longer grow at an exponential rate, as the following result shows.

Theorem 4.1. *Assume that $\mathbf{0} \ne \mu \in \mathcal{M}_c(D)$ and $\lambda_c = \infty$. Then, for any $\lambda \in \mathbb{R}$ and any open set $\varnothing \ne B \Subset D$,*

$$P_\mu\Big(\limsup_{t\to\infty} e^{-\lambda t} X_t(B) = \infty\Big) > 0. \tag{4.1}$$

Proof. We are following the proof of Theorem 3(ii) in [14].

We may assume without the loss of generality that $\lambda > 0$. Since $\lambda_c = \infty$, by standard theory (see Chapter 4 in [25]), there exists a large enough $B^* \Subset D$ with a smooth boundary so that

$$\lambda^* := \lambda_c\big(L + \beta, B^*\big) > \lambda.$$

In addition, we can choose B^* large enough so that $\operatorname{supp}(\mu) \Subset B^*$.

Let the eigenfunction ϕ^* satisfy $(L + \beta - \lambda^*)\phi^* = 0$, $\phi^* > 0$ in B^* and $\phi^* = 0$ on ∂B^*. Let X^{t,B^*} denote the exit measure[8] from $B^* \times [0,t)$. We would like to integrate ϕ^* against X^{t,B^*}, so formally we define for each fixed $t \ge 0$, $\phi^{*,t} : B^* \times [0,t] \to [0,\infty)$ such that $\phi^{*,t}(\cdot, u) = \phi^*(\cdot)$ for each $u \in [0,t]$. Then $\langle \phi^{*,t}, X^{t,B^*}\rangle$ is defined in the obvious way. Now define

$$M_t^{\phi^*} := e^{-\lambda^* t}\big\langle \phi^{*,t}, X^{t,B^*}\big\rangle / \big\langle \phi^*, \mu\big\rangle.$$

Since $\lambda^* > 0$, Lemma 6 in [14] implies that $M_t^{\phi^*}$ is a continuous $\mathbb{P}_\mu$-martingale with unit mean, and that $P_\mu(\lim_{t\to\infty} M_t > 0) > 0$. Since $\phi^* \ge 1/c > 0$ on B^*, we have

$$X_t\big(B^*\big) \ge c\big\langle \phi^*|_{B^*}, X_t\big\rangle \ge c\big\langle \phi^{*,t}, X^{t,B^*}\big\rangle, \quad \mathbb{P}_\mu\text{-a.s.}$$

Hence

$$\begin{aligned}\mathbb{P}_\mu\Big(\lim_{t\to\infty} e^{-\lambda t} X_t\big(B^*\big) = \infty\Big) &\ge \mathbb{P}_\mu\Big(\liminf_{t\to\infty} e^{-\lambda^* t} X_t\big(B^*\big) > 0\Big) \\ &\ge \mathbb{P}_\mu\Big(\lim_{t\to\infty} M_t > 0\Big) > 0.\end{aligned}$$

Now let B be *any* open set with $\varnothing \ne B \Subset D$. Then (4.1) follows exactly as in the end of the proof of Theorem 3(ii) in [14], on p. 93. □

The rest of the paper is devoted to the investigation of the super-exponential growth rate as well as the spread for certain superprocesses with infinite generalized principal eigenvalues.

[8] See [11] for more on the exit measure.

5. Conditions and examples

5.1. *Brownian motion with $|x|^\ell$ potential*

For the next example, we will need the following result.

Lemma 5.1. *Let B denote standard Brownian motion in $\mathbb{R}^d$ with $d \geq 1$ and let $\ell > 0$. Then there is a constant $c_{\ell,d} > 0$ so that*

$$\log \mathbb{P}\left(\int_0^1 |B_s|^\ell \, \mathrm{d}s \geq K\right) = -\frac{1}{2} c_{\ell,d} K^{2/\ell}\bigl(1+o(1)\bigr), \tag{5.1}$$

as $K \uparrow \infty$. Furthermore, $c_{1,1} = 3$.

Proof. First, the asymptotics (5.1) follows directly by taking $\varepsilon = K^{-2/\ell}$ in Schilder's Theorem (Theorem 5.2.3 in [8]) and using the Contraction Principle (Theorem 4.2.1 in [8]). We then get

$$c_{\ell,d} = \inf\left\{\int_0^1 \bigl|\dot{f}(s)\bigr|^2 \, \mathrm{d}s : f \in C\bigl([0,1], \mathbb{R}^d\bigr) \text{ with } f(0) = 0 \text{ and } \|f\|_\ell = 1\right\},$$

where $\|f\|_\ell := (\int_0^1 |f(s)|^\ell \, \mathrm{d}s)^{1/\ell}$. To determine the value of $c_{1,1}$, one can utilize the results in [4,5]: by taking $p := 2$ and $p' := \frac{1}{1-\frac{1}{p}} = 2$ in [5, p. 2311, line −8] and exploiting formula (1.7) there to show that $c_{1,1} = 3$. □

Remark 5.2. One can actually get a crude upper estimate for all $\ell > 0$ without using Schilder's Theorem but using the reflection principle for Brownian motion instead. For simplicity, we illustrate this for $d = 1$. Let $R_t := \max_{s\in[0,t]} |B_s|$. Then

$$\mathbb{P}\left(\int_0^1 |B_s|^\ell \, \mathrm{d}s \geq K\right) \leq \mathbb{P}\bigl(R_1^\ell \geq K\bigr) \leq 4\mathbb{P}\bigl(B_1 \geq K^{1/\ell}\bigr) \leq \frac{4}{K} e^{-\frac{1}{2}K^{2/\ell}}.$$

See, e.g., [9, Theorem 1.2.3] for the last inequality.

Example 5.3. Let $d \geq 1$ and $L = \frac{1}{2}\Delta$, $\beta(x) = a|x|^\ell$ with $a, \ell > 0$, and let $\alpha > 0$ be spatially constant. From Lemma 2.1, it is clear that (2.1) will not hold, no matter how slowly β grows. On the other hand, letting $h \equiv 1$, we have the following claim.

Claim 5.4. *There are three cases.*

(i) *If $0 < \ell < 2$, then $T_t^{\frac{1}{2}\Delta+\beta} 1(\cdot) < \infty$ for every $t > 0$.*
(ii) *If $\ell = 2$, then there is some function $t_0 = t_0(x)$ on $\mathbb{R}^d$ that is bounded between two positive constants so that $T_t^{\frac{1}{2}\Delta+\beta} 1(x) < \infty$ for every $t < t_0(x)$ and $T_t^{\frac{1}{2}\Delta+\beta} 1(x) \equiv \infty$ for every $t > t_0(x)$.*
(iii) *If $\ell > 2$, then $T_t^{\frac{1}{2}\Delta+\beta} 1 \equiv \infty$ for every $t > 0$.*

Consequently, when $0 < \ell < 2$, not only the construction of the superprocess is guaranteed by Theorem 3.1, but in fact that *the expected total mass remains finite for all times* for a compactly supported initial measure.

Proof of Claim 5.4. Under $\mathbb{P}_0$, by Brownian scaling, we have

$$\int_0^t |B_s|^\ell \, \mathrm{d}s = \int_0^1 |B_{tr}|^\ell t \, \mathrm{d}r \stackrel{d}{=} t^{1+\ell/2} \int_0^1 |B_r|^\ell \, \mathrm{d}r.$$

Hence we have from above and (5.1) that

$$\begin{aligned}\bigl(T_t^{\frac{1}{2}\Delta+\beta} 1\bigr)(0) &= \mathbb{E}_0\left[\exp\left(a \int_0^t |B_s|^\ell \, \mathrm{d}s\right)\right] \\ &= \int_1^\infty \mathbb{P}_0\bigl(e^{a\int_0^t |B_s|^\ell \, \mathrm{d}s} > x\bigr) \, \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \int_1^\infty \mathbb{P}_0\left(\int_0^t |B_s|^\ell \,\mathrm{d}s > (\log x)/a\right) \mathrm{d}x \\
&= \int_1^\infty \mathbb{P}_0\left(\int_0^1 |B_s|^\ell \,\mathrm{d}s > a^{-1} t^{-1-\ell/2} \log x\right) \mathrm{d}x \\
&= \int_0^\infty a t^{1+\ell/2} e^{aut^{1+\ell/2}} \mathbb{P}_0\left(\int_0^1 |B_s|^\ell \,\mathrm{d}s > u\right) \mathrm{d}u \\
&= \int_0^\infty a t^{1+\ell/2} e^{aut^{1+\ell/2}} \left(e^{-\frac{1}{2}c_\ell u^{2/\ell}(1+o(1))}\right) \mathrm{d}u.
\end{aligned}
\tag{5.2}
$$

The claims now clearly follow from the last integral expression.

For general $x \in \mathbb{R}^d$, observe that

$$
\left(T_t^{\frac{1}{2}\Delta+\beta} 1\right)(x) = \mathbb{E}_x\left[\exp\left(\int_0^t a|B_s|^\ell \,\mathrm{d}s\right)\right] = \mathbb{E}_0\left[\exp\left(\int_0^t a|x+B_s|^\ell \,\mathrm{d}s\right)\right],
$$

which is bounded between c_t and C_t, where

$$
\begin{aligned}
c_t &:= e^{-a|x|^\ell} \mathbb{E}_0\left[\exp\left(\int_0^t 2^{-\ell} a|x+B_s|^\ell \,\mathrm{d}s\right)\right]; \\
C_t &:= e^{2^\ell a|x|^\ell} \mathbb{E}_0\left[\exp\left(\int_0^t 2^\ell a|B_s|^\ell \,\mathrm{d}s\right)\right].
\end{aligned}
$$

The claim is thus proved. □

Remark 5.5. The statements of Claim 5.4 can be found in Sections 5.12–5.13 of [24], but since they follow very easily from Lemma 5.1 (which we need later anyway), we decided to present the above proof for the sake of being more self-contained.

When $\ell = 1$ we have the following estimate, which will be used later, in Example 5.22.

Claim 5.6. *Assume that $d = 1$ and $\beta(x) = |x|$. Then*

$$
e^{t^3/6} \le E_0|X_t| = \left(T_t^{\frac{1}{2}\Delta+\beta} 1\right)(0) = \mathbb{E}_0 \exp\left(\int_0^t |B_s| \,\mathrm{d}s\right) \le 4e^{t^3/2}. \tag{5.3}
$$

Proof. Recall that $R_t := \max_{s\le t} |B_s|$. By the symmetry and the reflection principle for Brownian motion,

$$
\mathbb{P}_0(R_t > x) \le 2\mathbb{P}_0\left(\max_{s\in[0,t]} B_s > x\right) = 4\mathbb{P}_0(B_t > x) \quad \text{for every } x > 0.
$$

Hence

$$
\begin{aligned}
\mathbb{E}_0 \exp\left(\int_0^t |B_s| \,\mathrm{d}s\right) &= \int_0^\infty \mathbb{P}_0\left(\exp\left(\int_0^t |B_s| \,\mathrm{d}s\right) > x\right) \mathrm{d}x \\
&= \int_0^\infty \mathbb{P}_0\left(\int_0^t |B_s| \,\mathrm{d}s > \log x\right) \mathrm{d}x \\
&\le 1 + \int_1^\infty \mathbb{P}_0(tR_t > \log x) \,\mathrm{d}x \\
&\le 1 + 4\int_1^\infty \mathbb{P}_0(tB_t > \log x) \,\mathrm{d}x \\
&= 1 + 4\int_1^\infty \mathbb{P}_0\left(e^{tB_t} > x\right) \mathrm{d}x \\
&\le 4\int_0^\infty \mathbb{P}_0\left(e^{tB_t} > x\right) \mathrm{d}x = 4\mathbb{E}_0 e^{tB_t} = 4e^{t^3/2}.
\end{aligned}
$$

Here in the last inequality we used the fact that $\mathbb{P}_0(B_t \geq 0) = 1/2$ and so $\mathbb{P}_x(e^{tB_t} > x) \geq 1/2$ for every $0 < x < 1$. For the lower bound, note that by Itô's formula,

$$\int_0^t B_s \,\mathrm{d}s = tB_t - \int_0^t s \,\mathrm{d}B_s = \int_0^t (t-s)\,\mathrm{d}B_s,$$

which is of centered Gaussian distribution with variance $t^3/3$. Hence

$$\mathbb{E}_0 \exp\left(\int_0^t |B_s|\,\mathrm{d}s\right) \geq \mathbb{E}_0 \exp\left(\int_0^t B_s\,\mathrm{d}s\right) = e^{t^3/6},$$

proving the claim. $\square$

Example 5.7. Let $L = \frac{1}{2}\Delta$, $\beta(x) = |x|^2$, and $\alpha \geq \beta$. We can define the superprocess even in this case, using an argument involving a *discrete* branching particle system as follows.

As noted in the proof of Theorem 3.1, one only needs that $S = \{S_t\}_{t\geq 0}$ satisfies the semigroup property (3.7) on the space $C_b^+(D)$ along with condition (3.6). We do not need to use H-transform in this case.

Strategy: to check these, along with the well-posedness of the nonlinear initial value problem, we will have the following ingredients.

1. We prove that the initial value problem has a solution (no blow-up) by using an approximation and by analytic arguments; the semigroup property will be a consequence of the minimality of the solution, which follows from the approximation procedure.
2. We will also consider the d-dimensional branching Brownian motion $(Z, \mathbf{P})$ with branching rate $\beta(x) = |x|^2$ (we show below that Z is well defined). The reason we bring Z into the discussion is that Z is related to the same semilinear elliptic operator as the superprocess we wish to define. Then condition (3.6) will easily follow from the probabilistic representation (5.4) of u.

Let us now carry out this program.

It is standard to show that $(Z, \mathbf{P})$ satisfies the following log-Laplace equation for a bounded measurable function $g \geq 0$:

$$\mathbf{E}_x e^{\langle -g, Z_t\rangle} = 1 - u(x,t), \tag{5.4}$$

where u is the minimal nonnegative solution to the initial value problem (the so-called 'FKPP-equation')

$$\begin{cases} \dot{u} = Lu + \beta u - \beta u^2, \\ \lim_{t\downarrow 0} u(\cdot, t) = 1 - e^{-g}(\cdot). \end{cases} \tag{5.5}$$

Here we have the advantage that we know a priori that Z 'does not blow up', that is $|Z_t| < \infty$ for all $t > 0$, a.s., although $|Z_t|$ has infinite expectation. This follows from (ii) of Claim 5.4. Indeed, write $(\mathbf{E}_x; x \in \mathbb{R}^d)$ for the expectation corresponding to Z. Then $\mathbf{E}_x|Z_t| < \infty$ for all $x \in \mathbb{R}^d$, if t is sufficiently small. But then, by the branching Markov property, $|Z_t| < \infty$ for *all* times, P_x-a.s. (Cf. [23].)

Of course, the fact that (5.5) has a solution does not require any discussion about Z. It is checked as follows. One approximates $\mathbb{R}^d$ by an increasing sequence of compact domains D_n, and for each n, considers the initial value problem (5.5), but on D_n instead of $\mathbb{R}^d$, and with zero boundary condition. Using Proposition 2.5, it follows that the solutions are increasing as n grows, and that their limit stays finite as $n \to \infty$, by comparison with the constant one function. It also follows by Proposition 2.5 that the limiting function is the *minimal* nonnegative solution. (To see that the limit is actually a solution, see Appendix B in [15] or [16].)

We have concluded that, when the initial function is bounded from above by one (as in the case above with the initial data $1 - e^{-g}$), the solution does not blow up. In fact, the same argument, using Proposition 2.5 shows that this is true if we replace $1 - e^{-g}$ by any bounded measurable function $g^* \geq 0$. Indeed, for $K > 1$, the function $h \equiv K$ is a super-solution if $g^* \leq K$. This argument is obviously still valid if the operator $u \mapsto Lu + \beta u - \beta u^2$ is replaced by $u \mapsto Lu + \beta u - \alpha u^2$, provided $\alpha \geq \beta$. Therefore, in this case the initial value problem is well-posed and can be considered the cumulant equation for the *superprocess*.

As mentioned at the beginning, to define the superprocess via the log-Laplace equation using minimal nonnegative solutions, we have to check two conditions. It is easy to see that (3.7) is a consequence of the minimality of the solution; as for condition (3.6), this is the point where we need Z in the argument: by (5.4), condition (3.6) follows by monotone convergence when $\beta = \alpha$; when $\alpha \geq \beta$, we are done by using Proposition 2.5.

Remark 5.8. Here we give a few comments regarding the significance of the condition $\alpha \ge \beta$.

(i) The argument in Example 5.7 shows that, in general, whenever the branching diffusion with L-motion on D and branching β is well defined and finite at all times, the $(L, \beta, \alpha; D)$-superdiffusion is also well defined and $\mathcal{M}_f(D)$-valued, provided that ($\alpha > 0$ and) $\alpha \ge \beta$.

(ii) In fact, here and in the sequel, the condition $\alpha \ge \beta$, whenever is assumed, can always be replaced by the assumption that $\alpha \ge \varepsilon\beta$ with some $\varepsilon > 0$. This is because an h-transform (re-weighting) of the superprocess with $h \equiv \varepsilon$ changes α to $\varepsilon\alpha$ but leaves L, β intact, while the mass (i.e. the measure X_t) has simply been multiplied by ε.

(iii) When $\alpha \ge \varepsilon\beta$ is not satisfied and the 'activity parameter' α is very small for large $|x|$ or for x close to ∂D, it is completely possible that the compact support property, invoked in the next subsection, breaks down [17]. In this case one can no longer talk about the spread of the superprocess.

The break down of the compact support property is a manifestation of the Law of Large Numbers, given the presence of 'infinitely many particles,' which holds if the variance (hence α) is small enough; it is actually the compact support property itself that can be considered 'pathological' i.e. a violation of LLN. This is explained in detail in [17].

As far as the growth of the superprocess is considered, we used the compact support property a number of times; these techniques also break down for small α. Intuitively, the instantaneous spread of the process, combined with a spatially fast growing mass creation term β may dramatically change the results on the growth of the total mass.

5.2. *The compact support property and an example*

Recall that X possesses the *compact support property* if $P(C_s \Subset D) = 1$ for all fixed $s \ge 0$, where

$$C_s(\omega) := \mathsf{closure}\left(\bigcup_{r \le s} \mathrm{supp}\big(X_r(\omega)\big)\right).$$

In this case, by the monotonicity in s, there exists an $\Omega_1 \subset \Omega$ with $P(\Omega_1) = 1$ such that for $\omega \in \Omega_1$,

$$C_s(\omega) \Subset D \quad \text{for every } s \ge 0. \tag{5.6}$$

It is easy to see that the criterion in [15] (see Theorem 3.4 and its proof in [15]) carries through for our more general superprocesses, that is,

Proposition 5.9 (Analytic criterion for CSP). *The compact support property holds if and only if the only non-negative function u satisfying*

$$\begin{cases} \dot{u} = Lu + \beta u - \alpha u^2, \\ \lim_{t \downarrow 0} u(\cdot, t) = 0, \end{cases} \tag{5.7}$$

is $u \equiv 0$; equivalently, if and only if $u_{\max}$, the maximal solution to (5.7) is identically zero.

We now apply this analytic criterion to a class of superdiffusions.

Claim 5.10. *Assume that L is conservative on D, that $T_t^{L+\beta}(1)(\cdot) < \infty$ and that $\alpha \ge \beta$. In addition, assume that the compact support property holds for the $(\beta^{-1}L, 1, 1; D)$-superprocess. Then the compact support property holds for X, the $(L, \beta, \beta; D)$-superprocess as well.*

Remark 5.11. Let X be the $(L, \beta, \beta; D)$-superprocess.

(i) Our assumption on $T^{L+\beta}$ guarantees that X is well defined. For example, by Claim 5.4 this assumption is satisfied when $L = \frac{1}{2}\Delta$ on $D = \mathbb{R}^d$ and $\beta(x) = |x|^p$, $0 < p < 2$; the same is true of course for $\beta(x) = C + |x|^p$, $C > 0$.

(ii) The last condition in Claim 5.10 is always satisfied as long as the coefficients of $\beta^{-1}L$ are sufficiently slowly growing. For example, when $D = \mathbb{R}^d$, it is enough, loosely speaking, that the growth (as $|x| \to \infty$) of the diffusion matrix is not more than quadratic, and the growth of the drift vector is not more than linear. (See Theorem EP2 in [17] for the precise statement.)

(iii) We also mention that by Theorem 3.6(i) of [15], if L is non-conservative on D and $\inf_D(\beta/\alpha) > 0$, then the compact support property fails for X.

Proof of Claim 5.10. By Propositions 2.5 and 5.9, it is enough to consider the case when $\alpha = \beta$, and show that $u_{\max}$ for (5.7) is identically zero.

We start with showing that $u_{\max} \le 1$. We are going to use several facts from Appendix B, so the reader should consult that appendix when reading this proof.

Note that the maximal nonnegative solution to the steady-state equation $Lu + \beta(u - u^2) = 0$ on D, denoted by $w_{\max}$, obviously coincides with the maximal nonnegative solution to $\beta^{-1}Lu + u - u^2 = 0$ on D. By assumption, the compact support property holds for the $(\beta^{-1}L, 1, 1; D)$-superprocess, hence $w_{\max} = w_{\text{ext}}$ for this latter superprocess. But $w_{\text{ext}} \le 1$ in this case and thus $w_{\max} \le 1$. By construction (or by the probabilistic representation), $u_{\max} \le w_{\max}$, hence $u_{\max} \le 1$.

In accordance with Remark 1.2 about the significance of the case $\alpha = \beta$, we are going to utilize a *discrete particle system*, just like in Example 5.7. Namely, consider the $(L, \beta; D)$-branching diffusion Z, and let $\{\mathbf{P}_x, \mathbf{E}_x x \in D\}$ denote the corresponding probabilities and expectations. Let $\hat{C}_s$ be defined similarly to C_s above (in (5.6)) but for Z in place of X, that is, $\hat{C}_s$ is the accumulated support of Z up to s. We then claim that $u_{\max}(x, t) = 1 - \mathbf{P}_x(\hat{C}_t \Subset D)$. Indeed, if the domains $D_n, n \ge 1$ satisfy $D_n \uparrow$ and $\bigcup_n D_n = D$, then (see Chapter 3 in [11])

$$\begin{aligned}\mathbf{P}_x(\hat{C}_t \Subset D) &= \lim_n \lim_m \mathbf{E}_x \exp\{\langle -m\mathbf{1}_{\partial D_n}, Z_t^{D_n}\rangle\}\\ &= 1 - \lim_n \lim_m u_{m,n}(x, t) =: 1 - u^*(x, t),\end{aligned}$$

where $Z_t^{D_n}$ is the exit measure on D_n up to t, and $u_{m,n}$ is the minimal nonnegative solution to

$$\begin{cases}\dot{u} = Lu + \beta(u - u^2), \quad t > 0, x \in D_n;\\ \lim_{x\in D_n, x\to\partial D_n} u(x, t) = 1 - e^{-m}, \quad t > 0;\\ \lim_{t\downarrow 0} u(x, t) = 1 - \exp(-m\mathbf{1}_{D_n^c}(x)), \quad x \in D_n.\end{cases} \tag{5.8}$$

The existence of u^* follows from monotonicity (using the maximum principle, or the obvious monotonicity for the lefthand side). That u^* solves (5.7) with $\beta = \alpha$, follows from standard PDE arguments.

Finally we verify maximality. Suppose that $0 \le v$ is another solution to (5.7) with $\beta = \alpha$. Since $u_{\max} \le 1$, it follows that $v \le 1$. Then $u^* \ge v$, because this is true on $D_n \times [0, \infty)$ (by the semilinear maximum principle and the fact that $v \le 1$, and by letting $m \to \infty$) for each $n \ge 1$. Hence $u^* = u_{\max}$.

Thus, we need to show that

$$\mathbf{P}_x(\hat{C}_t \Subset D) = 1 \quad \text{for all } x \in D \text{ and } t \ge 0.$$

Since we are dealing with a discrete system, this follows from the assumption that the underlying motion is conservative (that is, that particles never leave D) and from the fact that there are only finitely many particles around, i.e. $\mathbf{P}_x(|Z_t| < \infty) = 1$. The latter follows from the expectation formula, as we even have $\mathbf{E}_x(|Z_t|) = T_t^{L+\beta}(1)(x) < \infty$ by assumption. □

For super-Brownian motion with quadratic mass creation we still have the compact support property.

Claim 5.12 (CSP for quadratic mass creation). *Let $L = \frac{1}{2}\Delta$ on $D = \mathbb{R}^d$ and $\alpha(x) \ge \beta(x) > 0$ with $\beta(x) = |x|^2$ for $|x| \ge \varepsilon > 0$. Then the compact support property holds for X.*

Proof. We now show how to modify the proof of Claim 5.10 in this case. That the $(\frac{1}{2}\beta^{-1}\Delta, 1, 1; \mathbb{R}^d)$-superprocess satisfies the compact support property, follows from the fact that the coefficients of $\beta^{-1}\Delta$ stay bounded as $|x| \to \infty$. (See Theorem EP2 in [17].)

Even though, by Claim 5.4, the assumption of Claim 5.10 on the semigroup no longer holds, we know that the superprocess is well defined, as shown in Example 5.7. Furthermore, for the corresponding branching-Brownian motion, $\mathbf{P}_x(|Z_t| < \infty) = 1$ is still true – see [23].

The rest of the proof is exactly the same as in the case of Claim 5.10. □

5.3. *Semi-orbits*

In this part we discuss a method which is applicable in the absence of positive harmonic functions too. In this part, the assumption on the power of the nonlinearity (quadratic in (5.7)) is important as we are using the path continuity (in the weak topology of measures).

(I) ASSUME $\lambda_c < \infty$.

The almost sure upper estimate on the local growth is then based on the existence of positive harmonic functions. Indeed, let h be a positive harmonic function, that is, let $(L+\beta-\lambda_c)h = 0$, $h > 0$. (Such a function h always exists; see Chapter 4 in [25].) Define $H(x,t) := e^{-\lambda_c t}h(x)$; then for $t, s > 0$,

$$\big(T_t^{L+\beta} H(\cdot, t+s)\big)(x) \le H(x,s), \tag{5.9}$$

that is, $T_t^{L+\beta-\lambda_c} h \le h$, or equivalently, $T_t^{(L+\beta-\lambda_c)^h} 1 \le 1$. Here

$$\big\{T_t^{(L+\beta-\lambda_c)^h}; t \ge 0\big\}$$

is the semigroup obtained from $\{T_t^{L+\beta-\lambda_c}; t \ge 0\}$ through an h-transform.

Using the Markov and the branching properties together with h-transform theory, it then immediately follows that if $N_t := \langle H(\cdot,t), X_t\rangle$, then N is a continuous P_μ-supermartingale for $\mu \in \mathcal{M}_f(D)$ (where P_μ is the law of X with $X_0 = \mu$). Indeed, the fact that N is finite and has continuous paths follows since

$$N_t := e^{-\lambda_c t}\langle h, X_t\rangle = e^{-\lambda_c t}\big\langle 1, X_t^h\big\rangle,$$

where X^h is the $(L_0^h, \lambda_c, \alpha h; D)$-superdiffusion (see Lemma 3.5 and the comment following it) with continuous total mass process. Moreover,

$$\begin{aligned} E_\mu(N_t \mid \mathcal{F}_s) &= E_\mu(N_t \mid X_s) = E_{X_s} N_t = E_{X_s}\big\langle H(\cdot,t), X_t\big\rangle \\ &= \int_D E_{\delta_x}\big\langle H(\cdot,t), X_{t-s}\big\rangle X_s(\mathrm{d}s) = \int_D \big(T_{t-s}^{L+\beta} H(\cdot,t)\big)(x) X_s(\mathrm{d}x) \\ &\le \int_D H(x,s) X_s(\mathrm{d}x) = \big\langle H(\cdot,s), X_s\big\rangle = N_s. \end{aligned}$$

The above analysis also shows that if $(L+\beta-\lambda_c)^h$ is conservative, that is, if $T^{(L+\beta-\lambda_c)^h} 1 = 1$, then N is a continuous P_μ-martingale, as the inequality in the previous displayed formula becomes an equality.

The continuous non-negative supermartingale $N = \{N_t\}_{t\ge 0}$ has an almost sure limit N_∞ as $t \to \infty$. Note also that $N_t = \langle H(\cdot,t), X_t\rangle = e^{-\lambda_c t}\langle 1, hX_t\rangle$ and $h > 0$ is C^2 on D. Since for every $B \Subset D$, $\mathbf{1}_B \le ch$ with some $c > 0$, it follows that the local growth is $\mathcal{O}(e^{\lambda_c t})$; that is, for every $B \Subset D$,

$$X_t(B) = \mathcal{O}\big(e^{\lambda_c t}\big) \quad \text{a.s.}$$

(II) ASSUME $\lambda_c = \infty$.

In this case, there is no C^2-function $h > 0$ such that $(L+\beta-\lambda)h \le 0$ for some $\lambda \in \mathbb{R}$; see again [25, Chapter 4]. Can one still get an a.s. upper estimate for the local growth?

Assume that for some smooth positive space-time function F, inequality (5.9) holds with F in place of H there; that is, denoting

$$f^{(-t)}(\cdot) := F(\cdot, t),$$

we make the following assumption.

Assumption A. There exists a family $\{f^{(-t)}; t \ge 0\}$ of smooth positive functions, satisfying

$$T_t^{L+\beta} f^{(-t-s)} \le f^{(-s)}.$$

By smoothness we mean that $f^{(-t)}$ is a continuous spatial function for $t \ge 0$, and $t \mapsto f^{(-t)}(x)$ is continuous, uniformly on bounded spatial domains, at any $t_0 \ge 0$.

Remark 5.13. Note that, when $\lambda_c < \infty$, Assumption A holds with $f^{(-t)}(\cdot) := e^{-\lambda_c t}h(\cdot)$, where h is as before.

As we have seen, Assumption A implies the important property that for $N_t := \langle f^{(-t)}, X_t\rangle \ge 0$, N is a $\mathbb{P}_\mu$-supermartingale. In order to conclude that it has an almost sure limit, we make a short detour and investigate the continuity of this supermartingale.

Lemma 5.14. *Let $\{\mu_t, t \geq 0\}$ be a family in $\mathcal{M}_f(D)$ satisfying that $t \mapsto |\mu_t|$ is locally bounded, and assume that $t_0 > 0$ and $\mu_t \overset{v}{\Rightarrow} \mu_{t_0}$ as $t \to t_0$. Assume furthermore that*

$$C = C_{t_0,\varepsilon} := \text{closure}\left(\bigcup_{t=t_0-\varepsilon}^{t_0+\varepsilon} \operatorname{supp}(\mu_t)\right) \Subset D$$

with some $\varepsilon > 0$. Let $H : D \times \mathbb{R}_+ \to \mathbb{R}$ be a function that is continuous in $x \in D$ and continuous in time at t_0, uniformly on bounded spatial domains. Then $\lim_{t\to t_0}\langle H(\cdot, t), \mu_t\rangle = \langle H(\cdot, t_0), \mu_{t_0}\rangle$.

Proof. Using Urysohn's Lemma, there exists a continuous function $g : D \to \mathbb{R}$ such that $g(\cdot) = H(\cdot, t_0)$ on C and $g = 0$ on $D \setminus D_1$, where $C \Subset D_1 \Subset D$. Then,

$$\lim_{t\to t_0}\big\langle H(\cdot, t_0), \mu_t\big\rangle = \lim_{t\to t_0}\langle g, \mu_t\rangle = \langle g, \mu_{t_0}\rangle = \big\langle H(\cdot, t_0), \mu_{t_0}\big\rangle,$$

since $g \in C_c(D)$. Also, by the assumptions on μ and H, for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, one has

$$\big|\big\langle H(\cdot, t_0) - H(\cdot, t), \mu_t\big\rangle\big| \leq \sup_{x\in C}\big|H(x, t) - H(x, t_0)\big| \sup_{t\in(t_0-\varepsilon,t_0+\varepsilon)} |\mu_t|,$$

which tends to zero as $t \to t_0$. □

Recall that β is locally bounded and the branching is quadratic. We now need a path regularity result for superprocesses.

Claim 5.15 (Continuity of X). *Let $\mu \in \mathcal{M}_c(D)$. If the compact support property holds, then (X, P_μ) has an $\mathcal{M}_f(D)$-valued, continuous version. (Here continuity is meant in the weak topology of measures.)*

Note. In the sequel, we will work with a weakly continuous version of the superprocess whenever the compact support property holds.

Proof. Recall the definition of Ω_1 from (5.6); by the compact support property, we can in fact work on Ω_1 instead of Ω. Pick a sequence of domains $\{D_n\}_{n\geq 1}$ satisfying that $D_n \uparrow D$ and $D_n \Subset D$ for all $n \in \mathbb{N}$. Define

$$\tau_n := \inf\big\{t \geq 0 \mid X_t\big(D_n^c\big) > 0\big\},$$

and let $\mathcal{F}_{\tau_n}$ denote the σ-algebra up to τ_n, that is,

$$\mathcal{F}_{\tau_n} := \big\{A \subset \Omega_1 \mid A \cap \{\tau_n \leq t\} \in \mathcal{F}_t, \forall t \geq 0\big\}.$$

Let $X_t^{D_n}$ denote the exit measure from $D_n \times [0, t)$, which is a (random) measure on $(\partial D_n \times (0, t)) \cup (D_n \times \{t\})$. Since the coefficients are locally bounded, for any fixed $n \geq 1$, $t \to X_t^{D_n}$ has an $\mathcal{M}_f(D)$)-valued, weakly continuous version $t \to \widehat{X}_t^{D_n}$. If $P^{(n)}$ denotes their common distribution, then

$$P|_{\mathcal{F}_{\tau_n}} = P^{(n)}|_{\mathcal{F}_{\tau_n}}. \tag{5.10}$$

Let $\Omega^* := C([0, \infty), \mathcal{M}_f(D))$ be the space of weakly continuous functions from $[0, \infty)$ to $\mathcal{M}_f(D)$ and let $\mathcal{F}^*$ denote the Borels of Ω^*. By the definition of Ω_1,

$$\lim_{n\to\infty} \tau_n(\omega) = \infty, \quad \forall \omega \in \Omega_1, \tag{5.11}$$

and thus, it is standard to show that the measures-valued processes

$$\big\{\widehat{X}_t^{D_n}, t \in [0, \tau_n)\big\}_{n\geq 1}$$

with distributions $(P^{(n)}, \Omega^*, \mathcal{F}_{\tau_n}), n \geq 1$ have an extension to a process $(X_t^*, t \in [0, \infty))$ with distribution $(P^*, \Omega^*, \mathcal{F}^*)$. Since P^* is uniquely determined on the Borels of $\mathcal{M}_f(D)^{[0,\infty)}$ by the distributions $(P^{(n)}, \Omega^*, \mathcal{F}_{\tau_n}), n \geq 1$, (5.10) implies that $P^* = P$ on the Borels of $\mathcal{M}_f(D)^{[0,\infty)}$. Hence X^* is a weakly continuous version of X. □

Now it is easy to see that the supermartingale N has a continuous version: let us define a version of N using a weakly continuous version of X. By Assumption A, and letting $\mu_t = X_t(\omega)$, Lemma 5.14 implies the continuity of $N(\omega, t)$ at $\omega \in \Omega_1$, $t_0 > 0$. Then, since N is a continuous nonnegative supermartingale, we conclude that it has an almost sure limit.

In summary, since $\mathbf{1}_B \le \left(\sup_B \frac{1}{f^{(-t)}}\right) f^{(-t)}$, $B \Subset D$, we have obtained

Lemma 5.16 (Almost sure upper bound with f). *Under Assumption A and assuming the compact support property* (*or just the existence of continuous finite measure-valued trajectories*), *one has that almost surely, as $t \to \infty$,*

$$X_t(B) = \mathcal{O}\left(\sup_{x\in B} \frac{1}{f^{(-t)}(x)}\right), \quad \forall B \Subset D. \tag{5.12}$$

In particular, the martingale property would follow if we knew that for an appropriate $f \in C^+(D)$, the semi-orbit $t \mapsto T_t^{L+\beta}(f)$ can be extended from $[0,\infty)$ to $(-\infty,\infty)$. Indeed, we could then define

$$f^{(-t)}(x) = H(\cdot, t) := T_{-t}^{L+\beta}(f)(\cdot),$$

which implies the statement in Assumption A with equality. Hence, in this case, the local growth can be estimated from above as follows. Let $B \Subset D$ be nonempty and open. Then

$$N_t = \langle H(\cdot,t), X_t\rangle \ge \langle H(\cdot,t)\mathbf{1}_B, X_t\rangle \ge \inf_{x\in B} H(x,t) X_t(B). \tag{5.13}$$

Since N_t has an almost sure limit, therefore

$$X_t(B) = \mathcal{O}\left(\sup_{x\in B}\frac{1}{H(x,t)}\right) = \mathcal{O}\left(\sup_{x\in B}\frac{1}{T_{-t}^{L+\beta}(f)(x)}\right) \quad \text{a.s.}$$

Remark 5.17. It is of independent interest, that, using (5.13) one can always estimate the semigroup from above as follows:

$$(T_t\mathbf{1}_B)(x) = E_x X_t(B) \le \sup_{y\in B} H^{-1}(y,t)\cdot \big(T_t\big(H(\cdot,t)\big)\big)(x) = \sup_{y\in B} H^{-1}(y,t)\cdot f(x),$$

where H is as before.

5.4. *The 'p-generalized principal eigenvalue' and a sufficient condition*

The discussion in the previous subsection gives rise to the following questions:

(1) When is Assumption A satisfied?
(2) When can the semi-orbit $t \mapsto T_t^{L+\beta}(f)$ be extended?

We will focus on the first question. For simplicity, use the shorthand $T_t := T_t^{L+\beta}$. Assume that ϑ is a continuous non-decreasing function on $[0,\infty)$, satisfying $\vartheta(0) = 0$ and

$$\vartheta(s+t) \le C\big[\vartheta(s) + \vartheta(t)\big], \quad s,t \ge 0, \tag{5.14}$$

with some $C > 1$ (depending on ϑ) and that $\gamma := e^{-\vartheta}$ satisfies for all $g \in C_c^+$ that

$$I_g(B) := \int_0^\infty \gamma(s)\|\mathbf{1}_B T_s g\|_\infty \,\mathrm{d}s < \infty \tag{5.15}$$

for every $B \Subset D$. Then Assumption A is satisfied as well, since, using the monotonicity of γ, (5.15) and dominated convergence, the family

$$\mathcal{G}_g := \left\{ f^{(-t)} := \int_0^\infty \gamma(s+t) T_s g \,\mathrm{d}s;\, t \ge 0\right\}$$

is continuous in t, uniformly on bounded spatial domains, and a trivial computation shows that $T_t f^{(-t-s)} \le f^{(-s)}$. Assume now that the compact support property holds for X. By (5.12), for a nonempty open $B \Subset D$,

$$X_t(B) = \mathcal{O}\left(\left[\inf_{x\in B}\int_0^\infty \gamma(s+t)(T_s g)(x)\,\mathrm{d}s\right]^{-1}\right) \quad \text{a.s.},$$

and so by (5.14), and by the fact that $C > 1$,

$$\begin{aligned} X_t(B) &= \mathcal{O}\left(\gamma(t)^{-C}\left[\inf_{x\in B}\int_0^\infty \gamma(s)^C(T_s g)(x)\,\mathrm{d}s\right]^{-1}\right) \\ &= \mathcal{O}\big(\gamma(t)^{-C}\big) = \mathcal{O}\big(e^{C\vartheta(t)}\big) \quad \text{a.s.} \end{aligned} \tag{5.16}$$

Consider now the particular case when $\vartheta(t) := \lambda t^p$ with $\lambda > 0$, $p \ge 1$ and assume that condition (5.15) holds: there exists a non-trivial $g \ge 0$ so that

$$f^{(0)}(B) := I_g(B) = \int_0^\infty e^{-\lambda s^p}\|\mathbf{1}_B T_s g\|_\infty\,\mathrm{d}s < \infty \quad \text{for every } B \Subset D. \tag{5.17}$$

Then, by convexity, $C = C_p = 2^{p-1}$ satisfies (5.14), and so, using (5.16), one has

$$X_t(B) = \mathcal{O}\big(\exp\big(2^{p-1}\lambda t^p\big)\big) \quad P_\mu\text{-a.s.} \tag{5.18}$$

If (5.17) holds with some $\lambda > 0$, $p \ge 1$ and a non-trivial $g \ge 0$, then we will say that the '*p-generalized principal eigenvalue*' of $L+\beta$, denoted by $\lambda_c^{(p)}$, is finite and $\lambda_p \le \lambda$. More formally, we make the following definition.

Definition 5.18 (p-generalized principal eigenvalue). For a given $p \ge 1$ we define the p-generalized principal eigenvalue of $L+\beta$ on D by

$$\begin{aligned} \lambda_c^{(p)} := \inf\Big\{&\lambda \in \mathbb{R} : \exists \mathbf{0} \ne g \in B_b^+(D) \text{ so that} \\ &\int_0^\infty e^{-\lambda s^p}\|\mathbf{1}_B T_s g\|_\infty\,\mathrm{d}s < \infty \text{ for every } B \Subset D\Big\}. \end{aligned}$$

For more on the p-generalized principal eigenvalue, see Appendix A.

Consider the case when $D = \mathbb{R}^d$, $L = \frac{1}{2}\Delta$, $\beta(x) = |x|^\ell$, $\alpha > 0$ and $0 < \ell < 2$. In Section A.1 we will show the following.

Proposition 5.19. *Let $M := (2+\ell)/(2-\ell) \in (1,\infty)$. Then, with $p = M$, $\lambda_c^{(p)} < \infty$ and in fact $\lambda_c^{(p)} \le e^{c_1 2^\ell}$, where $c_1 = c_1(\ell)$ is an explicit constant. Furthermore, the exponent $(2+\ell)/(2-\ell)$ is sharp when $\ell = 1$.*

Let us now reformulate (5.18) in terms of the p-generalized principal eigenvalue. Let X be as in Definition 3.2.

Theorem 5.20 (Local growth with pgpe). *Assume the compact support property for X, and that $\lambda_c^{(p)} < \infty$ with some $p \ge 1$. Then, for $B \Subset D$, $\varepsilon > 0$, and $\mu \in \mathcal{M}_c(D)$, one has, as $t \to \infty$, that*

$$X_t(B) = \mathcal{O}\big(\exp\big(\big(2^{p-1}\lambda_c^{(p)} + \varepsilon\big)t^p\big)\big) \quad P_\mu\text{-a.s.}$$

Remark 5.21. The assumption that the compact support property holds is technical in nature. We only need it to guarantee the continuity of N. In fact, we suspect that this assumption can be dropped in Theorem 5.20.

We now revisit a previous example.

Example 5.22 (The $(\frac{1}{2}\Delta, |x|, \alpha, \mathbb{R}^d)$-superprocess). Let $D = \mathbb{R}^d$, $L = \frac{1}{2}\Delta$, $\beta(x) = |x|$, and $\alpha > 0$, and note that the compact support property holds for this example. Although by Lemma 2.1, $\lambda_c = \infty$, using (5.3), and the estimates preceding it, it follows that $\lambda_c^{(3+\varepsilon)} \le 0$ for all $\varepsilon > 0$. Also, (5.15) is satisfied with any $\vartheta(t) = -t^3/2 - f(t)$ and $\alpha > 0$, provided $e^{-f(t)}$ is integrable. Let $K > 0$ and $\widehat{C} := \max\{4, K\}$. Using the inequality $(t+s)^3 \leq 4(t^3+s^3)$, one obtains the estimate

$$X_t(B) = \mathcal{O}\big(\exp\big[\widehat{C}\big(t^3/2 + f(t)\big)\big]\big), \quad P_\mu\text{-a.s.},$$

for $\mu \in \mathcal{M}_c(D)$, $B \Subset \mathbb{R}^d$ and for any function $f \geq 0$ satisfying

$$f(t+s) \leq K\big(f(t) + f(s)\big).$$

For example, taking $f(t) := \varepsilon t^r$, $\varepsilon > 0$, $0 < r < 1$, one obtains that for $B \Subset \mathbb{R}^d$,

$$X_t(B) = \mathcal{O}\big(\exp\big[2t^3 + \varepsilon' t^r\big]\big), \quad P_\mu\text{-a.s.}$$

We conclude with an open problem.

Problem 5.23. In Example 5.22, what is the exact order of $X_t(B)$ as $t \to \infty$, when $B \Subset \mathbb{R}^d$? We believe that t^3 is the correct order in the exponent, however it is not at all clear if our constant is close to being optimal.

Note that Corollary 1.4 answers this question for the *global* mass when $\beta = \alpha$. See also Corollary 1.5.

6. Poisson-coupling method for the growth rate and for spatial spread estimates

In this section we will study the superdiffusion corresponding to the operator $u \mapsto \frac{1}{2}\Delta u + \beta u - \alpha u^2$ on $\mathbb{R}$ with $\beta(x) = |x|^p$ for $p \in (0, 2]$, and study the precise growth rate for its total mass by using the method of 'Poisson-coupling.' An upper bound for the spatial spread when $\beta(x) = |x|^2$ will also be given.

6.1. *Remarks on Poisson-coupling*

We start with explaining the Poisson-coupling method due to Fleischmann and Swart [21]. Let (X, P) be the superprocess corresponding to the operator $u \mapsto Lu + \beta u - \beta u^2$ on $D \subset \mathbb{R}^d$ and (Z, P) the branching diffusion on D with branching rate $\beta(\cdot) > 0$.

The more elementary version of Poisson-coupling is the fact that for a given $t > 0$, the following two spatial point processes are equal in law:

(a) the spatial point process Z_t under P_x;
(b) a spatial Poisson point process (PPP) Z_t^* with the random intensity measure X_t, where X_t is the superprocess at time t under P_{δ_x}.

(See Lemma 1 and Remark 2 in [21].)

Remark 6.1 (The $\alpha \geq \beta$ case). Fleischmann and Swart do not assume $\alpha = \beta$, only that $\alpha \geq \beta$. Then, in general, they have to include a 'death (or killing) coefficient' $\alpha - \beta$ in the discrete branching process (see Lemma 1 and Remark 2 in [21]). In fact, because of the death term, it is easy to see by comparison that the upper bounds we obtain with the Poisson-coupling method, hold when $\alpha \geq \beta$.

The above coupling is not 'at the process level,' as it only matches the one-dimensional distributions. However, Fleischmann and Swart provided a coupling of X and Z *as processes* too in [21].

Convention. Let us now introduce the following notation for convenience: when we write P_0, it denotes the law of the process, starting with measure δ_0, in case of X, and the law starting with a $\mathsf{Poisson}(1)$ number of particles at the origin, in case of Z. In particular, Z is the 'empty process' with P_0-probability $1/e$. (E_0 is meant similarly.)

Fleischmann and Swart proved that the two processes can be coupled (i.e., can be defined on the same probability space) in such a way that (with the same P_0 because of the coupling)

$$\mathsf{P}_0\big[Z_t \in \cdot \mid (X_s)_{0\leq s\leq t}\big] = \mathsf{P}_0\big[\mathsf{Pois}(X_t) \in \cdot \mid X_t\big], \quad \text{a.s. } \forall t \geq 0, \tag{6.1}$$

where $\mathsf{Pois}(\mu)$ denotes the PPP with intensity μ for a finite measure μ. (See [21, formula (1.2)] and note that in our case, the function h appearing in the formula is identically one.) Formula (6.1) says that the conditional law of Z_t, given the history of X up to t, is the law of a PPP with intensity X_t. (In fact they prove an even stronger version, involving *historical processes* in [21, Theorem 6].)

Remark 6.2 (Why (6.1) is applicable in our setting). Fleischmann and Swart assumed that either D is compact, or it is just locally compact but α and β are bounded (see Remark 2 in [21]). Why is then (6.1) applicable to the superdiffusion X corresponding to the operator $u \mapsto \frac{1}{2}\Delta u + \beta u - \alpha u^2$ on $\mathbb{R}$ with $\beta(x) = |x|^p$ for $p \in (0, 2]$?

The proof of (6.1) goes through as long as the superprocess and the h-transform are well defined. We have seen (Remark 3.10) that the latter is no problem even when using our more general definition of the superprocess under Assumption 2.2 (this is the case in particular when $p \in (0, 2)$), and even when the mere existence of the superprocess is verified only, without having Assumption 2.2 in force ($p = 2$).

6.2. *Upgrading the Fleischmann–Swart coupling from deterministic times to stopping times*

We need to upgrade the coupling result to nonnegative, finite stopping times, as follows. Let $\mathcal{F}^X$ denote the canonical filtration of X, that is, let $\mathcal{F}^X := \{\mathcal{F}^X_t; t \geq 0\}$.

Theorem 6.3 (Enhanced coupling). *Given the Fleischmann–Swart coupling, it also holds that for an almost surely finite and nonnegative $\mathcal{F}^X$-stopping time T,*

$$\mathsf{P}_0\big[Z_T \in \cdot \mid (X_s)_{0 \leq s \leq T}\big] = \mathsf{P}_0\big[\mathsf{Pois}(X_T) \in \cdot \mid X_T\big], \quad a.s.$$

Remark 6.4. Note that

(1) The lefthand side is just another notation for $\mathsf{P}_0[Z_T \in \cdot \mid \mathcal{F}^X_T]$. Actually, as the proof below reveals, a slightly stronger result is also true: $\mathcal{F}^X_T$ can be replaced even by $\mathcal{F}^X_{T+}$.
(2) For the time of extinction of X, the result is not applicable. Indeed, using that $\alpha = \beta$, it is easy to show that for this T, we have $T = \infty$ with positive probability.

Proof. As usual, we will approximate T with a decreasing sequence of countable range stopping times.

We need the facts that, as measure-valued processes, both X and Z are right-continuous, and X is in fact continuous. We proved weak continuity for X, see Claims 24 and 26. For Z, right-continuity is elementary.

We now turn to the proof of the statement of the theorem. Following pp. 56–58 in [6], take a general nonnegative $\mathcal{F}^X$-stopping time T, and let

$$\mathbb{T} := \big\{k/2^m \mid k, m \geq 0\big\}$$

be the dyadic set. For $n \geq 1$, define the $\mathbb{T}$-valued $\mathcal{F}^X$-stopping time (in [6], ‘strictly optional’ is used instead of ‘stopping’)

$$T_n := \frac{\lfloor 2^n T \rfloor + 1}{2^n}.$$

Then $T_n \downarrow T$ uniformly in ω. In fact (see [6]),

$$\mathcal{F}^X_{T+} = \bigwedge_{n=1}^{\infty} \mathcal{F}^X_{T_n}, \tag{6.2}$$

where the righthand side is the intersection of the σ-algebras.

Fix $n \geq 1$. Since T_n has countable range,

$$\mathsf{P}_0\big[Z_{T_n} \in \cdot \mid (X_s)_{0 \leq s \leq T_n}\big] = \mathsf{P}_0\big[\mathsf{Pois}(X_{T_n}) \in \cdot \mid X_{T_n}\big], \quad \text{a.s.}$$

Indeed, using Laplace-transforms and the Campbell formula for PPP, this is equivalent to the assertion that for all bounded and continuous $f \geq 0$,

$$\mathsf{E}_0\big[\exp\langle -f, Z_{T_n}\rangle \mid (X_s)_{0 \leq s \leq T_n}\big] = \exp\bigg(-\int_{\mathbb{R}^d} \big(1 - e^{-f(x)}\big) X_{T_n}(\mathrm{d}x)\bigg) \quad \text{a.s.} \tag{6.3}$$

To provide a rigorous proof for (6.3), let $A \in \mathcal{F}^X_{T_n}$ and for $t \in \mathbb{T}$, define

$$A_t := A \cap \{T_n = t\} \in \mathcal{F}^X_t.$$

Since T_n has countable range, we have almost surely,

$$\mathsf{E}_0\big[\exp\big(\langle -f, Z_{T_n}\rangle\big); A\big] = \sum_{t\in\mathbb{T}} \mathsf{E}_0\big[\exp\big(\langle Z_t, -f\rangle\big); A_t\big].$$

Since $A_t \in \mathcal{F}_t^X$, by the Fleischmann–Swart coupling, the last sum equals (a.s.)

$$\sum_{t\in\mathbb{T}} \exp\bigg[-\int_{\mathbb{R}^d}\big(1-e^{-f(x)}\big)X_t(\mathrm{d}x)\bigg]\mathsf{P}_0(A_t),$$

which is the same (a.s.) as

$$\mathsf{E}_0\bigg[\exp\bigg[-\int_{\mathbb{R}^d}\big(1-e^{-f(x)}\big)X_{T_n}(\mathrm{d}x)\bigg]; A\bigg].$$

This completes the proof of (6.3).

Now let $n\to\infty$. By the continuity of X, the a.s. limit of the righthand side in (6.3) is

$$\exp\bigg[-\int_{\mathbb{R}^d}\big(1-e^{-f(x)}\big)X_T(\mathrm{d}x)\bigg].$$

Thus, it remains to show that a.s.,

$$\lim_{n\to\infty} \mathsf{E}_0\big[\exp\langle -f, Z_{T_n}\rangle \mid \mathcal{F}_{T_n}^X\big] = \mathsf{E}_0\big[\exp\langle -f, Z_T\rangle \mid \mathcal{F}_T^X\big].$$

Note that we already know that the a.s. limit exists and just have to identify it. Hence, it is enough to prove that $\mathsf{E}_0[\exp\langle -f, Z_T\rangle \mid \mathcal{F}_T^X]$ is the limit in L^1, for example.

Clearly,

$$\begin{aligned}
&\mathsf{E}_0\big[\exp\langle -f, Z_{T_n}\rangle \mid \mathcal{F}_{T_n}^X\big]\\
&\quad= \mathsf{E}_0\big[\exp\langle -f, Z_T\rangle \mid \mathcal{F}_{T_n}^X\big] + \mathsf{E}_0\big[\exp\langle -f, Z_{T_n}\rangle - \exp\langle -f, Z_T\rangle \mid \mathcal{F}_{T_n}^X\big]\\
&\quad=: A_n + B_n.
\end{aligned}$$

Then $\lim_{n\to\infty} B_n = 0$ in L^1, because

$$\begin{aligned}
&\mathsf{E}_0\big(\big|\mathsf{E}_0\big[\exp\langle -f, Z_{T_n}\rangle - \exp\langle -f, Z_T\rangle \mid \mathcal{F}_{T_n}^X\big]\big|\big)\\
&\quad\le \mathsf{E}_0\big(\mathsf{E}_0\big[\big|\exp\langle -f, Z_{T_n}\rangle - \exp\langle -f, Z_T\rangle\big| \mid \mathcal{F}_{T_n}^X\big]\big)\\
&\quad= \mathsf{E}_0\big(\big|\exp\langle -f, Z_{T_n}\rangle - \exp\langle -f, Z_T\rangle\big|\big) \to 0 \quad \text{as } n\to\infty,
\end{aligned}$$

where the last step uses bounded convergence along with the ω-wise right continuity of Z.

Finally, since T_n is decreasing,

$$\lim_{n\to\infty} A_n = \mathsf{E}_0\big[\exp\langle -f, Z_T\rangle \mid \mathcal{F}_{T+}^X\big], \quad \text{a.s. and in } L^1$$

by (6.2) and the (reverse) Martingale Convergence Theorem for conditional expectations. □

6.3. *The growth of the total mass*; *proof of Theorem* 1.3 *and Corollary* 1.4

The almost sure growth rate of the total mass has been described in [1] for the discrete system Z on $\mathbb{R}$ with $\beta(x) = Cx^2$, $C > 0$, and in [2] for the case when $\beta(x) = |x|^p$, $0 \le p < 2$. For the first case, the authors have verified double-exponential growth:

$$\lim_{t\to\infty} \big(\log\log|Z_t|\big)/t = 2\sqrt{2}C, \quad \text{a.s.}$$

For $\beta(x) = |x|^p$, $0 \le p < 2$, it has been shown that

$$\lim_{t\to\infty} t^{-(2+p)/(2-p)} \log|Z_t| = K_p, \quad \text{a.s.},$$

where K_p is a positive constant, determined by a variational problem. (Also, for $p \in (0,2]$, right-most particle speeds are given.) Note that these proofs carry through for the case when Cx^2 (resp. $|x|^p$) is replaced by $1+Cx^2$ (resp. $1+|x|^p$), too.

We are going to utilize these results, as well as a general comparison result which produces an upper/lower bound on $|X|$ once one has an upper/lower bound on $|Z|$. This comparison result is based on Poisson-coupling.

We need some elementary Poissonian estimates first.

Lemma 6.5 (Poissonian tail estimates). *If Y is a Poisson random variable with parameter $\lambda > 0$, then*

$$P(Y \le y) \le e^{y-\lambda}\left(\frac{\lambda}{y}\right)^y, \quad \text{for } y < \lambda;$$

$$P(Y \ge y) \le e^{y-\lambda}\left(\frac{\lambda}{y}\right)^y, \quad \text{for } y > \lambda.$$

In particular, for $k < 1$ we have $P(Y \le k\lambda) \le C_k^\lambda$, and for $k > 1$ we have $P(Y \ge k\lambda) \le C_k^\lambda$, where

$$C_k := (e/k)^k \cdot (1/e) < 1.$$

Proof. Use the Chernoff-bound for the first part. The statement that $C_k < 1$, after taking logarithm and defining $z = \ln k$, becomes $1 - z < e^{-z}$. □

Proof of Theorem 1.3. We will utilize Lemma 6.5 and Theorem 6.3.

Keeping the Poisson-coupling method in mind, let both Z and X be defined on the probability space (Ω, P). As before, we will write P_0 to indicate that Z and X are started with a $\mathsf{Poisson}(1)$-number of particles at zero and with δ_0, respectively.

Notation. We will use the abbreviation FALT := 'for arbitrarily large times' = for some sequence of times tending to infinity.

(i) For $\varepsilon, t > 0$, define the events

$$E_\varepsilon^t := \{|X_t| > (1+\varepsilon) f(t)\};$$

$$F_{\varepsilon/2}^t := \{|Z_t|/f(t) \le 1 + \varepsilon/2\},$$

$$G_{\varepsilon/2}^t := \{|Z_t|/f(t) > 1 + \varepsilon/2\} = \left(F_{\varepsilon/2}^t\right)^c.$$

Define also

$$E_\varepsilon := \{|X_t| > (1+\varepsilon) f(t), \text{FALT}\};$$

$$H_{\varepsilon/2} := \{|Z_t|/f(t) > 1 + \varepsilon/2, \text{FALT}\}.$$

Since

$$\mathsf{P}_0\Big(\limsup_t |X_t|/f(t) > 1\Big) \le \sum_{m \ge 1} \mathsf{P}_0(E_{\frac{1}{m}}),$$

it is enough to show that for $\varepsilon > 0$, $\mathsf{P}_0(E_\varepsilon) = 0$.

Fix $\varepsilon > 0$. For $\omega \in E_\varepsilon$, define a sequence of random times $(t_n)_{n\ge 0} = (t_n(\omega))_{n \ge 0}$ recursively, by $t_0 := 0$ and

$$t_{n+1} := \inf\{t > t_n \mid |X_t| > (1+\varepsilon) f(t) \text{ and } f(t) \ge n+1\}, \quad n \ge 0.$$

(For convenience, define $t_n(\omega)$ for $\omega \in \Omega \setminus E_\varepsilon$ in an arbitrary way.) Recall that we have proved that X has weakly continuous trajectories, hence $|X|$ is continuous. Thus t_n is an $\mathcal{F}^X$-stopping time; let Q_n denote its distribution on $[0,\infty)$.

Clearly, $\liminf_n G_{\varepsilon/2}^{t_n} \subset H_{\varepsilon/2}$. Hence, if we show that

$$\begin{aligned} &\mathsf{P}_0\Big(E_\varepsilon \cap \Big(\liminf_n G_{\varepsilon/2}^{t_n}\Big)^c\Big) \\ &\quad = \mathsf{P}_0\Big(E_\varepsilon \cap \Big(\limsup_n F_{\varepsilon/2}^{t_n}\Big)\Big) = \mathsf{P}_0\Big(\limsup_n \big(F_{\varepsilon/2}^{t_n} \cap E_\varepsilon\big)\Big) = 0, \end{aligned} \tag{6.4}$$

then $\mathsf{P}_0(E_\varepsilon) > 0$ implies that $\mathsf{P}_0(H_{\varepsilon/2}) > 0$, which contradicts (1.1), and we are done.

To show (6.4), by Borel–Cantelli, it is sufficient to verify that

$$\sum_n \mathsf{P}_0\big(F^{t_n}_{\varepsilon/2} \cap E_\varepsilon\big) < \infty. \tag{6.5}$$

To achieve this, fix $n \geq 1$. Applying Theorem 6.3 with $T = t_n$, we have that

$$\begin{aligned} \mathsf{P}_0\big(F^{t_n}_{\varepsilon/2} \mid E_\varepsilon\big) &= \mathsf{E}_0\big[\mathsf{P}_0\big(F^{t_n}_{\varepsilon/2} \mid (X_s)_{0\leq s\leq t_n}\big) \mid E_\varepsilon\big] \\ &= \mathsf{E}_0\big[\mathsf{P}_0\big(\mathsf{Pois}\big(|X_{t_n}|\big) \leq (1+\varepsilon/2) f(t_n) \mid X_{t_n}\big) \mid E_\varepsilon\big]. \end{aligned} \tag{6.6}$$

Set $k = \frac{1+\varepsilon/2}{1+\varepsilon}$. By (6.6) along with Lemma 6.5 (recall $C_k < 1$ and that $f(t_n) \geq n$), it follows that, almost surely on E_ε,

$$\mathsf{P}_0\big(\mathsf{Pois}\big(|X_{t_n}|\big) \leq (1+\varepsilon/2) f(t_n) \mid X_{t_n}\big) \leq C_k^{(1+\varepsilon)n}.$$

Thus,

$$\mathsf{P}_0\big(F^{t_n}_{\varepsilon/2} \cap E_\varepsilon\big) \leq \mathsf{P}_0\big(F^{t_n}_{\varepsilon/2} \mid E_\varepsilon\big) \leq C_k^{(1+\varepsilon)n},$$

and since $C_k < 1$, (6.5) follows.

(ii) The proof is very similar to that of (i), except that we now work on S, the condition $|X_t| < (1-\varepsilon) f(t)$ has to be replaced by $|X_t| > (1+\varepsilon) f(t)$ throughout, and we now define

$$t_{n+1} := \inf\big\{t > t_n \mid n+1 < |X_t| < (1-\varepsilon) f(t)\big\}, \quad n \geq 0.$$

(In this case we set $k := (1-\varepsilon/2)/(1-\varepsilon) > 1$.) The summability at the end is still satisfied because of the $n+1 < |X_t|$ part in the definition.

Finally, we verify the statement given by the last sentence in (ii). At this point the reader should recall the definition of the function $w = w_{\text{ext}}$ and its basic properties from Appendix B at the end of this paper.

To this end, note that $\exp(-\langle w, X_t\rangle)$ is a martingale with expectation $e^{-w(0)}$. This, in turn, is a consequence of the Markov property and the fact that $\mathsf{P}_\mu(S^c) = e^{-\langle w,\mu\rangle}$. The martingale limit's expectation cannot be less than $e^{-w(0)}$, but on extinction, the limit is clearly one, and the probability of extinction is also $e^{-w(0)}$. Hence the limit must be zero on S, that is $\langle w, X_t\rangle \to \infty$. But $w \leq 1$ holds under the assumption by Claim B.1 in Appendix B. □

Corollary 1.4 in the Introduction is a consequence of Theorem 1.3, as shown below.

Proof of Corollary 1.4. We treat the non-quadratic case; the quadratic case is similar. Also, we only discuss the upper estimate; the lower estimate is similar.

Denote $h(t) := K t^{\frac{2+p}{2-p}}$. For the upper estimate, we need that

$$E := \left\{\limsup_t \frac{\log |X_t|}{h(t)} > 1\right\}$$

is a zero event. But E occurs if and only if

$$\exists \varepsilon > 0\colon \quad \frac{\log |X_t|}{h(t)} > (1+\varepsilon), \text{FALT} \quad \Leftrightarrow \quad \exists \varepsilon > 0\colon \quad |X_t| > \exp\big(h(t)(1+\varepsilon)\big), \text{FALT}.$$

Now

$$E \subset A := \left\{\exists \varepsilon > 0 : \limsup_t \frac{|X_t|}{\exp(h(t)(1+\varepsilon))} \geq 1\right\}.$$

Write

$$\frac{|X_t|}{\exp(h(t)(1+\varepsilon))} = \frac{|X_t|}{\exp(h(t))} \frac{1}{\exp(h(t)(\varepsilon))}.$$

The lim sup of the first term on the righthand side is almost surely bounded by one by Theorem 1.3 and by the corresponding result[9] on Z, while the second term tends to zero. Working with countably many ε's (say, $\varepsilon_m := 1/m$), we see that A is a zero event indeed. □

6.4. *Upper bound for the spatial spread*

Theorem 6.6 (Upper bound for the spread). *Let $\varepsilon > 0$. For $d = 1$ and $\beta(x) = \alpha(x) = 1 + |x|^2$, we have*

$$P_{\delta_0}\Big(\lim_{t\to\infty} X_t\big(B^c\big(0, \exp\big((\sqrt{2}+\varepsilon)t\big)\big)\big) = 0\Big) = 1.$$

Proof. Clearly, it is enough to prove that for any $\delta > 0$,

$$P_{\delta_0}\big(\exists T : X_t\big(B^c\big(0, \exp\big((\sqrt{2}+\varepsilon)t\big)\big)\big) \le \delta, \quad \text{for } t > T\big) = 1. \tag{6.7}$$

Harris and Harris [23] have shown for the (one-dimensional) discrete branching Brownian motion Z with branching rate β that

$$\mathsf{P}_0\left(\limsup_{t\to\infty} \frac{\log M_t}{t} \le \sqrt{2}\right) = 1,$$

where M_t is the rightmost particle's position. (Again, they considered $\beta(x) = |x|^2$, but the proof carries through for $\beta(x) = 1 + |x|^2$ as well.) By symmetry, it follows that

$$\mathsf{P}_0\left(\limsup_{t\to\infty} \frac{\log \rho_t}{t} \le \sqrt{2}\right) = 1,$$

where ρ_t is the radius of the minimal interval containing $\mathrm{supp}(Z_t)$. That is,

$$\mathsf{P}_0\big(\rho_t > \exp\big((\sqrt{2}+\varepsilon)t\big), \text{FALT}\big) = 0. \tag{6.8}$$

Returning to (6.7), we need to show that

$$p_\varepsilon := P\big(X_t\big(B^c\big(0, \exp\big((\sqrt{2}+\varepsilon)t\big)\big)\big) > \delta, \text{FALT}\big) = 0.$$

Indeed, suppose that $p_\varepsilon > 0$. Recall that for a PPP, the probability that a set with mass at least δ (by the intensity measure) is vacant is at most $\exp(-\delta)$.

As before, consider the 'Poisson-coupling' of the processes Z and X. By the reverse Fatou inequality,[10] on the event

$$\big\{X_t\big(B^c\big(0, \exp\big((\sqrt{2}+\varepsilon)t\big)\big)\big) > \delta, \text{FALT}\big\},$$

the discrete point process charges $B^c(0, \exp((\sqrt{2}+\varepsilon)t))$ FALT, with probability at least $e^{-\delta}$. It follows that the probability in (6.8) is positive; a contradiction. □

Remark 6.7. It is not difficult to see that this upper estimate remains valid if $\alpha \ge \beta$ instead of $\alpha = \beta$.

Problem 6.8. It is an interesting question whether one can have a sharper version of Theorem 6.6 formulated in terms of $\mathrm{supp}(X_t)$ as $t \to \infty$.

[9] The result for Z is true even if Z starts with $k \ge 1$ particles instead of a single one, as the process can be considered as an independent sum of k processes, each starting with a single particle.

[10] Which is $\limsup P(A_t) \le P(\limsup A_t)$.

Appendix A: Properties of $\lambda_c^{(p)}$

Recall the definition of the p-generalized principle eigenvalue, $\lambda_c^{(p)}$ from Definition 5.18. First note that $\lambda_c^{(1)} = \lambda_c$, because (c.f. Chapter 4 in [25])

$$\begin{aligned}\lambda_c^{(1)} &:= \inf\bigl\{\lambda \in \mathbb{R} : G_\lambda(x, B) \text{ is locally bounded in } D \text{ for some } B \Subset D\bigr\} \\ &= \lambda_c,\end{aligned}$$

where

$$G_\lambda(x, B) := G^{L+\beta-\lambda}(x, B) := \int_{t=0}^{\infty} p^{L+\beta-\lambda}(t, x, B)\,\mathrm{d}t,$$

and $p^{L+\beta-\lambda}(t, \cdot, \cdot)$ denotes the transition kernel for $L + \beta - \lambda$ on D. (When finite on compacts, the measure $G_\lambda(x, \cdot)$ is called the *Green-measure* for $L + \beta - \lambda$ on D.)

Next, note that if one replaces the semigroup in the definition of $\lambda_c^{(p)}$ by that of some compactly embedded ball in D (with zero boundary condition), then $\lambda_c^{(p)}$ will definitely not increase, while even this modified value is different from $-\infty$, as β is bounded on the ball. This leads to

Proposition A.1. *One has* $\lambda_c^{(p)} \in (-\infty, \infty]$.

Moreover, the following comparison principle holds:

Proposition A.2 (Comparison). *Let* $p > q \geq 1$.

(a) *If* $\lambda_c^{(q)} \geq 0$, *then* $\lambda_c^{(p)} \leq \lambda_c^{(q)}$.
(b) *If* $\lambda_c^{(q)} \leq 0$, *then* $\lambda_c^{(p)} \geq \lambda_c^{(q)}$.
(c) *If* $\lambda_c^{(q)} = 0$, *then* $\lambda_c^{(p)} = \lambda_c^{(q)} = 0$. *In particular, if* $\lambda_c = 0$, *then* $\lambda_c^{(p)} = 0$ *for all* $p > 0$.

Proof. (a): Suppose it is not true and pick a λ such that

$$\lambda_c^{(p)} > \lambda > \lambda_c^{(q)} \geq 0.$$

Then there is a non-trivial non-negative $g \in C_c(D)$ so that for every $B \Subset D$,

$$\int_0^\infty e^{-\lambda t^q} \|\mathbf{1}_B T_t g\|_\infty \,\mathrm{d}t < \infty,$$

but the same fails when t^q in the integral is replaced by t^p; contradiction.

(b): Suppose it is not true and pick a λ s.t.

$$\lambda_c^{(p)} < \lambda < \lambda_c^{(q)} \leq 0.$$

Then there is a non-trivial non-negative $g \in C_c(D)$ so that for every $B \Subset D$,

$$\int_0^\infty e^{-\lambda t^p} \|\mathbf{1}_B T_t g\|_\infty \,\mathrm{d}t < \infty,$$

but the same fails when t^p in the integral is replaced by t^q; contradiction.

(c): This is clear from (a) and (b). □

We can define $\lambda_c^{(p)}$ in a different way too.

Theorem A.3 (Equivalent definition). *Assume that* β *is bounded from below, that is,* $\inf_D \beta > -\infty$. *Then*

$$\lambda_c^{(p)} = \inf\left\{\lambda \in \mathbb{R} : \int_0^\infty e^{-\lambda s^p} \|\mathbf{1}_B T_s g\|_\infty \,\mathrm{d}s < \infty, \forall g \in C_c^+(D)\ \forall B \Subset D\right\}.$$

Proof. If suffices to show that for every $g \in C_c^+(D)$,

$$\lambda_c^{(p)} \geq \inf\left\{\lambda \in \mathbb{R} : \int_0^\infty e^{-\lambda s^p} \|\mathbf{1}_B T_s g\|_\infty \,\mathrm{d}s < \infty \text{ for every } B \Subset D\right\}. \tag{A.1}$$

For every $\varepsilon > 0$, by the definition of $\lambda_c^{(p)}$, there is some $g_0 \in B_b^+(D)$ with $g_0 \neq \mathbf{0}$ such that $\int_0^\infty e^{-(\lambda_c^{(p)}+\varepsilon)t^p} \|\mathbf{1}_B T_t g_0\|_\infty \,\mathrm{d}t < \infty$ for every $B \Subset D$. Let g be an arbitrary function in $C_c^+(D)$. We denote by f^- the negative part of a function f; that is, $f^-(x) := \max\{0, -f(x)\}$. Let $\{T_t^{(1)}; t \geq 0\}$ be the semigroup for the Schrödinger operator $L - \beta^-$; that is,

$$T_t^{(1)} f(x) := \mathbb{E}_x\left[\exp\left(-\int_0^t \beta^-(X_s)\,\mathrm{d}s\right) f(X_t); t < \tau_D\right].$$

Note that for any $f \geq 0$ and $t \geq 0$, one has $T_t f \geq T_t^{(1)} f$. Since by the strong Feller property and irreducibility of $\{T_t^{(1)}, t \geq 0\}$, $T_1^{(1)} g_0 \in C_b(D)$ and $T_1 g_0 > 0$, there is a constant $c > 0$ such that $0 \leq g \leq c T_1^{(1)} g_0 \leq c T_1 g_0$. Consequently, for any $B \Subset D$,

$$\begin{aligned}
\int_0^\infty e^{-(\lambda_c^{(p)}+2\varepsilon)t^p} \|\mathbf{1}_B T_t g\|_\infty \,\mathrm{d}t &\leq c \int_0^\infty e^{-(\lambda_c^{(p)}+2\varepsilon)t^p} \|\mathbf{1}_B T_{t+1} g_0\|_\infty \,\mathrm{d}t \\
&= c \int_1^\infty e^{-(\lambda_c^{(p)}+2\varepsilon)(t-1)^p} \|\mathbf{1}_B T_t g_0\|_\infty \,\mathrm{d}t \\
&\leq c_1 \int_0^\infty e^{-(\lambda_c^{(p)}+\varepsilon)t^p} \|\mathbf{1}_B T_t g_0\|_\infty \,\mathrm{d}t,
\end{aligned}$$

which is finite a.e. on D. This shows that

$$\inf\left\{\lambda \in \mathbb{R} : \int_0^\infty e^{-\lambda s^p} \|\mathbf{1}_B T_s g\|_\infty)\,\mathrm{d}s < \infty \text{ for every } B \Subset D\right\} \leq \lambda_c^{(p)} + 2\varepsilon.$$

Since this holds for every $\varepsilon > 0$, we conclude that (A.1) and hence the theorem holds. □

A.1. *The proofs of Proposition 5.19 and Theorem 1.5*

Consider the case when $D = \mathbb{R}^d$, $L = \frac{1}{2}\Delta$, $\beta(x) = |x|^\ell$, $\ell > 0$ and $\alpha > 0$. It is natural to ask for what p do we have $0 < \lambda_c^{(p)} < \infty$? Can we estimate it?

We start off by deriving the statements in Proposition 5.19 for the $0 < \ell < 2$ case. By (5.2) and the paragraph following it, if $(B, \mathbb{P})$ is a Brownian motion, then

$$\begin{aligned}
&e^{-|x|^\ell} \mathbb{E}_0\left[\exp\left(\int_0^t 2^{-\ell}|x + B_s|^\ell \,\mathrm{d}s\right)\right] \\
&\quad\leq \mathbb{E}_x\left[\exp\left(\int_0^t |B_s|^\ell \,\mathrm{d}s\right)\right] \leq e^{2^\ell |x|^\ell} \mathbb{E}_0\left[\exp\left(\int_0^t 2^\ell |B_s|^\ell \,\mathrm{d}s\right)\right],
\end{aligned} \tag{A.2}$$

while

$$\mathbb{E}_0\left[\exp\left(a \int_0^t |B_s|^\ell \,\mathrm{d}s\right)\right] = \int_0^\infty a t^{1+\ell/2} e^{aut^{1+\ell/2}} \left(e^{-\frac{1}{2}c_\ell u^{2/\ell}(1+o(1))}\right) \mathrm{d}u.$$

Hence there is a constant $c > 0$ so that

$$T_t 1(x) \leq e^{2^\ell |x|^\ell} 2^\ell t^{1+\ell/2} \int_0^\infty e^{2^\ell u t^{1+\ell/2}} e^{-cu^{2/\ell}} \,\mathrm{d}u.$$

Let c_1 be the solution of $(2^\ell + 1)v = cv^{2/\ell}$; that is, $c_1 = ((2^\ell + 1)/c)^{\ell/(2-\ell)}$. Note that for $v \geq c_1$, $cv^{2/\ell} \geq 2^\ell v + v$. Using the shorthands

$$M := (2 + \ell)/(2 - \ell) \in (1, \infty), \qquad k_\ell(x) := e^{2^\ell |x|^\ell} 2^\ell,$$

a change of variable $u = t^\eta v$ with $\eta = M\ell/2$ yields that

$$T_t 1(x) \le k_\ell(x) t^M \int_0^\infty \exp\big(t^M\big(2^\ell v - c v^{2/\ell}\big)\big)\,\mathrm{d}v$$
$$\le k_\ell(x) t^M \left(\int_0^{c_1} e^{v 2^\ell t^M}\,\mathrm{d}v + \int_{c_1}^\infty e^{-vt^M}\,\mathrm{d}v\right)$$
$$\le k_\ell(x)\big(\exp\big(c_1 2^\ell t^M\big) + 1\big).$$

Thus, for every $\gamma > e^{c_1 2^\ell}$,

$$\int_0^\infty e^{-\gamma t^M} T_t 1(x)\,\mathrm{d}t < \infty.$$

It follows that, with $p = M$, $\lambda_c^{(p)} \le \gamma$ and so

$$\lambda_c^{(p)} \le e^{c_1 2^\ell}. \tag{A.3}$$

The exponent $M = (2+\ell)/(2-\ell)$ is sharp when $\ell = 1$, as can be seen from (5.3).

Finally, as far as Theorem 1.5 is concerned, it follows from (A.3) along with Theorem 5.20.

Appendix B: The functions w_{ext} and $w_{\max}$ for the superprocess

In this section we review some basic properties of two specific functions, the functions w_{ext} and $w_{\max}$ for superprocesses and their probabilistic significance.

B.1. Two important functions

In what follows, we are going to use several results from [15], where the reader can find more elaboration and proofs.

Recall that we are working with Assumption 2.2. Although in [15] the more stringent assumption $\lambda_c < \infty$ was in force, the results are still applicable in our setting. The reason is that for all the results we are using in the $\lambda_c = \infty$ case, the proof only uses the local properties of the coefficients. So below we consider the general setting of Definition 3.2, without any particular assumption on the growth of β.

Let us start with the function we denote by w_{ext}. The reason for the notation is that this function corresponds to the event of extinction, S^c, where S stands for survival. Namely, $P_{\delta_x}(S^c) = e^{-w_{\mathrm{ext}}(x)}$, where w_{ext} is a particular nonnegative solution to the steady state equation

$$Lu + \beta u - \alpha u^2 = 0. \tag{B.1}$$

(See Theorem 3.1 in [15].) Its finiteness and the fact that it solves the equation, follows the same way as in [15]. Finiteness follows from Lemma 7.1 in [15]. More precisely, the content of Lemma 7.1 in [15] is that given any $t, R > 0$, with positive probability, the process may die out by time t without ever charging a ball of radius R around x. In the proof, all one needed was that locally, β (resp. α) was bounded from above (resp. bounded away from zero), hence, as mentioned above, only the local properties of the coefficients were used in [15]. Thus the argument carries through when $\lambda_c = \infty$ as well.

Another important function is what we denoted by $w_{\max}$: the *maximal* nonnegative solution to the steady state equation (B.1). Its construction is the same as in [15] or in [16], and again, only the local properties of the coefficients were used in [15,16], hence the construction carries through when $\lambda_c = \infty$ as well.

Clearly $w_{\mathrm{ext}} \le w_{\max}$, and it is natural to ask whether in fact $w_{\mathrm{ext}} = w_{\max}$. By Theorem 3.3 in [15], $w_{\mathrm{ext}} = w_{\max}$, whenever the compact support property holds, while in general, this is not necessarily the case. Recall that X satisfies the compact support property in a number of interesting cases; see Claims 5.10 and 5.12.

B.2. A particular case

After recalling these general facts, we now turn to the specific case, when $D = \mathbb{R}^d$, $0 < \alpha = \beta$. For simplicity, write w for w_{ext}.

The corresponding steady state equation is now $Lu + \alpha(u - u^2) = 0$, or equivalently, $\frac{1}{\alpha}Lu + u - u^2 = 0$. By definition, $w_{\max}$ is the maximal nonnegative solution to these two equations.

Denote the $(\frac{1}{\alpha}L, 1, 1; \mathbb{R}^d)$-superprocess by $\widehat{X}$. Write $\widehat{w}_{\text{ext}}$ and $\widehat{w}_{\max}$ in place of w_{ext} and $w_{\max}$ when X is replaced by $\widehat{X}$. The previous paragraph means that $\widehat{w}_{\max} = w_{\max}$.

Assume now in addition, that the coefficients of $\frac{1}{\alpha}L$ are bounded from above[11] (for example $L = \Delta/2$ and α is bounded away from zero).

Claim B.1. *Under these assumptions*, $w \le 1$.

Proof. By Theorem 3.5 in [15]), under the assumptions imposed on the coefficients of the semilinear operator $\frac{1}{\alpha}Lu + u - u^2$, the compact support property holds for $\widehat{X}$; therefore

$$\widehat{w}_{\text{ext}} = \widehat{w}_{\max} = w_{\max} \ge w.$$

Thus, $w \le 1$ follows from $\widehat{w}_{\text{ext}} \le 1$, which in turn follows from Proposition 3.1 in [15]. $\square$

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