

# Discrete rough paths and limit theorems

Yanghui Liu and Samy Tindel<sup>1</sup>

*Department of Mathematics, Purdue University, 150 N. University Street, W. Lafayette, IN 47907, USA.  
E-mail: [liu2048@purdue.edu](mailto:liu2048@purdue.edu); [stindel@purdue.edu](mailto:stindel@purdue.edu)*

Received 9 August 2018; revised 2 July 2019; accepted 3 July 2019

---

**Abstract.** In this article, we consider limit theorems for some weighted type random sums (or discrete rough integrals). We introduce a general transfer principle from limit theorems for unweighted sums to limit theorems for weighted sums via rough path techniques. As a by-product, we provide a natural explanation of the various new asymptotic behaviors in contrast with the classical unweighted random sum case. We apply our principle to derive some weighted type Breuer–Major theorems, which generalize previous results to random sums that do not have to be in a finite sum of chaos. In this context, a Breuer–Major type criterion in notion of Hermite rank is obtained. We also consider some applications to realized power variations and to Itô’s formulas in law. In the end, we study the asymptotic behavior of weighted quadratic variations for some multi-dimensional Gaussian processes.

**Résumé.** Dans cet article, nous étudions les théorèmes limite pour des sommes aléatoires pondérées (ou intégrales discrètes rugueuses). Nous introduisons un principe de transfert général entre les théorèmes limite pour les sommes non pondérées et pour les sommes pondérées, en utilisant des techniques de chemins rugueux. Comme conséquence, nous proposons une explication naturelle pour la diversité des nouveaux comportements asymptotiques par rapport aux cas des sommes aléatoires non pondérées. Nous appliquons notre principe pour obtenir des théorèmes de type Breuer–Major pondérés, qui généralisent des résultats précédents aux cas de sommes aléatoires qui ne sont pas dans une somme finie de chaos. Dans ce contexte, un critère de type Breuer–Major en termes de rang d’Hermite est obtenu. Nous considérons aussi des applications pour réaliser des variations de puissance et pour les formules d’Itô en loi. A cette fin, nous étudions le comportement asymptotique de variations quadratiques pondérées pour des processus Gaussiens multidimensionnels.

MSC: 60B10; 60G15; 60G22; 60H07

Keywords: Discrete rough paths; Discrete rough integrals; Weighted random sums; Limit theorems; Breuer–Major theorem

---

## 1. Introduction

For  $n \geq 1$ , we consider the uniform partition  $\mathcal{D}_n : 0 = t_0 < t_1 < \dots < t_n = 1$  of  $[0, 1]$  (notice that more general partitions could be considered, although this article is mostly restricted to the uniform case for sake of simplicity). Take a 1-increment process  $h_{st}^n$  defined for  $s, t \in \mathcal{D}_n$  such that  $s \leq t$  and a “weight” process  $y_t$  defined for  $t \in \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$ . We consider a “discrete integral” as a Riemann sum of the form:

$$\mathcal{J}_s^t(y; h^n) := \sum_{s \leq t_k < t} y_{t_k} h_{t_k t_{k+1}}^n. \quad (1.1)$$

Recall that a classical limit theorem for such a process is a statement of the type:

$$\frac{1}{a_n} \mathcal{J}(1; h^n) = \frac{h^n}{a_n} \longrightarrow \omega, \quad \text{as } n \rightarrow \infty. \quad (1.2)$$

Here  $a_n$  is an increasing sequence such that  $\lim_{n \rightarrow \infty} a_n = \infty$ ,  $\omega$  is a non-zero continuous process and the limit is usually understood as a finite dimensional distribution limit. A typical example of (1.2) is the convergence of a renormalized random walk to Brownian motion (Donsker’s theorem, see [26]), but a wide range of more complex situations can occur.

---

<sup>1</sup>S. Tindel is supported by the NSF grant DMS-1613163.

Indeed, it is well-known that the rate of growth of  $a_n$  and the nature of the limit process  $\omega$  are determined by both the marginal tails of  $h^n$  and its dependence structure; see e.g. [11,12,41,42].

In this paper we are interested in the following related problem:

**Problem 1.** Given that  $h^n$  converges to some “1-increment” process, say, the increment of a Wiener process, what is the asymptotic behavior of the discrete integral  $\mathcal{J}(y; h^n)$  for a general weight  $y$ , and when would (or would not) the asymptotic behavior of  $\mathcal{J}(y; h^n)$  be similar to that of  $h^n$ ?

This problem has drawn a lot of attention in recent articles due to its essential role in topics such as normal approximations (e.g. [33,34,36]), time-discretization based numerical approximations (e.g. [17,25,30]), parameter estimations (e.g. [3,10,29,32]), and the so-called *Itô’s formula in law* (e.g. [4,6,18,19,21–23,38,39]). Let us, however, point out several limitations in the existing results: (1) Each process  $h^n$  is usually a functional of a Gaussian process  $x$  with stationary increments, living in a fixed finite sum of chaos; (2) The underlying Gaussian process  $x$  is one-dimensional; (3) Only the special case  $y_t = f(x_t)$ ,  $t \geq 0$  is considered for the weight function. (4) To the best of our knowledge, there is no theoretical explanation for the various “unexpected” asymptotic behaviors of the discrete integral observed in e.g. [6,19,38,40] so far. (5) Satisfactory general criteria of convergence for sequences of discrete weighted integrals are still rare. This is in sharp contrast with the simple Breuer–Major type conditions in the unweighted case.

The aim of the current paper is thus to give an account on limit theorems for discrete integrals thanks to rough paths techniques combined with Gaussian analysis. In our setting, we will consider a general 1-increment process  $h^n$  and a general weight process  $(y, y', \dots, y^{(\ell-1)})$  with  $y_0 = 0$  which is controlled by the increments of some rough path  $x = (1, x^1, \dots, x^{\ell-1})$ . Here  $\ell$  is some constant in  $\mathbb{N}$ . Notice that we will define the notion of controlled process later in the paper, see Definition 2.3 below, but we can observe that this class of paths includes functions of the form  $y = f(x)$  or solutions of differential equations driven by  $x$ . Let us label the following hypothesis:

**Hypothesis 1.1.** Take  $i \in \{0, 1, \dots, \ell - 1\}$ . For any sequence of partitions  $0 \leq s_0 < s_1 < \dots < s_m \leq 1$  of  $[0, 1]$  with  $|s_{j+1} - s_j| \leq 1/m$ , we have

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \sum_{j=1}^m \mathcal{J}_{s_j}^{s_{j+1}}(x^i; h^n) \right| = 0, \tag{1.3}$$

where  $\mathcal{J}(x^i; h^n)$  is defined by (1.1) and the limit is understood as a limit in probability.

**Remark 1.2.** Hypothesis 1.1 specifies that the random sums  $\mathcal{J}(y, h^n)$  corresponding to the weights  $y = x^i$  are negligible with respect to a main contribution in sums like (1.1). In practice, Hypothesis 1.1 is easier to check for a simple weight like  $x^i$  than for a general weight  $y$ .

We will be able to prove that if Hypothesis 1.1 holds for  $i = 1, \dots, \ell - 1$  and  $y$  is a controlled process, then the following convergence holds in distribution (see Theorem 3.8 for a more precise statement):

$$\mathcal{J}(y; h^n) \xrightarrow{d} v, \tag{1.4}$$

where the integral  $v_{st} = \int_s^t y_u dW_u$  has to be understood as a conditional Wiener integral.

As alluded to above, our result can be seen as a general principle which allows to transfer limit theorems (1.3) taken on monomials of the rough path to the corresponding limit theorems involving controlled processes as weights. Therefore, potential applications of this result are numerous (see the aforementioned parameter estimation problem, Itô’s formula in law, or numerical schemes for rough differential equations), and will be detailed throughout the paper.

As has already been observed in [19,33,36], the asymptotic behavior of (1.1) can be completely different from (1.4). One of the first occurrences of this kind of result is provided by [33], where for a one-dimensional fractional Brownian motion  $x$  with Hurst parameter  $\nu \in (0, \frac{1}{4})$  the following limit theorem is obtained: consider the increment  $h_{st}^n = \sum_{s \leq t_k < t} [(n^\nu \delta x_{t_k t_{k+1}})^2 - 1]$ , where  $\delta x_{t_k t_{k+1}} = x_{t_{k+1}} - x_{t_k}$  and  $(t_k)_{k=0, \dots, n}$  stands for the uniform partition of  $[0, 1]$ . Let  $f$  be a continuous function with proper regularity. Then, as  $n \rightarrow \infty$ , we have:

$$n^{2\nu-1} \mathcal{J}_0^1(f(x); h^n) \xrightarrow{L_2} \frac{1}{4} \int_0^1 f''(x_s) ds. \tag{1.5}$$

Our approach allows to generalize our result to handle limits such as (1.5), weighted by controlled processes. In addition, our results provide an explanation of the appearance of  $f''$  in the right-hand side of (1.5), based on the struc-

tural understanding of the discrete integral from the rough path theory. Indeed, our next theorem shows that the limit  $\frac{1}{4} \int_0^1 f''(x_s) ds$  is the result of a “speed match” between different levels  $(1, x^1, \dots, x^{\ell-1})$  of the rough path  $x$  and the fact that  $f(x), f'(x), \dots, f^{(\ell-1)}(x)$  are the corresponding weight processes. Specifically, we shall show that if there is some  $\tau \in \{1, \dots, \ell - 1\}$  such that  $\mathcal{J}_s^t(x^\tau; h^n) \rightarrow (t - s)$  in probability, and  $\mathcal{J}_s^t(x^i; h^n)$  are negligible for all  $i \neq \tau$ , then we have the convergence in probability (see Theorem 3.11 for the precise statement):

$$\lim_{n \rightarrow \infty} \mathcal{J}(y; h^n) = \int y_t^{(\tau)} dt.$$

A more complicated situation of asymptotic behavior is observed in [6,34,38]. This usually corresponds to a transition in terms of roughness for the underlying rough path  $x$ . For example, in the critical cases when  $\nu = \frac{1}{4}$  in (1.5), and for the same  $f$  and  $h^n$  as in (1.5), one obtains the convergence in distribution:

$$n^{-1/2} \mathcal{J}_0^1(f(x); h^n) \xrightarrow{d} \sigma \int_0^1 f(x_s) dW_s + \frac{1}{4} \int_0^1 f''(x_s) ds, \tag{1.6}$$

where  $\sigma$  is some constant and recall that  $W$  is a standard Brownian motion independent of  $x$ . An explanation of the above asymptotic behavior according to the technique of rough path is that the two levels 1 and  $x^2$  give contributions of the same order in the limit theorem. This is then reflected into the fact that the components  $f(x)$  and  $f''(x)$  (respectively, 0th and 2nd derivatives of  $f(x)$  as a controlled process) give contributions of the same order. Our generalization of (1.6) is thus the “double” limit theorem: Suppose that there exists a  $\tau : 0 < \tau < \ell$  such that for any sequence of partitions  $0 \leq s_0 < s_1 < \dots < s_m \leq 1$  on  $[0, 1]$  with  $|s_{i+1} - s_i| \leq \frac{1}{m}$ , and  $s_0 = s, s_m = t$ , we have the convergence in probability:

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(x^\tau; h^n) = t - s. \tag{1.7}$$

Suppose further that Hypothesis 1.1 holds true for  $i \in \{1, \dots, \ell - 1\} \setminus \{\tau\}$ . Then

$$\mathcal{J}(y; h^n) \xrightarrow{d} \int y_t dW_t + \int y_t^{(\tau)} dt.$$

As mentioned previously, our results are abstract transfer principles from monomials of a rough path to a controlled process for limit theorems of the form (1.2). For sake of illustration, let us mention an important application of this transfer principle we will encounter in the article, namely a weighted type Breuer–Major theorem.

Recall that the Hermite polynomial of order  $q$  is defined as  $H_q(t) = (-1)^q e^{\frac{t^2}{2}} \frac{d^q}{dt^q} e^{-\frac{t^2}{2}}$ , and we denote by  $\gamma$  the standard normal distribution. We consider the following Breuer–Major type criterion:

**Hypothesis 1.3.** Take  $\ell \in \mathbb{N}$ . Let  $f \in L_2(\gamma)$  be a function such that we have the expansion  $f = \sum_{q=d}^\infty a_q H_q$  for a given  $d \geq 1$  and  $a_d \neq 0$ . We suppose that the coefficients  $a_q$  satisfy:

$$\sum_{q=d}^\infty a_q^2 q! q^{2(\ell-1)} < \infty. \tag{1.8}$$

Note that (1.8) is equivalent to the fact that  $f$  belongs to a specific Sobolev space, namely  $f \in W^{2\ell-2,2}(\mathbb{R}, \gamma)$ .

Following is our weighted type Breuer–Major theorem (see Theorem 4.7 and Theorem 4.14 for a more precise statement):

**Theorem 1.4.** Let  $x$  be a one-dimensional fBm with Hurst parameter  $\nu < \frac{1}{2}$ . Suppose that Hypothesis 1.3 holds true for  $f \in L_2(\gamma)$  with some  $\ell \in \mathbb{N}$  and  $d \geq 1$ . Let  $(y, y', \dots, y^{(\ell-1)})$  be a process controlled by  $x$ . We define a sequence  $\{h^n; n \geq 1\}$  of increments by  $h_{st}^n := n^{-1/2} \sum_{s \leq t_k < t} f(n^\nu \delta x_{t_k t_{k+1}})$  for  $s, t \in [0, 1]$  such that  $s < t$ .

(i) When  $d > \frac{1}{2\nu}$ , and  $\ell$  is the smallest integer such that  $\ell > \frac{1}{2\nu}$ , we have the convergence:

$$\mathcal{J}(y; h^n) \xrightarrow{d} \sigma_{\nu,d} \int y_t dW_t, \quad \text{as } n \rightarrow \infty,$$

where  $\sigma_{\nu,d}$  is a constant which can be computed explicitly and where we recall that  $\mathcal{J}(y; h^n)$  is defined by (1.1).

(ii) When  $d = \frac{1}{2\nu}$  and  $\ell = d + 1$ , the following convergence holds true:

$$\mathcal{J}(y; h^n) \xrightarrow{d} \sigma_{\nu,d} \int y_t dW_t + \left(-\frac{1}{2}\right)^d a_d \int y_u^{(d)} du, \quad \text{as } n \rightarrow \infty.$$

(iii) When  $d < \frac{1}{2\nu}$  and  $\ell = d + 1$ , we have the convergence in probability:

$$n^{-(\frac{1}{2}-\nu d)} \mathcal{J}_s^t(y; h^n) \longrightarrow \left(-\frac{1}{2}\right)^d a_d \int_s^t y_u^{(d)} du, \quad \text{as } n \rightarrow \infty.$$

The proof of Theorem 1.4 is based on the transfer principle results we have developed. Let us observe that Theorem 1.4 improves on the references on weighted Breuer–Major theorems quoted above in the following ways:

- (i) The function  $f$  is not assumed to be in a finite sum of chaos. In fact a convenient sufficient condition for (1.8) to be fulfilled is that the function  $f$  is an element of  $C_b^{2\ell-2}$ . Note that if the focus is on the rate of convergence of  $\mathcal{J}(y; h^n)$  (see e.g. [35]), it is generally assumed that  $f$  belongs to a finite sum of chaos.
- (ii) Multidimensional versions of Theorem 1.4 (based on [2]) are easily conceived, where  $f(n^\nu \delta x_{t_k t_{k+1}})$  in the definition of  $h^n$  is replaced by  $f(n^\nu \delta x_{t_k t_{k+1}}^1, \dots, n^\nu \delta x_{t_k t_{k+1}}^d)$ , for a  $d$ -dimensional Gaussian process  $(x^1, \dots, x^d)$ .
- (iii) The weight  $y$  in Theorem 1.4 is obviously a controlled process instead of a mere function of  $x$ . It is worth noting again that the class of controlled processes includes solutions of differential systems driven by  $x$ .
- (iv) As mentioned above, the single and double limiting phenomena in Theorem 1.4 can be explained in terms of speed match on different levels of the rough path above  $x$ .
- (v) The solution to Problem 1 above is expressed easily in terms of the Hurst parameter  $\nu$  of  $x$  and the *Hermite rank*  $d$  of  $f$ .

Throughout the paper we will give an account on other applications of our general transfer principle, such as real-ized power variations, convergence of trapezoidal Riemann sums and quadratic variations of multidimensional Gaussian processes. As the reader might see, the improvements (i)–(v) mentioned above will be a constant of our rough paths method.

Let us briefly explain the general methodology we have followed for our proofs, separating the general principle from the applications.

- (a) The proofs of our transfer principle results are mostly based on rough path type expansions for the weight process  $y$  and a more classical coarse graining argument (also called big block/small block in the literature). By handling the remainder terms thanks to rough paths techniques, the convergence of  $\mathcal{J}(y; h^n)$  is reduced to those of  $\mathcal{J}(y; \zeta^1)$ ,  $\mathcal{J}(y'; \zeta^2), \dots, \mathcal{J}(y^{(\ell-1)}; \zeta^\ell)$ , where each  $\zeta^i$  is a discrete process of the form  $\zeta_j^i = \mathcal{J}_s^{s_j+1}(x^i; h^n)$ . The convergence of these quantities are further reduced to those of  $\mathcal{J}(1; \zeta^1), \mathcal{J}(1; \zeta^2), \dots, \mathcal{J}(1; \zeta^\ell)$ , such as those in Hypothesis 1.1 and relation (1.7). The random processes  $\mathcal{J}(1; \zeta^1), \mathcal{J}(1; \zeta^2), \dots, \mathcal{J}(1; \zeta^\ell)$  will be the elementary bricks for our limiting procedures.
- (b) Our applications, such as the weighted type Breuer–Major Theorem 1.4, heavily rely on the criteria developed in the transfer principle results. This ingredient is combined with some Malliavin calculus techniques in order to handle the building bricks  $\mathcal{J}(1; \zeta^i)$ . More specifically, in case of the weighted Breuer–Major theorem 1.4, we shall invoke integration by parts on the Wiener space. This step is similar to what is done in [34]. However, due to our rough path reduction of the problem, we only have to consider integration by parts to compute moments of the elementary bricks  $x_{t_k}^i H_q(n^\nu \delta x_{t_k t_{k+1}})$  (as opposed to  $g(x_{t_k}) f(n^\nu \delta x_{t_k t_{k+1}})$  for a general nonlinear function  $g$ ). This reduction to computations in finite chaos is one of the crucial steps which allow to derive the Breuer–Major type criteria (1.8) for a general function  $f$ .

The paper is organized as follows. In Section 2 we introduce the concept of discrete rough paths and discrete rough integrals and recall some basic results of the rough paths theory. In Section 3, we prove our general limit theorems. In Section 4, we apply them to the one-dimensional fractional Brownian motion, which allows us to derive a weighted type Breuer–Major theorem. We also consider applications of the weighted type Breuer–Major theorem to parameter estimation and Itô’s formula in law. In Section 5, we consider the limit theorem of a weighted quadratic variation in the multi-dimensional Gaussian setting.

**Notation.** As mentioned above, for simplicity we mostly consider uniform partitions in the sequel. That is, we denote

$$t_k = \frac{k}{n}$$

for each  $k, n \in \mathbb{N}$ . Take  $s, t \in [0, 1]$ . We denote by  $\mathcal{S}_k(s, t)$  the simplex  $\{(r_1, \dots, r_k) \in [0, 1]^k; s \leq r_1 \leq \dots \leq r_k < t\}$ , and for simplicity we will write  $\mathcal{S}_k$  for  $\mathcal{S}_k(0, 1)$ . In contrast, whenever we deal with a discrete interval  $[s, t] \cap \mathcal{D}_n$ , we set  $\mathcal{S}'_k(s, t) = \{(r_1, \dots, r_k) \in \mathcal{D}_n^k; s \leq r_1 < \dots < r_k < t\}$ , and similarly, when  $s = 0$  and  $t = 1$  we simply write  $\mathcal{S}'_k$ .

Throughout the paper we work on a probability space  $(\Omega, \mathcal{F}, P)$ . If  $X$  is a random variable, we denote by  $\|X\|_{L_p}$  the  $L_p$ -norm of  $X$ . The letter  $K$  stands for a constant which can change from line to line. The letter  $G$  denotes a generic a.s. finite random variable. We denote by  $\lfloor a \rfloor$  the integer part of  $a$ .

## 2. Discrete rough paths

In this section, we introduce the concept of discrete rough paths and discrete rough integrals, and recall some basic results of the rough paths theory. Then we derive our main estimates on discrete rough integrals.

### 2.1. Definition and algebraic properties

This subsection is devoted to introduce the main rough paths notations which will be used in the sequel. The reader is referred to [15,16] for an introduction to the rough path theory.

Let  $\mathcal{V}$  be a finite dimensional vector space. We denote by  $\mathcal{C}_k(\mathcal{V})$  the set of functions  $g : \mathcal{S}_k \rightarrow \mathcal{V}$  such that  $g_{t_1 \dots t_k} = 0$  whenever  $t_i = t_{i+1}$  for  $i \leq k - 1$ . Such a function will be called a  $(k - 1)$ -*increment*. We define the operator  $\delta$  as follows:

$$\delta : \mathcal{C}_k(\mathcal{V}) \rightarrow \mathcal{C}_{k+1}(\mathcal{V}), \quad (\delta g)_{t_1 \dots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^i g_{t_1 \dots \hat{t}_i \dots t_{k+1}},$$

where  $\hat{t}_i$  means that this particular argument is omitted. For example, for  $f \in \mathcal{C}_1(\mathcal{V})$  and  $g \in \mathcal{C}_2(\mathcal{V})$  we have

$$\delta f_{st} = f_t - f_s \quad \text{and} \quad \delta g_{sut} = g_{st} - g_{su} - g_{ut}. \tag{2.1}$$

A fundamental property of  $\delta$ , which is easily verified, is that  $\delta\delta = 0$ , where  $\delta\delta$  is considered as an operator from  $\mathcal{C}_k(\mathcal{V})$  to  $\mathcal{C}_{k+2}(\mathcal{V})$ .

Let us now introduce the notion of rough path which will be used throughout the paper.

**Definition 2.1.** Consider  $\nu \in (0, 1)$ ,  $\ell \in \mathbb{N}$  such that  $\ell \leq \lfloor \frac{1}{\nu} \rfloor$  and  $p > 1$ . Let  $x = (x^1, \dots, x^\ell)$  be a continuous path on  $\mathcal{S}_2$  and with values in  $\bigoplus_{k=1}^\ell (\mathbb{R}^d)^{\otimes k}$ . For  $p > 1$  set

$$\|x^k\|_{[s,t], p, \nu} := \sup_{(u,v) \in \mathcal{S}_2([s,t])} \frac{|x^k_{uv}|_{L_p}^{1/k}}{|v - u|^\nu}. \tag{2.2}$$

For convenience, we denote  $\|x^k\|_{p, \nu} := \|x^k\|_{[0,1], p, \nu}$ . We define a  $\nu$ -Hölder semi-norm as follows:

$$\|x\|_{p, \nu} := \|x^1\|_{p, \nu} + \dots + \|x^\ell\|_{p, \nu}. \tag{2.3}$$

We call  $x$  a  $(L_p, \nu, \ell)$ -*rough path* (or simply a rough path) if the following properties holds true:

- (1) the semi-norms  $\|x^k\|_{[s,t], p, \nu}$  in (2.2) are finite. In this case we say that  $x^k$ ,  $k = 1, \dots, \ell$  are respectively in  $C^\nu(\mathcal{S}_2, (\mathbb{R}^m)^{\otimes k})$ .
- (2) For all  $k \in \{2, \dots, \ell\}$ ,  $x^k$  satisfies the identity

$$\delta x^k_{sut} = \sum_{j=1}^{k-1} x^{k-j}_{su} \otimes x^j_{ut}. \tag{2.4}$$

**Remark 2.2.** Our definition of rough path differs slightly from the usual one in several aspects:

- (i) We don't impose  $\ell = \lfloor \frac{1}{\nu} \rfloor$ , so that the order of our rough path might be lower than in the standard theory. In the sequel we will introduce another parameter  $\alpha \in (0, 1)$  such that  $\nu\ell + \alpha > 1$ .
- (ii) We consider a rough path with values in  $L_p$ , and measure its regularity by looking at increments of the form  $\|x^k_{st}\|_{L_p}$  for  $(s, t) \in \mathcal{S}_2$ .

(iii) The regular case  $\nu = 1$  could also be included in our considerations. We have refrained to do so since our limiting processes will mostly be non regular stochastic processes.

In this paper, we are mostly concerned with discrete sums. Recall that we are considering discrete simplexes related to uniform partitions of  $[0, 1]$ , which are denoted by  $\mathcal{S}'_2$ . We now introduce a general notion of discrete controlled process.

**Definition 2.3.** Fix  $\alpha > 0$  and let  $\ell$  be the smallest integer such that  $\nu\ell + \alpha > 1$ . Let  $\mathcal{V}$  be some finite dimensional vector space. Let  $y, y', y'', \dots, y^{(\ell-1)}$  be continuous processes on  $[0, 1]$  measurable with respect to  $\mathcal{F} := \sigma(x_t, t \in [0, 1])$  and with  $y_0 = y_0^{(0)} = 0$ . For convenience, we will also write:  $y^{(0)} = y, y^{(1)} = y', y^{(2)} = y'', \dots$ . Suppose that  $y$  takes values in  $\mathcal{V}$ , and  $y^{(k)}$  takes values in  $\mathcal{L}((\mathbb{R}^d)^{\otimes k}, \mathcal{V})$  for all  $k = 1, \dots, \ell - 1$ . For  $(s, t) \in \mathcal{S}_2$  and  $k = 0, 1, \dots, \ell - 2$  we denote

$$r_{st}^{(k)} = \delta y_{st}^{(k)} - y_s^{(k+1)} x_{st}^1 - \dots - y_s^{(\ell-1)} x_{st}^{\ell-k-1}, \tag{2.5}$$

and  $r_{st}^{(\ell-1)} = \delta y_{st}^{(\ell-1)}$ . We call  $(y^{(0)}, \dots, y^{(\ell-1)})$  a discrete  $\mathcal{V}$ -valued rough path in  $L_p$  controlled by  $(x, \alpha)$  if  $|r_{st}^{(k)}|_{L_p} \leq K(t-s)^{(\ell-k)\nu}$  for all  $k = 0, 1, \dots, \ell - 1$ . The discrete path  $(y^{(0)}, \dots, y^{(\ell-1)})$  is controlled by  $(x, \alpha)$  almost surely if  $|r_{st}^{(k)}| \leq G_y(t-s)^{(\ell-k)\nu}, k = 0, \dots, \ell - 1$  for some finite random variable  $G_y$ .

**Remark 2.4.** In some of our computations below we will rephrase (2.5) for  $k = 0$  as the following identity for  $(s, t) \in \mathcal{S}_2$ :

$$y_t = \sum_{i=0}^{\ell-1} y_s^{(i)} x_{st}^i + r_{st}^{(0)}, \tag{2.6}$$

where we take  $x^0 \equiv 1$  by convention. Note that while we call  $y$  a discrete controlled path, it is actually continuously defined on  $[0, 1]$ .

We label a simple algebraic property relating the remainders  $r^{(k)}$ .

**Lemma 2.5.** Let  $y = (y^{(0)}, \dots, y^{(\ell-1)})$  be a discrete rough path in  $L_p$  controlled by  $(x, \alpha)$  for all  $p > 1$ . Then the following identity holds true for all  $(s, u, t) \in \mathcal{S}'_3$ :

$$\delta r_{sut}^{(0)} = \sum_{i=1}^{\ell-1} r_{su}^{(i)} x_{ut}^i. \tag{2.7}$$

In particular, we have the following estimate for  $p > 1$ :

$$|\delta r_{sut}^{(0)}|_{L_p} \leq K(t-s)^{\nu\ell}. \tag{2.8}$$

**Proof.** By the definition of  $r^{(0)}$  in (2.5) and the expression (2.1) of  $\delta g$  for  $g \in \mathcal{C}_2(\mathcal{V})$ , some elementary computations yield:

$$\delta r_{sut}^{(0)} = - \sum_{i=1}^{\ell-1} y_s^{(i)} x_{st}^i + \sum_{i=1}^{\ell-1} y_u^{(i)} x_{ut}^i + \sum_{i=1}^{\ell-1} y_s^{(i)} x_{su}^i = \sum_{i=1}^{\ell-1} \delta y_{su}^{(i)} x_{ut}^i - \sum_{i=2}^{\ell-1} y_s^{(i)} \delta x_{sut}^i, \tag{2.9}$$

where we have used the fact that  $\delta x_{sut}^1 = 0$ . Therefore, invoking (2.5) and (2.4) again we obtain

$$\begin{aligned} \delta r_{sut}^{(0)} &= \sum_{i=1}^{\ell-1} r_{su}^{(i)} x_{ut}^i + \sum_{i=1}^{\ell-2} \sum_{j=i+1}^{\ell-1} y_s^{(j)} x_{su}^{j-i} \otimes x_{ut}^i - \sum_{i=2}^{\ell-1} y_s^{(i)} \sum_{j=1}^{i-1} x_{su}^{i-j} \otimes x_{ut}^j \\ &= \sum_{i=1}^{\ell-1} r_{su}^{(i)} x_{ut}^i. \end{aligned}$$

This concludes the identity (2.7). The inequality (2.8) follows by taking  $L_p$ -norm on both sides of (2.7) and taking into account the assumption that  $|x_{st}^i|_{L_p} \leq K(t-s)^{\nu i}$  and  $|r_{st}^{(i)}|_{L_p} \leq K(t-s)^{(\ell-i)\nu}$ . □

2.2. Discrete rough integrals

In this subsection, we derive upper-bound estimates for some “discrete” integrals defined as Riemann type sums. Namely, let  $f$  and  $g$  be functions on  $\mathcal{S}'_2$ . For a generic partition  $\mathcal{D}_n = \{0 = t_0 < \dots < t_n = 1\}$  of  $[0, 1]$ , we set

$$\varepsilon(t) = t_k \quad \text{for } t \in (t_{k-1}, t_k]. \tag{2.10}$$

We define the discrete integral of  $f$  with respect to  $g$  as:

$$\mathcal{J}_s^t(f; g) := \sum_{s \leq t_k < t} f_{\varepsilon(s)t_k} \otimes g_{t_k t_{k+1}}, \quad (s, t) \in \mathcal{S}_2, \tag{2.11}$$

where we highlight the fact that  $f$  is a function of two variables. Similarly, if  $f$  is a path on the grid  $0 = t_0 < \dots < t_n = 1$ , then we define the discrete integral of  $f$  with respect to  $g$  as:

$$\mathcal{J}_s^t(f; g) := \sum_{s \leq t_k < t} \delta f_{\varepsilon(s)t_k} \otimes g_{t_k t_{k+1}}, \quad (s, t) \in \mathcal{S}_2. \tag{2.12}$$

**Remark 2.6.** Notice that in (2.10),  $\varepsilon(t)$  is the upper endpoint of the partition when  $t \in (t_{k-1}, t_k]$ . As a result, the first term of the Riemann sum (2.12) is always vanishing. In addition, we also have  $\mathcal{J}_{t_k}^{t_{k+1}}(f; g) = 0$  for all  $(t_k, t_{k+1}) \in \mathcal{S}'_2$ .

The next proposition gives a basic estimate for discrete integrals. In the following,  $\mathcal{V}$  and  $\mathcal{V}'$  stand for some finite dimensional vector spaces.

**Proposition 2.7.** *Let  $y = (y^{(0)}, \dots, y^{(\ell-1)})$  be a discrete rough path on  $[0, 1]$ , controlled by  $(x, \alpha)$  in  $L_2$ , and let  $h$  be a 1-increment defined on  $\mathcal{S}'_2$  with values in  $\mathcal{V}$ . Suppose that  $h$  satisfies*

$$|\mathcal{J}_s^t(x^i; h)|_{L_2} \leq K(t - s)^{\alpha + \nu i}, \tag{2.13}$$

for  $i = 0, 1, \dots, \ell - 1$  and  $(s, t) \in \mathcal{S}'_2$ , where we recall that  $\ell$  is an integer such that  $\alpha + \nu \ell > 1$ . Then we have the estimate

$$|\mathcal{J}_s^t(r^{(0)}; h)|_{L_1} \leq K(t - s)^{\nu \ell + \alpha}, \tag{2.14}$$

which is valid for  $(s, t) \in \mathcal{S}'_2$ .

**Proof.** In order to bound the increment  $R_{st} := \mathcal{J}_s^t(r^{(0)}; h)$ , we first note that  $R_{t_k t_{k+1}} = 0$ , due to the fact that  $r_{t_k t_k}^{(0)} = 0$ . Let us now calculate  $\delta R$ : for  $(s, u, t) \in \mathcal{S}'_3$ , it is readily checked that

$$\begin{aligned} \delta R_{sut} &= \mathcal{J}_s^t(r^{(0)}; h) - \mathcal{J}_s^u(r^{(0)}; h) - \mathcal{J}_u^t(r^{(0)}; h) \\ &= \sum_{u \leq t_k < t} (r_{st_k}^{(0)} - r_{ut_k}^{(0)}) h_{t_k t_{k+1}}. \end{aligned}$$

Writing  $r_{st_k}^{(0)} - r_{ut_k}^{(0)} = \delta r_{sut_k}^{(0)} + r_{su}^{(0)}$  and invoking relation (2.7), we thus obtain

$$\delta R_{sut} = \sum_{i=0}^{\ell-1} r_{su}^{(i)} \mathcal{J}_u^t(x^i; h). \tag{2.15}$$

Now take the  $L_1$ -norm on both sides of (2.15), take into account condition (2.13) and the hypothesis  $\nu \ell + \alpha > 1$ , and then apply the discrete sewing Lemma 2.7 in [30]. This easily yields the desired estimate (2.14).  $\square$

3. Limit theorems

In this section, we first prove a general limit theorem for discrete integrals. Then we will handle two more specific situations which arise often in applications.



3.1. General limit theorem

Recall that the discrete integral  $\mathcal{J}_s^t(y; h)$  is defined in (2.11). In this subsection, we prove a general limit theorem for  $\mathcal{J}_s^t(y; h)$ .

**Theorem 3.1.** *Let  $\mathcal{V}$  and  $\mathcal{V}'$  be two finite-dimensional vector spaces. Let  $(y^{(0)}, \dots, y^{(\ell-1)})$  be a discrete  $\mathcal{V}$ -valued rough path on  $[0, 1]$  controlled by  $(x, \alpha)$  in  $L_2$  or almost surely (see Definition 2.3), and  $h^n$  be a 1-increment which satisfies (2.13) uniformly in  $n$ . Consider the family  $\mathcal{J}_s^t(x^i; h^n)$  defined by (2.11), and suppose that as  $n \rightarrow \infty$ :*

$$(\mathcal{J}(x^i; h^n), i \in \mathcal{I}') \xrightarrow{\text{stable f.d.d.}} (\omega^i, i \in \mathcal{I}'), \tag{3.1}$$

where  $(\omega^i, i \in \mathcal{I}')$  is a 1-increment independent of  $x$ , and  $\xrightarrow{\text{stable f.d.d.}}$  stands for  $\mathcal{F}$ -stable convergence of finite dimensional distributions; see e.g. [1,26] for the definition of stable convergence. Assume that for  $i \in \mathcal{I}''$  we have

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \sum_{j=0}^{m-1} y_{u_j}^{(i)} \mathcal{J}_{u_j}^{u_{j+1}}(x^i; h^n) \right| = 0 \tag{3.2}$$

in probability, where  $\{u_j, j = 0, 1, \dots, m\}$  is a uniform partition of  $[0, 1]$ . Here  $\mathcal{I}', \mathcal{I}''$  are disjoint subsets of  $\mathcal{I} := \{0, 1, \dots, \ell - 1\}$  such that  $\mathcal{I}' \cup \mathcal{I}'' = \mathcal{I}$ . Suppose further that if  $\mathcal{J}(y^{(i)}; \omega^i)$  is given by (2.11), we have

$$(\mathcal{J}(y^{(i)}; \omega^i), i \in \mathcal{I}') \xrightarrow{\text{stable f.d.d.}} (v^i, i \in \mathcal{I}'), \tag{3.3}$$

where  $v^i, i \in \mathcal{I}'$  are  $\mathcal{V} \otimes \mathcal{V}'$ -valued 1-increment. Then the following convergence holds true as  $n \rightarrow \infty$ :

$$\mathcal{J}(y; h^n) \xrightarrow{\text{stable f.d.d.}} \sum_{i \in \mathcal{I}'} v^i. \tag{3.4}$$

**Remark 3.2.** If we particularize our limit theorem to the level  $i = 0$  of  $\mathcal{J}(x^i; h^n)$ , we just get that  $h^n \xrightarrow{\text{stable f.d.d.}} \omega^0$  as part of the standing assumption. In return, we obtain that  $v^0 = \int_0^1 y_s d\omega_s^0$  in relation (3.4).

**Remark 3.3.** As the reader might have observed, Theorem 3.1 gives a general transfer principle from limit theorems for unweighted sums to limit theorems for weighted sums, within a rough paths framework.

**Remark 3.4.** Condition (3.1) is more demanding than condition (3.3) in Theorem 3.1. Indeed, condition (3.3) is usually reduced to the convergence of a Riemann sum to an Itô or Riemann type integral.

**Proof of Theorem 3.1.** For sake of conciseness we will only show the  $\mathcal{F}$ -stable convergence of  $\mathcal{J}_0^1(y; h^n)$  and for the case  $\mathcal{I}' = \mathcal{I}$ . The stable convergence of the finite dimensional distributions of  $\mathcal{J}(y; h^n)$  in the general case  $\mathcal{I}' \subset \mathcal{I}$  can be shown in a similar way. The proof is divided into several steps.

*Step 1: A decomposition of  $\mathcal{J}_0^1(y; h^n)$ .* Take two uniform partitions on  $[0, 1]$ :  $t_k = k/n$  for  $k, n \in \mathbb{N}$  and  $u_j = j/m$  for  $j, m \in \mathbb{N}$ , and  $m \ll n$ . Set:

$$D_j = \{t_k : u_{j+1} > t_k \geq u_j\} \quad \text{and} \quad \bar{u}_j = \varepsilon(u_j), \tag{3.5}$$

where the function  $\varepsilon$  has been introduced in (2.10). By definition (2.12) we have

$$\mathcal{J}_0^1(y; h^n) = \sum_{k=0}^{n-1} \delta y_{0t_k} \otimes h_{t_k t_{k+1}}^n = \sum_{k=0}^{n-1} y_{t_k} \otimes h_{t_k t_{k+1}}^n,$$

where the second identity is due to the fact that we have assumed  $y_0 = 0$  in Definition 2.3. Next we decompose the Riemann sum thanks to the sets  $D_j$ . We get

$$\mathcal{J}_0^1(y; h^n) = \sum_{j=0}^{m-1} \sum_{t_k \in D_j} y_{t_k} h_{t_k t_{k+1}}^n.$$



Now we invoke relation (2.6) with  $s = \bar{u}_j$  and  $t = t_k$  whenever  $t_k \in D_j$ . This yields:

$$\mathcal{J}_0^1(y; h^n) = \varphi^0 + \dots + \varphi^{\ell-1} + R^\varphi, \tag{3.6}$$

where

$$\begin{aligned} \varphi^i &= \sum_{j=0}^{m-1} \sum_{t_k \in D_j} y_{\bar{u}_j}^{(i)} x_{\bar{u}_j t_k}^i \otimes h_{t_k t_{k+1}}^n = \sum_{j=0}^{m-1} y_{\bar{u}_j}^{(i)} \mathcal{J}_{u_j}^{u_{j+1}}(x^i; h^n), \\ R^\varphi &= \sum_{j=0}^{m-1} \sum_{t_k \in D_j} r_{\bar{u}_j t_k}^{(0)} \otimes h_{t_k t_{k+1}}^n = \sum_{j=0}^{m-1} \mathcal{J}_{u_j}^{u_{j+1}}(r^{(0)}; h^n), \end{aligned} \tag{3.7}$$

and where we have set  $x_{st}^0 = 1$  by convention. Let us further decompose  $\varphi^i$  as follows:

$$\begin{aligned} \varphi^i &= \sum_{j=0}^{m-1} y_{u_j}^{(i)} \mathcal{J}_{u_j}^{u_{j+1}}(x^i; h^n) + \sum_{j=0}^{m-1} (y_{\bar{u}_j}^{(i)} - y_{u_j}^{(i)}) \mathcal{J}_{u_j}^{u_{j+1}}(x^i; h^n) \\ &:= \varphi_1^i + \varphi_2^i. \end{aligned}$$

We now study the convergence of  $\varphi_1^i$  and  $\varphi_2^i$  separately.

*Step 2: Convergences of  $\varphi_2^i$ .* In this step we show that for  $i = 1, \dots, \ell - 1$ , the random variable  $\varphi_2^i$  converges to zero in probability as  $n \rightarrow \infty$ . To this aim, it suffices to consider the case when:

$$(y^{(0)}, \dots, y^{(\ell-1)}) \text{ is controlled by } (x, \alpha) \text{ in } L_p, \text{ for an arbitrary } p > 1. \tag{3.8}$$

Indeed, for  $\varepsilon > 0$ , we can find a constant  $K$  such that  $P(G_y > K) \leq \varepsilon$  (see Definition 2.3 for the definition of  $G_y$ ). Define  $(\bar{y}^{(0)}, \dots, \bar{y}^{(\ell-1)})$  such that  $\bar{y}^i = y^i$  on  $\{G_y \leq K\}$  and  $\bar{y}^i \equiv 0$  on  $\{G_y > K\}$ . Then  $(\bar{y}^{(0)}, \dots, \bar{y}^{(\ell-1)})$  satisfies the condition (3.8), and we can write

$$\begin{aligned} P(|\varphi_2^i| > \varepsilon) &= P(|\varphi_2^i| > \varepsilon, G_y \leq K) + P(|\varphi_2^i| > \varepsilon, G_y > K) \\ &\leq P(|\tilde{\varphi}_2^i| > \varepsilon) + \varepsilon, \end{aligned}$$

where  $\tilde{\varphi}_2^i = \varphi_2^i$  when  $G_y \leq K$  and  $\tilde{\varphi}_2^i = 0$  when  $G_y > K$ . So if we can show that  $\tilde{\varphi}_2^i \rightarrow 0$  in probability, then the same convergence holds for  $\varphi_2^i$ .

Assume now that (3.8) is true. In this case we have

$$|\varphi_2^i| \leq \sum_{j=0}^{m-1} |y_{\bar{u}_j}^{(i)} - y_{u_j}^{(i)}| \cdot |\mathcal{J}_{u_j}^{u_{j+1}}(x^i; h^n)|. \tag{3.9}$$

Taking the  $L_1$ -norm on both sides of the inequality (3.9), invoking the fact that  $h^n$  satisfies relation (2.13) uniformly in  $n$ , and using the continuity of  $y^{(i)}$  given by (3.8), we easily obtain the following convergence in probability:

$$\lim_{n \rightarrow \infty} |\varphi_2^i| \rightarrow 0.$$

*Step 3: Convergences of  $\varphi_1^i$ .* The convergences of  $\sum_{i=0}^{\ell-1} \varphi_1^i$  follows immediately from the assumptions of the theorem. Indeed, fixing  $m$  and sending  $n$  to  $\infty$ , our assumption (3.1) directly yields the stable convergence:

$$\sum_{i=0}^{\ell-1} \varphi_1^i \rightarrow \sum_{i=0}^{\ell-1} y_{u_j}^i \omega_{u_j u_{j+1}}^i. \tag{3.10}$$

We now send  $m \rightarrow \infty$  in (3.10) and recall the convergence (3.3). This yields the stable convergence:

$$\sum_{i=0}^{\ell-1} \varphi_1^i \rightarrow \sum_{i=0}^{\ell-1} v^i,$$

as  $n \rightarrow \infty$  and  $m \rightarrow \infty$ .

Step 4: Convergences of the remainder term  $R^\varphi$ . Going back to equation (3.6) and summarizing our computations, our claim (3.4) is now reduced to show that we have

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |R^\varphi| = 0, \tag{3.11}$$

in probability. Moreover, as in Step 2 it suffices to show the convergence (3.11) under condition (3.8). Eventually, applying Proposition 2.7 to (3.7) we obtain:

$$|R^\varphi|_{L_1} \leq K \sum_{j=0}^{m-1} m^{-\nu\ell - \alpha} \leq K m^{1-\nu\ell - \alpha}. \tag{3.12}$$

The convergence (3.11) then follows from (3.12) and the fact that  $\nu\ell + \alpha > 1$ . □

Theorem 3.1 allows us to distinguish two predominant cases: (i) A usual asymptotic regime, for which only one level  $v^i$  remains. (ii) A critical case, for which more than one level survive as  $n$  goes to  $\infty$ .

The following definition captures those different behaviors.

**Definition 3.5.** We will call a limit theorem *single* if  $\mathcal{I}'$  in Theorem 3.1 has only one element. Similarly, a limit theorem is called *double* when  $\mathcal{I}'$  has two elements.

**Remark 3.6.** If the convergences in (3.1) and (3.3) hold true in probability, then in a similar way one can show that  $\mathcal{J}_s^t(y^{(0)}; h^n)$  converges to  $\sum_{i \in \mathcal{I}'} v_{st}^i$  in probability.

**Remark 3.7.** In the case  $\nu > \frac{1}{2}$  and  $\alpha \geq \frac{1}{2}$ , we have  $\ell = 1$  and  $\mathcal{I} = \{0\}$ . Therefore, conditions (2.13), (3.1), (3.2) are reduced to  $|h^n|_{L_2} \leq K(t-s)^\alpha$  and  $h^n \xrightarrow{\text{stable, f.d.d.}} \omega$ . If  $h^n \rightarrow \omega$  in  $L_p$  for all  $p \geq 1$ , then  $\mathcal{J}_0^1(y; h^n)$  also converges in  $L_q$  for  $q \geq 1$ . This situation allows to recover the results in [9] and [25, Proposition 7.1]. A more specific statement will be given in Proposition 4.9 below.

### 3.2. Single limit theorem I

An important case in Theorem 3.1 is when  $h^n$  converges in distribution to a Brownian motion and the discrete integral  $\mathcal{J}_0^1(y^{(0)}; h^n)$  converges to the Wiener integral  $\int_0^1 y_t \otimes dW_t$ . In this subsection we investigate this type of limit theorems.

**Theorem 3.8.** Let  $x$  be a  $(L_p, \nu, \ell)$ -rough path for  $p = 4, \nu \in (0, 1)$  and  $\ell$  such that  $\nu\ell + \frac{1}{2} > 1$ . Let  $y = (y^{(0)}, \dots, y^{(\ell-1)})$  be a process on  $[0, 1]$  controlled by  $(x, \frac{1}{2})$  in  $L_2$  or almost surely (see Definition 2.3), and assume that  $h^n$  satisfies the inequality (2.13) uniformly in  $n$ . Suppose that the following assumptions are fulfilled:

- (i) We have the convergence  $h^n \xrightarrow{\text{stable, f.d.d.}} W$  as  $n \rightarrow \infty$ , where  $W$  is a standard Brownian motion independent of  $x$ .
- (ii) For any sequence of partitions  $0 \leq s_0 < s_1 < \dots < s_m \leq 1$  on  $[0, 1]$  such that  $|s_{j+1} - s_j| \leq 1/m$ , we have

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(x^i; h^n) \right| = 0 \tag{3.13}$$

in probability for  $i = 1, \dots, \ell - 1$ .

Then we have the following convergence in distribution for the process  $y$ :

$$\mathcal{J}(y; h^n) \xrightarrow{\text{stable f.d.d.}} v, \tag{3.14}$$

where the integral  $v_{st} := \int_s^t y_u \otimes dW_u$  has to be understood as a conditional Wiener integral.

**Remark 3.9.** A classical result giving a convergence result similar to our Theorem 3.8 can be found in the seminal paper [27]. In this reference the authors consider convergences for general stochastic integrals of predictable processes with respect to a sequence of semi-martingales. Notice that we could certainly apply our methods to this context provided a rough path above the semi-martingale is given. However, the class of controlled paths is not as general as the class

of arbitrary predictable processes. We shall thus not delve deeper into this direction. Also notice that the references [19,33,36] deal with the fractional Brownian case (as already mentioned in the introduction). Our result generalizes the considerations therein.

**Remark 3.10.** Theorem 3.8 can be generalized to some other interesting situations. For example, suppose that  $h^n \xrightarrow{\text{stable f.d.d.}} \omega$ , where  $\omega$  is a continuous Gaussian process independent of  $x$ . Let  $\mathcal{H}$  be the Hilbert space corresponding to  $\omega$  and assume that  $C^\gamma \subset \mathcal{H}$  for  $\gamma < \nu$ . We assume that for any  $f \in C^\gamma$ , we have the following convergences for a generic partition  $0 \leq s_0 < \dots < s_m \leq 1$ :

$$\lim_{m \rightarrow \infty} \sum_{j,j'=0}^m \langle \delta f_{s_j, \cdot}, \delta f_{s_{j'}, \cdot} \rangle_{\mathcal{H}} = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \sum_{j=0}^{m-1} f_{t_j} \mathbf{1}_{[t_j, t_{j+1})} = f \tag{3.15}$$

where the second limit is a limit in  $\mathcal{H}$ . Then following the lines of the proof of Theorem 3.8 one can show that

$$\mathcal{J}_0^1(y; h^n) \xrightarrow{d} \int_0^1 y \otimes d\omega.$$

**Proof of Theorem 3.8.** Take  $\mathcal{I}' = \{0\}$  and  $\mathcal{I}'' = \{1, \dots, \ell\}$ . The theorem will be proved by applying Theorem 3.1 and verifying the convergences (3.1), (3.2) and (3.3). The proof is divided into several steps.

*Step 1:* We will show by induction that

$$\mathcal{J}_s^t(x^i; h^n) \xrightarrow{\text{stable f.d.d.}} \omega^i, \quad i = 0, 1, \dots, \ell - 1, \tag{3.16}$$

where we denote

$$\omega_{st}^i \equiv \int_s^t x_{su}^i \otimes dW_u.$$

Note that the convergence (3.16) is equivalent to that for any  $z \in \mathcal{F}$ :

$$(z, \mathcal{J}_s^t(x^i; h^n)) \xrightarrow{f.d.d.} (z, \omega^i), \quad i = 0, 1, \dots, \ell - 1,$$

a fact that we will use several times in our proof.

Since  $h^n \rightarrow W$  in *f.d.d.* sense, convergence (3.16) holds true when  $i = 0$ . Now assume that the convergence holds for  $i = 0, 1, \dots, \tau - 1$  with  $\tau < \ell$ . Take  $m \ll n$  and  $u_j = j/m$ , and set  $D_j = \{t_k : u_{j+1} > t_k \geq u_j\}$  as in (3.5). Take  $j_1$  such that  $s \in D_{j_1}$  and  $j_2$  such that  $t \in D_{j_2}$ . Then a small variant of (2.4) shows that for all  $t_k \in D_j$ ,

$$x_{\varepsilon(s)t_k}^\tau = \delta x_{\varepsilon(s), \bar{u}_j \vee \varepsilon(s), t_k}^\tau + x_{\varepsilon(s), \bar{u}_j \vee \varepsilon(s)}^\tau + x_{\bar{u}_j \vee \varepsilon(s), t_k}^\tau = \sum_{l=0}^{\tau} x_{\varepsilon(s), \bar{u}_j \vee \varepsilon(s)}^{\tau-l} \otimes x_{\bar{u}_j \vee \varepsilon(s), t_k}^l,$$

where recall that the function  $\varepsilon$  is defined in (2.10) and  $\bar{u}_j$  is given by (3.5). Hence it is readily checked that:

$$\mathcal{J}_s^t(x^\tau; h^n) = \sum_{l=0}^{\tau} \sum_{j=j_1}^{j_2} x_{\varepsilon(s), \bar{u}_j \vee \varepsilon(s)}^{\tau-l} \otimes \mathcal{J}_{u_j \vee s}^{u_{j+1} \wedge t} (x^l; h^n) := \sum_{l=0}^{\tau} A_l(s, t). \tag{3.17}$$

Let us change the name of our variables in order to match the notation of our theorem and use relation (3.13). Namely, set  $s_0 = s, s_1 = u_{j_1+1}, \dots, s_{j_2-j_1} = u_{j_2}, s_{j_2-j_1+1} = t$ . Then it is readily checked that  $A_\tau(s, t) = \sum_{j=0}^{j_2-j_1} \mathcal{J}_{s_j}^{s_{j+1}}(x^\tau; h^n)$ . Thus invoking assumption (3.13), we directly have the following convergence in probability:

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |A_\tau(s, t)| = 0. \tag{3.18}$$

In order to study the convergence of  $A_l$  for  $l < \tau$ , we first check that we can replace  $x_{\varepsilon(s), \bar{u}_j \vee \varepsilon(s)}^{\tau-l}$  by  $x_{s, u_j \vee s}^{\tau-l}$ . Indeed, we have the identity:

$$x_{\varepsilon(s), \bar{u}_j \vee \varepsilon(s)}^{\tau-l} - x_{s, u_j \vee s}^{\tau-l} = \delta x_{s, u_j \vee s, \bar{u}_j \vee \varepsilon(s)}^{\tau-l} - \delta x_{s, \varepsilon(s), \bar{u}_j \vee \varepsilon(s)}^{\tau-l} + x_{u_j \vee s, \bar{u}_j \vee \varepsilon(s)}^{\tau-l} - x_{s, \varepsilon(s)}^{\tau-l}.$$

Therefore, if we set:

$$\tilde{A}_l(s, t) := \sum_{j=j_1}^{j_2} x_{s, u_j \vee s}^{\tau-l} \otimes \mathcal{J}_{u_j \vee s}^{u_{j+1} \wedge t}(x^l; h^n) = \sum_{j=0}^{j_2-j_1} x_{ss_j}^{\tau-l} \otimes \mathcal{J}_{s_j}^{s_{j+1}}(x^l; h^n),$$

then it is readily checked that:

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |A_l(s, t) - \tilde{A}_l(s, t)| = 0 \quad \text{in probability.} \tag{3.19}$$

Let us now check the convergence for  $\tilde{A}_l$ . Sending  $n \rightarrow \infty$  and applying the induction assumption (3.16) with  $l < \tau$ , we get

$$(z, \tilde{A}_l) \xrightarrow{f.d.d.} (z, \bar{A}_l), \tag{3.20}$$

where

$$\bar{A}_l(s, t) := \sum_{j=0}^{j_2-j_1} x_{ss_j}^{\tau-l} \otimes \int_{s_j}^{s_{j+1}} x_{s_j u}^l \otimes dW_u.$$

We now separate the analysis of  $\tilde{A}_l$  in two cases.

(a) For  $0 < l < \tau$  the square of the  $L_2$ -norm of the right-hand side of (3.20) can be bounded thanks to Itô's isometry by:

$$K \mathbb{E} \left[ \sum_{j=0}^{j_2-j_1} |x_{ss_j}^{\tau-l}|^2 \int_{s_j}^{s_{j+1}} |x_{s_j u}^l|^2 du \right], \tag{3.21}$$

which by property (2.2) applied to  $p = 4$  and  $l \geq 1$  is less than

$$K \sum_{j=0}^{m-1} (s_{j+1} - s_j)^{2\nu+1}.$$

Owing to the fact that  $2\nu + 1 > 1$ , it is now trivially seen that as  $m \rightarrow \infty$ , the right-hand side of (3.20) converges to zero. Thus we get:

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |\tilde{A}_l(s, t)| = 0 \tag{3.22}$$

in probability. In summary of (3.19) and (3.22), we have the convergence

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |A_l(s, t)| = 0, \tag{3.23}$$

in probability for  $0 < l < \tau$ .

(b) When  $l = 0$ , the convergence (3.20) implies that, as  $n \rightarrow \infty$  and  $m \rightarrow \infty$  we obtain

$$(z, \tilde{A}_0) \xrightarrow{f.d.d.} (z, \omega^\tau). \tag{3.24}$$

Taking into account (3.19), the convergence (3.24) implies that

$$(z, A_0) \xrightarrow{f.d.d.} (z, \omega^\tau). \tag{3.25}$$

Putting together (a), (b) and the case  $l = \tau$ , we can now propagate our induction hypothesis. Indeed, applying (3.18), (3.23) and (3.25) to (3.17), we obtain

$$(z, \mathcal{J}(x^\tau; h^n)) \xrightarrow{f.d.d.} (z, \omega^\tau) \tag{3.26}$$

for  $\omega_{st}^\tau = \int_s^t x_{sr}^\tau \otimes dW_r$ . This completes the proof of (3.16) for  $i = 0, \dots, \ell - 1$ . Note that this shows that condition (3.1) in Theorem 3.1 holds true.

*Step 2:* In this step, we consider the convergence of  $\mathcal{J}_0^1(y^{(i)}; \omega^i)$ , which will yield condition (3.2) in Theorem 3.1.

We first show that the discrete integral  $\mathcal{J}_0^1(y^{(i)}; \omega^i)$ ,  $\ell > i > 0$  converges to zero in probability. As in the proof of Theorem 3.1, by a truncation argument, it suffices to show the convergence when  $(y^{(0)}, \dots, y^{(\ell-1)})$  is controlled by  $(x, \frac{1}{2})$  in  $L_p$  for  $p$  large enough. In this case, similarly to (3.21), we have

$$\begin{aligned} |\mathcal{J}_0^1(y^{(i)}; \omega^i)|_{L_2}^2 &\leq K \mathbb{E} \left[ \sum_{j=0}^{m-1} |y_{u_j}^{(i)}|^2 \int_{u_j}^{u_{j+1}} |x_{u_j u}^i|^2 du \right] \\ &\leq K \sum_{j=0}^{m-1} m^{-2\nu-1}. \end{aligned} \tag{3.27}$$

Therefore, we have  $\mathcal{J}_0^1(y^{(i)}; \omega^i) \rightarrow 0$  in probability. Combining this convergence with (3.26) for  $i = 1, \dots, \ell - 1$ , we obtain the convergence (3.2).

On the other hand, for the quantity  $\mathcal{J}_0^1(y^{(i)}; \omega^i)$  with  $i = 0$ , thanks to the convergences of Riemann sums related to Wiener integrals the following convergence holds in  $L^2$ :

$$\mathcal{J}_0^1(y; \omega^0) = \mathcal{J}_0^1(y; W) \rightarrow \int_0^1 y_t \otimes dW_t.$$

So the condition (3.3) holds true with  $v^0 = \int_0^1 y_t \otimes dW_t$ .

Summarizing our consideration, we can now apply Theorem 3.1 to  $\mathcal{J}_0^1(y; h^n)$  and we obtain the convergence (3.14). □

### 3.3. Single limit theorem II

In Section 3.2 we have investigated possible limit theorems under the assumption  $h^n \rightarrow W$ , which implies in particular  $\mathcal{J}_s^t(x^0; h^n) \rightarrow \delta W_{st}$ . In the current section we analyze situations for which the convergence of  $\mathcal{J}_s^t(x^i; h^n)$  is assumed for a more general  $i$ . Our results are summarized in the following theorem.

**Theorem 3.11.** *Let  $y = (y^{(0)}, \dots, y^{(\ell-1)})$  be a  $\mathcal{V}$ -valued rough path on  $[0, 1]$  controlled by  $(x, \alpha)$  in  $L_2$  or almost surely, and consider  $h^n$  satisfying the inequality (2.13). Recall that the increment  $\mathcal{J}(y; h^n) = \{\mathcal{J}_s^t(y; h^n); (s, t) \in \mathcal{S}_2\}$  is defined by (2.11). Suppose that  $x$  and  $h^n$  verify the following assumptions:*

- (i) *There is some  $\tau \in \mathcal{I}$  such that  $\mathcal{J}_s^t(x^\tau; h^n) \rightarrow (t - s)\varrho$  in probability for all  $(s, t) \in \mathcal{S}_2$ , where  $\varrho \in (\mathbb{R}^d)^{\otimes \tau} \otimes \mathcal{V}'$  is a constant matrix, and  $\mathcal{J}_s^t(x^i; h^n) \rightarrow 0$  in probability for all  $i < \tau$  and  $(s, t) \in \mathcal{S}_2$ .*
- (ii) *For any sequence of partitions  $0 \leq s_0 < s_1 < \dots < s_m \leq 1$  on  $[0, 1]$  such that  $|s_{j+1} - s_j| \leq 1/m$ , we have*

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(x^i; h^n) \right| = 0, \tag{3.28}$$

*in probability for  $i = \tau + 1, \dots, \ell - 1$ .*

*Then the following convergence holds true for  $y$ :*

$$\lim_{n \rightarrow \infty} \mathcal{J}(y; h^n) \longrightarrow \left( \int y_t^{(\tau)} dt \right) \otimes \varrho \tag{3.29}$$

*in probability.*

**Proof.** As for Theorem 3.8, we will prove our claim thanks to Theorem 3.1, and we are reduced to check (3.1), (3.2), and (3.3). The difference with Theorem 3.8 is that we now consider  $\mathcal{I}' = \{\tau\}$  and  $\mathcal{I}'' = \mathcal{I} \setminus \{\tau\}$ . We divide the proof in several steps.

*Step 1: Case  $i < \tau$ .* In this first situation it is immediate from our assumptions that (3.2) holds true for  $i < \tau$ .

Step 2: Case  $i \geq \tau$ . Similarly to the proof of Theorem 3.8 (Step 1), we prove by induction the following convergence in probability for all  $\ell > i \geq \tau$ :

$$\mathcal{J}(x^i; h^n) \longrightarrow \omega^i \quad \text{where } \omega_{st}^i = \left( \int_s^t x_{su}^{i-\tau} du \right) \otimes \varrho. \tag{3.30}$$

To this aim, notice that (3.30) is true for  $i = \tau$  by assumption. Next assume that (3.30) holds for  $i = \tau, \dots, \tau' - 1$ . We decompose the discrete interval  $\llbracket s, t \rrbracket$  into the subintervals  $D_j$  again (see (3.5)), with  $m \ll n$  and  $t_k = \frac{k}{n}$ ,  $u_j = \frac{j}{m}$ . Let  $s_0, \dots, s_{j_2-j_1+1}$  be as in Theorem 3.8 (Step 1). Then an approximation procedure similar to (3.19) allows to replace each  $x_{\varepsilon(s)t_k}^{\tau'-l}$  by an expression of the form:

$$\sum_{l=0}^{\tau'} x_{ss_j}^{\tau-l} \otimes x_{s_j t_k}^l,$$

in the sum defining  $\mathcal{J}_s^t(x^{\tau'}; h^n)$ . Therefore, we get an equivalent of (3.19) in our context:

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathcal{J}_s^t(x^{\tau'}; h^n) - \sum_{l=0}^{\tau'} \tilde{A}_l \right| = 0 \tag{3.31}$$

with

$$\tilde{A}_l = \sum_{j=0}^{j_2-j_1} x_{ss_j}^{\tau'-l} \otimes \mathcal{J}_{s_j}^{s_{j+1}}(x^l; h^n).$$

We now handle each  $\tilde{A}_l$ . For  $l < \tau$ , each  $\mathcal{J}_{s_j}^{s_{j+1}}(x^l; h^n)$  converges to 0 in probability as  $n \rightarrow \infty$  for all  $j$ , according to our assumption (i). Hence  $\tilde{A}_l \rightarrow 0$  in probability as  $n \rightarrow \infty$  and  $m \rightarrow \infty$ . When  $\tau < l < \tau'$ , we proceed along the same lines as for (3.20) and (3.21). Namely, we invoke the fact that  $\lim_{n \rightarrow \infty} \mathcal{J}_{s_j}^{s_{j+1}}(x^l; h^n) = \left( \int_{s_j}^{s_{j+1}} x_{s_j u}^{l-\tau} du \right) \otimes \varrho$  for each  $j_1 \leq j \leq j_2$  and then use the extra regularity given by  $x_{s_j u}^l$  on each  $[s_j, s_{j+1}]$ . This yields the following limit in probability:

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |\tilde{A}_l| \rightarrow 0. \tag{3.32}$$

Let now  $l = \tau'$ . Then

$$\tilde{A}_{\tau'} = \sum_{j=0}^{j_2-j_1} \mathcal{J}_{s_j}^{s_{j+1}}(x^{\tau'}; h^n),$$

and it is immediate from identity (3.28) that (3.32) holds true for  $l = \tau'$ . In summary, we have proved that for all  $l \in \{0, \dots, \tau'\} \setminus \{\tau\}$  we have:

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |\tilde{A}_l| = 0 \tag{3.33}$$

in probability.

In the case  $l = \tau$ , by sending  $n \rightarrow \infty$ , our assumption (i) allows to write:

$$\tilde{A}_\tau \longrightarrow \sum_{j=0}^{j_2-j_1} x_{ss_j}^{\tau'-\tau} (s_{j+1} - s_j) \otimes \varrho, \tag{3.34}$$

in probability, and thus as  $n \rightarrow \infty$  and  $m \rightarrow \infty$ , we obtain

$$\tilde{A}_\tau \longrightarrow \left( \int_s^t x_{su}^{\tau'-\tau} du \right) \otimes \varrho \tag{3.35}$$

in probability. Combining (3.33) and (3.35) and taking into account relation (3.31), we end up with:

$$\mathcal{J}_s^t(x^{\tau'}; h^n) \longrightarrow \left( \int_s^t x_{su}^{\tau'-\tau} du \right) \otimes \varrho$$

in probability. This completes our induction and the proof of (3.30).

*Step 3: Convergence of  $\mathcal{J}_0^1(y^{(i)}; \omega^i)$*  In a similar way as in the proof of Theorem 3.8 (see relation (3.27)), we can show the convergence

$$\mathcal{J}_0^1(y^{(i)}; \omega^i) \longrightarrow 0,$$

in probability for  $i \neq \tau$ , so the convergence (3.2) holds true. On the other hand, it is clear from classical integration theory that:

$$\mathcal{J}_0^1(y^{(\tau)}; \omega^\tau) \rightarrow \left( \int_0^1 y_u^{(\tau)} du \right) \otimes \varrho,$$

which implies the convergence (3.3). Summarizing, we have proved (3.1)–(3.3) and the convergence (3.29) follows immediately from Theorem 3.1. □

**Remark 3.12.** As in Remark 3.10, one can generalize Theorem 3.11 to some other interesting cases. For example, suppose that  $\mathcal{J}(x^\tau; h^n) \xrightarrow{\text{stable f.d.d.}} \omega$ , where  $\omega$  is a continuous Gaussian process independent of  $x$  and with values in  $(\mathbb{R}^d)^{\otimes \tau} \otimes \mathcal{V}'$ . As before, let  $\mathcal{H}$  be the Hilbert space corresponding to  $\omega$  and suppose that  $C^\gamma \subset \mathcal{H}$  for all  $\gamma < \nu$ . Suppose that (3.15) holds true for any  $f \in C^\gamma$ . Then one can show in a similar way as in Theorem 3.11 that

$$\mathcal{J}_0^1(y; h^n) \xrightarrow{d} \int_0^1 y^{(\tau)} d\omega.$$

### 3.4. Double limit theorem

In this subsection, we consider the double limit theorem case, which has been introduced in Definition 3.5. This usually corresponds to a transition in terms of roughness for the underlying noise  $x$ .

**Theorem 3.13.** *Let  $y = (y^{(0)}, \dots, y^{(\ell-1)})$  be a rough path on  $[0, 1]$  controlled by  $(x, \alpha)$  in  $L_2$  or almost surely, and suppose that  $h^n$  satisfies the inequality (2.13) uniformly in  $n$ . Furthermore, we assume that  $x$  and  $h^n$  fulfill the following conditions:*

- (i) *We have the convergence  $h^n \xrightarrow{\text{stable f.d.d.}} W$  as  $n \rightarrow \infty$ , where  $W$  is a standard Brownian motion independent of  $x$ .*
- (ii) *There exists a constant matrix  $\varrho \in (\mathbb{R}^d)^{\otimes \tau} \otimes \mathcal{V}'$  and some  $\tau : 0 < \tau < \ell$  such that for any sequence of partitions  $0 \leq s_0 < s_1 < \dots < s_m \leq 1$  on  $[0, 1]$  such that  $|s_{j+1} - s_j| \leq 1/m$  and  $s_0 = s, s_m = t$ , we have*

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(x^\tau; h^n) = (t - s)\varrho, \tag{3.36}$$

where the limit has to be understood as a limit in probability for all  $(s, t) \in \mathcal{S}_2$  and where  $\mathcal{J}_{s_j}^{s_{j+1}}(x^i; h^n)$  is defined by (2.11).

- (iii) *For any  $i \in \{1, \dots, \ell - 1\} \setminus \{\tau\}$  and any sequence of partitions  $0 \leq s_0 < s_1 < \dots < s_m \leq 1$  in (ii) above, we have the following convergence in probability:*

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(x^i; h^n) \right| = 0. \tag{3.37}$$

Then the following stable f.d.d. convergence holds true for  $\mathcal{J}(y; h^n)$ :

$$\mathcal{J}(y; h^n) \rightarrow \int y_t \otimes dW_t + \int \delta y_{0t}^{(\tau)} dt \otimes \varrho.$$



**Proof.** As in Theorem 3.8 and Theorem 3.11, we apply Theorem 3.1 and we are reduced to show relations (3.1), (3.2), and (3.3). In the current situation, we consider  $\mathcal{I}' = \{0, \tau\}$  and  $\mathcal{I}'' = \mathcal{I} \setminus \mathcal{I}'$ . We divide again the proof in several steps.

*Step 1: Case  $i < \tau$ .* As in the proof of Theorem 3.8, by induction we can show that for  $i < \tau$  we have the convergence

$$\mathcal{J}_s^t(x^i; h^n) \xrightarrow{d} \omega_{st}^i \equiv \int_s^t x_{su}^i \otimes dW_u. \tag{3.38}$$

*Step 2: Case  $i = \tau$ .* An approximation argument similar to (3.19) and (3.31) yields:

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathcal{J}_s^t(x^i; h^n) - \sum_{l=0}^i \tilde{A}_{l,i} \right| = 0 \tag{3.39}$$

with

$$\tilde{A}_{l,i} = \sum_{j=0}^{j_2-j_1} x_{ss_j}^{i-l} \otimes \mathcal{J}_{s_j}^{s_{j+1}}(x^l; h^n).$$

In the same way as in (3.22), and taking into account the convergence (3.38), for  $0 < l < \tau$  we have the convergence

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} |\tilde{A}_{l,\tau}| = 0$$

in probability. On the other hand, in the same way as for relation (3.24), the following limit holds true for  $l = 0$ :

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \tilde{A}_{0,\tau} \stackrel{(d)}{=} \int_s^t x_{su}^\tau \otimes dW_u.$$

In addition, owing to assumption (3.36) we have

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \tilde{A}_{\tau,\tau} = (t - s)\varrho,$$

where the limit holds in probability. In summary of the convergences of  $\tilde{A}_{l,\tau}$ ,  $l = 0, \dots, \tau$  and taking into account (3.39), we obtain

$$\mathcal{J}_s^t(x^\tau; h^n) \xrightarrow{d} \int_s^t x_{su}^\tau \otimes dW_u + (t - s)\varrho.$$

Notice that we can add up limits in distribution here, since one of the limits is deterministic.

*Step 3: Case  $i > \tau$ .* In the following, we show by induction the convergence

$$\mathcal{J}_s^t(x^i; h^n) \xrightarrow{d} \omega_{st}^i \equiv \int_s^t x_{su}^i \otimes dW_u + \left( \int_s^t x_{su}^{i-\tau} du \right) \otimes \varrho, \tag{3.40}$$

for  $\ell > i \geq \tau$ . Indeed, we have shown that convergence (3.40) holds when  $i = \tau$ . Now suppose that the convergence holds for  $i = \tau, \dots, \tau' - 1$ , and we wish to propagate the induction assumption. Thanks to the induction assumption and in a similar way as in (3.22) we can show that for  $l \in \{1, \dots, \tau' - 1\} \setminus \{\tau\}$  we have

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} |\tilde{A}_{l,\tau'}| = 0, \tag{3.41}$$

where the limit is understood in probability. Moreover, invoking assumption (3.37) we also have the following limit in probability:

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |\tilde{A}_{\tau',\tau'}| = 0. \tag{3.42}$$

On the other hand, we let the patient reader check that

$$\tilde{A}_{\tau,\tau'} - \left( \int_s^t x_{su}^{i-\tau} du \right) \otimes \varrho \rightarrow 0 \tag{3.43}$$

in probability, similarly to what has been done in (3.34) and (3.35). Taking into account (3.41), (3.42), (3.43) and (3.39), it is easily checked that (3.40) for  $i = \tau'$  is reduced to the following convergence:

$$\tilde{A}_{0,\tau'} + \left( \int_s^t x_{su}^{\tau'-\tau} du \right) \otimes \varrho \xrightarrow{d} \int_s^t x_u^{\tau'} dW_u + \left( \int_s^t x_{su}^{\tau'-\tau} du \right) \otimes \varrho, \tag{3.44}$$

as  $n \rightarrow \infty$  and then  $m \rightarrow \infty$ . In order to prove (3.44), we first fix  $m$  and let  $n$  go to  $\infty$ . Then, owing to the fact that  $(x, h^n) \xrightarrow{f.d.d.} (x, W)$ , we get that

$$\lim_{n \rightarrow \infty} \tilde{A}_{0,\tau'} + \left( \int_s^t x_{su}^{\tau'-\tau} du \right) \otimes \varrho \stackrel{(d)}{=} \sum_{j=0}^{m-1} x_{ss_j}^{\tau'} \otimes \delta W_{s_j s_{j+1}} + \left( \int_s^t x_{su}^{\tau'-\tau} du \right) \otimes \varrho.$$

Then, conditioning on  $x$  and considering limits of Riemann sums for Wiener integrals, we end up with:

$$\lim_{m \rightarrow \infty} \sum_{j=0}^{m-1} x_{ss_j}^{\tau'} \otimes \delta W_{s_j s_{j+1}} + \left( \int_s^t x_{su}^{\tau'-\tau} du \right) \otimes \varrho \stackrel{L_2}{=} \int_s^t x_{su}^{\tau'} dW_u + \left( \int_s^t x_{su}^{\tau'-\tau} du \right) \otimes \varrho,$$

from which (3.44), and thus (3.40) for  $i = \tau'$  are easily deduced. Therefore, we can conclude by induction that the convergence (3.40) holds for all  $i = \tau, \dots, \ell - 1$ .

*Step 4: Proof of (3.2) and (3.3).* Recall that  $\omega^i$  is defined by relation (3.38) when  $i < \tau$  and by (3.40) when  $i \geq \tau$ . For  $i = 1, \dots, \ell - 1, i \neq 0$  and  $i \neq \tau$ , as in the proof of Theorem 3.8 (see relation (3.27)) we can show that (3.2) holds. On the other hand, it is easy to show by classical integration arguments that

$$\mathcal{J}_0^1(y^{(\tau)}; \omega^\tau) \rightarrow \left( \int_0^1 y_u^{(\tau)} du \right) \otimes \varrho \tag{3.45}$$

in probability and

$$\mathcal{J}_0^1(y; \omega^0) \rightarrow \int_0^1 y_u \otimes dW_u \tag{3.46}$$

in probability, by convergence of Riemann sums for Wiener integrals. Putting together (3.45) and (3.46) and invoking the same arguments as in Step 3, we can conclude that (3.3) is satisfied.

In conclusion, we have checked conditions (3.1)–(3.3), and our result follows directly from Theorem 3.1. □

### 4. Breuer–Major theorem

In this section, we consider generalizations of Breuer–Major’s theorem [5]. Notice that recent contributions (see e.g. [33,34,36,38]) to this area involving weighted sums of stationary sequences mostly consider sequences of functionals of one-dimensional fractional Brownian motions (fBm). This is why we also stick to the one-dimensional fBm case, though multi-dimensional studies for more general Gaussian processes do not seem out of reach in our framework, given that we have the stable convergence of the corresponding unweighted random sum. Also observe that the aforementioned references focus on sequences in a fixed chaos or in a finite sum of chaos. In contrast, we will be able to handle general sequences in  $L_2$  with respect to a Gaussian measure.

#### 4.1. Weighted Breuer–Major theorem I

In this subsection, we consider the weighted type Breuer–Major theorem in the context of our single limit Theorem 3.8.

Let us first introduce some additional notation. Let  $d\gamma(t) = (2\pi)^{-1/2} e^{-t^2/2} dt$  be the standard Gaussian measure on the real line, and let  $f \in L_2(\gamma)$  be such that  $\int_{\mathbb{R}} f(t) d\gamma(t) = 0$ . It is well-known that the function  $f$  can be expanded into a series of Hermite polynomials as follows:

$$f(t) = \sum_{q=d}^{\infty} a_q H_q(t),$$

where  $d \geq 1$  is some integer and  $H_q(t) = (-1)^q e^{\frac{t^2}{2}} \frac{d^q}{dt^q} e^{-\frac{t^2}{2}}$  is the Hermite polynomial of order  $q$ . If  $a_d \neq 0$ , then  $d$  is called the *Hermite rank* of the function  $f$ . Note that since  $f \in L_2(\gamma)$ , we have  $\sum_{q=d}^\infty |a_q|^2 q! < \infty$ .

Our underlying process  $X$  is a one-dimensional Gaussian sequence. For such a process the basic tools to measure dependence are based on correlation functions. Throughout this subsection, we assume that the following hypothesis on correlations holds true.

**Hypothesis 4.1.** *Let  $X_k, k \in \mathbb{Z}$  be a centered stationary Gaussian sequence such that  $X_k$  has unit variance. Denote  $\rho(k) = \mathbb{E}(X_0 X_k)$ . We suppose that  $\sum_{k \in \mathbb{Z}} |\rho(k)|^d < \infty$  for some  $d \geq 1$ .*

For sake of conciseness we will not recall the basic notions of Gaussian analysis which will be used in this section. The interested reader is referred to [37] for further details.

We now recall a classical version of Breuer–Major’s theorem.

**Theorem 4.2.** *Consider  $f \in L_2(\gamma)$  with rank  $d \geq 1$ . Let  $\{X_k, k \in \mathbb{Z}\}$  be a centered stationary Gaussian sequence satisfying Hypothesis 4.1 for  $d$ . For  $n \geq 1$ , let  $0 = t_0 < \dots < t_n = 1$  be the uniform partition of  $[0, 1]$  defined in Section 1. We set  $h_{st}^n = \sum_{s \leq t_k < t} f(X_k)$  for all  $(s, t) \in \mathcal{S}_2$ . Then the following central limit theorem holds true:*

$$h^n / \sqrt{n} \xrightarrow{f.d.d.} \sigma W \quad \text{as } n \rightarrow \infty,$$

where the variance  $\sigma^2 \in [0, \infty)$  is defined by:

$$\sigma^2 = \sum_{q=d}^\infty q! a_q^2 \sum_{k \in \mathbb{Z}} \rho(k)^q. \tag{4.1}$$

In this subsection we specialize Theorem 4.2 to a situation where  $X_k = n^\nu \delta x_{t_k t_{k+1}}$ , where  $x$  is a fBm with Hurst parameter  $\nu$ . In this context we are interested in the following questions: (1) Do we have the convergence of the weighted sum

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} y_k f(X_k) \quad \text{as } n \rightarrow \infty, \tag{4.2}$$

for a general weight  $y_k$ ? (2) Does the central limit theorem for (4.2) still hold in general? We will give a complete answer to these two questions when the weight process  $y$  is a controlled process as introduced in Definition 2.3.

Before we start our discussions, let us recall some basic facts about fBm. (i) If  $x$  is a one-dimensional fBm with Hurst parameter  $\nu$ , then  $x$  is almost surely  $\gamma$ -Hölder continuous for all  $\gamma < \nu$ . (ii) For a fBm  $x$ , the covariance function  $\rho$  alluded to in Hypothesis 4.1 is defined by

$$\rho(k) = \mathbb{E}(\delta x_{01} \delta x_{k, k+1}). \tag{4.3}$$

Then, whenever  $\nu < \frac{1}{2}$ , we have  $\sum_{k \in \mathbb{Z}} \rho(k) = 0$ .

We also label the following notation for further use.

**Notation 4.3.** Let  $x$  be a one-dimensional fBm with Hurst parameter  $\nu$  on the probability space  $(\Omega, P, \mathcal{F})$ . We consider  $x$  as a  $(L_p, \nu, \ell)$  rough path according to Definition 2.1, where  $p$  is any real number in  $[1, \infty)$  and  $\ell$  is the smallest integer satisfying  $\nu \ell + \frac{1}{2} > 1$ . In addition, we will choose  $x_{st}^i = \frac{1}{i!} (\delta x_{st})^i$  for all  $(s, t) \in \mathcal{S}_2$  and  $i = 1, \dots, \ell$  for  $\ell \in \mathbb{N}$ .

Let us recall the following identity of multiple Wiener integrals. The reader is referred to e.g. [24,37,40] for more details:

**Lemma 4.4.** *Let  $f \in L^2([0, 1]^p)$  and  $g \in L^2([0, 1]^q)$  be symmetric functions. Then we have the identity*

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g), \tag{4.4}$$

where  $I_p(f)$  is the  $p$ th multiple Wiener integral of  $f$ , and  $\binom{p}{r} = \frac{p!}{r!(p-r)!}$ .

Let  $\mathcal{H}$  be the completion of the space of indicator functions with respect to the inner product  $(\mathbf{1}_{[u,v]}, \mathbf{1}_{[s,t]})_{\mathcal{H}} = \mathbb{E}(\delta x_{uv} \delta x_{st})$ . Let  $0 = t_0 < \dots < t_n = 1$  be the uniform partition of  $[0, 1]$  alluded to the above and  $0 \leq s_0 < \dots < s_m \leq 1$  be another partition of  $[0, 1]$  with  $m \ll n$ . In the following, we take  $\ell$  such that  $\ell - 1 \leq \frac{1}{2\nu} < \ell$  (or equivalently  $\ell$  is the smallest integer such that  $\nu\ell + \alpha > 1$  with  $\alpha = \frac{1}{2}$ ). We set

$$h_{st}^{n,q} = \sum_{s \leq t_k < t} H_q(n^\nu \delta x_{t_k, t_{k+1}}) \quad \text{and} \quad \zeta_j^{i,q} = \mathcal{J}_{s_j}^{s_{j+1}}(x^i; h^{n,q}) \tag{4.5}$$

for  $k = 0, \dots, n - 1, j = 0, \dots, m - 1, i = 0, \dots, \ell - 1$  and  $q \in \mathbb{N}$ , where  $\mathcal{J}_{s_j}^{s_{j+1}}(x^i; h^{n,q})$  is given by (2.11). We denote by  $\vartheta(q, q', i)$  the following quantity

$$\vartheta(q, q', i) := \mathbb{E} \left( \sum_{j, j'=0}^{m-1} \zeta_j^{i,q} \zeta_{j'}^{i,q'} \right). \tag{4.6}$$

We will need the following auxiliary result.

**Lemma 4.5.** *Let  $x$  be a one-dimensional fBm on  $[0, 1]$  with Hurst parameter  $\nu \leq \frac{1}{2}$ . Take  $i = 1, \dots, \ell - 1$ , where we recall that  $\ell$  satisfies  $\ell - 1 \leq \frac{1}{2\nu} < \ell$ . Then for  $q', q \geq \ell$  the following estimate holds true:*

(i) *When  $|q' - q| \leq 2i$ , we have*

$$\vartheta(q, q', i) \leq K(n^{1-2\nu} + nm^{-2i\nu} + n^{1-\nu}) \sum_{r=\frac{1}{2}(q+q')-i}^q r! \binom{q}{r} \binom{q'}{r}, \tag{4.7}$$

where  $\vartheta(q, q', i)$  is defined by (4.6) and  $K$  is a positive universal constant.

(ii) *When  $|q' - q| > 2i$ , we have*

$$\vartheta(q, q', i) = 0. \tag{4.8}$$

(iii) *When  $|q - q'| \leq 2i$ , the following inequality holds true for all  $(s, t) \in \mathcal{S}_2$ :*

$$\mathbb{E}(\mathcal{J}_s^t(x^i; h^{n,q}) \mathcal{J}_s^t(x^i; h^{n,q'})) \leq Kn(t-s)^{2i\nu+1} \sum_{r=\frac{1}{2}(q+q')-i}^{q \wedge q'} r! \binom{q}{r} \binom{q'}{r}. \tag{4.9}$$

(iv) *When  $|q - q'| > 2i$ , for all  $(s, t) \in \mathcal{S}_2$  we have:*

$$\mathbb{E}(\mathcal{J}_s^t(x^i; h^{n,q}) \mathcal{J}_s^t(x^i; h^{n,q'})) = 0.$$

**Remark 4.6.** Notice that our assumption imply in particular that  $q \wedge q' > \frac{1}{2\nu}$ . This is also the condition on the Hermite rank of  $f$  which will feature in Theorem 4.7 below.

**Proof of Lemma 4.5.** *Step 1:* Without loss of generality let us assume that  $q' \geq q$ . By the definition of  $\zeta_j^{i,q}$  we can write

$$\vartheta(q, q', i) = \sum_{j, j'=0}^{m-1} \sum_{s_j \leq t_k < s_{j+1}} \sum_{s_{j'} \leq t_{k'} < s_{j'+1}} a(j, j', k, k'), \tag{4.10}$$

where, recalling that  $\varepsilon(s_j)$  is defined by (2.11), we have

$$a(j, j', k, k') = \mathbb{E}(x_{\varepsilon(s_j)t_k}^i x_{\varepsilon(s_{j'})t_{k'}}^i H_q(n^\nu \delta x_{t_k, t_{k+1}}) H_{q'}(n^\nu \delta x_{t_{k'}, t_{k'+1}})).$$

Now set  $\beta_k = n^\nu \mathbf{1}_{[t_k, t_{k+1}]}$ . Recalling that  $H_q(n^\nu \delta x_{t_k, t_{k+1}}) = I_q(\beta_k^{\otimes q})$  and invoking identity (4.4), we easily obtain:

$$a(j, j', k, k') = \sum_{r=0}^q r! \binom{q}{r} \binom{q'}{r} \mathbb{E}(x_{\varepsilon(s_j)t_k}^i x_{\varepsilon(s_{j'})t_{k'}}^i I_{q+q'-2r}(\beta_k^{\otimes q-r} \otimes \beta_{k'}^{\otimes q'-r})) \langle \beta_k, \beta_{k'} \rangle_{\mathcal{H}}^r.$$

Now observe that  $\langle \beta_k, \beta_{k'} \rangle_{\mathcal{H}} = \rho(k - k')$ , where the covariance function  $\rho$  is defined by (4.3). Therefore, owing to an application of integration by parts, we end up with the following identity:

$$a(j, j', k, k') = \sum_{r=0}^q r! \binom{q}{r} \binom{q'}{r} b(r) \rho(k - k')^r, \tag{4.11}$$

where  $b(r)$  is the coefficient defined by:

$$b(r) = \mathbb{E} \langle D^{q+q'-2r} (x_{\varepsilon(s_j)t_k}^i x_{\varepsilon(s_{j'})t_{k'}}^i), \beta_k^{\otimes(q-r)} \otimes \beta_{k'}^{\otimes(q'-r)} \rangle_{\mathcal{H}^{\otimes(q+q'-2r)}}. \tag{4.12}$$

Step 2: Consider  $q \geq \ell$ . Due to the fact that  $x^i$  belongs to the sum of the first  $i$  chaos, when  $q' - q > 2i$ , it is easy to see that

$$D^{q+q'-2r} (x_{\varepsilon(s_j)t_k}^i x_{\varepsilon(s_{j'})t_{k'}}^i) = 0 \tag{4.13}$$

for all  $r = 0, \dots, q$ . Taking into account (4.11), this implies that whenever  $q' - q > 2i$  we have

$$a(j, j', k, k') = 0, \tag{4.14}$$

and thus the estimate in (4.8) holds.

In the following, we assume that  $0 \leq q' - q \leq 2i$  and we focus on inequality (4.7). Note first that since  $q' \geq q$  and  $q \geq \ell$ , we have  $\frac{1}{2}(q + q') - i \geq q - (\ell - 1) > 0$ .

We now recall that  $b(r)$  is defined by (4.12), and we separate the estimates on  $b(r)$  in several cases:

- (i) Case  $0 \leq r < \frac{1}{2}(q + q') - i$ . In this case, going back to the definition (4.12), it is readily checked that we differentiate the product  $x_{\varepsilon(s_j)t_k}^i x_{\varepsilon(s_{j'})t_{k'}}^i$  more than  $2i$  times, and hence  $b(r) = 0$ .
- (ii) Case  $\frac{1}{2}(q + q') - i \leq r \leq q - 1$ . In this case we still have  $q + q' - 2r > 0$ . Then we start from relation (4.12) again, taking into account the order of differentiation, and resorting to the relations

$$\mathbb{E}(|x_{\varepsilon(s_j)t_k}^l|^p)^{1/p} \leq Km^{-lv}$$

for any positive integer  $l$ , and

$$\langle \beta_k, \mathbf{1}_{[a,b]} \rangle_{\mathcal{H}} = n^{-v} \langle \mathbf{1}_{[k,k+1]}, \mathbf{1}_{[na,nb]} \rangle_{\mathcal{H}}, \quad \text{and} \quad |\langle \mathbf{1}_{[k,k+1]}, \mathbf{1}_{[na,nb]} \rangle_{\mathcal{H}}| \leq 1, \tag{4.15}$$

which are valid for all  $k \leq n$  and  $(a, b) \in \mathcal{S}'_2$  whenever  $v \leq \frac{1}{2}$ . This yields

$$|b(r)| \leq Kn^{-(q+q'-2r)v} m^{-(2i-(q+q'-2r)v)} \leq Kn^{-(q+q'-2r)v} \leq Kn^{-2v}. \tag{4.16}$$

- (iii) Case  $r = q$ . If  $q < q'$ , similarly to case (ii), we can get  $|b(r)| \leq Kn^{-v}$ . If  $r = q$  and  $q = q'$ , then  $|b(r)|$  becomes  $|b(r)| = |\mathbb{E}[x_{\varepsilon(s_j)t_k}^i x_{\varepsilon(s_{j'})t_{k'}}^i]|$ , from which is easily seen that this term is bounded by  $Km^{-2iv}$ .

Now gathering the estimates obtained in (i)–(iii) and plugging them in (4.11), we end up with:

$$|a(j, j', k, k')| \leq K \left( \sum_{r=\frac{1}{2}(q+q')-i}^{q-1} r! \binom{q}{r} \binom{q'}{r} n^{-2v} |\rho(k - k')|^r + q! \binom{q'}{q} (m^{-2iv} + n^{-v}) |\rho(k - k')|^q \right). \tag{4.17}$$

Furthermore, observe that

$$\mathcal{D}_n \cap [s, t] = \bigcup_{j=0}^{m-1} \{t_k; s_j \leq t_k < s_{j+1}\}. \tag{4.18}$$

Hence, substituting (4.17) into (4.10) and using the fact that  $\sum_{k \in \mathbb{N}} |\rho(k)| < \infty$ , we obtain

$$\begin{aligned}
 |\vartheta(q, q', i)| &\leq K \sum_{r=\frac{1}{2}(q+q')-i}^{q-1} r! \binom{q}{r} \binom{q'}{r} n^{-2v} \sum_{k, k'=0}^{n-1} |\rho(k-k')|^r \\
 &\quad + Kq! \binom{q'}{q} (m^{-2iv} + n^{-v}) \sum_{k, k'=0}^{n-1} |\rho(k-k')|^q \\
 &\leq K(n^{1-2v} + nm^{-2iv} + n^{1-v}) \sum_{r=\frac{1}{2}(q+q')-i}^q r! \binom{q}{r} \binom{q'}{r}.
 \end{aligned} \tag{4.19}$$

This completes the proof of inequality (4.7).

*Step 3:* In this step, we prove the estimates in (iii) and (iv). For  $(s, t) \in \mathcal{S}_2$  such that  $t - s < \frac{1}{n}$  and with Remark 2.6 in mind, we have  $\mathcal{J}_s^t(x^i; h^{n,q}) = 0$ . Therefore, in the following we assume that  $n(t - s) \geq 1$ . Suppose that  $q' \geq q$ . Then similarly to (4.10) and (4.11), we can derive the following expression:

$$\begin{aligned}
 &\mathbb{E}(\mathcal{J}_s^t(x^i; h^{n,q}) \mathcal{J}_s^t(x^i; h^{n,q'})) \\
 &= \sum_{s \leq t_k, t_{k'} < t} \sum_{r=0}^q r! \binom{q}{r} \binom{q'}{r} \mathbb{E} \langle D^{q'+q-2r} (x_{\varepsilon(s)t_k}^i x_{\varepsilon(s)t_{k'}}^i), \beta_k^{\otimes(q-r)} \otimes \beta_{k'}^{\otimes(q'-r)} \rangle_{\mathcal{H}^{\otimes(q'+q-2r)}} \rho(k-k')^r.
 \end{aligned} \tag{4.20}$$

As in the previous step, we now separate the case  $q' - q \leq 2i$  and  $q' - q > 2i$ . Indeed, when  $q' - q \leq 2i$ , we have seen that  $\frac{1}{2}(q + q') - i > 0$ . Hence, thanks to the assumption that  $q' \geq q, q \geq \ell$  and  $i \leq \ell - 1$ , we obtain the following estimate along the same lines as in the previous step:

$$\begin{aligned}
 |\mathbb{E}(\mathcal{J}_s^t(x^i; h^{n,q}) \mathcal{J}_s^t(x^i; h^{n,q'}))| &\leq \sum_{r=\frac{1}{2}(q+q')-i}^q r! \binom{q}{r} \binom{q'}{r} n^{-(q+q'-2r)v} (t-s)^{(2i-(q+q'-2r))v} \\
 &\quad \times \sum_{s \leq t_k, t_{k'} < t} |\rho(k-k')|^r.
 \end{aligned} \tag{4.21}$$

Thus, resorting to the inequality  $n(t - s) \geq 1$  and thanks to the fact that  $\sum_{k, k'=0}^{n-1} |\rho(k-k')|^r$  is of order  $n$ , we get

$$\mathbb{E}(\mathcal{J}_s^t(x^i; h^{n,q}) \mathcal{J}_s^t(x^i; h^{n,q'})) \leq Kn(t-s)^{2iv+1} \sum_{r=\frac{1}{2}(q+q')-i}^q r! \binom{q}{r} \binom{q'}{r},$$

which proves (iii).

The proof of (iv) is left to the reader. Indeed, it is done exactly as for (ii), taking advantage of the fact that  $D^p(x_{\varepsilon(s)t_k}^i x_{\varepsilon(s)t_{k'}}^i) = 0$  whenever  $p > 2i$ . The proof is now complete.  $\square$

We are ready to derive the first main result of this section, which is a Breuer–Major type central limit theorem.

**Theorem 4.7.** *Let  $x$  be a one-dimensional fBm with Hurst parameter  $v \leq \frac{1}{2}$ . Let  $\ell$  be an integer such that  $v\ell + \frac{1}{2} > 1$ . Let  $(y, y', \dots, y^{(\ell-1)})$  be a process controlled by  $(x, \frac{1}{2})$  in  $L_2$  or almost surely. Let  $f \in L_2(\gamma)$  with Hermite rank strictly bigger than  $\frac{1}{2}$ . Suppose that one of the following conditions holds true:*

(a) *We have the expansion  $f = \sum_{q=d}^{\infty} a_q H_q$ , and*

$$\sum_{q=d}^{\infty} a_q^2 q! q^{2(\ell-1)} < \infty, \tag{4.22}$$

*that is, the function  $f$  sits in the Sobolev space  $W^{2(\ell-1), 2}(\mathbb{R}, \gamma)$ , where recall that  $\gamma$  denotes the standard Gaussian measure on the real line; see e.g. Page 28 in [40].*

(b) The function  $f$  is an element of  $C^{2\ell-3}$  and  $f^{(2\ell-3)}$  is Lipschitz.

We define a family of increments  $\{h^n; n \geq 1\}$  by:

$$h_{st}^n := \frac{1}{\sqrt{n}} \sum_{s \leq t_k < t} f(n^\nu \delta x_{t_k t_{k+1}}), \quad (s, t) \in \mathcal{S}_2. \tag{4.23}$$

Then we have the stable f.d.d. convergence:

$$\mathcal{J}(y; h^n) \rightarrow \sigma \int y_t dW_t, \quad \text{as } n \rightarrow \infty,$$

where  $\sigma$  is given by (4.1).

**Remark 4.8.** As mentioned in the introduction, Theorem 4.7 can be seen as a generalization as well as a simplification of [33,36].

**Proof of Theorem 4.7.:** We first assume that condition (a) is true. We will prove the theorem thanks to our central limit Theorem 3.8 applied to  $h^n$ .

To this aim, it suffices to verify the  $\mathcal{F}$ -stable f.d.d. convergence  $h^n \rightarrow \omega$ , plus condition (2.13) in Proposition 2.7 and the convergence in (3.13). We now prove that those conditions are satisfied in separate steps.

*Step 1: Stable convergence of  $h^n$ .* The convergence in law of  $h^n$  to  $W$  is a direct consequence of Theorem 4.2. One can get the stable convergence by applying a multi-dimensional version of Theorem 6.3.1 in [37].

*Step 2: Proof of condition (2.13).* Recall that  $f = \sum_{q=d}^\infty a_q H_q$ , that  $h^n$  is defined by (4.23), and that  $h^{n,q}$  has been introduced in (4.5). Then  $|\mathcal{J}_s^t(x^i; h^n)|_{L_2}^2$  can be expressed as

$$|\mathcal{J}_s^t(x^i; h^n)|_{L_2}^2 = \frac{1}{n} \sum_{q, q'=d}^\infty a_q a_{q'} \mathbb{E}(\mathcal{J}_s^t(x^i; h^{n,q}) \mathcal{J}_s^t(x^i; h^{n,q'})).$$

We can now apply Lemma 4.5 (iii) and (iv) in order to get:

$$|\mathcal{J}_s^t(x^i; h^n)|_{L_2}^2 \leq K c_i (t - s)^{2iv+1}, \tag{4.24}$$

where  $K$  is defined by (4.9) and

$$c_i = \sum_{|q-q'| \leq 2i} a_q a_{q'} \sum_{r=\frac{1}{2}(q+q')-i}^{q \wedge q'} r! \binom{q}{r} \binom{q'}{r}. \tag{4.25}$$

In addition, we observe that  $r \geq (q \wedge q') - \ell + 1$  in the sum defining  $c_i$ . Hence, invoking the elementary bounds

$$\binom{q'}{r} \leq (q')^{q'-r} \quad \text{and} \quad r! \binom{q}{r} \leq q!,$$

plus an application of Cauchy–Schwarz’s inequality for the sum over  $r$ , it is readily checked that

$$c_i \leq \sum_{q=d}^\infty a_q^2 q! q^{2(\ell-1)}. \tag{4.26}$$

Taking square root in both sides of (4.24) and taking into account condition (4.22) we obtain the condition (2.13) in Proposition 2.7.

*Step 3: Proof of condition (3.13).* Recall that the increment  $h^n$  is defined by (4.23). For  $(s, t) \in \mathcal{S}_2$ , we set

$$\zeta_j^i = \mathcal{J}_{s_j}^{s_{j+1}}(x^i; h^n) = \frac{1}{\sqrt{n}} \sum_{q=d}^\infty a_q \mathcal{J}_{s_j}^{s_{j+1}}(x^i; h^{n,q}) = \frac{1}{\sqrt{n}} \sum_{q=d}^\infty a_q \zeta_j^{i,q},$$



where the last identity is due to our convention (4.5). Then according to our notation (4.6), the following relation holds true for  $i > 0$ :

$$\left| \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(x^i; h^n) \right|_{L_2}^2 = \frac{1}{n} \sum_{q, q'=d}^{\infty} a_q a_{q'} \vartheta(q, q', i).$$

According to Lemma 4.5, we have  $\vartheta(q, q', i) = 0$  when  $|q' - q| > 2i$ . Combining this with inequality (4.7), we obtain

$$\begin{aligned} \left| \sum_{j=0}^{m-1} \zeta_j^i \right|_{L_2}^2 &= \frac{1}{n} \sum_{|q-q'|\leq 2i} a_q a_{q'} \vartheta(q, q', i) \\ &\leq K(n^{-2\nu} + m^{-2\nu} + n^{-\nu}) \sum_{|q-q'|\leq 2i} a_q a_{q'} \sum_{r=\frac{1}{2}(q+q')-i}^{q \wedge q'} r! \binom{q}{r} \binom{q'}{r}. \end{aligned}$$

We now refer to our definition (4.25) of  $c_i$ , as well as inequality (4.26), which yields:

$$\left| \sum_{j=0}^{m-1} \zeta_j^i \right|_{L_2}^2 \leq K(n^{-2\nu} + m^{-2\nu} + n^{-\nu}) \sum_{q=d}^{\infty} a_q^2 q! q^{2(\ell-1)}.$$

Taking into account the assumption (4.22), this implies that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \sum_{j=0}^{m-1} \zeta_j^i \right|_{L_2} = 0. \tag{4.27}$$

This complete the proof of the theorem under condition (a).

*Step 4: Proof under conditions (b).* One can show that condition (b) implies condition (a). Indeed, by Proposition 1.2.4 in [40], we obtain that  $f^{(2\ell-3)} \in W^{1,2}(\mathbb{R}, \gamma)$ . It is then easy to show that  $f^{(2\ell-4)} \in L_2(\gamma)$  and  $(f^{(2\ell-4)})' = f^{(2\ell-3)}$ , which implies that  $f^{(2\ell-4)} \in W^{2,2}(\mathbb{R}, \gamma)$ . Repeating this argument, we obtain that  $f \in W^{2\ell-2,2}(\mathbb{R}, \gamma)$ . Our proof is now finished.  $\square$

We now consider a central limit theorem for weights  $y$  which satisfies the Young pairing condition with respect to a Brownian motion  $W$  (i.e.  $y$  is  $\nu'$ -Hölder continuous for  $\nu' > \frac{1}{2}$ ).

**Proposition 4.9.** *Let  $y$  be a  $\nu'$ -Hölder continuous path for some  $\nu' > \frac{1}{2}$  and let  $x$  be a fBm with Hurst parameter  $\nu \in (0, 1)$ . Suppose that  $f \in L_2(\gamma)$  has Hermite rank  $d$  such that  $\nu < 1 - \frac{1}{2d}$ . Then the following convergence holds true*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} y_{t_k} f(n^\nu x_{t_k t_{k+1}}) \xrightarrow{d} \sigma \int_0^1 y_t dW_t \quad \text{as } n \rightarrow \infty, \tag{4.28}$$

where  $\sigma$  is defined by (4.1).

**Remark 4.10.** It would be straightforward to generalize Proposition 4.9 to the case  $\nu > 1 - \frac{1}{2d}$ . Indeed, the proof would go exactly along the same lines as below, except for the fact that  $h^n$  would converge to a Hermite process. We have refrained to do so for sake of conciseness, since Hermite type limit theorems have already been obtained in [25] and also because Young situations like the  $\nu > 1 - \frac{1}{2d}$  case do not require rough paths techniques.

**Proof of Proposition 4.9.:** As in equation (4.23), for  $(s, t) \in \mathcal{S}_2$  we set

$$h_{st}^n = \frac{1}{\sqrt{n}} \sum_{s \leq t_k < t} f(n^\nu x_{t_k t_{k+1}}).$$

In a similar way as in (4.24), Lemma 4.5(iii) and (iv) we can show that

$$\|h_{st}^n\|_{L_2} \leq K(t - s)^{\frac{1}{2}}. \tag{4.29}$$

Notice that we are working here under the assumption  $v' + \frac{1}{2} > 1$ . Therefore, an application of Theorem 3.8 combined with Remark 3.7 yield our claim (4.28).  $\square$

4.2. Weighted Breuer–Major theorem II

In this subsection, we continue our discussion on the Breuer–Major theorem, handling situations with low order Hermite ranks. We first derive some auxiliary results on the discrete integral  $\mathcal{J}_s^t(y; h^{n,q})$ , where we recall that  $h^{n,q}$  is defined by (4.5).

**Lemma 4.11.** *Let  $x$  be a fBm with Hurst parameter  $v$  considered as a  $(L_p, v, \ell)$  rough path as in Notation 4.3. Take  $i = 1, \dots, \ell - 1$  and  $q \in \mathbb{N}$ .*

(i) *Let  $\vartheta(q, q, i)$  be defined by (4.6). Then for  $q < \frac{1}{2v}$  and  $q < i$ , we have*

$$\vartheta(q, q, i) \leq K(nm^{-4v} + n^{2-2qv}m^{-2v}). \tag{4.30}$$

*For  $q = \frac{1}{2v}$  and  $0 < i < q$ , we have*

$$\vartheta(q, q, i) \leq K(n^{1-2v} + nm^{-2iv}). \tag{4.31}$$

(ii) *Recall that  $h_{st}^{n,q} = \sum_{s \leq t_k < t} H_q(n^v \delta x_{t_k t_{k+1}})$  is defined by (4.5). Then for  $q < \frac{1}{2v}$  and  $q > i$ , we have*

$$\mathbb{E}(|\mathcal{J}_s^t(x^i; h^{n,q})|^2) \leq Kn(t-s)^{2iv+1}, \quad \text{for } (s, t) \in \mathcal{S}'_2. \tag{4.32}$$

*For  $q \geq \frac{1}{2v}$ , we have*

$$\mathbb{E}(|\mathcal{J}_s^t(x^i; h^{n,q})|^2) \leq Kn(t-s)^{2iv+1}, \quad \text{for } (s, t) \in \mathcal{S}'_2. \tag{4.33}$$

*For  $q < \frac{1}{2v}$ , and  $q \leq i$ , we have*

$$\mathbb{E}(|\mathcal{J}_s^t(x^i; h^{n,q})|^2) \leq Kn^{2-2qv}(t-s)^{2+2iv-2qv} \quad (s, t) \in \mathcal{S}'_2. \tag{4.34}$$

**Proof.** The proof is divided into several steps.

*Step 1: General estimate for  $\vartheta$ .* Recall that  $\vartheta(q, q, i)$  is given by (4.10). Next we use expression (4.11) for  $a(j, j', k, k')$ . We bound all the combination numbers by a constant and invoke the fact that  $b(r)$  (defined by (4.12)) satisfies (similarly to (4.16)):

$$|b(r)| \leq Kn^{-(2q-2r)v}m^{-(2i-(2q-2r))v}$$

for all  $r \geq q - i$ . Therefore, similarly to (4.17) we get

$$|a(j, j', k, k')| \leq K \left( \sum_{r=0 \vee (q-i)}^q n^{-(2q-2r)v}m^{-(2i-(2q-2r))v} |\rho(k-k')|^r \right). \tag{4.35}$$

*Step 2: Case  $q < \frac{1}{2v}$  and  $q < i$ .* In this situation, similarly to (4.19), substituting (4.35) into (4.10) we obtain

$$\begin{aligned} |\vartheta(q, q, i)| &\leq K \sum_{r=1}^q n^{-(2q-2r)v}m^{-(2i-(2q-2r))v} \sum_{k, k'=0}^{n-1} |\rho(k-k')|^r \\ &\quad + Kn^{-2qv}m^{-(2i-2q)v} \sum_{k, k'=0}^{n-1} |\rho(k-k')|^0. \end{aligned} \tag{4.36}$$

Therefore, owing to the fact that  $\sum_{k, k'=0}^{n-1} |\rho(k-k')|^r = O(n)$  whenever  $r \geq 1$  and properly bounding the exponents in (4.36), we end up with

$$|\vartheta(q, q, i)| \leq K(nm^{-4v} + n^{2-2qv}m^{-2v}).$$

This completes the proof of (4.30).

Step 3: Case  $q = \frac{1}{2v}$  and  $0 < i < q$ . If  $q = \frac{1}{2v}$  and  $\ell$  is the smallest integer such that  $v\ell > \frac{1}{2}$ , we have  $\ell = q + 1$ . Since  $0 < i < q$ , then substituting (4.35) into (4.10) we obtain the same inequality as (4.36), except for the fact that the term with  $\rho(k - k')^0$  is missing. We get

$$\begin{aligned} |\vartheta(q, q, i)| &\leq K \sum_{r=q-i}^q n^{-(2q-2r)v} m^{-(2i-(2q-2r)v)} \sum_{k, k'=0}^{n-1} |\rho(k - k')|^r \\ &\leq K(n^{1-2v} + nm^{-2iv}), \end{aligned}$$

where we have followed the same lines as in the previous step for the second inequality. This completes the proof of (4.31).

Step 4: General estimate for  $\mathcal{J}_s^t(x^i; h^{n,q})$ . By (4.1), we have the expression:

$$\begin{aligned} &\mathbb{E}(|\mathcal{J}_s^t(x^i; h^{n,q})|^2) \\ &= \sum_{s \leq t_k, t_{k'} < t} \sum_{r=0}^q r! \binom{q}{r}^2 \mathbb{E} \langle D^{2q-2r}(x_{\varepsilon(s)t_k}^i x_{\varepsilon(s)t_{k'}}^i), \beta_k^{\otimes(q-r)} \otimes \beta_{k'}^{\otimes(q-r)} \rangle_{\mathcal{H}^{\otimes(2q-2r)}} \rho(k - k')^r. \end{aligned}$$

Therefore, proceeding similarly to Step 1 and (4.21) and bounding all the combination numbers by a constant  $K$ , we obtain the estimate

$$\mathbb{E}(|\mathcal{J}_s^t(x^i; h^n)|^2) \leq K n^{-2iv} \sum_{r=0 \vee (q-i)}^q (n(t-s))^{(2i-(2q-2r)v)} \sum_{s \leq t_k, t_{k'} < t} |\rho(k - k')|^r. \tag{4.37}$$

Step 5: Proof of (4.32), (4.33) and (4.34). In order to prove (4.33), note that when  $q > \frac{1}{2v}$ , since  $i \leq \ell - 1 \leq \frac{1}{2v}$ , we have  $q > i$ , and so the estimate (4.37) implies (4.33) due to the fact that  $\sum_{s \leq t_k, t_{k'} < t} |\rho(k - k')|^r \leq Kn(t-s)$  when  $r \geq 1$ . Similarly, we can show that estimate (4.33) still holds when  $q = \frac{1}{2v}$ .

In a similar way, it stems from the inequality (4.37) that in the case  $q < \frac{1}{2v}$  and  $q > i$  we have (4.32), and that in the case  $q < \frac{1}{2v}$  and  $q \leq i$ , we have the estimate (4.34). □

In case of a low rank  $q$ , we now derive some deterministic limits for Riemann sums related to  $x^q$  and  $h^{n,q}$ .

**Lemma 4.12.** *Let  $n \geq 1$  and let  $0 = t_0 < \dots < t_n = 1$  be the uniform partition of  $[0, 1]$  of order  $n$ . Let  $x$  be a standard fBm with Hurst parameter  $v \in (0, \frac{1}{2})$ . Recall that  $h^{n,q}$  is defined by (4.5). Consider also  $n \gg m$ , and a partition  $0 \leq s_0 < \dots < s_m \leq 1$  of  $[0, 1]$ . We assume that  $|s_{i+1} - s_i| \leq \frac{1}{m}$ , and  $s_0 = s, s_m = t$ . Then the following limits hold true:*

(i) For  $v = \frac{1}{2q}$ , we have

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(x^q; h^{n,q}) = \left(-\frac{1}{2}\right)^q (t - s) \tag{4.38}$$

in  $L_2$ , for all  $(s, t) \in \mathcal{S}_2$ .

(ii) For  $v < \frac{1}{2q}$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1-qv}} \mathcal{J}_s^t(x^q; h^{n,q}) = \left(-\frac{1}{2}\right)^q (t - s) \tag{4.39}$$

in  $L_2$ , for all  $(s, t) \in \mathcal{S}_2$ .

**Proof.** We shall only prove item (i), since item (ii) can be treated along the same lines. Our global strategy is based on identity (4.10) and (4.11), as in the proof of Lemma 4.11, with a more in-depth analysis of the terms appearing in our decomposition.

Indeed, formula (4.10) together with (4.11) assert that

$$\mathbb{E} \left[ \left( \frac{1}{\sqrt{n}} \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(x^q; h^{n,q}) \right)^2 \right] = n^{-1} \vartheta(q, q, q) = \sum_{r=0}^q a(r),$$

where

$$a(r) = n^{-1} \sum_{j,j'=0}^{m-1} \sum_{s_j \leq t_k < s_{j+1}} \sum_{s_{j'} \leq t_{k'} < s_{j'+1}} r! \binom{q}{r}^2 \rho(k - k')^r b(r), \tag{4.40}$$

and where

$$b(r) = \mathbb{E} \langle D^{2q-2r} (x_{\varepsilon(s_j)t_k}^q x_{\varepsilon(s_{j'})t_{k'}}^q), \beta_k^{\otimes(q-r)} \otimes \beta_{k'}^{\otimes(q-r)} \rangle_{\mathcal{H}^{\otimes(2q-2r)}}. \tag{4.41}$$

We now split the analysis of the terms  $a(r)$  and  $b(r)$ .

*Step 1: Case  $r > 0$ .* The term  $a(r)$  for  $r > 0$  can be bounded as follows, along the same lines as in the proof of Lemma 4.5 and Lemma 4.11. Namely, we bound all the combination numbers by a constant, we use identity (4.18) and the fact that  $\sum_{k,k'=0}^{n-1} |\rho(k - k')|^r = O(n)$  in order to get

$$|a(r)| \leq K |b(r)|.$$

In order to bound  $b(r)$ , we resort to identity (4.41). Then we observe that each term  $D^{2q-2r} (x_{\varepsilon(s_j)t_k}^q x_{\varepsilon(s_{j'})t_{k'}}^q)$  is of order  $m^{-rv}$ , while each contribution of the form  $\langle \mathbf{1}_{[a,b]}^{\otimes(q-r)}, \beta_k^{\otimes(q-r)} \rangle_{\mathcal{H}^{\otimes(q-r)}}$  can be bounded by a constant (similarly to (4.15)). This yields

$$a(r) \leq K m^{-2v}.$$

Therefore, it is readily checked that  $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} a(r) = 0$ .

*Step 2: Decomposition of  $a(0)$  and  $b(0)$ .* When  $r = 0$ , formula (4.40) can be read as:

$$a(0) = n^{-1} \sum_{j,j'=0}^{m-1} \sum_{s_j \leq t_k < s_{j+1}} \sum_{s_{j'} \leq t_{k'} < s_{j'+1}} b(0).$$

Notice that  $D^{2q} (x_{\varepsilon(s_j)t_k}^q x_{\varepsilon(s_{j'})t_{k'}}^q)$  can be written as a sum of deterministic functions of the form  $h_{2q} = g_{1,q} \otimes g_{2,q}$ , where  $h_{2q}$  is a function of  $2q$  variables, and each  $g_{1,q}, g_{2,q}$  is a function of  $q$  variables. In addition, the reader can check that  $g_{1,q}$  contains  $q'$  (resp.  $q - q'$ ) tensor products of indicator functions  $\mathbf{1}_{[\varepsilon(s_j), t_k]}$  (resp.  $\mathbf{1}_{[\varepsilon(s_{j'}), t_{k'}]}$ ), and  $g_{2,q}$  contains  $q - q'$  tensor products of functions  $\mathbf{1}_{[\varepsilon(s_j), t_k]}$ , for some  $0 \leq q' \leq q$ . Pairing those functions with  $\beta_k$  and  $\beta_{k'}$ , we get the following identity:

$$b(0) = \sum_{q'=0}^q b(0, q'),$$

where

$$b(0, q') = \binom{q}{q'}^2 \langle \mathbf{1}_{[\varepsilon(s_j), t_k]}, \beta_k \rangle_{\mathcal{H}}^{q-q'} \langle \mathbf{1}_{[\varepsilon(s_{j'}), t_{k'}]}, \beta_{k'} \rangle_{\mathcal{H}}^{q-q'} \langle \mathbf{1}_{[\varepsilon(s_j), t_k]}, \beta_k \rangle_{\mathcal{H}}^{q'} \langle \mathbf{1}_{[\varepsilon(s_{j'}), t_{k'}]}, \beta_{k'} \rangle_{\mathcal{H}}^{q'}.$$

*Step 3: Study of  $b(0, q')$  for  $q' > 0$ .* Let us observe that, thanks to the fact that  $2qv = 1$ , we have

$$b(0, q') = \binom{q}{q'}^2 n^{-1} \hat{b}(0, q') \tilde{b}(0, q'), \tag{4.42}$$

where

$$\hat{b}(0, q') = \langle n^v \mathbf{1}_{[\varepsilon(s_j), t_k]}, \beta_k \rangle_{\mathcal{H}}^{q-q'} \langle n^v \mathbf{1}_{[\varepsilon(s_{j'}), t_{k'}]}, \beta_{k'} \rangle_{\mathcal{H}}^{q-q'}$$

and

$$\tilde{b}(0, q') = \langle n^v \mathbf{1}_{[\varepsilon(s_{j'}), t_{k'}]}, \beta_{k'} \rangle_{\mathcal{H}}^{q'} \langle n^v \mathbf{1}_{[\varepsilon(s_j), t_k]}, \beta_k \rangle_{\mathcal{H}}^{q'}.$$

In order to bound  $\hat{b}(0, q')$  we can proceed as in (4.15) and we just get

$$|\hat{b}(0, q')| \leq K. \tag{4.43}$$

We now turn to a bound on  $\tilde{b}(0, q')$ . Some scaling arguments similar to (4.15) reveal that

$$|\tilde{b}(0, q')| \leq \langle \mathbf{1}_{[\lceil ns_{j'} \rceil, k']}, \mathbf{1}_{[k, k+1]} \rangle_{\mathcal{H}}^{q'} \langle \mathbf{1}_{[\lceil ns_j \rceil, k]}, \mathbf{1}_{[k', k'+1]} \rangle_{\mathcal{H}}^{q'}. \tag{4.44}$$

We now obtain uniform bounds on  $\tilde{b}(0, q')$  according to the values of  $j, j'$ .

(i) If  $|j - j'| \geq 2$ , then we also have  $|k - k'| \geq \frac{n}{m}$  in (4.44). Hence it is readily checked that

$$\begin{aligned} |\langle \mathbf{1}_{[\lceil ns_{j'} \rceil, k']}, \mathbf{1}_{[k, k+1]} \rangle_{\mathcal{H}}| &\leq K_\nu \int_{[\lceil ns_{j'} \rceil, k'] \times [k, k+1]} \frac{du dv}{|u - v|^{2-2\nu}} \\ &\leq K_\nu \left(\frac{n}{m}\right)^{2\nu-2} \left(\frac{n}{m}\right) = \left(\frac{m}{n}\right)^{1-2\nu}, \end{aligned}$$

and the same bound holds true for  $\langle \mathbf{1}_{[\lceil ns_j \rceil, k]}, \mathbf{1}_{[k', k'+1]} \rangle_{\mathcal{H}}$ . Hence we have

$$|\tilde{b}(0, q')| \leq K_\nu \left(\frac{m}{n}\right)^{1-2\nu}.$$

(ii) If  $|j - j'| \leq 2$ , then we simply bound  $\tilde{b}(0, q')$  by a constant, just as in (4.15) and (4.43). Plugging those estimates into (4.42), it is now readily checked that

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{j, j'=0}^{m-1} \sum_{s_j \leq t_k < s_{j+1}} \sum_{s_{j'} \leq t_{k'} < s_{j'+1}} b(0, q') \leq K_\nu \lim_{n \rightarrow \infty} n^{-2} \sum_{k, k'=0}^{n-1} \left(\frac{m}{n}\right)^{1-2\nu} = 0.$$

Therefore, the limit of  $a(0)$  is equal to the limit of

$$\tilde{a}(0) := n^{-1} \sum_{j, j'=0}^{m-1} \sum_{s_j \leq t_k < s_{j+1}} \sum_{s_{j'} \leq t_{k'} < s_{j'+1}} b(0, 0).$$

*Step 4: Convergence of  $a(0)$ .* We use some notation of the previous step: we have  $b(0, 0) = \hat{b}(0, 0)$ , and we apply the same scaling arguments as before. We end up with

$$\begin{aligned} \tilde{a}(0) &= \frac{1}{n^2} \sum_{j, j'=0}^{m-1} \sum_{s_j \leq t_k < s_{j+1}} \sum_{s_{j'} \leq t_{k'} < s_{j'+1}} \langle \mathbf{1}_{[\lceil ns_j \rceil, k]}, \mathbf{1}_{[k, k+1]} \rangle_{\mathcal{H}}^q \langle \mathbf{1}_{[\lceil ns_{j'} \rceil, k']}, \mathbf{1}_{[k', k'+1]} \rangle_{\mathcal{H}}^q \\ &= \left( n^{-1} \sum_{j=0}^{m-1} \sum_{s_j \leq t_k < s_{j+1}} \langle \mathbf{1}_{[\lceil ns_j \rceil, k]}, \mathbf{1}_{[k, k+1]} \rangle_{\mathcal{H}}^q \right)^2 \\ &= \left( \sum_{j=0}^{m-1} \frac{m_j}{n} \frac{1}{m_j} \sum_{s_j \leq t_k < s_{j+1}} \langle \mathbf{1}_{[\lceil ns_j \rceil, k]}, \mathbf{1}_{[k, k+1]} \rangle_{\mathcal{H}}^q \right)^2, \end{aligned} \tag{4.45}$$

where  $m_j = \#\{t_k : s_j \leq t_k < s_{j+1}\}$ . Note that by the stationarity of increments of  $x$  and recalling that  $\rho(i) = \mathbb{E}[\delta x_{01} \delta x_{i, i+1}]$  we have

$$\begin{aligned} \frac{1}{m_j} \sum_{s_j \leq t_k < s_{j+1}} \langle \mathbf{1}_{[\lceil ns_j \rceil, k]}, \mathbf{1}_{[k, k+1]} \rangle_{\mathcal{H}}^q &= \frac{1}{m_j} \sum_{s_j < t_k < s_{j+1}} \left( \sum_{i=1}^{k-n\varepsilon(s_j)} \rho(i) \right)^q \\ &= \frac{1}{m_j} \sum_{0 < k < m_j} \left( \sum_{i=1}^k \rho(i) \right)^q. \end{aligned}$$

Taking into account the fact that  $\lim_{n \rightarrow \infty} \sum_{i=1}^k \rho(i) = -\frac{1}{2}\rho(0)$  (remember that  $\sum_{k \in \mathbb{Z}} \rho(k) = 0$ ), a Cesaro mean argument shows that

$$\lim_{n \rightarrow \infty} \frac{1}{m_j} \sum_{0 < k < m_j} \left( \sum_{i=1}^k \rho(i) \right)^q = \left( -\frac{\rho(0)}{2} \right)^q = \left( -\frac{1}{2} \right)^q.$$

Plugging this information back into (4.45) and taking into account the fact that  $\lim_{n \rightarrow \infty} \frac{m_j}{n} = (s_{j+1} - s_j)$ , we obtain that

$$\lim_{n \rightarrow \infty} \tilde{a}(0) = \left( \sum_{j=0}^{m-1} (s_{j+1} - s_j) \left( -\frac{1}{2} \right)^q \right)^2 = \left( (t - s) \left( -\frac{1}{2} \right)^q \right)^2.$$

We can thus conclude that

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E} \left[ \left( \frac{1}{\sqrt{n}} \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(x^q; h^{n,q}) \right)^2 \right] = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} a(0) = \left( -\frac{1}{2} \right)^{2q} (t - s)^2. \tag{4.46}$$

*Step 5: Conclusion.* With relation (4.46) in hand, the convergence (4.38) is reduced to show the convergence of the first moment of  $\mathcal{J}_{s_j}^{s_{j+1}}(x^q; h^{n,q})$ . Furthermore, we have

$$\frac{1}{\sqrt{n}} \mathbb{E}[\mathcal{J}_{s_j}^{s_{j+1}}(x^q; h^{n,q})] = n^{-\frac{1}{2}} \sum_{s_j \leq t_k < s_{j+1}} \mathbb{E}[x_{\varepsilon(s_j)t_k}^q H_q(n^\nu x_{t_k, t_{k+1}})].$$

Rescaling and integrating by parts we get:

$$\frac{1}{\sqrt{n}} \mathbb{E}[\mathcal{J}_{s_j}^{s_{j+1}}(x^q; h^{n,q})] = \frac{1}{n} \sum_{s_j \leq t_k < s_{j+1}} \langle \mathbf{1}_{[n\varepsilon(s_j), k]}, \mathbf{1}_{[k, k+1]} \rangle_{\mathcal{H}}^q.$$

With the same arguments as for (4.46), we end up with:

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \mathbb{E} \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(x^q; h^{n,q}) = (t - s) \left( -\frac{1}{2} \right)^q.$$

The proof of (4.38) is now complete. □

We are now ready to state a weighted type Breuer–Major theorem which generalizes [34, Theorem 5.3] and [38, Theorem 1.1] to weights given by a controlled process.

**Proposition 4.13.** *Let  $x$  be a fBm with Hurst parameter  $\nu \in (0, \frac{1}{2})$  considered as a  $(L_p, \nu, \ell)$  rough path as in Notation 4.3, and consider  $q > 0$ .*

*We define  $h^{n,q}$  by relation (4.5). Let  $y$  be a discrete process controlled by  $(x, 1 - q\nu)$  in  $L_2$  or almost surely. Then the following convergences hold true:*

(i) *When  $\nu = \frac{1}{2q}$ , we have*

$$n^{-\frac{1}{2}} \mathcal{J}_0^1(y; h^{n,q}) \xrightarrow{\text{stable f.d.d.}} \sigma \int y_u dW_u + \left( -\frac{1}{2} \right)^q \int y_u^{(q)} du,$$

where  $\sigma = q! \sum_{k \in \mathbb{Z}} \rho(k)^q$ .

(ii) *When  $\nu < \frac{1}{2q}$ , we get*

$$n^{-(1-q\nu)} \mathcal{J}_0^1(y; h^{n,q}) \longrightarrow \left( -\frac{1}{2} \right)^q \int_0^1 y_u^{(q)} du,$$

where the convergence holds in probability.

**Proof.** In the case  $\nu = \frac{1}{2q}$ , the condition  $\nu\ell + (1 - q\nu) = \nu\ell + \frac{1}{2} > 1$  can be read as  $\ell = q + 1$ . We now invoke Theorem 3.13. Indeed, condition (2.13) is ensured by (4.33), condition (i) in Theorem 3.13 is just Breuer–Major’s Theorem 4.2, and condition (3.36) has been proved in (4.38). Moreover, in our situation, condition (3.37) has to be checked for  $i < q$ , and is easily shown thanks to inequality (4.31). Therefore, a direct application of Theorem 3.13 yields our claim (i).

In order to get item (ii), we apply Theorem 3.11. In this case, condition (2.13) is a consequence of (4.32) and (4.34). Item (i) in Theorem 3.11 is a consequence of (4.39) and (4.32). Eventually, (3.28) is obtained through (4.30). This concludes the proof.  $\square$

We now go one step further in the generalization, and handle the case of a weighted sum in an infinite number of chaos.

**Theorem 4.14.** *Let  $x$  be a fBm with Hurst parameter  $\nu \in (0, \frac{1}{2})$  considered as a  $(L_p, \nu, \ell)$  rough path as in Notation 4.3. Let  $(y, y', \dots, y^{(\ell-1)})$  be a discrete process controlled by  $(x, 1 - \nu d)$  as in Proposition 4.13 and take  $\ell = d + 1$ . Let  $f = \sum_{q=d}^{\infty} a_q H_q \in L_2(\gamma)$  be a function with Hermite rank  $d > 0$  satisfying one of the conditions (a) and (b) of Theorem 4.7. Set*

$$h_{st}^n = n^{d\nu-1} \sum_{s \leq t_k < t} f(n^\nu \delta x_{t_k t_{k+1}}), \quad (s, t) \in \mathcal{S}_2.$$

Then the following limits hold true.

(i) When  $d = \frac{1}{2\nu}$  we have the stable f.d.d. convergence:

$$\mathcal{J}(y; h^n) \rightarrow \sigma \int y_t dW_t + \left(-\frac{1}{2}\right)^d a_d \int y_u^{(d)} du, \quad \text{as } n \rightarrow \infty, \tag{4.47}$$

where  $\sigma$  is given by (4.1).

(ii) When  $d < \frac{1}{2\nu}$  we get the following convergence in probability:

$$\mathcal{J}_s^t(y; h^n) \rightarrow \left(-\frac{1}{2}\right)^d a_d \int_s^t y_u^{(d)} du, \quad \text{as } n \rightarrow \infty.$$

**Proof.** *Step 1: A decomposition of  $f$ .* In order to prove the convergence (4.47) we invoke Theorem 3.13. It remains to verify that conditions in Theorem 3.13 are satisfied. To this aim, we define a new function

$$\tilde{f} := f - a_d H_d = \sum_{q=d+1}^{\infty} a_q H_q, \tag{4.48}$$

and denote

$$\tilde{h}_{st}^n = \frac{1}{\sqrt{n}} \sum_{s \leq t_k < t} \tilde{f}(n^\nu \delta x_{t_k t_{k+1}}), \quad (s, t) \in \mathcal{S}_2.$$

Now recalling that  $h^{n,d}$  is defined by (4.5), we write

$$\mathcal{J}_s^t(x^i; h^n) = a_d n^{-\frac{1}{2}} \mathcal{J}_s^t(x^i; h^{n,d}) + \mathcal{J}_s^t(x^i; \tilde{h}^n). \tag{4.49}$$

This decomposition will be used in order to verify the assumptions of Theorem 3.13.

*Step 2: Proof of condition (2.13).* We first note that relation (4.33) implies that the quantity  $\mathcal{J}_s^t(x^i; h^{n,d})$  on the right-hand side of (4.49) satisfies condition (2.13). On the other hand, since  $\tilde{f}$  satisfies the conditions of Theorem 4.7, it follows from the proof of Theorem 4.7 that  $\mathcal{J}_s^t(x^i; \tilde{h}^n)$  also satisfies (2.13). Combining these two observations and applying the triangle inequality for the  $L_2$ -norm to (4.49), we obtain (2.13) for  $\mathcal{J}_s^t(x^i; h^n)$ .

*Step 3: Stable convergence of  $h^n$ .* The proof of the stable convergence of  $h^n$  follows the same lines as in Theorem 4.7. It is omitted for sake of conciseness.

*Step 4: Proof of (3.36).* We have already noticed that  $\tilde{f}$  defined in (4.48) satisfies the conditions of Theorem 4.7. So it follows from the proof of Theorem 4.7 that  $\tilde{h}^n$  satisfies the relation (3.13). More precisely, the following convergence for  $i = 1, \dots, d$  is obtained similarly to (4.27):

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \left| \sum_{j=0}^{m-1} \mathcal{J}_s^{s_{j+1}}(x^i; \tilde{h}^n) \right|_{L_2} = 0. \tag{4.50}$$



On the other hand, it follows from (4.38) that  $n^{-\frac{1}{2}}h^{n,d}$  satisfies (3.36). Putting together (4.50) and the convergence of  $\mathcal{J}_{s_j}^{s_{j+1}}(x^i; h^{n,d})$  and taking into account (4.49) we obtain the convergence (3.36) for  $h^n$ .

*Step 5: Proof of (3.37).* As in the previous step, invoking relation (4.49), it suffices to consider the relation (3.37) for  $\tilde{h}^n$  and  $h^{n,d}$  separately. Notice that relation (3.37) for  $\tilde{h}^n$  follows directly from (4.50). On the other hand, relation (3.37) for  $h^n = h^{n,q}$  is obtained exactly as in the proof of Proposition 4.13, thanks to (4.31). This completes the proof of (3.37) for  $h^n$ .

*Step 6: Proof of item (ii).* Item (ii) is obtained by applying Theorem 3.11. We have to verify the same kind of conditions as in the previous steps. Resorting to our decomposition (4.49), this is done similarly to Step 2–5, applying Proposition 4.13 and Theorem 4.7. Details are left to the reader. The proof is now complete.  $\square$

### 4.3. Realized power variations and parameter estimations

The convergence of realized power variations is closely related to the parameter estimation problem of the volatility process (see e.g. [3] and [29] in a fBm context). Here we shall consider generalizations of realized power variations to rougher situations, and then discuss briefly the parameter estimation problem.

Let us start by introducing some additional notation. For  $p > -1$ , we denote

$$c_p = \mathbb{E}(|N|^p) = \frac{2^{p/2}}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right). \tag{4.51}$$

Notice that when  $p$  is an even integer we can also write  $c_p = \mathbb{E}(N^p) = (p-1)(p-3)\cdots 1$ . We also consider the function  $H : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $H(x) = |x|^p - c_p$ .

It is easy to see that  $H \in L_2(\gamma)$  when  $p > -\frac{1}{2}$  and  $H$  has Hermite rank  $d = 2$ . One can also verify that  $H$  has the decomposition  $H(x) = \sum_{q=1}^{\infty} a_{2q} H_{2q}(x)$ , where the constants  $a_{2q}$  are obtained by expanding the function  $|x|^p - c_p$  on the Hermite basis, and are expressed in terms of the  $c_p$ 's:

$$a_{2q} = \sum_{r=0}^q \frac{(-1)^r}{2^r r! (2q-2r)!} (c_{2q-2r+p} - c_p c_{2q-2r}). \tag{4.52}$$

For example, we will use the fact that  $a_2 = pc_p/2$ .

Our first result in this subsection concerns the weighted power variations of  $x$  by a controlled process  $y$ . We focus on the rough situation  $\nu \leq \frac{1}{2}$ , since more regular situations are handled in e.g. [3,29] and implied by our Proposition 4.9.

**Theorem 4.15.** *Let  $x$  be a fBm with Hurst parameter  $\nu \in (0, \frac{1}{2})$ , considered as a  $(L_p, \nu, \ell)$  rough path as in Notation 4.3. Let  $(y^{(0)}, \dots, y^{(\ell-1)})$  be a discrete process controlled by  $(x, \alpha)$ , in  $L_2$  or almost surely, with  $\nu\ell + \alpha > 1$ . Then the following limits for weighted power variations hold true:*

- (i) *Suppose that  $\frac{1}{2} \geq \nu > \frac{1}{4}$  and  $\alpha = \frac{1}{2}$ . Then for  $p \geq 2$  we have the convergence:*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} y_{t_k} (|n^\nu \delta x_{t_k t_{k+1}}|^p - c_p) \xrightarrow{d} \sigma \int_0^1 y_t dW_t, \tag{4.53}$$

where  $W$  is a Wiener process independent of  $x$  and  $\sigma^2$  is defined by (recall that  $a_{2q}$  is defined by (4.52)):

$$\sigma^2 = \sum_{q=1}^{\infty} (2q)! a_{2q}^2 \sum_{k \in \mathbb{Z}} \rho(k)^{2q}. \tag{4.54}$$

- (ii) *Suppose that  $\nu = \frac{1}{4}$  and  $\alpha = \frac{1}{2}$ . Then for  $p \geq 4$  we have the convergence:*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} y_{t_k} (|n^\nu \delta x_{t_k t_{k+1}}|^p - c_p) \xrightarrow{d} \sigma \int_0^1 y_t dW_t + \frac{a_2}{4} \int_0^1 y_t'' dt,$$

where  $\sigma$  is defined by (4.54) and where we recall that, according to (4.52), we have  $a_2 = \frac{pc_p}{2}$ .

(iii) Suppose that  $\nu < \frac{1}{4}$  and  $\alpha = 1 - 2\nu$ . Then for  $p \geq 4$  we have the following convergence in probability:

$$n^{2\nu-1} \sum_{k=0}^{n-1} y_{t_k} (|n^\nu \delta x_{t_k t_{k+1}}|^p - c_p) \rightarrow \frac{a_2}{4} \int_0^1 y_t'' dt.$$

**Proof.** Recall that we have set  $H(x) = |x|^p - c_p$  and that the Hermite rank of  $H$  is  $d = 2$ . Then item (i) follows immediately from Theorem 4.7. Indeed, since  $\nu > \frac{1}{4}$ , we have  $\frac{1}{2\nu} < 2$ , and so the Hermite rank of  $|x|^p - c_p$  is larger than  $\frac{1}{2\nu}$ . On the other hand, it is easy to see that  $\ell = 2$  for the definition of our controlled process  $y$  under the condition that  $\nu\ell + \alpha > 1$  and  $\alpha = \frac{1}{2}$ . So for  $p \geq 2$  we have  $H \in C^{2\ell-2}$  and thus  $H$  satisfies condition (c) in Theorem 4.7. Therefore, a direct application of Theorem 4.7 yields the convergence (4.53). Item (ii) and item (iii) follow from Theorem 4.14. The proof is similar and is omitted.  $\square$

We now consider a controlled process of order 2 with respect to  $x$ , called  $(z, z')$ . Recall that  $(z, z')$  satisfies:

$$|r_{st}^z| \leq G(t-s)^{2\nu}, \quad \text{with } r_{st}^z := \delta z_{st} - z'_s \delta x_{st}, \tag{4.55}$$

where  $G$  is some almost surely finite random variable. In the following we prove the convergence of the  $p$ -variation of  $z$  with the help of Theorem 4.15. We will see that, with a proper normalization, the  $p$ -variation of the (first-order) increments  $\sum_{k=0}^{n-1} |\delta z_{t_k t_{k+1}}|^p$  converges almost surely to the quantity  $c_p \int_0^1 |z'_s|^p ds$  (one can also use “longer filters”, i.e. replacing the increments  $\delta z_{t_k t_{k+1}}$  by the second-order increments  $\delta z_{t_k t_{k+1}} - \delta z_{t_{k-1} t_k}$  or higher-order increments for instance; see e.g. [43]). Observe that our motivation for this limit result is the parameter estimation of the diffusion coefficient for SDEs; see e.g. [3,29]. Indeed, consider the following equation governed by a fBm with Hurst parameter  $\nu \in (0, \frac{1}{2}]$ :

$$z_t = \int_0^t b(z_s) ds + \int_0^t v(z_s) dx_s. \tag{4.56}$$

In equation (4.56), the coefficient  $b$  and  $v$  are assumed to be  $C_b^2$  and  $C_b^3$ , respectively. The stochastic integral in (4.56) is understood thanks to the abstract rough paths theory (see e.g. [15,16,20]), by considering the rough path  $\{x^i, 1 \leq i \leq \lfloor \frac{1}{\nu} \rfloor\}$ , where  $x^i$  is given in Notation 4.3. Taking  $z' = v(z)$ , it is well-known that the pair  $(z, z')$  is a process controlled by  $x$ . Then our limit result for  $(z, z')$  implies that the  $p$ -variation of the solution of (4.56) converges almost surely to the average of the volatility  $c_p \int_0^1 |v(z_s)|^p ds$ .

**Corollary 4.16.** *Let  $x$  be a one-dimensional fBm with Hurst parameter  $0 < \nu \leq \frac{1}{2}$ , and let  $(z, z')$  be a controlled process of  $x$  satisfying (4.55). Let  $n \geq 1$  and consider the uniform partition  $0 = t_0 < \dots < t_n = 1$  of  $[0, 1]$ . Then for  $p > \frac{1}{2\nu}$ , we have almost surely the convergence of  $p$ -variations of  $z$ :*

$$\frac{1}{n} \sum_{k=0}^{n-1} |n^\nu \delta z_{t_k t_{k+1}}|^p \rightarrow c_p \int_0^1 |z'_t|^p dt. \tag{4.57}$$

**Proof.** Set  $\varphi := n^{p\nu-1} \sum_{k=0}^{n-1} |z'_{t_k}|^p |\delta x_{t_k t_{k+1}}|^p$ . We write

$$\frac{1}{n} \sum_{k=0}^{n-1} |n^\nu \delta z_{t_k t_{k+1}}|^p = \varphi + R_n, \tag{4.58}$$

where  $R_n$  is simply  $\frac{1}{n} \sum_{k=0}^{n-1} |n^\nu \delta z_{t_k t_{k+1}}|^p - \varphi$ . Using the inequality  $||a|^p - |b|^p| \leq p(|a|^{p-1} + |b|^{p-1})|a - b|$  for  $p > 1$  and the regularity of  $z$  and  $x$ , we obtain

$$\begin{aligned} |R_n| &\leq pn^{p\nu-1} \sum_{k=0}^{n-1} (|\delta z_{t_k t_{k+1}}|^{p-1} + |z'_{t_k} \delta x_{t_k t_{k+1}}|^{p-1}) |\delta z_{t_k t_{k+1}} - z'_{t_k} \delta x_{t_k t_{k+1}}| \\ &\leq Gn^{p\nu-1} \sum_{k=0}^{n-1} n^{-(p-1)\nu} n^{-2\nu} = Gn^{-\nu}. \end{aligned}$$

In particular, we have  $\lim_{n \rightarrow \infty} R_n = 0$  almost surely as  $n \rightarrow \infty$ . On the other hand, a direct application of Theorem 4.15 shows that  $\varphi \rightarrow c_p \int_0^1 |z'_t|^p dt$ . Putting together the convergence of  $R_n$  and  $\varphi$  and taking into account (4.58), we obtain the desired limit (4.57).  $\square$

#### 4.4. Stratonovich integrals

In this subsection we are shedding a new light on another problem which has drawn a lot of attention in the recent stochastic analysis literature. Namely, we are interested in the convergence of the following trapezoidal-rule Riemann sum:

$$\text{tr-}\mathcal{J}_0^1(y; x) := \sum_{k=0}^{n-1} \frac{y_{t_k} + y_{t_{k+1}}}{2} \delta x_{t_k t_{k+1}}. \tag{4.59}$$

This quantity has been considered by many authors (see e.g. [6,8,18,21–23,36,39]) in the case  $y_s = f(x_s)$ . Thanks to the rough paths technique developed in this paper, we will be able to get shorter proofs than in the aforementioned articles, and obtain results which are valid for a wider class of weight processes  $y$ . We will also see that the limit of (4.59) can be identified with the rough integral  $\int_0^1 y_s dx_s$  for  $\nu > \frac{1}{6}$  and that it is equal to the same rough integral plus a “correction” term when  $\nu = \frac{1}{6}$ .

Let us start by some preliminary results.

**Lemma 4.17.** *Let  $q$  be an integer such that  $q > 1$ , and assume that  $\nu \in (\frac{1}{2q}, \frac{1}{2})$ . Let  $x$  and  $y$  be as in Theorem 4.7. Then the following convergence holds as  $n \rightarrow \infty$ :*

$$n^{-\frac{1}{2}} \mathcal{J}_s^t(y; h^{n,q}) \xrightarrow{d} \sigma \int_s^t y_u dW_u, \tag{4.60}$$

where  $\sigma^2 = q! \sum_{k \in \mathbb{Z}} \rho(k)^q$ .

**Proof.** The lemma follows immediately from Theorem 4.7 with  $f = H_q$ . Notice that  $\nu > \frac{1}{2q}$  by assumption, thus we also have  $q > \frac{1}{2\nu}$ , which is one of the assumption in Theorem 4.7.  $\square$

Our second preliminary result concerns weighted power variations of the fBm  $x$ .

**Lemma 4.18.** *Let  $x$  be a fBm with Hurst parameter  $\nu \in (0, \frac{1}{2})$  considered as a  $(L_p, \nu, \ell)$  rough path as in Notation 4.3.*

(i) *Let  $(y, y')$  be a discrete process controlled by  $(x, 1 - \nu)$ . When  $q \geq 3$  is odd, we have the following convergence in probability:*

$$n^{(q+1)\nu-1} \sum_{k=0}^{n-1} y_{t_k} (\delta x_{t_k t_{k+1}})^q \longrightarrow -\frac{c_{q+1}}{2} \int_0^1 y'_s ds, \tag{4.61}$$

where the constants  $c_p$  are defined by (4.51).

(ii) *Let  $(y, \dots, y^{(\ell-1)})$  be a discrete process controlled by  $(x, \alpha)$ , in  $L_2$  or almost surely, with  $\nu\ell + \alpha > 1$ , where  $\alpha = \frac{1}{2}$  for  $\nu \in [\frac{1}{4}, \frac{1}{2})$  and  $\alpha = 1 - 2\nu$  for  $\nu \in (0, \frac{1}{4})$ . When  $q$  is even, we have the convergence in probability:*

$$n^{q\nu-1} \sum_{k=0}^{n-1} y_{t_k} (\delta x_{t_k t_{k+1}})^q \longrightarrow c_q \int_0^1 y_s ds. \tag{4.62}$$

**Proof.** We first show the convergence (4.61) with the help of Theorem 4.14. Note that whenever  $q$  is odd the function  $f(x) = x^q$  has rank  $d = 1$ , so we have  $d < \frac{1}{2\nu}$ . The convergence (4.61) then follows from Theorem 4.14 (ii). In order to prove (4.62) we start by observing that an easy consequence of (4.53) is that the following limit in probability holds true:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} y_{t_k} ((n^\nu \delta x_{t_k t_{k+1}})^q - c_q) = 0. \tag{4.63}$$

Then observe that  $y$  is a continuous process. Therefore, we trivially have the following limit in probability:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} y_{t_k} = \int_0^1 y_s ds. \tag{4.64}$$

Combining (4.63) and (4.64), the convergence (4.62) is established for  $\nu \in (\frac{1}{4}, \frac{1}{2})$ . The cases  $\nu = \frac{1}{4}$  and  $\nu < \frac{1}{4}$  are treated in the same way, thanks to (respectively) Theorem 4.15 (ii) and (iii).  $\square$

We can now state a convergence result for trapezoidal Riemann sums.

**Theorem 4.19.** *Let  $x$  be a one-dimensional fBm with Hurst parameter  $\nu \in (0, \frac{1}{2})$ . Let  $y$  be an almost sure controlled process of order  $\ell = 8$  (see Definition 2.3). Recall that the trapezoidal sums of  $y$  with respect to  $x$  are defined by (4.59), and we set*

$$\int_0^1 y_s d^{tr} x_s = \lim_{n \rightarrow \infty} tr\text{-}\mathcal{J}_0^1(y; x), \tag{4.65}$$

whenever the limit in the right-hand side is properly defined. Then the following assertions hold true:

(i) When  $\nu > \frac{1}{6}$ , the convergence (4.65) holds almost surely and we have the identity:

$$\int_0^1 y_s d^{tr} x_s = \int_0^1 y_s dx_s, \tag{4.66}$$

where  $\int_0^1 y_s dx_s$  stands for the rough path integral of  $y$  with respect to  $x$ .

(ii) When  $\nu = \frac{1}{6}$ , the convergence (4.65) holds in distribution and the following relation holds true:

$$\int_0^1 y_s d^{tr} x_s = \int_0^1 y_s dx_s + \frac{\sigma}{12} \int_0^1 y_s'' dW_s, \tag{4.67}$$

where  $\int_0^1 y_s dx_s$  is understood in the rough path sense and  $\sigma = 6 \sum_{k \in \mathbb{Z}} \rho(k)^3$ .

**Proof.** *Step 1: Decomposition of  $tr\text{-}\mathcal{J}_0^1(y; x)$ .* Owing to the Definition 2.3 of a controlled process, we have

$$\delta y_{st} = \sum_{i=1}^5 \frac{1}{i!} y_s^{(i)} (\delta x_{st})^i + r_{st}^y, \tag{4.68}$$

where the remainder  $r$  satisfies  $|r_{st}^y|_{L_2} \leq K(t-s)^{6\nu}$ . Plugging (4.68) into (4.59) we obtain

$$\begin{aligned} tr\text{-}\mathcal{J}_0^1(y; x) &= \sum_{k=0}^{n-1} y_{t_k} \delta x_{t_k t_{k+1}} + \frac{1}{2} \sum_{k=0}^{n-1} \delta y_{t_k t_{k+1}} \delta x_{t_k t_{k+1}} \\ &= \sum_{k=0}^{n-1} \left( y_{t_k} \delta x_{t_k t_{k+1}} + \frac{1}{2} \sum_{i=1}^5 \frac{1}{i!} y_{t_k}^{(i)} (\delta x_{t_k t_{k+1}})^{i+1} + \frac{1}{2} r_{t_k t_{k+1}}^y \delta x_{t_k t_{k+1}} \right), \end{aligned}$$

where we notice that our rough path type expansion is a natural generalization of the Taylor type expansions of  $f(x)$  performed in e.g. [21,39]. We now split the expansion as

$$tr\text{-}\mathcal{J}_0^1(y; x) = a_0 + a_1 + a_2 + a_3, \tag{4.69}$$

where

$$\begin{aligned} a_0 &= \sum_{k=0}^{n-1} \sum_{i=0}^5 \frac{1}{(i+1)!} y_{t_k}^{(i)} (\delta x_{t_k t_{k+1}})^{i+1}; \\ a_1 &= \frac{1}{12} \sum_{k=0}^{n-1} y_{t_k}^{(2)} (\delta x_{t_k t_{k+1}})^3 + \frac{1}{24} \sum_{k=0}^{n-1} y_{t_k}^{(3)} (\delta x_{t_k t_{k+1}})^4 := a_{11} + a_{12}, \end{aligned} \tag{4.70}$$

and

$$a_2 = \frac{1}{80} \sum_{k=0}^{n-1} y_{t_k}^{(4)} (\delta x_{t_k t_{k+1}})^5 + \frac{1}{360} \sum_{k=0}^{n-1} y_{t_k}^{(5)} (\delta x_{t_k t_{k+1}})^6; \quad a_3 = \frac{1}{2} \sum_{k=0}^{n-1} r_{t_k t_{k+1}}^{y} \delta x_{t_k t_{k+1}}. \tag{4.71}$$

We now consider these terms separately.

*Step 2: Terms  $a_0$  and  $a_3$ .* The remainder  $r^y$  has a Hölder regularity of order  $6\nu \geq 1$ . Therefore, it is readily checked that  $a_3 \rightarrow 0$  almost surely for  $\nu \geq \frac{1}{6}$ . In addition, we have the convergence  $a_0 \rightarrow \int_0^1 y_u dx_u$  almost surely whenever  $\nu \geq \frac{1}{6}$ , ensured by the abstract rough paths theory (see e.g. [15,20]). It is worth noticing at this point that the convergence of  $a_0$  is obtained in a much easier way in a rough path context than by means of integrations by parts as performed in e.g. [21,22,36].

*Step 3: Decomposition of  $a_1$  for  $\nu > \frac{1}{6}$ .* Among the terms defining  $a_1$  (4.70), we focus on the lower order term  $a_{11}$  (which potentially brings most difficulties). Thus we expand  $a_{11}$  by writing  $\xi^3 = H_3(\xi) - 3H_1(\xi)$ , where we recall that  $H_k$  stands for the Hermite polynomial of order  $k$ . This yields  $a_{11} = b_1 + b_2$ , where

$$b_1 = \frac{1}{12n^{3\nu}} \sum_{k=0}^{n-1} y_{t_k}^{(2)} H_3(n^\nu \delta x_{t_k t_{k+1}}), \quad \text{and} \quad b_2 = \frac{1}{4n^{2\nu}} \sum_{k=0}^{n-1} y_{t_k}^{(2)} \delta x_{t_k t_{k+1}}. \tag{4.72}$$

Moreover, thanks to Lemma 4.17, it is readily checked that  $b_1 \rightarrow 0$  in probability when  $\nu > \frac{1}{6}$ . We now focus on the term  $b_2$ . Since  $y^{(2)}$  is itself a controlled process of order 6, a slight elaboration of [16, Corollary 10.15] shows that

$$\int_{t_k}^{t_{k+1}} y_s^{(2)} dx_s - \sum_{i=0}^3 \frac{1}{(i+1)!} y_{t_k}^{(i+2)} (\delta x_{t_k t_{k+1}})^{i+1} := r_{t_k t_{k+1}}^{y^{(2)}},$$

where  $r_{t_k t_{k+1}}^{y^{(2)}}$  is a remainder of order  $5\nu$ :

$$|r_{t_k t_{k+1}}^{y^{(2)}}|_{L_2} \leq Kn^{-5\nu}. \tag{4.73}$$

Summing this identity over  $k$ , we thus get

$$b_2 = \frac{1}{4n^{2\nu}} \int_0^1 y_s^{(2)} dx_s - \sum_{i=1}^3 b_3^{(i)} - \frac{1}{4n^{2\nu}} \sum_{k=0}^{n-1} r_{t_k t_{k+1}}^{y^{(2)}}, \tag{4.74}$$

where each  $b_3^{(i)}$  is defined by

$$b_3^{(i)} = \frac{1}{4n^{2\nu}} \sum_{k=0}^{n-1} \frac{1}{(i+1)!} y_{t_k}^{(i+2)} (\delta x_{t_k t_{k+1}})^{i+1}. \tag{4.75}$$

In expression (4.74), it is easily seen that, thanks to (4.73), we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2\nu}} \sum_{k=0}^{n-1} r_{t_k t_{k+1}}^{y^{(2)}} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n^{2\nu}} \int_0^1 y_s^{(2)} dx_s = 0,$$

where the limits stand for limits in probability. Owing to (4.61) and (4.62), the reader can also check that  $\lim_{n \rightarrow \infty} b_3^{(i)} = 0$  for  $i = 2, 3$ . In order to analyze the right-hand side of (4.74) we are thus left with the term  $b_3^{(1)}$  defined by (4.75).

*Step 4: Terms  $b_3^{(1)}$  and  $a_{12}$ .* Comparing  $b_3^{(1)}$  with the expression (4.70) for  $a_{12}$ , we see that

$$a_{12} - b_3^{(1)} = \frac{1}{24n^{4\nu}} \sum_{k=0}^{n-1} y_{t_k}^{(3)} f(n^\nu \delta x_{t_k t_{k+1}}), \tag{4.76}$$

where the function  $f$  is given by  $f(\xi) = \xi^4 - 3\xi^2$ . In addition, invoking elementary properties of Hermite polynomials, it is easily seen that  $f$  has a Hermite rank of  $d = 2$ . Hence, according to the values of  $\nu$ , we can either apply Theorem 4.7

(for  $\frac{1}{4} < \nu < \frac{1}{2}$ ), Theorem 4.14(i) (for  $\nu = \frac{1}{4}$ ) or Theorem 4.14(ii) (for  $\frac{1}{6} < \nu < \frac{1}{4}$ ). As an example, when  $\frac{1}{6} < \nu < \frac{1}{4}$ , we get

$$|a_{12} - b_3^{(1)}|_{L_2} \leq \frac{K}{n^{6\nu-1}},$$

which obviously goes to 0 as  $n$  goes to  $\infty$ . In summary of the convergence of  $a_{12} - b_3^{(1)}$  and the analysis in Step 3, we obtain the convergence  $a_1 \rightarrow 0$  in probability as  $n \rightarrow \infty$ .

*Step 5: Terms  $a_2$  and conclusion for  $\nu > \frac{1}{6}$ .* The convergence in the case  $\nu > \frac{1}{6}$  is now easily obtained. Indeed, due to Lemma 4.18, it is readily checked that  $\lim_{n \rightarrow \infty} a_2 = 0$  in probability. Therefore, combining the convergence of  $a_0, a_1, a_2, a_3$  and taking into account (4.69) we obtain the convergence  $\text{tr-}\mathcal{J}_0^1(y; x) \rightarrow \int_0^1 y_u^{(0)} dx_u$  in probability, which identifies the two sides of equation (4.66).

*Step 6: Case  $\nu = \frac{1}{6}$ .* The proof for the case  $\nu = \frac{1}{6}$  follows the same arguments as for  $\nu > \frac{1}{6}$ . However, in the current situation more terms are contributing to the limit. Specifically, the terms  $a_2, b_1, b_3^{(2)}, b_3^{(2)}$  and  $a_{12} - b_3^{(1)}$ , respectively defined by (4.71), (4.72), (4.75) and (4.76), are now converging to non-zero limits. In order to handle the term  $b_1$ , we apply Proposition 4.13 (i) with  $q = 3$ . This yields the convergence:

$$(x, b_1) \xrightarrow{f.d.d.} \left( x, \frac{1}{12} \sigma \int_0^1 y_s^{(2)} dW_s - \frac{1}{96} \int_0^1 y_s^{(5)} ds \right),$$

where  $\sigma = 3! \sum_{k \in \mathbb{Z}} \rho(k)^3$ . On the other hand, applying (4.61), (4.62) respectively to the two terms of  $a_2$  in (4.71), we obtain the convergence in probability:  $\lim_{n \rightarrow \infty} a_2 = -\frac{5}{96} \int_0^1 y_s^{(5)} ds$ . Moreover, owing respectively to (4.61), (4.62) and Theorem 4.14(ii) (with  $d = 2$  and  $\nu = 3$ ) we obtain the convergence:

$$\lim_{n \rightarrow \infty} b_3^{(2)} = -\frac{1}{16} \int_0^1 y_t^{(5)} dt, \quad \lim_{n \rightarrow \infty} b_3^{(3)} = \frac{1}{32} \int_0^1 y_t^{(5)} dt, \quad \lim_{n \rightarrow \infty} (a_{12} - \tilde{b}_1) = \frac{1}{32} \int_0^1 y_t^{(5)} dt.$$

Putting together those additional convergences, and noticing that the terms involving  $y^{(5)}$  cancels, we end up with relation (4.67). The proof is now complete.  $\square$

### 5. Multi-dimensional Gaussian processes

Our method of analysis for limit theorems has potentially many applications in multidimensional settings. For sake of conciseness, we will restrict ourselves to an application concerning multidimensional quadratic variations. In this way we recover (in a more elementary way) a central limit theorem contained in [31] and used in [30]. We are also able to generalize this central limit theorem to a wide class of Gaussian processes (Section 5.1), and obtain a weighted version in Section 5.2.

#### 5.1. Preliminaries on Gaussian rough paths

Throughout the section we assume that  $X = (x^1, \dots, x^d)$  is a centered continuous Gaussian process with i.i.d. components. We shall write  $X_{uv}^1$  for the increments  $\delta X_{uv} = X_v - X_u$  of the process  $X$ . Then we define the covariance of the increments of  $X$  as:

$$\mathbb{E}(X_{uv}^{1,i} X_{st}^{1,i}) = R \begin{pmatrix} u & v \\ s & t \end{pmatrix},$$

where  $i$  stands for any of the components of  $X$ . We now recall some basic facts about the constructions of a rough path lift above  $X$ , borrowed from [16].

The basic assumption in order to be able to lift  $X$  as a rough path is that  $R$  admits a two-dimensional  $\rho$ -variation for  $\rho \in [1, 2)$ . Denote  $\nu = \frac{1}{2\rho}$ . For sake of simplicity, we will moreover assume that the  $\rho$ -variation of  $R$  satisfies:

$$|R|_{\rho\text{-var}, [s,t]} \leq K(t-s)^{2\nu}, \tag{5.1}$$

where  $|R|_{\rho\text{-var}, [s,t]}$  stands for the 2-dimensional  $\rho$ -variation of  $R$  in the interval  $[s, t]^2$ :

$$|R|_{\rho\text{-var}, [s,t]} = \sup_{(t_i), (t'_j) \in \mathcal{D}([s,t])} \left( \sum_{i,j} \left| R \begin{pmatrix} t_i & t_{i+1} \\ t'_j & t'_{j+1} \end{pmatrix} \right|^\rho \right)^{1/\rho},$$

where  $\mathcal{D}([s, t])$  denotes the set of partitions on the interval  $[s, t]$ . As mentioned in [7, Remark 2.4], the condition that  $R$  has finite  $\rho$ -variation for  $\rho \in [1, 2)$  is (essentially) equivalent to (5.1) up to a deterministic time change. Then it is shown in [16] that there exists a canonical continuous  $G^3(\mathbb{R}^d)$ -valued process  $\mathbf{X} = (1, X^1, X^2, X^3)$ , where  $G^3(\mathbb{R}^d)$  stands for the free nilpotent Lie group of order 3 endowed with the Carnot–Carathéodory distance  $d$  on  $G^3(\mathbb{R}^d)$ , such that:

- (i)  $\mathbf{X}$  “lifts” the Gaussian process  $X$  in the sense  $\pi_1(\mathbf{X}) = X_t^1 - X_0^1$ ;
- (ii) There exists  $C = C(\nu)$  such that for all  $s < t$  in  $[0, 1]$  and  $q \in [1, \infty)$ ,

$$|\mathbf{X}_{st}|_{L^q} \leq C\sqrt{q}|s - t|^\nu.$$

- (iii) For all  $\gamma < \nu$  there exists  $\epsilon = \epsilon(\nu, \gamma, C) > 0$  such that

$$\mathbb{E}(\exp(\epsilon|\mathbf{X}|_\gamma^2)) < \infty,$$

where  $|\mathbf{X}|_\gamma$  designates the  $\gamma$ -Hölder semi-norm of  $\mathbf{X}$ .

### 5.2. Unweighted limit theorem

With the construction of Section 5.1 in hand, let us consider the second level  $X^2$  of the rough path above  $X$  considered as a  $\mathbb{R}^{d \times d}$ -valued increment. In this subsection, we are interested in the convergence of the following random sum:

$$\sum_{k=0}^{n-1} \left( n^{2\nu} X_{t_k t_{k+1}}^2 - \frac{1}{2} \text{Id} \right). \tag{5.2}$$

Here  $\text{Id}$  stands for the identity matrix. It is clear that (5.2) is the generalization of the quadratic variation  $\sum_{k=0}^{n-1} (|n^\nu \delta \times X_{t_k t_{k+1}}|^2 - 1)$  to a multi-dimensional setting. Moreover, the quantity (5.2) is related to the analysis of numerical schemes for rough SDEs (see e.g. [30]).

The following assumption will be used heavily in our future computations.

**Hypothesis 5.1.** Consider a Gaussian process  $X$  whose covariance  $R$  satisfies (5.1). Suppose that  $X$  has stationary increments in the sense that the variance of its increments is given by

$$\mathbb{E}(|X_{st}^1|^2) = F(|t - s|) \geq 0,$$

with  $F$  continuous, nonnegative and with  $F(0) = 0$ . In addition, the following properties hold true:

- (i) Either  $F'' \geq 0$  or  $F'' \leq 0$ , in distributional sense on  $(0, T)$ . In other words, either  $F''$  or  $-F''$  is a nonnegative Radon measure on  $(0, T)$ .
- (ii) There exists a constant  $\theta : 2 - 2\nu \geq \theta > \frac{1}{2}$  such that

$$|F''| \leq C/t^\theta$$

holds true for  $t$  large, in distributional sense on  $(0, T)$  for some  $C > 0$ .

**Remark 5.2.** Condition (i) in Hypothesis 5.1 says that the Gaussian process  $X^1$  has either negative or positive correlation, that is, the covariance  $R\binom{u \ v}{s \ t}$  has the same sign for all disjoint intervals  $[u, v]$  and  $[s, t]$ . Condition (ii) implies that the correlation of two disjoint increments  $\mathbb{E}(X_{s, s+h}^1 X_{t, t+h}^1)$  decays at a rate of  $|t - s|^{-\theta}$ , where  $h, s, t$  are such that  $t > s + h$ . In terms of the covariance function, Condition (ii) implies the relation

$$R\binom{s \ s+h}{t \ t+h} \leq Ch^2/|t - s|^\theta \tag{5.3}$$

for  $|t - s|$  large. Examples of Gaussian processes satisfying Hypothesis 5.1 include (sums of) multi-dimensional fBMs with Hurst parameters  $\nu \in (\frac{1}{4}, \frac{3}{4})$ . The readers are referred to [14] for a discussion on the properties of this type of Gaussian processes.



Let us now define some parameters that will appear in the limit of (5.2). Namely, denote by  $X_{st}^{2,kl}$  the  $(k, l)$ th element of the matrix  $X^2$  for  $k, l \in \mathbb{N}$ , and set:

$$\lambda_{kl}^n = n^{4\nu} \mathbb{E}(X_{t_k t_{k+1}}^{2,12} X_{t_l t_{l+1}}^{2,12}), \quad \rho_{kl}^n = n^{4\nu} \mathbb{E}(X_{t_k t_{k+1}}^{2,12} X_{t_l t_{l+1}}^{2,21}). \tag{5.4}$$

We will need the following hypothesis:

**Hypothesis 5.3.** *Let  $\rho_{kl}, \lambda_{kl}$  be the sequences defined by (5.4). We assume that the following limit holds true:*

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k,l=0}^{n-1} \lambda_{kl}^n, \quad \rho = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k,l=0}^{n-1} \rho_{kl}^n, \tag{5.5}$$

where  $\rho$  and  $\lambda$  are finite constant. (This type of assumption also appears in [2], for instance.)

**Remark 5.4.** It is readily checked that Hypothesis 5.3 is satisfied for a 2-dimensional enhanced fractional Brownian motion with Hurst parameter  $\nu \in (\frac{1}{4}, \frac{1}{2})$ , and  $\lambda$  and  $\rho$  can be computed explicitly (see [30, Proposition 9.1], with  $\lambda$  and  $\rho$  respectively replaced by  $P$  and  $Q$ ).

With those preliminaries in hand, let us state the main result of this subsection.

**Proposition 5.5.** *Let  $X = (1, X^1, X^2, X^3)$  be the enhanced Gaussian process above the  $d$ -dimensional Gaussian process  $X^1$ . Suppose that Hypothesis 5.1 and Hypothesis 5.3 holds. Set*

$$h_{st}^n = \sum_{k=\lceil ns \rceil}^{\lfloor nt \rfloor - 1} \left( n^{2\nu} X_{t_k t_{k+1}}^2 - \frac{1}{2} Id \right) \tag{5.6}$$

for  $t \geq \frac{1}{n}$  and  $h_t^n = 0$  for  $t < \frac{1}{n}$ . Then the finite dimensional distributions of  $(n^{-\frac{1}{2}} h^n, X)$  converge weakly to those of  $(W, X)$ , where  $W = (W^{ij})$  is an  $m \times m$ -dimensional Brownian motion, independent of  $X$ , such that

$$\mathbb{E}[W_t^{ij} W_s^{i'j'}] = (\lambda \delta_{ii'} \delta_{jj'} + \rho \delta_{ij'} \delta_{ji'}) (t \wedge s). \tag{5.7}$$

In formula (5.7), we have set  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  if  $i \neq j$ . Furthermore, the quantities  $\rho$  and  $\lambda$  are defined by relation (5.5).

We will state and prove several intermediate results, and then prove Proposition 5.5 at the end of the subsection. The first of these lemmas concerns covariances of  $X^2$ , for which we introduce some additional notation. Namely, we consider the specific case when  $d = 2$ , and analyze the weak convergence of the two processes  $z^n$  and  $\tilde{z}^n$  defined by:

$$z_t^n = n^{2\nu} \sum_{k=0}^{\lfloor nt \rfloor} X_{t_k t_{k+1}}^{2,12}, \quad \text{and} \quad \tilde{z}_t^n = n^{2\nu} \sum_{k=0}^{\lfloor nt \rfloor} X_{t_k t_{k+1}}^{2,21}. \tag{5.8}$$

We first prove a lemma on the moment convergence of  $z^n, \tilde{z}^n$ .

**Lemma 5.6.** *Let  $X^{1,1}$  and  $X^{1,2}$  be two independent real-valued (incremental) Gaussian processes. We set the 2-dimensional process  $X^1 = (X^{1,1}, X^{1,2})$  and consider the rough path  $\mathbf{X}$  above  $X^1$ , as in Proposition 5.5. Let  $z^n$  and  $\tilde{z}^n$  be defined in (5.8). Suppose that Hypothesis 5.1 and Hypothesis 5.3 hold for  $X^1$ . Then the following limits hold true:*

$$\lim_{n \rightarrow \infty} n^{-1} \mathbb{E}[|z_t^n|^2] = \lambda t \quad \text{and} \quad \lim_{n \rightarrow \infty} n^{-1} \mathbb{E}[z_t^n \tilde{z}_t^n] = \rho t, \tag{5.9}$$

where  $\lambda$  and  $\rho$  are defined in (5.5).

**Proof.** First, by the definition of  $z^n$  and  $\lambda_{kl}^n$  it is readily checked that:

$$\mathbb{E}(|z_t^n|^2) = \sum_{k,l=0}^{\lfloor nt \rfloor} \lambda_{kl}^n. \tag{5.10}$$

Therefore, we can write

$$\frac{1}{n} \mathbb{E}(|z_t^n|^2) = t \frac{\lfloor nt \rfloor}{nt} \frac{1}{\lfloor nt \rfloor} \sum_{k,l=0}^{\lfloor nt \rfloor} \lambda_{kl}^n.$$

Sending  $n \rightarrow \infty$  we obtain the first point in (5.9) thanks to relation (5.5). In the same way, we can show the convergence of  $n^{-1} \mathbb{E}[z_t^n \bar{z}_t^n]$ . The proof is complete.  $\square$

In order to get our central limit theorem for the process  $z^n$ , we will apply a corollary of the fourth moment theorem. This relies on Malliavin calculus tools, for which we first introduce some basic notations.

**Notation 5.7.** We define the Hilbert space  $\mathcal{H}$  as the completion of indicator functions with respect to the inner product  $\langle \mathbf{1}_{[s,t]}, \mathbf{1}_{[u,v]} \rangle_{\mathcal{H}} = \mathbb{E}(X_{st}^{1,1} X_{uv}^{1,1})$ , where  $s, t, u, v \in [0, 1]$ . Denote by  $\mathfrak{H} := \{h : h(\cdot, i) \in \mathcal{H}, i = 1, 2\}$  the Hilbert space defined by the following inner product:

$$\langle h, \bar{h} \rangle_{\mathfrak{H}} = \langle h(\cdot, 1), \bar{h}(\cdot, 1) \rangle_{\mathcal{H}} + \langle h(\cdot, 2), \bar{h}(\cdot, 2) \rangle_{\mathcal{H}}. \tag{5.11}$$

Then, it is readily checked that the Gaussian family  $\{W(h) = \int h(\cdot, 1) \delta X^{1,1} + \int h(\cdot, 2) \delta X^{1,2} : h \in \mathfrak{H}\}$  is an isonormal Gaussian process, where we recall that  $(X^{1,1}, X^{1,2})$  is our couple of independent Gaussian process and where  $\int f \delta X^{1,1}$  stands for the Wiener integral. The random variable  $W(h)$  is called the (first-order) Wiener integral of  $h$  with respect to  $(X^{1,1}, X^{1,2})$  and is also denoted by  $I_1(h)$ .

The operator  $I_1$  can be generalized to  $\mathfrak{H}^{\otimes k}$ . Indeed, for  $h = \sum_{j=1}^n f_j \otimes g_j$ , where  $f_j \in \mathfrak{H}$  and  $g_j \in \mathfrak{H}^{\otimes(k-1)}$ , we set  $I_1(h) = \sum_{j=1}^n I_1(f_j)g_j$ . Since vectors in the form of  $h$  are dense in  $\mathfrak{H}^{\otimes k}$ , we see that  $I_1$  can be extended to a bounded operator from  $\mathfrak{H}^{\otimes k}$  into  $L^2(\Omega, \mathfrak{H}^{\otimes(k-1)})$ . The reader is referred to Page 35 in [37] for details on this construction.

Denote by  $I_k$  the  $k$ th iteration of the integration operator  $I_1$ , namely,  $I_k = I_1 \circ \dots \circ I_1$ . For  $h \in \mathfrak{H}^{\otimes q}$ ,  $I_q(h)$  is called the  $q$ th-order Wiener integral of  $h$ .

**Example 5.8.** Since  $X^{1,1}$  and  $X^{1,2}$  are independent, for  $t \in [0, 1]$  the random variable  $z_t^n$  can be represented as a 2nd-order Wiener integral. Indeed, define  $\phi^n \in \mathfrak{H}^{\otimes 2}$  as follows:

$$\begin{cases} \phi^n((u, 2), (s, 1)) = n^{2v} \sum_{k=0}^{\lfloor nt \rfloor} \mathbf{1}_{t_k \leq u \leq s \leq t_{k+1}} \\ \phi^n((u, i), (s, j)) = 0 \end{cases} \quad \text{for } (i, j) \neq (2, 1). \tag{5.12}$$

We also denote by  $\tilde{\phi}^n$  the symmetrization of  $\phi^n$ , that is,

$$\tilde{\phi}^n((u, i), (s, j)) = \frac{1}{2} (\phi^n((u, i), (s, j)) + \phi^n((s, j), (u, i))). \tag{5.13}$$

Then it is easily checked (see e.g. [13] and Page 23 in [40]) that

$$z_t^n = I_2(\phi^n) = I_2(\tilde{\phi}^n). \tag{5.14}$$

Now that we have expressed  $z_t^n$  as a multiple Wiener integral, we can use the 4th moment theorem in order to study its limiting law. We thus recall the following result borrowed from Theorem 5.2.7 in [37]:

**Proposition 5.9.** Fix  $q \geq 1$ . Let  $\{z^n = I_q(f_n) = I_q(\tilde{f}_n); n \geq 1\}$  be a sequence of centered random variables belonging to the  $q$ th chaos of  $X^1 = (X^{1,1}, X^{1,2})$ , where  $\tilde{f}_n$  denotes the symmetrization of  $f_n$  in  $\mathfrak{H}^{\otimes q}$ . Assume that

$$\lim_{n \rightarrow \infty} \mathbb{E}[|z^n|^2] = 1.$$

Then  $z^n$  converges in distribution to a centered Gaussian random variable if and only if the following condition is met:

$$\lim_{n \rightarrow \infty} \|\tilde{f}_n \otimes_r \tilde{f}_n\|_{\mathfrak{H}^{\otimes(2q-2r)}} = 0, \quad \text{for all } r = 1, \dots, q - 1.$$

The reader is referred to [37, Appendix B.4] for the definition of the contraction  $\tilde{f}_n \otimes_r \tilde{f}_n$ .

In view of Proposition 5.9, Lemma 5.6 and Example 5.8, we are reduced to the analysis of the contraction  $\|\tilde{\phi} \otimes_1 \tilde{\phi}\|_{\mathcal{H}^{\otimes 2}}$  in order to get our central limit theorem for  $z^n$ , where  $\tilde{\phi}^n$  is defined by (5.13). This is what is done in the next lemma.

**Lemma 5.10.** *Let the assumptions of Lemma 5.6 prevail, and consider  $z_t^n = I_2(\tilde{\phi}^n)$  defined by (5.14). Then we have the convergence*

$$\lim_{n \rightarrow \infty} n^{-2} \|\tilde{\phi}^n \otimes_1 \tilde{\phi}^n\|_{\mathfrak{H}^{\otimes 2}}^2 = 0. \tag{5.15}$$

**Proof.** We will divide the proof in several steps. We denote  $e = \|\tilde{\phi}^n \otimes_1 \tilde{\phi}^n\|_{\mathfrak{H}^{\otimes 2}}^2$ .

*Step 1:* An expression for  $e$ . Owing to relation (5.11) for the inner product in  $\mathfrak{H}$ , we have

$$\tilde{\phi}^n \otimes_1 \tilde{\phi}^n = \varphi_1^n + \varphi_2^n, \tag{5.16}$$

where

$$\varphi_1^n((c, 2), (d, 2)) = \tilde{\phi}^n((c, 2), (a, 1)) \otimes_1 \tilde{\phi}^n((d, 2), (a, 1))$$

and

$$\varphi_2^n((c, 1), (d, 1)) = \tilde{\phi}^n((c, 1), (a, 2)) \otimes_1 \tilde{\phi}^n((d, 1), (a, 2)),$$

and the other terms of  $\varphi_1^n$  and  $\varphi_2^n$  are null. Here the letter  $a$  designates the pairing for our inner product in  $\mathcal{H}$ . Moreover, owing to the definition (5.12) of  $\phi^n$  and (5.13) of  $\tilde{\phi}^n$  one can check that:

$$\begin{aligned} \varphi_1^n((c, 2), (d, 2)) &= \frac{1}{4} \phi^n((c, 2), (a, 1)) \otimes_1 \phi^n((d, 2), (a, 1)) \\ \varphi_2^n((c, 1), (d, 1)) &= \frac{1}{4} \phi^n((a, 2), (c, 1)) \otimes_1 \phi^n((a, 2), (d, 1)). \end{aligned} \tag{5.17}$$

Taking the operation  $\|\cdot\|_{\mathcal{H}^{\otimes 2}}$  on both sides of (5.16) and taking into account the expressions of  $\varphi_1$  and  $\varphi_2$  we obtain

$$\begin{aligned} e &= \frac{1}{16} \|\varphi_1^n((c, 2), (d, 2))\|_{\mathcal{H}^{\otimes 2}}^2 + \frac{1}{16} \|\varphi_2^n((c, 1), (d, 1))\|_{\mathcal{H}^{\otimes 2}}^2 \\ &= \frac{1}{8} \|\varphi_1^n((c, 2), (d, 2))\|_{\mathcal{H}^{\otimes 2}}^2. \end{aligned} \tag{5.18}$$

Here the letters  $c, d$  designate the pairing for our inner products in  $\mathcal{H}^{\otimes 2}$ .

We now decompose the term  $\varphi_1^n$  in (5.18). To this aim, denote  $\phi_k^n(u, s) = n^{2\nu} \mathbf{1}_{t_k \leq u \leq s \leq t_{k+1}}$ . Then by the definition (5.12) of  $\phi^n$  we have

$$\phi^n((u, 2), (s, 1)) = \sum_{k=0}^{\lfloor nt \rfloor} \phi_k^n(u, s).$$

Plugging this formula into the expression (5.17) of  $\varphi_1^n$  we obtain:

$$\varphi_1^n((c, 2), (d, 2)) = \frac{1}{4} \sum_{k, k'=0}^{\lfloor nt \rfloor} \phi_k^n(c, a) \otimes_1 \phi_{k'}^n(d, a), \tag{5.19}$$

where we recall that  $a$  is the letter used for the pairing in  $\mathcal{H}$ . Next we compute the  $\mathcal{H}^{\otimes 2}$ -norm of  $\varphi_1^n$  thanks to relation (5.19). Taking into account formula (5.18), this yields:

$$e = \frac{1}{128} \sum_{(k_1, k_2, k_3, k_4) \in M} c(k_1, k_2, k_3, k_4), \tag{5.20}$$

where we denote

$$c(k_1, k_2, k_3, k_4) = \langle \phi_{k_1}^n(d, a) \otimes_1 \phi_{k_4}^n(c, a), \phi_{k_3}^n(d, b) \otimes_1 \phi_{k_2}^n(c, b) \rangle_{\mathcal{H}^{\otimes 2}}, \tag{5.21}$$

and where  $M$  is the set of indices  $M = \{0, 1, \dots, \lfloor nt \rfloor\}^4$ .

*Step 2: Decomposition of e.* We will now split the summation in (5.20) according to convenient subsets of  $M$ . We thus introduce an additional notation, valid for all subsets  $M' \subset M$ :

$$e(M') = \frac{1}{128} \sum_{(k_1, k_2, k_3, k_4) \in M'} c(k_1, k_2, k_3, k_4). \tag{5.22}$$

Next for  $i = 0, \dots, 4$  we define the following subsets of indices:

$$M_i = \{(k_1, k_2, k_3, k_4) \in M : \text{exactly } i \text{ of the pairs } (j, j') \in \mathcal{P} \text{ satisfy } |k_j - k_{j'}| \leq 2\},$$

where we denote  $\mathcal{P} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ . Then we can decompose  $e$  as:

$$e = \sum_{i=0}^4 e(M_i). \tag{5.23}$$

So to prove (5.15), we are now reduced to show that  $n^{-2}e(M_i)$  tends to 0 for  $i = 0, \dots, 4$ .

*Step 3: Computations for  $e(M_1)$ .* Let us approximate the functions  $n^{2\nu} \mathbf{1}_{t_k \leq u < s \leq t_{k+1}}$  in the definition of  $\phi_k^n$  by sums of indicators of rectangles. Namely, for  $k \leq \lfloor nt \rfloor$  we set

$$\phi_k^{n, \ell}(u, s) = n^{2\nu} \sum_{i=0}^{\ell-1} \mathbf{1}_{[t_k, t_k + \frac{i}{n\ell}]}(u) \times \mathbf{1}_{[t_k + \frac{i}{n\ell}, t_k + \frac{i+1}{n\ell}]}(s) \tag{5.24}$$

$$= n^{2\nu} \sum_{i=0}^{\ell-1} \mathbf{1}_{[t_k + \frac{i}{n\ell}, t_k + \frac{i+1}{n\ell}]}(u) \times \mathbf{1}_{[t_k + \frac{i+1}{n\ell}, t_{k+1}]}(s). \tag{5.25}$$

Note that we have  $\phi_k^n \in \mathcal{H}^{\otimes 2}$  and the following approximation result holds true:

$$\lim_{\ell \rightarrow \infty} \|\phi_k^n - \phi_k^{n, \ell}\|_{\mathcal{H}^{\otimes 2}} = 0. \tag{5.26}$$

We now compute  $e(M_{11})$  for a given subset  $M_{11} \subset M_1$ . Namely, denote

$$I_{ij} = \{(k_1, k_2, k_3, k_4) \in M : |k_i - k_j| > 2\} \tag{5.27}$$

and set  $M_{11} = I_{13} \cap I_{14} \cap I_{23} \cap I_{24}^c$ . It is clear that  $M_{11} \subset M_1$ . Now consider  $(k_1, k_2, k_3, k_4) \in M_{11}$ . By (5.26), the expression (5.24) of  $\phi_k^{n, \ell}$  and (5.21) we have

$$\begin{aligned} |c(k_1, k_2, k_3, k_4)| &= \lim_{\ell \rightarrow \infty} \lim_{\ell' \rightarrow \infty} \left| \langle \phi_{k_1}^n(d, a) \otimes \phi_{k_4}^{n, \ell}(c, a), \phi_{k_3}^n(d, b) \otimes \phi_{k_2}^{n, \ell'}(c, b) \rangle_{\mathcal{H}^{\otimes 2}} \right| \\ &\leq n^{2\nu} \lim_{\ell \rightarrow \infty} \lim_{\ell' \rightarrow \infty} \sum_{i=1}^{\ell-1} \sum_{i'=1}^{\ell'-1} |\tilde{c}(i, i')| \left| \langle \mathbf{1}_{[t_{k_4}, t_{k_4} + \frac{i}{n\ell}]}(c), \mathbf{1}_{[t_{k_2}, t_{k_2} + \frac{i'}{n\ell'}]}(c) \rangle_{\mathcal{H}} \right|, \end{aligned}$$

where

$$\tilde{c}(i, i') = n^{2\nu} \langle \phi_{k_1}^n(d, a) \otimes \mathbf{1}_{[t_{k_4} + \frac{i}{n\ell}, t_{k_4} + \frac{i'+1}{n\ell}]}(a), \mathbf{1}_{[t_{k_2} + \frac{i}{n\ell'}, t_{k_2} + \frac{i'+1}{n\ell'}]}(b) \otimes \phi_{k_3}^n(d, b) \rangle_{\mathcal{H}}.$$

Notice that, thanks to Cauchy–Schwarz inequality, for all  $i, i' \leq \ell - 1$  we have

$$\left| \langle n^{2\nu} \mathbf{1}_{[t_{k_4}, t_{k_4} + \frac{i}{n\ell}], \mathbf{1}_{[t_{k_2}, t_{k_2} + \frac{i'}{n\ell'}]} \rangle_{\mathcal{H}} \right| \leq 1. \tag{5.28}$$

We thus get

$$|c(k_1, k_2, k_3, k_4)| \leq \lim_{\ell \rightarrow \infty} \lim_{\ell' \rightarrow \infty} \sum_{i=1}^{\ell-1} \sum_{i'=1}^{\ell'-1} |\tilde{c}(i, i')|.$$

In order to evaluate  $\tilde{c}(i, i')$ , observe that, thanks to Hypothesis 5.1(i) and the fact that  $(k_1, k_2, k_3, k_4) \in M_{11}$ , the quantities  $\tilde{c}(i, i')$  have the same sign for all  $i, i' = 1, \dots, \ell - 1$ . Denoting  $\psi_k = \mathbf{1}_{[k, k+1]}$  and  $\phi_k = \mathbf{1}_{k \leq u \leq s \leq k+1}$ , we thus get

$$\begin{aligned} |c(k_1, k_2, k_3, k_4)| &\leq \lim_{\ell \rightarrow \infty} \lim_{\ell' \rightarrow \infty} \left| \sum_{i=1}^{\ell-1} \sum_{i'=1}^{\ell'-1} \tilde{c}(i, i') \right| \\ &= \left| \langle \phi_{k_1}(d, a) \otimes_1 \psi_{k_4}(a), \psi_{k_2}(b) \otimes_1 \phi_{k_3}(d, b) \rangle_{\mathcal{H}} \right| \\ &\leq \left| \langle \psi_{k_1}(d) \psi_{k_1}(a), \psi_{k_4}(a) \rangle_{\mathcal{H}} \cdot \langle \psi_{k_2}(b), \psi_{k_3}(d) \psi_{k_3}(b) \rangle_{\mathcal{H}} \right| \\ &= |\langle \psi_{k_4}, \psi_{k_1} \rangle_{\mathcal{H}}| \cdot |\langle \psi_{k_3}, \psi_{k_1} \rangle_{\mathcal{H}}| \cdot |\langle \psi_{k_2}, \psi_{k_3} \rangle_{\mathcal{H}}|. \end{aligned} \tag{5.29}$$

In (5.29), notice that we can replace the simplex indicator  $\phi_k(u, s)$  by  $\psi_k^n \otimes \psi_k^n(s, u) = \mathbf{1}_{[k, k+1]^2}(s, u)$  due to the fact that each of the three pairs  $(k_1, k_4)$ ,  $(k_2, k_3)$ , and  $(k_1, k_3)$  are disjoint and also Hypothesis 5.1(i). Furthermore, applying (5.3) to relation (5.29) with  $u = 1$  we obtain

$$|c(k_1, k_2, k_3, k_4)| \leq K |k_1 - k_4|^{-\theta} |k_1 - k_3|^{-\theta} |k_2 - k_3|^{-\theta}, \tag{5.30}$$

where  $\frac{1}{2} < \theta \leq 2 - 2\nu$ . Applying this estimate to (5.22) with  $M' = M_{11}$  we obtain

$$e(M_{11}) \leq K \sum_{(k_1, k_2, k_3, k_4) \in M_{11}} |k_1 - k_4|^{-\theta} |k_1 - k_3|^{-\theta} |k_2 - k_3|^{-\theta}.$$

It is now easy to show from this estimate that

$$n^{-2} e(M_{11}) \leq K n^{-\theta} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \tag{5.31}$$

which is our desired estimate for  $e(M_{11})$ .

In order to conclude for the term  $e(M_1)$ , set

$$M_{12} = I_{13} \cap I_{14} \cap I_{23}^c \cap I_{24}, \quad M_{13} = I_{13} \cap I_{14}^c \cap I_{23} \cap I_{24}, \quad M_{14} = I_{13}^c \cap I_{14} \cap I_{23} \cap I_{24}.$$

Similarly to what we have done above, we can show that the convergence (5.31) still holds when  $M_{11}$  is replaced by  $M_{1i}$ , for  $i = 2, 3, 4$ . Noticing that  $M_1 = \bigcup_{i=1}^4 M_{1i}$ , we conclude that

$$\lim_{n \rightarrow \infty} n^{-2} e(M_1) = 0.$$

*Step 4: Computations for  $e(M_2)$  – Part 1.* As in the case of  $M_1$ , we will decompose  $e(M_2)$  in several terms and analyze them individually. To start with, set  $M_{21} = I_{13}^c \cap I_{14} \cap I_{23} \cap I_{24}^c$ , where we recall that  $I_{ij}$  is defined by (5.27). Along the same lines as the proof of Step 3, for  $(k_1, k_2, k_3, k_4) \in M_{21}$  we can show that

$$|c(k_1, k_2, k_3, k_4)| \leq K |k_2 - k_3|^{-\theta} |k_1 - k_4|^{-\theta} \leq K |k_2 - k_3|^{-2\theta}, \tag{5.32}$$

where the last relation stems from the fact that  $|k_1 - k_3| \leq 2$  and  $|k_2 - k_4| \leq 2$ . Let us highlight the following difference between the term  $e(M_{21})$  and  $e(M_{11})$ : in order to handle  $e(M_{21})$ , since now both  $|k_1 - k_3|$  and  $|k_2 - k_4|$  are smaller than 3, we need to apply the approximation (5.24) for each of  $\phi_{k_1}^n, \phi_{k_2}^n, \phi_{k_3}^n$  and  $\phi_{k_4}^n$ . Then applying relation (5.32) to (5.22) with  $M = M_{21}$  and invoking the fact that  $\#M_{21} = O(n^2)$  and  $\sum_{j \geq 1} |j|^{-2\theta} < \infty$ , we obtain

$$n^{-2} e(M_{21}) \leq K n^{-1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In a similar way we can show that this convergence still holds for  $M_{22} := I_{13} \cap I_{14}^c \cap I_{23}^c \cap I_{24}$ .

*Step 5: Computations for  $e(M_2)$ -Part 2.* We now deal with a slightly different kind of term involved in  $e(M_2)$ . Namely, set  $M_{23} = I_{13}^c \cap I_{14} \cap I_{23}^c \cap I_{24}$  and take  $(k_1, k_2, k_3, k_4) \in M_{23}$ . Owing to relation (5.26) we have

$$|c(k_1, k_2, k_3, k_4)| = \lim_{\ell \rightarrow \infty} \lim_{\ell' \rightarrow \infty} \left| \langle \phi_{k_1}^{n, \ell}(d, a) \otimes_1 \phi_{k_4}^n(c, a), \phi_{k_2}^{n, \ell'}(c, b) \otimes_1 \phi_{k_3}^n(d, b) \rangle_{\mathcal{H} \otimes 2} \right|.$$

We now use expression (5.24) for  $\phi_{k_1}^{n,\ell}$  and expression (5.25) for  $\phi_{k_2}^{n,\ell'}$ . This yields:

$$|c(k_1, k_2, k_3, k_4)| \leq n^{2\nu} \lim_{\ell \rightarrow \infty} \lim_{\ell' \rightarrow \infty} \sum_{i=0}^{\ell-1} \sum_{i'=0}^{\ell'-1} |\hat{c}(i, i')| \cdot \left| \langle \mathbf{1}_{[t_{k_1}, t_{k_1} + \frac{i'}{n\ell}]}(d) \mathbf{1}_{[t_{k_2} + \frac{i+1}{n\ell'}, t_{k_2+1}]}(b), \phi_{k_3}^n(d, b) \rangle_{\mathcal{H}^{\otimes 2}} \right|, \quad (5.33)$$

where

$$\hat{c}(i, i') = n^{2\nu} \langle \langle \mathbf{1}_{[t_{k_1} + \frac{i'}{n\ell}, t_{k_1} + \frac{i'+1}{n\ell}]}(a), \phi_{k_4}^n(c, a) \rangle_{\mathcal{H}}, \mathbf{1}_{[t_{k_2} + \frac{i}{n\ell'}, t_{k_2} + \frac{i+1}{n\ell'}]}(c) \rangle_{\mathcal{H}}.$$

We now observe two facts:

- (i) Since  $|k_1 - k_4| > 2$  and  $|k_2 - k_4| > 2$ , and resorting to Hypothesis 5.1(i), we have  $\hat{c}(i, i') \geq 0$  for all  $i = 1, \dots, \ell - 1, i' = 1, \dots, \ell' - 1$ .
- (ii) Similarly to (5.28), we can apply Cauchy–Schwarz inequality in order to get

$$\left| \langle n^{2\nu} \mathbf{1}_{[t_{k_1}, t_{k_1} + \frac{i'}{n\ell}]}(d) \mathbf{1}_{[t_{k_2} + \frac{i+1}{n\ell'}, t_{k_2+1}]}(b), \phi_{k_3}^n(d, b) \rangle_{\mathcal{H}^{\otimes 2}} \right| \leq 1.$$

Plugging this information into (5.33) we obtain

$$|c(k_1, k_2, k_3, k_4)| \leq \lim_{\ell \rightarrow \infty} \lim_{\ell' \rightarrow \infty} \left| \sum_{i=0}^{\ell-1} \sum_{i'=0}^{\ell'-1} \hat{c}(i, i') \right| = \left| \langle \phi_{k_1}(a), \phi_{k_4}(c, a) \rangle_{\mathcal{H}}, \phi_{k_2}(c) \rangle_{\mathcal{H}} \right|.$$

We can now proceed by enlarging the simplex  $\{t_k \leq u \leq s \leq t_{k+1}\}$  to a rectangle  $[t_k, t_{k+1}]^2$  as in (5.29), and using the bound (5.3) as in (5.30). We end up with:

$$|c(k_1, k_2, k_3, k_4)| \leq K |k_2 - k_4|^{-\theta} |k_1 - k_4|^{-\theta}.$$

It is now easy to show by this estimate, expression (5.22), and the fact that  $|k_i - k_j| \leq 4$  for  $i, j \in \{1, 2, 3\}$  that

$$n^{-2}e(M_{23}) \leq Kn^{-1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We can easily extend the considerations above in order to get a similar convergence for  $e(M_{2i}), i = 4, 5, 6$ , where

$$M_{24} = I_{13}^c \cap I_{14}^c \cap I_{23} \cap I_{24}, \quad M_{25} = I_{13} \cap I_{14}^c \cap I_{23} \cap I_{24}^c, \quad M_{26} = I_{13} \cap I_{14} \cap I_{23}^c \cap I_{24}^c.$$

In summary of Step 4 and 5 and noticing that  $M_2 = \bigcup_{i=1}^6 M_{2i}$ , we obtain the convergence:

$$\lim_{n \rightarrow \infty} n^{-2}e(M_2) = \sum_{i=1}^6 \lim_{n \rightarrow \infty} n^{-2}e(M_{2i}) = 0.$$

*Step 6: Computations for  $e(M_0)$ .* Take now  $(k_1, k_2, k_3, k_4) \in M_0$ . Then as before, by assumption (5.3) we obtain

$$|c(k_1, k_2, k_3, k_4)| \leq K |k_1 - k_4|^{-\theta} |k_1 - k_3|^{-\theta} |k_2 - k_3|^{-\theta} |k_2 - k_4|^{-\theta}.$$

It is easy to show from this estimate and expression (5.22) that

$$n^{-2}e(M_0) \leq Kn^{1-2\theta} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

*Step 7: Computations for  $e(M_3 \cup M_4)$ .* Finally, we consider the case when  $(k_1, k_2, k_3, k_4) \in M_3 \cup M_4$ . In order to get an estimate for  $e(M_3 \cup M_4)$ , we first note that  $\#(M_3 \cup M_4) \leq 19n$ . On the other hand, a simple application of Cauchy–Schwarz inequality yields the relation  $|c(k_1, k_2, k_3, k_4)| \leq 1$  for all  $(k_1, k_2, k_3, k_4) \in M_3 \cup M_4$ . Therefore, we obtain

$$n^{-2}e(M_3 \cup M_4) \leq Kn^{-1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Gathering the estimates we have obtained in Steps 3 to 7 and recalling the decomposition (5.23), the proof of our claim (5.15) is now complete. □

**Proof of Proposition 5.5.** According to the fourth moment method applied to the second chaos  $\mathfrak{H}^{\otimes 2}$  introduced in Notation 5.7, we are reduced to show the following facts:

(i) For any  $L \geq 1$ , the covariance matrix of

$$(n^{-\frac{1}{2}}(h_{0r_1}^n, \dots, h_{0r_L}^n), X_{r_1}^1, \dots, X_{r_L}^1)$$

converges to that of

$$((W_{r_1}, \dots, W_{r_L}), X_{r_1}^1, \dots, X_{r_L}^1).$$

(ii) The following weak convergence holds true for all  $i, j = 1, \dots, m, l = 1, \dots, L$ :

$$n^{-\frac{1}{2}}h_{0r_l}^{n,ij} \Rightarrow W_{r_l}^{ij}.$$

Note that we have recalled the fourth moment method for 1-d sequences of random variables in Proposition 5.9. We refer to [37] for more details about the fourth moment method for random vectors in a fixed chaos, and we now focus on the proof of item (i) and (ii).

The weak convergence (ii) of  $h_{0r_l}^{n,ij}$  for  $i \neq j$  follows immediately from Lemma 5.10 and Proposition 5.9. In the case when  $i = j$ , (ii) follows from the classical results in [5], see also Section 7.4 in [37]. In the following, we show the convergence of the covariance  $\mathbb{E}(h_{0r_l}^{n,ij} h_{0r_{l'}}^{n,i'j'})$ .

We start by studying  $\mathbb{E}(h_{0r_l}^{n,ij} h_{0r_{l'}}^{n,i'j'})$  when  $r_l = r_{l'}$ . In this case, whenever  $(i, j) = (i', j')$  or  $(i, j) = (j', i')$ , the convergence of  $\mathbb{E}(h_{0r_l}^{n,ij} h_{0r_{l'}}^{n,i'j'})$  follows from Lemma 5.6. In the case  $(i, j) \neq (i', j')$  and  $(i, j) \neq (j', i')$ , the covariance  $\mathbb{E}(h_{0r_l}^{n,ij} h_{0r_{l'}}^{n,i'j'})$  is simply equal to 0.

Let us now assume that  $r_l > r_{l'}$ . Since  $\mathbb{E}(h_{0r_l}^{n,ij} h_{0r_{l'}}^{n,i'j'}) = \frac{1}{2}(\mathbb{E}(h_{0r_l}^{n,ij} h_{0r_{l'}}^{n,i'j'}) + \mathbb{E}(h_{0r_{l'}}^{n,ij} h_{0r_l}^{n,i'j'}))$ , we can reduce this case to the previous study by invoking the following identity:

$$\mathbb{E}(h_{0r_l}^{n,ij} h_{0r_{l'}}^{n,i'j'}) = \frac{1}{2}(\mathbb{E}[h_{0r_l}^{n,ij} h_{0r_l}^{n,i'j'}] + \mathbb{E}[h_{0r_{l'}}^{n,ij} h_{0r_{l'}}^{n,i'j'}] - \mathbb{E}[\delta h_{r_{l'}r_l}^{n,ij} \delta h_{r_{l'}r_l}^{n,i'j'}]). \tag{5.34}$$

Then thanks to Lemma 5.6, the first two terms on the right-hand side of (5.34) converge to  $(\lambda \delta_{ii'} \delta_{jj'} + \rho \delta_{ij'} \delta_{ji'})r_l$  and  $(\lambda \delta_{ii'} \delta_{jj'} + \rho \delta_{ij'} \delta_{ji'})r_{l'}$ . In order to treat the term  $\mathbb{E}[\delta h_{r_{l'}r_l}^{n,ij} \delta h_{r_{l'}r_l}^{n,i'j'}]$ , note that  $\delta h_{r_{l'}r_l}^{n,ij} \delta h_{r_{l'}r_l}^{n,i'j'}$  is equal to  $h_{[r_l]-[r_{l'}]}^{n,ij} h_{[r_l]-[r_{l'}]}^{n,i'j'}$  in distribution, where recall that  $[r_l]$  and  $[r_{l'}]$  denote respectively the integer part of  $r_l$  and  $r_{l'}$ . So by Lemma 5.6 the third term converges to  $(\lambda \delta_{ii'} \delta_{jj'} + \rho \delta_{ij'} \delta_{ji'})(r_l - r_{l'})$ . Summarizing our last considerations, we easily get:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(h_{r_l}^{ij} h_{r_{l'}}^{i'j'}) = (\lambda \delta_{ii'} \delta_{jj'} + \rho \delta_{ij'} \delta_{ji'})r_{l'}.$$

The proof is complete. □

### 5.3. Weighted limit theorem

Let  $X$  be the enhanced Gaussian process defined as in Section 5.1. With the preparation in the previous subsection, we now consider the convergence of the discrete integral  $n^{-\frac{1}{2}}\mathcal{J}_0^1(y; h^n)$  with  $h^n$  defined in (5.6), where  $(y, y', \dots, y^{(\ell-1)})$  is a discrete process controlled by  $(X, \alpha)$ .

Let us recall some basic facts about the range of our parameters. First, the covariance  $R$  satisfies (5.1), and we consider a parameter  $\nu = \frac{1}{2\rho}$ . Since we assume that  $\rho = [1, 2)$ , we also have  $\nu \in (\frac{1}{4}, \frac{1}{2}]$ . Then the coefficient  $\alpha$  is dictated by the regularity type estimate (5.9), namely  $\alpha = \frac{1}{2}$ . Eventually the order  $\ell$  of the controlled process  $y$  is such that  $\nu\ell + \frac{1}{2} > 1$ , which yields  $\ell = 2$  in our setting.

We start by giving some uniform bounds on  $\mathcal{J}(X^1; h^n)$ .

**Lemma 5.11.** *The following relations holds true:*

$$n^{-\frac{1}{2}}|\mathcal{J}_s^t(X^1; h^n)|_{L_2} \leq K(t-s)^{\nu+\frac{1}{2}}, \quad n^{-\frac{1}{2}}\left|\sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(X^1; h^n)\right|_{L_2} \leq Km^{-\nu}. \tag{5.35}$$

**Proof.** The estimate (5.35) is obtained in a similar way as in those in Lemma 4.5. We just observe that the non diagonal terms of the matrix  $h^n$  will be handled by approximating the indicator function of the simplex by indicator functions of rectangles, similarly to what we did in the proof of Lemma 5.10. The details are omitted. □



**Theorem 5.12.** Let  $X$  and  $h^n$  be as in Proposition 5.5, with  $v \in (\frac{1}{4}, \frac{1}{2}]$ . Let  $y$  be a controlled process of order  $\ell = 2$ . Then the following stable f.d.d. convergence holds true:

$$n^{-\frac{1}{2}} \mathcal{J}(y; h^n) \rightarrow \int y_r \otimes dW_r, \quad (5.36)$$

where  $W$  is the Wiener process introduced in Proposition 5.5, and where the integral  $\int_s^t y_r \otimes dW_r$  has to be understood in the Wiener sense.

**Proof.** In order to show the convergence (5.36) we invoke Theorem 3.8. We first note that inequality (2.13) holds true thanks to the first relation in (5.35). Furthermore, the convergence of  $(X, n^{-\frac{1}{2}}h^n)$  follows from Proposition 5.5. Finally, relation (3.13) is a consequence of the second relation in (5.35). Therefore, applying Theorem 3.8 we obtain the desired convergence (5.36).  $\square$

## References

- [1] D. J. Aldous and G. K. Eagleson. On mixing and stability of limit theorems. *Ann. Probab.* **6** (2) (1978) 325–331. MR0517416 <https://doi.org/10.1214/aop/1176995577>
- [2] M. Arcones. Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors. *Ann. Probab.* **22** (4) (1994) 2242–2274. MR1331224
- [3] O. Barndorff-Nielsen, J. Corcuera and M. Podolskij. Power variation for Gaussian processes with stationary increments. *Stochastic Process. Appl.* **119** (2009) 1845–1865. MR2519347 <https://doi.org/10.1016/j.spa.2008.09.004>
- [4] G. Binotto, I. Nourdin and D. Nualart. Weak symmetric integrals with respect to the fractional Brownian motion. *Ann. Probab.* **46** (4) (2018) 2243–2267. MR3813991 <https://doi.org/10.1214/17-AOP1227>
- [5] P. Breuer and P. Major. Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.* **13** (3) (1983) 425–441. MR0716933 [https://doi.org/10.1016/0047-259X\(83\)90019-2](https://doi.org/10.1016/0047-259X(83)90019-2)
- [6] K. Burdzy and J. Swanson. A change of variable formula with Itô correction term. *Ann. Probab.* **38** (5) (2010) 1817–1869. MR2722787 <https://doi.org/10.1214/09-AOP523>
- [7] T. Cass, M. Hairer, C. Litterer and S. Tindel. Smoothness of the density for solutions to Gaussian Rough Differential Equations. *Ann. Probab.* **43** (1) (2015) 188–239. MR3298472 <https://doi.org/10.1214/13-AOP896>
- [8] P. Cheridito and D. Nualart. Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter  $H \in (0, \frac{1}{2})$ . *Ann. Inst. Henri Poincaré Probab. Stat.* **41** (6) (2005) 1049–1081. MR2172209 <https://doi.org/10.1016/j.anihpb.2004.09.004>
- [9] J. M. Corcuera, D. Nualart and M. Podolskij. Asymptotics of weighted random sums. *Commun. Appl. Ind. Math.* **6** (1) (2014) e486. MR3277312 <https://doi.org/10.1685/journal.caim.486>
- [10] J. M. Corcuera, D. Nualart and J. Woerner. Power variation of some integral fractional processes. *Bernoulli* **12** (4) (2006) 713–735. MR2248234 <https://doi.org/10.3150/bj/1155735933>
- [11] R. Davis and S. Resnick. Limit theory for moving averages of random variables with regularly varying tail probabilities. *Ann. Probab.* **13** (1) (1985) 179–195. MR0770636
- [12] R. L. Dobrushin and P. Major. Non-central limit theorems for nonlinear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete* **50** (1) (1979) 27–52. MR0550122 <https://doi.org/10.1007/BF00535673>
- [13] A. Ferreiro-Castilla and F. Utzet. Lévy area for Gaussian processes: A double Wiener–Itô integral approach. *Statist. Probab. Lett.* **81** (9) (2011) 1380–1391. MR2811853 <https://doi.org/10.1016/j.spl.2011.04.015>
- [14] P. K. Friz, B. Gess, A. Gulisashvili and S. Riedel. The Jain–Monrad criterion for rough paths and applications to random Fourier series and non-Markovian Hörmander theory. *Ann. Probab.* **44** (1) (2016) 684–738. MR3456349 <https://doi.org/10.1214/14-AOP986>
- [15] P. K. Friz and M. Hairer. *A Course on Rough Paths: With an Introduction to Regularity Structures*. Springer, Berlin, 2014. MR3289027 <https://doi.org/10.1007/978-3-319-08332-2>
- [16] P. K. Friz and N. B. Victoir. *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications*, **120**. Cambridge University Press, Cambridge, 2010. MR2604669 <https://doi.org/10.1017/CBO9780511845079>
- [17] M. Gradinaru and I. Nourdin. Milstein’s type schemes for fractional SDEs. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** (4) (2009) 1085–1098. MR2572165 <https://doi.org/10.1214/08-AIHP196>
- [18] M. Gradinaru, I. Nourdin, F. Russo and P. Vallois.  $m$ -order integrals and Itô’s formula for non-semimartingale processes; the case of a fractional Brownian motion with any Hurst index. *Ann. Inst. Henri Poincaré Probab. Stat.* **41** (2005) 781–806. MR2144234 <https://doi.org/10.1016/j.anihpb.2004.06.002>
- [19] M. Gradinaru, F. Russo and P. Vallois. Generalized covariations, local time and Stratonovich Itô’s formula for fractional Brownian motion with Hurst index  $H \geq \frac{1}{4}$ . *Ann. Probab.* **31** (2001) 1772–1820. MR2016600 <https://doi.org/10.1214/aop/1068646366>
- [20] M. Gubinelli. Controlling rough paths. *J. Funct. Anal.* **216** (1) (2004) 86–140. MR2091358 <https://doi.org/10.1016/j.jfa.2004.01.002>
- [21] D. Harnett and D. Nualart. Weak convergence of the Stratonovich integral with respect to a class of Gaussian processes. *Stochastic Process. Appl.* **122** (10) (2012) 3460–3505. MR2956113 <https://doi.org/10.1016/j.spa.2012.06.008>
- [22] D. Harnett and D. Nualart. Central limit theorem for a Stratonovich integral with Malliavin calculus. *Ann. Probab.* **41** (4) (2013) 2820–2879. MR3112933 <https://doi.org/10.1214/12-AOP769>
- [23] D. Harnett and D. Nualart. On Simpson’s rule and fractional Brownian motion with  $H = 1/10$ . *J. Theoret. Probab.* **28** (4) (2013) 1651–1688. MR3422945 <https://doi.org/10.1007/s10959-014-0552-1>
- [24] Y. Hu. *Analysis on Gaussian Spaces*. World Scientific, Hackensack, NJ, 2017. MR3585910



- [25] Y. Hu, Y. Liu and D. Nualart. Rate of convergence and asymptotic error distribution of Euler approximation schemes for fractional diffusions. *Ann. Appl. Probab.* **26** (2) (2016) 1147–1207. MR3476635 <https://doi.org/10.1214/15-AAP1114>
- [26] J. Jacod and A. Shiryaev. *Limit Theorems for Stochastic Processes*, **288**, 2nd edition. Springer, Berlin, 2002. MR1943877 <https://doi.org/10.1007/978-3-662-05265-5>
- [27] A. Jakubowski, J. Mémin and G. Pagès. Convergence in law of sequences of stochastic integrals on the Skorokhod space  $\mathbf{D}^1$ . *Probab. Theory Related Fields* **81** (1) (1989) 111–137. MR0981569 <https://doi.org/10.1007/BF00343739>
- [28] J. Lamperti. Semi-stable stochastic processes. *Trans. Amer. Math. Soc.* **104** (1962) 62–78. MR0138128 <https://doi.org/10.2307/1993933>
- [29] J. León and C. Ludeña. Limits for weighted  $p$ -variations and likewise functionals of fractional diffusions with drift. *Stochastic Process. Appl.* **117** (3) (2007) 271–296. MR2290877 <https://doi.org/10.1016/j.spa.2006.05.016>
- [30] Y. Liu and S. Tindel. First-order Euler scheme for SDEs driven by fractional Brownian motions: The rough case. *Ann. Appl. Probab.* **29** (2) (2019) 758–826. MR3910017 <https://doi.org/10.1214/17-AAP1374>
- [31] A. Neuenkirch, S. Tindel and J. Unterberger. Discretizing the fractional Lévy area. *Stochastic Process. Appl.* **120** (2) (2010) 223–254. MR2576888 <https://doi.org/10.1016/j.spa.2009.10.007>
- [32] R. Norvaisa. Weighted power variation of integrals with respect to a Gaussian process. *Bernoulli* **21** (2) (2015) 1260–1288. MR3338664 <https://doi.org/10.3150/14-BEJ606>
- [33] I. Nourdin. Asymptotic behavior of some weighted quadratic and cubic variations of the fractional Brownian motion. *Ann. Probab.* **36** (2008) 2159–2175. MR2478679 <https://doi.org/10.1214/07-AOP385>
- [34] I. Nourdin and D. Nualart. Central limit theorems for multiple Skorohod integrals. *J. Theoret. Probab.* **23** (1) (2010) 39–64. MR2591903 <https://doi.org/10.1007/s10959-009-0258-y>
- [35] I. Nourdin, D. Nualart and G. Peccati. Quantitative stable limit theorems on the Wiener space. *Ann. Probab.* **44** (1) (2016) 1–41. MR3456331 <https://doi.org/10.1214/14-AOP965>
- [36] I. Nourdin, D. Nualart and C. Tudor. Central and non-central limit theorems for weighted power variations of fractional Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** (4) (2010) 1055–1079. MR2744886 <https://doi.org/10.1214/09-AIHP342>
- [37] I. Nourdin and G. Peccati. *Normal Approximations with Malliavin Calculus: From Stein's Method to Universality*, **192**. Cambridge University Press, Cambridge, 2012. MR2962301 <https://doi.org/10.1017/CBO9781139084659>
- [38] I. Nourdin and A. Réveillac. Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: The critical case  $H = 1/4$ . *Ann. Probab.* **37** (6) (2008) 2200–2230. MR2573556 <https://doi.org/10.1214/09-AOP473>
- [39] I. Nourdin, A. Réveillac and J. Swanson. The weak Stratonovich integral with respect to fractional Brownian motion with Hurst parameter  $1/6$ . *Electron. J. Probab.* **15** (70) (2010) 2117–2162. MR2745728 <https://doi.org/10.1214/EJP.v15-843>
- [40] D. Nualart. *The Malliavin Calculus and Related Topics*, 2nd edition. Springer, Berlin, 2006. MR2200233
- [41] M. Rosenblatt. A central limit theorem and a strong mixing condition. *Proc. Natl. Acad. Sci. USA* **42** (1956) 43–47. MR0074711 <https://doi.org/10.1073/pnas.42.1.43>
- [42] M. Taqqu. Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete* **50** (1979) 53–83. MR0550123 <https://doi.org/10.1007/BF00535674>
- [43] C. A. Tudor. *Analysis of Variations for Self-Similar Processes. A Stochastic Calculus Approach. Probability and Its Applications*. Springer, Cham, 2013. MR3112799 <https://doi.org/10.1007/978-3-319-00936-0>